Things I Have Forgotten Since College

This is meant to summarize most everything I learned in college, down to a couple pages.

Calculus

$\frac{d}{dx}[f] = \frac{df}{dx} = f'$. If x is time, then $f' = \dot{f}$.

$\frac{d}{dx}[x^b] = bx^{b-1}$ $\frac{d}{dx}[f(x) * g(x)] = f'g + fg'$ $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'g - fg'}{g^2}$ $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

$\frac{d}{dx}[a^{bx}] = a^{bx} \ln a \cdot b$ $\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$ $\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[e^{ax}] = ae^{x}$ $\int e^{ax} dx = \frac{1}{a}e^{ax} + C$ $\frac{d}{dx}[\ln x] = \frac{1}{x}$ $\int \frac{1}{x} dx = \ln |x| + C$

$\int_b^a f(x)\, dx = [F(x)]_b^a = F(a) - F(b)$ $\int x^b\, dx = \frac{1}{b+1}x^{b+1} + C$

Integration by Parts

Get a hard equation into one you know how to integrate, by playing games with canceling. Find a u that you can use to rewrite the integral into a form you'll be able to solve, then rewrite using that, and sub x back in at the end. Starting with $\int xe^{x^2} dx$.

$$u = x^2, du = 2xdx, x = \sqrt{u}, dx = \frac{1}{2\sqrt{u}}du : \int xe^{x^2} dx = \int \sqrt{u}e^u \frac{1}{2\sqrt{u}}du = \frac{1}{2}\int e^u du = \frac{e^u}{2} + C = \frac{e^{x^2}}{2} + C$$

Vector Calculus

This is calculus when applied to vectors and matricies. Basic multivariable calculus just requires doing the same math as for one variable, but repeated. This type of calculus uses the interactions between dimensions that can arise.

Cross Product

$$
\begin{aligned}
\vec{a} \times \vec{b} &= a_1 b_2 \vec{e}_3 - a_1 b_3 \vec{e}_2 - a_2 b_1 \vec{e}_3 + a_2 b_3 \vec{e}_1 + a_3 b_1 \vec{e}_2 - a_3 b_2 \vec{e}_1 \\
&= (a_2 b_3 - a_3 b_2)\vec{e}_1 + (a_3 b_1 - a_1 b_3)\vec{e}_2 + (a_1 b_2 - a_2 b_1)\vec{e}_3
\end{aligned}
$$

Scalar Triple Product

$$
\begin{aligned}
\vec{a} \cdot \vec{b} \times \vec{c} &= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1) \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \vec{b} \cdot \vec{c} \times \vec{a} = \vec{c} \cdot \vec{a} \times \vec{b}
\end{aligned}
$$

That is, the scalar triple product is a particular form, whose expansion happens to be the determinant of the three vectors involved. Notice that the first line is simply the three pieces of the cross product, each times the corresponding piece of $\vec{a}$.

The magnitude of the cross product of two vectors is the area of the parallelogram they define. Similarly, the scalar triple product is the volume of the parallelepiped the three vectors define.

There is also a **Vector Triple Product**, $\vec{a} \times \vec{b} \times \vec{c}$.

Fields

A function of position is a *field*, whether scalar or vector.

Line Integrals

A line integral is the integral of a field along a path. With a simple function, this could give the length of a line or curve.

Given a particle in a field which exerts force based on position, a line integral gives the work needed to make the particle move along a path.

$$\text{Work} = -\text{Force} \cdot \text{distance}$$

The negative is to account for the force from the field acting one way, and the work that has to be input to move the particle being the other.

$$W = \sum_{i=1}^{N} \vec{F}(r_i) \cdot \Delta r_i, \text{ or in limit: } \int_{t=a}^{t=b} \vec{F}(\vec{r}) \cdot d\vec{r}$$

Example

Path: $x = t, y = t, z = 2t^2, 0 \leq t \leq 1$
Force: $F = (y, x, z)$

$$
\begin{aligned}
W &= \int_C (y, x, z) \cdot d\vec{r}, \qquad \vec{r} = (t, t, 2t^2) \\
&= \int_C (y, x, z) \cdot \frac{d\vec{r}}{dt} dt, \qquad \frac{d\vec{r}}{dt} = (1, 1, 4t) \\
&= \int_0^1 (t, t, 2t^2) \cdot (1, 1, 4t) dt = \int_0^1 (2t + 8t^3) dt = 3
\end{aligned}
$$

- Conservative vector field: integral around closed loop = 0
- Surface Integrals, like Line but More D
- Volume Integrals, see a pattern?
- Gradients, Divergence, Curl: The Good Stuff
- Laplacian (of a twice-differentiable scalar field)
- Suffix Notation (Einstein notation)
- Kronecker Delta "substitution tensor"
- Alternating/Antisymmetric Tensor
- Expressing cross products with these
- Gradiant, Divergence, and Curl in Suffix Notation
- Also Laplacian
- Some useful identities
- Divergance Theorem
- Polar Coordinate Systems
- Using Scalars in Suffix Notation
- Tensors, Quotient Rule
- Symmetric and Anti-Symmetric Tensors
- Isotropic Tensors
- Ohm's Law
- Inertia Tensor
- Maxwell's Equations

Differential Equations

Ordinary differential equation (ODE) only uses x, y(x), y'(x). Trying to get standard $y = f(x)$ form. Look at start, see what form you can get into.

y' = X(x)Y(y)

$$y' = X(x)Y(y) = \frac{dy}{dx}$$

$$\frac{1}{Y(y)}dy = X(x)dx$$

$$\int \frac{1}{Y(y)}dy = \int X(x)dx$$

, then solve for y.

Example

$$y' = \frac{4y}{x}$$

$$X(x) = \frac{1}{x}, Y(y) = 4y$$

$$\int \frac{1}{4y}dy = \int \frac{1}{x}dx$$

$$\frac{1}{4}\ln|y| = \ln|x| + C$$

$$\ln|y| = 4\ln|x| + C$$

$$y = Cx^4$$

y' + p(x)y = q(x)

Make up a useful u(x), multiply both sides with it and pull a fast one with the multiplication rule, only divide out at the end.

$$u(x) = e^{\int p(x)dx}, \frac{du}{dx} = p(x)u(x)$$

(because of the e^x)

$$u(x)(y' + p(x)y) = u(x)q(x)$$

$$(u(x)y)' = u(x)q(x)$$

(because $(fg)' = f'g * fg'$)

$$\int (u(x)y)'dx = \int u(x)q(x)dx$$

$$y = \frac{1}{u(x)} \int u(x)q(x)dx$$

Example

$$y' - 2xy = x$$

$$p(x) = -2x, q(x) = 1, u(x) = e^{\int -2xdx} = Ce^{-x^2}$$

$$y = Ce^{x^2} \int Cxe^{-x^2}dx$$

use integration by parts

$$y = Ce^{x^2}(-\frac{1}{2}e^{-x^2} + C)$$

$$y = Ce^{x^2} - \frac{1}{2}$$

Second Order ODEs

$y'' + p(x)y' + q(x)y = g(x)$ is easiest. If $g(x) = 0$, it's "homogeneous" and is even easier. If $p(x)$ and $q(x)$ are constants, then y'' can also have a constant factor, and the solution is $y = C_1e^x + C_2e^{-x}$. If there's no y term, set $u = y', u' = y''$ to make a 1st-order ODE.

Partial Differential Equations

Assume PDE's solution will have separated variables, so $f(x, y) = X(x)Y(y)$. Feed that assumption through the given equation, to try to get functions of any variable x together with the corresponding dx. Set the equation equal to a constant. Now you can break the equation into two differential equations in just one variable each.