Homework 1 – STAT 642

Due Thursday, January 21 (in class);

You can also turn it in to our TA during his office hour on January 22.

1. Let $\Omega = (0, 1]$ and suppose $\mathcal{F}$ consists of all finite unions of disjoint intervals of the form $(a, a'], 0 \leq a \leq a' \leq 1$. Show that $\mathcal{F}$ is an algebra. Show that $\mathcal{F}$ is NOT a $\sigma-$algebra.

2. Let Ω be a non-empty set and let $\mathcal{L}$ be a class of subsets of Ω. If $\mathcal{L}$ is a $\lambda-$class and also a $\pi-$class, show that $\mathcal{L}$ is a σ-algebra.

   [Hint: To verify $\mathcal{L}$ is a σ-algebra, the obstacle is to show that, if sets $A_1, A_2, ... \in \mathcal{L}$ (where the A_i's may not be disjoint), then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$. Starting from $A_1, A_2, ... \in \mathcal{L}$, construct disjoint $B_1, B_2 \ldots, \in \mathcal{L}$ where $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.]

3. Problem 2.1 Athreya & Lahiri text

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function on a measurable space $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

   (a) Using a set $A \subset \mathbb{R}$, define an indicator function $f(x) = \mathbb{I}_A(x) \equiv \mathbb{I}(x \in A)$, $x \in \mathbb{R}$. Show f is a Borel measurable function (i.e., $\langle \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}) \rangle$-measurable) if and only if A is Borel measurable (i.e., $A \in \mathcal{B}(\mathbb{R})$).

   (b) Show that if f is increasing ($f(y) \leq f(x)$ if $y \leq x$) then f is Borel measurable.

5. On a measure space $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, where m denotes the Lebesgue measure, define a sequence $f_n : \mathbb{R} \to \mathbb{R}$ of nonnegative measurable functions by

   $$f_n(x) = \begin{cases} 1/n & \text{if } x \in [n, 2n] \\ 0 & \text{otherwise.} \end{cases} \qquad x \in \mathbb{R}.$$

   Find $\lim_{n \to \infty} f_n(x) = f(x)$, for each $x \in \mathbb{R}$, and show that

   $$\lim_{n \to \infty} \int f_n(x) dx \neq \int f(x) dx.$$

   This implies that changing the order of integration and limits is generally not valid, unless other conditions are in place (e.g., MCT, DCT).

6. On a measure space $(\Omega, \mathcal{F}, \mu)$, suppose $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable and μ-integrable function. Show that, given $\epsilon > 0$, there is a simple function ψ on $\mathbb{R}$ such that

   $$\int |f - \psi| d\mu < \epsilon$$

   [Consider f^+ and f^- separately; you should have also seen (somewhere) that if h is a nonnegative measurable function, there exist nonnegative simple functions ϕ_n where $\phi_n(\omega) \uparrow h(\omega)$ for each point $\omega \in \Omega$.]

7. Let P and Q denote two probability measures defined on measurable space $(\Omega, \mathcal{F}, \mu)$. Show that

   $$\sup_{A \in \mathcal{F}} |P(A) - Q(A)| = \frac{1}{2} \int |p - q| d\mu$$

where p and q denote densities for measures P and Q, respectively, with respect to some dominating measure μ, e.g., $P(A) = \int_A p d\mu$ for $A \in \mathcal{F}$. (The form of μ isn't important but $\mu = P + Q$ is one example of a dominating measure for both P and Q.)

[Hint: For $f = p - q$, write $f = f^+ - f^-$ where $|f| = f^+ + f^-$; all functions here are μ-integrable because p, q are. Use that, for any $A \in \mathcal{F}$, $|P(A) - Q(A)| = |\int_A f d\mu| \leq \max\{\int_A f^+ d\mu, \int_A f^-\} \leq \max\{\int f^+ d\mu, \int f^-\}$ along with $\int(f^+ - f^-)d\mu = \int f d\mu = 0$ (why?) and $\{\omega \in \Omega : f^+(\omega) > 0\}$ is a measurable set.]

8. Suppose X_n, $n \geq 0$, are random variables on a psp $(\Omega, \mathcal{F}, \mu)$ (i.e, measurable $X_n : \Omega \to \mathbb{R}$) such that $X_n \to X_0$ a.s.(P). For any integer $m \geq 1$, an expected value in our notation is

$$\mathrm{E}(X_n^m) = \int X_n^m dP = \int [X_n(\omega)]^m dP(\omega).$$

If $\sup_{n\geq 1} \mathrm{E}(X_n^2) < \infty$ show that $\mathrm{E}(X_n) \to \mathrm{E}(X_0)$.

[Hint: It's a short proof upon using uniform integrability.]