Spring 2016 Math 511 Assignment 2 Solution

1. Evaluate $\oint_\gamma \dfrac{\overline{z}}{8+z}\,dz$, where γ is the rectangle with vertices $\pm 3 \pm i$ with counter-clockwise direction.

Solution: Let $\gamma_1(t) = t - i,\ -3 \le t \le 3,\ \gamma_2(t) = 3 + ti,\ -1 \le t \le 1,\ \gamma_3(t) = -t + i,\ -3 \le t \le 3,\ \gamma_4(t) = -3 - ti,\ -1 \le t \le 1.$

$$\int_{\gamma_1} \frac{\overline{z}}{8+z}\,dz = \int_{-3}^{3} \frac{t+i}{8+t-i}\,dt = \int_{-3}^{3} \frac{8+t-i-8+2i}{8+t-i}\,dt$$

$$= \int_{-3}^{3} 1 - \frac{8-2i}{8+t-i}\,dt = 6 - (8-2i)\int_{-3}^{3} \frac{8+t+i}{(8+t)^2+1}\,dt$$

$$= 6 - (8-2i)\left(\frac{1}{2}\log\left((8+t)^2+1\right) + i\tan^{-1}(8+t)\Big]_{-3}^{3}\right)$$

$$= 6 - (8-2i)\left(\frac{1}{2}\log\left(\frac{61}{13}\right) + i\left(\tan^{-1}(11) - \tan^{-1}(5)\right)\right)$$

Similarly,

$$\int_{\gamma_3} \frac{\overline{z}}{8+z}\,dz = -6 + (8+2i)\left(\frac{1}{2}\log\left(\frac{61}{13}\right) + i\left(\tan^{-1}(5) - \tan^{-1}(11)\right)\right)$$

$$\int_{\gamma_2} \frac{\overline{z}}{8+z}\,dz = 2i\left(-1 + 7\pi - 14\tan^{-1}(11)\right),\ \int_{\gamma_4} \frac{\overline{z}}{8+z}\,dz = 2i\left(1 - \pi + 2\tan^{-1}(5)\right)$$

$$\Rightarrow \int_{\gamma} \frac{\overline{z}}{8+z}\,dz = 2i\left(\log\left(\frac{61}{13}\right) + 6\pi + 10\tan^{-1}(5) - 22\tan^{-1}(11)\right)$$

2. Let $f(x+iy) = x^3y^2 + i\,x^2y^3$. Find all the points P in $\mathbf{C}$ where $f'(P)$ exists.

Solution: Since $u(x,y) = x^3y^2$ and $v(x,y) = x^2y^3$ are polynomials in $x,\ y$, they are in C^∞. So, for the existence of $f'(P)$, we only need to check the Cauchy-Riemann equations.

Since $u_x = 3x^2y^2 = v_y$, $u_y = 2x^3y$ and $v_x = 2xy^3$, $f'(P)$ exists $\Leftrightarrow u_y = -v_x$ $\Leftrightarrow 2x^3y = -2xy^3 \Leftrightarrow 2x^3y + 2xy^3 = 0 \Leftrightarrow 2xy(x^2+y^2) = 0 \Leftrightarrow x = 0$ or $y = 0$.

3. Let $U \subset \mathbf{C}$ be an open set and $f \in C^1(U)$. Let $z_0 \in U$.

   (a) Suppose $|D_{e^{i\theta_1}} f(z_0)| = |D_{e^{i\theta_2}} f(z_0)|$ for all $\theta_1,\ \theta_2$. Prove that either $f'(z_0)$ or $(\overline{f})'(z_0)$ exists. (Note: $(\overline{f})' \ne \overline{(f')}$.)

   (b) Suppose $D_{e^{i\theta_1}} f(z_0)\overline{D_{e^{i\theta_2}} f(z)} = |D_{e^{i\theta_1}} f(z_0) D_{e^{i\theta_2}} f(z_0)| e^{i(\theta_1-\theta_2)}$ for all $\theta_1,\ \theta_2$. Prove that $f'(z_0)$ exists.

Solution: Given $\theta \in \mathbf{R}$, let $\gamma(t) = z_0 + te^{i\theta}$ for $t \in [0,\ 1]$. Then

$$\begin{aligned} D_{e^{i\theta}} f(z_0) &= f_z(z_0)e^{i\theta} + f_{\overline{z}}(z_0)e^{-i\theta} && \text{(I)} \\ &= (u_x(z_0)\cos\theta + u_y(z_0)\sin\theta) + i\,(v_x(z_0)\cos\theta + v_y(z_0)\sin\theta) && \text{(II)} \end{aligned}$$

(a) Suppose $|D_{e^{i\theta_1}} f(z_0)| = |D_{e^{i\theta_2}} f(z_0)|$ for all $\theta_1,\ \theta_2$.

We will use (I) for the proof of this part. Then

$$|f_z(z_0)e^{i\theta_1} + f_{\overline{z}}(z_0)e^{-i\theta_1}| = |f_z(z_0)e^{i\theta_2} + f_{\overline{z}}(z_0)e^{-i\theta_2}| \text{ for all } \theta_1,\ \theta_2\,.$$

Let $f_z(z_0) = |f_z(z_0)|e^{i\phi}$ and $f_{\overline{z}}(z_0) = |f_{\overline{z}}(z_0)|e^{i\psi}$. Choose $\theta_1 = \dfrac{\psi - \phi}{2}$ and $\theta_2 = \dfrac{\psi - \phi - \pi}{2}$. Then we have

$$\begin{aligned} & |f_z(z_0)e^{i\theta_1} + f_{\overline{z}}(z_0)e^{-i\theta_1}| = |f_z(z_0)e^{i\theta_2} + f_{\overline{z}}(z_0)e^{-i\theta_2}| \\ \Rightarrow\ & ||f_z(z_0)|e^{i(\phi+\theta_1)} + |f_{\overline{z}}(z_0)|e^{i(\psi-\theta_1)}| = ||f_z(z_0)|e^{i(\phi+\theta_2)} + |f_{\overline{z}}(z_0)|e^{i(\psi-\theta_2)}| \\ \Rightarrow\ & ||f_z(z_0)| + |f_{\overline{z}}(z_0)|e^{i(\psi-\phi-2\theta_1)}| = ||f_z(z_0)| + |f_{\overline{z}}(z_0)|e^{i(\psi-\phi-2\theta_2)}| \\ \Rightarrow\ & ||f_z(z_0)| + |f_{\overline{z}}(z_0)|| = ||f_z(z_0)| - |f_{\overline{z}}(z_0)|| \\ \Rightarrow\ & |f_z(z_0)| + |f_{\overline{z}}(z_0)| = |f_z(z_0)| - |f_{\overline{z}}(z_0)| \text{ or } |f_{\overline{z}}(z_0)| - |f_z(z_0)| \\ \Rightarrow\ & |f_{\overline{z}}(z_0)| \text{ or } |f_z(z_0)| = 0 \\ \Rightarrow\ & f_{\overline{z}}(z_0) \text{ or } f_z(z_0) = 0 \\ \Rightarrow\ & \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x}\right)(u+iv)(z_0) = 0 \text{ or } \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial x}\right)(u+iv)(z_0) = 0 \\ \Rightarrow\ & (u_x(z_0) = v_y(z_0) \text{ and } u_y(z_0) = -v_x(z_0)) \text{ or } (u_x(z_0) = -v_y(z_0) \text{ and } u_y(z_0) = v_x(z_0)) \\ \Rightarrow\ & f'(z_0) \text{ or } (\overline{f})'(z_0) \text{ exists.} \end{aligned}$$

(b)Suppose $D_{e^{i\theta_1}} f(z)\overline{D_{e^{i\theta_2}} f(z)} = |D_{e^{i\theta_1}} f(z) D_{e^{i\theta_2}} f(z)|e^{i(\theta_1-\theta_2)}$ for all $\theta_1,\ \theta_2$.

We will use (II) for the proof of this part. Then for every $\theta \in \mathbf{R}$, we have

$$D_{e^{i\theta}} f(z_0) = (u_x(z_0)\cos\theta + u_y(z_0)\sin\theta) + i\,(v_x(z_0)\cos\theta + v_y(z_0)\sin\theta)$$

For convenience of notation, we will write $u_x,\ u_y,\ v_x$ and v_y for $u_x(z_0),\ u_y(z_0),\ v_x(z_0)$ and $v_y(z_0)$.

Condition (b) is equivalent to

$$D_{e^{i\theta_1}} f(z)\overline{D_{e^{i\theta_2}} f(z)} e^{i(\theta_2-\theta_1)} \in \mathbf{R}$$

$$\Leftrightarrow \quad \operatorname{Im}\left(D_{e^{i\theta_1}} f(z)\overline{D_{e^{i\theta_2}} f(z)} e^{i(\theta_2-\theta_1)}\right) = 0$$

$$\begin{aligned}\Leftrightarrow \sin(\theta_2-\theta_1)\Big((u_x^2+v_x^2)\cos\theta_1\cos\theta_2 + (u_y^2+v_y^2)\sin\theta_1\sin\theta_2 \\ +(u_xu_y+v_xv_y)\sin(\theta_1+\theta_2)+(v_xu_y-v_yu_x)\cos(\theta_1-\theta_2)\Big)=0\end{aligned} \tag{1}$$

for all $\theta_1,\ \theta_2 \in \mathbf{R}$.

Putting different values for θ_1 and θ_2 into (1), we have the following

(a) $\theta_1 = 0,\ \theta_2 = \frac{\pi}{4}$, we have

$$u_x^2+v_x^2+(u_xu_y+v_xv_y)+(v_xu_y-v_yu_x)=0 \tag{2}$$

(b) $\theta_1 = \frac{\pi}{4},\ \theta_2 = \frac{\pi}{2}$, we have

$$u_y^2+v_y^2+(u_xu_y+v_xv_y)+(v_xu_y-v_yu_x)=0 \tag{3}$$

(c) $\theta_1 = \frac{\pi}{2},\ \theta_2 = 0$, we have

$$u_xu_y+v_xv_y=0 \tag{4}$$

(2) + (3) - 2(4) gives

$$(u_x-v_y)^2+(u_y+v_x)^2=0$$

$$\Rightarrow \quad u_x = v_y \quad \text{and } u_y = -v_x$$

Since $f \in C^1(U)$, by Lemma 1.4.2, f is holomorphic on U.

4. Let u be a real-valued C^1 function on an open disc U with center 0. Assume that u is harmonic on $U \setminus \{0\}$. Prove that u is the real part of a holomorphic function on U.

Solution: u harmonic on $U \setminus \{0\} \Rightarrow u_{xx}+u_{yy}=0 \Rightarrow (-u_y)_y = (u_x)_x$. By Theorem 2.3.2, there exists a C^1 function $v : U \to \mathbf{R}$ such that $v_x = -u_y$ and $v_y = u_x$ on U. Therefore, $u+iv$ is holomorphic on U.

5. Evaluate $\oint_\gamma \frac{\zeta^2+8i}{(\zeta+i)(\zeta-8)}\,d\zeta$, where γ is the circle with center $2+i$ and radius 3 with counter-clockwise direction.

Solution: Since $|-i-(2+i)| = \sqrt{8} < 3$ and $|8-(2+i)| = \sqrt{37} > 3$, we have

$$\oint_\gamma \frac{\zeta^2+8i}{(\zeta+i)(\zeta-8)}\,d\zeta = \oint_\gamma \frac{\frac{\zeta^2+8i}{(\zeta-8)}}{(\zeta+i)}\,d\zeta = 2\pi i\frac{((-i)^2+8i)}{((-i)-8)} = 2\pi$$

6. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ and ψ be a complex continuous function on γ. Show that

$$\overline{\oint_\gamma \psi(z)\, dz} = -\oint_\gamma \overline{\psi(z)\, z^2}\, dz.$$

Solution:

$$-\oint_\gamma \overline{\psi(z)\, z^2}\, dz = -\int_0^{2\pi} \overline{\psi\left(e^{it}\right)\left(e^{it}\right)^2}\, ie^{it}dt = -\int_0^{2\pi} \overline{\psi\left(e^{it}\right)}\left(e^{-2it}\right) ie^{it}dt$$
$$= \int_0^{2\pi} \overline{\psi\left(e^{it}\right)}\left(-ie^{-it}\right) dt = \int_0^{2\pi} \overline{\psi\left(e^{it}\right)}\,\overline{\left(ie^{it}\right)}dt = \overline{\oint_\gamma \psi(z)\, dz}$$

7. Let $f(z) = z^2$. Show that the integral of f around the circle $\partial D(2,1)$ given by

$$\int_0^{2\pi} f\left(2+e^{i\theta}\right)\, d\theta$$

is not zero. Yet the Cauchy integral theorem asserts that

$$\oint_{\partial D(2,1)} f(\zeta)\, d\zeta = 0\,.$$

Give an explanation.

Solution:

$$\int_0^{2\pi} f\left(2+e^{i\theta}\right)\, d\theta = \int_0^{2\pi} \left(2+e^{i\theta}\right)^2\, d\theta = \int_0^{2\pi} 4+4e^{i\theta}+e^{2i\theta}\, d\theta = 4\theta + \frac{4}{i}e^{i\theta} + \frac{1}{2i}e^{2i\theta}\Big]_0^{2\pi} = 8\pi$$

but $\oint_{\partial D(2,1)} f(\zeta)\, d\zeta = 0 = \int_0^{2\pi} \left(2+e^{i\theta}\right)^2 ie^{i\theta}\, d\theta \ne \int_0^{2\pi} \left(2+e^{i\theta}\right)^2\, d\theta.$

8. **a)** Evaluate the integral $\oint_{\partial D(1+i,\,2)}(\overline{z}+1)^2 dz$ directly. **b)** Then evaluate it using the Cauchy integral formula and Cauchy integral theorem.

Solution: a) On $\partial D(1+i,\,2)$, $z = 1+i+2e^{i\theta} \Rightarrow \overline{z} = 1-i+2e^{-i\theta}$
$\Rightarrow (\overline{z}+1)^2 = \left(2-i+2e^{-i\theta}\right)^2$. Also, $dz = 2ie^{i\theta}d\theta$.

$$\oint_{\partial D(1+i,\,2)}(\overline{z}+1)^2 dz = \int_0^{2\pi}\left(2-i+2e^{-i\theta}\right)^2 2ie^{i\theta}d\theta$$
$$= 2i\int_0^{2\pi}(2-i)^2e^{i\theta} + 4(2-i) + 4e^{-i\theta}\, d\theta = 2i(4(2-i))2\pi = 16(1+2i)\pi$$

b) On $\partial D(1+i,\,2)$, $z = 1+i+2e^{i\theta} \Rightarrow e^{i\theta} = \dfrac{z-(1+i)}{2} \Rightarrow e^{-i\theta} = \dfrac{2}{z-(1+i)} \Rightarrow$

$\overline{z} = 1-i+2e^{-i\theta} = 1-i+\dfrac{4}{z-(1+i)} \Rightarrow (\overline{z}+1)^2 = \left(2-i+\dfrac{4}{z-(1+i)}\right)^2.$

$$\oint_{\partial D(1+i,2)}(\overline{z}+1)^2dz = \oint_{\partial D(1+i,2)}\left(2-i+\frac{4}{z-(1+i)}\right)^2 dz$$

$$= \oint_{\partial D(1+i,2)}(2-i)^2 + \frac{8(2-i)}{z-(1+i)} + \left(\frac{4}{z-(1+i)}\right)^2 dz = 8(2-i)(2\pi i) = 16(1+2i)\pi$$

because $(2-i)$ is holomorphic and $\left(\frac{4}{z-(1+i)}\right)^2 = \left[-\frac{4}{z-(1+i)}\right]'$ on $\mathbf{C}\setminus\{0\}$.

9. Let γ be the unit circle equipped with **clockwise** orientation. For each real number λ, give an example of a nonconstant holomorphic function F on the annulus $\{z : \frac{1}{2} < |z| < 2\}$ such that

$$\frac{1}{2\pi i}\oint_\gamma F(z)\,dz = \lambda\,.$$

Solution: Let $F(z) = -\frac{\lambda}{z}$. Then $\frac{1}{2\pi i}\oint_\gamma F(z)\,dz = \frac{1}{2\pi i}\oint_\gamma -\frac{\lambda}{z}\,dz = \lambda\,.$

10. Let γ_1 be the curve $\partial D(0,1)$ and let γ_2 be the curve $\partial D(0,3)$, both equipped with counterclockwise direction. Evaluate

$$\frac{1}{2\pi i}\oint_{\gamma_2}\frac{\zeta^3-3\zeta-6}{\zeta(\zeta+2)(\zeta+4)}\,d\zeta - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{\zeta^3-3\zeta-6}{\zeta(\zeta+2)(\zeta+4)}\,d\zeta$$

Solution: Note that $\gamma_2 \cup (-\gamma_1)$ is the boundary of the region $U = \{z : 1 < |z| < 3\}$ and the function $f(z) = \frac{z^3-3z-6}{z(z+4)}$ is holomorphic on the set $\mathbf{C}\setminus\{0,-4\}$ which contains $\overline{U}$. Therefore,

$$\frac{1}{2\pi i}\oint_{\gamma_2}\frac{\zeta^3-3\zeta-6}{\zeta(\zeta+2)(\zeta+4)}\,d\zeta - \frac{1}{2\pi i}\oint_{\gamma_1}\frac{\zeta^3-3\zeta-6}{\zeta(\zeta+2)(\zeta+4)}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_2\cup(-\gamma_1)}\frac{f(\zeta)}{(\zeta-(-2))}\,d\zeta = f(-2) = 2\,.$$