MATH/STAT 521, AUTUMN 2018
FINAL EXAM, DUE DECEMBER 12

RULES AND REGULATIONS:

(i) You cannot collaborate with any other person.

(ii) You can use results from any published book or research article in a journal. You must include detailed citation if you use a result from any source. You cannot cite materials from the Web. You can search the Web but if you want to use a result, you must find it in a book or research article.

The following rule is different from those applying to the midterm: If you use a result or an argument (proof) from a book or published paper then you have to copy the proof, preferably in your own words, or supply your own proof.

You do not have to supply proofs of those results that were discussed in class or assigned as homework problems. You do not have to supply proofs of those results that can be found in books by Durrett or Mörters-Peres.

(iii) Submit the pdf file by e-mail just like homework. Latex is preferable but scanned handwritten notes are acceptable.

(iv) Final exams will not be posted in Canvas. If I have comments, the pdf's will be returned to students individually.

Definition 1. Suppose that $\{B_t^k, t \geq 0\}$ is a Brownian motion for $k = 1, 2, \dots, d$, where $d \geq 2$. Assume that all these processes are defined on the same probability space and (jointly) independent, that is, if $A_k \in \sigma(B_t^k, t \geq 0)$ for $k = 1, 2, \dots, d$, then events $A_1, A_2, \dots, A_d$ are jointly independent. Let $B_t = (B_t^1, B_t^2, \dots, B_t^d)$. The process $\{B_t, t \geq 0\}$ is called a d-dimensional Brownian motion.

Problem 1. (10 points) Suppose that $\{B_t, t \geq 0\}$ is a two-dimensional Brownian motion. We will use complex notation, that is, (B_t^1, B_t^2) has the same meaning as $B_t^1 + iB_t^2$. Hence, $B_t = B_t^1 + iB_t^2$.

(i) Fix any $\theta \in \mathbb{R}$ and let $X_t = e^{i\theta} B_t$. In other words, X_t is the rotation of B_t by the angle θ. Prove that $\{X_t, t \geq 0\}$ is a two-dimensional Brownian motion. In other words, prove that processes $\{X_t, t \geq 0\}$ and $\{B_t, t \geq 0\}$ have the same distributions.

(ii) Prove the following version of the Law of Iterated Logarithm, for two-dimensional Brownian motion $\{B_t, t \geq 0\}$,

$$\limsup_{t \to \infty} \frac{|B(t)|}{\sqrt{2t \log \log t}} = 1, \qquad \text{a.s.}$$

Problem 2. (10 points) Suppose that $\{B_t, t \geq 0\}$ is a three-dimensional Brownian motion.

(i) Find an explicit formula for the density of B_t (with respect to the three-dimensional Lebesgue measure), for every fixed $t > 0$.

(ii) Find $\alpha > 0$, $c < \infty$ and $\beta < -1$ such that $\mathbb{P}(|B_n| \leq n^\alpha) \leq cn^\beta$ for integer $n \geq 1$.

(iii) Fix $\alpha > 0$. Let $A_n = \{\max_{t\in[0,1]} |B_n - B_{n+t}| \geq n^\alpha/2\}$ for integer $n \geq 1$. Prove that A_n^c occur eventually, a.s.

(iv) Prove that the three-dimensional Brownian motion is transient, that is, with probability 1,

$$\lim_{t\to\infty} |B_t| = \infty.$$

Definition 2. Suppose that $\{B_t, t \geq 0\}$ is a one-dimensional Brownian motion and let $X_t = e^{-t}B(e^{2t})$ for $t \in \mathbb{R}$. The process $\{X_t, t \in \mathbb{R}\}$ is called Ornstein-Uhlenbeck.

Problem 3. (10 points) Suppose that $\{X_t, t \in \mathbb{R}\}$ is an Ornstein-Uhlenbeck process.

(i) Prove that $\{X_t, t \in \mathbb{R}\}$ is a Gaussian process, that is, for any fixed real numbers $-\infty < t_1 < t_2 < \cdots < t_n < \infty$, the vector $(X(t_1), X(t_2), \ldots, X(t_n))$ has a Gaussian distribution. Prove that $\mathbb{E}\, X_t = 0$ for all $t \in \mathbb{R}$. Find the covariance structure of $\{X_t, t \in \mathbb{R}\}$, that is, find $\mathrm{Cov}(X_s, X_t)$ for all $s, t \in \mathbb{R}$ (including the case $s = t$).

(ii) Prove that $\{X_t, t \in \mathbb{R}\}$ is stationary in the following sense. Fix any $s \in \mathbb{R}$ and let $Y_t = X_{s+t}$ for all $t \in \mathbb{R}$. Prove that $\{X_t, t \in \mathbb{R}\}$ and $\{Y_t, t \in \mathbb{R}\}$ have the same distributions.

(iii) Find Fourier coefficients of the Ornstein-Uhlenbeck process, that is, find the distributions of

$$\int_0^{2\pi} \sin(nt)X_t dt, \qquad \int_0^{2\pi} \cos(nt)X_t dt,$$

for all integers $n \geq 0$. Your answer should be simplified, that is, the distributions should be identified by name and their parameters should be presented as explicit simplified expressions.

Hint: If the integrand is continuous, the Lebesgue and Riemann integrals agree. Form a finite Riemann sum. Identify the distribution of the Riemann sum and find its mean and variance. At this point, the formulas do not have to be simplified. Pass to the limit with the Riemann sums, simplify the formulas, and give a rigorous proof that passing to the limit yields the correct answer.

You can use a computer algebra system (such as Maple, Matlab or Mathematica) to evaluate an integral if a difficult integration problem is a part of your solution. If you use such a system, include a copy of your input and output as an appendix to your solutions. State what system you used and indicate in the main body of your solution which results were obtained in this way.