Spring 2016 Math 511 Assignment 3 Solution

1. Suppose for $m, n \in \mathbb{N}$, $|a_{m,n}| \leq 1$. Define

$$K(w, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} w^m z^n.$$

Prove that for every $w \in D(0,\ 1)$, the function $f_w(z) = K(w, z)$ is well-defined, holomorphic on $D(0,\ 1)$.

(Hint: Prove that for every $w \in D(0,\ 1)$ and $m \geq 1$, $\sum_{n=1}^{\infty} a_{m,n} w^m z^n$ converges to a function $f_m(z)$ holomorphic on $D(0, 1)$. Then show that $\sum_{m=1}^{\infty} f_m(z)$ converges to a function $f(z)$ holomorphic on $D(0, 1)$.)

Solution: Let $w \in D(0,\ 1)$ and $m \geq 1$. Then $|w| < 1$. For every $0 < r < 1$ and $z \in \overline{D(0, r)}$, we have $|a_{m,n} w^m z^n| \leq |w|^m r^n \leq r^n$. Since $\sum_{n=1}^{\infty} r^n$ converges, $\sum_{n=1}^{\infty} a_{m,n} w^m z^n$ converges uniformly on $\overline{D(0, r)}$. Therefore, $\sum_{n=1}^{\infty} a_{m,n} w^m z^n$ converges to a function $f_m(z)$ holomorphic on $D(0, 1)$.

For every $0 < r < 1$ and $z \in \overline{D(0, r)}$, we have

$$|f_m(z)| = |\sum_{n=1}^{\infty} a_{m,n} w^m z^n| \leq \sum_{n=1}^{\infty} |a_{m,n} w^m z^n| \leq \sum_{n=1}^{\infty} |w|^m r^n = \frac{r|w|^m}{1 - r}.$$

Since $\sum_{m=1}^{\infty} \frac{r|w|^m}{1 - r} = \frac{r|w|}{(1 - r)(1 - |w|)}$, the series $\sum_{m=1}^{\infty} f_m(z)$ converges uniformly on $\overline{D(0, r)}$. Hence, $\sum_{m=1}^{\infty} f_m(z)$ converges to a function $f(z)$ holomorphic on $D(0, 1)$.

2. Find the power series expansion of the following holomorphic functions about the given point and find the radius of convergence.

(a) $f(z) = \frac{1}{z}$ at $z_0 = 2 - i$.

Solution:

$$\frac{1}{z} = \frac{1}{(2 - i) + (z - (2 - i))} = \frac{1}{(2 - i)} \frac{1}{\left(1 + \frac{z - (2 - i)}{2 - i}\right)}$$

$$= \frac{1}{(2 - i)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - (2 - i)}{2 - i}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(z - (2 - i))^n}{(2 - i)^{n+1}}$$

with radius of convergence $|2 - i| = \sqrt{5}$.

(b) $f(z) = \frac{z - \frac{1}{2}}{1 - \frac{z}{2}}$ at $z_0 = 0$.

Solution:

$$\frac{z - \frac{1}{2}}{1 - \frac{z}{2}} = -2 + \frac{3}{2}\left(\frac{1}{1 - \frac{z}{2}}\right) = -2 + \frac{3}{2}\left(\sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n\right) \text{ for } \left|\frac{z}{2}\right| < 1 \Leftrightarrow |z| < 2$$

3. Suppose that $f : D(0,2) \to \mathbf{C}$ is holomorphic and that $|f(z)| \leq 9$ for all $z \in D(0,2)$. Prove that

$$\left|\frac{\partial^3}{\partial z^3} f\left(\frac{i}{2}\right)\right| \leq 16$$

Solution: For every $0 < r < \dfrac{3}{2}$, $\overline{D\left(\frac{i}{2}, r\right)} \subset D(0,2)$. Therefore, $|f(z)| \leq 9$ for all $z \in \overline{D\left(\frac{i}{2}, r\right)}$. By Theorem 3.4.1, we have

$$\left|\frac{\partial^3}{\partial z^3} f\left(\frac{i}{2}\right)\right| \leq \frac{9(3!)}{r^3} \qquad \text{for all } r < \frac{3}{2}$$

$$\Rightarrow \quad \left|\frac{\partial^3}{\partial z^3} f\left(\frac{i}{2}\right)\right| \leq \frac{9(3!)}{\left(\frac{3}{2}\right)^3} = 16\,.$$

4. Suppose $a_0 \geq a_1 \geq a_2 \geq \cdots a_n \geq \cdots$ and $\lim_{n\to\infty} a_n = 0$. Show that $\sum_{n=0}^{\infty} a_n z^n$ converges for all z, with $|z| = 1$ and $z \neq 1$.

Solution: Suppose $|z| = 1$ and $z \neq 1$. Let $S_n(z) = \sum_{k=0}^{n} z^k = \dfrac{1 - z^{n+1}}{1 - z}$. Then $|S_n(z)| = \dfrac{|1 - z^{n+1}|}{|1 - z|} \leq \dfrac{2}{|1 - z|}$. Given $\epsilon > 0$, choose $N > 1$ such that $|a_n| < \dfrac{|1 - z|\epsilon}{4}$ for all $n \geq N$. Then for all $m > n \geq N$, we have

$$\begin{aligned}
&|\textstyle\sum_{k=n}^{m} a_n z^n| = |\sum_{k=n}^{m} a_n (S_n(z) - S_{n-1}(z))| \\
= \;& \left|a_m S_m(z) + \textstyle\sum_{k=n}^{m-1} (a_n - a_{n+1})\, S_n(z) - a_n S_{n-1}(z)\right| \\
\leq \;& |a_m|\,|S_m(z)| + \textstyle\sum_{k=n}^{m-1} |a_n - a_{n+1}|\,|S_n(z)| + |a_n|\,|S_{n-1}(z)| \\
\leq \;& (a_m + (a_n - a_m) + a_n)\left(\frac{2}{|1 - z|}\right) = \frac{4a_n}{|1 - z|} < \epsilon
\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} a_n z^n$ converges.

5. Determine the radius of convergence of the series $\sum_{k=0}^{\infty} \dfrac{k}{k^2 + 4} z^k$ and the points (including those on the boundary of the disk of convergence) at which the series converge.

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)\left(\frac{k^2 + 4}{(k+1)^2 + 4}\right) = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)\left(\frac{1 + \frac{4}{k^2}}{\left(1 + \frac{1}{k}\right)^2 + \frac{4}{k^2}}\right) = 1$$

Therefore, the radius of convergence is 1. For $k \geq 2$, $\dfrac{k}{k^2 + 4} \geq \dfrac{k}{k^2 + k^2} = \dfrac{1}{2k}$. Since the harmonic series $\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges, the power series $\sum_{k=0}^{\infty} \dfrac{k}{k^2 + 4} z^k$ diverges at $z = 1$.

Let $f(x) = \dfrac{x}{x^2 + 4}$. Then $f'(x) = \dfrac{x^2 + 4 - x(2x)}{(x^2 + 4)^2} = \dfrac{4 - x^2}{(x^2 + 4)^2} < 0$ for $x > 2$. Therefore, $a_2 > a_3 > \cdots$. Also, $\lim_{k\to\infty} a_k = 0$. We can apply the result in 4. So, the set of points on which the series converges is $\overline{D(0,1)} \setminus \{1\}$.

6. Suppose $f \not\equiv 0$ is an entire function such that for some B, $K > 0$, $|f(z)| \leq B|z|^K$ for all $z \in \mathbf{C}$. Prove that K is an integer and $f(z) = Cz^K$ for some $C \in \mathbf{C}$, with $|C| \leq B$. (Hint: Theorem 3.4.4 can be used but is not enough.)

   Solution: Let k be the smallest integer $\geq K$. Then $k-1 < K \leq k$. We have $|f(z)| \leq B|z|^K \leq B|z|^k$ for $|z| \geq 1$. From Theorem 3.4.4, we have $f(z)$ is a non-zero polynomial in z of degree at most k. Let $f(z) = \sum_{n=0}^{k} a_n z^n$. Then for all $z \neq 0$, we have $|f(z)| \leq B|z|^K \Rightarrow \lim_{z\to 0} |f(z)| = 0 \Rightarrow a_0 = 0$. Suppose we have proven $a_0 = \cdots = a_n = 0$ for some $n < k-1$. Then we have

   $$|a_{n+1}| = \lim_{z\to 0} |\sum_{n=0}^{k} \frac{|f(z)|}{|z^{n+1}|} \leq \lim_{z\to 0} B|z|^{K-n-1} = 0 \Rightarrow a_{n+1} = 0$$

   Therefore, $a_0 = \cdots = a_{k-1} = 0$ and $f(z) = a_k z^k$ and $a_k \neq 0$. Hence, $|a_k z^k| \leq B|z|^K \Rightarrow |a_k| \leq B|z|^{(K-k)}$ for all $z \in \mathbf{C}$. Since $k \geq K$ and $a_k \neq 0$, we have $K = k$ and $|a_k| \leq B$.

7. Suppose f is a holomorphic function on $D(0,1)$ such that f^2 is a holomorphic **polynomial** on $D(0,1)$. Must f be a holomorphic polynomial on $D(0,1)$? Explain your answer.

   Solution: No. For $z = re^{i\theta}$, $r > 0$, $-\pi < \theta < \pi$. Define $g(z) = \sqrt{z} = r^{1/2}e^{i\theta/2}$. Then g is well defined and continuous on $U = \mathbf{C} \setminus (-\infty, 0]$. For $z_0 \in U$, we have :

   $$g'(z_0) = \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{\sqrt{z} - \sqrt{z_0}}{(\sqrt{z} - \sqrt{z_0})(\sqrt{z} + \sqrt{z_0})} = \frac{1}{2\sqrt{z_0}}$$

   Since $g'(z) = \dfrac{1}{2\sqrt{z}}$ is continuous on U, g is holomorphic on U. Let $f(z) = \sqrt{1+z}$. Since $1 + D(0,1) \subset U$, f is holomorphic on $D(0,1)$ and $(f(z))^2 = 1+z$ is a holomorphic polynomial. Note that for all $k \geq 1$,

   $$f^{(k)}(z) = \frac{(-1)^{k-1}\Pi_{j=1}^{k}(2j-3)}{2^k(1+z)^{\frac{2k-1}{2}}} \neq 0$$

   Therefore, f is not a holomorphic polynomial.

8. Suppose $U \subseteq \mathbf{C}$ is open $f : U \to \mathbf{C}$ is a function such that both f^2 and f^3 are holomorphic on U. Prove that f is holomorphic on U. (Warning: Beware of the zeros of f.)

   Solution: Let $p \in U$. We are going to show that there exists $r > 0$ such that f is holomorphic in $D(p,r) \subset U$. Consider the following cases:

   If $f(p) \neq 0$, then there exists $r > 0$ such that $D(p,r) \subset U$ and $f(z) \neq 0$ for all $z \in D(p,r)$. Therefore, $f(z) = \dfrac{(f(z))^3}{(f(z))^2}$ is holomorphic in $D(p,r)$.

   Suppose $f(p) = 0$. Choose $r_0 > 0$ such that $D(p,r_0) \subset U$.

   If for some $0 < r \leq r_0$, $f \equiv 0$ on $D(p,r)$, then $f \equiv 0$ on $D(p,r_0)$. Therefore, f is holomorphic in $D(p,r)$.

If $f \not\equiv 0$ on $D(p, r_0)$, then, f^2, $f^3 \not\equiv 0$ on $D(p, r_0)$. By Theorem 3.6.1, we can choose $0 < r < r_0$ such that $f(z) \neq 0$ for all $z \in D(p, r) \setminus \{0\}$. Let $f^2 = (z-p)^n g(z)$, $f^3(z) = (z-p)^m h(z)$, where n, $m > 0$ and $g(z)$, $h(z)$ are non-zero and holomorphic in $D(p, r)$. Then

$$\begin{aligned} & ((z-p)^n g(z))^3 = f^6(z) = \left(f^3(z)\right)^2 = ((z-p)^m h(z))^2 \\ \Rightarrow \quad & 3n = 2m \Rightarrow 2(m-n) = n > 0 \Rightarrow m - n > 0 \end{aligned}$$

Therefore, $f(z) = \dfrac{f^3(z)}{f^2(z)} = \dfrac{(z-p)^m h(z)}{(z-p)^n g(z)} = \dfrac{(z-p)^{m-n} h(z)}{g(z)}$ is holomorphic in $D(p, r)$.

9. Suppose f is bounded and holomorphic on $\mathbf{C} \setminus \{0\}$. Prove that f is constant on $\mathbf{C} \setminus \{0\}$. (Note : $f(0)$ is not defined. Hint: Consider $g(z) = z^2 f(z)$.)

   Solution: Suppose $|f(z)| \leq M$ for all $z \in \mathbf{C} \setminus \{0\}$. Then $|g(z)| \leq M|z|^2$ for all $z \in \mathbf{C} \setminus \{0\}$ and we can define $g(0) = \lim_{z \to 0} g(z) = 0$. $g'(0) = \lim_{z \to 0} \dfrac{g(z) - g(0)}{z - 0} = \lim\limits_{z \to 0} z f(z) = 0$. Therefore, g is holomorphic on $\mathbf{C}$. By the result in 6., we have $g(z) = Cz^2$ for some C. Therefore, $f(z) = C$ for all $z \in \mathbf{C} \setminus \{0\}$

10. In each of the following cases, determine if there exists f holomorphic on $D(0, 1)$ satisfying the condition. If so, find f. If not, explain why.

    (a) $f\left(\dfrac{1}{2n+1}\right) = \dfrac{1}{n}$.

    Solution: Let $f(z) = \dfrac{2z}{1-z}$. Then $f\left(\dfrac{1}{2n+1}\right) = \dfrac{\dfrac{2}{2n+1}}{\left(1 - \dfrac{1}{2n+1}\right)} = \dfrac{1}{n}$.

    (b) $f\left(\dfrac{(-1)^n}{n}\right) = \dfrac{1}{n}$.

    Solution: Suppose f is holomorphic on $D(0, 1)$ satisfying $f\left(\dfrac{(-1)^n}{n}\right) = \dfrac{1}{n}$. Then $g(z) = f(z) + z = 0$ for all $z = \dfrac{-1}{2n+1}$, $n > 1$. Therefore, the zeros of g has an accumulation point 0 in $D(0, 1)$. Hence, $g(z) \equiv 0$ on $D(0, 1)$. But $g\left(\dfrac{1}{2}\right) = 1 \neq 0$, a contradiction. Hence, no such f exists.