Spring 2016 Math 511 Assignment 5
Due date: Wednesday, April 6, 2016

1. Let g be a holomorphic on an open set $U \subset \mathbf{C}$ and f a meromorphic function on U. Suppose $\overline{D(P,r)} \subset U$ such that f has zeros $z_1, \ldots, z_p$ and poles $w_1, \ldots, w_q$ in $D(P,r)$ and f has neither zeros nor poles on $\partial D(P,r)$. Prove that

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} g(\zeta) \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = \sum_{j=1}^{p} n_j g(z_j) - \sum_{k=1}^{q} m_k g(w_k) \,,$$

where n_j is the multiplicity of z_j, $1 \le j \le p$ and m_k is the order of w_k, $1 \le k \le q$.

Solution: Let $F(z) = g(z)\dfrac{f'(z)}{f(z)}$. By the results in Lemma 5.1.1 to Theorem 5.1.4. $\dfrac{f'(z)}{f(z)}$ has a pole of order 1 at each z_j and w_k with residue n_j and $-m_k$ respectively. If $g(z_j) \neq 0$ (or $g(w_k) \neq 0$, respectively), then $F(z)$ also has a pole of order 1 at z_j (or w_k respectively). We have

$$\mathrm{Res}_F(z_j) = \lim_{z \to z_j} g(z) \frac{f'(z)}{f(z)}(z - z_j) = g(z_j) \lim_{z \to z_j} \frac{f'(z)}{f(z)}(z - z_j) = g(z_j) n_j$$

$$\mathrm{Res}_F(w_k) = \lim_{z \to w_k} g(z) \frac{f'(z)}{f(z)}(z - w_k) = g(z_j) \lim_{z \to w_k} \frac{f'(z)}{f(z)}(z - w_k) = -g(w_k) m_k$$

If $g(z_j) = 0$ (or $g(w_k) = 0$, respectively), then $F(z)$ has a removable singularity at z_j (or w_k respectively). Therefore, $\mathrm{Res}_F(z_j) = 0$ (or $\mathrm{Res}_F(w_k) = 0$, respectively).

Since F(z) has no other poles in $\overline{D(P,r)}$, we have

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} g(\zeta) \frac{f'(\zeta)}{f(\zeta)} \, d\zeta = \sum_{j=1}^{p} \mathrm{Res}_F(z_j) + \sum_{k=1}^{q} \mathrm{Res}_F(w_k) = \sum_{j=1}^{p} n_j g(z_j) - \sum_{k=1}^{q} m_k g(w_k) \,.$$

2. Find the number of zeros of $f(z) = z^{10} + 10ze^{z+1} - 9$ in $\{z : |z| < 1\}$.

Solution: On $\partial D(0,1)$, let $g(z) = 10ze^{z+1}$, then

$$|f(z) - g(z)| = |z^{10} - 9| \underset{(1)}{\le} |z^{10}| + 9 = 10 \underset{(2)}{\le} 10e^{\Re z + 1} = |10ze^{z+1}| = |g(z)| \,.$$

For $|z| = 1$, equality holds in (1) only when $z^{10} = -1$ and equality holds in (2) only when
$\Re(z+1) = 0 \Leftrightarrow \Re z = -1 \Leftrightarrow z = -1$. Since $(-1)^{10} \neq -1$, we have $|f(z) - g(z)| < |g(z)|$ on $\partial D(0,1)$. g has only 1 zero in $D(0,1)$. Therefore, f has only 1 zero in $D(0,1)$.

3. Find the number of zeros of $f(z) = 2z^5 - 6z^2 + z + 1$ in $\{z : 1 < |z| < 2\}$.

Solution: On $\partial D(0,1)$, let $g(z) = -6z^2$, then $|f(z) - g(z)| = |2z^5 + z + 1| \leq 2|z^5| + |z| + 1 = 4 < 6 = |-6z^2| = |g(z)|$ and g has 2 zeros in $D(0,1)$. Therefore, f has 2 zeros in $D(0,1)$.

On $\partial D(0,2)$, let $g(z) = 2z^5$, then

$$|f(z) - g(z)| = |-6z^2 + z + 1| \leq 6|z^2| + |z| + 1 = 8 < 2^6 = |g(z)|$$

and g has 5 zeros (counting multiplicity) in $D(0,2)$. Therefore, f has 5 zeros in $D(0,2)$.

Hence, f has 3 zeros in $\{z : 1 < |z| < 2\}$. (Note that $f(z) \neq 0$ on $\partial D(0,1)$, f has 2 zeros in $\overline{D(0,1)}$)

4. Find the number of zeros of $f(z) = z^8 + 3z^3 + 7z + 5$ in $\{x + iy : x > 0, y > 0\}$.

Solution: Let $Q = \{x + iy : x > 0, y > 0\}$. Choose $R > 5^{1/8}$ such that $R^8 > 3R^3 + 7R + 5$. Let Γ_1 be the line segment from iR to 0, Γ_2 the line segment from 0 to R and Γ_3 be along the circle centered at 0 from R to iR. Let $g(z) = z^8 + 5$.

On Γ_1, $z = iy$ with $0 \leq y \leq R$, therefore

$$|f(z)-g(z)| = |i(3y^3+7y)| = |3y^3+7y| < |y^8+5+i(3y^3+7y)| = |f(z)| \leq |f(z)|+|g(z)|\,.$$

On Γ_2, $z = x$ with $0 \leq x \leq R$, therefore

$$|f(z) - g(z)| = 3x^3 + 7x < x^8 + 3x^3 + 7x + 5 = |f(z)| \leq |f(z)| + |g(z)|\,.$$

On Γ_3, $|z| = R$, therefore

$$|f(z) - g(z)| = |3z^3 + 7z| \leq 3R^3 + 7R < R^8 - 5 \leq |z^8 + 5| = |g(z)|\,.$$

Since g has 2 zeros ($5^{1/8}e^{ik\pi/8}, k = 1, 3$) inside $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, so does f. Any point in Q would lie inside $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ for sufficiently large R. Hence, f has 2 zeros in $\{x + iy : x > 0, y > 0\}$.

5. Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Show that all the zeros of P lie inside $D(0,R)$, where $R = 1 + \max\{|a_k| : 0 \leq k \leq n-1\}$.

Let $g(z) = z^n$. On $|z| = R$, we have

$$\begin{aligned} & |P(z) - g(z)| \\ \leq\ & |a_{n-1}|R^{n-1} + \cdots + |a_0| \\ \leq\ & (R-1)(R^{n-1} + R^{n-2} \cdots + 1) \\ =\ & R^n - 1 < R^n = |g(z)| \leq f(z) + |g(z)| \end{aligned}$$

and $g(z)$ has n zeros inside $D(0,R)$. Therefore, $f(z)$ has n zeros inside $D(0,R)$.

6. Suppose $\{f_j\}$ is a sequence of holomorphic function on on $D(0,1)$ such that each f_j has **at most** k zeros (counting multiplicity) in $D(0,1)$ and $\{f_j\}$ converges uniformly on compact subsets of $D(0,1)$ to f. Prove that if $f \not\equiv 0$, then f has **at most** k zeros (counting multiplicity) in $D(0,1)$.

Solution: Suppose $f \not\equiv 0$ and f has more than k zeros in $D(0,1)$. Since the zeros of f are isolated in $D(0,1)$, we can choose $0 < r < 1$ such that $z_i \in D(0,r)$ for $i = 1, \ldots m$, with $m > k$ and f has no zeros on $\partial D(0,r)$. Let $\epsilon = \min\{|f(z)| : |z| = r\} > 0$. Since $\{f_j\}$ converges uniformly on $\partial D(0,r)$ to f, there exists N such that for all $j \geq N$ and $z \in \partial D(0,r)$, we have

$$\frac{\epsilon}{2} > |f_j(z) - f(z)| \geq |f(z)| - |f_j(z)| \geq \epsilon - |f_j(z)|$$

$$\Rightarrow \quad |f_j(z)| > \frac{\epsilon}{2} > 0$$

From our discussion in class, $\{f_j'\}$ also converges uniformly on $\partial D(0,r)$ to f'. Hence,

$$\text{the number of zeros of } f_j \text{ in } D(0,r) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f_j'(z)}{f_j(z)}\, dz$$

$$\to \quad \text{the number of zeros of } f \text{ in } D(0,r) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f'(z)}{f(z)}\, dz$$

Therefore, there exists N such that for $j \geq N$, we have

$$\begin{aligned} k \geq &\quad \text{the number of zeros of } f_j \text{ in } D(0,r) \text{ (counting multiplicity)} \\ = &\quad \text{the number of zeros of } f \text{ in } D(0,r) \text{ counting multiplicity)} \\ \geq &\quad m > k \end{aligned}$$

a contradiction.

7. Let $k > 0$. For each $0 \leq \ell \leq k$, construct a sequence $\{f_j\}$ on $D(0,1)$ such that each f_j has **at least** k zeros (counting multiplicity) in $D(0,1)$ and $\{f_j\}$ converges uniformly on compact subsets of $D(0,1)$ to f but f has only ℓ zeros in $D(0,1)$.

Solution: For $j \geq 1$, let $f_j(z) = z^\ell \left(z - \dfrac{j}{j+1}\right)^{k-\ell}$ for $z \in D(0,1)$. Then every f_j

has exactly k zeros in $D(0,1)$. For $z \in D(0,1)$, we have

$$\left|f_j(z) - z^\ell (z-1)^{k-\ell}\right| = \left|z^\ell \left(z - \frac{j}{j+1}\right)^{k-\ell} - z^\ell (z-1)^{k-\ell}\right|$$

$$\leq \left|\left(z - \frac{j}{j+1}\right)^{k-\ell} - (z-1)^{k-\ell}\right|$$

$$\leq \left|\left(z - \frac{j}{j+1}\right) - (z-1)\right| \textstyle\sum_{i=0}^{k-\ell-1} \left|\left(z - \frac{j}{j+1}\right)\right|^i |(z-1)|^{k-\ell-1-i}$$

$$\leq \left|1 - \frac{j}{j+1}\right| \textstyle\sum_{i=0}^{k-\ell-1} \left(|z| + \left|\frac{j}{j+1}\right|\right)^i (|z|+1)^{k-\ell-1-i}$$

$$\leq \frac{1}{j+1} \textstyle\sum_{i=0}^{k-\ell-1} 2^{k-\ell-1} = \frac{(k-\ell)2^{k-\ell-1}}{j+1}$$

Therefore,$\{f_j\}$ converges uniformly on $D(0,1)$ to $f(z) = z^\ell (z-1)^{k-\ell}$, which has only ℓ zeros in $D(0,1)$.

8. Let $R > 0$ and $U = \mathbf{C} \setminus \overline{D(0,\, R)}$. Suppose f is holomorphic on U, continuous and bounded on $\overline{U}$. Prove that there exists $z_0 \in \partial D(0,\, R)$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \overline{U}$. (Caution: $\overline{U}$ is not compact.)

Solution: Define $g : \overline{D\left(0, \frac{1}{R}\right)} \setminus \{0\} \to \mathbf{C}$ by $g(z) = f\left(\frac{1}{z}\right)$. Since f is holomorphic on U, continuous and bounded on $\overline{U}$, g has a removable singularity at 0 and can be extended to a holomorphic function on $D\left(0, \frac{1}{R}\right)$ which is continuous on $\overline{D\left(0, \frac{1}{R}\right)}$. So there exists $p \in \partial D\left(0, \frac{1}{R}\right)$ $\left(\Rightarrow |p| = \frac{1}{R}\right)$ such that $|g(p)| \geq |g(z)|$ for all $z \in \overline{D\left(0, \frac{1}{R}\right)}$. Let $z_0 = \frac{1}{p}$. Then $|z_0| = R \Rightarrow z_0 \in \partial D(0,\, R)$. For all $z \in \overline{U}$, we have $|z| \geq R \Rightarrow \frac{1}{z} \in \overline{D\left(0, \frac{1}{R}\right)}$. Therefore,

$$|f(z)| = \left|g\left(\frac{1}{z}\right)\right| \leq |g(p)| = |f(z_0)|$$

9. Suppose f is 1-1 and holomorphic on $D(0,\,1)$, $D(0,\,1) \subseteq f(D(0,\,1))$ and $f(0) = 0$. Show that $|f'(0)| \geq 1$ and the equality holds if and only if $f(z) = \alpha z$ for some α with $|\alpha| = 1$.

   Let $g = f^{-1}|_{D(0,1)}$. Then $g : D(0,1) \to D(0,1)$ and $g(0) = 0$. Therefore, by Schwarz's Lemma, $|g'(0)| \leq 1$. Since $f(g(z)) = z$ for all $z \in D(0,1)$, we have

$$f'(g(0))g'(0) = 1 \Rightarrow |f'(g(0))||g'(0)| = 1 \Rightarrow |f'(0)| = \frac{1}{|g'(0)|} \geq 1.$$

$$|f'(0)| = 1 \Leftrightarrow |g'(0)| = 1 \Leftrightarrow g(z) = az \text{ for some } |a| = 1 \Leftrightarrow f(z) = \frac{1}{a}z \text{ for some } \left|\frac{1}{a}\right| = 1$$

10. Let f be a holomorphic map on $D(0,1)$ such that $f(-\frac{1}{2}) = 0$ and $|f(z)| \leq |1 + z^2|$ for all $z \in D(0,1)$. Show that $|f(\frac{1}{2})| \leq 1$.

   Solution: Let $g(z) = \dfrac{f(z)}{1+z^2}$ for $z \in D(0,1)$. Then $|g(z)| \leq 1$ for all $z \in D(0,1)$. Hence, by the Schwarz-Pick Theorem, we have

$$\left|\frac{g\left(\frac{1}{2}\right) - g\left(-\frac{1}{2}\right)}{1 - g\left(\frac{1}{2}\right)g\left(-\frac{1}{2}\right)}\right| \leq \left|\frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}\right| \Rightarrow \frac{\left|\dfrac{f\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)^2} - 0\right|}{|1-0|} \leq \left|\frac{1}{1+\left(\frac{1}{2}\right)^2}\right| \Rightarrow |f(\frac{1}{2})| \leq 1$$