Spring 2016 Math 511 Assignment 4 Solution

1. Let $R(z) = \dfrac{P(z)}{Q(z)}$, where P and Q are two polynomials with no zeros in common. Let $P_1, P_2, \ldots, P_k$ be the zeros of Q. Suppose f is holomorphic on $\mathbf{C} \setminus \{P_1, P_2, \ldots, P_k\}$ such that $|f(z)| \leq |R(z)|$ for all $z \in \mathbf{C} \setminus \{P_1, P_2, \ldots, P_k\}$. Prove that $f(z) = CR(z)$ for some constant C.

   Solution: Let $\{P_1', P_2', \ldots, P_\ell'\}$ be the zeros of $P(z)$ and $S = (\{P_1, P_2, \ldots, P_k\} \cup \{P_1', P_2', \ldots, P_\ell'\})$. Let $g(z) = f(z)/R(z)$ for all $z \in \mathbf{C} \setminus S$. Then $|g(z)| \leq 1$ for all $z \in \mathbf{C} \setminus S$. Therefore, $|g(z)|$ is bounded near each point in S. So, $g(z)$ has a removable singularity at each point in S. . Hence, g can be extended to an entire function and $|g(z)| \leq 1$ for all $z \in \mathbf{C}$. By Liouville's Theorem, there exists a constant C, such that $f(z)/R(z) = g(z) = C$ for all $z \in \mathbf{C} \setminus S \Rightarrow f(z) = CR(z)$ for all $z \in \mathbf{C} \setminus S$. For $1 \leq i \leq \ell$, $|f(P_i')| \leq |R(P_i')| = 0 \Rightarrow f(P_i') = R(P_i') = 0$. Therefore, $f(z) = CR(z)$ for all $z \in \mathbf{C} \setminus \{P_1, P_2, \ldots, P_k\}$.

2. Prove that $f(z) = z \cdot e^{\frac{1}{z}} \cdot e^{-\frac{1}{z^2}}$ has an essential singularity at $z = 0$.

   Solution:
$$\lim_{z \to 0^+} f(z) = \lim_{z \to 0^+} z \cdot e^{\frac{1}{z}} \cdot e^{-\frac{1}{z^2}} = \lim_{z \to 0^+} z \cdot e^{\frac{1}{z^2}(z-1)} = 0$$
   because $\frac{1}{z^2}(z-1) \to -\infty$ as $z \to 0^+$. Therefore, f does not have a pole at 0.
$$\lim_{z \to 0\ \Im z > 0} |f(z)| = \lim_{y \to 0^+} \left| iy \cdot e^{\frac{1}{iy}} \cdot e^{-\frac{1}{(iy)^2}} \right| = \lim_{y \to 0^+} y \cdot e^{\frac{1}{y^2}} = \lim_{t \to \infty} \frac{e^{t^2}}{t} = \lim_{t \to \infty} \frac{2te^{t^2}}{1} = \infty$$
   Therefore, f does not have a removable singularity at 0. Hence, f has an essential singularity at 0.

3. Let P and Q be two polynomials with no zeros in common and let a be a zero of Q. Express the residue of $P(z)/Q(z)$ at a in terms of $P^{(k)}(a)$ and $Q^{(k)}(a)$, $k = 0, 1, 2, \ldots$ Let P and Q be two polynomials with no zeros in common and let a be a zero of Q. Express the residue of $P(z)/Q(z)$ at a in terms of $P^{(k)}(a)$ and $Q^{(k)}(a)$, $k = 0, 1, 2, \ldots$ if Q has a zero of order 2 at a.

   Solution: Suppose Q has a zero of order 2 at a. Then $Q(z) = (z-a)^2 Q_1(z)$, where $Q_1(z)$ is a polynomial with $Q_1(a) \neq 0$. The residue of $P(z)/Q(z)$ at a is equal to
$$\left(\frac{P}{Q_1} \right)'(a) = \frac{P'(a)Q_1(a) - P(a)Q_1'(a)}{Q_1(a)^2}$$

$$
\begin{aligned}
& Q(z) = (z-a)^2 Q_1(z) \\
\Rightarrow\quad & Q'(z) = 2(z-a)Q_1(z) + (z-a)^2 Q_1'(z) \\
\Rightarrow\quad & Q^{(2)}(z) = 2Q_1(z) + 4(z-a)Q_1'(z) + (z-a)^2 Q_1^{(2)}(z) \\
\Rightarrow\quad & Q^{(2)}(a) = 2Q_1(a) \\
& Q^{(3)}(z) = 6Q_1'(z) + 6(z-a)Q_1^{(2)}(z) + (z-a)^2 Q_1^{(3)}(z) \\
\Rightarrow\quad & Q^{(3)}(a) = 6Q_1'(a)
\end{aligned}
$$

Therefore, the residue of $P(z)/Q(z)$ at a is equal to

$$
\frac{P'(a)\frac{Q^{(2)}(a)}{2} - P(a)\frac{Q^{(3)}(a)}{6}}{\left(\frac{Q^{(2)}(a)}{2}\right)^2} = \frac{6P'(a)Q^{(2)}(a) - 2P(a)Q^{(3)}(a)}{3\left(Q^{(2)}(a)\right)^2}
$$

4. Find the Laurent series for

$$
f(z) = \frac{1}{z(z-1)(z-2)}
$$

centered at $z = 0$ and converging in the annulus $\{z : 1 < |z| < 2\}$.

Solution: By partial fraction,

$$
\frac{1}{z(z-1)(z-2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{2}{z-1} + \frac{1}{z-2}\right)
$$

For $1 < |z| < 2$,

$$
\begin{aligned}
\frac{2}{z-1} &= \frac{2}{z\left(1-\frac{1}{z}\right)} = \frac{2}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = 2\sum_{n=1}^{\infty}\frac{1}{z^n} \\
\frac{1}{z-2} &= -\frac{1}{2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad \frac{1}{z(z-1)(z-2)} &= \frac{1}{2}\left(\frac{1}{z} - 2\sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right) \\
&= -\left(\frac{1}{2z} + \sum_{n=2}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}\right)
\end{aligned}
$$

5. Find the function $f(z)$ satisfying the following conditions:

   (a) f has a pole of order 2 at $z = 0$ with residue 2, a simple pole at $z = 1$ with residue 2.

   (b) f is holomorphic on $\mathbf{C} \setminus \{0, 1\}$.

   (c) There exists R and $M > 0$ such that $|f(z)| < M$ for all z, with $|z| \geq R$.

   (d) f(2)=5, $f(-1) = 2$.

   Solution:

$$(a) \text{ and } (b) \Rightarrow \quad f(z) = \frac{a}{z^2} + \frac{2}{z} + \frac{2}{(z-1)} + g(z), \text{ where } g \text{ is holomorphic on } \mathbf{C}$$

$$(c) \Rightarrow \quad g(z) = b \text{ for some constant } b$$

$$\Rightarrow \quad f(z) = \frac{a}{z^2} + \frac{2}{z} + \frac{2}{(z-1)} + b$$

$$(d) \Rightarrow \begin{cases} \dfrac{a}{2^2} + \dfrac{2}{2} + \dfrac{2}{(2-1)} + b = 5 \\ \dfrac{a}{(-1)^2} + \dfrac{2}{(-1)} + \dfrac{2}{(-1-1)} + b = 2 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{a}{4} + b = 2 \\ a + b = 5 \end{cases} \Rightarrow a = 5, \; b = 1$$

$$\Rightarrow \quad f(z) = \frac{5}{z^2} + \frac{2}{z} + \frac{2}{(z-1)} + 1$$

6. Let $f(z)$ be meromorphic in the plane. Assume that $f(z) = f(z+2i) = f(z+(1-3i))$ for all complex z. Let $\mathcal{P}$ be the parallelogram with vertices 0, $2i$, $1-i$, and $1-3i$, and assume that f has no poles on $\mathcal{P}$. Let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be the poles of f inside $\mathcal{P}$. Prove that

$$\sum_{k=1}^{n} \mathrm{Res}_f(\zeta_k) = 0.$$

   Solution: The boundary of $\mathcal{P}$ can be decomposed int 4 parts:

$$0 \to 1-3i: \quad \gamma_1(t) = (1-3i)t, \; 0 \leq t \leq 1 \qquad \Rightarrow \gamma_1'(t) = (1-3i)$$

$$1+3i \to 1-i: \quad \gamma_2(t) = 1-3i+2it, \; 0 \leq t \leq 1 \qquad \Rightarrow \gamma_2'(t) = 2i$$

$$1-i \to 2i: \quad \gamma_3(t) = 1-i-(1-3i)t, \; 0 \leq t \leq 1 \quad \Rightarrow \gamma_3'(t) = -(1-3i)$$

$$2i \to 0: \quad \gamma_4(t) = 2i(1-t), \; 0 \leq t \leq 1 \qquad \Rightarrow \gamma_4(t) = -2i$$

$$
\begin{array}{rl|rl}
& \int_{\gamma_3} f(z)\,dz & & \int_{\gamma_4} f(z)\,dz \\
= & \int_0^1 f(1-i-(1-3i)t)(-(1-3i))\,dt & = & \int_0^1 f(2i(1-t))(-2i)\,dt \\
= & \int_0^1 f(1-i-(1-3i)t-2i)(-(1-3i))\,dt & = & \int_0^1 f(2i(1-t)+1-3i)(-2i)\,dt \\
= & \int_0^1 f((1-3i)(1-t))(-(1-3i))\,dt & = & \int_0^1 f(1-3i+2i(1-t))(-2i)\,dt \\
& \quad (\text{ use } s=1-t) & & \quad (\text{ use } s=1-t) \\
= & \int_1^0 f((1-3i)s)(1-3i)\,ds & = & \int_1^0 f(1-3i+2is)(2i)\,ds \\
= & -\int_{\gamma_1} f(z)\,dz & = & -\int_{\gamma_2} f(z)\,dz
\end{array}
$$

$$
\Rightarrow \sum_{k=1}^{n} \mathrm{Res}_f\,(\zeta_k) = \frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4} f(z)\,dz = 0
$$

7. Evaluate the integral

$$
\int_0^{\pi/2} \frac{dx}{9+7\sin^2 x}\,,
$$

(Hint: Use compound angle and change the integral to one over the interval $[0,\ 2\pi]$.)

Solution:

$$
\begin{aligned}
\int_0^{\pi/2} \frac{dx}{9+7\sin^2 x} &= \int_0^{\pi/2} \frac{dx}{9+7\left(\dfrac{1-\cos 2x}{2}\right)} = \int_0^{\pi/2} \frac{2}{25-7\cos 2x}dx \\
&= \int_0^{\pi} \frac{1}{25-7\cos t}dt \ (\text{ use } t=2x) \\
&= \int_{2\pi}^{\pi} \frac{-1}{25-7\cos s}ds \ (\text{ use } t=2\pi-s) \\
&= \int_{\pi}^{2\pi} \frac{1}{25-7\cos s}ds \\
\Rightarrow \int_0^{\pi/2} \frac{dx}{9+7\sin^2 x} &= \frac{1}{2}\int_0^{2\pi} \frac{1}{25-7\cos t}dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{2\pi} \frac{1}{25-7\left(\dfrac{e^{it}+e^{-it}}{2}\right)}dt = \int_0^{2\pi} \frac{e^{it}}{\left(25-7\left(\dfrac{e^{it}+e^{-it}}{2}\right)\right)2e^{it}}dt \\
&= \int_0^{2\pi} \frac{e^{it}}{-7e^{2it}+50e^{it}-7}dt = \frac{1}{i}\int_0^{2\pi} \frac{2ie^{it}}{-7e^{2it}+50e^{it}-7}dt \\
&= \frac{1}{i}\int_{|z|=1} \frac{dz}{-7z^2+50z-7} = \frac{1}{i}\int_{|z|=1} \frac{dz}{(7z-1)(z-7)} \\
&= (2\pi)\text{Residue of } \frac{1}{(7z-1)(z-7)} \text{ at } z=\frac{1}{7} \\
&= 2\pi \lim_{z\to\frac{1}{7}} \frac{\left(z-\frac{1}{7}\right)}{(7z-1)(z-7)} = \frac{2\pi}{48} = \frac{\pi}{24}
\end{aligned}
$$

8. Let $a > 0$. Compute the value of the integral $\int_0^\infty \dfrac{x\sin(x)}{x^2+a^2}\,dx$.

Solution: Let $R > 1$. Define $\gamma_1^R(x) = x,\ -R \le x \le R,\ \gamma_2^R(t) = Re^{it},\ 0 \le t \le \pi$. Then

$$
\int_{\gamma_1} \frac{ze^{iz}}{z^2+a^2}\,dz + \int_{\gamma_2} \frac{ze^{iz}}{z^2+a^2}\,dz
$$

$$
\begin{aligned}
&= (2\pi i)\ \text{ residual of } \frac{ze^{iz}}{z^2+a^2} \text{ at } z = ai \\
&= (2\pi i)\ \lim_{z\to ai} \frac{ze^{iz}}{(z-ai)(z+ai)}(z-ai) \\
&= (2\pi i)\frac{(ai)e^{i(ai)}}{(ai+ai)} \\
&= i\pi e^{-a}
\end{aligned}
$$

$$\int_{\gamma_1} \frac{ze^{iz}}{z^2+a^2}\,dz$$

$$= \int_{-R}^{R} \frac{x(\cos(x)+i\sin(x))}{x^2+a^2}\,dx$$

$$= \int_{-R}^{R} \frac{x\cos(x)}{x^2+a^2}\,dx + i\int_{-R}^{R} \frac{x\sin(x)}{x^2+a^2}\,dx$$

$$= 2i\int_{0}^{R} \frac{x\sin(x)}{x^2+a^2}\,dx$$

$$\to 2i\int_{0}^{\infty} \frac{x\sin(x)}{x^2+a^2}\,dx \qquad \text{as } R\to\infty$$

$$\left|\int_{\gamma_2} \frac{ze^{iz}}{z^2+a^2}\,dz\right| = \left|\int_0^{\pi} \frac{Re^{it}e^{iRe^{it}}iRe^{it}}{(Re^{it})^2+a^2}\,dt\right|$$

$$\le \int_0^{\pi} \frac{R^2|e^{iR(\cos t+i\sin t)}|}{R^2-a^2}\,dt = \frac{R^2}{R^2-a^2}\int_0^{\pi} e^{-R\sin t}\,dt = \frac{2R^2}{R^2-a^2}\int_0^{\pi/2} e^{-R\sin t}\,dt$$

$$\le \frac{2R^2}{R^2-a^2}\int_0^{\pi/2} e^{-\frac{2Rt}{\pi}}\,dt = \frac{2R^2}{R^2-a^2}\left[-\frac{\pi}{2R}e^{-\frac{2Rt}{\pi}}\right]_0^{\pi/2} = \frac{\pi R}{R^2-a^2}\left[\left(1-e^{-R}\right)\right] \to 0 \ \text{ as } R\to\infty$$

Therefore, $2i\int_0^{\infty} \frac{x\sin(x)}{x^2+a^2}\,dx = i\pi e^{-a} \Rightarrow \int_0^{\infty} \frac{x\sin(x)}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2}$.

9. Evaluate the integral

$$\int_0^{\infty} \frac{x^{1/2}}{1+x^4}\,dx.$$

Solution:

Let $R>1$. Define $\gamma_1^R(x)=x,\ \frac{1}{R}\le x\le R,\ \gamma_2^R(t)=Re^{it},\ 0\le t\le \frac{\pi}{2}$, $\gamma_3^R(x)=ix,\ \frac{1}{R}\le x\le R,\ \gamma_4^R(t)=\frac{1}{R}e^{it},\ 0\le t\le\frac{\pi}{2}$. Then

$$\left(\int_{\gamma_1}+\int_{\gamma_2}-\int_{\gamma_3}-\int_{\gamma_4}\right)\frac{z^{1/2}}{1+z^4}\,dz = (2\pi i)\ \text{ residual of } \frac{z^{1/2}}{1+z^4} \text{ at } z=e^{i\pi/4}$$

$$\int_{\gamma_1} \frac{z^{1/2}}{1+z^4}\,dz = \int_{\frac{1}{R}}^{R} \frac{x^{1/2}}{1+x^4}\,dx,\ \int_{\gamma_2} \frac{z^{1/2}}{1+z^4}\,dz = \int_0^{\frac{\pi}{2}} \frac{R^{1/2}e^{it/2}i}{1+R^4e^{4it}}\,dt,$$

$$\int_{\gamma_3} \frac{z^{1/2}}{1+z^4}\,dz = \int_{\frac{1}{R}}^{R} \frac{x^{1/2}e^{i\pi/4}}{1+x^4}\,dx,\ \int_{\gamma_4} \frac{z^{1/2}}{1+z^4}\,dz = \int_0^{\frac{\pi}{2}} \frac{R^{-1/2}e^{it/2}i}{1+R^{-4}e^{4it}}\,dt\,.$$

$$\left|\int_{\gamma_2} \frac{z^{1/2}}{1+z^4}\,dz\right| = \left|\int_0^{\frac{\pi}{2}} \frac{R^{1/2}e^{it/2}R(ie^{it})}{1+(Re^{it})^4}\,dt\right| \leq \int_0^{\frac{\pi}{2}} \frac{R^{3/2}}{R^4-1}\,dt = \frac{\pi R^{3/2}}{2(R^4-1)} \to 0 \text{ as } R \to \infty$$

$$\left|\int_{\gamma_4} \frac{z^{1/2}}{1+z^4}\,dz\right| = \left|\int_0^{\frac{\pi}{2}} \frac{R^{-1/2}e^{it/2}R^{-1}(ie^{it})}{1+R^{-4}(e^{it}))^4}\,dt\right| \leq \int_0^{\frac{\pi}{2}} \frac{R^{-3/2}}{1-R^{-4}}\,dt = \frac{\pi R^{-3/2}}{2(1-R^{-4})} \to 0 \text{ as } R \to \infty$$

$$\int_{\gamma_3} \frac{z^{1/2}}{1+z^4}\,dz = ie^{i\pi/4}\int_{\frac{1}{R}}^{R} \frac{x^{1/2}}{1+x^4}\,dx \to ie^{i\pi/4}\int_0^{\infty} \frac{x^{1/2}}{1+x^4}\,dx \text{ as } R \to \infty$$

Therefore,

$$\left(1-ie^{i\pi/4}\right)\int_0^{\infty} \frac{x^{1/2}}{1+x^4}\,dx = (2\pi i) \;\text{ residual of }\; \frac{z^{1/2}}{1+z^4} \text{ at } z = e^{i\pi/4}$$

$$\begin{aligned}
\Rightarrow \quad \int_0^{\infty} \frac{x^{1/2}}{1+x^4}\,dx &= \frac{2\pi i}{(1-ie^{i\pi/4})}\lim_{z\to e^{i\pi/4}} \frac{z^{1/2}\left(z-e^{i\pi/4}\right)}{1+z^4} \\
&= \frac{2\pi i}{(1-ie^{i\pi/4})}\lim_{z\to e^{i\pi/4}} \frac{z^{1/2}}{4z^3} = \frac{2\pi i}{(1-ie^{i\pi/4})}\frac{e^{i\pi/8}}{4e^{3i\pi/4}} \\
&= \frac{\pi i}{2\left(e^{5i\pi/8}-e^{11i\pi/8}\right)} = \frac{\pi}{4\sin(5\pi/8)}
\end{aligned}$$

10. Evaluate

$$\sum_{j=-\infty}^{\infty} \frac{1}{j^3+2} \, .$$

Solution: Let $f(z) = \dfrac{\cot z}{z^3 + 2\pi^3}$. Then f has a simple pole at $z = 2^{1/3}\pi e^{i(2k+1)\pi/3}$ for $k = 0, 1, 2$ and $z = j\pi$, $j = 0, \pm 1, \pm 2, \ldots$. The residue of f at $z = 2^{1/3}\pi e^{i(2k+1)\pi/3}$ is

$$\lim_{z \to 2^{1/3}\pi e^{i(2k+1)\pi/3}} \frac{\left(z - 2^{1/3} e^{i(2k+1)\pi/3}\right) \cot z}{z^3 + 2\pi^3} = \lim_{z \to 2^{1/3}\pi e^{i(2k+1)\pi/3}} \frac{\cot z}{3z^2} = \frac{\cot\left(2^{1/3}\pi e^{i(2k+1)\pi/3}\right)}{3\left(2^{1/3}\pi e^{i(2k+1)\pi/3}\right)^2}$$

The residue of f at $z = j\pi$ is

$$\lim_{z \to j\pi} \frac{(z - j\pi)\cot z}{z^3 + 2\pi^3} = \frac{1}{(j^3+2)\pi^3}$$

Let γ_n be the square with center at the origin and sides of length $(2n+1)\pi$ parallel to the coordinate axes. On the side parallel to the imaginary axis,

$$z = \pm\left(n + \frac{1}{2}\right)\pi + iy$$

and hence

$$|\cot z| = \left| \frac{\cos\left[\pm\left(n+\frac{1}{2}\right)\pi + iy\right]}{\sin\left[\pm\left(n+\frac{1}{2}\right)\pi + iy\right]} \right| = \left| \frac{\sin(iy)}{\cos(iy)} \right| = \left| \frac{e^{-y} - e^{y}}{e^{-y} + e^{y}} \right| \le 1$$

On the side parallel to the real axis,

$$z = x \pm i\left(n + \frac{1}{2}\right)\pi$$

and hence

$$|\cot z| = \left| \frac{\cos\left[x \pm i\left(n+\frac{1}{2}\right)\pi\right]}{\sin\left[x \pm i\left(n+\frac{1}{2}\right)\pi\right]} \right| = \left| \frac{e^{\left(n+\frac{1}{2}\right)\pi} + e^{-\left(n+\frac{1}{2}\right)\pi}}{e^{\left(n+\frac{1}{2}\right)\pi} - e^{-\left(n+\frac{1}{2}\right)\pi}} \right| = \left| \frac{1 + e^{-(2n+1)\pi}}{1 - e^{-(2n+1)\pi}} \right| \le \frac{1+e^{-\pi}}{1-e^{-\pi}}$$

Therefore, for $n > 2^{1/3}\pi$, $\left|\int_{\gamma_n} f(z)\,dz\right| = \left|\int_{\gamma_n} \dfrac{\cot z}{z^3 + 2\pi^3}\,dz\right| \le \dfrac{1+e^{-\pi}}{1-e^{-\pi}} \dfrac{4(2n+1\pi)}{n^3 - 2\pi^3} \to 0$ as $n \to \infty$. Hence, the sum of all residue of f is 0.

$$\textstyle\sum_{j=-\infty}^{\infty} \dfrac{1}{(j^3+2)\pi^3} + \sum_{k=0}^{2} \dfrac{\cot\left(2^{1/3}\pi e^{i(2k+1)\pi/3}\right)}{3\left(2^{1/3}\pi e^{i(2k+1)\pi/3}\right)^2} = 0$$

$$\Rightarrow \quad \textstyle\sum_{j=-\infty}^{\infty} \dfrac{1}{(j^3+2)} = -\pi^3 \sum_{k=0}^{2} \dfrac{\cot\left(2^{1/3}\pi e^{i(2k+1)\pi/3}\right)}{3\left(2^{1/3}\pi e^{i(2k+1)\pi/3}\right)^2} \approx 1.75976$$