Spring 2016 Math 511 Homework 1 Solution

1. Show that a cubic equation $E_1 : t^3 + at^2 + bt + c = 0$ can be transformed into an equation of the form $E_2 : x^3 + px = q$ by a substitution of $t = x - a/3$. Suppose α^3 and β^3 are the roots of the equation $z^2 - qz - p^3/27 = 0$ with $\alpha\beta = -p/3$. Show that $x = \alpha + \beta$ is a solution of E_2. Hence, give the solutions of E_1 in terms of a, b and c.

   Solution: By the substituting $t = (x - a/3)$ into E_1, we have

$$
\begin{array}{rl}
 & t^3 + at^2 + bt + c = 0 \\
\Leftrightarrow & (x - a/3)^3 + a(x - a/3)^2 + b(x - a/3) + c = 0 \\
\Leftrightarrow & (x^2 - 3(a/3)x^2 + 3(a/3)^2x - (a/3)^3) + a\,(x^2 - 2(a/3)x + (a/3)^2) + b(x - a/3) + c = 0 \\
\Leftrightarrow & x^3 + px = q
\end{array}
$$

   with $p = b - \dfrac{a^2}{3}$ and $q = \dfrac{ab}{3} - \dfrac{2a^3}{27} - c$.

   Suppose α^3 and β^3 are the roots of the equation $z^2 - qz - p^3/27 = 0$ with $\alpha\beta = -\dfrac{p}{3}$. Then we have $\alpha^3 + \beta^3 = q$. Putting $x = \alpha + \beta$, we have

$$
\begin{array}{rl}
 & x^3 + px \\
= & (\alpha + \beta)^3 + p(\alpha + \beta) \\
= & \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta) + p(\alpha + \beta) \\
= & q - p(\alpha + \beta) + p(\alpha + \beta) \\
= & q
\end{array}
$$

   The solution of the equation $z^2 - qz - p^3/27 = 0$ are

$$
\begin{array}{rl}
z = & \dfrac{q \pm \sqrt{q^2 + 4p^3/27}}{2} \\
= & \dfrac{q \pm \sqrt{q^2 + 4p^3/27}}{2} \\
= & \dfrac{9ab - 2a^3 - 27c \pm 3^{3/2}\sqrt{4a^3c + 4b^3 + 27c^2 - a^2b^2 - 18abc}}{54}
\end{array}
$$

   Therefore,

$$
\begin{array}{rl}
\alpha = & \sqrt[3]{\dfrac{9ab - 2a^3 - 27c + 3^{3/2}\sqrt{4a^3c + 4b^3 + 27c^2 - a^2b^2 - 18abc}}{54}} \text{ and} \\
\beta = & \sqrt[3]{\dfrac{9ab - 2a^3 - 27c - 3^{3/2}\sqrt{4a^3c + 4b^3 + 27c^2 - a^2b^2 - 18abc}}{54}}
\end{array}
\tag{1}
$$

   Note that if α^3 and β^3 are the roots of the equation $z^2 - qz - p^3/27 = 0$, then we have $\alpha^3\beta^3 = -p^3/27$. So the condition $\alpha\beta = -\dfrac{p}{3}$ represents a specific choice of the cube roots of α^3 and β^3 in (1). If $(\alpha,\ \beta)$ satisfies the above conditions, then the pairs $(\alpha e^{i2\pi/3},\ \beta e^{-i2\pi/3})$ and $(\alpha e^{-i2\pi/3},\ \beta e^{i2\pi/3})$ will also satisfy the above conditions

Hence, the solution for t is given by

$$\begin{aligned} t = & \sqrt[3]{\frac{9ab-2a^3-27c+3^{3/2}\sqrt{4a^3c+4b^3+27c^2-a^2b^2-18abc}}{54}} \\ & +\sqrt[3]{\frac{9ab-2a^3-27c-3^{3/2}\sqrt{4a^3c+4b^3+27c^2-a^2b^2-18abc}}{54}}-\frac{a}{3} \end{aligned}$$

where the cube roots are chosen as explained above.

2. Find all solutions of the equation $x^3-15x-4=0$ directly. (Hint: The equation has one integer root.) Then compare them with the solutions using the formula found in 1. For each solution, find the corresponding α and β as given in 1.

   Solution: The integer solution of $x^3-15x-4=0$ must be a factor of 4. Therefore, the only possible choices are $\pm1,\ \pm2,\ \pm4$. Since $4^3-15(4)-4=0$, $x=4$ is a solution. We have

   $$x^3-15x-4=(x-4)(x^2+4x+1)=(x-4)(x+2+\sqrt{3})(x+2-\sqrt{3})$$

   Therefore, $x=4,\ -2-\sqrt{3}, 2+\sqrt{3}$ are the solutions of the equation.

   By the result in 1, we consider the equation $z^2-4z-\dfrac{(-15)^3}{27}=0 \Leftrightarrow z^2-4z+125=0$.

   Solution of this quadratic equation is given by $z=\dfrac{4\pm\sqrt{4^2-4(125)}}{2}=2\pm 11i$.

   To find α and β satisfying $\alpha^3=2+11i$ and $\beta^3=2-11i$ with $\alpha\beta=15/3=5$, let $\alpha=a+ib$, with $a,\ b\in\mathbb{R}$. Then we have

   $$(a+ib)^3=2+11i \Rightarrow \begin{cases} a^3-3ab^2 = 2 & (1) \quad \text{and} \\ 3a^2b-b^3 = 11 & (2) \end{cases}$$

   Also,

   $$a^2+b^2=|\alpha|^2=(2^2+11^2)^{1/3}=5 \qquad (3)$$

   (3) $\Rightarrow a^2=5-b^2$. Putting back to (2), we have

   $$3(5-b^2)b-b^3=11 \Rightarrow 4b^3-15b+11=0$$

   Clearly, $b=1$ is a solution. Then by (3), we have $a^2=5-1=4$.Then (1) implies

   $$4a-3a=2 \Rightarrow a=2$$

Check that $\alpha = 2+i$ satisfies $\alpha^3 = 2+11i$ and $\beta = 2-i$ satisfies $\beta^3 = 2-11i$ and $\alpha\beta = 5$. So we have

α	$=$	$2+i$	$(2+i)e^{2\pi i/3}$	$(2+i)e^{4\pi i/3}$
β	$=$	$2-i$	$(2-i)e^{-2\pi i/3}$	$(2-i)e^{-4\pi i/3}$
$\alpha+\beta$	$=$	4	$-2-\sqrt{3}$	$-2+\sqrt{3}$

3. Let S be the sphere of radius 1, centered at the origin in $\mathbb{R}^3$ and $N = (0,0,1)$. For each point $P(x,y,z)$ on $S \setminus \{N\}$, let the ray (half line) from N through P intersects the X-Y plane at $Q(u,v,0)$. Express **1)** $x,\ y$, and z in terms of u and v; and **2)** u and v in terms of $x,\ y$, and z.

   Solution: The ray from N through Q is given by the parametric equations:

$$(X,Y,Z) = (0,0,1) + ((u,v,0)-(0,0,1))t \Rightarrow \left\{\begin{array}{lcl} X &=& ut \\ Y &=& vt \\ Z &=& 1-t \end{array}\right.$$

   This ray intersects S at

$$\begin{aligned} & (ut)^2 + (vt)^2 + (1-t)^2 = 1 \\ \Rightarrow\ & (u^2+v^2+1)t^2 + 1 - 2t = 1 \\ \Rightarrow\ & (u^2+v^2+1)t^2 = 2t \\ \Rightarrow\ & t = \frac{2}{(u^2+v^2+1)} \end{aligned}$$

   because $t=0$ corresponds to N. Therefore, we have

$$x = \frac{2u}{(u^2+v^2+1)},\ y = \frac{2v}{(u^2+v^2+1)},\ z = 1 - \frac{2}{(u^2+v^2+1)} = \frac{(u^2+v^2-1)}{(u^2+v^2+1)}$$

   From $z = 1-t \Rightarrow t = 1-z$, we have

$$u = \frac{x}{t} = \frac{x}{1-z}, \quad \text{and} \quad v = \frac{y}{t} = \frac{y}{1-z}$$

4. Prove Lagrange's Identity:

$$\left|\sum_{j=1}^{n} z_j w_j\right|^2 = \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right) - \sum_{1\le j<k\le n} |z_j\overline{w_k} - \overline{w_j}z_k|^2$$

and from this deduce the Cauchy-Schwarz Inequality in Proposition 1.2.4.

Solution:

$$\begin{aligned}
& \left|\sum_{j=1}^{n} z_j w_j\right|^2 \\
= & \left|\sum_{j=1}^{n} z_j w_j\right| \left|\sum_{j=1}^{n} \overline{z_j w_j}\right| \\
= & \sum_{j=1}^{n} |z_j w_j|^2 + \sum_{1\le j<k\le n} z_j w_j \overline{z_k w_k} + \overline{z_j w_j} z_k w_k \\
= & \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right) - \sum_{1\le j<k\le n} \left(|z_j|^2 |w_k|^2 + |z_k|^2 |w_j|^2\right) + \sum_{1\le j<k\le n} \left(z_j w_j \overline{z_k w_k} + \overline{z_j w_j} z_k w_k\right) \\
= & \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right) - \sum_{1\le j<k\le n} \left(|z_j|^2 |w_k|^2 + |z_k|^2 |w_j|^2 - z_j w_j \overline{z_k w_k} - \overline{z_j w_j} z_k w_k\right) \\
= & \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right) - \sum_{1\le j<k\le n} \left(z_j \overline{w_k} - \overline{w_j} z_k\right)\left(\overline{z_j} w_k - w_j \overline{z_k}\right) \\
= & \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right) - \sum_{1\le j<k\le n} \left|z_j \overline{w_k} - \overline{w_j} z_k\right|^2
\end{aligned}$$

Since $\sum_{1\le j<k\le n} |z_j \overline{w_k} - \overline{w_j} z_k|^2 \ge 0$, we have $\left|\sum_{j=1}^{n} z_j w_j\right|^2 \le \left(\sum_{j=1}^{n} |z_j|^2\right)\left(\sum_{j=1}^{n} |w_j|^2\right)$.

5. Compute the following derivatives:

   (a) $\dfrac{\partial^3}{\partial x^2 \partial y}(3z^2\overline{z}^4 - 2z^3\overline{z} + z^4 - \overline{z}^5)$

   Solution: Using $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \overline{z}}$ and $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \overline{z}}\right)$, we have

$$\begin{aligned}
& \frac{\partial^3}{\partial x^2 \partial y}\left(3z^2\overline{z}^4 - 2z^3\overline{z} + z^4 - \overline{z}^5\right) \\
= & \; i\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\right)^2 \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)(3z^2\overline{z}^4 - 2z^3\overline{z} + z^4 - \overline{z}^5) \\
= & \; i\left(\frac{\partial^3}{\partial z^3} + \frac{\partial^3}{\partial z^2 \partial \overline{z}} - \frac{\partial^3}{\partial z \partial \overline{z}^2} - \frac{\partial^3}{\partial \overline{z}^3}\right)(3z^2\overline{z}^4 - 2z^3\overline{z} + 4z^4 - \overline{z}^5) \\
= & \; i\left((-12\overline{z} + 24z) + (24\overline{z}^3 - 12z) - (72z\overline{z}^2) - (72z^2\overline{z} - 60\overline{z}^2)\right) \\
= & \; 12i\left(z - 6z\overline{z}^2 - 6z^2\overline{z} - \overline{z} + 5\overline{z}^2 + 2\overline{z}^3\right) \\
= & \; 12\left((6x^2y + 10xy - 2y - 2y^3) + i(5x^2 - 10x^3 - 18xy^2 - 5y^2)\right)
\end{aligned}$$

(b) $\dfrac{\partial^4}{\partial z \partial \overline{z}^3}(xy^2)$

Solution: Using $x = \dfrac{z+\overline{z}}{2}$, $y = \dfrac{z-\overline{z}}{2i}$, we have $xy^2 = -\dfrac{z^3 - z^2\overline{z} - z\overline{z}^2 + \overline{z}^3}{8}$. Therefore,

$$\frac{\partial^4}{\partial z \partial \overline{z}^3}\left(xy^2\right) = \frac{\partial^4}{\partial z \partial \overline{z}^3}\left(-\frac{z^3 - z^2\overline{z} - z\overline{z}^2 + \overline{z}^3}{8}\right) = 0$$

6. Find two real valued harmonic functions u and v on $\mathbb{C}$ such that $u \cdot v$ is not harmonic.

   Solution: Take $u(x,y) = x = v(x,y)$. Then $\Delta u = 0 = \Delta v$ but $\Delta uv = 2 \neq 0$.

7. Suppose u and v are two real valued harmonic functions on $\mathbb{C}$ such that $u + iv$ is holomorphic. Prove that $u \cdot v$ is harmonic.

   Solution: $u+iv$ is holomorphic $\Rightarrow (u+iv)^2 = u^2 - v^2 + i(2uv)$ holomorphic. Therefore, $2uv$ is harmonic $\Rightarrow uv$ is harmonic.

   You can also prove it directly:

   Suppose $u + iv$ is holomorphic. Then $u_x = v_y$ and $u_y = -v_x$. We have

$$(uv)_x = uv_x + vu_x \Rightarrow (uv)_{xx} = uv_{xx} + 2u_xv_x + vu_{xx}$$

   Similarly, $(uv)_{yy} = uv_{yy} + 2u_yv_y + vu_{yy}$. Hence,

$$\begin{aligned}
\Delta(uv) &= (uv)_{xx} + (uv)_{yy} \\
&= uv_{xx} + vu_{xx} + 2u_xv_x + uv_{yy} + vu_{yy} + 2u_yv_y \\
&= u\left(v_{xx} + v_{yy}\right) + v\left(u_{xx} + u_{yy}\right) + 2u_xv_x - 2v_xu_x \\
&= 0
\end{aligned}$$

8. Let $U \subseteq \mathbb{C}$ be an open set. Let $z_0 \in U$ and $r > 0$ and assume that $\{z : |z - z_0| \leq r\} \subseteq U$. For j a positive integer, compute

$$\frac{1}{2\pi}\int_0^{2\pi} \left(z_0 + re^{i\theta}\right)^j \, d\theta \qquad \text{and} \qquad \frac{1}{2\pi}\int_0^{2\pi} \overline{\left(z_0 + re^{i\theta}\right)^j} \, d\theta\,.$$

   Use these results to prove that if u is a harmonic polynomial on U, then

$$\frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\theta}\right) \, d\theta = u(z_0)\,.$$

   Solution: We are going to prove by induction on j that for $j \geq 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} \left(z_0 + re^{i\theta}\right)^j d\theta = z_0^j \;\text{ and }\; \frac{1}{2\pi}\int_0^{2\pi} \overline{(z_0 + re^{i\theta})^j} d\theta = \overline{z}_0^j$$

For $j=1$, $\frac{1}{2\pi}\int_0^{2\pi}\left(z_0+re^{i\theta}\right)d\theta=\frac{1}{2\pi}\left(z_0\theta-ire^{i\theta}\big]_0^{2\pi}\right)=z_0$.

Suppose the result holds for some $j\geq 1$. Then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\left(z_0+re^{i\theta}\right)^{j+1}d\theta\\
=&\ \frac{1}{2\pi}\left(\int_0^{2\pi}z_0\left(z_0+re^{i\theta}\right)^j d\theta+\int_0^{2\pi}re^{i\theta}\left(z_0+re^{i\theta}\right)^j d\theta\right)\\
=&\ z_0\left(\frac{1}{2\pi}\int_0^{2\pi}\left(z_0+re^{i\theta}\right)^j d\theta\right)+\frac{1}{2\pi}\left((-i)\frac{\left(z_0+re^{i\theta}\right)^{j+1}}{j+1}\Bigg]_0^{2\pi}\right)\\
=&\ z_0\left(z_0^j\right)+0\\
=&\ z_0^{j+1}
\end{aligned}$$

The proof for the second integral is similar.

Suppose u is a harmonic polynomial on U. Then from the proof of Lemma 1.4.5 , we have $a_j,\ b_k\in\mathbb{C}$, $0\leq j\leq m$ and $1\leq k\leq n$ such that,, $u(z,\overline{z})=a_0+\sum\limits_{j=1}^m a_jz^j+\sum\limits_{k=1}^n b_k\overline{z}^k$.

Therefore,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}u\left(z_0+re^{i\theta}\right)d\theta\\
=&\ \frac{1}{2\pi}\int_0^{2\pi}a_0+\sum_{j=1}^m a_j\left(z_0+re^{i\theta}\right)^j+\sum_{k=1}^n b_k\overline{\left(z_0+re^{i\theta}\right)}^k d\theta\\
=&\ \frac{1}{2\pi}\left(\int_0^{2\pi}a_0 d\theta+\sum_{j=1}^m a_j\int_0^{2\pi}\left(z_0+re^{i\theta}\right)^j d\theta+\sum_{k=1}^n b_k\int_0^{2\pi}\overline{\left(z_0+re^{i\theta}\right)}^k d\theta\right)\\
=&\ a_0+\sum_{j=1}^m a_jz_0^j+\sum_{k=1}^n b_k\overline{z}_0^k\\
=&\ u(z_0)
\end{aligned}$$

9. Prove that if f is holomorphic on an open set $U\subseteq\mathbb{C}$ and f is nonvanishing, then

$$\Delta\left(|f|^p\right)=p^2|f|^{p-2}\left|\frac{\partial f}{\partial z}\right|^2\ ,\ \text{for all } p>0\,.$$

Solution: Since $(|f|^p)_x=p|f|^{p-1}|f|_x$ and $(|f|^p)_y=p|f|^{p-1}|f|_y$, by linearity, we have $(|f|^p)_z=p|f|^{p-1}|f|_z$.

For $p=2$, we have $2|f||f|_z=(|f|^2)_z=(f\overline{f})_z=f(\overline{f})_z+\overline{f}f_z=\overline{f}f_z$ because f is holomorphic $\Rightarrow(\overline{f})_z=\overline{(f_{\overline{z}})}=0$.

Therefore we have

$$(|f|^p)_z=p|f|^{p-1}|f|_z=p|f|^{p-1}\frac{\overline{f}f_z}{2|f|}=\frac{p|f|^{p-2}\overline{f}f_z}{2}$$

Then, taking complex conjugate, we have

$$(|f|^p)_{\overline{z}} = \frac{p|f|^{p-2}f\overline{f_z}}{2}$$

Therefore,

$$\begin{aligned}
& \Delta\left(|f|^p\right) \\
= \;& 4\left((|f|^p)_z\right)_{\overline{z}} \\
= \;& 4\left(\frac{p|f|^{p-2}\overline{f}f_z}{2}\right)_{\overline{z}} \\
= \;& 2p\left((|f|^{p-2})_{\overline{z}}\,\overline{f}f_z + |f|^{p-2}\left(\overline{f}\right)_{\overline{z}} f_z + |f|^{p-2}\overline{f}\left(f_z\right)_{\overline{z}}\right) \quad \text{by product rule} \\
= \;& 2p\left(\frac{(p-2)|f|^{p-4}f\overline{f_z}}{2}\overline{f}f_z + |f|^{p-2}\overline{(f)_z}f_z + 0\right) \quad \text{because } \left(f_z\right)_{\overline{z}} = \left(f_{\overline{z}}\right)_z = 0 \\
= \;& p\left((p-2)|f|^{p-4}|f|^2|f_z|^2 + 2|f|^{p-2}|f_z|^2\right) \\
= \;& p^2|f|^{p-2}|f_z|^2
\end{aligned}$$

You can also do it as follows:

Let $f = u + iv$. Then

$$|f|^2 = u^2 + v^2 \Rightarrow 2|f||f|_x = 2uu_x + 2vv_x \Rightarrow |f|_x = \frac{uu_x + vv_x}{|f|}$$

Similarly, $|f|_y = \dfrac{uu_y + vv_y}{|f|}$. Therefore,

$$\begin{aligned}
& (|f|^p)_x = p|f|^{p-1}|f|_x = p|f|^{p-2}\left(uu_x + vv_x\right) \\
\Rightarrow\;& (|f|^p)_{xx} = p(p-2)|f|^{p-4}\left(uu_x + vv_x\right)^2 + p|f|^{p-2}\left(uu_{xx} + u_x^2 + vv_{xx} + v_x^2\right)
\end{aligned}$$

Similarly, $(|f|^p)_{yy} = p(p-2)|f|^{p-4}\left(uu_y + vv_y\right)^2 + p|f|^{p-2}\left(uu_{yy} + u_x^2 + vv_{yy} + v_y^2\right)$.

Therefore,

$$
\begin{aligned}
& \Delta\left(|f|^p\right) \\
= \; & p(p-2)|f|^{p-4}\left(\left(u u_x+v v_x\right)^2+\left(u u_y+v v_y\right)^2\right) \\
& +p|f|^{p-2}\left(u\left(u_{xx}+u_{yy}\right)+v\left(v_{xx}+v_{yy}\right)+\left(u_x^2+v_x^2+u_y^2+v_y^2\right)\right) \\
= \; & p(p-2)|f|^{p-4}\left(\left(u u_x+v v_x\right)^2+\left(-u v_x+v u_x\right)^2\right)+2 p|f|^{p-2}\left(u_x^2+v_x^2\right) \\
= \; & p|f|^{p-4}\left[(p-2)\left(u^2 u_x^2+2 u v u_x v_x+v^2 v_x^2+u^2 v_x^2-2 u v u_x v_x+v^2 u_x^2\right)\right. \\
& \left.+2\left(u^2+v^2\right)\left(u_x^2+v_x^2\right)\right] \\
= \; & p|f|^{p-4}\left[(p-2)\left(u^2+v^2\right)\left(u_x^2+v_x^2\right)+2\left(u^2+v^2\right)\left(u_x^2+v_x^2\right)\right] \\
= \; & p|f|^{p-4}\left[p\left(u^2+v^2\right)\left(u_x+i v_x\right)\left(u_x-i v_x\right)\right] \\
= \; & p^2|f|^{p-2}\left|\frac{\partial f}{\partial z}\right|^2 \qquad \left(f \text{ holomorphic } \Rightarrow \frac{\partial f}{\partial z}=\frac{\partial f}{\partial x}=u_x+i v_x\right)
\end{aligned}
$$

10. An open subset U of $\mathbb{R}^2$ is said to be **star-shaped** if there exists $(a, b) \in U$ such that for every $(x, y) \in U$, the line segment joining (a, b) to (x, y)

$$r(t)=(a+t(x-a),\, b+t(y-b)),\ 0 \leq t \leq 1$$

is contained in U. Suppose U is star-shaped and $f,\, g \in C^1(U)$ satisfies $f_y=g_x$. Show that the function $h(x, y)=\int_0^1(f(r(t)), g(r(t))) \cdot r^{\prime}(t)\, dt$ satisfies $h_x=f$ and $h_y=g$. (Hint: Show that $h_x=\int_0^1 \frac{d(t\, f(r(t)))}{dt}\, dt$ and $h_y=\int_0^1 \frac{d(t\, g(r(t)))}{dt}\, dt$.)

Solution: Let

$$
\begin{aligned}
& h(x, y) \\
= \; & \textstyle\int_0^1(f(r(t)), g(r(t))) \cdot r^{\prime}(t)\, dt \\
= \; & \textstyle\int_0^1(f(a+t(x-a),\, b+t(y-b)), g(a+t(x-a),\, b+t(y-b))) \cdot((x-a),\,(y-b))\, dt \\
= \; & \textstyle\int_0^1(x-a) f(a+t(x-a),\, b+t(y-b))+(y-b) g(a+t(x-a),\, b+t(y-b))\, dt
\end{aligned}
$$

Then

$$
\begin{aligned}
& h_x \\
= \;& \textstyle\int_0^1 f(a+t(x-a),\, b+t(y-b)) + (x-a)tf_x(a+t(x-a),\, b+t(y-b))\, dt \\
& + \textstyle\int_0^1 (y-b)tg_x(a+t(x-a),\, b+t(y-b))\, dt \\
= \;& \textstyle\int_0^1 f(a+t(x-a),\, b+t(y-b)) + (x-a)tf_x(a+t(x-a),\, b+t(y-b))\, dt \\
& + \textstyle\int_0^1 (y-b)tf_y(a+t(x-a),\, b+t(y-b))\, dt \\
= \;& \textstyle\int_0^1 \dfrac{d\left(t\cdot f(a+t(x-a),\, b+t(y-b))\right)}{dt}\, dt \\
= \;& t\cdot f(a+t(x-a),\, b+t(y-b))]_0^1 \\
= \;& f(x,y)
\end{aligned}
$$

$$
\begin{aligned}
& h_y \\
= \;& \textstyle\int_0^1 t(x-a)f_y(a+t(x-a),\, b+t(y-b))\, dt \\
& + \textstyle\int_0^1 g(a+t(x-a),\, b+t(y-b)) + (y-b)tg_y(a+t(x-a),\, b+t(y-b))\, dt \\
= \;& \textstyle\int_0^1 t(x-a)g_x(a+t(x-a),\, b+t(y-b))\, dt \\
& + \textstyle\int_0^1 g(a+t(x-a),\, b+t(y-b)) + (y-b)tg_y(a+t(x-a),\, b+t(y-b))\, dt \\
= \;& \textstyle\int_0^1 \dfrac{d\left(t\cdot g(a+t(x-a),\, b+t(y-b))\right)}{dt}\, dt \\
= \;& t\cdot g(a+t(x-a),\, b+t(y-b))]_0^1 \\
= \;& g(x,y)
\end{aligned}
$$