

**502 ASSIGNMENTS**  
**SPRING 2017**

Assignment I

1. Let  $\mathcal{V}$  be a vector space over the field  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ . A *seminorm* on  $\mathcal{V}$  is a map  $\rho : \mathcal{V} \rightarrow \mathbb{R}$  satisfying

- (i)  $\rho(v) \geq 0$  for all  $v \in \mathcal{V}$
- (ii)  $\rho(\alpha v) = |\alpha| \rho(v)$  for all  $\alpha \in \mathbb{F}$  and  $v \in \mathcal{V}$ .
- (iii)  $\rho(v + w) \leq \rho(v) + \rho(w)$  for all  $v, w \in \mathcal{V}$ .

Let  $\varphi$  be a continuous function  $[0, 1] \rightarrow \mathbb{R}$ .

For  $f \in C[0, 1]$  define  $\rho(f) = \int_0^1 |f(x)| \varphi(x) dx$ . What conditions must  $\varphi$  satisfy so that  $\rho$  is a seminorm? What conditions must  $\varphi$  satisfy so that  $\rho$  is a norm?

2. Let  $C^1[0, 1]$  denote the vector space of functions defined on the interval  $[0, 1]$  which have continuous derivatives. (The derivative at an endpoint is the one-sided derivative.) For  $f \in C^1[0, 1]$ , let  $\rho(f) = \max_{0 \leq x \leq 1} |f'(x)|$ . Is  $\rho$  a norm? Is it a seminorm?

3. Determine which of the following formulas define a metric.

- (i) On  $\mathbb{R}$ ,  $d(x, y) = \sqrt{|x - y|}$
- (ii) On  $\mathbb{R}$ ,  $d(x, y) = (x - y)^2$
- (iii) On  $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ ,  $d(x, y) = |\log(y/x)|$

4. Do exercises 1, 2, and 4 in section 2.1, p. 27

due 1/18

Assignment II

1. In  $(\mathbb{Q}, |\cdot|)$ , let  $a \in \mathbb{Q}$ , describe

- (i)  $\partial B_r(a)$  if  $r \in \mathbb{Q}$ ,  $r > 0$ .
- (ii)  $\partial B_r(a)$  if  $r \in \mathbb{R} \setminus \mathbb{Q}$ ,  $r > 0$ .

2. Do problems 1, 2, 4, and 5 in Sec. 2.2, p. 34.

3. Do problems 1 and 3 in Sec. 2.3, p. 40.

due 1/25

Assignment III

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ x^2, & \text{if } x \text{ is irrational} \end{cases}$$

Using the definition of continuity (i.e., involving convergent sequences), verify that  $f$  is continuous at  $x = 0$  and at  $x = 1$ .

Show that for  $0 < \epsilon < 1$ ,  $f^{-1}(B_\epsilon(1))$  is a neighborhood of 1, but it is not an open neighborhood.

Recall the notation:  $B_\epsilon(1) = (1 - \epsilon, 1 + \epsilon)$ .

2. Let  $(X, d)$  be a metric space.

- (i) If  $\{x_n\}_{n=1}^\infty$  is a convergent sequence with  $\lim_{n \rightarrow \infty} x_n = x_0$ , then the sequence  $x_1, x_0, x_2, x_0, x_3, x_0, \dots$  is Cauchy.
- (ii) If  $\{x_n\}_{n=1}^\infty$  is a sequence, and the sequence  $x_1, x_0, x_2, x_0, x_3, x_0, \dots$  is Cauchy, then  $\lim_{n \rightarrow \infty} x_n = x_0$ .
- (iii) If  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence, and  $x_0 \in X$  is such that  $x_n = x_0$  for infinitely many  $n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} x_n = x_0$ .

3. Let  $d, d'$  be two metrics on a (non-empty) set  $X$ . The metrics  $d, d'$  are *equivalent* if they satisfy the condition of Definition 2.3.12 in our text. We say the metrics  $d, d'$  are *Cauchy equivalent* if whenever a sequence  $\{x_n\}$  is Cauchy in one of the metrics, it is Cauchy in the other. We say that the metrics  $d, d'$  are *strongly equivalent* if there exist positive constants  $m, M$  such that, for  $x, y \in X$ ,

$$m d(x, y) \leq d'(x, y) \leq M d(x, y)$$

- (i) Show that each of these definitions gives an equivalence relation on the set of all metrics on the space  $X$ .
- (ii) Show that  $d, d'$  strongly equivalent implies  $d, d'$  Cauchy equivalent.
- (iii) Show that  $d, d'$  Cauchy equivalent implies  $d, d'$  equivalent.
- (iv) Show that on  $X = \mathbb{R}_+$ , the positive reals, the metrics  $d(x, y) = |y - x|$  and  $d'(x, y) = |\log(y/x)|$  are equivalent metrics, but not Cauchy equivalent.
- (v) On  $X = \mathbb{R}$ , let  $d(x, y) = |y - x|$  and  $d'(x, y) = \sqrt{|y - x|}$ . Show that the two metrics are Cauchy equivalent but not strongly equivalent.

Recall, an *equivalence relation*  $\sim$  on a set  $S$  is a relation which satisfies

- (1)  $s \sim s \ \forall s \in S$
- (2) If  $s \sim t$  then  $t \sim s$

- (3) If  $s \sim t$  and  $t \sim u$  then  $s \sim u$

Apply this definition to the set  $S$  of all metrics on a space  $X$ .

4. Do problem 6 in Sec. 2.3, p. 40.

5. Do problems 2 and 4 in Sec. 2.4, p. 51.

Hint: For problem 2, show that  $d(x_{n+1}, x_n) \leq \theta^{n-1}d(x_2, x_1)$ . Use this to estimate  $d(x_m, x_n)$ .

due 2/1

#### Assignment IV

1. The space  $X = C_b(\mathbb{R}, \mathbb{R})$  is equipped with the metric  $D$  given by

$$D(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

Which of the following subspaces  $Y$  is a complete metric space?

- (i)  $Y = \{f \in X : \lim_{x \rightarrow \infty} f(x) = 0\}$
- (ii)  $Y = \{f \in X : f \text{ is differentiable}\}$
- (iii)  $Y = \{f \in X : \int_0^1 f(x) dx = 0\}$

2. Let  $X = \mathbb{R}_+$ , the set of positive real numbers, equipped with the metrics  $d(x, y) = |y - x| + |1/y - 1/x|$  and  $d'(x, y) = |\log(y/x)|$ . Show that these metrics are Cauchy equivalent, and in fact  $\mathbb{R}_+$  is complete in both metrics. Show furthermore that the metrics are not strongly equivalent.

3. Let  $(X, d_X)$ ,  $(Y, d_Y)$  be metric space, and suppose  $(X, d_X)$  is complete. Further suppose that there is a continuous surjection  $f : X \rightarrow Y$ . Does it follow that  $(Y, d_Y)$  is complete? Either prove, or give a counterexample.

4. Consider the following sequence of functions  $f_n : [0, 1] \rightarrow \mathbb{R}$ , defined by

$$f_n(x) = \begin{cases} 0, & \text{if } x = 0, \\ 0, & \text{if } x \text{ is irrational,} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ with } q \leq n, \\ 0, & \text{if } x = \frac{p}{q}, \text{ with } q > n \end{cases}$$

where when we write  $x = \frac{p}{q}$  we mean that the  $p, q$  have no common divisor greater than 1.

It is clear that the sequence  $\{f_n\}$  converges *pointwise* to a function  $f : [0, 1] \rightarrow \mathbb{R}$ . Does the sequence  $\{f_n\}$  converge uniformly to  $f$ , in other words, does  $\|f_n - f\|_\infty \rightarrow 0$ ?

At what points is the function  $f$  continuous?

5. Do problems 5 and 7(all parts) in Sec. 2.4

due 2/8

#### Assignment V

1. Do problems 3, 4 and 6 in Sec. 2.5

2. Let  $\{r_n\}_{n=1}^\infty$  be an enumeration of the rationals in the interval  $[0, 1]$ , and define  $f : [0, 1] \rightarrow \mathbb{R}$  by

$$f(x) = \sum_{r_n < x} 2^{-n}$$

Show that  $f$  is discontinuous at every rational number in  $[0, 1]$ , and continuous at every irrational number in  $[0, 1]$ . In fact, show that  $f$  is left-continuous at every rational, and discontinuous from the right; that is, for  $r \in (0, 1)$

$$\lim_{x \rightarrow r^-} f(x) = f(r) \quad \text{and} \quad \lim_{x \rightarrow r^+} f(x) < f(r)$$

Define  $F : [0, 1] \rightarrow \mathbb{R}$ ,  $F(x) = \int_0^x f(t) dt$ . By the Fundamental Theorem of Calculus,  $F$  is differentiable at points  $x$  at which  $f$  is continuous, and at those points  $F'(x) = f(x)$ . But the FTC does not say anything if  $f$  is not continuous at  $x$ . So, it could happen that  $F$  is differentiable at  $x$  or not. Thus it follows from the FTC that  $F$  is differentiable at the irrational numbers in  $(0, 1)$ . Show that  $F$  is not differentiable at rational numbers in  $(0, 1)$ .

3. Let  $a < b \in \mathbb{R}$ , and let  $\text{Lip}[a, b]$  denote the set of functions  $f : [a, b] \rightarrow \mathbb{R}$ , such that there is a positive constant  $M$  for which

$$|f(y) - f(x)| \leq M|y - x| \text{ for all } x, y \in [a, b].$$

$M$  is said to be a Lipschitz constant for the function  $f$ , and  $f$  is said to be a Lipschitz function. Show that  $\text{Lip}[a, b]$  is a subspace of  $C[a, b]$ .

Fix a positive constant  $M_0$ . Let

$$A = \{f \in \text{Lip}[a, b] : M_0 \text{ is a Lipschitz constant for } f, \text{ and } |f(a)| \leq M_0\}$$

Show that  $A$  is compact in  $C[a, b]$ .

4. Let  $C^1[0, 1]$  denote the set of continuous functions on the interval  $[0, 1]$  which are differentiable, and such that the derivative is continuous. (Note: the derivative at the endpoints is the one-sided derivative.) Define a norm on  $C^1[0, 1]$  by

$$\|f\| = \sup_{0 \leq x \leq 1} |f(x)| + \sup_{0 \leq x \leq 1} |f'(x)|$$

Show that  $C^1[0, 1]$  is a complete metric space.

Hint: Let  $\{f_n\}$  be a Cauchy sequence in  $C^1[0, 1]$ . Then both  $\{f_n\}$  and  $\{f'_n\}$  are Cauchy sequences in  $C[a, b]$ , which is complete, so they have limits, say,  $f, g$  respectively. Apply the Fundamental Theorem of Calculus to show that  $f \in C^1[0, 1]$ .

Let  $B_1 = \{f \in C^1[0, 1] : \|f\| \leq 1\}$ . Is  $B_1$  compact  $C^1[0, 1]$ ?

due 2/15

### Assignment VI

1. A *partition* of  $[0, 1]$  is a set of points  $0 = x_0 < x_1 < \cdots < x_n = 1$ . If  $X = C([0, 1], \mathbb{R})$  with the usual supremum metric, and if  $f \in X$ , define the variation of  $f$  over the partition  $\mathcal{P}$  to be  $V(f, \mathcal{P}) = \sum_{j=1}^n |f(x_j) - f(x_{j-1})|$ . The variation of  $f$  (also called the total variation of  $f$ ) is defined to be  $V(f) = \sup_{\mathcal{P}} V(f, \mathcal{P})$ , where the supremum is taken over all partitions of  $[0, 1]$ . Of course, the supremum could be  $+\infty$ . Consider the set

$$\{f \in X : f(0) = 1 \text{ and } V(f) \leq 1\}$$

Is this set complete in  $X$ ? Is it compact?

2. Let  $X = C([0, 1], \mathbb{R})$  with the usual supremum metric, and  $B_1[0] = \{f \in X : \|f\|_{\infty} \leq 1\}$ . Define the Volterra integral operator on  $X$  by

$$Vf(x) = \int_0^x f(t) dt$$

Show that the closure of the set  $V(B_1[0]) = \{Vf : f \in B_1[0]\}$  is compact in  $X$ .

3. Do problems 2, 4, and 6 in Sec. 3.1.

4. Let  $\tau$  be the topology on the integers defined in problem 6. Is this topology Hausdorff?

5. Let  $(\mathbb{R}^2, \tau)$  be the topological space of example 3.1.26.

(a) Is this topology Hausdorff?

(b) Let  $(a, b) \in \mathbb{R}^2$ . Is the one-point set  $\{(a, b)\}$  closed? If not, what is its closure?

- (c) Let  $S$  be the square  $S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ . Describe  $\partial S$ .

due 2/22

### Assignment VII

1. We define a topology on the natural numbers as follows: a set  $U$  is open if either  $U = \emptyset$  or else

$$\liminf_n \frac{\text{card}(U \cap \{1, 2, \dots, n\})}{n} = 1$$

Verify that this defines a topology on  $\mathbb{N}$ . Is this topology Hausdorff?

2. On the set  $\mathbb{R}$  of real numbers, let  $\tau$  be the topology in which the open sets are complements of countable sets, together with the empty set. Let  $|\cdot|$  refer to the metric topology on  $\mathbb{R}$  defined by the absolute value metric.

- (a) Let  $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)$ ,  $f(x) = x^2$ . Is  $f$  continuous?
- (b) Let  $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, |\cdot|)$ ,  $f(x) = x^2$ . Is  $f$  continuous?
- (c) Let  $f : (\mathbb{R}, |\cdot|) \rightarrow (\mathbb{R}, \tau)$ ,  $f(x) = x^2$ . Is  $f$  continuous?

3. Do problem 4 in Sec. 1.3 on page 20.

4. Do problems 4, 5 and 11 in Sec. 3.2, page 78.

5. Put a topology  $\tau$  on  $\mathbb{R}$  as follows: a base  $\mathcal{B}$  for the topology  $\tau$  consists of

$$\mathcal{B} = \{U_{a,b} = [a, b), a < b \in \mathbb{R}\}$$

- (a) Show that this topology is finer than the usual topology on  $\mathbb{R}$ .
- (b) Is this topology first countable?
- (c) Is this topology second countable?
- (d) Is this topology separable?

6. Let  $\tau'$  be a topology on  $\mathbb{R}$  as follows: a base  $\mathcal{B}'$  for this topology consists of

$$\mathcal{B}' = \{U_{a,b} = [a, b), a < b, \text{ and } a \in \mathbb{Q}, b \in \mathbb{R}\}$$

Answer the same questions as in problem 5 for the topology  $\tau'$ .

due 3/1

### Assignment VIII

1. Do problems 2, 4, 7 and 8 in Sec. 3.3, page 88.

2. Let  $\tau$  be the topology on  $\mathbb{R}$  which was defined in problem 6 of Assignment VII. Show that this topology is finer than the usual metric topology on  $\mathbb{R}$ . Show that the topology is second countable. Is  $[0, 1]$  compact in this topology? Is  $\mathbb{R}$  locally compact in this topology?

3. Let  $\{r_n\}$  be an enumeration of the rational numbers, and consider the functions

$$f(x) = \sum_{r_n < x} 2^{-n} \text{ and } g(x) = \sum_{r_n \leq x} 2^{-n}$$

Let  $\tau$  be the topology on  $\mathbb{R}$  defined in problem 6 of Assignment VII. Let  $(\mathbb{R}, |\cdot|)$  denote the reals with the usual topology. Viewing the functions  $f, g$  as functions from  $(\mathbb{R}, \tau) \rightarrow (\mathbb{R}, |\cdot|)$ , one of these functions is continuous, the other is not. Explain.

due 3/8

### Assignment IX

1. Do problems 2, 4, 6, and 8 in Sec. 3.4, p. 99.

2. Let  $\mathcal{R}$  be a commutative ring with identity, which is an integral domain. Let  $X$  be the space of prime ideals of  $\mathcal{R}$ , equipped with the Zariski topology. Show that  $X$  is connected.

3. Let  $X$  be the space of all strictly increasing functions  $f : [0, 1] \rightarrow [0, 1]$  such that  $f(0) = 0, f(1) = 1$ . Put the supremum metric on  $X$ , i.e.,  $d(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$ .

- (a) Is  $X$  compact?
- (b) Is  $X$  locally compact?
- (c) Is  $X$  separable; either find a countable dense subset, or show it is not separable.
- (d) Is  $X$  connected? Is it path connected?

4. Do problems 1, 3 and 8 in Sec. 3.5, p. 106.

due 3/29

### Assignment X

Recall the definition of a regular topological space:  $(X, \tau)$  is regular if it is  $T_1$  and if given any point  $x_0 \in X$ , and any closed set  $F \subset X$  such that  $x_0 \notin F$ , there are open sets  $U, V$  in  $X$  such that  $x_0 \in U, F \subset V$ , and  $U \cap V = \emptyset$ .

Note: This definition is only mentioned in the chapter notes at the end of chapter 4.

1. Prove that if  $(X, \tau)$  is a regular space, and if  $Y \subset X$  is a subspace, then  $Y$  is regular.

2. Let  $g, h$  be continuous functions,  $[0, 1] \rightarrow \mathbb{R}$  such that  $g(0) = h(0)$ . Prove that there is a continuous function  $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  with “boundary values”  $g$  and  $h$ . That is,

$$f(x, 0) = g(x), \quad x \in [0, 1], \quad \text{and} \quad f(0, y) = h(y), \quad y \in [0, 1].$$

Indeed, give *two* proofs of this fact: one which uses Theorem 4.1.13 (Tietze’s extension theorem), and a second, constructive proof (which does not use any theorem in this chapter).

3. Do problems 2, 3, 4 and 7 in Sec. 4.1, pages 115 - 116.

4. Prove that if  $(X, \tau_X)$  and  $(Y, \tau_Y)$  are regular spaces, and  $Z = X \times Y$  (with the product topology), then  $Z$  is regular.

due 4/5

#### Assignment XI

1. Do problems 2, 3 and 4 in Sec. 4.2, pp. 120 - 121.

2.(a) Let  $\tau$  be the topology on  $\mathbb{R}$  defined by the base  $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ . Is  $(\mathbb{R}, \tau)$  metrizable?

(b) Let  $\tau'$  be the topology on  $\mathbb{R}$  defined by the base  $\mathcal{B}' = \{[a, b) : a, b \in \mathbb{Q}\}$ . Is  $(\mathbb{R}, \tau')$  metrizable?

3. Let  $(X, \tau)$  be a locally compact Hausdorff topological space which is not compact. Since any locally compact space is completely regular, there are two compactifications available: the one-point compactification  $X_\infty$  and the Stone-Čech compactification,  $\beta X$ .

Let  $\kappa : X \rightarrow X_\infty$  be the map  $\kappa(x) = x$ , where  $X$  is viewed as a subspace of  $X_\infty$ . If  $\iota : X \rightarrow \beta X$  is the embedding of Theorem 4.2.4, then the theorem states there is a unique continuous map  $\hat{\kappa} : \beta X \rightarrow X_\infty$  such that  $\hat{\kappa} \circ \iota = \kappa$ . Let  $\omega$  be a point of  $\beta X \setminus \iota(X)$ . What is  $\hat{\kappa}(\omega)$ ?

due 4/12



## Assignment XII

1. Let  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $f(x) = 2x + 1$ . Let  $\mathcal{A}$  be the algebra over  $\mathbb{R}$  generated by  $f$ . Thus, a typical element of  $\mathcal{A}$  is a function  $g = a_1f + a_2f^2 + \cdots + a_nf^n$  for  $a_j \in \mathbb{R}$ ,  $1 \leq j \leq n$ , and  $n \in \mathbb{N}$ . Is  $\mathcal{A}$  dense in  $C([0, 1], \mathbb{R})$ ?

2. Let  $f : [0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = e^{-x}$ . Is the algebra generated by  $f$  dense in  $C_0([0, \infty), \mathbb{R})$ ?

3. Let  $S \subset \mathbb{R}^2$ , with topology induced from the metric topology of  $\mathbb{R}^2$ ,

$$S = \{(x, y) : 1 \leq x \leq 2, 1 \leq y \leq 2\}$$

Let  $f : S \rightarrow \mathbb{R}$ ,  $f(x, y) = 1/x$ , and  $g : S \rightarrow \mathbb{R}$ ,  $g(x, y) = 1/y$ . Let  $\mathcal{A}$  be the algebra of functions generated by  $f$  and  $g$ . Is  $\mathcal{A}$  dense in  $C(S, \mathbb{R})$ ?

4. The unit circle  $\mathbb{T}$  can be expressed as the set of complex numbers  $z$  such that  $|z| = 1$ . Let  $\mathcal{A}$  be the algebra of functions  $f : \mathbb{T} \rightarrow \mathbb{C}$  such that  $f(z) = \sum_{-n}^n a_j z^j$ ,  $a_j \in \mathbb{C}$ ,  $n \in \mathbb{N}$ . Is  $\mathcal{A}$  dense in  $C(\mathbb{T}, \mathbb{C})$ ?

5. Do problems 2, 5, 6 and 8 in Sec. 4.3, p. 129.

due 4/21