MATH/STAT 521, AUTUMN 2018
MIDTERM 1, DUE NOVEMBER 5

RULES AND REGULATIONS:

(i) You cannot collaborate with any other person.

(ii) You can use results from any published book or research article in a journal. You must include detailed citation if you use a result from any source. You cannot cite materials from the Web. You can search the Web but if you want to use a result, you must find it in a book or research article.

(iii) Submit the pdf by e-mail just like homework.

(iv) Midterms will not be posted in Canvas. If I have comments, the pdf's will be returned to students individually.

Problem 1. (5 points) Suppose that X is a uniform random variable on $[0, 1]$, that is, it has a density that is equal to 1 on this interval.

Prove that there exists a function $f : [0, 1] \to (-\infty, \infty)$ with the following properties.

(i) f is continuous.

(ii) f is strictly increasing, that is, if $x < y$ then $f(x) < f(y)$.

(iii) Let $Y = f(X)$. Random variable Y does not have a density, that is, there is no measurable function $g : (-\infty, \infty) \to [0, \infty)$ such that for all $a, b \in (-\infty, \infty)$ we have

$$\mathbb{P}(Y \in (a, b)) = \int_a^b g(x)dx,$$

where the integral should be interpreted in the Lebesgue sense.

Problem 2. (10 points) This problem is concerned with a modified St Petersburg paradox.

Suppose that $X_1, X_2, \dots$ are i.i.d. random variables with the following distribution,

$$\mathbb{P}(X_k = 2^{2j}) = 2^{-j},$$

for $j = 1, 2, \dots$.

(i) Let N_n be the number of X_k's with $1 \le k \le 2^n$ such that $X_k = 2^{2n}$. Prove that for every $m_1 < \infty$ there exist $p_1 > 0$ and $n_1 < \infty$ such that for every $n \ge n_1$,

$$\mathbb{P}(N_n \ge m_1) \ge p_1.$$

(ii) Let M_n be the number of X_k's with $1 \le k \le 2^n$ such that $X_k > 2^{2n}$. Prove that there exist $p_2 > 0$ and $n_2 < \infty$ such that for every $n \ge n_2$,

$$\mathbb{P}(M_n = 0) \ge p_2.$$

(iii) Let

$$S_n^* = \sum_{k=1}^{2^n} X_k \mathbf{1}_{\{X_k < 2^{2n}\}}.$$

Prove that for some $a < \infty$ there exist $p_3 < 1$ and $n_3 < \infty$ such that for every $n \geq n_3$,

$$\mathbb{P}(S_n^* \geq a2^{2n}) \leq p_3.$$

(iv) Let

$$S_n = \sum_{k=1}^{2^n} X_k.$$

Prove that there is no sequence of real numbers μ_n such that

$$\frac{S_n}{\mu_n} \to 1,$$

in distribution.

Hint: One can base the proof of (iv) on (i)-(iii) but this requires some care because random variables N_n, M_n and S_n^* are not independent. You are not required to use this hint to solve (iv).

Problem 3. (10 points) For a random variable X, we will call the function

$$L_X(t) = \mathbb{E}(e^{-tX}), \qquad -\infty < t < \infty,$$

the Laplace transform of X. The Laplace transform may exist for only some values of t (or none at all).

(i) Prove that if X and Y are independent then $L_{X+Y}(t) = L_X(t)L_Y(t)$, provided that the Laplace transforms exist.

(ii) Suppose that S_n has the binomial distribution with parameters n and $1/2$, that is,

$$\mathbb{P}(S_n = k) = \binom{n}{k}(1/2)^n, \qquad 0 \leq k \leq n.$$

Compute $L_{S_n}(t)$.

(iii) Suppose that Z is standard normal, that is, the density of Z is

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Compute $L_Z(t)$.

(iv) Let $R_n = (S_n - n/2)/\sqrt{n}$. Prove that for some constant c and every real t,

$$L_{cR_n}(t) \to L_Z(t).$$

Find c.