Part II: Real Analysis

5

Construct a function on $[0,1]$ which is monotone increasing and discontinuous precisely at the rationals. Rigorously prove that your function has the desired properties.

Solution

Let $\{\alpha_n\}_{n=0}^{\infty}$ be an enumeration of the rationals in $[0,1]$. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) := \sum_{\{n:\alpha_n \leq x\}} 2^{-n}, \qquad \forall\, x \in [0,1].$$

Claim 1: f is monotone increasing.

Proof of claim 1: Trivial by the construction of f. ■ Claim 1

Claim 2: f is discontinuous at every rational in $[0,1]$ and continuous otherwise.

Proof of claim 2: Let $x \in [0,1]$

Case 1 $x \in [0,1] - \mathbb{Q}$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$. By the denseness of the rationals in $[0,1]$, we can choose $\delta > 0$ such that $n > N$ for all $\alpha_n \in \mathbb{Q} \cap (x-\delta, x+\delta)$. Now let $y \in (x-\delta, x+\delta)$. Without loss of generality suppose $y > x$. Then

$$|f(x) - f(y)| = \sum_{\{n:x<\alpha_n \leq y_n\}} 2^{-n} \leq \sum_{n=N+1}^{\infty} 2^{-n} = 2^{-N} < \epsilon.$$

Therefore f is continuous at x.

Case 2 $x \in \mathbb{Q}$.

There exists $N \in \mathbb{N}$ such that $\alpha_N = x$. So clearly $f(x) - f(y) > 2^{-N}$ for all $y < x$ (assuming $x \neq 0$ of course). Therefore f is discontinuous at x.

■ Claim 2