502 ASSIGNMENTS
SPRING 2017

Assignment I

1. Let $\mathcal{V}$ be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A *seminorm* on $\mathcal{V}$ is a map $\rho : \mathcal{V} \to \mathbb{R}$ satisfying

(i) $\rho(v) \geq 0$ for all $v \in \mathcal{V}$
(ii) $\rho(\alpha v) = |\alpha|\rho(v)$ for all $\alpha \in \mathbb{F}$ and $v \in \mathcal{V}$.
(iii) $\rho(v + w) \leq \rho(v) + \rho(w)$ for all $v, w \in \mathcal{V}$.

Let φ be a continuous function $[0, 1] \to \mathbb{R}$.

For $f \in C[0, 1]$ define $\rho(f) = \int_0^1 |f(x)|\varphi(x)\, dx$. What conditions must φ satisfy so that ρ is a seminorm? What conditions must φ satisfy so that ρ is a norm?

2. Let $C^1[0, 1]$ denote the vector space of functions defined on the interval $[0, 1]$ which have continuous derivatives. (The derivative at an endpoint is the one-sided derivative.) For $f \in C^1[0, 1]$, let $\rho(f) = \max_{0 \leq x \leq 1} |f'(x)|$. Is ρ a norm? Is it a seminorm?

3, Determine which of the following formulas define a metric.

(i) On $\mathbb{R}$, $d(x, y) = \sqrt{|x - y|}$
(ii) On $\mathbb{R}$, $d(x, y) = (x - y)^2$
(iii) On $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, $d(x, y) = |\log(y/x)|$

4. Do exercises 1, 2, and 4 in section 2.1, p. 27

due 1/18

Assignment II

1. In $(\mathbb{Q}, |\cdot|)$, let $a \in \mathbb{Q}$, describe

(i) $\partial B_r(a)$ if $r \in \mathbb{Q}$, $r > 0$.
(ii) $\partial B_r(a)$ if $r \in \mathbb{R} \setminus \mathbb{Q}$, $r > 0$.

2. Do problems 1, 2, 4, and 5 in Sec. 2.2, p. 34.

3. Do problems 1 and 3 in Sec. 2.3, p. 40.

due 1/25

Assignment III

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ x^2, & \text{if } x \text{ is irrational} \end{cases}$$

Using the definition of continuity (i.e., involving convergent sequences), verify that f is continuous at $x = 0$ and at $x = 1$.

Show that for $0 < \epsilon < 1$, $f^{-1}(B_\epsilon(1))$ is a neighborhood of 1, but it is not an open neighborhood.

Recall the notation: $B_\epsilon(1) = (1 - \epsilon, 1 + \epsilon)$.

2. Let (X, d) be a metric space.

(i) If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} x_n = x_0$, then the sequence $x_1, x_0, x_2, x_0, x_3, x_0, \ldots$ is Cauchy.

(ii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence, and the sequence $x_1, x_0, x_2, x_0, x_3, x_0, \ldots$ is Cauchy, then $\lim_{n\to\infty} x_n = x_0$.

(iii) If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and $x_0 \in X$ is such that $x_n = x_0$ for infinitely many $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = x_0$.

3. Let d, d' be two metrics on a (non-empty) set X. The metrics d, d' are *equivalent* if they satisfy the condition of Definition 2.3.12 in our text. We say the metrics d, d' are *Cauchy equivalent* if whenever a sequence $\{x_n\}$ is Cauchy in one of the metrics, it is Cauchy in the other. We say that the metrics d, d' are *strongly equivalent* if there exist positive constants m, M such that, for $x, y \in X$,

$$m\, d(x, y) \leq d'(x, y) \leq M\, d(x, y)$$

(i) Show that each of these definitions gives an equivalence relation on the set of all metrics on the space X.

(ii) Show that d, d' strongly equivalent implies d, d' Cauchy equivalent.

(iii) Show that d, d' Cauchy equivalent implies d, d' equivalent.

(iv) Show that on $X = \mathbb{R}_+$, the positive reals, the metrix $d(x, y) = |y - x|$ and $d'(x, y) = |\log(y/x)|$ are equivalent metrics, but not Cauchy equivalent.

(v) On $X = \mathbb{R}$, let $d(x, y) = |y - x|$ and $d'(x, y) = \sqrt{|y - x|}$. Show that the two metrics are Cauchy equivalent but not strongly equivalent.

Recall, an *equivalence relation* $\sim$ on a set S is a relation which satisfies

(1) $s \sim s \; \forall s \in S$

(2) If $s \sim t$ then $t \sim s$

(3) If $s \sim t$ and $t \sim u$ then $s \sim u$

Apply this definition to the set S of all metrics on a space X.

4. Do problem 6 in Sec. 2.3, p. 40.

5. Do problems 2 and 4 in Sec. 2.4, p. 51.
Hint: For problem 2, show that $d(x_{n+1}, x_n) \leq \theta^{n-1} d(x_2, x_1)$. Use this to estimate $d(x_m, x_n)$.

due 2/1

Assignment IV

1. The space $X = C_b(\mathbb{R}, \mathbb{R})$ is equipped with the metric D given by

$$D(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$$

Which of the following subspaces Y is a complete metric space?

(i) $Y = \{f \in X : \lim_{x\to\infty} f(x) = 0\}$
(ii) $Y = \{f \in X : f \text{ is differentiable}\}$
(iii) $Y = \{f \in X : \int_0^1 f(x)\, dx = 0\}$

2. Let $X = \mathbb{R}_+$, the set of positive real numbers, equipped with the metrics $d(x, y) = |y - x| + |1/y - 1/x|$ and $d'(x, y) = |\log(y/x)|$. Show that these metrics are Cauchy equivalent, and in fact $\mathbb{R}_+$ is complete in both metrics. Show furthermore that the metrics are not strongly equivalent.

3. Let (X, d_X), (Y, d_Y) be metric space, and suppose (X, d_X) is complete. Further suppose that there is a continuous surjection $f : X \to Y$. Does it follow that (Y, d_Y) is complete? Either prove, or give a counterexample.

4. Consider the following sequence of functions $f_n : [0, 1] \to \mathbb{R}$, defined by

$$f_n(x) = \begin{cases} 0, & \text{if } x = 0, \\ 0, & \text{if } x \text{ is irrational}, \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ with } q \leq n, \\ 0, & \text{if } x = \frac{p}{q}, \text{ with } q > n \end{cases}$$

where when we write $x = \frac{p}{q}$ we mean that the p, q have no common divisor greater than 1.

It is clear that the sequence $\{f_n\}$ converges *pointwise* to a function $f : [0,1] \to \mathbb{R}$. Does the sequence $\{f_n\}$ converge uniformly to f, in other words, does $||f_n - f||_\infty \to 0$?

At what points is the function f continuous?

5. Do problems 5 and 7(all parts) in Sec. 2.4

due 2/8

Assignment V

1. Do problems 3, 4 and 6 in Sec. 2.5

2. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in the interval $[0,1]$, and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{r_n < x} 2^{-n}$$

Show that f is discontinuous at every rational number in $[0,1]$, and continuous at every irrational number in $[0,1]$. In fact, show that f is left-continuous at every rational, and discontinuous from the right; that is, for $r \in (0,1)$

$$\lim_{x \to r^-} f(x) = f(r) \quad \text{and} \quad \lim_{x \to r^+} f(x) < f(r)$$

Define $F : [0,1] \to \mathbb{R},\ F(x) = \int_0^x f(t)\, dt$. By the Fundamental Theorem of Calculus, F is differentiable at points x at which f is continuous, and at those points $F'(x) = f(x)$. But the FTC does not say anything if f is not continuous at x. So, it could happen that F is differentialbe at x or not. Thus it follows from the FTC that F is differentiable at the irrational numbers in $(0,1)$. Show that F is not differentiable at at rational numbers in $(0,1)$.

3. Let $a < b \in \mathbb{R}$, and let $\mathrm{Lip}[a,b]$ denote the set of functions $f : [a,b] \to \mathbb{R}$, such that there is a positive constant M for which

$$|f(y) - f(x)| \le M|y - x| \text{ for all } x, y \in [a,b].$$

M is said to be a Lipschitz constant for the function f, and f is said to be a Lipschitz function. Show that $\mathrm{Lip}[a,b]$ is a subspace of $C[a,b]$.

Fix a positive constant M_0. Let

$$A = \{f \in \mathrm{Lip}[a,b] : M_0 \text{ is a Lipschitz constant for } f, \text{ and } |f(a)| \le M_0\}$$

Show that A is compact in $C[a,b]$.

4. Let $C^1[0,1]$ denote the set of continuous functions on the interval $[0,1]$ which are differentiable, and such that the derivative is continuous. (Note: the derivative at the endpoints is the one-sided derivative.) Define a norm on $C^1[0,1]$ by

$$||f|| = \sup_{0\le x\le 1} |f(x)| + \sup_{0\le x\le 1} |f'(x)|$$

Show that $C^1[0,1]$ is a complete metric space.
Hint: Let $\{f_n\}$ be a Cauchy sequence in $C^1[0,1]$. Then both $\{f_n\}$ and $\{f_n'\}$ are Cauchy sequences in $C[a,b]$, which is complete, so they have limits, say, f, g respectively. Apply the Fundamental Theorrem of Calculus to show that $f \in C^1[0,1]$.

Let $B_1 = \{f \in C^1[0,1] : ||f|| \le 1\}$. Is B_1 compact $C^1[0,1]$?

due 2/15

Assignment VI

1. A *partition* of $[0,1]$ is a set of points $0 = x_0 < x_1 < \cdots < x_n = 1$. If $X = C([0,1],\mathbb{R})$ with the usual supremum metric, and if $f \in X$, define the variation of f over the partition $\mathcal{P}$ to be $V(f,\mathcal{P}) = \sum_{j=1}^n |f(x_j) - f(x_{j-1})|$. The variation of f (also called the total variation of f) is defined to be $V(f) = \sup_{\mathcal{P}} V(f,\mathcal{P})$, where the supremum is taken over all partitions of $[0,1]$. Of course, the supremum could be $+\infty$. Consider the set

$$\{f \in X : f(0) = 1 \text{ and } V(f) \le 1\}$$

Is this set complete in X? Is it compact?

2. Let $X = C([0,1],\mathbb{R})$ with the usual supremum metric, and $B_1[0] = \{f \in X : ||f||_\infty \le 1\}$. Define the Volterra integral operator on X by

$$Vf(x) = \int_0^x f(t)\,dt$$

Show that the closure of the set $V(B_1[0]) = \{Vf : \ f \in B_1[0]\}$ is compact in X.

3. Do problems 2, 4, and 6 in Sec. 3.1.

4. Let τ be the topology on the integers defined in problem 6. Is this topology Hausdorff?

5. Let $(\mathbb{R}^2, \tau)$ be the topological space of example 3.1.26.

(a) Is this topology Hausdorff?
(b) Let $(a,b) \in \mathbb{R}^2$. Is the one-point set $\{(a,b)\}$ closed? If not, what is its closure?

(c) Let S be the square $S = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$. Describe ∂S.

due 2/22

Assignment VII

1. We define a topology on the natural numbers as follows: a set U is open if either $U = \emptyset$ or else

$$\liminf_n \frac{\mathrm{card}(U \cap \{1, 2, \ldots, n\})}{n} = 1$$

Verify that this defines a topology on $\mathbb{N}$. Is this topology Hausdorff?

2. On the set $\mathbb{R}$ of real numbers, let τ be the topology in which the open sets are complements of countable sets, together with the empty set. Let $|\cdot|$ refer to the metric topology on $\mathbb{R}$ defined by the absolute value metric.

(a) Let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau),\ f(x) = x^2$. Is f continuous?
(b) Let $f : (\mathbb{R}, \tau) \to (\mathbb{R}, |\cdot|),\ f(x) = x^2$. Is f continuous?
(c) Let $f : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, \tau),\ f(x) = x^2$. Is f continuous?

3. Do problem 4 in Sec. 1.3 on page 20.

4. Do problems 4, 5 and 11 in Sec. 3.2, page 78.

5. Put a topology τ on $\mathbb{R}$ as follows: a base $\mathcal{B}$ for the topology τ consists of

$$\mathcal{B} = \{U_{a,b} = [a, b),\ a < b \in \mathbb{R}\}$$

(a) Show that this topology is finer than the ususal topology on $\mathbb{R}$.
(b) Is this topology first countable?
(c) Is this topology second countable?
(d) Is this topology separable?

6. Let τ' be a topology on $\mathbb{R}$ as follows: a base $\mathcal{B}'$ for this topology consists of

$$\mathcal{B}' = \{U_{a,b} = [a, b),\ a < b,\ \text{and } a \in \mathbb{Q}, b \in \mathbb{R}\}$$

Answer the same questions as in problem 5 for the topology τ'.

due 3/1

Assignment VIII

1. Do problems 2, 4, 7 and 8 in Sec. 3.3, page 88.

2. Let τ be the topology on $\mathbb{R}$ which was defined in problem 6 of Assignment VII. Show that this topology is finer than the usual metric topology on $\mathbb{R}$. Show that the topology is second countable. Is $[0,1]$ compact in this topology? Is $\mathbb{R}$ locally compact in this topology?

3. Let $\{r_n\}$ be an enumeration of the rational numbers, and consider the functions

$$f(x) = \sum_{r_n < x} 2^{-n} \text{ and } g(x) = \sum_{r_n \leq x} 2^{-n}$$

Let τ be the topology on $\mathbb{R}$ defined in problem 6 of Assignment VII. Let $(\mathbb{R}, |\cdot|)$ denote the reals with the usual topology. Viewing the functions $f,\ g$ as functions from $(\mathbb{R}, \tau) \to (\mathbb{R}, |\cdot|)$, one of these functions is continuous, the other is not. Explain.

due 3/8