Spring 2016 Math 511 Homework 6
Due date: Friday, April 22, 2016

1. Suppose $f : \mathbf{C} \to \mathbf{C}$ is 1-1 (no assumption of onto) and holomorphic. Prove that $f(z) = az + b$ for some $a \neq 0$, $b \in \mathbf{C}$. (Hint: Consider the power series of f and check the behavior at ∞.)

   Solution: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbf{C}$. Define $g : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ by

   $$g(z) = f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n \qquad (1)$$

   Since f is holomorphic on $\mathbf{C}$, (1) converges for all $z \neq 0$. We will prove that g does not have an essential singularity at 0.

   Since f is 1-1, g is also 1-1. We have

   $$(D(0,1) \setminus \{0\}) \cap D(3,1) = \emptyset \Rightarrow g(D(0,1) \setminus \{0\}) \cap g(D(3,1)) = \emptyset$$

   By the Open Mapping Theorem, $g(D(3,1))$ is a non-empty open subset. Therefore, $g(D(0,1) \setminus \{0\})$ is not dense in $\mathbf{C}$. By Theorem 4.1.4, 0 is not an essential singularity of g. So $a_n \neq 0$ for at most a finite number of n. Hence, f is a polynomial. Since f is 1-1, f has degree 1.

2. Let S be the sphere of radius 1, centered at the origin in $\mathbf{R}^3$ and $N = (0,0,1)$. For each point (x,y,z), let $p(x,y,z)$ be the point in $\mathbf{C} \cup \{\infty\}$ under the stereographic projection (see Assignment 1, Q.3). Let γ be a circle on S. Prove that **a)** $p(\gamma)$ is a line (respectively, circle) if $N \in \gamma$ (respectively, $N \notin \gamma$) and **b)** every line or circle in $\mathbf{C} \cup \{\infty\}$ is equal to $p(\gamma)$ for some circle γ on S.

   Solution: Suppose $p(x,y,z) = u + iv$. From Assignment 1, Q. 3, we have

   $$x = \frac{2u}{(u^2+v^2+1)}, \ y = \frac{2v}{(u^2+v^2+1)}, \ z = \frac{(u^2+v^2-1)}{(u^2+v^2+1)} \qquad (2)$$

   a) Suppose γ is a circle on S. Then γ is the intersection of S with a plane given by an equation $Ax + By + Cz = D$, where at least one of A, B or C is nonzero. For the plane to cut out a circle on S, the distance from the origin to the plane has to be less than 1. Therefore, we have

   $$\frac{|0 - D|}{\sqrt{A^2+B^2+C^2}} < 1 \ \Leftrightarrow \ A^2 + B^2 + C^2 > D^2$$

$u + iv$ lies on $p(\gamma)$ if and only if

$$A\left(\frac{2u}{(u^2+v^2+1)}\right) + B\left(\frac{2v}{(u^2+v^2+1)}\right) + C\left(\frac{(u^2+v^2-1)}{(u^2+v^2+1)}\right) = D$$

$$\Leftrightarrow \quad 2Au + 2Bv + C(u^2+v^2-1) = D(u^2+v^2+1)$$

$$\Leftrightarrow \quad (C-D)(u^2+v^2) + 2Au + 2Bv = (C+D) \qquad (3)$$

If $N \in \gamma$, then $A(0) + B(0) + C(1) = D \Rightarrow C = D$ and (3) becomes $2Au + 2Bv = (C+D)$, which is the equation of a line.

If $N \notin \gamma$, then $A(0) + B(0) + C(1) \neq D \Rightarrow C - D \neq 0$ and (3) becomes

$$(u^2+v^2) + \frac{2Au}{(C-D)} + \frac{2Bv}{(C-D)} = \frac{(C+D)}{(C-D)}$$

$$\Leftrightarrow \quad \left(u + \frac{A}{(C-D)}\right)^2 + \left(v + \frac{B}{(C-D)}\right)^2 = \frac{(C+D)}{(C-D)} + \frac{A^2+B^2}{(C-D)^2}$$

$$\Leftrightarrow \quad \left(u + \frac{A}{(C-D)}\right)^2 + \left(v + \frac{B}{(C-D)}\right)^2 = \frac{A^2+B^2+C^2-D^2}{(C-D)^2}$$

Therefore, $f(\gamma)$ is a circle.

b) Suppose L is a line on the uv plane with equation $Au + Bv = C$. Then let γ be the intersection of S with the plane $Ax + By + Cz = C$. Then by the calculation in **a)**, we have $L = p(\gamma)$.

Suppose L is a circle on the uv plane with equation $(u+h)^2 + (v+k)^2 = r^2$. Then let γ be the intersection of S with the plane $Ax + By + Cz = D$, where

$$A = h, \ B = k, \ C = \frac{r^2 - h^2 - k^2 + 1}{2} \quad \text{and} \quad D = \frac{r^2 - h^2 - k^2 - 1}{2}$$

Then by the calculation in **a)**, we have $L = p(\gamma)$.

3. Suppose $f : \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$ is conformal. Prove that there exists $a \in \mathbf{C} \setminus \{0\}$ such that $f(z) = az$ or $f(z) = \dfrac{a}{z}$ for all $z \in \mathbf{C} \setminus \{0\}$. (Hint: First consider the case when 0 is a removable singularity of f. Show that $\lim\limits_{z \to 0} f(z) = 0$.)

Solution: Suppose 0 is a removable singularity of f. Then we can extend $f : \mathbf{C} \to \mathbf{C}$. We will show that $\lim\limits_{z \to 0} f(z) = 0$. Suppose not. Let $f(0) = q \in \mathbf{C} \setminus \{0\}$. Then there exists $p \in \mathbf{C} \setminus \{0\}$ such that $f(p) = q$. Let $\Gamma = \{z \in \mathbf{C} : |z| = 2|p|\}$ and

$r = \min\{|f(z) - q| : z \in \Gamma\}/2$. Let $N(z) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(\zeta)}{f(\zeta) - z}\, d\zeta$. Then $N(z)$ is an integer valued continuous function on $D(q, r)$, with $N(q) = 2$ and $N(z) \le 1$ for all $z \in D(q, r) \setminus \{q\}$, a contradiction. Thus $f : \mathbf{C} \to \mathbf{C}$ is (1-1 and onto $\Rightarrow$) conformal. By Theorem 6.1.1 and $f(0) = 0$, we have $f(z) = az$ for some $a \in \mathbf{C} \setminus \{0\}$

If f has a pole at 0, then $g(z) = \dfrac{1}{f(z)}$ is also a conformal map on $\mathbf{C} \setminus \{0\}$. Since $\lim\limits_{z\to 0} g(z) = 0$, there exists $a \in \mathbf{C} \setminus \{0\}$ such that $g(z) = az \Rightarrow f(z) = \dfrac{1/a}{z}$.

It remains to show that f cannot have an essential singularity at 0. Since f is 1-1, we have $f(D(0, 1) \setminus \{0\}) \cap f(D(3, 1)) = \emptyset$. By the Open Mapping Theorem, $f(D(3, 1))$ is a non-empty open subset. Therefore, $f(D(0, 1) \setminus \{0\})$ is not dense in $\mathbf{C}$. By Theorem 4.1.4, 0 is not an essential singularity of f.

4. Describe $f(U)$ for each of the following domain U under the indicated maps.

   (a) $U = \{z \in \mathbf{C} : \operatorname{Im}(z) > 0\}$, $f(z) = \dfrac{2z - i}{2 + iz}$.

   Solution:

$$\begin{aligned} f(-1) &= \frac{-2 - i}{2 - i} = \frac{-(2 + i)^2}{(2 - i)(2 + i)} = \frac{-3 - 4i}{5} \\ f(0) &= \frac{-i}{2} \\ f(1) &= \frac{2 - i}{2 + i} = \frac{(2 - i)^2}{(2 + i)(2 - i)} = \frac{3 - 4i}{5} \end{aligned}$$

   So the real line is mapped to the circle passing through $\dfrac{-3 - 4i}{5}$, $\dfrac{-i}{2}$, $\dfrac{3 - 4i}{5}$, which has center at $-\dfrac{5i}{4}$ and radius $\dfrac{3}{4}$. Therefore, $f(U) = \{z \in \mathbf{C} : \left|z + \dfrac{5i}{4}\right| > \dfrac{3}{4}\}$.

   Note: The last assertion can be obtained by the checking the orientation of the images of the boundaries or $f(\frac{i}{2}) = 0$.

   (b) $U = \{z \in \mathbf{C} : 0 < \operatorname{Re}(z) < 1\}$, $f(z) = \dfrac{z - 1}{z}$.

Solution:

$$\begin{aligned} f(0) &= \infty \\ f(i) &= \frac{i-1}{i} = 1+i \\ f(\infty) &= 1 \\ f(1) &= 0 \\ f(1+i) &= \frac{1+i-1}{1+i} = \frac{1+i}{2} \end{aligned}$$

So f takes the line $\mathrm{Re}(z) = 0$ to the line $\mathrm{Re}(z) = 1$ and the line $\mathrm{Re}(z) = 1$ to the circle passing through $1,\ 0,\ \dfrac{1+i}{2}$, which has center at $\dfrac{1}{2}$ and radius $\dfrac{1}{2}$. Therefore, $f(U) = \{z \in \mathbf{C} : \mathrm{Re}(z) < 1 \text{ and } \left|z - \dfrac{1}{2}\right| > \dfrac{1}{2}\}$.

Note: The last assertion can be obtained by the checking the orientation of the images of the boundaries or $f(\frac{1}{2}) = -1$.

5. Let $f : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ be a linear fractional transformation. Prove that f has a unique fixed point z_0 with $z_0 \in \mathbf{C}$ if and only if $\dfrac{1}{f(z) - z_0} = \dfrac{1}{z - z_0} + h$, with $h \neq 0$.

Solution: " $\Leftarrow$ " If $\dfrac{1}{f(z) - z_0} = \dfrac{1}{z - z_0} + h$, with $h \neq 0$ then

$$f(z) = z \;\Leftrightarrow\; \frac{1}{z - z_0} = \frac{1}{z - z_0} + h \;\Leftrightarrow\; z = z_0$$

"$\Rightarrow$" Let $f(z) = \dfrac{az+b}{cz+d}$. First consider the case when $z_0 = 0$. Then $b = 0 \Rightarrow ad \neq 0$. Since ∞ is not a fixed point, $c \neq 0$. $z \in \mathbf{C}$ is a fixed point if and only if z satisfies

$$\frac{az}{cz+d} = z \;\Leftrightarrow\; cz^2 + (d-a)z = 0$$

f has a unique fixed point at 0 if and only if $a = d$. Let $h = \dfrac{c}{a} \neq 0$. We have

$$f(z) = \frac{z}{hz+1} \Rightarrow \frac{1}{f(z) - 0} = \frac{1}{z - 0} + h$$

Suppose f is a linear fractional transformation with a unique fixed point $z_0 \in \mathbf{C}$. Then let $g(z) = f(z + z_0) - z_0$. Then

$$g(z) = z \;\Leftrightarrow\; f(z+z_0) - z_0 = z \;\Leftrightarrow\; f(z+z_0) = z + z_0 \;\Leftrightarrow\; z + z_0 = z_0 \;\Leftrightarrow\; z = 0$$

So g has a unique fixed point at 0. Thus we have a non-zero $h \in \mathbf{C}$, such that, for

$$\begin{aligned}
& \frac{1}{g(z)} = \frac{1}{z} + h \quad \text{for all } z \in \mathbf{C} \\
\Rightarrow \quad & \frac{1}{f(z+z_0) - z_0} = \frac{1}{z} + h \quad \text{for all } z \in \mathbf{C} \\
\Rightarrow \quad & \frac{1}{f(z) - z_0} = \frac{1}{z - z_0} + h \quad \text{for all } z \in \mathbf{C}
\end{aligned}$$

6. Find a linear fractional transformation f such that i is the only fixed point and $f(1) = \infty$.

   Solution: By the result in the previous problem, we have $\dfrac{1}{f(z) - i} = \dfrac{1}{z - i} + h$ for some $h \neq 0$.

   $$f(1) = \infty \Rightarrow \frac{1}{1-i} + h = 0 \Rightarrow h = -\frac{1}{1-i}$$

   Therefore,

   $$\begin{aligned}
   & \frac{1}{f(z) - i} = \frac{1}{z-i} - \frac{1}{1-i} = \frac{(1-i)-(z-i)}{(1-i)(z-i)} = \frac{(1-z)}{(1-i)(z-i)} \\
   \Rightarrow \quad & f(z) = \frac{(1-i)(z-i)}{1-z} + i = \frac{(1-2i)z - 1}{1-z}
   \end{aligned}$$

7. Find a linear fractional transformation f that takes the upper half plane to itself and satisfies $f(0) = 1$, $f(i) = 2i$.

   Solution:

   By the symmetry principle, we have $f(-i) = -2i$. Hence, we can solve w in terms of z in

   $$\frac{(w-1)}{(w-2i)} \frac{(-2i-2i)}{(-2i-1)} = \frac{(z-0)}{(z-i)} \frac{(-i-i)}{(-i-0)} \Rightarrow w = \frac{4z+2}{2-z}$$

8. Find a linear fractional transformation that takes the half plane $P = \{x+iy : x+y > 2\}$ to $D(0, 1)$.

   Solution: Let $w = T(z)$ be the linear fractional transformation that takes 2 to $-i$, $1+i$ to -1 and $2i$ to i. Then w can be solved from

   $$\frac{(w+1)}{(w-i)} \frac{(-i-i)}{(-i+1)} = \frac{(z-(i+1))}{(z-2i)} \frac{(2-2i)}{(2-(i+1))} \Rightarrow w = \frac{(z-(2+2i))}{z}$$

Remark: There are many linear fractional transformations S taking P to $D(0,\ 1)$. Since $S \circ T^{-1}$ takes $D(0,\ 1)$ to $D(0,\ 1)$, we have $S \circ T^{-1} = e^{i\theta}\phi_a$ for some $a \in D(0,1)$ and $\theta \in [0,\ 2\pi)$. Therefore, the most general form of S is

$$S(z) = e^{i\theta}\phi_a \circ T(z) = e^{i\theta}\frac{((1-a)z - (2+2i))}{((1-\overline{a})z + \overline{a}(2+2i))}$$

9. Find a conformal map $f : \{z \in \mathbf{C} : |z-1| > 1 \ \text{ and } |z-2| < 2\} \to D(0,1)$.

Solution: Let $U = \{z \in \mathbf{C} : |z-1| > 1 \ \text{ and } |z-2| < 2\}$.

(a) Find a linear fractional transformation f such that $f_1(0) = \infty,\ f_1(2) = 0,\ f_1(4) = 1$ by solving

$$\frac{1-0}{w-0} = \frac{(z-0)}{(z-2)}\frac{(4-2)}{(4-0)} \Rightarrow w = \frac{2(z-2)}{z}$$

Let $f_1(z) = \dfrac{2(z-2)}{z}$. Then $f_1(1+i) = 2i$ and $f_1(2+2i) = 1+i$. Therefore, f_1 takes the circles $\partial D(1,1)$ and $\partial D(2,2)$ to the vertical lines passing 0 and 1 respectively. So, $f_1(U) = U_1 = \{x+iy \in \mathbf{C} : 0 < x < 1\}$.

(b) Let $f_2(z) = \pi i z$. Then $f_2(U_1) = U_2 = \{x+iy \in \mathbf{C} : 0 < y < \pi\}$.

(c) Let $f_3(z) = e^z$. Then $f_3(U_2) = U_3 = \{x+iy \in \mathbf{C} : 0 < y\}$.

(d) Let $f_4(z) = \dfrac{z-i}{z+i}$. Then $f_4(U_3) = D(0,1)$.

Therefore, we can take $f(z) = \dfrac{e^{\frac{2\pi i(z-2)}{z}} - i}{e^{\frac{2\pi i(z-2)}{z}} + i}$.

10. Find a conformal map $f : \{x+iy \in \mathbf{C} : x > 0 \ \text{ and } 0 < y < 1\} \to D(0,1)$.

Let $U = \{x+iy \in \mathbf{C} : x > 0 \ \text{ and } 0 < y < 1\}$.

(a) Let $f_1(z) = \pi z$. Then $f_1(U) = U_1 = \{x+iy \in \mathbf{C} : x > 0 \ \text{ and } 0 < y < \pi\}$.

(b) Let $f_2(z) = e^z$. Then $f_2(U_1) = U_2 = \{re^{i\theta} : r > 1,\ 0 < \theta < \pi\}$.

(c) Let $f_3(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$. Then $f_3(U_2) = U_3 = \{x+iy \in \mathbf{C} : y > 0\}$.

(d) Let $f_4(z) = \dfrac{z-i}{z+i}$. Then $f_4(U_3) = D(0,1)$.

Therefore, we can take $f(z) = \dfrac{\frac{1}{2}\left(e^{\pi z} + e^{-\pi z}\right) - i}{\frac{1}{2}\left(e^{\pi z} + e^{-\pi z}\right) + i} = \dfrac{(e^{\pi z} + e^{-\pi z} - 2i)}{(e^{\pi z} + e^{-\pi z} + 2i)}$.