QRW Search: Example Calculation

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1 Introduction

The purpose of this document is to show what the Quantum Random Walk search algorithm proposed by Shenvi et al is doing in an explicit case. The random walk corresponds to a walk on an n-dimensional cube. We will show in detail what occurs in the two dimensional case.

Figure 1 is the the 2D Hypercube ($n = 2$) which corresponds to 4 (2^n) states of the state space S. The change from one state to another on the hypercube is one-bit flip. Thus two bit states are connected if the change between them is one bit ($|\vec{x} - \vec{y}| = 1$ where $\vec{x}$ and $\vec{y}$ are bit strings). The algorithm in general operates in a hilbert space defined by a coin

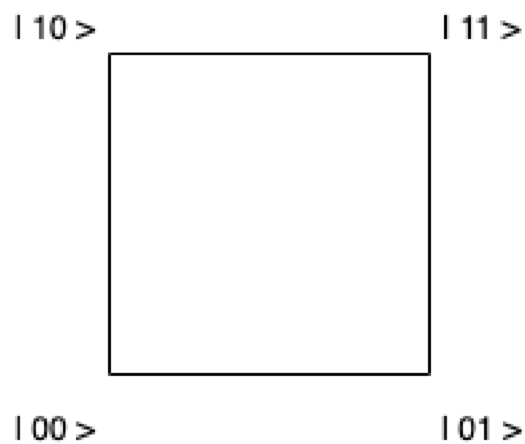

Figure 1: Bit state mapping on a 2D hypercube

space C and a state space S. Figure 1 is how the state space is mapped to the hypercube. The coin space determines the direction of the walk along the states of this cube. Thus the coin state will have dimension n and the state space will have dimension 2^n thus the alogrithm will operate in the hilbert space $\mathcal{H}^C \otimes \mathcal{H}^S$

2 The Algorithm

1. First, Initialize the quantum registry in an equal superposition over all states $|\psi_0\rangle = |s^c\rangle \otimes |s^s\rangle$ in our example of 2-dimensions this corresponds to the following state (recall from the paper that $\left|s^C\right\rangle = 1/\sqrt{n}\sum_{d=1}^{n}|d\rangle \rightarrow$ Our 2D case corresponds to: $\left|s^C\right\rangle = 1/\sqrt{2}\sum_{d=1}^{2}|d\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ note that we encoded the two directions as 0 and 1 rather than 1 and 2):
$$|\psi_0\rangle = (1/\sqrt{2}(|0\rangle + |1\rangle)) \otimes (1/2(|00\rangle + |01\rangle + |10\rangle + |11\rangle)) \tag{1}$$
$$|\psi_0\rangle = (1/2\sqrt{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \tag{2}$$

2. Next we apply the unitary operator $U' = SC'$ (where $S = \sum_{d=0}^{n-1}\sum_{\vec{x}}|d, \vec{x} \oplus \vec{e_d}\rangle\langle d, \vec{x}|$ and $C' = C_0 \otimes \mathcal{I} + (C_1 - C_0) \otimes \left|\vec{0}\right\rangle\left\langle\vec{0}\right|$) $\pi/2\sqrt{2^n}$ times. In our two dimensional case this corresponds to π times or ≈ 3 times.

3. Finally, we measure the quantum registry in the $|d, \vec{x}\rangle$ basis

2.1 Expansion of U'

We want to put the operator U' in the simplest form possible. Thus this section is to convert the operator to just outer products. From equation 8 in the shenvi et al paper this is the simplified operator U' for the search of the $|00\rangle$ in state space.

$$U' = SC' = U - 2S(|S^c\rangle\langle S^c| \otimes \left|\vec{0}\right\rangle\left\langle\vec{0}\right|)$$

focusing on the outer product of the coin space in the second term:
$|S^c\rangle\langle S^c| = (1/\sqrt{2}(|0\rangle + |1\rangle))(1/\sqrt{2}(\langle 0| + \langle 1|))$
$= 1/2(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)$

The second term in the operator U' can be written as
$U' = U - 2S(1/2(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) \otimes |00\rangle\langle 00|)$
$U' = U - 2S(1/2(|000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| + |100\rangle\langle 100|)$
$U' = U - S((|000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| + |100\rangle\langle 100|)$

Now lets further expand the S operator
$$S = \sum_{d=0}^{n-1}\sum_{\vec{x}} |d, \vec{x} \oplus \vec{e_d}\rangle\langle d, \vec{x}| = \sum_{\vec{x}} |0, \vec{x} \oplus \vec{e_0}\rangle\langle 0, \vec{x}| + \sum_{\vec{y}} |1, \vec{y} \oplus \vec{e_1}\rangle\langle 1, \vec{y}|$$
Considering just a single sum over the 0^{th} direction
$$\sum_{\vec{x}} |0, \vec{x} \oplus \vec{e_0}\rangle\langle 0, \vec{x}| = |0, 00 \oplus \vec{e_0}\rangle\langle 0, 00| + |0, 01 \oplus \vec{e_0}\rangle\langle 0, 01| + |0, 10 \oplus \vec{e_0}\rangle\langle 0, 10| + |0, 11 \oplus \vec{e_0}\rangle\langle 0, 11|$$
the $\vec{e_0}$ vector corresponds to a bit flip in the 2^0 place of the $\vec{x}$ bit string
more generally, we will let $\vec{e_i}$ correspond to a bit flip of the 2^i place of the $\vec{x}$ bit string
$$\sum_{\vec{x}} |0, \vec{x} \oplus \vec{e_0}\rangle\langle 0, \vec{x}| = |0, 01\rangle\langle 0, 00| + |0, 00\rangle\langle 0, 01| + |0, 11\rangle\langle 0, 10| + |0, 10\rangle\langle 0, 11|$$
Similary,
$$\sum_{\vec{x}} |1, \vec{x} \oplus \vec{e_1}\rangle\langle 1, \vec{x}| = |1, 00 \oplus \vec{e_1}\rangle\langle 1, 00| + |1, 01 \oplus \vec{e_1}\rangle\langle 1, 01| + |1, 10 \oplus \vec{e_1}\rangle\langle 1, 10| + |1, 11 \oplus \vec{e_1}\rangle\langle 1, 11|$$
$$\sum_{\vec{x}} |1, \vec{x} \oplus \vec{e_1}\rangle\langle 1, \vec{x}| = |1, 10\rangle\langle 1, 00| + |1, 11\rangle\langle 1, 01| + |1, 00\rangle\langle 1, 10| + |1, 01\rangle\langle 1, 11|$$
Thus,
$S = |001\rangle\langle 000| + |000\rangle\langle 001| + |011\rangle\langle 010| + |010\rangle\langle 011| + |110\rangle\langle 100| + |111\rangle\langle 101| + |100\rangle\langle 110| + |101\rangle\langle 111|$

Now can we can further simplify the second term of U', recall:
$U' = U - S((|000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| + |100\rangle\langle 100|)$
the only outer products in S that will survive will be the ones that have $\langle 000|$ or $\langle 100|$ in the outer product
This makes sense because by doing so the state $|00\rangle$ in the state space is 'marked'
Thus, U' becomes:
$\underline{U' = U - (|001\rangle\langle 000| + |001\rangle\langle 100| + |110\rangle\langle 000| + |110\rangle\langle 100|)}$

Now we will focus on simplifying the U operator, recall:

$U = SC = S(C_0 \otimes \mathcal{I}^{\mathcal{S}}) = S((2\left|s^c\right\rangle\left\langle s^c\right| - \mathcal{I}^{\mathcal{C}}) \otimes \mathcal{I}^{\mathcal{S}})$

Obviously,

$\mathcal{I}^{\mathcal{S}} = \left|00\right\rangle\left\langle 00\right| + \left|01\right\rangle\left\langle 01\right| + \left|10\right\rangle\left\langle 10\right| + \left|11\right\rangle\left\langle 11\right|$ and $\mathcal{I}^{\mathcal{C}} = \left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|$

we know that: $\left|S^c\right\rangle\left\langle S^c\right| = 1/2(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|)$

$(2\left|s^c\right\rangle\left\langle s^c\right| - \mathcal{I}^{\mathcal{C}}) = (\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|) - (\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|)$

$(2\left|s^c\right\rangle\left\langle s^c\right| - \mathcal{I}^{\mathcal{C}}) = \cancel{\left|0\right\rangle\left\langle 0\right| - \left|0\right\rangle\left\langle 0\right|} + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| + \cancel{\left|1\right\rangle\left\langle 1\right| - \left|1\right\rangle\left\langle 1\right|}$

Thus,

$(2\left|s^c\right\rangle\left\langle s^c\right| - \mathcal{I}^{\mathcal{C}}) = \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| \rightarrow U = S(\left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|) \otimes \mathcal{I}^{\mathcal{S}})$

We are getting closer to fully simplifying the U' operator

$C = (\left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|) \otimes \mathcal{I}^{\mathcal{S}} = \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|) \otimes (\left|00\right\rangle\left\langle 00\right| + \left|01\right\rangle\left\langle 01\right| + \left|10\right\rangle\left\langle 10\right| + \left|11\right\rangle\left\langle 11\right|)$

$C = \left|000\right\rangle\left\langle 100\right| + \left|001\right\rangle\left\langle 101\right| + \left|010\right\rangle\left\langle 110\right| + \left|011\right\rangle\left\langle 111\right| + \left|100\right\rangle\left\langle 000\right| + \left|101\right\rangle\left\langle 001\right| + \left|110\right\rangle\left\langle 010\right| + \left|111\right\rangle\left\langle 011\right|$

Now S can act C to give us U. Recall:

$S = \left|001\right\rangle\left\langle 000\right| + \left|000\right\rangle\left\langle 001\right| + \left|011\right\rangle\left\langle 010\right| + \left|010\right\rangle\left\langle 011\right| + \left|110\right\rangle\left\langle 100\right| + \left|111\right\rangle\left\langle 101\right| + \left|100\right\rangle\left\langle 110\right| + \left|101\right\rangle\left\langle 111\right|$

$U = SC = \left|001\right\rangle\left\langle 000|000\right\rangle\left\langle 100\right| + \left|000\right\rangle\left\langle 001|001\right\rangle\left\langle 101\right| + \left|011\right\rangle\left\langle 010|010\right\rangle\left\langle 110\right| + \left|010\right\rangle\left\langle 011|011\right\rangle\left\langle 111\right|$

$+ \left|110\right\rangle\left\langle 100|100\right\rangle\left\langle 000\right| + \left|111\right\rangle\left\langle 101|101\right\rangle\left\langle 001\right| + \left|100\right\rangle\left\langle 110|110\right\rangle\left\langle 010\right| + \left|101\right\rangle\left\langle 111|111\right\rangle\left\langle 011\right|$

$\underline{U = \left|001\right\rangle\left\langle 100\right| + \left|000\right\rangle\left\langle 101\right| + \left|011\right\rangle\left\langle 110\right| + \left|010\right\rangle\left\langle 111\right| + \left|110\right\rangle\left\langle 000\right| + \left|111\right\rangle\left\langle 001\right| + \left|100\right\rangle\left\langle 010\right| + \left|101\right\rangle\left\langle 011\right|}$

Now we want to combine our outer product representation for U to complete our derivation of U'

$U' = U - (\left|001\right\rangle\left\langle 000\right| + \left|001\right\rangle\left\langle 100\right| + \left|110\right\rangle\left\langle 000\right| + \left|110\right\rangle\left\langle 100\right|)$

$U' = \cancel{\left|001\right\rangle\left\langle 100\right|} + \left|000\right\rangle\left\langle 101\right| + \left|011\right\rangle\left\langle 110\right| + \left|010\right\rangle\left\langle 111\right| + \cancel{\left|110\right\rangle\left\langle 000\right|} + \left|111\right\rangle\left\langle 001\right| + \left|100\right\rangle\left\langle 010\right| + \left|101\right\rangle\left\langle 011\right|$

$- (\left|001\right\rangle\left\langle 000\right| + \cancel{\left|001\right\rangle\left\langle 100\right|} + \cancel{\left|110\right\rangle\left\langle 000\right|} + \left|110\right\rangle\left\langle 100\right|)$

$$\boxed{U' = \left|000\right\rangle\left\langle 101\right| + \left|011\right\rangle\left\langle 110\right| + \left|010\right\rangle\left\langle 111\right| + \left|111\right\rangle\left\langle 001\right| + \left|100\right\rangle\left\langle 010\right| + \left|101\right\rangle\left\langle 011\right| - (\left|001\right\rangle\left\langle 000\right| + \left|110\right\rangle\left\langle 100\right|)} \tag{3}$$

2.2 First Application of U'

Assuming we have the quantum registry in an equal superposition (can easily be done by applying $n-$single bit Hadamard operators to the $\left|000\right\rangle$ state). We can perform the operator U' on the state $\left|\psi_0\right\rangle$

$\left|\psi_0\right\rangle = (\frac{1}{2\sqrt{2}}(\left|000\right\rangle + \left|001\right\rangle + \left|010\right\rangle + \left|011\right\rangle + \left|100\right\rangle + \left|101\right\rangle + \left|110\right\rangle + \left|111\right\rangle)$

$U'\left|\psi_0\right\rangle = (\left|000\right\rangle\left\langle 101\right| + \left|011\right\rangle\left\langle 110\right| + \left|010\right\rangle\left\langle 111\right| + \left|111\right\rangle\left\langle 001\right| + \left|100\right\rangle\left\langle 010\right| + \left|101\right\rangle\left\langle 011\right| - \left|001\right\rangle\left\langle 000\right| - \left|110\right\rangle\left\langle 100\right|)\left|\psi_0\right\rangle$

$\left|\psi_1\right\rangle = \frac{1}{2\sqrt{2}}(\left|000\right\rangle\left\langle 101|101\right\rangle + \left|011\right\rangle\left\langle 110|110\right\rangle + \left|010\right\rangle\left\langle 111|111\right\rangle + \left|111\right\rangle\left\langle 001|001\right\rangle + \left|100\right\rangle\left\langle 010|010\right\rangle$

$+ \left|101\right\rangle\left\langle 011|011\right\rangle - \left|001\right\rangle\left\langle 000|000\right\rangle - \left|110\right\rangle\left\langle 100|100\right\rangle)$

$\left|\psi_1\right\rangle = \frac{1}{2\sqrt{2}}(\left|000\right\rangle + \left|011\right\rangle + \left|010\right\rangle + \left|111\right\rangle + \left|100\right\rangle + \left|101\right\rangle - \left|001\right\rangle - \left|110\right\rangle$

$\left|\psi_1\right\rangle = \left|\psi_0\right\rangle - \frac{1}{\sqrt{2}}(\left|001\right\rangle + \left|110\right\rangle)$

2.3 Second Application of U'

$$|\psi_1\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |011\rangle + |010\rangle + |111\rangle + |100\rangle + |101\rangle - |001\rangle - |110\rangle$$
$$|\psi_1\rangle = |\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)$$
$$|\psi_2\rangle = U'\,|\psi_1\rangle = U'(|\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle))$$
$$|\psi_2\rangle = |\psi_1\rangle - U'(\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle))$$
$$U'(\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)) = |000\rangle\langle 101| + |011\rangle\langle 110| + |010\rangle\langle 111| + |111\rangle\langle 001|$$
$$+ |100\rangle\langle 010| + |101\rangle\langle 011| - (|001\rangle\langle 000| + |110\rangle\langle 100|)(\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle))$$
$$U'(\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)) = \frac{1}{\sqrt{2}}(|111\rangle + |011\rangle)$$
$$|\psi_2\rangle = |\psi_1\rangle - U'(\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)) = |\psi_1\rangle - \frac{1}{\sqrt{2}}(|111\rangle + |011\rangle)$$
$$|\psi_2\rangle = |\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle) - \frac{1}{\sqrt{2}}(|111\rangle + |011\rangle)$$
$$|\psi_2\rangle = |\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle)$$

2.4 Third Application of U'

$$|\psi_2\rangle = |\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle)$$
$$|\psi_3\rangle = U'\,|\psi_2\rangle = U'(|\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle))$$
$$|\psi_3\rangle = |\psi_1\rangle - U'(\frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle))$$
$$U'(\frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle)) =$$
$$|000\rangle\langle 101| + |011\rangle\langle 110| + |010\rangle\langle 111| + |111\rangle\langle 001| + |100\rangle\langle 010|$$
$$+ |101\rangle\langle 011| - (|001\rangle\langle 000| + |110\rangle\langle 100|)(\frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle))$$
$$U'(\frac{1}{\sqrt{2}}(|001\rangle + |011\rangle + |110\rangle + |111\rangle)) = \frac{1}{\sqrt{2}}(|111\rangle + |101\rangle + |011\rangle + |010\rangle)$$
$$|\psi_3\rangle = |\psi_1\rangle - \frac{1}{\sqrt{2}}(|111\rangle + |101\rangle + |011\rangle + |010\rangle)$$
$$|\psi_3\rangle = |\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle) - \frac{1}{\sqrt{2}}(|111\rangle + |101\rangle + |011\rangle + |010\rangle)$$
$$|\psi_3\rangle = |\psi_0\rangle - \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle + |111\rangle + |101\rangle + |011\rangle + |010\rangle)$$
$$|\psi_3\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |011\rangle - |010\rangle - |111\rangle + |100\rangle - |101\rangle - |001\rangle - |110\rangle)$$

The result after applying $U' \approx 3$ times results in the states $|000\rangle$ and $|100\rangle$ (the marked states) being π radians out of phase.