Infinite Number of Twin Primes

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One of the main problems of prime numbers theory that has not been solved for over 2,000 years, is a Proof of an Infinite Number of Twin Primes. The problem was designated by the 5th International Mathematical Congress. Currently, this problem is known as the second Landau problem.

In 2013, the American mathematician Yitang Zhang from the University of New Hampshire proved that there is an infinite number of prime pairs, a distance between which is less than 70 million. Later, James Maynard improved the result to 600. In 2014 the Polymath Project supervised by Terence Tao slightly improved last method and obtained an estimate as 246 [1].

To solve this problem, we have suggested a new method, which allows to empirically estimate an infinite number of twin primes [2]. In this paper we propose another simple proof, which provides more correct result for twin primes infinity.

First, for convenience we introduce the following notation. As known, a sequential multiplication of natural numbers is called factorial: $\prod_{i=1}^{n} i = n!$ Hereafter, a sequential multiplication of prime numbers will occur frequently, therefore for such cases, we use the following notation:

$$2 * 3 * 5 * 7 * 11 * \dots * \ p_n = \prod_{i=1}^{n} p_i = p_n!'$$

Here p_i is a prime number with indexing number i. Combination of symbols $p_n!'$ means a sequential multiplication of prime numbers only from 2 to p_n. We shall call it as **a special factorial of a prime number $\boldsymbol{p_n}$.** For example, $p_4!'$ is a special factorial of prime number $p_4 = 7$ or $p_4!' = 7!' = 2 * 3 * 5 * 7 = 210$.

Matrices of prime numbers.

Let we represent a set of natural numbers in a form of A_k matrices family with elements *a(k,i,j)*, where I – an indexing number of row, j – an indexing number of column, and k – an indexing number of A_k matrix.

Here, a maximum number of rows of A_k matrix must be equal to the special factorial $p_k!'$, i.e. $i_{k,max} = p_k!'$. This means, that for every matrix with indexing number k there is a specific set of prime sequence: p_1, p_2, p_3, … , p_k (note that the last prime number, which corresponds to this matrix is p_k). A number of columns can be arbitrarily large, up to infinity.

Here and further it is supposed that we don't know any prime number. Prime numbers will be generated in the course of construction of A_k matrices.

First, we show how a matrix A_1 is formed. For this, we consider a series of natural numbers from 1 to infinity (*Figure 1,* A_0). In this series, a number 1 is followed by a number 2. So a number 2 is divided by 1 and 2 only. Therefore, the first prime number is 2, i.e. $p_1 = 2$.. Then, the first matrix A_1, built in view of the first prime number, has only 2 rows ($i_{k,max} = i_{1,max} = p_1!' = 2!' = 2$). A number of columns is infinite (*Figure 1,* A_1*-a)*. In the first row of A_1 matrix numbers 2, 4, 6,... are located. These numbers form an arithmetic progression. The first term and difference (step) of this progression are equal to 2, i.e. terms of the first row are respectively equal to:

$$a(k,i,j) = a(1,1,j) = 2 + 2 * (j - 1), \text{ where } j=1, 2, 3,...$$

Numbers located in the second row of the new matrix, also form an arithmetic progression. The first term and difference of this progression are 3 and 2 respectively, i.e.

$$a(k,i,j) = a(1,2,j) = 3 + 2 * (j - 1), \text{ where } j=1, 2, 3,...$$

As previously defined, number 2 is a prime number. Therefore, all numbers divisible by 2, are the composite numbers. In view of this, all numbers, except 2, located in the first row of the considered matrix (*Figure 1,* A_1-*в*), for clarity are painted by a dark color. Thus, with help of A_1 matrix all composite numbers, which should be divisible by 2, are defined.

This implies that number 1 is not a prime number, otherwise all numbers divisible by 1 would be composite numbers. Number 1 is also not a composite number, since it is not divided by other numbers. That is why number 1 in this and other matrices is located separately in the upper left corner.

Algorithm for matrix transformation from one to another type.

It is seen from A_1 matrix *(Figure, 1,* A_1*-в)*, that not painted number next to number 2 is number 3 and it is not divisible by 2 and therefore it is the second prime number, i.e. $p_2 = 3$.

Thus, we transform A_1matrix (*Figure. 1,* A_2-*в*) into the next matrix A_2 (*Figure.1*, A_2-*a*). A maximum number of rows of this matrix must be equal to a special factorial of the second prime number $p_2 = 3$, i.e. $i_{k,max} = i_{2,max} = p_2!' = 3!' = 6$. A number of columns as in the first case can be arbitrary.

For transforming matrix from one to another type a simple method is used. Implementation of this method lies in simple transposition of numbers of certain rows and columns of the original matrix into corresponding rows and columns of the new matrix. For example, for forming a first column of A_2 matrix, first numbers 2 and 3, located in the first column of the original matrix A_1 are transposed to the first and second rows of the new matrix A_2, then, numbers 4 and 5 of the first matrix are transposed to the third and fourth rows of the new matrix. Then numbers 6 and 7 are also transposed to the fifth and sixth rows of the new matrix. This completes formation of the first column of A_2 matrix.

To form the second column of A_2 matrix, we similarly transpose in pairs the numbers (8 and 9), (10 and 11), and (12 and 13) into the second column of A_2 matrix. Next, other columns are formedin a similar way.

Note, that in the new matrix (*Figure 1*, A_2-*a*) all numbers located in the third and fifth rows are dark painted as they have been already defined as composite numbers due to A_1 matrix.

In the new matrix all numbers located in each row, as in the case of A_1 matrix, form an arithmetic progression, which in general takes the following form:

$$a(k,i,j) = (i+1) + p_k!'(j-1), \text{ where } j = 1,2,\dots,\infty; \quad i = 1,2,\dots,p_k!' \quad (1)$$

The difference of this arithmetic progression is $p_k!'$. Expression (1) for A_2 matrix, particularly for its second row, appears as follows:

$$a(2,2,j) = 3 + 6(j-1), \text{ where } j = 1,2,\dots,\infty; \quad i = 1,2,\dots,p_2!'$$

As you can see, all the numbers of this row are is divisible by 3. Therefore, they (except number 3) are composite numbers. In a view of this, they are painted a dark color in A_2 matrix (*Figure.1*, A_2-*в*). In regard to the numbers located in the fifth row, they are also divided by 3. However, these numbers, as mentioned above when considering A_1 matrix, have been already defined as composite numbers. A set of numbers of fourth row (and sixth row) also form an arithmetic progression. But among them there are both prime and composite numbers. Therefore, numbers of these rows are not painted yet. Note, that rows containing only composite numbers are dark painted.

Note that here and in all next figures, letter *a* denotes those matrices (e.g. A_1-*a*, A_2-*a*, A_3-*a)*, which are formed after transformation of the previous matrix. Letter *в* denotes those matrices (e.g., A_1-*в*, A_2-*в*, A_1-*в*), which are obtained as a result of processing of already transformed matrices.

Natural number series

1	2	3	4	5	6	7	8	9	10	…	…	…	…

A0

Column numbers

		1	2	3	4	…
	1					
1		2	4	6	8	…
2		3	5	7	9	…

A1-a

Column numbers

		1	2	3	4	…
	1					
1		2	4	6	8	…
2		3	5	7	9	…

A1-в

Column numbers

		1	2	3	4	…
	1					
1		2	8	14	20	…
2		3	9	15	21	…
3		4	10	16	22	…
4		5	11	17	23	…
5		6	12	18	24	…
6		7	13	19	25	…

A2-a

Column numbers

		1	2	3	4	…
	1					
1		2	8	14	20	…
2		3	9	15	21	…
3		4	10	16	22	…
4		5	11	17	23	…
5		6	12	18	24	…
6		7	13	19	25	…

A2-в

Column numbers

Row Number		1	2	3	4	…
	1					
1		2	32	62	92	…
2		3	33	63	93	…
3		4	34	64	94	…
4		5	35	65	95	…
5		6	36	66	96	…
6		7	37	67	97	…
7		8	38	68	98	…
8		9	39	69	99	…
9		10	40	70	100	…
10		11	41	71	101	…
11		12	42	72	102	…
12		13	43	73	103	…
13		14	44	74	104	…
14		15	45	75	105	…
15		16	46	76	106	…
16		17	47	77	107	…
17		18	48	78	108	…
18		19	49	79	109	…
19		20	50	80	110	…
20		21	51	81	111	…
21		22	52	82	112	…
22		23	53	83	113	…
23		24	54	84	114	…
24		25	55	85	115	…
25		26	56	86	116	…
26		27	57	87	117	…
27		28	58	88	118	…
28		29	59	89	119	…
29		30	60	90	120	…
30		31	61	91	121	…

A3-a

Column numbers

Row Number		1	2	3	4	…
	1					
1		2	32	62	92	…
2		3	33	63	93	…
3		4	34	64	94	…
4		5	35	65	95	…
5		6	36	66	96	…
6		7	37	67	97	…
7		8	38	68	98	…
8		9	39	69	99	…
9		10	40	70	100	…
10		11	41	71	101	…
11		12	42	72	102	…
12		13	43	73	103	…
13		14	44	74	104	…
14		15	45	75	105	…
15		16	46	76	106	…
16		17	47	77	107	…
17		18	48	78	108	…
18		19	49	79	109	…
19		20	50	80	110	…
20		21	51	81	111	…
21		22	52	82	112	…
22		23	53	83	113	…
23		24	54	84	114	…
24		25	55	85	115	…
25		26	56	86	116	…
26		27	57	87	117	…
27		28	58	88	118	…
28		29	59	89	119	…
29		30	60	90	120	…
30		31	61	91	121	…

A3-в

Figure 1. **Matrices of prime numbers**

It should be noted that during processing of the second matrix, all numbers, which are divisible by 3, are finally determined, and they are also properly painted in dark color.

It is seen from A_2 matrix *(Figure 1, A_2-в)* that, number 5 next to numbers 2 and 3 remains unpainted. Thus it is a third prime number $p_3 = 5$. Therefore, the second matrix *(Figure 1, A_2-в)* is transformed into third matrix A_3 *(Figure 1, A_3-a)*. For this purpose, a similar procedure used for transforming A_1 matrix into A_2 matrix is applied.

For example, for forming the first column of A_3 matrix at first, numbers (2 ÷ 7) located in the first column of the original matrix A_2 are transposed into (1st ÷ 6th) row of new matrix A_3. Then, numbers (8 ÷ 13) from the second column of A_2 matrix are transposed into (7th ÷ 12th) row of the new matrix. After this number (14 ÷ 19) are transposed into (13th ÷ 18th) row of the new matrix. Then the numbers (20 ÷ 25) are transposed into (19th ÷ 24th) row of the new matrix and finally, the numbers (26 ÷ 31) are transposed into (25th ÷ 30th) row of the new matrix. This completes formation of the first column of A_3 matrix.

For forming the second column of A_3 matrix, in a similar way numbers (32 ÷ 37), (38 ÷ 43), (44 ÷ 49), (50 ÷ 55) and (56 ÷ 61), located in A_2, matrix are gradually transposed into second column of the A_3 matrix. After that, other columns are similarly formed.

A maximum number of third matrix rows should be equal to a special factorial of the third prime number $p_3 = 3$, i.e.

$$i_{k,max} = i_{3,max} = p_3!' = 5!' = 30.$$

For A_3 matrix expression (1) appears as follows:

$$a(3, i, j) = (i + 1) + 30(j - 1), \text{where } j = 1, 2, \dots, \infty; \; i = 1, 2, \dots, p_3!'$$

From this expression, we obtain that all the numbers located in the fourth row is divisible by 5, and those numbers that located in the 24th row are also divisible by 5. In this context, all of them are accordingly transposed into the series of composite numbers and repainted into dark color (except prime number 5). Here, in the case of the third matrix *(Figure 1, A_3-в)* it should be also noted that all numbers which must be divisible by 5, are finally revealed and repainted into dark color (for example, in particular a row with an indexing number 24).

Note that in all cases, location of painted and unpainted numbers within one column of any considered matrix is not amenable to any strict law. But a picture of mutual location of these numbers within one column is repeated with perfect precision in the next columns (starting from the second column). This law of repeating pictures by columns is obtained when each previous matrix A_n with a number of rows equal to $p_n!'$ is transformed into the next matrix A_{n+1} with a number of rows equal to $p_{n+1}!'$.

In A_3 matrix *(Figure 1, A_3-в)* number 7 which is next to the numbers 2, 3, and 5 is not painted. Therefore, it is ae fourth prime number $p_4 = 7$. Now, knowing the fourth prime number 7, matrix A_3 can be similarly transformed into the next forth matrix A_4. In this matrix a maximum number of rows must be equal to the special factorial of prime number 7, i.e. 7! = 210.

In this case, carrying out series of similar operations, as in the previous cases, we can finally identify a set of all composite numbers, which should be divisible by 7. Similarly, we can build other matrices.

Now, using matrices A_k, we will try to determine a number of prime twins.

Infinite number of twin primes.

First, we set a number of definitions:

Definition 1. If in a certain row of a matrix there are only composite numbers, then for illustration purposes the row is painted into dark color and for convenience we call it as *a painted row.*

Definition 2. If in a certain row of a matrix there are both prime and composite numbers, the row is not painted and for convenience we call such rows as *not painted row.*

Definition 3. If in a first number of a row is not painted and rest of numbers are painted, then this not painted number is a prime number and the rest numbers are composite.

Definition 4. If a difference between indexing numbers of two neighbor not painted rows is equal to 2, then such rows we call *a pair of twin rows or twin rows.* For numbers located in different rows but in one column of twin rows pairs, the equation $|a(k,i,j) - a(k,i \mp 2,j)| = 2$ is always satisfied.

Definition 5. If an indexing number of a certain painted row differs from an indexing number of the nearest not painted row larger than 2, then the raw is called as *a single row.*

From these definitions it follows that twin prime numbers can be *only* in twin rows.

A goal ff the research is determine a total number of prime numbers. Therefore, hereafter we *will put main emphasis on* pairs of twin rows.

Theorem 1. ***A number of twin rows pairs in A_k matrix is monotonically increased with a growth of indexing number k of the matrix and also in each row of any twin rows pair there are an infinite number of prime numbers.***

As known, all twin prime numbers can be located in paired twin rows only. Moreover, if at some point, for example when considering A_k matrix, all pairs of twin rows are disappeared, then it obviously that they will not appear in next next matrices. In that case, it means that a number of twin prime numbers should be limited.

We will analyze whether such case is possible and conjointly prove Theorem 1.

Proof of Theorem 1.

Let suppose that in some matrix there is only one single pair of twin rows (for example, as in the case of *Figure 1*, A_2-*в*). Here, there is reason to assume that in the course of further transformation of the matrix into the next matrices, pairs of twin rows in question may disappear. But, in fact the opposite is true. When transforming the matrix into the next matrix a number of twin rows pairs, as shown above, becomes **larger.**

For example, in A_2 matrix there is only one single pair of twin rows *(Figure 1*, A_2-*в)*. From the expression (1) it follows that terms of the arithmetic progression, which are located in the rows of this single pair of twin rows are defined by the expression:

$$(6 \mp 1) + 6(j-1) = (p_2!' \mp 1) + p_2!'(j-1), \text{ where } j = 1, 2, \dots, \infty \quad (2)$$

Here signs «-» and «+» correspond to upper and lower row of the pair of twin rows respectively.

But this only one pair of rows generates 5 ($p_3 = 5$) new pairs of rows during transformation of this matrix into A_3. That is, the original unique pair of twin rows is ungrouped by 5 new pairs of rows.

A set of numbers located in each row of 5 new pairs of rows of A_3 matrix also forms an arithmetic progression with a constant $p_3!' = 30$ which is defined by the expression:

$$(6m \mp 1) + 30(j-1) = (p_2!'m \mp 1) + p_3!'(j-1), \quad (3)$$

where $j = 1, 2, 3, \dots, \infty$; $m = 1, 2, \dots, p_3$.

From expression (3) we obtain that if at some value of $m = 1, 2, \dots, p_3$ the following equation is satisfied

$$\frac{p_2!'m \mp 1}{p_3} = \text{integer}, \quad (4)$$

then all numbers of this row are divided by p_3 exactly. Therefore, the numbers are composite. In fact, it is known that within interval of 0 <m <p_3 the equation (4) with regard to parameter m has unique solution [3], [4] and [5]. For example, equation (4) for the case of $(p_2!'m - 1)$ is satisfied for *m* = 1 and for the case of $(p_2!'m + 1)$ at *m* = 4. That is at *m* = 1 and *m* = 4 a pair of rows in question is not a pair of twin rows and corresponding row for which the equation (4) is satisfied, is repainted into dark color. As a result only 3 of 5 newly formed pairs of rows are twin rows.

All the numbers of each row of 3 newly formed pairs of twin rows, as stated above, form an arithmetic progression and in each of them first term and difference of the arithmetic progression are coprimes, i.e.:

$$(p_2!'m \mp 1, p_3!') \equiv 1, \text{where } m = 2, 3 \text{ и } 5, m \neq 1, m \neq 4$$

In virtue of this, from Dirichlet theorem for prime numbers in arithmetic progression, it follows that in each row of these three pairs of twin rows there is an infinite number of prime numbers.

Now we consider a transformation of A_3 matrix into A_4 matrix. In this case, each pair of twin rows of A_3 matrix generates 7 ($p_4 = 7$) new pairs of rows and totally 21 new pairs of rows are formed in new A_4 matrix. Values of numbers in the rows of these pairs, are defined by the expression:

$$[p_2!'p_i + p_3!'(m-1) \mp 1] + p_4!'(j-1), \tag{5}$$

where $j = 1, 2, 3, \dots, \infty$; $m = 1,2, \dots, p_4$; $i = 1, 2, 3$.

It can be seen from (5), that a set of numbers lying in each row of newly created 21 pairs of rows, separately forms an arithmetic progression with difference of $p_4!'$ and first terms as $[p_2!'p_i + p_3!'(m-1) \mp 1]$.

Now we will see which of these 21 pairs of rows of A_4 matrix are pairs of twin rows. For this purpose we analyze the divisibility of the first terms of the aforementioned arithmetic progressions by $p_4 = 7$. In addition, for convenience and visualization, we consider a case when *i* = 3, but at the same time we mean the cases when *i* = 1 and *i* = 2. Then from (5) we find that values of numbers lying in 7 new rows A_4 matrix generated by the last pair of twin rows of A_3 matrix are determined by the expression:

$$p_3!'m \mp 1 + p_4!'(j-1),$$

where $j = 1, 2, 3, \dots, \infty$; $m = 1,2, \dots, p_4$.

Let we consider the divisibility of the first terms $p_3!'m \mp 1$ of the arithmetic progression in question by p_4, i.e. the satisfiability of the equation:

$$\frac{p_3!'m \mp 1}{p_4} = \text{integer}$$

In this case, as in case (4), we find that within the interval of $0 < m < p_4$ this equation in reference to parameter *m* (for the case of $p_3!'m - 1$, and also for the case of $p_3!'m + 1$) has an unique solution. Therefore, in this case 2 pairs of rows in question cease to be a pair of twin rows. If in addition we consider cases when $i = 1$ and $i = 2$, then we finally obtain that 6 pairs of rows of 21 newly formed pairs of rows cease to be pairs of twin rows and corresponding rows, as shown above, are repainted into dark color. As a result, a number of new pairs of twin rows in A_4 matrix is equal to 15.

Moreover, all first terms and difference of the arithmetic progression derived from the numbers located in each row of the newly formed 15 pairs of twin rows of A_4 matrix, are coprimes, i.e.:

$$([p_2!'p_i + p_3!'(m-1) \mp 1], p_4!') \equiv 1, \qquad (6)$$

where $i = 1, 2, 3$; $m \in (1, p_4)$ и $m \neq 5,6,9,10,17,18$.

In virtue of this, from Dirichlet theorem for prime numbers in arithmetic progression, it follows that in each row of 15 newly created pairs of twin rows there is an infinite number of prime numbers.

If we consider further similar transformation of matrices into the the next matrices, for example, matrix A_{k-1} into matrix A_k, then every time we verify that any pair of twin rows of the original matrix generates p_k new pairs of rows in new matrix. In addition 2 pairs of them will not be pairs of twin rows and corresponding rows are moving into the ranks of painted and single rows. From this we see that in any A_k matrix a total number of rows $(i_{k,max})$ and a total number of twin row pairs (m_k) are respectively equal to:

$$i_{k,max} = i_{k-1,max} * p_k \quad \text{and} \quad m_k = m_{k-1}(p_k - 2) \qquad (7-1)$$

or

$$i_{k,max} = p_k!' \quad \text{and} \quad m_k = (p_k - 2)!', \quad k \geq 2 \qquad (7-2)$$

where $k -$ an indexing number of A_k matrix and/or prime number p_k and

$$(p_k - 2)!' = (p_2 - 2)(p_3 - 2) * \ldots * (p_k - 2) =$$
$$= 1 * 3 * 5 * 9 * 11 * \ldots * (p_k - 2)$$

From (7) it follows that a number of twin rows pairs while moving to the next matrices, i.e. with increasing of indexing number k of A_k matrix, increases monotonically. On the other hand, a set of numbers, that are in each row of these m_k pairs of twin rows, forms an arithmetic progression. And the first term and difference of each these progressions are coprimes. Because of this, from Dirichlet's theorem it follows that in each row of any pair of twin rows there are an infinite number of primes.

Theorem 1 is proved.

As shown in (7), while moving to the next matrices a number of twin row pairs will be progressively increasing. But a number of general rows of each next matrix is increased as a special factorial $p_k!'$. Therefore **a density** of twin row pairs is progressively decreased since with rising k the ratio $\frac{(p_k-2)!'}{p_k!'}$ is reduced progressively.

Theorem 2. ***I in any pair of twin rows of any matrix A_k имеются there are prime twins.***

As shown above, all twin primes can be located in twin rows **<u>only</u>**. But whether there are cases where in some pair of twin rows no any pair of two prime numbers can be located in one column. Then, due to the asymmetry (i.e. due to the skewness in prime numbers location) there will be no any pair of prime twins in this pair of twin rows. If this general skewness happens in all pairs of twin rows of this matrix, then twin primes will no longer be in this and all next matrices. Therefore we can say clearly that a number of prime twins should be limited.

We will analyze this case and conjointly prove Theorem 2.

Проанализируем этот случай и заодно докажем теорему 2.

Proof of Theorem 2.

Let consider A_2 matrix which contains only one unique pair of twin rows *(Figure 1, A_2-в)*. On the other hand, as shown above, twin prime numbers can be located in pairs of twin rows **<u>only</u>**. This means that all existing twin prime numbers are located in this only one pair of twin rows.

A simple analysis shows that pairs of twin prime numbers conform to some simple rules. In particular, the last digit of any prime number (except 2 and 5) can not be an even number and it can not be equal to 5 as well. This means that last digits of the first and second number of any twin pair should be respectively (1 and 3) and (7 and 9), and (9 and 1) as well. Therefore, a set of twin prime numbers should be divided into 3 subsets on these grounds. In fact, when we form A_3 matrix, a single pair of twin rows of A_2 matrix generates new three pairs of twin rows in A_3 matrix. Moreover, the last digits of each number located in the rows of individually selected pair of twin rows are respectively equal (1 and 3) and (7 and 9) and (9 and 1), this is readily illustrated by *Figure 1, A_3-в*.

Consequently, the fact that in each pair of twin rows of A_3 matrix there is a set twin primes, which is sorted with regard to the value of the latest digit, is beyond question. That is, in pairs of twin rows of A_3 matrix primes a general skewness will not appear.

Now we consider next A_4 matrix. The analysis performed shows that all the members of each pair of twin rows of this matrix are more ordered than in the case of A_3 matrix. For example, the last two digits of each number of the arithmetic progression, located in any row of any twin row pair of A_4 matrix form cyclically increasing sequence: 21, 31, 41, ..., 81, 91, 01, 11, 21, 31, ...

Here we suppose that an average distance <u>*by columns*</u> between cells, where neighbor prime numbers are located in *one* pair of twin rows of A_4 matrix, must be less than an average distance of cells, where neighbor primes in *one* pair of twin rows of A_3 matrix are located, If this is the case, then in each pair of twin rows of A_4 matrix there inevitably are twin primes. Since in this case, due to the tightness, if above-mentioned skewness of primes appears in A_4 matrix then it will be to a lesser extent then in case of A_3 matrix. At least in A_4 matrix there will not be a general skewness of prime numbers.

To verify this, first we enter new parameter $d_{i,i+1}$ which is <u>*a distance by columns between the cells*</u>, where neighbor prime numbers with indexing numbers i and $i+1$ (*Figure 2*) are located:

$$d_{i,i+1} = |M_{i+1} - M_i|, \qquad (8)$$

where M_i is indexing number of the column in which a cell of i-th prime number is located. Moreover, if two neighbor primes are located in two mutually adjacent cells along one row (i.e., horizontally, as shown in *Figure 2*, *a*), then a distance between these cells is equal to 1:

$$d_{i,i+1} = |M_{i+1} - M_i| = 1$$

On the other hand, if two neighbor prime numbers are located in two neighboring and adjoining cells lying in the same column (i.e., vertically as shown in *Figure 2*, *в*), the distance between these cells is zero :

$$d_{i,i+1} = |M_i - M_i| = 0$$

In this case these two primes are twins.

Now, for example, we consider a fragment of <u>one</u> pair of twin rows of any matrix. Let suppose that this fragment contains *n* prime number, as shown in *Figure 2*, *c*. In this figure, light circles denote cells, in which composite numbers are located and dark circles mark cells with prime numbers. Then, above mentioned <u>*average distance by columns*</u> d_{av} between cells, where neighbor primes are located, will be equal to:

$$d_{av} = \frac{\sum_{i=1}^{n-1} d_{i,i+1}}{n-1} \qquad (9)$$

In this case, as shown above, a distance by columns between cells, where neighbor prime numbers with indexing numbers (1 and 3) and (4 and 6) are located, is equal 1, i.e.: $d_{1,3} = |M_3 - M_2| = 1$; $d_{4,6} = |M_6 - M_5| = 1$.

On the other hand a distance by columns between cells of neighbor prime numbers with indexing (1 and 2), (5 and 6) and (7 and 8) is equal to zero, that is: $d_{1,2} = M_2 - M_2 = 0$; $d_{5,6} = M_6 - M_6 = 0$; $d_{7,8} = M_9 - M_9 = 0$. Therefore, if we consider a prime number with indexing number 1, then its nearest neighbor prime number on the right side is a prime number with indexing number 2, rather than a number with indexing number 3 (*Figure 2, c*). If we consider a prime number with indexing number 4, its nearest neighbor to the right side is a prime number with indexing number 5, not a number with indexing number 6. This is easily seen, if we calculate the difference between values of these numbers by formula (1). Similarly, it can be easily determined that the nearest neighbor of a prime number with indexing number 9 on the left side is a prime number with a indexing number 8, rather than a number with indexing number 7, although these two prime numbers (7 and 9) are located in one row.

Note, that numbers with indexing numbers (1 and 2), (5 and 6), and (7 and 8) are twins *(Figure.2,c)*.

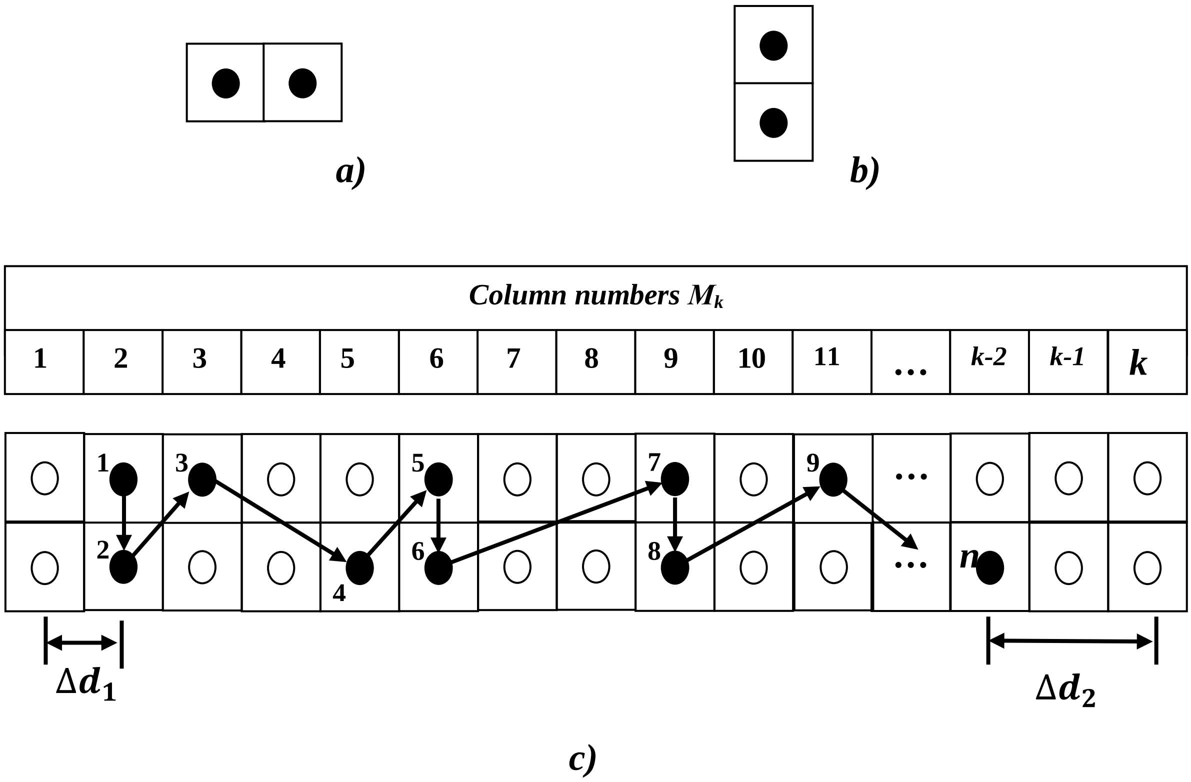

***Figure 2.* Location of prime numbers in a pair of twins rows**

It should be said here that the first prime number can take a cell that will be located not in the first column of the considered fragment (*Figure 2, c*). Similarly, it can be said that a cell in which the last prime number with indexing number *n* is located, may also located not in the last column. These cases are not taken into account in (9), i.e. there are no parameters Δd_1 and Δd_2 in the expression (9), where Δd_1 is a number of columns from the beginning of the considered fragment to a cell of the first prime number, Δd_2 is a number of columns from a cell of the last prime number to the end of the considered fragment *(Figure 2, c)* . To account for this, let we consider a sum:

$$\Delta d = \Delta d_1 + \Delta d_2.$$

It should be said that a value of the considered parameter Δd is comparable to and most likely equal to the distance by columns from the cell of the last prime number with an indexing number n to the cell, where the next prime number with indexing number $n+1$ will be located. The last number, as follows from the figure 2-c, is located outside the analyzed fragment, but it will be the nearest neighbor prime number to the right for n-th prime number. From this it follows that:

$$\Delta d \simeq d_{n,n+1}$$

If we also add parameter Δd into the sum in numerator of expression (9) then we exactly obtain «a length» of the considered fragment *(рис.2,c):*

$$\sum_{i=1}^{n-1} d_{i,i+1} + \Delta d = M_k$$

Here, a number of items in the numerator of expression (9) will be greater by 1, i.e. a number of the prime numbers in question will be as it increased by 1 and becomes equal to $n+1$. Accordingly expression (9) takes the following form:

$$d_{av} = \frac{\sum_{i=1}^{n-1} d_{i,i+1} + \Delta d}{(n+1)-1} = \frac{M_k}{n} \tag{10}$$

Now consider a real case - a case of A_{k-1} matrix fragment with sizes (m_{k-1}, M_{k-1}). Here m_{k-1} is a number of pairs of twin rows, M_{k-1} a number of columns in given fragment of A_{k-1}matrix, which corresponds to a prime number p_{k-1}. It should be said here that a value of M_{k-1} should be sufficiently large to make sense for statistics. For example, it should be:

$$M_{k-1} \gg p_k!' \tag{11}$$

Let π_{k-1} is a total number of all prime numbers located in all pairs of twin rows of the considered fragment of A_{k-1}, matrix, (k-1), $\pi_{k-1,l}$ is a number of prime numbers contained in one selected pair of twin rows with indexing number *l*. Here a total number of twin rows pairs should be equal m_{k-1}, i.e. *l* takes values from 1 to m_{k-1} , in short $l = 1 \div m_{k-1}$.

Then from (10) applied to the case of matrix A_{k-1} we obtain:

$$d_{k-1,l} = \frac{M_{k-1}}{\pi_{k-1,l}} \tag{12}$$

where $d_{k-1,l}$ – average distance *by columns* between cells where neighbor prime numbers, lying in one pair of twin rows with indexing number l, are located.

Note, here and further, that first index of the parameter in question (in this case index $k-1$) will correspond to indexing number of the considered matrix, and second index of this parameter (in this case index l) is an indexing number of the analyzed pair of twin rows.

Above we made an assumption that prime numbers in pairs of twin rows of each new matrix must be located more closely than in pairs of twin rows of previous matrix. For the analysis and evaluation of the assumption we will analyze a value of $d_{k-1,l}$ in pairs of twin rows of the considered fragment.

From the works of Siegel [6], [7], [8] and other researchers [9], [10], [11], [12] it follows that if the constants (the difference) of different arithmetic progressions are equal to each other, and also the first term and difference of each arithmetic progressions are co-primes, then in these progressions prime numbers are distributed equally and identically. On the other hand, as shown above, a sequence of numbers located in any row of any pair of twin rows of A_k matrix forms an arithmetic progression with a difference equal to equal $p_k!'$. . In addition the first term and difference of these arithmetic progressions are co-primes. These progressions differ from each other only by a value of the first term, and in all other respects they are identical. This means that within the considered fragment a number of prime numbers in any pair of twin rows is approximately equal to each other, i.e.:

$$\pi_{k-1,av} \simeq \pi_{k-1,1} \simeq \pi_{k-1,2} \simeq \cdots \simeq \pi_{k-1,l} \simeq \cdots \simeq \pi_{k-1,m_{k-1}}, \tag{13}$$

where $\pi_{k-1,av}$ is an average number of prime numbers, containing in a pair of twin rows of the considered fragment of A_{k-1} matrix.

Then from this and expression (12) we obtain:

$$d_{k-1,av} \simeq d_{k-1,1} \simeq d_{k-1,2} \simeq \cdots \simeq d_{k-1,l} \simeq \cdots \simeq d_{k-1,m_{k-1}},$$

or

$$d_{k-1,av} = \frac{M_{k-1}}{\pi_{k-1,\text{cp}}} \tag{14}$$

где $d_{k-1,av}$ - a mean distance *by columns* between cells where neighbor prime numbers, containing in a pair of twin rows of the considered fragment of A_{k-1} matrix, are located.

On the other side, it follows from (13) that

$$\pi_{k-1,av} = \frac{\sum_{j=1}^{m_{k-1}} \pi_{k-1,j}}{m_{k-1}} = \frac{\pi_{k-1}}{m_{k-1}} \tag{15}$$

Here, we call attention to the following:

1. In separately selected row of any pair of twin rows in any matrix, excepting matrices A_0 and A_1 *(Figure 1*, A_0 and *Figure.1*, A_1), there can not be prime twins as shown above.
2. If two prime numbers, as shown above, are located in one column within one pair of twin rows, then they are twins. In this case a distance *by columns* between cells, where these prime twin numbers located should be equal to zero.
3. Here, considered distance $d_{k,av}$ *by columns* between cells, where neighbor prime numbers are located, should not be confused with a difference, i.e. with a distance between values of neighbor prime numbers. It is obvious, that in one cell of considered fragment of any matrix only one number can be located. In addition, a difference of values of neighbor prime numbers, which are located in different cells of the fragment, albeit adjacent by row to each other, can possess any integer value

As shown in case of the proving Theorem 1, it follows from (7) that while transforming A_{k-1} matrix into $A_k$matrix, each pair of twin rows of A_{k-1} matrix generates $p_k - 2$ new pairs of twin rows and additionally 2 single rows in a new matrix. As a result, each pair of twin rows of A_{k-1} matrix creates $p_k -$ pairs of new rows. Totally, m_{k-1} pairs of twin rows generate $m_k = m_{k-1}\,(p_k - 2)$ new pairs of twin rows and additionally $m_k' = m_{k-1}$ pairs of single rows in new A_k. matrix.

In short, a set of prime numbers in an amount of $\pi_{k-1} = m_{k-1}M_{k-1}/d_{k-1,av}$, located in all pairs of twin rows of the considered fragment of A_{k-1} matrix is redistributed by $m_k' = m_{k-1}\,(p_k - 1)$ new pairs of rows of a new fragment corresponding to the new matrix A_k. Then a number of prime number located in all newly created m_k pairs of twin rows of A_k matrix fragment is defined by the expression:

$$\pi_k = \frac{m_k\,\pi_{k-1}}{m_k + m_k'} = \frac{m_k \pi_{k-1}}{m_{k-1}\,(p_k - 1)}$$

Therefore, a number of prime numbers, located in one selected pair of twin rows of the considered fragment of new matrix A_k, is equal on average to:

$$\pi_{k,av} = \frac{\pi_k}{m_k} = \frac{\pi_{k-1}}{m_{k-1}\,(p_k - 1)} \tag{16}$$

On other hand, while transforming a fragment of A_{k-1}matrix into a fragment of A_k matrix a number of columns in new fragment, as shown above, is reduced p_k times, i.e. $M_k = M_{k-1}/p_k$.

Then from (14), (15) and (16) we obtain an average distance by columns between the cells, where neighbor prime numbers, lying in one pair of twin rows of considered fragment of A_k matrix, are located

$$d_{k,av} = \frac{M_k}{\pi_{k,av}} = \frac{m_{k-1}M_{k-1}}{\pi_{k-1}} * \frac{p_k - 1}{p_k} = d_{k-1,av}\frac{p_k - 1}{p_k} \tag{17}$$

It should be noted here, that parameter Δd in the numerator of expression (10) is considered as one summand although it, as shown above, consists of two parameters Δd_1 and Δd_2. If in (10) these two parameters will be considered separately, then a number of summands in the numerator of (10) will be equal to $n + 2$. Taking this into account expression (10) takes the following form:

$$d_{av} = \frac{\sum_{i=1}^{n-1} d_{i,i+1} + \Delta d_1 + \Delta d_2}{(n+2) - 1} = \frac{M_k}{n+1}$$

If using this expression, we make appropriate simple changes in (12) and (14), then expression (17) finally takes the following form:

$$d_{k,av} = \frac{M_k}{\pi_{k,av} + 1} = \frac{1}{\frac{1}{d_{k-1,av}} + \frac{p_k - 1}{M_{k-1}}} * \frac{(p_k - 1)}{p_k}$$

It can be seen from this expression that the second summand located in its denominator in virtue of inequation (11) becomes equal to zero, i.e.:

$$\frac{p_k - 1}{M_{k-1}} = 0$$

Therefore, as a result of this we obtain, that expression for the considered parameter $d_{k,av}$ will have the same form as in (17). In short, separate consideration of Δd_1 and Δd_2 parameters provides the same result which we could obtain considering only one parameter Δd instead of two parameters Δd_1 and Δd_2.

From this it follows that with increasing indexing number k of A_k matrix an average distance $d_{k,av}$ between the cells of neighbor prime numbers in any pair of twin rows of A_k matrix decreases continuously.

Therefore, because of the growing tightness of an infinite number of prime numbers as well as the identity of their distribution in any pair of twin rows of A_k matrix, twin primes rather "are formed" than in a case of the previous A_{k-1}matrix.

In particular, due to the fact that, as shown above, the general skewness of prime numbers in pairs of twin rows of A_3 matrix does not exist, then it will not appear in pairs of twin rows of the next matrices A_4, A_5, A_6, …, A_k, ... So, in each pair of twin rows of any A_k matrix there are twin prime numbers.

Theorem 2 is proved.

Now we consider a problem posed in front of this research.

Theorem 3. ***Twin primes number is infinite.***

Proof of Theorem 3.

As shown above, each matrix A_k corresponds to a certain prime number p_k. It is known that a number of primes is infinite, Therefore, a number of A_k matrices variations is also infinite. On the other hand twin prime numbers can be only in pairs of twin rows.

It follows from Theorem 1 and expression (7) that with the growth of A_k matrix indexing number, a number of pairs of twin rows in this matrix is steadily increased. i.e.

$$\lim_{k\to\infty} m_k = \lim_{k\to\infty} (p_k - 2)!' = \infty$$

It also follows from Theorem 2, that in any pair of twin rows there are prime twins. All this means that a number of prime twins is infinite.

This conclusion is also unavoidably followed from expression (17).

If in (17) we express the parameter $d_{k-1,av}$ in terms of $d_{k-2,av}$, which is a mean distance by columns between cells, where neighbor prime numbers in a pair of twin rows of matrix A_{k-2} are located, then we obtain

$$d_{k-1,av} = d_{k-2,av} \frac{p_{k-1} - 1}{p_{k-1}}$$

then

$$d_{k,av} = d_{k-2,av} \frac{p_{k-1} - 1}{p_{k-1}} * \frac{p_k - 1}{p_k}$$

Next, in the same manner, we transform parameter $d_{k-2,av}$ through $d_{k-3,av}$, then $d_{k-3,av}$ through $d_{k-4,av}$, etc. Continuing the transformation up to $d_{1,\text{cp}}$, we obtain, that an average distance by columns between cells of prime numbers in any pair of twin rows of matrix A_k can be generally determined by expression:

$$d_{k,av} = d_{1,av} \frac{(p_1 - 1)(p_2 - 1) \dots (p_k - 1)}{p_1 p_2 * \dots * p_k} = d_{1,av} \frac{(p_k - 1)!'}{p_k!'} \qquad (18)$$

where $k \geq 2$ and $d_{1,av}$ is a mean distance between cells of neighbor prime numbers lying in only one not painted row of A_1 matrix *(Figure.1, A_1-в).*

It appears from (18), that with increasing of indexing number k of matrix A_k an average distance $d_{k,av}$ between cells of prime numbers in any pair of twin rows of A_k matrix is continuously reducing. And if $k \to \infty$ we obtain that $d_{k,av} \to 0$. In fact, having done small transformation we obtain from (18) that

$$\ln d_{k,av} = \ln d_{1,av} - \left(\sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i=1}^{k} \sum_{m=2}^{\infty} \frac{1}{m * p_i^m} \right)$$

Here, the infinite series, containing reciprocals of prime numbers, diverges, as was shown by Euler [13], i.e.:

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$$

Therefore,

$$\lim_{k \to \infty} d_{k,av} = 0 \qquad (19)$$

.

This means that with the growth of the indexing number of A_k matrix an average distance $d_{k,av}$ between the cells, where neighbor primes are located, tend to zero. In addition, the infinity of twins is not only inevitable, but also obvious because it follows from (7) that as $k \to \infty$ a number of pairs of twin rows tends to infinity: $\lim_{k \to \infty} m_k = \infty$, on the other hand as it follows from the Theorem 2, in each of these pairs of twin rows there are twin prime numbers.

Theorem 3 is proved.

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Abstract

The research proposes prime numbers matrices by which prime numbers series can be easily generated. The paper also suggests a number of theorems based on which infinity of a twin primes number is proved.

Keywords: *prime numbers, twin primes, composite numbers, natural numbers, algorithms, arithmetic progression, prime numbers matrix, special factorial, generation of prime numbers.*