Symmetries in Physics

Lecture Notes

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The current version of this work can be found at: http://people.phys.ethz.ch/~nbeisert/lectures/.

Contents

0 Overview

Some keywords. Group theory in connection to physics is an incredibly rich topic:

- symmetry
- geometry
- topology
- solid state lattices
- special relativity
- quantum mechanics
- spin
- exchange statistics spin
- quantum Hall effect
- gauge theory
- quantum anomalies
- supersymmetry
- integrable systems
- string theory
- exceptional groups

One might argue that physics of the 20th century is all about group theory. The mathematics involved in this is not quite as recent though; modern mathematical formulations of groups are very general and also abstract.

We will learn about many mathematical concepts of group theory in a physicist's or classical mathematics language. This is intended to be a (theoretical) physicist's course. I will try to provide examples from various areas of physics and use them as a motivation for the mathematical concepts.

0.1 Prerequisites

Prerequisites for this course are the core courses in mathematics and theoretical physics of the bachelor syllabus:

- linear algebra (basic concepts, linear maps)
- quantum mechanics (representations of SO(3) or SU(2))
- classical mechanics (formalism)
- electrodynamics (fields)
- mathematical methods in physics (HO, Fourier transforms, ...)

0.2 Contents

1. Two-Dimensional Rotations (125 min)
2. Three-Dimensional Symmetries (245 min)
3. Finite Group Theory (155 min)
4. Point and Space Groups (130 min)
5. Structure of Simple Lie Algebras (100 min)
6. Finite-Dimensional Representations (240 min)
7. Representations of SU(N) (skipped)
8. Classification of Simple Lie Algebras (135 min)
9. Conformal Symmetry (195 min)

Indicated are the approximate number of lecture minutes for each chapter. Altogether, the course consists of 26 lectures.

0.3 References

There are many text books and lecture notes on group theory, representation and physics. Here are some which I will refer to in my preparation:

- J. F. Cornwell, "Group Theory in Physics, An Introduction", Academic Press (1997)
- S. Sternberg, "Group theory and physics", Cambridge University Press (1994)
- H. F. Jones, "Groups, Representations and Physics", CRC Press (1998)
- M. Hamermesh, "Group Theory and Its Application to Physical Problems", Dover Publications (1989)
- online: M. Gaberdiel, "Symmetries in Physics", lecture notes (HS13), `http://edu.itp.phys.ethz.ch/hs13/Symmetries/`
- ...

1 Two-Dimensional Rotations

Symmetries in physics are typically expressed by mathematical groups acting in some specific way on some objects or spaces.

In the first chapter we introduce the basic notions of group theory using the example of rotations in two spatial dimensions. We shall proceed slowly and treat some auxiliary details carefully so that we can rely on these discussions in more complex situations to be encountered in the following chapters.

1.1 Group Basics

We start with rotations in two-dimensional space. A rotation by angle φ is defined by the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \end{pmatrix}. \tag{1.1}$$

This simple example already contains several notions of group theory which we shall explore in the following. Most of them will be intuitive (for anyone who has studied physics for a while), but it will be good to know the appropriate mathematical terms and concepts and to associate them to physical applications right away. We shall also highlight a few subtleties that are resolved by the mathematical framework.

Definition of Group. Let us collect a few properties of the above maps:

- Two consecutive rotations by angles φ_1 and φ_2 yield another rotation, namely by the angle $\varphi = \varphi_1 + \varphi_2$. This statement can be confirmed by direct computation applying the addition theorem for the trigonometric functions.
- The rotation by the angle $\varphi = 0$ (or more generally by $\varphi \in 2\pi\mathbb{Z}$) is distinguished in that it maps all points (x, y) to themselves (identity map).
- For the rotation by any angle φ there exists the inverse rotation by the angle $-\varphi$ such that their composition is the trivial rotation.
- The order in which two or more rotations are performed does not matter.
- Rotations whose angles differ by an integer multiple of 2π are equivalent.

The former three of the above properties match nicely with the definition of a group: A *group* is a set G with a composition rule $\mu : \mathrm{G} \times \mathrm{G} \to \mathrm{G}$, which has the following three properties:

- It is associative

$$\mu\bigl(\mu(a, b), c\bigr) = \mu\bigl(a, \mu(b, c)\bigr) \quad \text{for all } a, b, c \in \mathrm{G}. \tag{1.2}$$

- There exists a (unique) identity element $e \in \mathrm{G}$ such that

$$\mu(e, a) = \mu(a, e) = a \quad \text{for all } a \in \mathrm{G}. \tag{1.3}$$

- There exists a (unique) inverse map $\eta : \mathrm{G} \to \mathrm{G}$ such that

$$\mu\big(a, \eta(a)\big) = \mu\big(\eta(a), a\big) = e \quad \text{for all } a \in \mathrm{G}. \tag{1.4}$$

Usually one considers the concept of a group as the generalisation of multiplication of numbers. Therefore one writes the composition rule $\mu(a, b)$ as multiplication $a{\cdot}b$ or even shorter as ab. The inverse $\eta(a)$ of an element a is denoted by a^{-1}. In this course, we will also denote the identity element as 1 instead of e.[1] The group axioms in multiplicative notation read

$$(ab)c = a(bc), \qquad 1a = a1 = a, \qquad aa^{-1} = a^{-1}a = 1. \tag{1.5}$$

In a general group, the order of the elements to be multiplied matters. In our example, this is apparently not the case. A group with a symmetric composition rule

$$\mu(a, b) = \mu(b, a) \qquad \text{or} \qquad ab = ba \tag{1.6}$$

is called *abelian*. Abelian groups are largely boring, up to some subtleties to be discussed. This course will mostly be about *non-abelian* groups where the composition rule is non-commutative.

Group Action. The group axioms translate to the properties of the rotations, however, there is no immediate match for associativity. Associativity comes about by considering the above rotation as a group action. In mathematics, one clearly distinguishes between the group elements as abstract rotations and the group action as a transformation rule acting on some set. The *action* of a group G on a set M is a map $\alpha : \mathrm{G} \times M \to M$ with the following properties

- Consecutive group actions can be combined, i.e. the action is compatible with the group composition rule[2]

$$\alpha\big(b, \alpha(c, m)\big) = \alpha\big(\mu(b, c), m\big) \quad \text{for all } b, c \in \mathrm{G},\ m \in M. \tag{1.7}$$

- The identity element acts trivially

$$\alpha(e, m) = m \quad \text{for all } m \in M. \tag{1.8}$$

In physics one usually assumes a unique/natural action of a given group on a given set, and in multiplicative notation it suffices to denote the group action $\alpha(b, m)$ by $b{\cdot}m$ or just bm. Thus

$$b(cm) = (bc)m, \qquad 1m = m. \tag{1.9}$$

[1] Also the identity elements of various other algebraic structures will be denoted by 1 hoping that no ambiguities will arise. 1 is one is **1** is id.

[2] Note that the order of the actions is from right to left (as for matrices acting on vectors), i.e. c acts first in the product bc, which can be confusing.

The requirement that the composition of group actions is again a group action in fact hinges on associativity of the group:[3]

$$\big((ab)c\big)m = (ab)(cm) = a\big(b(cm)\big) = a\big((bc)m\big) = \big(a(bc)\big)m. \tag{1.10}$$

Furthermore, a simple corollary of the definition is that the group action α_b of a given group element b is a bijection on M, i.e. the map $\alpha_b : M \to M$ with $\alpha_b(m) := \alpha(b, m)$ is one-to-one. The inverse group action $(\alpha_b)^{-1}$ is given by the inverse group element $\alpha_{b^{-1}}$

$$b^{-1}(bm) = (b^{-1}b)m = 1m = m. \tag{1.11}$$

Initially, we merely considered concrete rotational transformations on two-dimensional space $\mathbb{R}^2$. For the group theoretical treatment we have to split up this concept into a group element and a group action. The natural choice is to introduce an abstract rotation by the angle $\varphi \in \mathbb{R}$ denoted by $R_\varphi \in \mathrm{G}$. The group composition rule, identity and inverse read

$$R_{\varphi_2}R_{\varphi_1} = R_{\varphi_1+\varphi_2}, \qquad R_0 = 1, \qquad (R_\varphi)^{-1} = R_{-\varphi}. \tag{1.12}$$

The action of R_φ on a point $\vec{x} \in \mathbb{R}^2$ is then denoted by $R_\varphi \vec{x}$. More concretely,

$$R_\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \end{pmatrix}. \tag{1.13}$$

Group Topology. In fact, the above definition of the rotational group is not yet uniquely determined by the discussion, and it makes sense to philosophise on the appropriate definition in order to understand better the relationship between a physical symmetry and an abstract group: For example, one could easily embed the above group elements R_φ into a larger group and still obtain the same rotational action on $\mathbb{R}^2$. To this end, a physically reasonable setup is to consider the group of rotations together with time translations ($t \mapsto t + \tau$). However, time translations are independent of spatial rotations, and the rotational action of a group element would simply ignore the time translation. There surely is no need to do this, but it can be done with the only penalty of having a somewhat bigger abstract group. Such a group can be physically relevant, e.g. to describe a constantly rotating reference frame where the angle of rotation is linked to time ($\varphi = \varphi_0 + \omega\tau$). Equally, it makes sense to ask what is the minimal abstract group with the above properties, and how to understand its extensions.

The abstract rotation R_φ is not yet uniquely defined because we have not specified the suitable range of φ which is evidently a subset of the real numbers. We know that rotations which differ by multiples of 2π act equivalently

$$R_\varphi \simeq R_{\varphi+2\pi m} \qquad \text{or} \qquad R_\varphi\vec{x} = R_{\varphi+2\pi m}\vec{x}. \tag{1.14}$$

[3] In principle, $(ab)c$ and $a(bc)$ could differ while their actions on m are identical. This logical possibility is avoided by associativity of the group.

However, that does not imply that the corresponding abstract rotations are the same. In other words, it may or may not make physical sense to keep track of how many full turns a given rotation contains. Let us identify group elements which differ by $n \in \mathbb{Z}^+$ full turns using the symbols $R^{(n)}_\varphi$

$$R^{(n)}_\varphi = R^{(n)}_{\varphi+2\pi n}. \tag{1.15}$$

The corresponding group can be defined as

$$\mathrm{G}^{(n)} := \{R^{(n)}_\varphi; \varphi \in \mathbb{R}/2\pi n\mathbb{Z}\}. \tag{1.16}$$

The group without identifications can be denoted as

$$\mathrm{G}^{(\infty)} := \{R^{(\infty)}_\varphi; \varphi \in \mathbb{R}\}. \tag{1.17}$$

Thus $\mathrm{G}^{(1)}$ is the minimal definition for the group of rotations in two dimensions. However, in quantum theory, $\mathrm{G}^{(2)}$ also has a natural meaning: Here, a rotation by 4π is the identity element e, but a rotation by 2π is a non-trivial element. For quantum mechanical states with Fermi statistics, an odd number of full turns should flip the sign of the state. Thus one would naturally identify this element with the operator $(-1)^F$

$$R^{(2)}_{2\pi} = (-1)^F. \tag{1.18}$$

Finally, $\mathrm{G}^{(\infty)}$ keeps track of all full rotations, which may be beneficial in some cases. For instance, the function $\log(x + \mathring{\imath}y)$ which measures the logarithmic distance and the angle w.r.t. the origin is not uniquely defined on plain $\mathbb{R}^2$. If considered as a multi-valued function one could say that its value shifts by $\mathring{\imath}\varphi$ under the action of $R^{(\infty)}_\varphi$ (if the multi-valued function is analytically continued without jumps).

A further subtlety is that all groups $\mathrm{G}^{(n)}$ with finite n are equivalent, i.e. *isomorphic*. In other words, one can identify the elements

$$R^{(n)}_\varphi \equiv R^{(1)}_{\varphi/n} \tag{1.19}$$

such that the group composition rule is respected (with all applicable identifications between the elements). From an abstract group theoretical point of view, it suffices to consider $\mathrm{G}^{(1)}$ and $\mathrm{G}^{(\infty)}$.

In this picture, the above ambiguities in choosing the periodicity of R_φ translate to the existence of inequivalent group actions of $\mathrm{G}^{(1)}$ on $\mathbb{R}^2$ labelled by $n \in \mathbb{Z}$

$$\alpha_n(R^{(1)}_\varphi, \vec{x}) = \begin{pmatrix} x\cos(n\varphi) - y\sin(n\varphi) \\ x\sin(n\varphi) + y\cos(n\varphi) \end{pmatrix}. \tag{1.20}$$

In other words $R^{(1)}_\varphi$ could act as a rotation by n times the angle specified by the group element. Note that α_0 is the trivial group action which acts as the identity map for all group elements. The difference for $\mathrm{G}^{(\infty)}$ as compared to $\mathrm{G}^{(1)}$ is that the parameter n of the action α_n is not restricted to the integers, but can rather be any real number, $n \in \mathbb{R}$, because of the absence of the periodicity constraint.

Furthermore, the groups are isomorphic to the additive groups on a circle or on the real line

$$\mathrm{G}^{(n)} \equiv \mathbb{R}/2\pi n\mathbb{Z}, \qquad \mathrm{G}^{(\infty)} \equiv \mathbb{R}, \tag{1.21}$$

via the trivial identification $R_\varphi^{(n)} \equiv \varphi$. This shows that the set of rotations has the topology of either the circle or the real line

$$\mathrm{G}^{(n)} \equiv \mathrm{S}^1, \qquad \mathrm{G}^{(\infty)} \equiv \mathbb{R}. \tag{1.22}$$

Thus $\mathrm{G}^{(n)}$ is compact while $\mathrm{G}^{(\infty)}$ is non-compact in a topological sense. Compact groups have some convenient features which simplifies their treatment substantially. Furthermore, they play an important role in quantum physics. Nevertheless, non-compact groups are very relevant for physics as well.

1.2 Representations

The final topic in connection to two-dimensional rotations are representations. Representations are perhaps the most important objects in group theory relevant to physics, in particular quantum physics. A representation is essentially a group action which acts linearly on a set. For this to make sense, the set should have the structure of a vector space. Useful examples of representations correspond to the notions of vectors, tensor, spinors, but also momentum eigenstates, spherical harmonics, and many other physics concepts.

Definition. A *representation* ρ [4] of a group G on a vector space $\mathbb{V}$ is a map

$$\rho : \mathrm{G} \to \mathrm{Aut}(\mathbb{V}), \tag{1.23}$$

which respects the group composition law and the identity element

$$\rho(a) \circ \rho(b) = \rho(\mu(a, b)), \qquad \rho(e) = \mathrm{id}_{\mathbb{V}}, \tag{1.24}$$

or more concisely in the multiplicative notation

$$\rho(a)\rho(b) = \rho(ab), \qquad \rho(e) = 1. \tag{1.25}$$

Note that the formal definition of the representation is slightly different from a linear group action $\alpha : \mathrm{G} \times \mathbb{V} \to \mathbb{V}$ in that it maps a group element to an *automorphism* of $\mathbb{V}$, i.e. an invertible linear transformation from $\mathbb{V}$ to itself. The space $\mathrm{Aut}(\mathbb{V})$ is equipped with a natural multiplication which is composition of the linear transformations. In the finite-dimensional case, and given a certain basis of $\mathbb{V}$, a linear transformation can be viewed as a square matrix, and multiplication corresponds to the matrix product.

In the physics context, as in the case of the group action, the representation symbol ρ is often omitted, and $\rho(a)\vec{v}$ becomes $a\vec{v}$. This is valid because in many cases,

[4] Other common symbols are R, D, r, $\mathit{\Gamma}$, ...

there is a distinguished representation for a particular vector space. Furthermore, in the physics literature, no distinction is made between the representation as defined above and the space $\mathbb{V}$ it acts upon. Often the context reveals which of the two concepts is referred to. More correctly, the vector space can be called a G-*module*. We shall refer to the space $\mathbb{V}$ as the *representation space*.

Types of Representations. The group actions discussed above are linear and thus correspond to representations on the space $\mathbb{V} = \mathbb{R}^2$. For the plain rotation we can define the representation

$$\rho(R_\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{1.26}$$

The resulting matrices are orthogonal, $(R_\varphi)^\mathsf{T} R_\varphi = 1$ and have unit determinant. Matrices of this kind form the group SO(2). For the compact group $\mathrm{G}^{(1)}$ the representation $\rho : \mathrm{G}^{(1)} \to \mathrm{SO}(2)$ is one-to-one which shows the group isomorphism

$$\mathrm{G}^{(1)} \equiv \mathrm{SO}(2). \tag{1.27}$$

The latter is a common name for the group of rotations in two-dimensional space. In this case, the identity map on SO(2) in fact serves as a representation, namely the *defining* or *fundamental* representation. The above representation is *real*, *two-dimensional* and *orthogonal*, i.e. it acts on a vector space $\mathbb{R}^2$ over the field $\mathbb{R}$ of real numbers by orthogonal transformations (w.r.t. the canonical symmetric bilinear form on $\mathbb{R}^2$).

It can equivalently be written as a *complex*, *one-dimensional*, *unitary* representation which acts on a vector space $\mathbb{C}^1$ over the field $\mathbb{C}$ of complex numbers by unitary transformations (w.r.t. any hermitian form on $\mathbb{C}^1$).[5] To that end, one embeds the vector $(x, y) \in \mathbb{R}^2$ into $\mathbb{C}^1$ as $x + \mathring{\imath} y$ and obtains[6]

$$\rho_\mathbb{C}(R_\varphi) = \left(\mathrm{e}^{\mathring{\imath}\varphi}\right). \tag{1.28}$$

Again, this map is one-to-one and thus we have a group isomorphism to the group of unitary 1×1 matrices

$$\mathrm{G}^{(1)} \equiv \mathrm{U}(1). \tag{1.29}$$

Beyond real and complex representations, one often encounters *quaternionic* or *pseudo-real* representations over the field $\mathbb{H}$ of quaternions.[7] [8]

Similarly, a representation is called *symplectic* if all group elements are mapped to symplectic transformations (w.r.t. some anti-symmetric bilinear form on $\mathbb{V}$).

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[5] In the ordinary (strict) sense, the notion of unitarity requires that the underlying hermitian form is positive definite.

[6] The real representation matrix has a particular form which allows the interpretation as complex multiplication: $(c + \mathring{\imath}s)(x + \mathring{\imath}y) = (cx - sy) + \mathring{\imath}(sx + cy)$, where $c = \cos\varphi$ and $s = \sin\varphi$.

[7] Even though quaternions are non-commutative, matrices with quaternionic entries are perfectly admissible (in contradistinction to the remaining division algebra of octonions, where associativity is lost).

[8] For example, spinors in three-dimensional space (and correspondingly the Pauli matrices) have a quaternionic structure rather than a complex one. Nevertheless, physicists usually avoid thinking about quaternions and instead consider a complex vector space of twice the dimension.

Representation Theory. The *representation theory* of a given group describes its representations as well as their relations. For our example, the compact abelian group $\mathrm{G}^{(1)}$,[9] we have already found further group actions which translate to representations on $\mathbb{R}^2$

$$\rho_n(R_\varphi) = \begin{pmatrix} \cos(n\varphi) & -\sin(n\varphi) \\ \sin(n\varphi) & \cos(n\varphi) \end{pmatrix}. \tag{1.30}$$

Equivalently, we can write these as complex one-dimensional representations[10]

$$\rho_{\mathbb{C},n}(R_\varphi) = \mathrm{e}^{\mathring{\imath}n\varphi}\,. \tag{1.31}$$

Given some representation(s), there are three standard tools to construct further ones, namely similarity transformations, direct sums and tensor products. A similarity transformation by some invertible linear map $T \in \mathrm{Aut}(\mathbb{V})$ applied to a representation ρ on $\mathbb{V}$ yields a new representation ρ' on $\mathbb{V}$ [11]

$$\rho'(a) := T\rho(a)T^{-1}. \tag{1.32}$$

It is straight-forward to convince oneself of this fact. This new representation ρ' behaves for all purposes like ρ. Therefore, one hardly distinguishes between ρ and ρ' in representation theory and calls the representations *equivalent*

$$\rho' \equiv \rho. \tag{1.33}$$

For example, the real representations ρ_n and ρ_{-n} are equivalent

$$\rho_n \equiv \rho_{-n} \qquad \text{via } T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.34}$$

The *direct sum* of two or more representations ρ_j on the spaces $\mathbb{V}_j$ is a representation $\rho_\oplus := \rho_1 \oplus \ldots \oplus \rho_N$ on the direct sum of spaces $\mathbb{V}_\oplus := \mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_N$

$$\rho_1 \oplus \ldots \oplus \rho_N : \mathrm{G} \to \mathrm{Aut}(\mathbb{V}_1 \oplus \ldots \oplus \mathbb{V}_N). \tag{1.35}$$

The direct sum of vector spaces is easily explained if the individual vector spaces are equipped with some bases. A basis of the direct sum is given by the union of all the bases, where all basis vectors are treated as linearly independent. This implies that the dimension of the direct sum equals the sum of dimensions of its components. The direct sum of vector spaces is the same as the Cartesian product $\mathbb{V}_\oplus = \mathbb{V}_1 \times \ldots \times \mathbb{V}_N$. One can write the direct sum of representations as a matrix in block form acting on the individual spaces $\mathbb{V}_j$

$$\rho_\oplus(a) = \begin{pmatrix} \rho_1(a) & 0 & \ldots & 0 \\ 0 & \rho_2(a) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & \rho_N(a) \end{pmatrix}. \tag{1.36}$$

[9]The representation theory for the non-compact abelian group $\mathrm{G}^{(\infty)}$ is analogous, but the parameters m, n can take arbitrary real values instead of integer values.

[10]It is not a coincidence that this function appears in Fourier series.

[11]More generally, the new space for ρ' could also be a different but isomorphic space $\mathbb{V}'$.

In other words, direct sums of representations are block-diagonal and vice versa. One notable example is that the direct sum of $\rho_{\mathbb{C},n}$ and $\rho_{\mathbb{C},-n}$ is the two-dimensional representation

$$\rho_{\mathbb{C},\pm n}(R_\varphi) = \begin{pmatrix} e^{\mathring{\imath} n\varphi} & 0 \\ 0 & e^{-\mathring{\imath} n\varphi} \end{pmatrix}. \tag{1.37}$$

This matrix is in fact equivalent to

$$\rho_n(R_\varphi) = T\rho_{\mathbb{C},\pm n}(R_\varphi)T^{-1} \qquad \text{with } T = \begin{pmatrix} 1 & -\mathring{\imath} \\ -\mathring{\imath} & 1 \end{pmatrix}. \tag{1.38}$$

This shows the statement[12]

$$\rho_n \equiv \rho_{\mathbb{C},n} \oplus \rho_{\mathbb{C},-n}. \tag{1.39}$$

Similarly, the *tensor product* of two or more representations ρ_j on the spaces $\mathbb{V}_j$ is a representation $\rho_\otimes := \rho_1 \otimes \ldots \otimes \rho_N$ on the tensor product of spaces $\mathbb{V}_\otimes := \mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_N$ [13]

$$\rho_1 \otimes \ldots \otimes \rho_N : \mathrm{G} \to \mathrm{Aut}(\mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_N). \tag{1.40}$$

A tensor product of vector spaces is conveniently defined via their bases. The basis of the tensor product is the Cartesian product of the bases of the individual vector spaces. As such, the dimension of the tensor product equals the product of the dimensions of its components. The tensor product representation is defined by

$$\rho_\otimes(a) := \rho_1(a) \otimes \ldots \otimes \rho_N(a). \tag{1.41}$$

The representation axioms follow immediately from the multiplication rule for tensor products[14]

$$(m_1 \otimes \ldots \otimes m_N)(n_1 \otimes \ldots \otimes n_N) = (m_1 n_1 \otimes \ldots \otimes m_N n_N). \tag{1.42}$$

As an example, the tensor product of two generic complex one-dimensional representations is another such representation

$$\rho_{\mathbb{C},m} \otimes \rho_{\mathbb{C},n} = \rho_{\mathbb{C},m+n}. \tag{1.43}$$

The situation for two generic real representations is more involved

$$(\rho_m \otimes \rho_n)(R_\varphi) = \begin{pmatrix} c_m c_n & -c_m s_n & -s_m c_n & s_m s_n \\ c_m s_n & c_m c_n & -s_m s_n & -s_m c_n \\ s_m c_n & -s_m c_n & c_m c_n & -c_m s_n \\ s_m s_n & s_m c_n & c_m s_n & c_m c_n \end{pmatrix} \tag{1.44}$$

[12]Note that the direct sum of a complex representation and its complex conjugate is a real representation. In our case, the complex conjugate of $\rho_{\mathbb{C},n}$ is $\rho_{\mathbb{C},-n}$. This statement is based on the equivalence $\mathbb{R}^2 \equiv \mathbb{C} \oplus \bar{\mathbb{C}}$ where $\bar{\mathbb{C}}$ is understood as the complex conjugate space of $\mathbb{C}$.

[13]In quantum physics, the tensor product of two systems describes the combined system which can take on entangled states.

[14]A tensor product of two matrices $A \otimes B$ can be viewed as a block matrix of type A whose blocks are of type B, i.e. a nesting of two matrices. The elements of $A \otimes B$ are the elements of A times the elements of B (where every combination appears precisely once).

with the abbreviations $c_k := \cos(k\varphi)$ and $s_k := \sin(k\varphi)$. A similarity transformation by the matrix

$$T = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{1.45}$$

and use of the addition theorem for trigonometric functions brings the above representation to the block form $\mathrm{diag}(\rho_{m+n}, \rho_{m-n})$. Consequently, we have the equivalence

$$\rho_m \otimes \rho_n \equiv \rho_{m+n} \oplus \rho_{m-n}. \tag{1.46}$$

Note that this statement also follows from the above equivalences.

We observe that some representations, in particular tensor products of representations, are equivalent to the direct sum of representations. Direct sum representations are easy to set up, therefore they can be discarded towards understanding the representation theory of a given group; it suffices to focus on atomic building blocks. There are three types of representations in this regard:

A representation ρ on $\mathbb{V}$ which takes a block-diagonal form on the direct sum of two (suitably chosen) subspaces $\mathbb{V}_1 \oplus \mathbb{V}_2$

$$\rho \equiv \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \tag{1.47}$$

is called *decomposable*; otherwise it is *indecomposable*. For example, the representation ρ_n is indecomposable over $\mathbb{R}$ (but decomposable over $\mathbb{C}$ as we have seen above) for generic n.[15] For $n = 0$, however, the representation is trivial

$$\rho_0(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.48}$$

and can be decomposed into two trivial one-dimensional representations.

A representation with an invariant subspace is called *reducible*; otherwise it called *irreducible*. An *invariant subspace* of ρ is a space $\mathbb{V}_1 \subset \mathbb{V}$ for which $\rho(a)\mathbb{V}_1 \subset \mathbb{V}_1$ for all $a \in \mathrm{G}$. A reducible representation in a basis on $\mathbb{V}_1 \oplus \mathbb{V}_2$ has the block form

$$\rho \equiv \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}. \tag{1.49}$$

The restriction of the representation to the invariant subspace $\mathbb{V}_1$ can easily be shown to be a sub-representation ρ_1 of ρ.[16] Note that a decomposable representation is clearly reducible, while the converse is not necessarily true: there

[15] To show this statement, note that a real 1-dimensional representation is necessarily trivial, and thus the direct sum of two 1-dimensional representations is also trivial, but ρ_n is clearly not.

[16] Also ρ_2 is a representation on $\mathbb{V}_2$, but this is not a sub-representation of ρ because $\rho : \mathbb{V}_2 \to \mathbb{V}$ as compared to $\rho : \mathbb{V}_1 \to \mathbb{V}_1$. In fact, this implies that there is some freedom to choose $\mathbb{V}_2$ and this choice has an influence on the representation ρ_2.

exist *reducible but indecomposable* representations. However, for certain classes of representations (such as the physically relevant case of unitary representations) they can be excluded. In representation theory, the irreducible representations serve as the atomic building blocks from which other representations can be constructed. Due to their importance the term irreducible representation is often abbreviated as *irrep*.

Altogether, we observe that the set of considered representations of $\mathrm{G}^{(1)} \equiv \mathrm{SO}(2) \equiv \mathrm{U}(1)$ closes under all the above elementary operations on representations. All complex irreducible representations are one-dimensional and unitary. In our case, they are labelled by an integer $n \in \mathbb{Z}$. All non-trivial real irreducible representations are two-dimensional and orthogonal. In our case, they are labelled by a positive integer $n \in \mathbb{Z}^+$. The trivial irreducible representation is clearly one-dimensional.[17] We have also seen how to translate between real and complex representations. In fact, the above classification of irreducible representations (up to the precise labelling) is universal for compact abelian groups: All complex representations are one-dimensional and all real representations are two-dimensional except for the trivial one. Thus, the representation theory of compact abelian groups is hardly exciting. For non-abelian groups the representation theory has much richer structures by far.

[17]Note again the similarities to complex and real Fourier series.

2 Three-Dimensional Symmetries

In the following we consider the more exciting case of symmetries in three dimensions. The group of rotations SO(3) and its double cover SU(2) serve as the prototype of continuous groups.

2.1 Lie Group

Let us start by setting up the group. Then we shall discuss its intrinsic geometry and relate it to physics.

Elements of the Group. Rotations are linear transformations of $\mathbb{R}^3$ which leave the scalar product $\vec{x}\cdot\vec{y} := \vec{x}^{\mathsf{T}}\vec{y} = \sum_{i=1}^{3} x_i y_i$ between two vectors $\vec{x}$, $\vec{y}$ invariant. We can thus write a rotation as a 3×3 matrix which is orthogonal

$$R^{\mathsf{T}} = R^{-1}. \tag{2.1}$$

Reflections share the above properties, and we can exclude them by the further requirement that the matrix has positive determinant

$$\det R = +1. \tag{2.2}$$

The above constraints are compatible with the group axioms, and hence the set of all such matrices forms a group

$$\mathrm{SO}(3) = \bigl\{R \in \mathrm{Aut}(\mathbb{R}^3);\, R^{\mathsf{T}} R = 1, \det R = 1\bigr\}. \tag{2.3}$$

There are several ways of parametrising rotations in three dimensions. A prominent one is Euler angles

$$R_{\phi,\theta,\psi} = R^z_\phi R^y_\theta R^z_\psi, \tag{2.4}$$

where $R^{x,y,z}_\psi$ denote rotations about the corresponding axis,

$$R^z_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R^y_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{2.5}$$

The angles ϕ and ψ are 2π-periodic and $0 \leq \theta \leq \pi$. Note that the individual rotations do not commute (in general); rotations in three dimensions form a non-abelian group. Therefore the first and last rotations cannot be combined into a single rotation about the z-axis. This feature allows to parametrise arbitrary rotations in three dimensions, even about the x-axis.

A different way of parametrising rotations in three dimensions makes use of the fact that any rotation leaves a one-dimensional subspace invariant.[1] We can specify this subspace by a unit vector $\vec{n} \in \mathrm{S}^2 \subset \mathbb{R}^3$. The rotation then acts on the orthogonal two-dimensional subspace as a two-dimensional rotation about some angle ψ. This matrix of rotation can be written as

$$R_{\vec{n},\psi} = \vec{n}\vec{n}^{\mathsf{T}} + \vec{n}^{\times} \sin\psi + (1 - \vec{n}\vec{n}^{\mathsf{T}}) \cos\psi. \tag{2.6}$$

Here, $\vec{v}^{\times}$ denotes the 3×3 anti-symmetric matrix that defines the cross product of $\vec{v}$ with an arbitrary vector $\vec{w}$ via matrix multiplication, $\vec{v}^{\times}\vec{w} := \vec{v} \times \vec{w}$,[2]

$$\vec{v}^{\times} := \begin{pmatrix} 0 & -v_z & +v_y \\ +v_z & 0 & -v_x \\ -v_y & +v_x & 0 \end{pmatrix}, \qquad (\vec{v}^{\times})_{ij} = \varepsilon_{ikj} v^k. \tag{2.7}$$

Without loss of generality we can assume that $0 \leq \psi \leq \pi$. It is straight-forward to confirm that the above form of $R_{\vec{n},\psi}$ describes a rotation that leaves the axis $\vec{n}$ invariant. In particular, R^z_ϕ and R^y_θ are easily reproduced.

The group is defined as a subgroup of $\mathrm{Aut}(\mathbb{R}^3)$: Therefore, the identity map on $\mathrm{Aut}(\mathbb{R}^3)$ restricted to $\mathrm{SO}(3)$

$$\mathrm{id} : \mathrm{SO}(3) \subset \mathrm{Aut}(\mathbb{R}^3) \to \mathrm{Aut}(\mathbb{R}^3) \tag{2.8}$$

is in fact a representation. It is called the fundamental, defining or vector representation of SO(3).[3]

Group Manifold and Topology. Let us discuss the geometric properties of the group itself. The group elements are parametrised by three continuous numbers: $(\phi, \theta, \psi) \in \mathbb{R}^3$ or $(\vec{n}, \psi) \in \mathrm{S}^2 \times \mathbb{R}$. The neighbourhood of a generic point is a patch of $\mathbb{R}^3$; we shall discuss special points further below. Furthermore, the group composition rule and inversion are apparently smooth functions of the coordinates. These are the defining properties of a Lie group.

A *Lie group* is a group whose set G is a differentiable manifold and whose composition rule and inversion are smooth maps on this manifold.

How about the special points? Let us discuss the parametrisation $R_{\vec{n},\psi}$. The parameter $\vec{n} \in \mathrm{S}^2$ has no distinguished point; S^2 is a symmetric space and treated

[1]The spectrum of $R^{-1} = R^{\mathsf{T}}$ coincides with the one of R. Consequently, the eigenvalues of orthogonal matrices come in inverse pairs $(\mathrm{e}^{+i\psi}, \mathrm{e}^{-i\psi})$ or in singlets ($+1$ or -1). As there are three eigenvalues whose overall product is 1, one of them must be 1.

[2]We assume the Einstein summation convention where for each pair of (upper and lower) equal indices within a term there is an implicit sum over the range of allowable values, e.g. $v_k w^k := \sum_{k=1}^{3} v_k w^k$.

[3]Many relevant examples of groups in physics are defined in terms of matrices. The proximity of the group and its fundamental representation may be a reason for an occasional confusion of terminology between groups, representations and, later on, algebras.

as such in the matrix $R_{\vec{n},\psi}$. The remaining parameter ψ can be restricted to the interval $0 \leq \psi \leq \pi$, both of whose end-points are special. For $\psi = 0$, the matrix $R_{\vec{n},0}$ is the identity and the dependence on $\vec{n}$ becomes trivial. This situation is analogous to polar coordinates at the coordinate origin. We could thus introduce a non-unit vector to describe a general rotation

$$\vec{\psi} := \vec{n}\psi \in \mathbb{R}^3, \qquad \|\vec{\psi}\| \leq \pi. \tag{2.9}$$

Its direction describes the axis of rotation, its magnitude the angle of rotation. The dependence of the group element on $\vec{\psi}$ becomes non-singular at the coordinate origin $\vec{\psi} = 0$, which describes the unit element.

It remains to discuss the boundary $\psi = \pi$ of the parameter space. These are rotations by half of a full turn. In this special case, it does not matter whether a rotation is towards the left or the right; the two rotations are identical

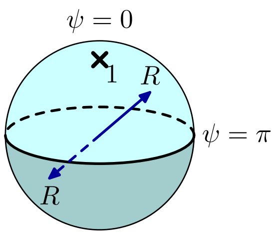
$$\vec{n} \equiv -\vec{n} \qquad \text{at} \qquad \psi = \pi. \tag{2.10}$$

More generally, there is the following identification of group elements

$$R_{\vec{n},\psi} = R_{-\vec{n},2\pi-\psi}. \tag{2.11}$$

This shows that the neighbourhood of a point at $\psi = \pi$ is not special and still a patch of $\mathbb{R}^3$. The boundary at $\psi = \pi$ simply arises by identifying rotations about an angle $\psi > \pi$ with rotations about an angle $\psi < \pi$.

Our findings are summarised in the figure:



(2.12)

It makes sense to view $(\vec{n}, \psi)$ as polar coordinates around the north-pole on a three-dimensional sphere S^3. The north-pole is the point $\psi = 0$ while $\psi = \pi$ describes points on the equator. The above relation between the rotations then simply identifies antipodes. This shows that the group manifold has the topology of three-dimensional real projective space

$$SO(3) \equiv S^3/\mathbb{Z}_2 \equiv \mathbb{RP}^3. \tag{2.13}$$

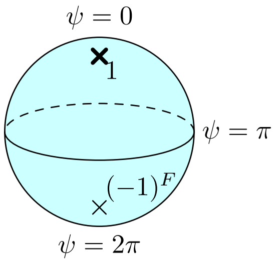
Now the $\mathbb{Z}_2$-quotient complicates the topology of the manifold somewhat. The group manifold is connected, but not simply connected; its fundamental group is $\mathbb{Z}_2$ because S^3 is simply connected. It thus makes sense to consider a bigger group which contains the original group and which is simply connected. This is the *universal cover* of a group.[4] For SO(3) it is a *double cover* known as the spin

[4] In the case of two-dimensional rotations, the fundamental group of $G^{(1)} = SO(2)$ is $\mathbb{Z}$ and its universal cover is $G^{(\infty)} = \mathbb{R}$.

group Spin(3). It is isomorphic to a unitary group or a symplectic group, and it has the topology of a three-sphere

$$\mathrm{Spin}(3) \equiv \mathrm{SU}(2) \equiv \mathrm{Sp}(1) \equiv \mathrm{S}^3. \tag{2.14}$$

The extension to Spin(3) lifts the identification between ψ and $2\pi - \psi$ and makes ψ a 4π-periodic parameter. The south pole of the group manifold becomes a rotation by 2π, i.e. it corresponds to the operator $(-1)^F$ in quantum mechanics. This is summarised in the following figure:



(2.15)

In fact, Lie theory provides a natural metric on the group manifold; here it is the canonical metric on the round S^3. The latter is a curved space, however, with a large amount of symmetry, a so-called symmetric space. More on these issues will follow later.

Parity. A physically meaningful extension of the group of rotations is to include reflections. This yields the matrix group

$$\mathrm{O}(3) = \big\{R \in \mathrm{Aut}(\mathbb{R}^3); R^{\mathsf{T}} R = 1\big\}. \tag{2.16}$$

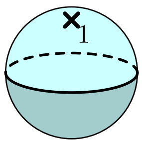
This group consists of the rotational elements $R_{\vec{n},\psi}$ and elements which combine a reflection P with a rotation $PR_{\vec{n},\psi}$. For three dimensions, a reasonable choice for the elementary reflection is the overall parity operation[5]

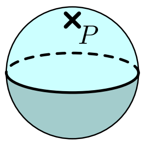
$$P = -\,\mathrm{diag}(1, 1, 1), \tag{2.17}$$

which obeys the rules $P^2 = 1$ and $PR = RP$. Thus parity decouples from the rotations and extends the rotations by a discrete group $\mathbb{Z}_2$

$$\mathrm{O}(3) = \mathrm{SO}(3) \times \mathbb{Z}_2. \tag{2.18}$$

For the group manifold we obtain two disconnected copies of the projective space

$\mathrm{O}(3) \equiv \mathbb{R}\mathrm{P}^3 + \mathbb{R}\mathrm{P}^3 =$ 

$+$ 

. (2.19)

[5]The choice $P = -\mathrm{id}$ is a reflection for any odd number of dimensions. For even dimensions, it is a (distinguished) rotation. This demonstrates that the group of rotations has some qualitative differences for even and odd dimensions.

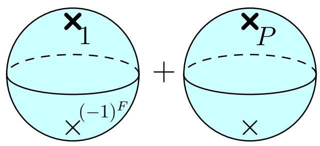
The situation becomes somewhat more interesting when we go to the double cover; let us explore for the fun of it: Here, there are essentially two distinct choices for a meaningful parity operation. One may assume parity to square to the identity $P^2 = 1$. In this case, the group is a direct product

$$\mathrm{Spin}(3) \times \mathbb{Z}_2. \tag{2.20}$$

However, we also have the almost trivial rotation by 2π at our disposal. It reduces to the identity in SO(3), but it is a non-trivial element of Spin(3). Thus we can set $P^2 = (-1)^F$ such that only the fourth power of parity is trivial $P^4 = 1$. In this case the group can be written as a semi-direct product

$$\mathbb{Z}_2 \ltimes \mathrm{Spin}(3) \tag{2.21}$$

meaning that $\mathbb{Z}_2$ interacts non-trivially with Spin(3). More concretely, parity combines with the centre $\mathbb{Z}_2$ of Spin(3) consisting of 1 and $(-1)^F$ and forms the group $\mathbb{Z}_4$. In both of the above cases, the group manifold is two copies of S^3.

$\mathrm{S}^3 + \mathrm{S}^3 =$  (2.22)

The interesting observation for physics is that there is a choice in how to define the parity operation in the presence of fermions. Note that the interactions of parity and spin will become even more complicated once Lorentz transformations are included for a spacetime with indefinite signature.

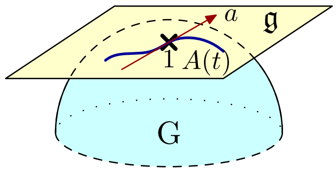
2.2 Lie Algebra

We have seen (by means of example) that a non-abelian Lie group is based on a curved manifold. Its elements and thus its representations were described by a well-chosen combination of trigonometric functions. For more elaborate groups such a direct approach becomes rather difficult. In physics we often use the approach of series expansion in order to treat problems at least approximately. For Lie groups this treatment is particularly fruitful because the linearisation of the problem covers most aspects of these groups exactly. Merely questions concerning topology require the full non-linear treatment.

A full treatment of Lie groups would require familiarity with differential geometry. Here we shall try to go without differential geometry and merely touch on some elementary concepts in passing. To this end and for conciseness, we shall restrict to matrix groups, i.e. $\mathrm{G} \subset \mathrm{Aut}(\mathbb{V})$ for some vector space $\mathbb{V}$. This generalises the case of $\mathrm{G} = \mathrm{SO}(3)$ with $\mathbb{V} = \mathbb{R}^3$ which we shall use as the main example. Later on we will discuss more general, abstract Lie groups where we can resort to our previously gained intuition in geometry in the context of algebra.

Tangent Space. Let us set up the perturbative treatment. The one point we understand very well for any Lie group is the identity element 1. It therefore makes sense to investigate its neighbourhood containing group elements which we may declare as 'small'. They are small in the sense that multiplying two such elements will again yield another (not necessarily quite as) small element. To set up perturbation theory, we consider differentiable curves on the group manifold which pass through the identity element. In particular, let $I_k \subset \mathbb{R}$ be some sufficiently small intervals containing the number 0, and let $A_k : I_k \to \mathrm{G}$ be some differentiable curves on G such that $A_k(0) = 1$. For small $t \in I_k$ the most relevant information about the curve A_k is the derivative at $t = 0$

$$a_k = A'_k(0) \in \mathfrak{g} \subset \mathrm{End}(\mathbb{V}). \tag{2.23}$$



Note that $A_k(t) \in \mathrm{Aut}(\mathbb{V}) \subset \mathrm{End}(\mathbb{V})$ is an *endomorphism* (linear map from $\mathbb{V}$ to $\mathbb{V}$; square matrix), and the space of endomorphisms $\mathrm{End}(\mathbb{V})$ is a vector space. As such, the derivative w.r.t. t is canonically defined[6] and yields another endomorphism. The space of all permissible a_k is called the *tangent space* $\mathrm{T_1G}$ of G at the unit element 1. We denote it by a lowercase $\mathfrak{gothic}$ letter $\mathfrak{g} := \mathrm{T_1G}$ corresponding to the uppercase letter G that labels the group. It naturally carries the structure of a vector space for the composite path $A_3(t) = A_1(\lambda_1 t) A_2(\lambda_2 t)$ has the derivative

$$a_3 = \lambda_1 a_1 + \lambda_2 a_2 \in \mathfrak{g}. \tag{2.24}$$

For the sample group SO(3), the corresponding tangent space $\mathfrak{so}(3)$ follows straight-forwardly from the form $R_{\vec{n},\psi}$.[7] The unit element is parametrised as $R_{\vec{n}_0,0} = 1$ with arbitrary $\vec{n}_0$. We thus define a path $\vec{n}(t) = \vec{n}_0 + \ldots$ and $\psi = 0 + t\psi_1 + \ldots$ and expand

$$R_{\vec{n},\psi} = 1 + \psi_1 \vec{n}_0^{\times} t + \ldots . \tag{2.25}$$

Here $\vec{n}_0^{\times}$ is an anti-symmetric matrix by definition, and arbitrary 3×3 anti-symmetric matrices can be written as $\psi_1 \vec{n}_0^{\times}$. This shows that $\mathfrak{so}(3)$ is the space of anti-symmetric 3×3 matrices

$$\mathfrak{so}(3) = \{r \in \mathrm{End}(\mathbb{R}^3); r^{\mathsf{T}} = -r\}. \tag{2.26}$$

Adjoint Actions. The composition rule of the group induces a corresponding algebraic structure on the vector space $\mathfrak{g}$. Before defining it, let us introduce the concept of adjoint action.

We first define the adjoint action $\mathrm{Ad}(R) : \mathfrak{g} \to \mathfrak{g}$ of a group element $R \in \mathrm{G}$ on $\mathfrak{g}$. We can combine R and a curve $A_1(t)$ to define another curve $A_2(t) = RA_1(t)R^{-1}$

[6] In any given basis, the derivative is determined element by element.

[7] The parametrisation $R_{\phi,\theta,\psi}$ using Euler angles is not immediately suitable because in first order only θ and $\phi + \psi$ contribute.

which passes through the identity element. In the neighbourhood of $t = 0$ this defines a relationship between two elements of $\mathfrak{g}$:

$$\mathrm{Ad}(R)(a_1) := R a_1 R^{-1} = a_2 \in \mathfrak{g}. \tag{2.27}$$

Since $\mathfrak{g}$ is a vector space and the adjoint action is linear in a_1, the latter in fact defines a representation $\mathrm{Ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$ of the Lie group G on $\mathfrak{g}$, called the adjoint representation.

With this we can go one step further, and consider the adjoint action of a curve $A_2(t)$ on $\mathfrak{g}$. Near $t = 0$, this defines the adjoint action $\mathrm{ad}(a) : \mathfrak{g} \to \mathfrak{g}$ of an element $a \in \mathfrak{g}$ on $\mathfrak{g}$. We set $a_3(t) := \mathrm{Ad}(A_2(t))(a_1)$ and obtain[8]

$$\mathrm{ad}(a_2)(a_1) := a_3'(0) = a_2 a_1 - a_1 a_2 \in \mathfrak{g}. \tag{2.28}$$

Clearly, the adjoint action $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a linear map.

Algebra. Altogether, the adjoint action $\mathrm{ad}(a)b \in \mathfrak{g}$ is linear in both arguments $a, b \in \mathfrak{g}$. As such, it equips the vector space $\mathfrak{g}$ with the structure of an *algebra*, the so-called *Lie algebra*. The composition rule of a Lie algebra is called the *Lie bracket* $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$,

$$[\![a, b]\!] := \mathrm{ad}(a)b = ab - ba =: [a, b]. \tag{2.29}$$

Thus for any Lie group G there is a corresponding Lie algebra $\mathfrak{g}$.[9] In the case of a Lie algebra, the composition rule is anti-symmetric[10]

$$[\![a, b]\!] = -[\![b, a]\!] \quad \text{for all } a, b \in \mathfrak{g}. \tag{2.30}$$

Moreover, it is not even associative. Instead, it satisfies the *Jacobi identity*[11]

$$\big[\!\big[a, [\![b, c]\!]\big]\!\big] + \big[\!\big[b, [\![c, a]\!]\big]\!\big] + \big[\!\big[c, [\![a, b]\!]\big]\!\big] = 0 \quad \text{for all } a, b, c \in \mathfrak{g}. \tag{2.31}$$

In our case, the anti-symmetry and Jacobi identity follow from the corresponding identities of matrices

$$[a, b] = -[b, a] \quad \text{for all } a, b \in \mathrm{End}(\mathbb{V}) \tag{2.32}$$

and

$$\big[a, [b, c]\big] + \big[b, [c, a]\big] + \big[c, [a, b]\big] = 0 \quad \text{for all } a, b, c \in \mathrm{End}(\mathbb{V}). \tag{2.33}$$

Somewhat confusingly, the Lie bracket $[\![\cdot,\cdot]\!]$ typically uses the same symbol as the matrix commutator $[\cdot,\cdot]$. One should bear in mind that a bilinear map

[8] The terms $a_2 a_1$ and $a_1 a_2$ are merely elements of $\mathrm{End}(\mathbb{V})$, but typically not of $\mathfrak{g}$; only their difference is an element of the Lie algebra $\mathfrak{g} \subset \mathrm{End}(\mathbb{V})$.

[9] We have shown this statement only for matrix Lie groups $\mathrm{G} \subset \mathrm{End}(\mathbb{V})$, but it holds in general.

[10] It measures by how much the composition rule of the Lie algebra deviates from being abelian.

[11] The Jacobi identity is a consequence of associativity of the associated Lie group.

$[\![\cdot,\cdot]\!] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ refers to Lie brackets while a bilinear map $[\cdot,\cdot] : \mathrm{End}(\mathbb{V})\times\mathrm{End}(\mathbb{V})\to\mathrm{End}(\mathbb{V})$ refers to the matrix commutator. In cases where the Lie algebra is given in terms of matrices, $\mathfrak{g}\subset\mathrm{End}(\mathbb{V})$, the Lie brackets and the matrix commutator coincide.

The major difference between a Lie algebra $\mathfrak{g}$ and some space of matrices $\mathrm{End}(\mathbb{V})$ is that the former is equipped with the Lie bracket $[\![\cdot,\cdot]\!]$ as the only algebraic operation, whereas the latter uses composition of matrices. Note that the Lie bracket is anti-symmetric and non-associative whereas composition has indefinite symmetry, but is associative. Clearly, composition of matrices is more general because the commutator of matrices can be used to define a Lie bracket on $\mathrm{End}(\mathbb{V})$.

The algebra $\mathfrak{so}(3)$. To apply the above concept to $\mathfrak{so}(3)$, it makes sense to introduce a basis J_k for this vector space. The elements of $\mathfrak{so}(3)$ are anti-symmetric matrices. However, (quantum) physicists are obsessed with hermitian matrices because their spectrum of eigenvalues is real. Therefore, we introduce a conventional factor of $\mathring{\imath}$ for the basis elements of $\mathfrak{g}$ to make the anti-symmetric matrices hermitian[12] [13]

$$\mathrm{J}_k := \mathring{\imath}\vec{e}_k^{\,\times} \in \mathring{\imath}\,\mathfrak{so}(3)\subset\mathfrak{so}(3,\mathbb{C}), \quad k=x,y,z. \tag{2.34}$$

More explicitly

$$\mathrm{J}_{x,y,z} = \begin{pmatrix}0&0&0\\0&0&-\mathring{\imath}\\0&+\mathring{\imath}&0\end{pmatrix}, \begin{pmatrix}0&0&+\mathring{\imath}\\0&0&0\\-\mathring{\imath}&0&0\end{pmatrix}, \begin{pmatrix}0&-\mathring{\imath}&0\\+\mathring{\imath}&0&0\\0&0&0\end{pmatrix}. \tag{2.35}$$

The Lie bracket on this space can now be expanded as follows

$$[\![\mathrm{J}_i,\mathrm{J}_j]\!] = [\mathring{\imath}\vec{e}_i^{\,\times},\mathring{\imath}\vec{e}_j^{\,\times}] = -(\vec{e}_i\times\vec{e}_j)^\times = -\varepsilon_{ijk}\vec{e}_k^{\,\times} = \mathring{\imath}\varepsilon_{ijk}\mathrm{J}_k. \tag{2.36}$$

Above, we have claimed that the double cover of SO(3) is the group SU(2) of unitary 2×2 matrices with unit determinant. Its Lie algebra $\mathfrak{su}(2)$ consists of anti-hermitian, traceless 2×2 matrices

$$\mathfrak{su}(2) = \{m\in\mathrm{End}(\mathbb{C}^2); m=-m^\dagger, \operatorname{tr} m = 0\}. \tag{2.37}$$

An (imaginary) basis $\mathrm{J}_k\in\mathring{\imath}\,\mathfrak{su}(2)$ for such matrices is provided by the well-known Pauli matrices

$$\mathrm{J}_k = \tfrac{1}{2}\sigma_k. \tag{2.38}$$

The Lie algebra follows by direct computation

$$[\![\mathrm{J}_i,\mathrm{J}_j]\!] = [\tfrac{1}{2}\sigma_i,\tfrac{1}{2}\sigma_j] = \tfrac{\mathring{\imath}}{2}\varepsilon_{ijk}\sigma_k = \mathring{\imath}\varepsilon_{ijk}\mathrm{J}_k. \tag{2.39}$$

[12] Mathematicians prefer to avoid cluttering many expressions by factors of $\mathring{\imath}$ and work with a real basis $\mathrm{J}_k = \vec{e}_k^{\,\times}$ instead.

[13] Note that the elements of the Lie algebra remain real, $a\in\mathfrak{so}(3)$, merely our basis is complex (purely imaginary), $\mathrm{J}_k\in\mathring{\imath}\,\mathfrak{so}(3)$. A generic element thus expands as $a=\mathring{\imath}a_k\mathrm{J}_k$. Later on we shall work with complex(ified) Lie algebras such as $\mathfrak{so}(3,\mathbb{C})$ where factors of $\mathring{\imath}$ are not an issue.

It agrees perfectly with the Lie algebra $\mathfrak{so}(3)$, thus the two are isomorphic

$$\mathfrak{so}(3) \equiv \mathfrak{su}(2). \tag{2.40}$$

Note that the isomorphism of Lie algebras is blind to the topological structure of the group manifold ($\mathrm{SO}(3) \equiv \mathrm{SU}(2)/\mathbb{Z}_2$).

Finally, we point out that the Lie algebra for the group O(3) of reflections is the same $\mathfrak{so}(3)$. The point here is that O(3) extends SO(3) by a disconnected component. The neighbourhood of the identity element is the same in both groups, and thus they share the same Lie algebra.

Exponential Map. We have defined the Lie algebra $\mathfrak{g}$ as the tangent space of a Lie group G at the identity element. Conversely, we can construct a map from the Lie algebra to the Lie group, the so-called *exponential map*

$$\exp : \mathfrak{g} \to \mathrm{G}. \tag{2.41}$$

For matrix groups $\mathrm{G} \subset \mathrm{Aut}(\mathbb{V})$ or for representations $\rho : \mathrm{G} \to \mathrm{Aut}(\mathbb{V})$ the map is given by the matrix exponential $\exp : \mathrm{End}(\mathbb{V}) \to \mathrm{Aut}(\mathbb{V})$ defined as a power series

$$\exp a = \sum_{n=0}^{\infty} \frac{1}{n!}\, a^n. \tag{2.42}$$

Like the ordinary exponential function, this series converges for all a.

Importantly, the exponential map has the group property that

$$\exp a \exp b = \exp C(a, b), \tag{2.43}$$

where $C(a, b) \in \mathfrak{g}$ is determined by the Baker–Campbell–Hausdorff formula in terms of iterated Lie brackets (or commutators where applicable)

$$C(a, b) = a + b + \tfrac{1}{2}[\![a, b]\!] + \tfrac{1}{12}\big[\!\big[a, [\![a, b]\!]\big]\!\big] + \tfrac{1}{12}\big[\!\big[b, [\![b, a]\!]\big]\!\big] + \ldots\,. \tag{2.44}$$

Furthermore, the inverse is given by

$$(\exp a)^{-1} = \exp(-a). \tag{2.45}$$

For a connected Lie group, the exponential map is typically surjective. If there is more than one connection component, however, only the component containing the identity element can be reached. As physicists generally prefer to work with the linearised Lie algebra, they frequently refer to the exponential map to denote elements of the Lie group. A prominent example in physics is the coefficient $\mathrm{e}^{\mathring{\imath}px}$ in the Fourier transformation. Here, p refers to the eigenvalue of the momentum operator P (representation of the algebra of translations) and x to the position (shift from the origin). In terms of algebra, if P is the generator of infinitesimal translations (Lie algebra), $\mathrm{e}^{\mathring{\imath}\mathrm{P}x}$ generates a finite translation (Lie group). Note that as usual the factor of $\mathring{\imath}$ is due to the choice of an imaginary basis.

2.3 Representations

We want to understand the representation theory of the rotation group SO(3). To this end, the Lie algebra $\mathfrak{so}(3)$ comes in handy because the linearisation makes the problem much easier to handle.

Lie Algebra Representations. First of all, representations of a Lie group G straight-forwardly translate to representations of the corresponding Lie algebra $\mathfrak{g}$. A *representation of a Lie algebra* is defined as a linear map

$$\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V}) \tag{2.46}$$

such that the Lie brackets are represented by the commutator of matrices

$$\rho\big([\![a,b]\!]\big) = \big[\rho(a), \rho(b)\big] \qquad \text{for all } a, b \in \mathfrak{g}. \tag{2.47}$$

For example, we already know two representations of $\mathfrak{su}(2) \equiv \mathfrak{so}(3)$ as the defining representations of these two algebras[14]

$$\begin{aligned} \rho_{\mathfrak{su}(2)}(\mathrm{J}_k) &= \tfrac{1}{2}\sigma_k \in \mathring{\imath}\,\mathfrak{su}(2) \subset \mathrm{End}(\mathbb{C}^2), \\ \rho_{\mathfrak{so}(3)}(\mathrm{J}_k) &= \mathring{\imath}\vec{e}_k^{\times} \in \mathring{\imath}\,\mathfrak{so}(3) \subset \mathring{\imath}\,\mathrm{End}(\mathbb{R}^3). \end{aligned} \tag{2.48}$$

Furthermore, for every Lie algebra there is the *trivial representation* (it exist for any vector space $\mathbb{V}$, but typically one assumes $\mathbb{V} = \mathbb{R}^1$)

$$\rho_0 : \mathfrak{g} \to \mathrm{End}(\mathbb{V}), \qquad \rho_0(\mathrm{J}_k) = 0, \tag{2.49}$$

as well as the *adjoint representation* ad

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \qquad [\![a,b]\!] = \mathrm{ad}(a)b. \tag{2.50}$$

It is straight-forward to prove that the latter is a representation via the Jacobi identity: for all $a, b, c \in \mathfrak{g}$

$$\begin{aligned} \mathrm{ad}\big([\![a,b]\!]\big)c = \big[\![[\![a,b]\!], c\big]\!] &= \big[\![a, [\![b,c]\!]\big]\!] - \big[\![b, [\![a,c]\!]\big]\!] \\ &= \mathrm{ad}(a)\,\mathrm{ad}(b)c - \mathrm{ad}(b)\,\mathrm{ad}(a)c = \big[\mathrm{ad}(a), \mathrm{ad}(b)\big]c. \end{aligned} \tag{2.51}$$

The adjoint representation is equivalent to the defining representation $\rho_{\mathfrak{so}(3)}$ of $\mathfrak{so}(3)$ noting that $\mathfrak{g} \equiv \mathbb{R}^3$ as a vector space

$$\mathrm{ad} \equiv \rho_{\mathrm{so}(3)}. \tag{2.52}$$

To show this statement, identify $\mathrm{J}_k \equiv \mathring{\imath}\vec{e}_k$ and compare

$$\begin{aligned} \mathrm{ad}(\mathrm{J}_i)\mathrm{J}_j &= [\![\mathrm{J}_i, \mathrm{J}_j]\!] = \mathring{\imath}\varepsilon_{ijk}\mathrm{J}_k, \\ \rho_{\mathfrak{so}(3)}(\mathrm{J}_i)\mathrm{J}_j &= -\vec{e}_i^{\times}\vec{e}_j = -\vec{e}_i \times \vec{e}_j = -\varepsilon_{ijk}\vec{e}_k = \mathring{\imath}\varepsilon_{ijk}\mathrm{J}_k. \end{aligned} \tag{2.53}$$

[14] The appearance of the imaginary unit $\mathring{\imath}$ is due to choice of imaginary basis.

The above representations can be characterised as complex, two-dimensional, unitary ($\rho_{\mathfrak{su}(2)}$) and real, three-dimensional, orthogonal ($\rho_{\mathfrak{so}(3)}$).

A third type of number field suitable for representations is given by the quaternions $\mathbb{H}$.[15] We are now in a good position to discuss this option. The quaternions are a generalisation of the complex numbers spanned by the real unit 1 and three imaginary units $\hat{\imath}, \hat{\jmath}, \hat{k}$. Each imaginary unit on its own behaves as the complex imaginary unit $\mathring{\imath}$. Their products are given by $\hat{\imath}\hat{\jmath} = -\hat{\jmath}\hat{\imath} = \hat{k}$ and cyclic permutations of $\hat{\imath}, \hat{\jmath}, \hat{k}$. As such they are non-commutative numbers. Quaternions are closely related to the Pauli matrices because they can be represented in terms of 2×2 complex matrices as follows

$$(1, \hat{\imath}, \hat{\jmath}, \hat{k}) \equiv (1, -\mathring{\imath}\sigma_x, -\mathring{\imath}\sigma_y, -\mathring{\imath}\sigma_z). \tag{2.54}$$

It is straight-forward to show the equivalence. This equivalence implies that the representation $\rho_{\mathfrak{su}(2)}$ can be written as the quaternionic representation $\rho_{\mathfrak{sp}(1)} : \mathfrak{g} \to \mathrm{End}(\mathbb{H}^1)$ [16]

$$\rho_{\mathfrak{sp}(1)}(-\mathring{\imath}\mathrm{J}_x) = \tfrac{1}{2}\hat{\imath}, \qquad \rho_{\mathfrak{sp}(1)}(-\mathring{\imath}\mathrm{J}_y) = \tfrac{1}{2}\hat{\jmath}, \qquad \rho_{\mathfrak{sp}(1)}(-\mathring{\imath}\mathrm{J}_z) = \tfrac{1}{2}\hat{k}. \tag{2.55}$$

Note that this representation is symplectic because

$$\rho_{\mathfrak{sp}(1)}(a)^\dagger = -\rho_{\mathfrak{sp}(1)}(a), \tag{2.56}$$

where the adjoint operation $\dagger$ flips the order and sign of the imaginary units $\hat{\imath}, \hat{\jmath}, \hat{k}$.[17] This shows the equivalence of Lie algebras

$$\mathfrak{su}(2) \equiv \mathfrak{sp}(1). \tag{2.57}$$

In terms of physics, the defining representation of $\mathfrak{su}(2)$ is in fact quaternionic, even though hardly any physicist would think of it in this way.

Construction of Irreducible Representations. We now want to construct general finite-dimensional irreps of $\mathfrak{so}(3)$ from scratch. For simplicity, the representation is assumed to be complex N-dimensional

$$\rho : \mathfrak{so}(3) \to \mathrm{End}(\mathbb{V}), \qquad \mathbb{V} = \mathbb{C}^N. \tag{2.58}$$

[15] Quaternionic representations are also called ‘pseudo-real’.

[16] The imaginary basis J_k is not suitable for the quaternionic representation because of the clash between the complex imaginary unit $\mathring{\imath}$ (which is meant to commute with everything) and the the quaternionic imaginary unit $\hat{\imath}$ (which does not commute with $\hat{\jmath}, \hat{k}$). Therefore we express the representation in terms of the real basis $-\mathring{\imath}\mathrm{J}_k$.

[17] This does not explain the term ‘symplectic’ which for representations of the Lie algebra means anti-symmetric w.r.t. an anti-symmetric metric. To that end, define the order-inverting transpose operation as $1^\mathsf{T} = 1$, $\hat{\imath}^\mathsf{T} = \hat{\imath}$, $\hat{\jmath}^\mathsf{T} = -\hat{\jmath}$, $\hat{k}^\mathsf{T} = \hat{k}$. Thus, $\hat{\jmath}$ is the only anti-symmetric number and we define the symplectic conjugate as $x \mapsto -\hat{\jmath}x^\mathsf{T}\hat{\jmath}$. This operation is the same as $x \mapsto x^\dagger$, and $-\hat{\jmath}x^\mathsf{T}\hat{\jmath} = x^\dagger = -x$ is solved by the basis $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

Let us single out the generator $\mathrm{J}_z \in \mathring{\imath}\,\mathfrak{so}(3)$ and split the space $\mathbb{V}$ into eigenspaces[18] $\mathbb{V}_m$ corresponding to the eigenvalue $m \in \mathbb{C}$ of $\rho(\mathrm{J}_z)$

$$\mathbb{V} = \bigoplus_m \mathbb{V}_m. \tag{2.59}$$

Furthermore, consider the generators $\mathrm{J}_\pm$ defined as

$$\mathrm{J}_\pm := \mathrm{J}_x \pm \mathring{\imath}\mathrm{J}_y \in \mathfrak{so}(3,\mathbb{C}). \tag{2.60}$$

These generators obey the following Lie brackets

$$\begin{aligned}
[\![\mathrm{J}_z, \mathrm{J}_\pm]\!] &= \pm \mathrm{J}_\pm, \\
[\![\mathrm{J}_+, \mathrm{J}_-]\!] &= 2\mathrm{J}_z.
\end{aligned} \tag{2.61}$$

The former relation implies that $\rho(\mathrm{J}_+)$ maps a vector of the space $\mathbb{V}_m$ to a vector of the space $\mathbb{V}_{m\pm1}$ (if the latter exists, otherwise to 0): Suppose that $|\psi\rangle \in \mathbb{V}_m$, then define $|\psi'\rangle := \rho(\mathrm{J}_\pm)|\psi\rangle$ and show that the latter belongs to $\mathbb{V}_{m\pm1}$

$$\begin{aligned}
\rho(\mathrm{J}_z)|\psi'\rangle &= \rho(\mathrm{J}_z)\rho(\mathrm{J}_\pm)|\psi\rangle \\
&= \rho(\mathrm{J}_\pm)\rho(\mathrm{J}_z)|\psi\rangle + \big[\rho(\mathrm{J}_z), \rho(\mathrm{J}_\pm)\big]|\psi\rangle \\
&= m\rho(\mathrm{J}_\pm)|\psi\rangle \pm \rho(\mathrm{J}_\pm)|\psi\rangle \\
&= (m \pm 1)|\psi'\rangle.
\end{aligned} \tag{2.62}$$

Now, the generators J_z and $\mathrm{J}_\pm$ form a basis of $\mathfrak{so}(3,\mathbb{C})$. For an irrep, the space $\mathbb{V}$ must be spanned by the vectors which are obtained by applying a sequence of $\rho(\mathrm{J}_k)$ on any particular state $|\psi\rangle \in \mathbb{V}$. This implies that the labels m of $\mathbb{V}_m$ must form an uninterrupted sequence

$$m \in \{m_-, m_- + 1, \ldots, m_+ - 1, m_+\}. \tag{2.63}$$

Let us choose some state $|\psi\rangle$ from the 'highest' eigenspace $\mathbb{V}_{m_+}$. We claim that the sequence of states

$$|\psi_k\rangle := \rho(\mathrm{J}_-)^k|\psi\rangle, \qquad k = 0, \ldots, N-1 = m^+ - m^-, \tag{2.64}$$

spans the complete space $\mathbb{V}$. It suffices to show that all the states are non-trivial and that the action of all generators $\rho(\mathrm{J}_k)$ closes on this space: By construction $\rho(\mathrm{J}_z)$ maps all states back to themselves. Furthermore, $\rho(\mathrm{J}_-)$ maps state $|\psi_k\rangle$ to state $|\psi_{k+1}\rangle$. The last state $|\psi_{N-1}\rangle$ is mapped to the zero vector $|\psi_N\rangle = 0$ because there is no space $\mathbb{V}_{m_- -1}$. Likewise, the state $|\psi_0\rangle$ is annihilated by $\rho(\mathrm{J}_+)$ because there is no space $\mathbb{V}_{m_+ +1}$. Let us act on any other state:

$$\begin{aligned}
\rho(\mathrm{J}_+)|\psi_k\rangle &= \rho(\mathrm{J}_+)\rho(\mathrm{J}_-)|\psi_{k-1}\rangle \\
&= \rho(\mathrm{J}_-)\rho(\mathrm{J}_+)|\psi_{k-1}\rangle + [\rho(\mathrm{J}_+), \rho(\mathrm{J}_-)]|\psi_{k-1}\rangle \\
&= \rho(\mathrm{J}_-)\rho(\mathrm{J}_+)|\psi_{k-1}\rangle + 2(m^+ - k + 1)|\psi_{k-1}\rangle \\
&= 2\sum_{j=1}^{k}(m^+ - k + j)|\psi_{k-1}\rangle \\
&= k\big(2m^+ - k + 1\big)|\psi_{k-1}\rangle.
\end{aligned} \tag{2.65}$$

[18] A priori, the matrix $\rho(\mathrm{J}_z)$ could be non-diagonalisable. The following discussion nevertheless works with minor adjustments. Eventually, we will see that $\rho(\mathrm{J}_z)$ is in fact diagonalisable.

The last step follows by induction terminated by the condition $\rho(\mathrm{J}_+)|\psi_0\rangle = 0$. Most importantly, the calculation shows that no additional states for any of the subspaces $\mathbb{V}_m$ are generated. As a consequence, all subspaces $\mathbb{V}_m$ for the irreducible representation are one-dimensional, and the dimension of the representation is $N = m_+ - m_- + 1$.

We can derive one further relationship: The state $|\psi_N\rangle = 0$ does not exist. By the above derivation, however, $\rho(\mathrm{J}_+)$ maps it to $N(2m^+ - N + 1)|\psi_{N-1}\rangle$. This contradiction is resolved by demanding the coefficient to vanish,

$$m^\pm = \pm\tfrac{1}{2}(N-1). \tag{2.66}$$

This concludes the construction of the irreducible representation. We see that there exists one irrep for every positive integer N.

Spin Representations. Let us summarise the findings: An N-dimensional irrep of $\mathfrak{so}(3)$ is labelled by a non-negative half-integer $j = \frac{1}{2}(N-1)$, the so-called spin

$$\rho_{\mathbb{C},j} : \mathfrak{so}(3,\mathbb{C}) \to \mathrm{End}(\mathbb{C}^{2j+1}), \qquad j \in \tfrac{1}{2}\mathbb{Z}_0^+. \tag{2.67}$$

The representation space is spanned by the vectors

$$|m\rangle, \qquad m \in \{-j, -j+1, \ldots, j-1, j\}. \tag{2.68}$$

The generators acts as follows on these states[19]

$$\begin{aligned} \rho(\mathrm{J}_z)|m\rangle &= m|m\rangle, \\ \rho(\mathrm{J}_\pm)|m\rangle &= c_m^\pm|m \pm 1\rangle. \end{aligned} \tag{2.69}$$

The algebra implies the relationships

$$c_{m-1}^+ c_m^- - c_{m+1}^- c_m^+ = 2m, \qquad c_j^+ = c_{-j}^- = 0. \tag{2.70}$$

With the combination $\gamma_m := c_{m-1}^+ c_m^-$ these are of the form $\gamma_m - \gamma_{m+1} = 2m$, $\gamma_{m+1} = \gamma_{-m} = 0$, which are solved by

$$c_{m-1}^+ c_m^- = (j+1)j - m(m-1). \tag{2.71}$$

Furthermore, we can investigate whether the representation is unitarity. Unitarity is the statement that the representation maps elements of the real algebra to anti-hermitian matrices. Our choice of complex basis leads to the following unitarity relationships

$$\rho(\mathrm{J}_z)^\dagger = \rho(\mathrm{J}_z), \qquad \rho(\mathrm{J}_+)^\dagger = \rho(\mathrm{J}_-). \tag{2.72}$$

The former is manifestly obeyed because $m \in \mathbb{R}$. The latter implies

$$(c_m^+)^* = c_{m+1}^-. \tag{2.73}$$

[19]It is straight-forward to convert these relationships into a matrix notation. Then $\rho(\mathrm{J}_z)$ is a diagonal matrix while the $\rho(\mathrm{J}_\pm)$ have non-zero elements next to the diagonal.

Together with the above relationship we have

$$|c^+_{m-1}|^2 = |c^-_m|^2 = (j+1)j - m(m-1). \tag{2.74}$$

Importantly, the left hand side is non-negative for $|m| \leq j$, i.e. unitarity can be achieved.[20] We can thus write the coefficients explicitly as the real numbers

$$c^\pm_m = \sqrt{(j+1)j - m(m\pm 1)} = \sqrt{(j\mp m)(j\pm m+1)}. \tag{2.75}$$

Reality of these coefficients, however, does not imply reality of the representation because the states and generators were chosen to be complex. In fact, one finds that representations with odd N, or equivalently integer j, are real

$$\rho_j : \mathfrak{so}(3) \to \mathrm{End}(\mathbb{R}^{2j+1}), \qquad j \in \mathbb{Z}^+_0. \tag{2.76}$$

Conversely, representations with even N, or equivalently half-integers j, are quaternionic

$$\rho_j : \mathfrak{so}(3) \to \mathrm{End}(\mathbb{H}^{j+1/2}), \qquad j \in \mathbb{Z}^+_0 + \tfrac{1}{2}. \tag{2.77}$$

We shall not show these statements, but merely refer to the examples $j = 1$ and $j = \frac{1}{2}$ discussed above.

We have already discussed that the algebras $\mathfrak{so}(3)$, $\mathfrak{su}(2)$ and $\mathfrak{sp}(1)$ are isomorphic, and thus their representation theory is identical. However, the corresponding Lie groups SO(3) and SU(2) ≡ Sp(1) are not isomorphic because the latter are double covers of the former. The representation theory for the larger group SU(2) consists of all of the above representations. The smaller group SO(3), however, can accommodate only a subset of representations. It identifies the unit element with the antipode $(-1)^F$ on the three-sphere SU(2). Thus only such representations of SU(2) where the antipode $(-1)^F$ is mapped to the identity matrix can be lifted to representations of SO(3). These are precisely the representations with integer j and odd dimension.

Spherical Harmonics. One of the main applications of the general finite-dimensional representation of $\mathfrak{so}(3)$ is *spherical harmonics*. The spherical harmonics $Y_{\ell,m}$ with $\ell \in \mathbb{Z}^+_0$ and $m \in \{-\ell, \ldots, +\ell\}$ provide an orthogonal basis of (square integrable) functions on the two-sphere S^2

$$F(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell,m} Y_{\ell,m}(\theta,\phi). \tag{2.78}$$

This decomposition is the analog of the Fourier series for periodic functions, i.e. functions on the circle S^1. We have already related the Fourier series to representation theory of the group SO(2). Here the relevant group is SO(3) which

[20] Note that the sequence of states $|m\rangle, |m+1\rangle, \ldots$ breaks precisely at the point where otherwise unitarity would be violated. This coincidence plays an important role in the representation theory of more general algebras.

serves as the group of isometries of the round two-sphere S^2. Note that S^2 is the orbit of a point in $\mathbb{R}^3$ under the action of SO(3).

Spherical harmonics have been introduced in the context of electrodynamics (multipole expansion) and quantum mechanics (orbital angular momentum) and have been discussed extensively there. Here we note that the spherical harmonics $Y_{\ell,m}$ provide a basis for functions which transform in the representation ρ_ℓ of $\mathfrak{so}(3)$ or SO(3).[21] The label m corresponds to the state $|m\rangle$. In other words, performing a rotation on the function $F(\theta,\phi)$ is equivalent to transforming the coefficients $f_{\ell,m}$ by means of the representation ρ_ℓ. Alternatively, one can also say that a function on S^2 transforms in the representation

$$\rho_{S^2} = \bigoplus_{\ell=0}^{\infty} \rho_\ell. \tag{2.79}$$

An interesting question is how to extend spherical harmonics to higher dimensions. For the next higher-dimensional case $S^3 \subset \mathbb{R}^4$ the relevant group is SO(4) whose double cover is $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}$.[22] The expansion of a function F on S^3 into spherical harmonics can be expressed as

$$F(\theta,\phi,\psi) = \sum_{\ell=0}^{\infty} \sum_{m_\mathrm{R}=-\ell}^{\ell} \sum_{m_\mathrm{L}=-\ell}^{\ell} f_{\ell,m_\mathrm{L},m_\mathrm{R}} Y_{\ell,m_\mathrm{L},m_\mathrm{R}}(\theta,\phi,\psi). \tag{2.80}$$

The coefficients $f_{\ell,m_\mathrm{L},m_\mathrm{R}}$ for fixed ℓ then naturally transform in irreps of both SU(2) group factors with common overall spin $j = \ell/2$

$$\rho_{S^3} = \bigoplus_{\ell=0}^{\infty} \rho^\mathrm{L}_{\ell/2} \otimes \rho^\mathrm{R}_{\ell/2}. \tag{2.81}$$

Casimir Operator. A useful way to classify representations is based on the Casimir invariants, most importantly, the quadratic invariant. The latter is an element of the tensor product of two copies of the Lie algebra

$$C \in \mathfrak{g} \otimes \mathfrak{g}. \tag{2.82}$$

It has the special property of being invariant

$$[\![a, C]\!] = 0, \qquad \text{for all } a \in \mathfrak{g}, \tag{2.83}$$

where the action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ is defined as $\mathrm{ad} \otimes \mathrm{ad}$

$$[\![a, b \otimes c]\!] := [\![a, b]\!] \otimes c + b \otimes [\![a, c]\!]. \tag{2.84}$$

[21]Note that only representations ρ_j with integer j appear in the decomposition. The half-integer rotations possess non-trivial rotations by an angle of 2π which evidently cannot be represented on the space $\mathbb{R}^3$ containing the two-sphere S^2.

[22]The indices L/R are used to distinguish the two isomorphic components SU(2)'s. They also indicate the chirality of the corresponding spin.

This definition has the convenient feature that a representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ can be lifted to a representation of the tensor product algebra $\rho : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ with[23]

$$\rho(a \otimes b) := \rho(a)\rho(b) \tag{2.85}$$

such that

$$\begin{aligned}
\rho\bigl(\llbracket a, b \otimes c\rrbracket\bigr) &= \rho\bigl(\llbracket a, b\rrbracket \otimes c + b \otimes \llbracket a, c\rrbracket\bigr) \\
&= \rho\bigl(\llbracket a, b\rrbracket\bigr)\rho(c) + \rho(b)\rho\bigl(\llbracket a, c\rrbracket\bigr) \\
&= \bigl[\rho(a), \rho(b)\bigr]\rho(c) + \rho(b)\bigl[\rho(a), \rho(c)\bigr] \\
&= \bigl[\rho(a), \rho(b)\rho(c)\bigr] \\
&= \bigl[\rho(a), \rho(b \otimes c)\bigr].
\end{aligned} \tag{2.86}$$

Without further ado, the Casimir invariant for $\mathfrak{so}(3)$ reads

$$C = \mathrm{J}_k \otimes \mathrm{J}_k = \mathrm{J}_z \otimes \mathrm{J}_z + \tfrac{1}{2}\mathrm{J}_+ \otimes \mathrm{J}_- + \tfrac{1}{2}\mathrm{J}_- \otimes \mathrm{J}_+. \tag{2.87}$$

It obeys the invariance property

$$\begin{aligned}
[\mathrm{J}_j, C] &= \llbracket \mathrm{J}_j, \mathrm{J}_k\rrbracket \otimes \mathrm{J}_k + \mathrm{J}_\mathrm{k} \otimes \llbracket \mathrm{J}_j, \mathrm{J}_k\rrbracket \\
&= \mathring{\imath}\varepsilon_{jkm}\mathrm{J}_m \otimes \mathrm{J}_k + \mathring{\imath}\varepsilon_{jkm}\mathrm{J}_k \otimes \mathrm{J}_m = 0.
\end{aligned} \tag{2.88}$$

Its representation thus takes the form

$$\rho(C) = \rho(\mathrm{J}_k)\rho(\mathrm{J}_k). \tag{2.89}$$

By construction it commutes with the representation of all generators

$$\bigl[\rho(\mathrm{J}_k), \rho(C)\bigr] = 0. \tag{2.90}$$

This is a useful property because $\rho(C)$ must act as a number on any irrep, e.g.

$$\rho_j(C) = j(j+1). \tag{2.91}$$

To show this result, we act on the state $|j\rangle$ with highest eigenvalue of $\rho(\mathrm{J}_z)$. Before applying $\rho(C)$ blindly, we adjust the representation slightly for the purposes of our state which obeys $\rho(\mathrm{J}_+)|j\rangle = 0$

$$\begin{aligned}
\rho(C) &= \rho(\mathrm{J}_z)\rho(\mathrm{J}_z) + \tfrac{1}{2}\rho(\mathrm{J}_+)\rho(\mathrm{J}_-) + \tfrac{1}{2}\rho(\mathrm{J}_-)\rho(\mathrm{J}_+) \\
&= \rho(\mathrm{J}_z)\rho(\mathrm{J}_z) + \rho(\mathrm{J}_z) + \rho(\mathrm{J}_-)\rho(\mathrm{J}_+).
\end{aligned} \tag{2.92}$$

The action on the state thus yields

$$\rho(C)|j\rangle = (j^2 + j)|j\rangle = j(j+1)|j\rangle. \tag{2.93}$$

By construction then $\rho(C)|m\rangle = j(j+1)|m\rangle$ irrespectively of the value of m.

[23] Elements of the Lie algebra cannot be multiplied, but their representations can. The concept of universal enveloping algebra will later allow to multiply Lie elements directly.

Tensor Product Decomposition. An important element of representation theory is the decomposition of tensor products. The tensor product $\rho_\otimes : \mathfrak{g} \to \mathrm{End}(\mathbb{V}_\otimes)$ of Lie algebra representations $\rho_k : \mathfrak{g} \to \mathrm{End}(\mathbb{V}_k)$, $k = 1, \ldots, N$, is defined as a representation

$$\rho_\otimes := \sum_{k=1}^{N} 1 \otimes \ldots \otimes 1 \otimes \rho_k \otimes 1 \otimes \ldots \otimes 1. \tag{2.94}$$

This definition is compatible with the Lie algebra as the tangent space of a Lie group and the exponential map. The direct sum $\rho_\oplus : \mathfrak{g} \to \mathrm{End}(\mathbb{V}_\oplus)$ of representations is, however, defined analogously to the Lie group

$$\rho_\oplus := \rho_1 \oplus \ldots \oplus \rho_N. \tag{2.95}$$

The central question which we want to answer is, how does the tensor product of two representations with spin j and j' decompose? The tensor product has dimension $(2j+1)(2j'+1)$. However, the maximum eigenvalue of $\rho_\otimes(\mathrm{J}_z)$ is $j+j'$ as can be seen from the above formula. This means that the tensor product cannot be irreducible (unless $j = 0$ or $j' = 0$), but it must contain a representation of spin $j+j'$ among others.

In order to determine the decomposition, we can use a shortcut of group theory, namely *character polynomials*. We first define a group element $g(q)$ depending on a formal variable q as follows[24] [25]

$$g(q) = \exp\bigl[2\log(q)\mathrm{J}_z\bigr] = q^{2\mathrm{J}_z}. \tag{2.96}$$

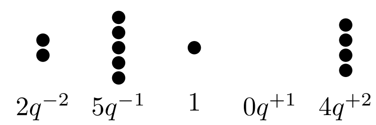
We define the *character* of this group element in a certain representation ρ as

$$P_\rho(q) = \operatorname{tr} \rho\bigl(g(q)\bigr). \tag{2.97}$$

The character is most conveniently determined in a basis where $\rho(\mathrm{J}_z)$ is a diagonal matrix. Then by construction, $P_\rho(q)$ evaluates to

$$P_\rho(q) = \sum_k n_k q^{2m_k}. \tag{2.98}$$

Here, the numbers m_k are the eigenvalues of $\rho(\mathrm{J}_z)$ and the integers n_k are their corresponding multiplicities. For a finite representation, $P_\rho(q)$ is a Laurent polynomial and it is a convenient tool to summarise the quantum numbers carried by the representation. For example, a hypothetical representation with 2 states at $m = -1$, 5 states at $m = -\frac{1}{2}$, 1 state at $m = 0$ and 4 states at $m = +1$ corresponds to the polynomial $2q^{-2} + 5q^{-1} + 1 + 4q^2$. This situation can also be visualised by a collection of dots as follows:



(2.99)

[24] If reality conditions for group elements are to be respected, the variable q should be on the unit circle in the complex plane so that $\log q$ is imaginary.

[25] The conventional factor of 2 ensures that the exponent of q is integer-valued whenever J_z is a half-integer as is the case for finite-dimensional representations of $\mathfrak{so}(3)$.

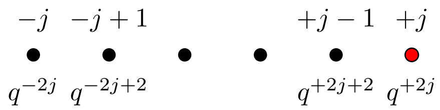
Quite clearly, setting $q = 1$ determines the dimension of some representation as

$$P_\rho(1) = \dim \rho. \tag{2.100}$$

For the representation of spin j we have[26]

$$P_j(q) = \sum_{k=0}^{2j} q^{2k-2j} = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}} \tag{2.101}$$

or in terms of dots:



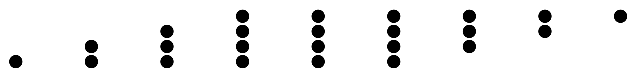
(2.102)

Now the character polynomial is useful because tensor products and direct sums of characters are easily computed as the products and sums of characters

$$\begin{aligned} P_{\rho\otimes\rho'}(q) &= P_\rho(q) P_{\rho'}(q), \\ P_{\rho\oplus\rho'}(q) &= P_\rho(q) + P_{\rho'}(q). \end{aligned} \tag{2.103}$$

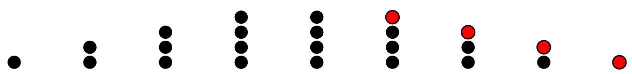
Thus the tensor product of two representations with spin j and j' yields

$$P_{j\otimes j'}(q) = P_j(q) P_{j'}(q) = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}} \, \frac{q^{2j'+1} - q^{-2j'-1}}{q - q^{-1}} \,. \tag{2.104}$$


(2.105)

We should write this as a sum of polynomials $P_j(q)$. A standard algorithm to determine the decomposition is analogous to long division:[27] Find the highest power of q and subtract as many of the corresponding irreps with the corresponding spin. The residual polynomial has lower degree and the above step can be iterated until there is nothing left.[28] We find an identity which is easy to confirm[29]

$$P_{j\otimes j'}(q) = \sum_{k=0}^{2j'} P_{j+j'-k}(q). \tag{2.106}$$


(2.107)

It translates to the tensor product decomposition of two generic finite-dimensional representations of $\mathfrak{so}(3)$

$$\rho_j \otimes \rho_{j'} = \bigoplus_{k=0}^{2\min(j,j')} \rho_{j+j'-k}. \tag{2.108}$$

[26]In some sense, the denominator determines the spacing of the states whereas the numerator determines the highest and lowest states.

[27]An alternative algorithm for finite-dimensional representations is to identify monomials with negative powers of m as $q^m \mapsto -q^{-2-m}$ (and $q^{-1} \mapsto 0$). The resulting polynomial then counts complete representations rather than states.

[28]A shortcut is to write out the polynomial $(q - q^{-1}) P_{j\otimes j'}(q)$. The non-negative powers of q correspond to irreps in the tensor product.

[29]We should assume that $j' \leq j$. However, the formula holds even if $j' > j$ in which case we need to use $P_{-k} = -P_{k-1}$ which cancels some terms.

The above technique has many further applications and generalisations. For instance, we can apply it to determine (anti)-symmetric tensor products. In quantum mechanics, these describe the wave function of identical bosonic (or fermionic) particles. They are obtained by projecting the tensor product of two alike representations onto the (anti)-symmetric part

$$\rho_\pm := (\rho \otimes \rho)\pi_\pm, \tag{2.109}$$

where $\pi_\pm$ are defined in terms of the permutation operator σ as the (anti)-symmetric projectors

$$\pi_\pm = \tfrac{1}{2}(1 \pm \sigma). \tag{2.110}$$

One can convince oneself that these are projectors and that $\rho_\pm$ are indeed proper representations. Now the character polynomial for the (anti)-symmetric tensor product can be computed

$$\begin{aligned} P_\pm(q) &= (\mathrm{tr} \otimes \mathrm{tr})\bigl[\rho(g(q)) \otimes \rho(g(q))\tfrac{1}{2}(1 \pm \sigma)\bigr] \\ &= \tfrac{1}{2}\bigl[\mathrm{tr}\, g(q)\bigr]^2 \pm \tfrac{1}{2}\,\mathrm{tr}\bigl[g(q)^2\bigr] \\ &= \tfrac{1}{2}P_\rho(q)^2 \pm \tfrac{1}{2}P_\rho(q^2). \end{aligned} \tag{2.111}$$

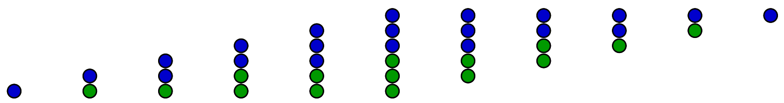
Note that the result is conveniently expressed in terms of the character polynomial for the tensor factor.

The tensor product of two spin-j representations is described by the function

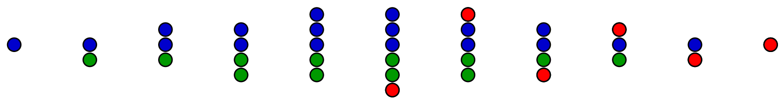
$$P_\pm(q) = \frac{(q^{2j+1} - q^{-2j-1})^2}{2(q - q^{-1})^2} \pm \frac{q^{4j+2} - q^{-4j-2}}{2(q^2 - q^{-2})}\,. \tag{2.112}$$

The decomposition yields for the symmetric and anti-symmetric parts

$$\rho_j^{\vee 2} = \bigoplus_{n=0}^{\lfloor j \rfloor} \rho_{2j-2n}, \qquad \rho_j^{\wedge 2} = \bigoplus_{n=0}^{\lfloor j-1/2 \rfloor} \rho_{2j-2n-1}. \tag{2.113}$$



(2.114)



(2.115)

Finally, let us remark that the totally symmetric product of k spin-$\frac{1}{2}$ representations is the irrep of spin $k/2$. Thus, all the representation theory of $\mathfrak{so}(3)$ follows from the smallest non-trivial representation and its tensor products.

3 Finite Group Theory

We switch gears and consider groups with finitely many elements for a while. In physics they are relevant when there are some preferred points or axes in space, for example in a solid with a crystal lattice structure. Finite groups are also an exciting topic on their own.

Many of the concepts we have discussed carry over to finite groups.[1] On the one hand, we can derive some stronger results based on finiteness. On the other hand, the discrete nature of these groups complicates some issues because we have no linearisation at our disposal as in the case of continuous groups.

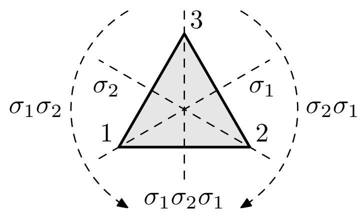
3.1 Finite Group Basics

Before we start, let us introduce a few sample groups, representations and some further basic notions of group theory.

Sample Groups and Representations. In order to get acquainted with finite groups and their representations, let us present a few elementary examples:

- The *trivial group* consists of the unit element alone.
- The *cyclic group* $\mathrm{C}_n = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, $n > 1$, consists of the integers modulo n under addition. This group is abelian.
- The *symmetric group* S_n, $n > 1$, consists of all $n!$ permutations of a set of n elements.
- The *alternating group* A_n, $n > 1$, consists of all $n!/2$ even permutations of a set of n elements.

Throughout this chapter we shall use the symmetric group S_3 as the main example. It has $|\mathrm{S}_3| = 3! = 6$ elements which can be written in terms of two elementary permutations σ_1, σ_2 obeying the relations $(\sigma_1)^2 = (\sigma_2)^2 = (\sigma_1\sigma_2)^3 = 1$. It can be viewed as the symmetry group of an equilateral triangle consisting of the identity, 2 rotations $\sigma_1\sigma_2, \sigma_2\sigma_1$ and 3 reflections $\sigma_1, \sigma_2, \sigma_1\sigma_2\sigma_1$



(3.1)

[1]We shall only consider finite-dimensional representations in this chapter.

A few elementary representations are listed in the following table:

g	ρ_1	$\rho_{1'}$	ρ_2	ρ_3
1	1	$+1$	$\begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$	$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$
$\sigma_1\sigma_2$	1	$+1$	$\begin{pmatrix} 0 & - \\ + & - \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$
$\sigma_2\sigma_1$	1	$+1$	$\begin{pmatrix} - & + \\ - & 0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$
σ_1	1	-1	$\begin{pmatrix} 0 & + \\ + & 0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}$
σ_2	1	-1	$\begin{pmatrix} + & - \\ 0 & - \end{pmatrix}$	$\begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$
$\sigma_1\sigma_2\sigma_1$	1	-1	$\begin{pmatrix} - & 0 \\ - & + \end{pmatrix}$	$\begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$

(3.2)

Of these, ρ_1, $\rho_{1'}$ and ρ_2 are irreducible while $\rho_3 \equiv \rho_1 \oplus \rho_2$.

Orbits and Cosets. In the following we will introduce four relevant notions of group theory in connection to geometry, i.e. when a group G acts on a set M by a group action $\alpha : \mathrm{G} \times M \to M$. These are orbits, stabilisers, group cosets and quotients, and we shall see how they are related.

The *orbit* M_m of a point $m \in M$ is the image of m under the group action of all group elements, i.e. the set

$$M_m := \alpha(\mathrm{G}, m) = \{\alpha(g, m); g \in \mathrm{G}\}. \tag{3.3}$$

By the group properties we have that $m \in M_m$ and $M_n = M_m$ if $n \in M_m$. This is sufficient to ensure that the sets M_m partition the set M into disjoint subsets and therefore define an equivalence relation on M.

The *stabiliser* G_m (sometimes *isotropy group* or in physics *little group*) of a point $m \in M$ is the subgroup of G which leaves the point m fixed

$$\mathrm{G}_m := \{g \in \mathrm{G}; \alpha(g, m) = m\}. \tag{3.4}$$

Similarly, one can define the stabiliser of a subset of points $X \subset M$ as $\mathrm{G}_X := \{g \in \mathrm{G}; \alpha(g, X) = X\}$. Then for example, the stabiliser of an orbit is the full group by construction.

Finally, a *left coset* $g\mathrm{H}$ of an element $g \in \mathrm{G}$ and a subgroup $\mathrm{H} \subset \mathrm{G}$ is the set[2]

$$g\mathrm{H} := \{gh; h \in \mathrm{H}\}. \tag{3.5}$$

Due to the group properties of H, the cosets of a fixed subgroup H partition the group G into disjoint subsets. Consequently, they define an equivalence relation in G, and the *quotient set* G/H is the set of cosets or equivalence classes.[3]

[2]Right cosets $\mathrm{H}g$ are defined analogously, and they possess similar properties.

[3]Consistent group multiplication does not necessarily extend from G to the quotient set G/H. The latter has a group structure only if H is a so-called normal subgroup.

In the following, we shall show that the group action is a bijection between the quotient G/G_m by the stabiliser G_m of some point $m \in M$ and the orbit M_m of this point,

$$\mathrm{G}/\mathrm{G}_m \equiv M_m. \tag{3.6}$$

First, we realise that all the elements gh of some left coset $g\mathrm{G}_m$ of the stabiliser G_m map m to a common point $n \in M_m$ in the same orbit:

$$\alpha(gh, m) = \alpha\big(g, \alpha(h, m)\big) = \alpha(g, m) = n \in M_m. \tag{3.7}$$

In other words, the group action is a map $\mathrm{G}/\mathrm{G}_m \to M_m$, and it is surjective by construction. Conversely, the image $n \in M_m$ specifies the coset uniquely: Assuming that $\alpha(g, m) = \alpha(g', m) = n$ we infer

$$m = \alpha(g^{-1}, n) = \alpha\big(g^{-1}, \alpha(g', m)\big) = \alpha(g^{-1}g', m), \tag{3.8}$$

in other words $g^{-1}g'$ is in the stabiliser G_m or g' is in the coset $g\mathrm{G}_m$. This implies that the map $g\mathrm{G}_m \mapsto \alpha(g, m)$ bijective (*orbit-stabiliser theorem*). As all the cosets $g\mathrm{G}_m$ have the cardinality of the stabiliser G_m, we deduce that

$$|M_m| \cdot |\mathrm{G}_m| = |\mathrm{G}|. \tag{3.9}$$

In particular, both $|M_m|$ and $|\mathrm{G}_m|$ must be divisors of the order of the group $|\mathrm{G}|$.

For example, we can consider the action of the group S_3 on the set M of vertices of an equilateral triangle. Then the orbit of any point $m \in M$ is $M_m = M$. The stabiliser is a $\mathbb{Z}_2$-subgroup of G consisting of the identity element and a reflection passing through the point m. We confirm that $|M_m||\mathrm{G}_m| = 2 \cdot 3 = 6 = |\mathrm{G}|$.

Characters and Conjugacy Classes. We have already encountered the character of a group element in a representation. It turned out to be a useful concept towards decomposing a representation into its irreducible components. For finite groups, characters of representations have an even more pronounced role.

The *character* $\chi : \mathrm{G} \to \mathbb{C}$ of a representation $\rho : \mathrm{G} \to \mathrm{Aut}(\mathbb{V})$ on a complex vector space $\mathbb{V}$ is defined as the trace of the representation

$$\chi(g) := \operatorname{tr} \rho(g) \qquad \text{for all } g \in \mathrm{G}. \tag{3.10}$$

The character reduces the information contained in a representation to an essential minimum. First, the character of the identity element describes the dimension of the representation

$$\chi(1) = \operatorname{tr} \mathrm{id} = \dim \mathbb{V} = \dim \rho. \tag{3.11}$$

Second, the characters of two equivalent representations $\rho_2 = R\rho_1 R^{-1}$ are identical maps

$$\chi_2(g) = \operatorname{tr} R\rho_1(g)R^{-1} = \operatorname{tr} \rho_1(g) = \chi_1(g) \qquad \text{for all } g \in \mathrm{G}. \tag{3.12}$$

Third, characters for direct sums and tensor products are computed easily as the sums and products of the characters

$$\chi_\oplus(g) = \sum\nolimits_k \chi_k(g), \qquad \chi_\otimes(g) = \prod\nolimits_k \chi_k(g). \tag{3.13}$$

These properties together with an orthogonality relation allow to describe the decomposition of a representation into irreps in terms of its character as we shall see later.

Let us calculate the characters of the elements of S_3 in various representations:

$$\begin{array}{c||ccc|c} g & \chi_1 & \chi_{1'} & \chi_2 & \chi_3 \\ \hline\hline 1 & 1 & 1 & 2 & 3 \\ \hline \sigma_1\sigma_2 & 1 & 1 & -1 & 0 \\ \sigma_2\sigma_1 & 1 & 1 & -1 & 0 \\ \hline \sigma_1 & 1 & -1 & 0 & 1 \\ \sigma_2 & 1 & -1 & 0 & 1 \\ \sigma_1\sigma_2\sigma_1 & 1 & -1 & 0 & 1 \end{array} \qquad (3.14)$$

Clearly, $\chi_3 = \chi_1 + \chi_2$ because $\rho_3 \equiv \rho_1 \oplus \rho_2$. One can also confirm the relationship $\chi_1 \cdot \chi_2 = \chi_{1'} \cdot \chi_2 = \chi_2$ which at the level of representations means that the two tensor products are equivalent (but not equal). Finally, $\chi_2 \cdot \chi_2 = \chi_1 + \chi_{1'} + \chi_2$ which also holds at the level of representations. The tools that we will develop in the following allow us in general to promote such relationships among the characters to relationship among representations.

From the table we can observe that the characters of several elements are the same. In fact, the characters of g and bgb^{-1} coincide for any $g, b \in \mathrm{G}$ in any group G

$$\chi(bgb^{-1}) = \operatorname{tr} \rho(b)\rho(g)\rho(b)^{-1} = \operatorname{tr} \rho(g) = \chi(g). \qquad (3.15)$$

Therefore it makes sense to collect all elements bgb^{-1} for a given $g \in \mathrm{G}$ into a so-called *conjugacy class* $[g]$. The reduced table for the characters of irreps vs. the conjugacy classes is called the *character table*:

$$\begin{array}{c|ccc} [g] & \chi_1 & \chi_{1'} & \chi_2 \\ \hline [1] & 1 & 1 & 2 \\ [\sigma_1\sigma_2] & 1 & 1 & -1 \\ [\sigma_1] & 1 & -1 & 0 \end{array} \qquad (3.16)$$

Conjugacy classes can be viewed as the orbits of the action of G on itself by conjugation $(b, g) \mapsto bgb^{-1}$. This implies that the order of each conjugacy class must be a divisor of the order of the group. Furthermore, there is an associated group to each conjugacy class acting as the stabiliser of some representative. In our example S_3 we find

$$\begin{array}{lll} \text{identity}, & \text{rotations}, & \text{reflections}, \\ |[1]| = 1, & |[\sigma_1\sigma_2]| = 2, & |[\sigma_1]| = 3, \\ \mathrm{G}_{[1]} = \mathrm{G}, & \mathrm{G}_{[\sigma_1\sigma_2]} = \mathbb{Z}_3, & \mathrm{G}_{[\sigma_1]} = \mathbb{Z}_2. \end{array} \qquad (3.17)$$

3.2 Complete Reducibility

We have seen that any representation can be reduced to its irreducible components, and the irreps can be viewed as elementary building blocks. However, there also exist representations which are indecomposable in the sense that they cannot be written as a direct sum of their components. A convenient feature of finite groups is that all representations are *completely reducible*; any representation is a direct sum of irreps. Therefore, the classification of representations reduces to the classification of irreps.

The key ingredients for complete reducibility are unitary representations and averaging over the group.

Unitary Representations. Recall that a unitary representation $\rho : \mathrm{G} \to \mathrm{Aut}(\mathbb{V})$ to a complex vector space $\mathbb{V}$ [4] is a representation which obeys

$$\langle v, w\rangle = \big\langle \rho(g)v, \rho(g)w\big\rangle \qquad \text{for all } g \in \mathrm{G},\ v, w \in \mathbb{V} \tag{3.18}$$

for some positive-definite hermitian form $\langle\cdot,\cdot\rangle$ on $\mathbb{V}$, i.e.

$$\langle v, w\rangle = \langle w, v\rangle^*, \qquad \langle v, v\rangle \geq 0 \qquad \text{with} \qquad \langle v, v\rangle = 0 \iff v = 0. \tag{3.19}$$

We want to show that every unitary representation is either completely decomposable or irreducible.

Suppose there is some non-trivial invariant subspace $\mathbb{W}$ of $\mathbb{V}$, i.e.

$$\rho(g)w \in \mathbb{W} \qquad \text{for all } g \in \mathrm{G},\ w \in \mathbb{W}. \tag{3.20}$$

Define the hermitian complement $\mathbb{W}^\perp$ of $\mathbb{W}$ as

$$\mathbb{W}^\perp := \{v \in \mathbb{V}; \langle v, w\rangle = 0 \text{ for all } w \in \mathbb{W}\}. \tag{3.21}$$

We want to show that $\mathbb{W}^\perp$ is an invariant subspace as well, that is we need to show that $\langle \rho(g)v, w\rangle = 0$ for any $w \in \mathbb{W}$, $v \in \mathbb{W}^\perp$ and $g \in \mathrm{G}$ so that $\rho(g)\mathbb{W}^\perp \subset \mathbb{W}^\perp$. Indeed by unitarity of ρ we have that

$$\big\langle \rho(g)v, w\big\rangle = \big\langle v, \rho(g)^{-1}w\big\rangle = 0 \tag{3.22}$$

due to the assumption that $\mathbb{W}$ is an invariant subspace. This shows that also $\mathbb{W}^\perp$ is an invariant subspace of $\mathbb{V}$.

Finally, we show that the full space $\mathbb{V}$ is the direct sum

$$\mathbb{V} = \mathbb{W} \oplus \mathbb{W}^\perp. \tag{3.23}$$

This is because for finite-dimensional spaces, we can easily deduce the dimensionality from the definition of $\mathbb{W}^\perp$ as $\dim \mathbb{W}^\perp = \dim \mathbb{V} - \dim \mathbb{W}$. It remains to show that the intersection $\mathbb{W} \cap \mathbb{W}^\perp$ is trivial: Consider a $v \in \mathbb{W}^\perp$ which also satisfies $v \in \mathbb{W}$. The former implies that $\langle v, w\rangle = 0$ for all $w \in \mathbb{W}$, and in particular, $\langle v, v\rangle = 0$. By positive definiteness of the hermitian form, we deduce that $v = 0$ so the intersection $\mathbb{W} \cap \mathbb{W}^\perp$ is trivial. This shows that the representation decomposes to the sub-representations on $\mathbb{W}$ and $\mathbb{W}^\perp$

$$\rho = \rho|_{\mathbb{W}} \oplus \rho|_{\mathbb{W}^\perp}. \tag{3.24}$$

[4]The following discussion equally applies to real representations which are orthogonal w.r.t. a positive-definite symmetric bilinear form.

Averaging over the Group. The above argument applies to unitary representations of generic groups. The important feature of finite groups is that all representations are unitary w.r.t. some hermitian form. To see this, we start with an arbitrary positive-definite hermitian form $\langle\cdot,\cdot\rangle_0$ on $\mathbb{V}$. Now define another hermitian form as follows

$$\langle v, w\rangle := \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}}\big\langle\rho(g)v, \rho(g)w\big\rangle_0, \tag{3.25}$$

This above sum divided by the order $1/|\mathrm{G}|$ represents an average over all group elements. Finiteness of the group implies that the sum is well-defined. Hermiticity of $\langle\cdot,\cdot\rangle$ follows by linearity and positivity from the fact that for $v \neq 0$ all summands in $\langle v, v\rangle$ are strictly positive.

The representation ρ is unitary w.r.t. the averaged hermitian form

$$\begin{aligned}\big\langle\rho(g)v, \rho(g)w\big\rangle &= \frac{1}{|\mathrm{G}|}\sum_{b\in\mathrm{G}}\big\langle\rho(b)\rho(g)v, \rho(b)\rho(g)w\big\rangle_0\\ &= \frac{1}{|\mathrm{G}|}\sum_{bg^{-1}\in\mathrm{G}}\big\langle\rho(b)v, \rho(b)w\big\rangle_0\\ &= \langle v, w\rangle.\end{aligned} \tag{3.26}$$

Here we made use of the fact that $\mathrm{G}g = \mathrm{G}$ for any $g \in \mathrm{G}$ to relabel the summation variable. This proves that every representation of a finite group is unitary with respect to some positive-definite hermitian form and thus completely decomposable according to our previous result.

In our main example S_3, all representations but ρ_2 are already unitary w.r.t. the canonical hermitian form $\langle v, w\rangle_0 = v^\dagger w$. For ρ_2 we construct a suitable hermitian form as described above

$$\langle v, w\rangle := v^\dagger H w, \qquad H := \frac{1}{6}\sum_{g\in\mathrm{G}}\rho_2(g)^\dagger\rho_2(g) = \frac{2}{3}\begin{pmatrix}+2 & -1\\ -1 & +2\end{pmatrix}. \tag{3.27}$$

The representation is unitary w.r.t. the form $\langle\cdot,\cdot\rangle$. By conjugating the representation with $\sqrt{H}$, we can also make it unitary (orthogonal) w.r.t. the canonical hermitian (symmetric) form. We find[5]

$$\rho_{2'}(\sigma_1) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \rho_{2'}(\sigma_2) = \frac{1}{2}\begin{pmatrix}+\sqrt{3} & -1\\ -1 & -\sqrt{3}\end{pmatrix}. \tag{3.28}$$

3.3 Orthogonality Relations

A central tool for representation theory is Schur's lemma which tightly constrains the form of module homomorphisms or invariant linear maps. A *module*

[5]These are elements of O(2) which map an equilateral triangle centred at the origin with one vertex located at 45° to itself.

homomorphism is a linear map $T : \mathbb{V}_1 \to \mathbb{V}_2$ between two vector spaces (modules) $\mathbb{V}_1, \mathbb{V}_2$ which is compatible with the representations ρ_1, ρ_2 of the group G on these spaces

$$T\rho_1(g)v = \rho_2(g)Tv \qquad \text{for all } g \in \mathrm{G} \text{ and } v \in \mathbb{V}_1. \tag{3.29}$$

Schur's Lemma. In the case where the two representations are irreducible, a module homomorphism is almost completely determined by *Schur's lemma*: Suppose $T : \mathbb{V}_1 \to \mathbb{V}_2$ is a module homomorphism for two irreps ρ_1, ρ_2. Then:

- Either $T = 0$ or T is invertible.
- For equal representations T is proportional to the identity map, $T \sim \mathrm{id}$.

Let us prove this lemma: First, we show that the kernel of T is an invariant subspace of $\mathbb{V}_1$

$$T\,\rho_1(g)\ker T = \rho_2(g)\,T\ker T = 0 \qquad \text{for all } g \in \mathrm{G}. \tag{3.30}$$

In other words, ρ_1 maps any vector from $\ker T$ to a vector which is annihilated by T, i.e. back to $\ker T$

$$\rho_1(g)\ker T \subset \ker T. \tag{3.31}$$

Since ρ_1 is irreducible, $\ker T$ can either be trivial or $\mathbb{V}_1$ itself. In other words, either T is injective or $T = 0$. Secondly, we show that the image $T\,\mathbb{V}_1$ of T is an invariant subspace of $\mathbb{V}_2$

$$\rho_2(g)\,T\,\mathbb{V}_1 = T\,\rho_1(g)\,\mathbb{V}_1 \subset T\,\mathbb{V}_1. \tag{3.32}$$

Since ρ_2 is irreducible, $T\,\mathbb{V}_1$ can either be trivial or $\mathbb{V}_2$ itself. In other words, either $T = 0$ or T is surjective. Altogether this shows that either $T = 0$ or T is invertible.

Finally, it is easy to show that for equal representations $\rho := \rho_1 = \rho_2$ we must have $T = \lambda\,\mathrm{id}$: The map T has at least one eigenvalue λ. Now the map $T' = T - \lambda\,\mathrm{id}$ is also a module homomorphism. It is not invertible and by the above results it must be zero, $T' = T - \lambda\,\mathrm{id} = 0$. This completes the proof of the lemma.

Some notes and corollaries are as follows:

- If T is invertible, the two representations are equivalent. Then T must be proportional to the map that relates the irreps.
- For inequivalent irreps necessarily $T = 0$.

We can also generalise the results of Schur's lemma to direct sums of irreps: A module homomorphism T takes the form of a block matrix where

- the blocks corresponding to two inequivalent irreps must be zero,
- the blocks corresponding to two equivalent irreps must be proportional to their similarity transformation map.

Effectively, one can view each block as a single numerical entry of the matrix. For example, a module homomorphism $\mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3 \oplus \mathbb{V}_3 \oplus \mathbb{V}_3 \to \mathbb{V}_3 \oplus \mathbb{V}_3 \oplus \mathbb{V}_2$ (for inequivalent irreps on the spaces $\mathbb{V}_k$) has the most general form

$$\begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & * & 0 & 0 & 0 \end{pmatrix}. \tag{3.33}$$

Orthogonality Relations. We can combine Schur's lemma with averaging over the group to obtain a very useful relationship for working with representations. For example, it can be used to construct projectors on selected irreps as we shall see later.

Suppose again that we have two irreps ρ_1, ρ_2 on $\mathbb{V}_1, \mathbb{V}_2$ and a linear map $S_0 : \mathbb{V}_1 \to \mathbb{V}_2$. Define another map $S : \mathbb{V}_1 \to \mathbb{V}_2$ as follows

$$S := \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \rho_2(g)^{-1} S_0 \rho_1(g). \tag{3.34}$$

This map is a module homomorphism because

$$\begin{aligned} S\rho_1(g) &= \frac{1}{|\mathrm{G}|} \sum_{b \in \mathrm{G}} \rho_2(b)^{-1} S_0 \rho_1(b) \rho_1(g) \\ &= \frac{1}{|\mathrm{G}|} \sum_{bg^{-1} \in \mathrm{G}} \rho_2(bg^{-1})^{-1} S_0 \rho_1(b) \\ &= \rho_2(g) S. \end{aligned} \tag{3.35}$$

Schur's lemma then tells us that:

- If ρ_1 and ρ_2 are inequivalent, the map S is zero, $S = 0$.
- If $\rho_1 = \rho_2$ are equal representations on $\mathbb{V}$, the map S is proportional to the identity.
- If ρ_1 and ρ_2 are equivalent representations on $\mathbb{V}$, the map S is proportional to the map that relates ρ_1 to ρ_2.

More concretely, we can find the factors of proportionality by taking the trace on both sides of the defining relation. For identical representations we find

$$S = \frac{\operatorname{tr} S}{\dim \mathbb{V}} \operatorname{id} = \frac{\operatorname{tr} S_0}{\dim \mathbb{V}} \operatorname{id}. \tag{3.36}$$

For equivalent representations $\rho_2 = R^{-1}\rho_1 R$, the combination $RS : \mathbb{V}_1 \to \mathbb{V}_1$ is a module homomorphism. By matching traces one finds

$$S = \frac{\operatorname{tr}(RS)}{\dim \mathbb{V}} R^{-1} = \frac{\operatorname{tr}(RS_0)}{\dim \mathbb{V}} R^{-1}. \tag{3.37}$$

It makes sense to rephrase the above result using tensor products by dividing out by the arbitrary map S_0 [6]

$$T_{12} := \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \rho_1(g) \otimes \rho_2(g^{-1}) \in \operatorname{End}(\mathbb{V}_1 \otimes \mathbb{V}_2). \tag{3.38}$$

If the representations are inequivalent one finds $T_{12} = 0$. Only for two equivalent representations ρ_1, ρ_2 on $\mathbb{V}$ with $\rho_2 = R^{-1}\rho_1 R$ the above operator takes a non-trivial form

$$T_{12} = \frac{1}{\dim \mathbb{V}} (R \otimes R^{-1}) \sigma \tag{3.39}$$

[6] Using the components of T_{12} to be introduced below, the defining relationship between S and S_0 reads $S^i{}_j = T^{ki}{}_{jl} S_0{}^l{}_k$.

with σ the permutation operator on the tensor square $\mathbb{V}\otimes\mathbb{V}$. Alternatively, the result can be expanded using two bases for the spaces $\mathbb{V}_1, \mathbb{V}_2$. In the case of equivalent representations, the component expansion reads

$$(T_{12})^{ij}{}_{kl} = \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \rho_1(g)^i{}_k\, \rho_2(g^{-1})^j{}_l = \frac{1}{\dim\mathbb{V}}\, R^i{}_l (R^{-1})^j{}_k. \tag{3.40}$$

Relations like these frequently appear in the context of group theory, and they act as *completeness relations* which allow to expand quantities in terms of a given basis of group elements.

Projectors and Orthogonality. The above formula is very useful when combined with characters. To that end, define the representation $\rho[f]\in\mathrm{End}(\mathbb{V})$ of a group function $f:\mathrm{G}\to\mathbb{C}$ as

$$\rho[f] := \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} f(g^{-1})\rho(g). \tag{3.41}$$

When we take the trace of the above tensor T_{12} over $\mathbb{V}_2$ we find

$$\mathrm{tr}_2\, T_{12} = \rho_1[\chi_2] = \begin{cases} 0 & \text{if } \rho_1 \not\equiv \rho_2, \\ (\dim\mathbb{V}_1)^{-1}\,\mathrm{id}_1 & \text{if } \rho_1\equiv\rho_2, \end{cases} \tag{3.42}$$

where χ_2 is the character of ρ_2 and thus a group function. This statement holds for irreps ρ_1, ρ_2, and it can serve as a useful test for equivalence of representations because a potential similarity transformation between the two representations plays no role. Now, all representations are completely reducible for finite groups. If ρ_1 is reducible, the above describes a projector (with weight $1/\dim\mathbb{V}_1$) to all of its irreducible components equivalent to ρ_2. A useful corollary is that characters of inequivalent irreps must be linear independent functions.

We can go one step further and also take the trace over the other space in the above formula (for irreps)

$$\langle\chi_1,\chi_2\rangle := \frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} \chi_2(g^{-1})\chi_1(g) = \begin{cases} 0 & \text{if } \rho_1\not\equiv\rho_2, \\ 1 & \text{if } \rho_1\equiv\rho_2. \end{cases} \tag{3.43}$$

This shows that the characters of irreps are orthonormal w.r.t. the symmetric bilinear form $\langle\cdot,\cdot\rangle$.[7]

This feature is extremely useful for determining the representation content of a reducible representation ρ. It tells us that

$$\langle\chi,\chi_k\rangle = n_k, \tag{3.44}$$

[7]Noting that all representations of finite groups are unitary (in some basis), we have $\chi(g^{-1}) = \chi(g)^*$ and thus the above bilinear form can also be viewed as a hermitian form.

where n_k is the multiplicity of the representation ρ_k in ρ. Knowing the character of ρ and the characters of all irreps ρ_k (character table of G), it is straight-forward to deduce the complete decomposition of ρ!

Conveniently, the sums over the group reduce to sums over the conjugacy classes with their order as the weight factor

$$\langle \chi_1, \chi_2 \rangle := \sum_{[g]\subset \mathrm{G}} \frac{|[g]|}{|\mathrm{G}|} \chi_2([g^{-1}])\chi_1([g]) \tag{3.45}$$

For example, recall the character table for S_3:

$[g]$	$\lvert[g]\rvert$	χ_1	$\chi_{1'}$	χ_2
$[1]$	1	1	1	2
$[\sigma_1\sigma_2]$	2	1	1	-1
$[\sigma_1]$	3	1	-1	0

(3.46)

It is straight-forward to verify that the characters of the irreps are orthonormal

$$\langle \chi_j, \chi_k \rangle = \delta_{jk}, \qquad j, k = 1, 1', 2. \tag{3.47}$$

Moreover, consider the reducible representation ρ_3 with character

$$\chi_3([1]) = 3, \qquad \chi_3([\sigma_1\sigma_2]) = 0, \qquad \chi_3([\sigma_1]) = 1. \tag{3.48}$$

We find

$$\langle \chi_1, \chi_3 \rangle = 1, \qquad \langle \chi_{1'}, \chi_3 \rangle = 0, \qquad \langle \chi_2, \chi_3 \rangle = 1, \tag{3.49}$$

and consequently $\rho_3 \equiv \rho_1 \oplus \rho_2$.

Group Algebra and Regular Representation. A few useful results for representations and characters follow by considering the group algebra and the regular representation.

The group algebra lifts a group G to a vector space $\mathbb{C}[\mathrm{G}]$. For each element $g \in \mathrm{G}$ there is a basis vector $e_g \in \mathbb{C}[\mathrm{G}]$ and multiplication in the algebra is defined by multiplication in the group via the basis vectors $e_g e_b = e_{gb}$.[8] The canonical representation of the group on the group algebra, the so-called *regular representation* ρ_{reg},[9] is defined as

$$\rho_{\mathrm{reg}}(g)e_b := e_{gb}. \tag{3.50}$$

By applying the above concepts to the regular representation, we find some useful identities.

The character of the regular representation is given by

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |\mathrm{G}| & \text{for } g = 1, \\ 0 & \text{for } g \neq 1. \end{cases} \tag{3.51}$$

[8]Note that the group algebra is the dual concept of a group function.

[9]Another canonical representation is by conjugation $\rho_{\mathrm{conj}}(g)e_b := e_{gbg^{-1}}$. Note that this representation has very different properties from the regular representation.

This follows from the fact that the identity element maps all $|G|$ basis vectors to themselves while any other element g maps any basis vector to some different one with no contribution to the trace.

Let us now decompose the regular representation into irreps using characters. We find

$$\langle \chi_{\text{reg}}, \chi_k \rangle = \chi_k(1) = \dim \rho_k. \tag{3.52}$$

Thus, remarkably, any irrep ρ_k of G appears in the regular representation with multiplicity equals its dimension

$$\rho_{\text{reg}} \equiv \bigoplus_k \rho_k^{\oplus \dim \rho_k}. \tag{3.53}$$

A corollary is that the order of the group equals the sum of squares of the dimensions of its irreps

$$\dim \rho_{\text{reg}} = |G| = \sum_k (\dim \rho_k)^2. \tag{3.54}$$

In our example of S_3, we see that $6 = 1 + 1 + 2^2$ and thus ρ_1, $\rho_{1'}$, ρ_2 form a complete set of irreps up to equivalence.

Another useful statement that follows from considerations of the regular representation is that the characters form a basis of the class functions. A *class function* is a function $f : G \to \mathbb{C}$ which is constant on the conjugacy classes, i.e. $f(g) = f(bgb^{-1}) = f([g])$ for all $g, b \in G$.

Let us prove the statement: First show that a representation $\rho[f] \in \text{End}(\mathbb{V})$ of the class function f

$$\rho[f] = \frac{1}{|G|} \sum_{g \in G} f(g^{-1}) \rho(g) \tag{3.55}$$

is a $\mathbb{V} \to \mathbb{V}$ module homomorphism because

$$\begin{aligned} \rho[f]\rho(g) &= \frac{1}{|G|} \sum_{b \in G} f(b^{-1}) \rho(bg) \\ &= \frac{1}{|G|} \sum_{gbg^{-1} \in G} f(gb^{-1}g^{-1}) \rho(gb) \\ &= \frac{1}{|G|} \sum_{b \in G} f(b^{-1}) \rho(g) \rho(b) = \rho(g)\rho[f]. \end{aligned} \tag{3.56}$$

By Schur's lemma this map $\rho[f]$ is proportional to the identity if ρ is irreducible. By taking the trace of $\rho[f]$ we can evaluate the coefficient

$$\operatorname{tr} \rho[f] = \frac{1}{|G|} \sum_{g \in G} f(g^{-1}) \chi(g) = \langle f, \chi \rangle, \qquad \rho[f] = \frac{\langle f, \chi \rangle}{\dim \rho} \operatorname{id}. \tag{3.57}$$

Suppose now that f is orthogonal to all characters χ_k w.r.t. the bilinear form $\langle \cdot, \cdot \rangle$. Then $\rho_k[f] = 0$ for all irreducible representations. As the regular representation is a direct sum of irreducible representations, and by linearity of $\rho[f]$ in ρ, we can extend the statement to the regular representation, $\rho_{\text{reg}}[f] = 0$. However, the regular representation is faithful because every group element is mapped to a different permutation of the basis vectors. In other words, the map $f \mapsto \rho_{\text{reg}}[f]$ is injective and therefore $f = 0$. This proves that the characters of the irreps form an (orthonormal) basis of class functions.

A corollary is that there are as many irreps as there are conjugacy classes. In the example of S_3 this number is 3.

4 Point and Space Groups

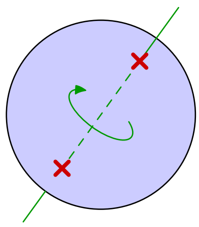
In this chapter we will investigate discrete subgroups of the euclidean group (consisting of rotations and translations in three dimensions) which play a role in the presence of matter (crystal lattices). Regular lattices (in higher dimensions) also play an important role in the theory of Lie algebras and their representations.

4.1 Point Groups

Suppose we put some matter in the form of a crystal into space. If the matter has a macroscopic shape which is rotationally symmetric, the residual symmetry group of the body is SO(2) for a single symmetry axis or SO(3) for a spherically symmetric object like a ball. However, this residual symmetry does not hold at the microscopic level which is relevant for orbital energies of individual atoms or energy bands of the body. Here, the crystal nature comes into play, and breaks the group of rotations even further. This group must map special axes of the crystal among themselves allowing only for a discrete set of rotations.[1] Thus the group SO(3) of rotations in three-dimensional space is broken by the crystal lattice to a discrete (and thus finite) subgroup.

Discrete Subgroups of SO(3). Let us therefore find all the finite subgroups of the group SO(3) (up to equivalence). Assume G is a finite subgroup. The central tool to classify the subgroups is a consistency argument about the action on the unit sphere S^2, fixed points, orbits and stabiliser subgroups.

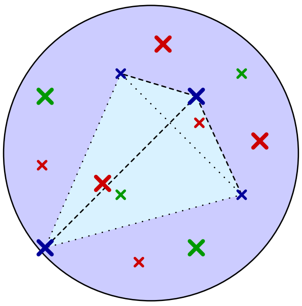
First we consider a non-trivial element $g \in G$. We know that every rotation in SO(3) is given by an axis and an angle. This implies that g has exactly two fixed points on the unit sphere.



(4.1)

[1]The macroscopic shape of the body has negligible influence on the microscopic properties. Therefore we will assume the crystal to have infinite extent in all directions so that the macroscopic shape does not spoil any symmetry.

Next consider the set P of all fixed points on S^2 of the (non-trivial) elements in G.



(4.2)

It is evident that G must map these fixed points to themselves,[2] thus it has a closed action on P. The set P furthermore splits into orbits P_i under G, let us suppose there are r such orbits.

Now let us consider the number $N := 2|\mathrm{G}| - 2$, and express it using the above fixed points and orbits. Evidently, N counts twice the number of non-trivial group elements

$$N = \sum_{g \neq 1 \in \mathrm{G}} 2. \tag{4.3}$$

As every group element has precisely two associated fixed points, we can recast N as a sum over fixed points m where each fixed point contributes its multiplicity $|\mathrm{G}_m| - 1$

$$N = \sum_{m \in P} \big(|\mathrm{G}_m| - 1\big). \tag{4.4}$$

We can then split the above sum into orbits P_i where each of the $|P_i| = |\mathrm{G}|/|\mathrm{G}_m|$ points m contributes the same amount $|\mathrm{G}_m| - 1$

$$N = \sum_{i=1}^{r} \frac{|\mathrm{G}|}{|\mathrm{G}_m|} \big(|\mathrm{G}_m| - 1\big). \tag{4.5}$$

Altogether, we obtain a simple equation involving $n := |\mathrm{G}|$ and r further unknowns $n_i := |\mathrm{G}_m|$ with $m \in P_i$

$$2 - \frac{2}{n} = \sum_{i=1}^{r} \left(1 - \frac{1}{n_i}\right). \tag{4.6}$$

This equation is very useful because it constrains the allowable values of r and n_i substantially. We know that $n_i \geq 2$ and thus each summand on the r.h.s. is at least $1/2$. So we find $2 > r/2$ or $r < 4$. On the other hand, $r = 1$ can be excluded because $n_i \leq n$ and the equation has no solution. It remains to discuss the cases $r = 2$ and $r = 3$.

[2] Fixed points correspond to rotational axes and group composition maps the allowable axes to themselves.

Cyclic Group. The easiest case is $r = 2$ where the equation reduces to

$$\frac{2}{n} = \frac{1}{n_1} + \frac{1}{n_2}\,. \tag{4.7}$$

We know that $n_i \leq n$ which only leaves the solution $n_1 = n_2 = n$. In this case there are only two fixed points and thus a single axis. The group consists of rotations by multiples of $2\pi/n$. This is the cyclic group C_n of order n. It is in fact a subgroup of SO(2) rotations in the plane.

Dihedral Group. For $r = 3$ we have

$$1 + \frac{2}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\,, \tag{4.8}$$

where we can assume $n_1 \leq n_2 \leq n_3$. We can argue that for $n_1 > 2$ the equation has no solution because the r.h.s. is less or equal 1. Similarly, $n_2 > 3$ yields a r.h.s. less or equal 1. The simplest solution is $(n_1, n_2, n_3) = (2, 2, k)$. This group is the *dihedral group* D_k of order $2k$. It describes the symmetries of a dihedron which is a flat polyhedron with two faces (front and back) each being a regular k-sided polygon. The group consists of the identity element, $k - 1$ rotations around the central axis of the polygon by multiples of $2\pi/k$ as well as k rotations by 180° around axes within the plane of the polygon. From a two-dimensional point of view the latter can also be viewed as reflections in the plane: As such the dihedral group is a subgroup of O(2), but the action on the coordinate orthogonal to the plane is different from the embedding into SO(3).

Polyhedral Groups. The remaining solutions for $(n_1, n_2, n_3; n)$ are

$$\begin{aligned} (2,3,3;12) &\to \mathrm{T}, \\ (2,3,4;24) &\to \mathrm{O}, \\ (2,3,5;60) &\to \mathrm{I}. \end{aligned} \tag{4.9}$$

One can figure out that these groups are the rotational symmetry groups of the regular polyhedra, the five Platonic solids.

- The *tetrahedral group* T maps a regular tetrahedron to itself. Apart from the identity element, there are $8 = 4 \cdot 2$ rotations about an axis joining a vertex and the centre of the opposite face as well as 3 rotations about an axis joining the centres of two opposite edges.
- The *octahedral group* O maps a regular octahedron to itself. Likewise, it maps a cube (regular hexahedron) to itself. Since the prototype of a three-dimensional lattice is cubic, this group plays a dominant role for crystals. The symmetry group of the octahedron (cube) has $8 = 4 \cdot 2$ rotations about the faces (vertices), 6 rotations about the edges and $9 = 3 \cdot 3$ rotations about the faces (vertices).
- The *icosahedral group* I maps a regular icosahedron to itself; similarly, it maps a regular dodecahedron to itself. Even though these polyhedra have the largest amount of symmetry, the group does not apply to crystals as we shall see

below.[3] As a symmetry group of the icosahedron (dodecahedron) it has $24 = 6 \cdot 4$ rotations of vertices (faces), 15 rotations of edges and $20 = 10 \cdot 2$ rotations of faces (vertices).

The platonic solids can also be viewed as regular tilings of the two-sphere. It is therefore natural that they are associated to the finite subgroups of $\mathrm{SO}(3)$.

Reflections. So far we have restricted our attention to rotations, but we can also consider reflections. Knowing that the determinant of such a transformation is either $+1$ or -1 one can argue that there must be as many rotations as reflections in a finite subgroup $\mathrm{G} \subset \mathrm{O}(3)$. However, extending a finite rotational group G_0 is not necessarily unique, but can yield up to three alternatives. There are two main classes to be distinguished:

- In the easier case, the reflection group contains the complete reflection $-\mathrm{id}$. Since $-\mathrm{id}$ commutes with all elements, also $-g \in \mathrm{G}$ for all rotational $g \in \mathrm{G}_0$. The reflection group is the direct product $\mathrm{G} = \mathrm{G}_\mathrm{h} := \mathrm{G}_0 \times \mathbb{Z}_2$ with $\mathbb{Z}_2$.
- In the other case $-\mathrm{id}$ is not an element. Then one can argue that the reflection group is isomorphic to a rotational group G' which contains G_0 as a subgroup: Simply multiply the reflections in G by the overall inversion $-\mathrm{id}$ to obtain a finite subgroup of $\mathrm{SO}(3)$. Furthermore, the subgroup G_0 is normal and has index 2. In order to classify reflection groups of this kind, we therefore look for rotation groups G' which have a suitable subgroup G_0. This turns out to yield at most two reflection group extensions G for each rotation group G_0.

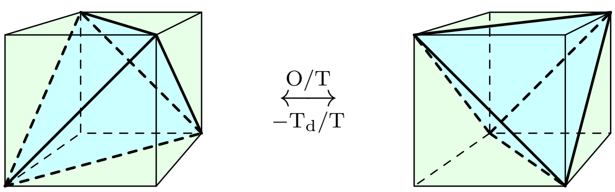
An instructive example is the extension of the tetrahedral group T. The direct product

$$\mathrm{T}_\mathrm{h} := \mathrm{T} \times \mathbb{Z}_2 \tag{4.10}$$

is straight-forward to understand. Note that the reflections in T_h are not symmetries of a tetrahedron because they map points to centres of faces and vice versa. The reflectional symmetries of a tetrahedron are given by the non-trivial extension T_d. This is isomorphic to a rotational group of twice as many elements as T. One can also argue that it must be among the polyhedral groups, hence it can only be the octahedral group

$$\mathrm{T}_\mathrm{d} \equiv \mathrm{O}. \tag{4.11}$$

Alternatively one can argue by embedding the tetrahedron into alternating vertices of a cube.



(4.12)

Those rotations in O which do not map vertices of the tetrahedron among themselves should be multiplied by an overall inversion. This combination

[3] Nevertheless, this and related groups can appear in nature in the form of quasi-groups.

preserves the tetrahedron, but curiously the group multiplication remains as in O. The resolution to the apparent puzzle is that $\mathrm{T_d} \equiv \mathrm{O}$ has two inequivalent representations of dimension 3. One is the defining representation of $\mathrm{T_d}$ describing the reflectional symmetries of the tetrahedron. The other one is the defining representation of O describing the rotational symmetries of the cube or octahedron. Finally, one can say that the latter groups are isomorphic to the symmetric group S_4 describing permutations of the 4 vertices of the tetrahedron. Equally, the tetrahedral group T is isomorphic to the alternating group A_4

$$\mathrm{T} \equiv \mathrm{A}_4, \qquad \mathrm{T_d} \equiv \mathrm{O} \equiv \mathrm{S}_4. \tag{4.13}$$

The other polyhedral groups O and I only have the trivial extensions $\mathrm{O} \times \mathbb{Z}_2$ and $\mathrm{I} \times \mathbb{Z}_2$. Conversely, the cyclic and dihedral groups C_n and D_k have up to three reflectional extensions.

Discrete Subgroups of SU(2). In the presence of half-integer spin particles, the double cover $\mathrm{Spin}(3) \equiv \mathrm{SU}(2)$ of SO(3) becomes relevant. Just like SO(3) this group has discrete subgroups. Here each subgroup $\mathrm{G} \subset \mathrm{SO}(3)$ has a unique double cover $\mathrm{G}^* \subset \mathrm{SU}(2)$ which includes the element $(-1)^F$:[4] There are the two infinite sequences C^*_n and D^*_k of order $2n$ and $4k$, respectively as well as the three polyhedral cases:

- *binary tetrahedral group* T^* of order 24,
- *binary octahedral group* O^* of order 48,
- *binary icosahedral group* I^* of order 120.

Recalling that $\mathrm{Spin}(3) \equiv \mathrm{Sp}(1)$, there is a curious quaternionic presentation of the above groups: The group elements are given by unit quaternions distributed in a regular pattern over the three-sphere S^3.

4.2 Representations

Let us discuss a few aspects of the representations theory of the point groups introduced above.

Irreducible Representations. The representation theory of the cyclic groups C_n is trivial. The irreducible representations of the dihedral groups D_k are one-dimensional (2 for odd k and 4 for even k) or two-dimensional (all remaining). We will not discuss the representation theory of these groups further.

[4]There are also finite subgroups of SU(2) which do not include the element $(-1)^F$; these are the cyclic groups of odd order.

For the tetrahedral group A_4 one finds as the character table:

	$[\cdot]$	$[2]$	$[3]$	$[3']$
$\#$	1	3	4	4
1	1	1	1	1
$1'$	1	1	$e^{+2\pi i/3}$	$e^{-2\pi i/3}$
$\bar{1}'$	1	1	$e^{-2\pi i/3}$	$e^{+2\pi i/3}$
3	3	-1	0	0

(4.14)

Note that the representations $1'$ and $\bar{1}'$ form a complex conjugate pair of representations while the others are real. It is straight-forward to infer the decomposition of tensor products into irreducibles. Only $3 \otimes 3 \equiv 3 \oplus 3 \oplus 1 \oplus 1' \oplus \bar{1}'$ requires a brief calculation.

The character table for the octahedral group S_4 reads:

	$[\cdot]$	$[2]$	$[3]$	$[4]$	$[2,2]$
$\#$	1	6	8	6	3
1	1	1	1	1	1
$1'$	1	-1	1	-1	1
2	2	0	-1	0	2
3	3	-1	0	1	-1
$3'$	3	1	0	-1	-1

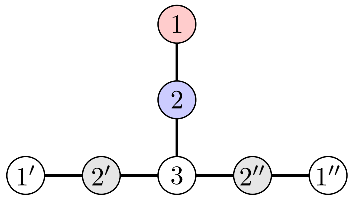
(4.15)

Note that the irreducible representations 3 and $3'$ are the defining geometrical representations of O and T_d, respectively. Furthermore, the representation $1'$ is the determinant of the defining representation of T_d; the subgroup T is specified by restricting to the group elements with positive determinant.

The icosahedral group is isomorphic to the alternating group A_5. This group has 5 irreducible representations of dimensions $1, 3, 3, 4, 5$ as well as 5 conjugacy classes of sizes $1, 12, 12, 15, 20$. As this group does not play a role for lattices, we will not discuss it further.

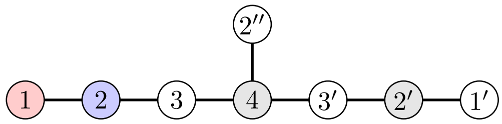
Representations of the Binary Polyhedral Groups. The representation theory of the binary polyhedral groups has a curious feature known as the *McKay correspondence*. Let us discuss the representation theory using the *McKay graph* of the group: Draw a node for each irreducible representation of the group. For each node, consider the tensor product with the 2-dimensional defining representation (as a subgroup of SU(2)). Draw an edge from this node to any node that appears in the tensor product decomposition. It turns out that for the binary polyhedral groups, each irreducible representation appears at most once and that the connectivity is symmetric (hence no multiple lines or arrows are needed as decorations).

The group T^* has 7 irreducible representations of dimensions $1, 1, 1, 2, 2, 2, 3$ and 7 conjugacy classes of sizes $1, 1, 4, 4, 4, 4, 6$. Compared to the group T there are three additional representations of dimension $2, 2, 2$. The McKay graph takes the form:



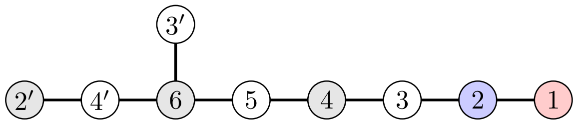
(4.16)

The group O* has 8 irreducible representations of dimensions $1, 1, 2, 2, 2, 2, 3, 3, 4$ and 8 conjugacy classes of sizes $1, 1, 6, 6, 6, 8, 8, 12$. Compared to the group O there are three additional representations of dimension $2, 2, 4$. The McKay graph takes the form:



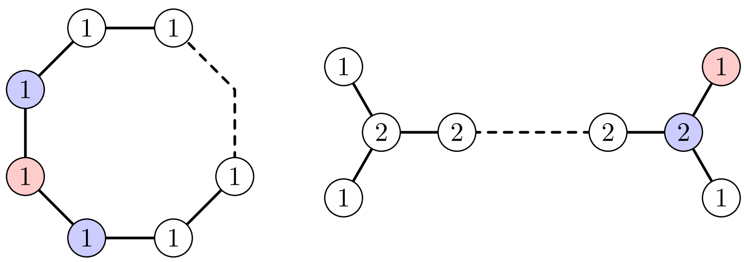
(4.17)

The group I* has 9 irreducible representations of dimensions $1, 2, 2, 3, 3, 4, 4, 5, 6$ and 9 conjugacy classes of sizes $1, 1, 12, 12, 12, 12, 20, 20, 30$. Compared to the group I there are four additional representations of dimension $2, 2, 4, 6$. The McKay graph takes the form:



(4.18)

Finally, the McKay graph for the cyclic groups and the binary dihedral groups take the form:



(4.19)

The above graphs are the so-called extended ADE-graphs.[5] They also play a role in the classification of simple finite-dimensional Lie algebras.

Splitting of Representations. In quantum mechanics the orbital motion of particles (electrons) around a central point (nucleus) is given in terms of spherical harmonics $Y_{l,m}$. As we have seen above, the set of functions $Y_{l,m}$ with fixed l correspond to a representation of SO(3) with spin l. Rotational symmetry of the potential then implies that the energy levels must not depend on m.[6]

[5] The cyclic, dihedral and polyhedral groups have McKay graphs of type A, D and E, respectively.

[6] For the Coulomb potential, they actually also do not depend on l. The reason for the further degeneracy is a hidden SO(4) symmetry related to the Runge–Lenz vector.

If, however, the atom resides within actual matter, the potentials of the surrounding atoms break the rotational symmetry. Consequently, the degeneracies of energy levels are also broken. If the disturbance is small, the symmetry still holds approximately, and only minor deviations in the energy levels are expected. Conversely, large disturbances may completely distort the spectrum and obscure the symmetry of free space. The situation in a crystal is different because there can be residual symmetries which still hold (at least to a good approximation). Then the continuous rotational symmetries reduce to the finite point group of the lattice. This will break some degeneracies of energy levels while others are preserved. Here, representation theory gives a precise answer for the expected degeneracies.

Let us discuss the example of a simple cubic lattice where the octahedral group O is the relevant residual symmetry. We start with a particle in an orbit with angular momentum L. In a spherically symmetric potential all $2L+1$ states within this orbit have the same energy because they transform in an irreducible representation of SO(3). Within a lattice with orthogonal symmetry this representation is clearly reducible for $L > 1$ because there are no irreducible representations of O of dimension greater than 3. We have learned above how to decompose representations using characters.

Let us therefore compute the character of the spin-L representation of SO(3). The resulting formula remains valid for all subgroups when the group element is restricted to the subgroup. We know that the character depends only on the conjugacy class. Therefore we need to understand the conjugacy classes of SO(3). A rotational element is specified by an axis $\vec{n} \in \mathrm{S}^2$ and an angle $0 \leq \psi \leq \pi$. Conjugation of this element changes the axis to some other direction on S^2, but it does not alter the angle. Therefore the conjugacy classes are formed by rotations with equal angles. In order to compute the character we choose a rotation around the z-axis. By construction the states $|m\rangle$ are eigenstates under this rotation with eigenvalue $\mathrm{e}^{\imath m\psi}$. The spin-$L$ character is thus given by the sum

$$\begin{aligned}\chi_L(\psi) &= \sum_{m=-L}^{L} \mathrm{e}^{\imath m\psi} = \frac{\mathrm{e}^{\imath(L+1/2)\psi} - \mathrm{e}^{-\imath(L+1/2)\psi}}{\mathrm{e}^{\imath\psi/2} - \mathrm{e}^{-\imath\psi/2}} \\ &= \frac{\sin\big((L+\tfrac{1}{2})\psi\big)}{\sin\big(\tfrac{1}{2}\psi\big)}\,. \end{aligned} \tag{4.20}$$

We have to evaluate the representation for the 5 conjugacy classes of O. These are elements with rotational angles $\psi = 0, \pi, \frac{2}{3}\pi, \frac{1}{2}\pi, \pi$ for $[\cdot], [2], [3], [4], [2,2]$, respectively. The results reads

$$\begin{aligned}\chi_L(0) &= 2L+1, \\ \chi_L(\pi) &= 1, -1, \qquad \text{for } L \equiv 0,1 \pmod 2, \\ \chi_L(\tfrac{2}{3}\pi) &= 1, 0, -1, \qquad \text{for } L \equiv 0,1,2 \pmod 3, \\ \chi_L(\tfrac{1}{2}\pi) &= 1, 1, -1, -1, \qquad \text{for } L \equiv 0,1,2,3 \pmod 4. \end{aligned} \tag{4.21}$$

Using orthonormality and the character table of O one finds the branching rules for SO(3) → O where the spin-L representations are labelled by their dimension

$2L + 1$:

$$\begin{aligned} 1 &\to 1, \\ 3 &\to 3, \\ 5 &\to 2 \oplus 3', \\ 7 &\to 1' \oplus 3 \oplus 3, \\ 9 &\to 1 \oplus 2 \oplus 3 \oplus 3, \\ &\ldots . \end{aligned} \tag{4.22}$$

This table repeats after 12 lines with a certain shift of the multiplicities. In terms of physics, one expects pairwise and triple degeneracies of energy levels in a crystal with a cubic structure.

Suppose further, the cubic symmetry O is just approximate and breaks further to dihedral symmetry D_3. In order to understand the splitting of irreducible representations, note that D_3 is isomorphic to the symmetric group S_3 and it consists of elements from the conjugacy classes $[\cdot]$, $[2]$, $[3]$ of O. Comparing the character tables of both groups one finds

$$\begin{aligned} 1 &\to 1, \\ 1' &\to 1', \\ 2 &\to 2, \\ 3 &\to 1' \oplus 2, \\ 3' &\to 1 \oplus 2. \end{aligned} \tag{4.23}$$

This list tells us that all the pairwise degeneracies in the cubic crystal are preserved while all threefold degeneracies are expected to split into pairs and singlets.

4.3 Crystallographic Groups

Finally, we discuss the implications of the lattice on the allowable discrete symmetry groups.

Crystallographic Point Groups. We have found several infinite families of discrete point symmetry groups as well as some special cases related to the regular polyhedra. Gladly, only finitely many are suitable for lattice structures. There is a simple argument to prove this fact.

A lattice is described by three vectors which span an elementary cell. Under a rotation these vectors are not necessarily mapped to themselves because the elementary cell could be mapped to another elementary cell. This implies that the basis vectors should be mapped to some vectors on the lattice. In other words, their image must be an *integer* linear combination of the basis vectors. When the rotation matrix is expressed in the lattice basis, it must have integer coefficients only, i.e. it must belong to $\mathrm{SL}(3, \mathbb{Z})$. Now the lattice basis is typically not orthonormal, and it is not easy to understand which matrices are both orthogonal and integer. Here the character is of help because it is independent of the choice of basis. The trace of an integer matrix must be integer. As we have seen above, the trace of a 3×3 orthogonal matrix (spin-1 representation of SO(3)) equals

$$\chi_1(\psi) = \frac{\sin\left(\frac{3}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)} = 1 + 2\cos\psi. \tag{4.24}$$

There are only 5 integer solutions, namely $\psi = 0°, 60°, 90°, 120°, 180°$ or

$$\psi = \frac{2\pi}{n} \qquad \text{with} \qquad n = 1, 2, 3, 4, 6. \tag{4.25}$$

The order of the group elements cannot be 5 or greater than 6. This reduces the rotational point groups suitable for crystals to just 11

$$\{1\}, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4, \mathrm{C}_6, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4, \mathrm{D}_6, \mathrm{T}, \mathrm{O}. \tag{4.26}$$

In particular, the 5-fold symmetry of the dodecahedron and icosahedron excludes the group I. If reflections are included, one obtains the 32 *crystallographic point groups* or *crystal classes*.[7]

Space Groups. So far we have only discussed the symmetries that leave a point fixed. However, in an (infinite) lattice there are also translational symmetries. The combination of rotations and translations is called a *space group* G_∞. A space group is based on one of the 32 crystallographic point groups G_0. Group elements take the form $(R, \vec{t})$ with the group multiplication

$$(R_1, \vec{t}_1)\cdot(R_2, \vec{t}_2) = (R_1R_2, \vec{t}_1 + R_1\vec{t}_2). \tag{4.27}$$

Translations by any lattice vector are admissible, i.e.

$$(R, \vec{t}) \in \mathrm{G}_\infty \qquad \Longrightarrow \qquad (R, \vec{t} + n_k\vec{\ell}_k) \in \mathrm{G}_\infty, \quad n_k \in \mathbb{Z}, \tag{4.28}$$

and therefore space groups are infinite.[8] We have seen above that the basis vectors must be mapped in a particular way by the rotations, but this restriction still leaves some choices. Moreover, it is not even guaranteed that $(R, \vec{0}) \in \mathrm{G}_\infty$ for all $R \in \mathrm{G}_0$; some space groups have this property, others not. In other words, G_0 is not necessarily a subgroup of G_∞. The classification of all distinct space groups is a tedious case-by-case study and leads to 230 cases altogether.

The representation theory of these space groups imposes constraints on the electronic band structure of the crystals which we will not discuss in this course.

[7] Some of these groups are isomorphic as groups but they have inequivalent defining representations which describe the action on three-dimensional space.

[8] In order to remain with finite groups and their favourable properties, one can impose periodic boundary conditions.

5 Structure of Simple Lie Algebras

In this chapter we will discuss the structure of a general simple Lie algebra $\mathfrak{g}$ as preparation for the representation theory in the following chapter. A *simple* Lie algebra is defined as having no non-trivial ideals, i.e. it is minimal in a certain sense and its adjoint representation is irreducible.[1]

5.1 The Algebra $\mathfrak{su}(3)$

Throughout this chapter and the following we will use the Lie algebra $\mathfrak{su}(3)$ to illustrate the general results by means of a simple example.

The Lie group SU(3) is the group of all 3×3 matrices U which are unitary ($U^\dagger = U^{-1}$) and have unit determinant ($\det U = 1$). This group plays an important role in low-energy hadronic physics where it serves as the approximate symmetry group of the hadronic particles made from the lightest 3 of the 6 elementary quark flavours. It also serves as the gauge group for Quantum Chromodynamics where it related the three colour degrees of freedom of the quarks.

The Lie algebra $\mathfrak{su}(3)$ is the infinitesimal form of the Lie group SU(3). It is spanned by all 3×3 traceless anti-hermitian matrices L

$$\mathfrak{su}(3) = \{L \in \mathrm{End}(\mathbb{C}^3); L^\dagger = -L, \operatorname{tr} L = 0\}. \tag{5.1}$$

In physics, the so-called *Gell-Mann matrices* λ_k, $k = 1, \ldots, 8$, are often used as an imaginary basis for $\mathfrak{su}(3)$. They are a straight-forward generalisation of the 2×2 Pauli matrices to 3×3 matrices. For our analysis of the representation theory they will not be immediately applicable.

5.2 Cartan–Weyl Basis

First, we introduce a basis of generators suitable for the construction of representations.

Complexification. We will proceed in analogy to $\mathfrak{so}(3)$ in Section 2.3 and introduce a basis of generators which measure, raise or lower certain charges.[2] This

[1] Furthermore, one commonly excludes the one-dimensional abelian Lie algebra.

[2] We will rely on some notion of charges which is based on symmetries in quantum mechanics. This notion generalises the z-component of spin in $\mathfrak{su}(2)$. Such charges are well-defined in the sense that they can be measured simultaneously, and they are additive under compositions of charged objects.

basis involved complex combinations $\mathrm{J}_z, \mathrm{J}_\pm$ of the generators, and more accurately we discussed the complex Lie algebra $\mathfrak{so}(3,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})$ and its representation theory. Along the same lines it is convenient to consider the Lie algebra $\mathfrak{g}$ to be *complex*, such that we can naturally take arbitrary complex linear combinations of basis elements. After the work is done, one can reduce to a real form of the algebra and obtain certain restrictions on its representation theory. In our example, the *complexification* of $\mathfrak{su}(3)$ is the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. Let us for the time being assume all Lie algebras to be complex, and abbreviate $\mathfrak{sl}(N,\mathbb{C})$ as $\mathfrak{sl}(N)$.

Cartan Subalgebra. For $\mathfrak{sl}(2)$, the generator J_z was used to measure a certain charge which allowed us to partition the representation space into subspaces with fixed eigenvalue of the charge. For the bigger algebra $\mathfrak{sl}(3)$ there is no single such element, but rather a two-dimensional subalgebra $\mathfrak{h}$ of diagonal (traceless) matrices

$$\mathfrak{h} = \{\mathrm{diag}(a_1, a_2, a_3); a_k \in \mathbb{C},\ a_1 + a_2 + a_3 = 0\}. \tag{5.2}$$

The elements of $\mathfrak{h}$ allow to decompose a space into subspaces specified by two charge eigenvalues. In general, the *Cartan subalgebra* $\mathfrak{h}$ is the maximal abelian subalgebra of the Lie algebra $\mathfrak{g}$ that is self-normalising. The latter is the requirement that provided some $L \in \mathfrak{g}$ with $[\![H, L]\!] \in \mathfrak{h}$ for all $H \in \mathfrak{h}$ implies $L \in \mathfrak{h}$.[3] The requirement that all elements of $\mathfrak{h}$ have trivial Lie brackets

$$[\![H, H']\!] = 0 \qquad \text{for all } H, H' \in \mathfrak{h}, \tag{5.3}$$

is essential because their representations mutually commute and can be diagonalised simultaneously allowing to determine several charges at the same time. Note that the choice of Cartan subalgebra is not unique; for example, one can apply some similarity transformation to $\mathfrak{h}$. However, the dimension of $\mathfrak{h}$ is well-defined, it is called the *rank* $r = \dim \mathfrak{h}$ of the Lie algebra. The earlier example $\mathfrak{sl}(2)$ is the unique simple Lie algebra of rank $r = 1$. The present example $\mathfrak{sl}(3)$ has rank $r = 2$ which adds several complications to the analysis. Conceptually, this example is as difficult as it gets among the simple Lie algebras, and the treatment of all higher-rank simple Lie algebras follows along the same lines.

Cartan–Weyl Basis. The Cartan subalgebra acts on the Lie algebra by the adjoint representation, and thus we can decompose the Lie algebra into eigenspaces of the Cartan algebra.[4] This is achieved via the eigenvalue equation

$$\mathrm{ad}(H)L = [\![H, L]\!] = \alpha_L(H)\, L \qquad \text{for all } H \in \mathfrak{h}, \tag{5.4}$$

where α_L is a linear function $\mathfrak{h} \to \mathbb{C}$ which describes the charges of the eigenvector $L \in \mathfrak{g}$.[5] The charges of the elements of the Cartan subalgebra are zero by construction.

[3] The actual definition of Cartan subalgebras is in fact slightly different. Our definition refers to the maximal Cartan subalgebra of a simple Lie algebra.

[4] In fact, this statement relies on diagonalisability of the adjoint representation of the Cartan subalgebra which holds for simple Lie algebras.

[5] In physics one would pick a basis for $\mathfrak{h}$ and define the charges as the eigenvalues of the $r = 2$ basis elements. The above statement is equivalent but independent of a choice of basis.

For concreteness, we perform this decomposition for $\mathfrak{sl}(3)$: Consider a Cartan element $H \in \mathfrak{h}$ and a generic algebra element $L \in \mathfrak{g}$

$$H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad L = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{5.5}$$

with $a_1 + a_2 + a_3 = b_{11} + b_{22} + b_{33} = 0$. Then

$$[\![H, L]\!] = \begin{pmatrix} 0 & (a_1 - a_2)b_{12} & (a_1 - a_3)b_{13} \\ (a_2 - a_1)b_{21} & 0 & (a_2 - a_3)b_{23} \\ (a_3 - a_1)b_{31} & (a_3 - a_2)b_{32} & 0 \end{pmatrix}, \tag{5.6}$$

and we can immediately see that the eigenvectors are given by the matrices L_{ij} with a 1 in row i and column j and 0 everywhere else. Noting that the matrices obey the algebra $L_{ij}L_{kl} = \delta_{jk}L_{il}$ we can write more abstractly

$$[\![H, L_{ij}]\!] = \alpha_{ij}(H)L_{ij}, \qquad \text{where } \alpha_{ij}(H) = a_i - a_j. \tag{5.7}$$

Analogously we can pick a basis for $\mathfrak{h}$ consisting of the diagonal matrices

$$L_{11} - L_{22} = \operatorname{diag}(1, -1, 0), \qquad L_{22} - L_{33} = \operatorname{diag}(0, 1, -1). \tag{5.8}$$

Altogether a so-called *Cartan–Weyl basis* for $\mathfrak{sl}(3)$ is given by the generators

$$\big\{L_{11} - L_{22}, L_{22} - L_{33}, L_{12}, L_{13}, L_{21}, L_{23}, L_{31}, L_{32}\big\}. \tag{5.9}$$

5.3 Root System

Having identified a set of generators with well-defined charges, we now proceed to analyse the charges and their relationships.

Roots. We have decomposed our Lie algebra as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{5.10}$$

Here $\mathfrak{g}_\alpha$ is the subspace of $\mathfrak{g}$ defined by the above eigenvalue equation

$$[\![H, L]\!] = \alpha(H)L \qquad \text{for all } H \in \mathfrak{h}, L \in \mathfrak{g}_\alpha, \tag{5.11}$$

and Δ defines the set of permissible non-zero eigenvalues. The linear function $\alpha : \mathfrak{h} \to \mathbb{C}$ is by definition an element of the dual space $\mathfrak{h}^*$ of the Cartan algebra. The permissible non-zero eigenvalues $\alpha \in \Delta$ are called the *roots* of the Lie algebra $\mathfrak{g}$, and the subspace $\mathfrak{g}_\alpha$ is the corresponding *root space.* The subspace $\mathfrak{g}_0$ corresponding to the zero vector in $\mathfrak{h}^*$ is the Cartan subalgebra $\mathfrak{h}$. However, the zero vector is usually not called a root and $0 \notin \Delta$.

A very useful feature is that the Lie brackets preserve the charges. For two generators L, L' with charges α, α' the Jacobi identity implies

$$[\![H, [\![L, L']\!]]\!] = \big(\alpha(H) + \alpha'(H)\big)[\![L, L']\!] \tag{5.12}$$

meaning that the Lie bracket $[\![L, L']\!]$ carries the sum of charges $\alpha + \alpha'$. For the root subspace this implies the relationship

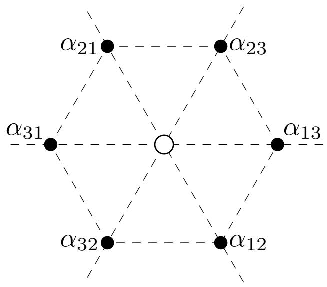
$$[\![\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}]\!] \subset \mathfrak{g}_{\alpha+\alpha'}, \tag{5.13}$$

where the Lie bracket has to be zero when $\alpha + \alpha' \notin \Delta \cup \{0\}$.

In our example, the set of roots is given by

$$\Delta = \{\alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{23}, \alpha_{31}, \alpha_{32}\}. \tag{5.14}$$

The corresponding root spaces $\mathfrak{g}_{ij} = \mathbb{C}L_{ij}$ are all one-dimensional. Moreover, the negative of every root is a root as well, $\Delta = -\Delta$. The latter two facts conveniently extend to all simple Lie algebras. Only the subspace $\mathfrak{g}_0 = \mathfrak{h}$ corresponding to $0 \in \mathfrak{h}^*$ has a higher dimension equalling the rank r. The Cartan–Weyl basis thus consists of a basis for the Cartan subalgebra together with one generator L_α for each root $\alpha \in \Delta$. For the example of $\mathfrak{sl}(3)$ with two-dimensional Cartan subalgebra, we can conveninently plot the configuration of roots in a two-dimensional root diagram:



(5.15)

Positive and Negative Roots. Our construction of finite-dimensional irreps of $\mathfrak{sl}(2)$ made use of a state which was distinguished by having the highest charge. For rank $r > 1$ the charge is specified by more than one number, and thus there is no canonical ordering principle. However, it suffices to specify some non-zero element $H_0 \in \mathfrak{h}$ [6] [7] with which a partial ordering for $\alpha, \beta \in \mathfrak{h}^*$ can be established via

$$\alpha \leq \beta \qquad \Longrightarrow \qquad \alpha(H_0) \leq \beta(H_0). \tag{5.16}$$

It will be of no concern that two unequal $\alpha \neq \beta$ may have equal distinguished charges $\alpha(H_0) = \beta(H_0)$. However, the choice must ensure that all roots have non-zero distinguished charge

$$\alpha(H_0) \neq 0 \qquad \text{for all } \alpha \in \Delta. \tag{5.17}$$

[6] It does not matter much which element is chosen.

[7] Strictly speaking, the spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ are complex and ordering would make no sense, but in fact for the finite-dimensional simple Lie algebras there are canonical real slices for these spaces.

This allows us to classify the roots as either *positive* or *negative* according to

$$\mathit{\Delta} = \mathit{\Delta}_+ \cup \mathit{\Delta}_-, \qquad \mathit{\Delta}_- = -\mathit{\Delta}_+ \tag{5.18}$$

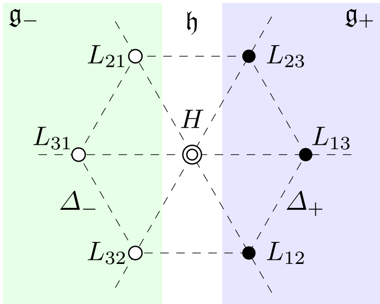
with

$$\alpha(H_0) > 0 \qquad \text{for all } \alpha \in \mathit{\Delta}_+. \tag{5.19}$$

In terms of the distinguished charge, one can understand the positive and negative roots to correspond to *raising and lowering generators.* Altogether the Lie algebra splits into a positive part $\mathfrak{g}_+$, a negative part $\mathfrak{g}_-$ and the Cartan subalgebra

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_- \qquad \text{with} \qquad \mathfrak{g}_\pm := \bigoplus_{\alpha \in \mathit{\Delta}_\pm} \mathfrak{g}_\alpha. \tag{5.20}$$

Note that all of these form subalgebras of $\mathfrak{g}$ as well as the so-called *Borel subalgebras* $\mathfrak{b}_\pm := \mathfrak{g}_\pm \oplus \mathfrak{h}$.



(5.21)

In our example, a useful choice for the distinguished Cartan element is

$$H_0 = \operatorname{diag}(+1, 0, -1). \tag{5.22}$$

With this choice the positive roots are

$$\mathit{\Delta}_+ = \{\alpha_{12}, \alpha_{13}, \alpha_{23}\}. \tag{5.23}$$

These correspond to a basis L_{12}, L_{13}, L_{23} for strictly upper triangular 3×3 matrices. The corresponding negative generators L_{21}, L_{31}, L_{32} are the strictly lower triangular matrices, while the diagonal matrices belong to the Cartan subalgebra.

Simple Roots. Furthermore, it makes sense to distinguish some basis for $\mathfrak{h}^*$ in terms of the roots. Usually the number of positive roots is larger than the rank $r = \dim \mathfrak{h}^*$ of the algebra, and there must be linear dependencies among the positive roots. The charge relationship $[\![\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}]\!] \subset \mathfrak{g}_{\alpha+\alpha'}$ paired with simplicity of the Lie algebra then implies that some positive roots should be linear combinations of other positive roots with non-negative integer coefficients. The remaining positive roots which cannot be expressed in terms of others in this way are called *simple.* There are always precisely r simple roots and they form a basis for $\mathfrak{h}^*$.

In our example, the simple roots are α_{12} and α_{23} while $\alpha_{13} = \alpha_{12} + \alpha_{23}$ is composite. The corresponding simple generators L_{12} and L_{23} obey the Lie bracket

$$[\![L_{12}, L_{23}]\!] = L_{13}, \tag{5.24}$$

and thus all relations involving L_{13} can in principle be expressed in terms of the simple generators. Note that the simple generators are next-to-diagonal matrices while the non-simple generator is further away from the diagonal.

Chevalley–Serre Generators. Following these lines, we can reduce the algebraic relations to some minimal set. First, we denote the simple roots by $(k = 1, 2)$

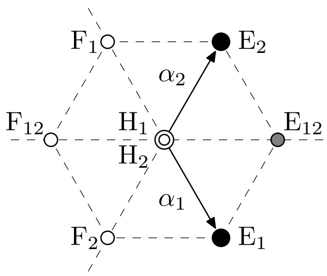
$$\alpha_k := \alpha_{k,k+1}. \tag{5.25}$$

We then introduce a notation for the generators corresponding to these simple roots $(k = 1, 2)$

$$\mathrm{E}_k := L_{k,k+1}, \qquad \mathrm{F}_k := L_{k+1,k}, \qquad \mathrm{H}_k := L_{k,k} - L_{k+1,k+1}. \tag{5.26}$$

These so-called *Chevalley–Serre generators* form a basic set of Lie generators which are reminiscent of $r = 2$ copies of the raising, lowering and charge generators $J_\pm$ and J_z for $\mathfrak{sl}(2)$. The two remaining composite generators are given by

$$\mathrm{E}_{12} := [\![\mathrm{E}_1, \mathrm{E}_2]\!] = L_{13}, \qquad \mathrm{F}_{12} := [\![\mathrm{F}_2, \mathrm{F}_1]\!] = L_{31}. \tag{5.27}$$



(5.28)

The Lie algebra relations can be expressed in a uniform way $(j, k = 1, 2)$

$$\begin{aligned} [\![\mathrm{H}_j, \mathrm{H}_k]\!] &= 0, & [\![\mathrm{E}_j, \mathrm{F}_k]\!] &= \delta_{jk}\mathrm{H}_k, \\ [\![\mathrm{H}_j, \mathrm{E}_k]\!] &= \alpha_k(\mathrm{H}_j)\mathrm{E}_k, & [\![\mathrm{H}_j, \mathrm{F}_k]\!] &= -\alpha_k(\mathrm{H}_j)\mathrm{F}_k, \end{aligned} \tag{5.29}$$

where the simple roots evaluated on the basis of the Cartan algebra read

$$\alpha_k(\mathrm{H}_j) = \begin{pmatrix} +2 & -1 \\ -1 & +2 \end{pmatrix}. \tag{5.30}$$

The Lie brackets among the positive and among the negative generators are not yet specified allowing for composites like E_{12} and F_{12} to form. Nevertheless, constraints are needed to remove unwanted generators. These are the *Serre relations* which read for $\mathfrak{sl}(3)$

$$\begin{aligned} \big[\![\mathrm{E}_1, [\![\mathrm{E}_1, \mathrm{E}_2]\!]\big]\!] &= 0, & \big[\![\mathrm{F}_1, [\![\mathrm{F}_1, \mathrm{F}_2]\!]\big]\!] &= 0, \\ \big[\![\mathrm{E}_2, [\![\mathrm{E}_2, \mathrm{E}_1]\!]\big]\!] &= 0, & \big[\![\mathrm{F}_2, [\![\mathrm{F}_2, \mathrm{F}_1]\!]\big]\!] &= 0. \end{aligned} \tag{5.31}$$

5.4 Invariant Bi-Linear Forms

We can equip the various vector spaces we have encountered with canonical bi-linear forms. This will be of use for various relations later on.[8]

[8]. . . and cheer up the physicist who would not know how to handle a vector spaces without a scalar product.

Killing Form. A Lie algebra has a canonical symmetric bi-linear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Up to a prefactor this form is given by the *Killing form* which is defined via the adjoint representation as

$$\kappa(L, L') \sim \operatorname{tr}\bigl[\operatorname{ad}(L)\operatorname{ad}(L')\bigr]. \tag{5.32}$$

For a simple algebra we can alternatively use any other irreducible representation ρ

$$\kappa(L, L') \sim \operatorname{tr}\bigl[\rho(L)\rho(L')\bigr]. \tag{5.33}$$

By construction this form is invariant under the adjoint action of $\mathfrak{g}$ as follows

$$\kappa\bigl(\operatorname{ad}(L'')L, L'\bigr) + \kappa\bigl(L, \operatorname{ad}(L'')L'\bigr) = 0. \tag{5.34}$$

The bi-linear form is non-degenerate for simple Lie algebras. Furthermore, for a real Lie algebra, the bi-linear form has a definite signature if and only if the Lie group corresponding to the Lie algebra is compact. Finally, the invariance property ensures that the Killing form is non-zero only for opposite root spaces, $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}) = 0$ for $\alpha + \alpha' \neq 0$. As an aside, note that the Killing form is the inverse of the quadratic Casimir invariant which we encountered earlier.

For $\mathfrak{sl}(3)$ the bi-linear form is determined by providing the non-zero components. We choose a normalisation based on the defining representation and obtain

$$\kappa(\mathrm{E}_1, \mathrm{F}_1) = \kappa(\mathrm{E}_2, \mathrm{F}_2) = \kappa(\mathrm{E}_{12}, \mathrm{F}_{12}) = 1 \tag{5.35}$$

and

$$\kappa(\mathrm{H}_1, \mathrm{H}_1) = \kappa(\mathrm{H}_2, \mathrm{H}_2) = 2, \qquad \kappa(\mathrm{H}_1, \mathrm{H}_2) = -1. \tag{5.36}$$

Upon inversion we obtain the quadratic Casimir invariant which reads for $\mathfrak{sl}(3)$

$$\begin{aligned} C = {}& \tfrac{2}{3}\mathrm{H}_1 \otimes \mathrm{H}_1 + \tfrac{1}{3}\mathrm{H}_1 \otimes \mathrm{H}_2 + \tfrac{1}{3}\mathrm{H}_2 \otimes \mathrm{H}_1 + \tfrac{2}{3}\mathrm{H}_2 \otimes \mathrm{H}_2 \\ & + \mathrm{E}_1 \otimes \mathrm{F}_1 + \mathrm{E}_2 \otimes \mathrm{F}_2 + \mathrm{E}_{12} \otimes \mathrm{F}_{12} \\ & + \mathrm{F}_1 \otimes \mathrm{E}_1 + \mathrm{F}_2 \otimes \mathrm{E}_2 + \mathrm{F}_{12} \otimes \mathrm{E}_{12} \end{aligned} \tag{5.37}$$

or more concisely

$$C = L_{jk} \otimes L_{kj} - \tfrac{1}{3} L_{jj} \otimes L_{kk}. \tag{5.38}$$

Scalar Product on $\mathfrak{h}^*$. We can use the Killing form on $\mathfrak{g}$ to obtain a scalar product for the space $\mathfrak{h}^*$. We thus restrict the bi-linear form to the Cartan subalgebra where it remains invertible. By inverting and dualising the restricted form we obtain a scalar product $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ on $\mathfrak{h}^*$.

In our example, the restricted Killing form reads

$$\kappa(\mathrm{H}_j, \mathrm{H}_k) = A_{jk}, \qquad A = \begin{pmatrix} +2 & -1 \\ -1 & +2 \end{pmatrix}. \tag{5.39}$$

The scalar product on $\mathfrak{h}^*$ thus takes the form

$$\langle \mathrm{H}_j^*, \mathrm{H}_k^* \rangle = A^{-1}_{jk}, \qquad A^{-1} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}, \tag{5.40}$$

where the dual elements $\mathrm{H}^*_k \in \mathfrak{h}^*$ are defined by the relationship $\mathrm{H}^*_j(\mathrm{H}_k) = \delta_{jk}$. However, we normally do not express elements of $\mathfrak{h}^*$ in the basis of H^*_k, but rather in terms of the simple roots α_k which are related to H_j by

$$\alpha_k(\mathrm{H}_j) = A_{jk}. \tag{5.41}$$

Combining the various relationships we find $\alpha_k = A_{kj}\mathrm{H}^*_j$ and finally

$$\langle \alpha_j, \alpha_k \rangle = A_{jk} = (AA^{-1}A)_{jk}. \tag{5.42}$$

This implies that the simple roots do not form an orthonormal basis of $\mathfrak{h}^*$.[9]

The matrix A which appears frequently in the above is called the *Cartan matrix*. For more general Lie algebras, the Cartan matrix is defined as[10]

$$A_{jk} := \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_j, \alpha_j \rangle} = \alpha_k(\mathrm{H}_j). \tag{5.43}$$

The Cartan matrix will later be used to construct general simple Lie algebras from scratch and to classify them.

[9]The above figures of root configurations use an orthonormal coordinate system, and one can observe that the roots are not orthogonal.

[10]In our example all roots have the same length, $\langle \alpha_j, \alpha_j \rangle = 2$, and thus A_{jk} equals the scalar product of roots.

6 Finite-Dimensional Representations

We are now in a good position to investigate the representation theory of a generic simple Lie algebra $\mathfrak{g}$, and construct finite-dimensional irreducible representations. Again, $\mathfrak{su}(3)$ and its complexification $\mathfrak{sl}(3)$ will serve as the main example to illustrate the mostly general results.

6.1 Representations of $\mathfrak{su}(3)$

We already know at least 4 representations of $\mathfrak{su}(3)$: There is the trivial one-dimensional representation and the adjoint representation which is eight-dimensional. These two representations are evidently real. Then there is the defining three-dimensional representation. As a (truly) complex representation it has a complex conjugate representation which is distinct. The latter two are also called the fundamental and anti-fundamental representations. All of these representations are irreducible, and often irreps are labelled by a bold number giving their dimensionality:

$$\mathbf{1} = \mathbf{triv}, \qquad \mathbf{8} = \mathbf{adj}, \qquad \mathbf{3} = \mathbf{fund}, \qquad \bar{\mathbf{3}} = \mathbf{3}^* = \mathbf{fund}^*. \tag{6.1}$$

Here the two fundamental representations have the same dimension and thus they should be distinguished somehow.

Clearly, $\mathfrak{su}(3)$ has many more distinct irreps and the goal of this chapter is to understand their representation theory. In order to construct further representations, we can use the tensor product of the above representations and then decompose them into irreps. For example, one finds

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} &= \mathbf{6} \oplus \mathbf{3}^*, \\ \mathbf{3} \otimes \mathbf{3}^* &= \mathbf{8} \oplus \mathbf{1}, \\ \mathbf{3}^* \otimes \mathbf{3}^* &= \mathbf{6}^* \oplus \mathbf{3}. \end{aligned} \tag{6.2}$$

Apart from the previously known four representations, there are is a new pair of complex conjugate six-dimensional irreps. When they are used within tensor products, further irreps can be produced.

One may wonder why the tensor product $\mathbf{3} \otimes \mathbf{3}$ decomposes and how. The crucial point is that the two tensor factors are equivalent. In this case the tensor product space and the representation can be decomposed (at least) into symmetric and anti-symmetric contributions. The symmetric product forms the new irrep $\mathbf{6}$ while the anti-symmetric product happens to yield the anti-fundamental representation $\mathbf{3}^*$ in the case of $\mathfrak{su}(3)$. In fact, there is a general principle governing the tensor products of equivalent representations. Applied to the defining representation, this will actually determine the representation theory of the algebras $\mathfrak{su}(N)$ in terms of the representation theory of the symmetric group. A general finite-dimensional irrep of $\mathfrak{su}(3)$ turns out to be labelled by two non-negative integers $[n_1, n_2]$. This is analogous to the representation theory of $\mathfrak{su}(2)$ which uses one non-negative label $[n]$ related to the spin $j = n/2$ of an irrep.

6.2 Weights

States of a finite-dimensional irrep of a simple Lie algebra can be classified by their charges under the Cartan subalgebra, the so-called weights, which generalise the spin quantum number of $\mathfrak{su}(2)$. Irreps have a distinct pattern of weights, and there is a unique state of highest weight which is often used to identify the irrep.

Weights. Consider a finite-dimensional irrep $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$. We shall assume that the representation of the Cartan subalgebra $\mathfrak{h}$ on $\mathbb{V}$ is diagonalisable.[1] We proceed as in the case of $\mathfrak{sl}(2)$ and decompose the representation space into eigenspaces $\mathbb{V}_\lambda$

$$\mathbb{V} = \bigoplus_{\lambda \in \Lambda_\rho} \mathbb{V}_\lambda \tag{6.3}$$

under the representation of the Cartan subalgebra[2]

$$\rho(H)|v\rangle = \lambda(H)|v\rangle \qquad \text{for all } H \in \mathfrak{h},\ |v\rangle \in \mathbb{V}_\lambda. \tag{6.4}$$

The charges $\lambda \in \Lambda_\rho \subset \mathfrak{h}^*$ are called *weights*,[3] the corresponding eigenspaces $\mathbb{V}_\lambda \subset \mathbb{V}$ are called *weight spaces*, and their dimension $m_\lambda := \dim \mathbb{V}_\lambda$ is called the *multiplicity* of the weight λ. Clearly, the roots of the Lie algebra (including the zero weight) are also weights, namely those of the adjoint representation,

$$\Lambda_{\mathrm{ad}} = \Delta \cup \{0\}, \qquad m_\lambda = \begin{cases} 1 & \text{for } \lambda \in \Delta, \\ r & \text{for } \lambda = 0. \end{cases} \tag{6.5}$$

The weights are additive in the sense[4]

$$\rho(L_\alpha)\mathbb{V}_\lambda \subset \mathbb{V}_{\alpha+\lambda} \qquad \text{for all } L_\alpha \in \mathfrak{g}_\alpha, \tag{6.6}$$

[1] The latter requirement in fact follows from our setup, but the subsequent analysis is simplified substantially if we can assume it from the start.

[2] It would suffice to abbreviate the representation of a generator $L \in \mathfrak{g}$ on a state $|v\rangle \in \mathbb{V}$ by $L|v\rangle := \rho(L)|v\rangle$ whenever there is a canonical representation ρ on $\mathbb{V}$ and no ambiguities arise. This is common practice at least in the physics literature.

[3] The weights are vectors of the dual Cartan subalgebra $\mathfrak{h}^*$, but note that the term *weight vector* usually refers to vectors of the representation space $|v\rangle \in \mathbb{V}_\lambda$ with definite charges. In order to avoid potential ambiguities, we will refer to vectors of representation spaces as states, i.e. $|v\rangle \in \mathbb{V}_\lambda$ is a *weight state*.

[4] As usual we declare $\mathbb{V}_\lambda := \{0\}$ if $\lambda \notin \Lambda_\rho$.

as can be confirmed straight-forwardly using the representation property involving the Cartan subalgebra. This implies that the weighs of an irreducible representation must be evenly spaced, i.e. differences should be integer linear combinations of the (simple) roots. The weights of a representation lie on a lattice

$$\Lambda_\rho \subset \lambda + \mathbb{Z}\Delta = \lambda + \sum_{\alpha\in\Delta} \mathbb{Z}\alpha \qquad \text{with } \lambda \in \Lambda_\rho. \tag{6.7}$$

In our example $\mathfrak{sl}(3)$, we already understand the adjoint representation, and the trivial representation is trivial. Let us therefore consider the two three-dimensional fundamental representations. For the defining representation we declared

$$\rho(\mathrm{H}_1) = \mathrm{diag}(1, -1, 0), \qquad \rho(\mathrm{H}_2) = \mathrm{diag}(0, 1, -1). \tag{6.8}$$

The matrices are already diagonalised and therefore the three weights of the representation read ($H = a_1\mathrm{H}_1 + a_2\mathrm{H}_2$)

$$\lambda_1(H) = a_1, \qquad \lambda_2(H) = a_2 - a_1, \qquad \lambda_3(H) = -a_2. \tag{6.9}$$

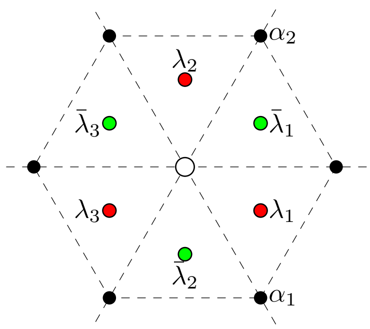
The weights of the conjugate representation are just the opposites

$$\bar\lambda_1(H) = -a_1, \qquad \bar\lambda_2(H) = a_1 - a_2, \qquad \bar\lambda_3(H) = a_2. \tag{6.10}$$

We can express the above weights as linear combinations of the simple roots (recall $\alpha_1(H) = 2a_1 - a_2$, $\alpha_2(H) = 2a_2 - a_1$)

$$\begin{aligned} \lambda_1 &= +\tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2, & \bar\lambda_1 &= -\tfrac{2}{3}\alpha_1 - \tfrac{1}{3}\alpha_2,\\ \lambda_2 &= -\tfrac{1}{3}\alpha_1 + \tfrac{1}{3}\alpha_2, & \bar\lambda_2 &= +\tfrac{1}{3}\alpha_1 - \tfrac{1}{3}\alpha_2,\\ \lambda_3 &= -\tfrac{1}{3}\alpha_1 - \tfrac{2}{3}\alpha_2, & \bar\lambda_3 &= +\tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2, \end{aligned} \tag{6.11}$$

and plot their configuration in a so-called *weight diagram* along with the roots:



(6.12)

Highest Weight and Descendants. In the construction of irreps for $\mathfrak{sl}(2)$ we singled out the state with the highest z-component of spin, and derived all other states from it. We have no canonical ordering on the set Λ_ρ of higher-dimensional charges, but we already introduced a partial ordering based on some Cartan element H_0. This partial ordering singles out a *highest weight* $\mu \in \Lambda_\rho$ with the maximum value of $\lambda(H_0)$. A priori, the highest weight is not necessarily unique, nor does it have multiplicity 1. We are free to choose any *highest-weight state* $|\mu\rangle \in \mathbb{V}_\mu$ and we shall later see that this state is indeed unique for an irrep.

We know by construction that all the positive roots $\alpha \in \Delta_+$ obey $\alpha(H_0) > 0$. Therefore, the highest-weight state must be annihilated by all positive generators

$$\rho(L)|\mu\rangle = 0, \qquad \text{for all } L \in \{L_\alpha; \alpha \in \Delta_+\}. \tag{6.13}$$

In fact, it suffices to demand that all simple positive generators E_j, $j = 1, \ldots, r$ annihilate the state, $\rho(\mathrm{E}_j)|\mu\rangle = 0$, because all other positive generators are composed from the E_j. Furthermore, the action of the Cartan subalgebra is determined by the weight μ

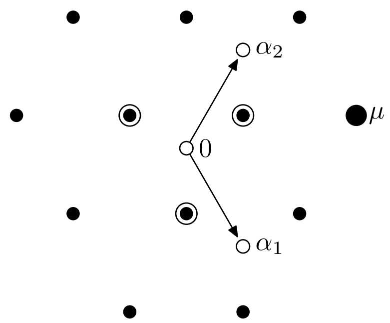
$$\rho(H)|\mu\rangle = \mu(H)|\mu\rangle. \tag{6.14}$$

All other states of the representation must be generated by acting repeatedly with the negative generators $L_{-\alpha}$, $\alpha \in \Delta_+$ on $|\mu\rangle$. The representation space $\mathbb{V}$ is spanned by (not necessarily independent or non-trivial) states

$$\rho(L_n) \cdots \rho(L_1)|\mu\rangle, \qquad \text{where } L_k \in \{L_{-\alpha}; \alpha \in \Delta_+\}. \tag{6.15}$$

These states can be called *descendants* of $|\mu\rangle$. Again it would suffice to restrict the L_k to the negative simple generators F_j, $j = 1, \ldots, r$ because the non-simple generators can be written as commutators which are also accounted for by longer chains of operators acting on $|\mu\rangle$.

A more elaborate example of a finite-dimensional irrep of $\mathfrak{su}(3)$ can be depicted by its weight diagram:



(6.16)

It consists of 9 weights of multiplicity 1 and 3 weights of multiplicity 2. The rightmost weight labelled by μ is the highest weight, it can be specified in terms of the simple roots as $\mu = \frac{4}{3}\alpha_1 + \frac{5}{3}\alpha_2$. Later, we shall use this example to discuss the structure of finite-dimensional irreps.

6.3 Highest-Weight Representations

Our aim is to construct finite-dimensional irreps. The above discussion outlines their structure in terms of weight states, but it does not tell which of these weight states exist or are independent. In analogy to our construction of finite-dimensional irreps of $\mathfrak{su}(2)$ it makes sense to take a detour: We assume that all weight states exist and are maximally independent. This leads us to a class of infinite-dimensional representations which are conceptually simpler and from which the finite-dimensional irreps can be deduced.

Representation. Let us first show that the space spanned by the above highest-weight state $|\mu\rangle$ and descendants $\rho(L_n)\cdots\rho(L_1)|\mu\rangle$ with negative generators L_k can accommodate a consistent representation. We thus apply the operator $\rho(L)$ and show that we obtain a linear combination of states of the same form:

- For all negative contributions to L we clearly obtain a state of the same form but with one additional generator.
- Our aim is then to bring the remainder of the operator closer to the highest-weight state $|\mu\rangle$. In each iteration, the operator is commuted one step closer to $|\mu\rangle$.
- Each commutator produces a new term with a new operator $\rho(L')$. All negative contributions to L' merely produce a state of the above form which we do not have to consider further. The remainder is also one step closer to $|\mu\rangle$ than the original operator $\rho(L)$.
- After a finite number of iterations, all non-negative operators $\rho(L'')$ reside right next to $|\mu\rangle$. The positive contributions to L'' annihilate the state, and the remaining Cartan contribution multiply the state by a number.
- At the end of the day we obtain a linear combination of states of the above form.

This argument also shows that there is a unique highest-weight state $|\mu\rangle$ in an irreducible representation.

Basis. We now have a set of states $\rho(L_n)\cdots\rho(L_1)|\mu\rangle$ with negative L_k to span the representation space, but we do not know which ones are trivial or linearly dependent. For the individual states, the ordering of the L_k is relevant. However, changing the ordering amounts to adding a state where two adjacent L_k are replaced by their Lie bracket. The latter is another negative generator and hence it is a state of the same form but with lower n. Here we are interested in the span of all states, where the latter state is already accounted for. For our purposes the ordering therefore does not matter, and we can reduce the relevant states to

$$|\mu;\ell\rangle := \left[\prod_{\alpha\in\Delta_+}\rho(L_{-\alpha})^{\ell_\alpha}\right]|\mu\rangle, \qquad \ell_\alpha\in\mathbb{Z}_0^+. \tag{6.17}$$

One can show that all of these states are algebraically independent supposing that they exist. Our assumption of a finite-dimensional representation implies that almost all of them do not in fact exist, i.e. there are only finitely many non-trivial weight spaces $\mathbb{V}_\lambda$.

However, this assumption complicates the analysis, and for the time being we shall assume all of the above states to be non-trivial and independent. In other words, the above states $|\mu;\ell\rangle$ with $\ell_k\in\mathbb{Z}_0^+$ form a basis for an infinite-dimensional space $\mathbb{V}_{\mathrm{hw}}$. On this space, we have established a representation $\rho_\mu:\mathfrak{g}\to\mathbb{V}_{\mathrm{hw}}$ which is a so-called *highest-weight representation.* For a generic highest weight μ this representation is irreducible, but if μ satisfies certain integrality constraints, it becomes reducible. After factoring out all sub-representations we will later obtain a finite-dimensional representation.

Multiplicities. We thus consider the complete highest-weight representation space where all states $|\mu;\ell\rangle$ are independent. Then we can more easily count the multiplicities of the weights. The weight λ_ℓ of the state $|\mu;\ell\rangle$ is given by

$$\lambda_\ell = \mu - \sum_{\alpha\in\Delta_+} \ell_\alpha \alpha. \tag{6.18}$$

Now the elements of Δ_+ are dependent and therefore some of the λ_ℓ coincide and lead to multiplicities. How these degeneracies come about depends on the details of the Lie algebra $\mathfrak{g}$ and its set of positive roots.

We shall therefore continue the analysis for $\mathfrak{sl}(3)$. Here the only relationship among the positive roots is

$$\alpha_{12} + \alpha_{23} = \alpha_{13}. \tag{6.19}$$

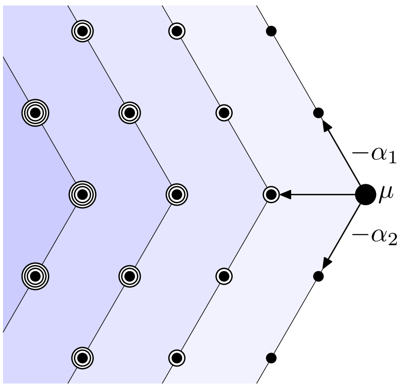
Given a weight

$$\lambda_{k_1,k_2} = \mu - k_1\alpha_1 - k_2\alpha_2, \tag{6.20}$$

there are $\min(k_1, k_2) + 1$ ways of writing it in terms of a positive integer linear combination of $\alpha_{12} = \alpha_1$, $\alpha_{23} = \alpha_2$ and $\alpha_{13} = \alpha_1 + \alpha_2$. In other words, the multiplicity of λ_{k_1,k_2} reads

$$m_{k_1,k_2} = \min(k_1, k_2) + 1. \tag{6.21}$$

For $\mathfrak{sl}(3)$ the highest-weight representations have a pattern of linearly increasing multiplicities starting at the highest weight and along the direction $-\alpha_1 - \alpha_2$:



(6.22)

This completes the description of highest-weight representations.

6.4 Finite-Dimensional Representations

The next step is to reduce an infinite-dimensional highest-weight representation to a finite-dimensional irrep.

Quotient Representation. The above construction produces infinitely many states which are related by a representation of $\mathfrak{g}$. To obtain a finite-dimensional irrep with the same highest-weight state, we have to declare infinitely many of these states (or linear combinations thereof) to be trivial. Now, the representation on the reduced space is only consistent if the states to be projected out transform under a sub-representation. The resulting representation is the *quotient* of the highest-weight representation by all of its non-trivial sub-representations. Furthermore, we have to make sure that the resulting representation is finite-dimensional.

The standard approach to eliminate the unwanted states in highest-weight representations is to look for further states $|\mu'\rangle$ which are annihilated by all positive roots so that one cannot go back to the original state $|\mu\rangle$. Clearly such states $|\mu'\rangle$ have a definite weight and thus they satisfy all the above properties of highest-weight states. They therefore reside at the top of another highest-weight representation which is also a sub-representation of the original highest-weight representation. If we declare this highest-weight state to be zero, then all of its descendants will have to be set to zero as well due to the representation property. This leads to various dependencies among the basis states $|\mu;\ell\rangle$ in the reduced representation. Gladly, the sub-representation with the new highest weight μ' has very much the same structure as the original highest-weight representation based at μ, only with the highest weight shifted to a different location. Therefore it will be sufficient to know all secondary highest weights μ' to understand the structure of the finite-dimensional irrep.

Weight Lattice. As emphasised above, a generic highest-weight representation is irreducible; it becomes reducible only for specific highest weights μ. In order to derive the constraints on μ it is convenient to rely on our earlier results for the representation theory of $\mathfrak{sl}(2)$. The latter is a subalgebra of $\mathfrak{sl}(3)$ and as such, the irreps of the latter should decompose into a direct sum of irreps of the former. Moreover, there are inequivalent ways of embedding $\mathfrak{sl}(2)$ into $\mathfrak{sl}(3)$. This provides us with several constraints which are necessary for our multi-dimensional weight space.

An $\mathfrak{sl}(2)$ subalgebra is generated by $\mathrm{H}_1, \mathrm{E}_1, \mathrm{F}_1$. In our normalisation

$$[\![\mathrm{H}_1, \mathrm{E}_1]\!] = \alpha_1(\mathrm{H}_1)\mathrm{E}_1 \qquad \text{with} \qquad \alpha_1(\mathrm{H}_1) = 2. \tag{6.23}$$

We know that for $\mathfrak{sl}(2)$ the weights of the finite-dimensional representations are integer or half-integer multiples of the simple root. Therefore

$$\lambda(\mathrm{H}_1) \in \tfrac{1}{2}\alpha_1(\mathrm{H}_1)\mathbb{Z} = \mathbb{Z} \qquad \text{for all } \lambda \in \mathit{\Lambda}_\rho. \tag{6.24}$$

This statement derives from the representation of some generator in $\mathfrak{sl}(3)$ and therefore it applies to the representation as a whole. If the highest-weight representation is to contain finite-dimensional components, the above constraint will have to be satisfied.

A second choice of $\mathfrak{sl}(2)$ subalgebra is generated by $\mathrm{H}_2, \mathrm{E}_2, \mathrm{F}_2$. Likewise we obtain the integrality constraint $\lambda(\mathrm{H}_2) \in \mathbb{Z}$. A third choice would be the subalgebra generated by $\mathrm{H}_1 + \mathrm{H}_2$, E_{12}, F_{21}, but this merely implies $\lambda(\mathrm{H}_1 + \mathrm{H}_2) \in \mathbb{Z}$ which does not correspond to an independent constraint.

Altogether we find one constraint for each basis vector of $\mathfrak{h}^*$

$$\lambda(\mathrm{H}_j) \in \mathbb{Z} \qquad \text{for } j = 1, \ldots, r \text{ and all } \lambda \in \Lambda_\rho. \tag{6.25}$$

Alternatively, the constraints can be formulated via the scalar product on $\mathfrak{h}^*$

$$\frac{2\langle \lambda, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \in \mathbb{Z} \qquad \text{for } j = 1, \ldots, r. \tag{6.26}$$

The above implies that the weights of finite-dimensional representations reside on a lattice fixed by the algebra $\mathfrak{g}$, the so-called *weight lattice* Ω.[5]

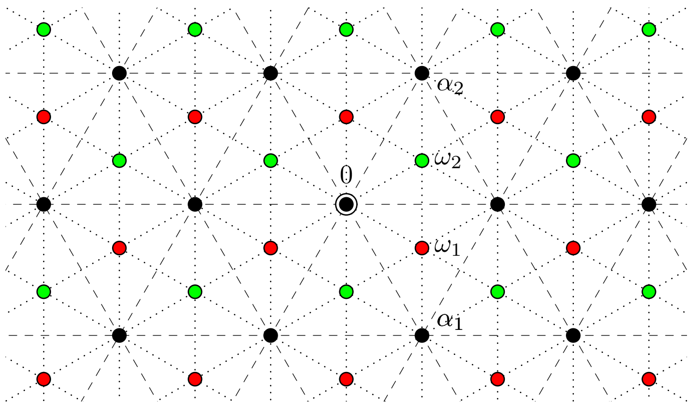
$$\Omega := \sum_{k=1}^{r} \omega_k \mathbb{Z}, \tag{6.27}$$

where the lattice vectors ω_k, $k = 1, \ldots, r$ are determined by being dual to the simple roots α_j w.r.t. the scalar product on $\mathfrak{h}^*$

$$\frac{2\langle \omega_k, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \omega_k(\mathrm{H}_j) = \delta_{jk}. \tag{6.28}$$

For $\mathfrak{sl}(3)$ the weight lattice vectors reads

$$\omega_1 := \tfrac{2}{3}\alpha_1 + \tfrac{1}{3}\alpha_2, \qquad \omega_2 := \tfrac{1}{3}\alpha_1 + \tfrac{2}{3}\alpha_2. \tag{6.29}$$



(6.30)

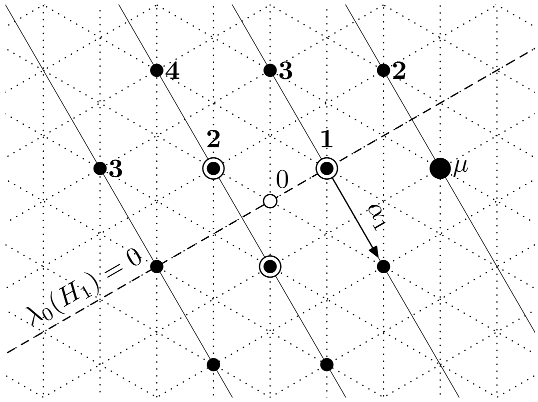
Notice that the weight lattice is not necessarily spanned by the simple roots. As a consequence, the weights of a given irrep only populate a sub-lattice of the weight lattice which is spanned by the simple roots. In our example $\mathfrak{sl}(3)$ we have three sub-lattices while the earlier example $\mathfrak{sl}(2)$ led to two sub-lattices (integer and half-integer spin). In fact, this discrepancy of lattices translates to the centre of the simply connected compact Lie group corresponding to the Lie algebra. We will

[5]Although the algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$ are complex, the weight lattice resides in a real slice of $\mathfrak{h}^*$. This is related to the fact that finite-dimensional representations are unitary in the compact real form of $\mathfrak{g}$.

not prove this fact, but merely note that the centre of SU(2) is $\mathbb{Z}_2$ while the centre of SU(3) is $\mathbb{Z}_3$.[6] The centre acts on all states of the representation with a common eigenvalue and this eigenvalue distinguishes the sub-lattices. Note that the roots reside on the sub-lattice based at zero.

Discrete Symmetries. Next, we will demonstrate that all weights are symmetrically distributed about the origin in some way. We can deduce this by considering $\mathfrak{sl}(2)$ subalgebras.

More concretely, we consider subsets of weights related by the generators H_1, E_1, F_1 spanning an $\mathfrak{sl}(2)$ subalgebra. As such, each subset must reside on the one-dimensional lattice $\lambda + \alpha_1\mathbb{Z}$, where λ is one of the weights of the subset.



(6.31)

However, the distribution of weights is further constrained by the fact that these weights form finite-dimensional representations of $\mathfrak{sl}(2)$ whose structure we understand well. In particular, we know that the weights of finite-dimensional irreps $\mathfrak{sl}(2)$ are distributed symmetrically about the origin: The origin w.r.t. the $\mathfrak{sl}(2)$ subalgebra is given by some weight λ_0 with $\lambda_0(\mathrm{H}_1) = 0$. A symmetric distribution is expressed in terms of the multiplicities as

$$\dim \mathbb{V}_{\lambda+n\alpha_1/2} = \dim \mathbb{V}_{\lambda-n\alpha_1/2}. \tag{6.32}$$

Equivalent statements hold for the other $\mathfrak{sl}(2)$ subalgebra based on the second set of Chevalley–Serre generators.

It makes sense to formulate the symmetry in terms of a reflection in $\mathfrak{h}^*$

$$\sigma_1 : \lambda_0 + x\alpha_1 \mapsto \lambda_0 - x\alpha_1, \qquad \text{where } \lambda_0(\mathrm{H}_1) = 0. \tag{6.33}$$

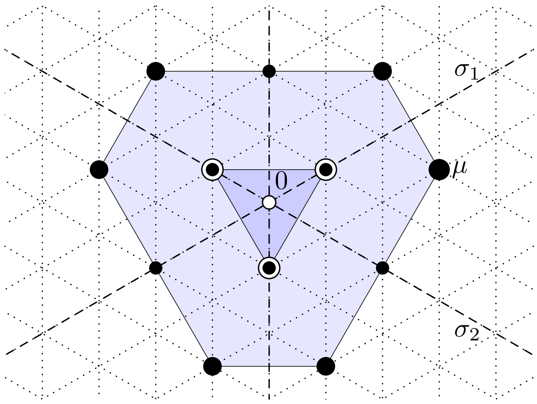
The second $\mathfrak{sl}(2)$ subalgebra yields a second reflection

$$\sigma_2 : \lambda_0 + x\alpha_2 \mapsto \lambda_0 - x\alpha_2, \qquad \text{where } \lambda_0(\mathrm{H}_2) = 0. \tag{6.34}$$

These reflections generate a group W called the *Weyl group*. For $\mathfrak{sl}(3)$ it is the symmetric group $\mathrm{W} = \mathrm{S}_3$ which is also the reflectional symmetry group of the

[6]The centre of SU(3) is given by the identity matrix times the third roots of unity 1, $\mathrm{e}^{2\pi\hat{\imath}/3}$, $\mathrm{e}^{-2\pi\hat{\imath}/3}$.

equilateral triangle. With reference to our discussions of crystal lattices one can also say that the Weyl group is the point group of the lattice.



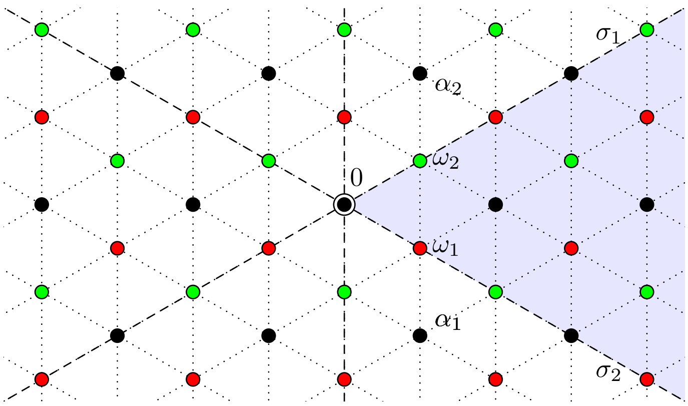
(6.35)

In fact, we can deduce a further characteristic property of the weight diagram from the structure of finite-dimensional $\mathfrak{sl}(2)$ representations: The latter are direct sums of irreps whose weights are distributed uniformly and symmetrically. This implies that the multiplicities of weights must not decrease towards the centre

$$\dim \mathbb{V}_{\lambda+m\alpha_1/2} \geq \dim \mathbb{V}_{\lambda+n\alpha_1/2} \qquad \text{for } |m|<|n|,\ m-n\in 2\mathbb{Z}. \tag{6.36}$$

This equally applies to all $\mathfrak{sl}(2)$ subalgebras, and thus the density of weights should not decrease towards the centre of $\mathfrak{h}^*$, as can be observed in the sample weight diagram.

The highest weight μ_ρ now has up to five images under the Weyl group. However, these must be all smaller than μ_ρ itself. This implies that the highest weight must be in the highest of the 6 *Weyl chambers* for $\mathfrak{sl}(3)$ which are fundamental domains of $\mathfrak{h}^*$ under the action of the Weyl group.



(6.37)

Together with the fact that $\mu_\rho \in \Omega$ we can deduce that

$$\mu_\rho = n_1\omega_1 + n_2\omega_2 =: [n_1, n_2] \qquad \text{with} \qquad n_1, n_2 \in \mathbb{Z}_0^+. \tag{6.38}$$

A finite-dimensional irrep of $\mathfrak{sl}(3)$ is thus uniquely specified by a pair of non-negative integers $[n_1, n_2]$. The labels of the trivial, adjoint and fundamental representations read

$$\begin{aligned} &\rho_0 : [0,0], \qquad &&\rho_{\mathrm{ad}} : [1,1], \\ &\rho_{\mathbf{3}} : [1,0], \qquad &&\rho_{\mathbf{3}^*} : [0,1], \\ &\rho_{\mathbf{6}} : [2,0], \qquad &&\rho_{\mathbf{6}^*} : [0,2]. \end{aligned} \tag{6.39}$$

The sample representation in the above figures has labels $[1,2]$.

For a general rank-r algebra, a finite-dimensional irrep is specified by r non-negative integers $[n_1,\ldots,n_r]$, the so-called *Dynkin labels* of the representation, which are related to the highest weight μ as follows

$$\mu = \sum_{j=1}^{r} n_j\omega_j \qquad \text{with} \qquad n_j = \frac{2\langle\mu,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \in \mathbb{Z}_0^+. \tag{6.40}$$

Sub-Representations. We know so far that a finite-dimensional irrep is described by a number of weights which are arranged symmetrically w.r.t. the Weyl group around the origin of $\mathfrak{h}^*$. We also know that there is a unique highest weight μ and that the multiplicities can only increase towards the origin of $\mathfrak{h}^*$. In some cases, this may suffice to determine the structure of the representation, otherwise we must be more specific about the multiplicities.

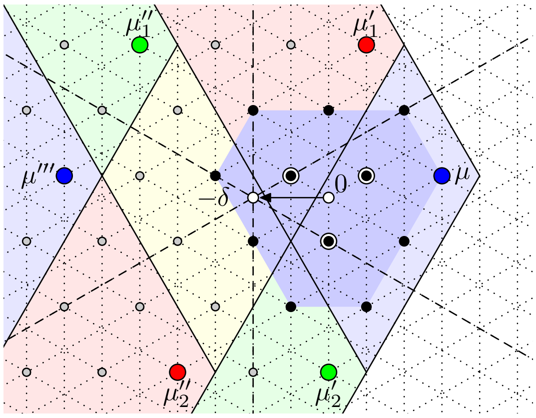
Here we can again make use of the $\mathfrak{sl}(2)$ subalgebras and their representation theory. Let us consider the k-th subalgebra generated by $\mathrm{E}_k, \mathrm{F}_k, \mathrm{H}_k$. The highest-weight state carries the charge $\mu(\mathrm{H}_k) = n_k$. We thus know that acting n_k+1 times with $\rho(\mathrm{F}_k)$ on the highest-weight state $|\mu\rangle$ yields a state

$$|\mu'_k\rangle = \rho(\mathrm{F}_k)^{n_k+1}|\mu\rangle \tag{6.41}$$

which is not within the finite-dimensional $\mathfrak{sl}(2)$ irrep. According to our above discussions, it must satisfy $\rho(\mathrm{E}_k)|\mu'_k\rangle = 0$. Furthermore, it trivially satisfies $\rho(\mathrm{E}_j)|\mu'_k\rangle = 0$ for all other $j \neq k$. Therefore, $|\mu'_k\rangle$ is a secondary highest-weight state, Moreover, there is one such secondary highest weight μ'_k for each of the r $\mathfrak{sl}(2)$ subalgebras.

The remaining complication is that the sub-representations can have further highest-weight states and sub-representations. In other words, we have to find all the would-be highest-weight states among the $|\mu;\ell\rangle$. Noting that the above map from $\mu \mapsto \mu'$ can be understood as a Weyl reflection, the full Weyl group W will be of help. The complete set of would-be highest weights μ' is given by the Weyl reflections acting on the original highest weight

$$\mu' \in \{\sigma(\mu+\delta) - \delta; \sigma \in \mathrm{W}\}. \tag{6.42}$$



$$(6.43)$$

More accurately, these Weyl reflections are not centred at the origin, but rather at the point $-\delta$ where $\delta \in \mathfrak{h}^*$ is given by half the sum of all positive roots

$$\delta := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{6.44}$$

After quotienting out all sub-representations, the remaining irrep turns out to be finite-dimensional.[7]

For the example $\mathfrak{sl}(3)$, there are 5 additional highest-weight states in a finite-dimensional highest-weight representation

$$\begin{aligned} \mu &= [n_1, n_2], \\ \mu_1' &= [-n_1 - 2, n_1 + n_2 + 1], \\ \mu_2' &= [n_1 + n_2 + 1, -n_2 - 2], \\ \mu_1'' &= [-n_1 - n_2 - 3, n_1], \\ \mu_2'' &= [n_2, -n_1 - n_2 - 3], \\ \mu''' &= [-n_2 - 2, -n_1 - 2], \end{aligned} \tag{6.45}$$

where $[k_1, k_2] := k_1\omega_1 + k_2\omega_2$. Observing which of the highest weights are within the representations of some other highest weights, cf. the figure above, we can establish the full highest-weight spaces $\mathbb{W}$ in terms of the irreducible components $\mathbb{V}$

$$\begin{aligned} \mathbb{W}''' &= \mathbb{V}''', \\ \mathbb{W}_j'' &= \mathbb{V}_j'' \oplus \mathbb{V}''', \\ \mathbb{W}_j' &= \mathbb{V}_j' \oplus \mathbb{V}_1'' \oplus \mathbb{V}_2'' \oplus \mathbb{V}''', \\ \mathbb{W} &= \mathbb{V} \oplus \mathbb{V}_1' \oplus \mathbb{V}_2' \oplus \mathbb{V}_1'' \oplus \mathbb{V}_2'' \oplus \mathbb{V}'''. \end{aligned} \tag{6.46}$$

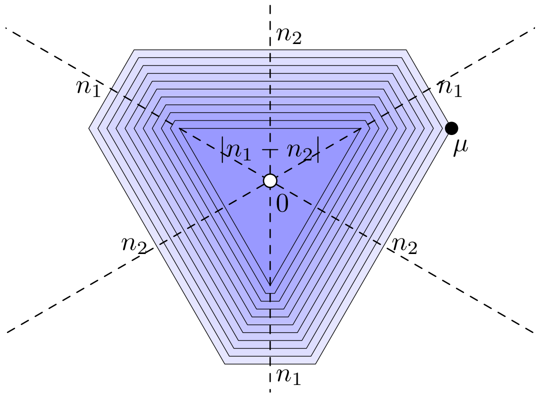
We are interested in the irreducible vector spaces, and therefore we have to

[7] One can also rephrase the integrality conditions for finite-dimensional irreps as follows: All images of the highest weight μ under the shifted Weyl reflections must be descendants of μ (so that their sub-representations can be quotiented out).

“invert” these relations:

$$\begin{aligned}\mathbb{V}''' &= \mathbb{W}''',\\ \mathbb{V}''_j &= \mathbb{W}''_j \ominus \mathbb{W}''',\\ \mathbb{V}'_j &= \mathbb{W}'_j \ominus \mathbb{W}''_1 \ominus \mathbb{W}''_2 \oplus \mathbb{W}''',\\ \mathbb{V} &= \mathbb{W} \ominus \mathbb{W}'_1 \ominus \mathbb{W}'_2 \oplus \mathbb{W}''_1 \oplus \mathbb{W}''_2 \ominus \mathbb{W}'''. \end{aligned} \tag{6.47}$$

Here the subtraction of vector spaces can be understood as a subtraction of multiplicities of corresponding weights. We obtain the following structure for the weights of a finite-dimensional $\mathfrak{sl}(3)$ representation with highest weight $[n_1, n_2]$ and their multiplicities:



(6.48)

- The weights are bounded by a hexagonal shape with triangular symmetry centred at the origin.
- The length of the sides alternates between n_1 and n_2.
- The multiplicity along the perimeter of the hexagon is 1.
- The multiplicity increases linearly by one unit per step towards the centre.
- The multiplicity reaches a plateau once the level set has degenerated from a hexagonal shape to a triangular shape. The maximum multiplicity is $\min(n_1, n_2) + 1$ on a triangle of side length $|n_1 - n_2|$.

Furthermore, we can compute the dimension of the representation with highest weight $\mu = [n_1, n_2]$: A direct computation based on the above structure yields the following results (we assume $n_1 \leq n_2$)

$$\begin{aligned}\dim \rho_\mu &= \sum_{j=0}^{n_1-1} 3(n_1 + n_2 - 2j)(j+1)\\ &\quad + \tfrac{1}{2}(n_1+1)(n_2-n_1+1)(n_2-n_1+2)\\ &= \tfrac{1}{2}(n_1+1)(n_2+1)(n_1+n_2+2). \end{aligned} \tag{6.49}$$

Here the first term accounts for the hexagonal perimeter while the second term accounts for the central triangular plateau. It is straightforward to confirm the dimensions of the representations encountered above

$$\begin{aligned} &\dim \rho_{[0,0]} = 1, \qquad \dim \rho_{[1,1]} = 8,\\ &\dim \rho_{[1,0]} = 3, \qquad \dim \rho_{[0,1]} = 3,\\ &\dim \rho_{[2,0]} = 6, \qquad \dim \rho_{[0,2]} = 6,\\ &\dim \rho_{[1,2]} = 15. \end{aligned} \tag{6.50}$$

As a side remark, one can also find similar expressions for the quadratic and cubic Casimir invariants evaluated on the finite-dimensional irreps[8]

$$\begin{aligned} C_2 &\sim n_1^2 + n_1 n_2 + n_2^2 + 3n_1 + 3n_2, \\ C_3 &\sim (n_1 - n_2)(n_1 + 2n_2 + 3)(n_2 + 2n_1 + 3). \end{aligned} \tag{6.51}$$

Character Polynomials. We can phrase our above results on the structure of the finite-dimensional irreps of $\mathfrak{sl}(3)$ in terms of character polynomials. Since we have two charges to describe states and generators, we introduce a pair of variables $q := (q_1, q_2)$ and compute the character

$$P_\rho(q) := \chi(q_1^{\mathrm{H}_1} q_2^{\mathrm{H}_2}) = \operatorname{tr} \rho(q_1^{\mathrm{H}_1} q_2^{\mathrm{H}_2}). \tag{6.52}$$

Within this polynomial a monomial mq^λ describes a state with weight λ and multiplicity m. Here we have introduced the notation

$$q^{[k_1,k_2]} := q_1^{k_1} q_2^{k_2} \qquad \text{where} \qquad [k_1, k_2] := k_1\omega_1 + k_2\omega_2. \tag{6.53}$$

As described above, the infinite-dimensional highest-weight representation space is spanned by states

$$\rho(\mathrm{F}_1)^{k_1} \rho(\mathrm{F}_2)^{k_2} \rho(\mathrm{F}_{12})^{k_{12}} |\mu\rangle \tag{6.54}$$

with one independent state for each combination of non-negative integers k_1, k_2, k_{12}. Using the properties of formal geometric series one can figure out that the character function for the highest-weight representation reads

$$\begin{aligned} P_\mu^{\mathrm{hw}}(q) &= \sum_{k_1,k_2,k_{12}=0}^{\infty} q^{\mu - k_1\alpha_1 - k_2\alpha_2 - k_{12}(\alpha_1+\alpha_2)} \\ &= \frac{q^\mu}{(1-q^{-\alpha_1})(1-q^{-\alpha_2})(1-q^{-\alpha_1-\alpha_2})}\,. \end{aligned} \tag{6.55}$$

After reducing to the finite-dimensional components we obtain

$$P_\mu(q) = \frac{q^\mu - q^{\mu_1'} - q^{\mu_2'} + q^{\mu_1''} + q^{\mu_2''} - q^{\mu'''}}{(1-q^{-\alpha_1})(1-q^{-\alpha_2})(1-q^{-\alpha_1-\alpha_2})}\,. \tag{6.56}$$

This formula agrees perfectly with the structure of the representation discussed above, and it serves as a formal expression for the weight diagram of finite-dimensional irreps of $\mathfrak{su}(3)$.

As usual, we can deduce the dimension of the representation as the characters of the unit element at $q_1 = q_2 = 1$. The character formula is singular at this point, but the rational function can be regularised to yield

$$\dim \rho_\mu = P_\mu(1) = \tfrac{1}{2}(n_1+1)(n_2+1)(n_1+n_2+2). \tag{6.57}$$

[8]These combinations are invariant under the Weyl reflections based at $-\delta$, whereas the above expression for the dimension flips the sign. In fact, this property singles them out.

The character and dimension formulas generalise to arbitrary simple Lie algebras. The character polynomial is given by the *Weyl character formula*[9]

$$P_\mu(q) = \frac{\sum_{\sigma\in \mathrm{W}} \operatorname{sign}(\sigma) q^{\sigma(\mu+\delta)-\delta}}{\prod_{\alpha\in\Delta_+}(1-q^{-\alpha})}\,. \tag{6.58}$$

Note that the Weyl group is a group of reflections and thus there is an associated $\mathbb{Z}_2$-grading to define $\operatorname{sign}(\sigma)$. The dimension of this representation follows by setting $q_j = 1$. The character polynomial formula is singular, and by properly taking the limit one obtains the *Weyl dimension formula*

$$\dim \rho_\mu = \prod_{\alpha\in\Delta_+} \frac{\langle \mu+\delta, \alpha\rangle}{\langle \delta,\alpha\rangle}\,. \tag{6.59}$$

6.5 Hadronic Physics

The group SU(3) plays an important role in hadronic physics and (in a different way) for the underlying quarks and the strong interactions among them. In fact, the example of the SU(3) flavour group in hadronic physics paved the way for Lie groups in physics.

Isospin. The lightest (bosonic and fermionic) hadronic particles are the pion triplet (π^-, π^0, π^+) of pseudo-scalar mesons and the nucleon doublet $N := (n^0, p^+)$ consisting of the proton p^+ and the neutron n^0. Although these particles all carry different electrical charges, they are also similar in their masses: The meson masses are around 135–140 MeV while the nucleon masses are around 938–940 MeV. This suggests that the mesons transform in a spin $I = 1$ irrep of some approximate SU(2) symmetry, while the nucleons transform in a spin $I = \frac{1}{2}$ irrep. Note that the former is a real representation, and indeed the anti-particles of the pion triplet are given by the pion triplet itself (with the charged pions interchanged). Conversely, the latter irrep is complex (or quaternionic) which means that the anti-particles of the nucleons are distinct particles: anti-proton and anti-neutron.

This SU(2) group is called the isospin symmetry. It is in fact part of the gauge group of the weak interactions which is spontaneously broken. For the purposes of the strong interaction the breaking effects are small and the masses of the particles within the multiplets are similar. The most evident difference between the particles in each multiplet is the electrical charge Q. After symmetry breaking, it can be expressed as a combination of the isospin component I_3 and the baryon number B

$$Q = I_3 + \tfrac{1}{2}B. \tag{6.60}$$

The nucleons are baryons and thus carry baryon number $B = 1$ while the mesons are uncharged, $B = 0$. Conservation of the baryon number charge is expressed by

[9] Note that the denominator equals the numerator at $\mu = 0$ in order to reproduce the appropriate character $P_0(q) = 1$ of the trivial representation.

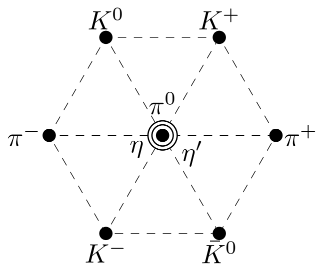
the stability of the proton. As the lightest charged particle, it cannot decay into anything else.[10]

Hadronic SU(3). Given more powerful particle physics experiments, new hadronic particles and resonances were discovered. They come along with an additional charge named strangeness. This charge is not exactly conserved, but strangeness-violating decays have a substantially longer decay time than those which preserve the charge. Moreover, processes which violate strangeness by more than one unit are even further suppressed. This fact was attributed to conservation of strangeness by the strong interactions whereas weak interactions can violate strangeness by one unit. The strong interactions take place on a time scale which is several orders of magnitude faster than the typical weak interaction time scale. The introduction of particles carrying strangeness necessitates to adapt the formula for the electrical charge

$$Q = I_3 + \tfrac{1}{2}Y, \qquad Y = B + S. \tag{6.61}$$

Here Y is called the hypercharge.

Now the various hadronic particles and resonances can be classified by their spin, parity, baryon number, isospin component I_3 and hypercharge Y. For each given set of quantum numbers, several hadrons with different masses and decay widths have been identified. Here we consider only the lightest few of these particles. Let us fix the spin, parity and baryon number and plot the lightest particles in a diagram with horizontal axis I_3 and vertical axis Y. For the pseudo-scalar mesons (spin 0, negative parity, baryon number 0) we obtain the following diagram:



π	137 MeV
K	496 MeV
η	548 MeV
η'	958 MeV

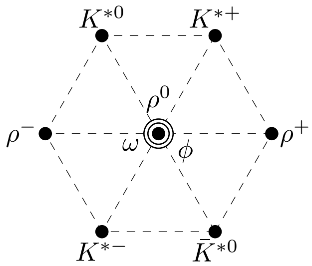
(6.62)

Note that the provided masses are approximate. One observes that the new particles, in this case the kaons K, also form SU(2) multiplets because their masses agree up to a few percent. The eta particle η and its heaver cousin η' form singlets of SU(2). Therefore isospin continues to be an approximate symmetry.[11] The new particles, however, have a substantially higher mass, and thus they are not related to the pions by some approximate symmetry.

[10] In the presence of anti-protons with $B = -1$, it could decay into lighter uncharged particles such as the pions. However, there are hardly any anti-protons around at our present location within the universe.

[11] However, it not longer coincides with the SU(2) group of the weak interactions which also acts on strangeness.

The next set of particles are the vector mesons (spin 1, positive parity, baryon number 0):

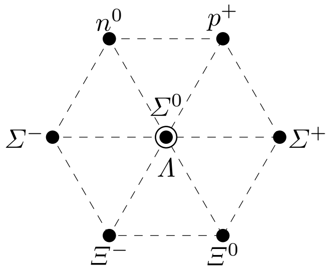


ρ	775 MeV
K^*	892 MeV
ω	782 MeV
ϕ	1019 MeV

(6.63)

Again isospin is a good approximate symmetry, and also 8 of the 9 particle masses are reasonably nearby so that they might be related by a broken symmetry. In any case, one observes precisely the same hexagonal pattern of isospin I_3 and hypercharge Y as for the pseudo-scalar mesons.

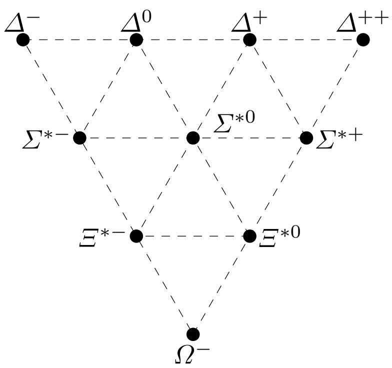
Finally, there are also generalisations of the nucleons with half-integer spin. There are 8 light baryons with spin $\frac{1}{2}$, positive parity and baryon number 1:



N	939 MeV
Λ	1116 MeV
Σ	1193 MeV
Ξ	1318 MeV

(6.64)

Furthermore, there are 10 light baryons with spin $\frac{3}{2}$, positive parity and baryon number 1:



Δ	1232 MeV
Σ	1385 MeV
Ξ	1533 MeV
Ω	1672 MeV

(6.65)

Again their structure is reminiscent of the mesons.

We observe an ordering pattern which we can clearly identify as the Lie group SU(3).[12] The above particles appear to form multiplets which follow the structure of SU(3) irreps. In particular we find octets ($\mathbf{8} = [1,1]$) sometimes joined by singlets ($\mathbf{1} = [0,0]$) as well as a decuplet ($\mathbf{10} = [3,0]$). This hadronic SU(3) is broken to the SU(2) hadronic isospin at an energy scale of several 100 MeV. In other words, for light particles this symmetry is hardly apparent, only its subgroup

[12] In reality this ordering pattern was not so easy to identify: The particles were identified one by one, masses were not accurately determined at first, the decay widths are substantial, several other particles with comparable energies exist and some quantum numbers are not easy to read out from collider experiment.

SU(2) is more manifest. Conversely, for heavy particles SU(3) becomes more of an approximate symmetry. However, at all energy levels the charge distribution is reminiscent of SU(3) (provided that the particles are grouped correctly).

Masses. The masses in the multiplets also follow a certain pattern which derives from the breaking of SU(3) symmetry to SU(2). We align the basis of $\mathfrak{sl}(3)$ such that the residual $\mathfrak{sl}(2)$ algebra is given by $\mathrm{E}_1, \mathrm{F}_1, \mathrm{H}_1$, i.e. it acts on the first two elements of the defining vectors space $\mathbb{C}^3$. The symmetry breaking can be achieved by an element in the adjoint representation, namely $\mathrm{diag}(1, 1, -2)$ which is the defining representation of $\mathrm{H}_1 + 2\mathrm{H}_2$. Incidentally this combination is proportional to the hypercharge

$$Y = \tfrac{1}{3}\mathrm{H}_1 + \tfrac{2}{3}\mathrm{H}_2. \tag{6.66}$$

All the symmetry breaking effects should be explained by using this algebra element.

Let us consider the baryon decuplet first, where the symmetry is most apparent. One finds that the mass decreases with the strangeness S with around 150 MeV for each unit of strangeness. This dependency can be expressed in a simple ansatz for the mass matrix M. In a situation where the symmetry breaking effect is small, the symmetry breaking element Y should appear only in first order. The mass matrix for particles transforming in a representation ρ can be expressed as

$$M = m_0 + m_1\rho(Y) + \ldots. \tag{6.67}$$

The two coefficients m_0 and m_1 determine the approximate mass structure in the multiplet. For particles transforming in the decuplet this is indeed the only linear term because the tensor product of a decuplet with an octet (Y) yields only a single decuplet

$$\mathbf{10} \otimes \mathbf{8} = \mathbf{35} \oplus \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{8}. \tag{6.68}$$

For the baryon octet one can admit one further linear term in the mass matrix because the decomposition of the tensor product of the octet (particles) with another octet (Y) yields two octets:

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}. \tag{6.69}$$

The elements of the mass matrix can be expressed in terms of the structure constants f_{abc} and the cubic Casimir coefficients d_{abc}

$$M_{ab} = m_0\delta_{ab} + m_1 f_{abc}Y^c + m_1' d_{abc}Y^c + \ldots. \tag{6.70}$$

These determine the 4 isospin multiplets in the octet in terms of 3 undetermined constants. Furthermore, the relationship is linear, and one can extract one constraint, the so-called Gell-Mann–Okubo (GMO) relation

$$2M_2 + 2M_{2'} = 3M_1 + M_3. \tag{6.71}$$

Here M_k denotes the mass of the isospin multiplet consisting of k states. Adding up both sides for the baryon octet yields the masses 4514 MeV and 4541 MeV which are indeed very close and which confirms the results of the approximate symmetry.

The situation for the vector mesons is more complicated: First of all this multiplet contains its own anti-particles. Therefore the mass matrix must be symmetric which excludes the term involving the structure constants f_{abc}. Secondly, there is a ninth vector meson which plays an important role. It is an SU(3) singlet, and normally one would not expect any influence from it. Here, however the singlet appears in the tensor product of the octet (particles) with another octet (Y), see above. This allows for mixing effects between the two multiplets. The resulting mass matrix reads (the index $a = 0$ denotes the singlet)

$$M_{ab} = m_0\delta_{ab} + m_0'\delta_{a0}\delta_{b0} + m_1' d_{abc}Y^c + m_1''(\delta_{a0}Y_b + \delta_{b0}Y_a) + \ldots . \tag{6.72}$$

Here we find 4 undetermined constants for 4 independent masses and therefore the symmetry allows almost arbitrary mass configurations. However, one might still consider the parameter values and find that they all have a natural magnitude.

Finally, there are the pseudo-scalar mesons. Here the mixing of the octet and the ninth state η' is not important, and one might expect the GMO relation to hold. However, this yields a bad approximation. Instead one finds better GMO relation for the squares of the masses

$$2M_2^2 + 2M_{2'}^2 = 3M_1^2 + M_3^2. \tag{6.73}$$

The numbers on both sides of the equation read $(959\,\mathrm{MeV})^2$ and $(992\,\mathrm{MeV})^2$ which are not too far apart. In fact, there are several good reasons why the squared masses appear in the relation:

- The Lagrangian for scalar fields contains the squared masses rather than the masses, and thus it is natural to expand the matrix of squared masses.
- The pseudo-scalar multiplet is the lightest of all multiplets. It is well below the symmetry breaking scale and non-linear effects become important. Above the symmetry breaking scale, linearisation is typically a good approximation.
- These particles can be viewed as the Goldstone bosons for the breaking of our (approximate) symmetry. As such, they require a special treatment.

The latter point also explains to some extent why there is little mixing with the ninth singlet particle η'. Namely the U(1) symmetry for which it serves as the Goldstone boson is broken more violently and by different means than the SU(3). This implies the rather large mass observed for the η' and suppressed mixing with η.

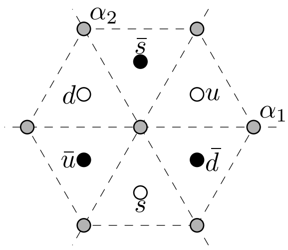
Strong SU(3). Just three types of hadron multiplets were observed in nature (**1**, **8** and **10**) while there are many more finite-dimensional irreps in SU(3). One may wonder why this is the case. An obvious guess is that the hadrons are composite particles made as bound states of some more elementary particles, the so-called *quarks*. Assuming that there are three reasonably light types of quarks which transform in the defining representation (**3**) of some SU(3) and its conjugate

($\mathbf{3}^*$) one can accommodate for all the meson and baryon multiplets by the two tensor products

$$\begin{aligned}\mathbf{3}\otimes\mathbf{3}^* &= \mathbf{8}\oplus\mathbf{1},\\ \mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} &= \mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1}.\end{aligned} \tag{6.74}$$

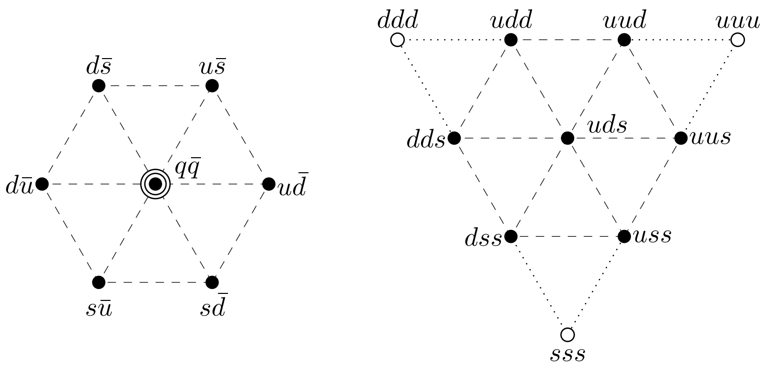
The three light quark flavours are called up, down and strange

$$q = u, d, s. \tag{6.75}$$



The particle content of the mesons and baryons is given in the following weight diagrams:

(6.76)



Moreover the arising hadron spins are explained by declaring the quark to be spin-$\frac{1}{2}$ particles. Then the tensor products of the SU(2) representations naturally yield the observed meson and baryon spins

$$\begin{aligned}\tfrac{1}{2}\otimes\tfrac{1}{2} &= 1\oplus 0,\\ \tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2} &= \tfrac{3}{2}\oplus\tfrac{1}{2}\oplus\tfrac{1}{2}.\end{aligned} \tag{6.77}$$

In this model, the masses of the up and down quarks are approximately the same while the strange quark is substantially heavier. This means that the quark mass matrix is a linear combination of the identity matrix and of $\rho_{\mathrm{def}}(Y)$. Therefore one can view the symmetry breaking to be induced by the quark mass matrix.

However, there is one surprise: On the one hand, the decuplet wave function is both symmetric in the quark flavours and in the spin degrees of freedom. In fact, also the baryon octet wave function is totally symmetric under interchange of the quarks

$$(\mathbf{3}, \tfrac{1}{2})^{\odot 3} = (\mathbf{10}, \tfrac{3}{2})\oplus(\mathbf{8}, \tfrac{1}{2}). \tag{6.78}$$

On the other hand, the quarks are spin-$\frac{1}{2}$ particles and the spin-statistics theorem demands that their wave function is totally anti-symmetric.

The resolution to this mismatch is to introduce additional degrees of freedom for the quarks. In addition to flavour and spin, a quark also carries a colour degree of freedom, which can take one of three values (red, green, blue). The anti-quarks carry the conjugate colours. The colour degrees of freedom transform under a new SU(3) symmetry, which is the gauge symmetry of the strong interaction alias quantum chromodynamics. This symmetry confines, and thus only singlets can be observed at sufficiently low energies. For the baryons, the colour degrees of freedom transform in the triple tensor product of the defining representation

$$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{10}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{1},\qquad \mathbf{3}^{\wedge 3}=\mathbf{1}. \tag{6.79}$$

This contains one singlet which is in fact totally anti-symmetric. Taking these degrees of freedom into account, the baryon wave function is totally anti-symmetric as it should. Also the tensor product for mesons contains a singlet

$$\mathbf{3}\otimes\mathbf{3}^*=\mathbf{8}\oplus\mathbf{1}, \tag{6.80}$$

so mesons can exist, but there are no further constraints on the particle content. Finally, the additional degrees of freedom account for some factors of three required in places to match theory with experiment, and all is well.

7 Representations of SU(N)

Finally we will consider the representation theory of the unitary group SU(N) of general rank $N-1$ and its complexified Lie algebra $\mathfrak{sl}(N)$. We will introduce a description of irreps in terms of certain diagrams.

7.1 Tensor Powers and Permutations

The complete finite representation theory for the algebras $\mathfrak{sl}(N)$ can be obtained by repeated tensor products of the defining representation. We therefore consider the m-th tensor power

$$\mathbb{V}^m := (\mathbb{V}_{\text{def}})^{\otimes m}, \qquad \rho^m := \sum_{j=1}^{m} \rho_{\text{def},j}, \tag{7.1}$$

where $\rho_{\text{def},j}$ denotes the defining representation ρ_{def} acting on the j-th factor in $(\mathbb{V}_{\text{def}})^{\otimes m}$ (and by the identity on the remaining factors).

The definition of the tensor power representation is manifestly symmetric under any permutation of the tensor factors

$$\rho(\pi)\rho^m = \sum_{j=1}^{m} \rho_{\text{def},\pi(j)}\rho(\pi) = \rho^m \rho(\pi). \tag{7.2}$$

Here $\pi \in \mathrm{S}_m$ is a permutation of a set of m elements and $\rho(\pi) \in \mathrm{Aut}(\mathbb{V}^m)$ is its representation on the tensor power space.[1] This means that $\rho(\pi)$ is a *module endomorphism*[2] of $\mathbb{V}^m$ for all permutations $\pi \in \mathrm{S}_m$. As there is more than one permutation, Schur's lemma (which does apply to finite-dimensional representations of Lie algebras) tells us that the tensor power representation is reducible.

By taking linear combinations of the permutations, we can construct projectors to certain symmetric components of the tensor power space. For instance, the available projectors for two tensor factors are the symmetriser and the anti-symmetriser

$$\tfrac{1}{2}\big(\rho(1) \pm \rho(\sigma)\big). \tag{7.3}$$

These are module endomorphism of ρ^m as well, but they map the symmetric and anti-symmetric subspaces $\mathbb{V}^{\vee 2}$ and $\mathbb{V}^{\wedge 2}$ of $\mathbb{V}^m$ to themselves. This implies that

[1] As before, one might abbreviate the representation of π on $\mathbb{V}^m$ by π itself because it does not lead to ambiguities.

[2] A module homomorphism from the vector space to itself.

$\rho^2 \equiv \rho^{\vee 2} \oplus \rho^{\wedge 2}$ decomposes into a symmetric and an anti-symmetric sub-representation.

A characteristic feature of the defining representation of $\mathfrak{sl}(N)$ is that the sub-representations obtained by complete symmetrisations of the tensor power are in fact all irreducible. In other words, the representation theory of the symmetric group S_m, determines the irreps in the m-th tensor power ρ^m of ρ_{def} in $\mathfrak{sl}(N)$.

7.2 Orthogonal Symmetrisers

As the projectors are linear combinations of representations of group elements, it makes sense phrase them in terms of the group algebra $\mathbb{C}[\mathrm{S}_m]$. For instance, the above projectors on ρ^2 can be written more generally as

$$\tfrac{1}{2}\big(e_1 \pm e_\sigma\big) \in \mathbb{C}[\mathrm{S}_2]. \tag{7.4}$$

Moreover, the two elements form a basis of $\mathbb{C}[\mathrm{S}_2]$.

We can easily construct a set of projectors for m tensor factors given some irrep ρ_μ of S_m

$$P^{m\mu}_{ab} := \frac{\dim \rho_\mu}{m!} \sum_{\pi \in \mathrm{S}_m} \langle a|\rho_\mu(\pi)|b\rangle \, e_\pi. \tag{7.5}$$

Here $|b\rangle$ and $\langle a|$ are some states in the representation space of ρ_μ and its dual and they are used to extract a matrix element of ρ_μ.

Using the orthogonality relations of S_m we can show

$$\begin{aligned}
&P^{m\mu}_{ab} P^{m\nu}_{cd} \\
&= \frac{(\dim \rho_\mu)(\dim \rho_\nu)}{m!^2} \sum_{\pi_1,\pi_2 \in \mathrm{S}_m} \langle a|\rho_\mu(\pi_1)|b\rangle \langle c|\rho_\nu(\pi_2)|d\rangle \, e_{\pi_1\pi_2} \\
&= \frac{(\dim \rho_\mu)(\dim \rho_\nu)}{m!^2} \sum_{\pi_1,\pi \in \mathrm{S}_m} \langle a|\rho_\mu(\pi_1)|b\rangle \langle c|\rho_\nu(\pi_1^{-1})\rho_\nu(\pi)|d\rangle \, e_\pi \\
&= \delta_{\mu\nu}\delta_{bc} \frac{\dim \rho_\nu}{m!} \sum_{\pi \in \mathrm{S}_m} \langle a|\rho_\nu(\pi)|d\rangle \, e_\pi \\
&= \delta_{\mu\nu}\delta_{bc} P^{m\mu}_{ad}.
\end{aligned} \tag{7.6}$$

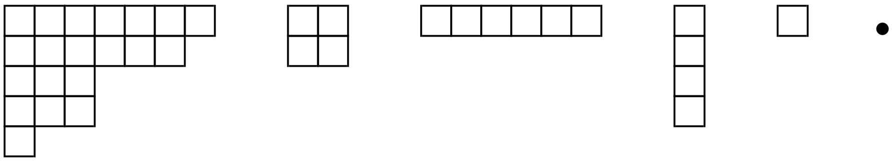
This orthogonality relation implies that the projectors $P^{m\mu}_{ab}$ are independent. Moreover there are $\sum_\mu (\dim \rho_\mu)^2 = m!$ such projectors which therefore form a basis of $\mathbb{C}[\mathrm{S}_m]$. Consequently, for each irrep ρ_μ of S_m, there are $\dim \rho_\mu$ diagonal projectors $\rho(P^{m\mu}_{aa})$ acting on the space $\mathbb{V}^m$. Provided that these do not map the whole of $\mathbb{V}^m$ to zero, one obtains an irreducible representation of $\mathfrak{sl}(N)$

$$\rho^m_{\mu,a} := \rho(P^{m\mu}_{aa})\rho^m. \tag{7.7}$$

7.3 Young Tableaux

There is a useful alternative to enumerate the representations and a basis for their representation spaces in terms of diagrams.

Young Diagrams and Standard Fillings. Let us first introduce the diagrams: Consider a configuration of m boxes arranged into rows and columns. All the rows should be filled starting from a straight line on the left. Likewise, all the columns should be filled from a straight line on the top. In other words, the lower-left boundary of the collection of boxes takes the form of a staircase.[3] Such a diagram is called a *Young diagram*. For example, the following are Young diagrams:[4]

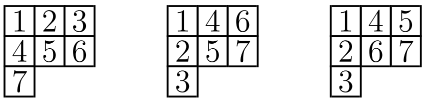


(7.8)

We furthermore introduce a filling of the Young diagram by some numbers called a *Young tableau*. Many of the relevant Young tableaux respect a certain ordering (increasing or decreasing) among the rows as well as among the columns. The ordering is typically strict among the columns whereas the rows may have identical entries. A *standard filling* is a filling with the following three properties:

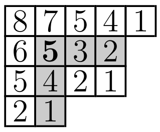
- All numbers $1, \ldots, m$ appear once.
- The numbers in each row are increasing from left to right.
- The numbers in each column are increasing from top to bottom.

For example, the following are standard fillings of a given Young diagram:



(7.9)

There exists a simple algorithm for computing the number $|\mu|$ of standard fillings for a given Young diagram μ: Fill all boxes with the hook length of the box. The *hook length* of a box is defined as the number of boxes in the hook based at the former box



(7.10)

In other words, the hook length equals 1 plus the number of boxes directly below and directly to the right of a given box. The number of standard fillings then equals

$$|\mu| = \frac{m!}{\prod\limits_{j\in\text{boxes}(\mu)} \text{hook-length}(j,\mu)}\,. \tag{7.11}$$

[3]... defying the laws of gravity.

[4]The empty diagram might be denoted by a dot.

For example, for a particular Young diagram of 5 boxes we obtain the following 5 standard fillings:

$$\mu = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \qquad |\mu| = \frac{5!}{\begin{smallmatrix} 4 & 2 \\ 3 & 1 \\ 1 & \end{smallmatrix}} = 5 : \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\; \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\; \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\; \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \tag{7.12}$$

Young Symmetriser. For each standard filling $a = 1, \ldots, |\mu|$ of a Young diagram μ we can construct a *Young symmetriser* as the product

$$Y_{\mu,a} := \frac{|\mu|}{m!}\, S_{\mu,a} A_{\mu,a} \in \mathbb{C}[\mathrm{S}_m] \tag{7.13}$$

with the total symmetriser of rows $S_{\mu,a}$ and total anti-symmetriser of columns $A_{\mu,a}$

$$\begin{aligned} S_{\mu,a} &:= \prod_{j \in \mathrm{rows}(\mu,a)} \sum_{\pi \in \mathrm{S}(j)} e_\pi, \\ A_{\mu,a} &:= \prod_{j \in \mathrm{columns}(\mu,a)} \sum_{\pi \in \mathrm{S}(j)} \mathrm{sign}(\pi)\, e_\pi. \end{aligned} \tag{7.14}$$

Here $\mathrm{S}(j) \subset \mathrm{S}_m$ denotes the subgroup of permutations acting on the subset defined by a row or column j. The Young symmetrisers have the following properties (without proof):

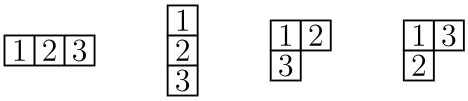
- Every $Y_{\mu,a}$ is a projector,
$$Y_{\mu,a} Y_{\mu,a} = Y_{\mu,a}. \tag{7.15}$$
- The product of symmetrisers for different Young diagrams $\mu \neq \mu'$ is zero,
$$Y_{\mu,a} Y_{\mu',a'} = 0 \qquad \text{if} \qquad \mu \neq \mu'. \tag{7.16}$$
Note that the product of symmetrisers for different fillings a, a' of the same Young diagram μ is not necessarily zero, although for many examples of small Young tableaux this is the case.[5]
- The image of each symmetriser is the space of an irreducible sub-representation of the regular representation ρ_{reg} acting on the group algebra $\mathbb{C}[\mathrm{S}_m]$. Two of these sub-representations are equivalent if and only if the Young diagrams match.
- The group algebra $\mathbb{C}[\mathrm{S}_m]$ is the direct sum of all images of the symmetrisers
$$\mathbb{C}[\mathrm{S}_m] = \bigoplus_{\mu,a} Y_{\mu,a} \mathbb{C}[\mathrm{S}_m]. \tag{7.17}$$

The correspondence between the diagrams and the representation theory of S_m is therefore as follows:

[5]The minimal example of a non-zero product is between the fillings $1,2,3/4,5$ and $1,3,5/2,4$ for $m = 5$ boxes.

- The Young diagrams are in one-to-one correspondence to the irreps of S_m.[6]
- The number of standard fillings equals the dimension of the corresponding representation. One might use the standard fillings to label a basis for the representation space.

For example, for $m = 3$ there are four standard Young tableaux:



(7.18)

The former two correspond to the trivial and determinant representation. The latter two correspond to the two-dimensional representation (which appears twice in the regular representation). Correspondingly, there are four irreducible representations in the tensor product of three defining representations of $\mathfrak{sl}(N)$ (for $N \geq 3$).

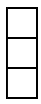
7.4 Young Tableaux for $\mathfrak{sl}(N)$

Above, we have established a correspondence between the irreps of S_m and those of $\mathfrak{sl}(N)$. This is used to label the irreps of $\mathfrak{sl}(N)$ by Young diagrams. Based on these, there are simple algorithms to compute the irrep dimensions and to perform other tasks.

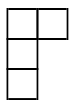
Correspondence. Let us first of all establish the correspondence for the most basic representations: By construction, a single box corresponds to the defining representation

$\rho_{\mathrm{def}} \equiv$  . (7.19)

A vertical stack of $N-1$ boxes corresponds to total anti-symmetrisation of $N-1$ defining representations. This yields the N-dimension dual defining representation, e.g. for $N = 4$

$\rho^*_{\mathrm{def}} \equiv$  . (7.20)

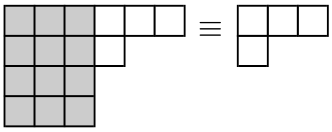
Finally, the adjoint representation is given by a hook of $N-1$ in the first column and 1 box in the second

$\rho_{\mathrm{ad}} \equiv$  . (7.21)

Next we have to limit the set of applicable Young diagrams. It is obvious that a Young diagram with a column of k boxes yields a symmetriser which involves an anti-symmetrisation of k tensor factors. As there are only N linearly independent

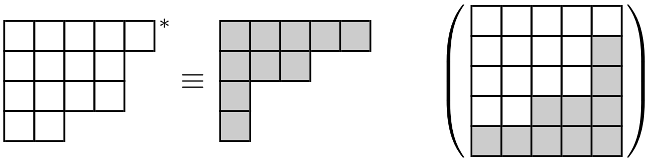
[6]Curiously, the equivalence classes of S_m can be enumerated by the same diagrams. An equivalence class is specified by its cycle structure. Ordering the cycles by their lengths, we associate to it a row of boxes of the same length and stacks these rows into Young diagram. At least this shows that the number of Young diagrams agrees with the number of irreps of S_m.

states in the defining representation space $\mathbb{C}^N$ one cannot anti-symmetrise more than N tensor factors. The height of the Young diagram should therefore be bounded by N. Moreover, anti-symmetrisation of N boxes projects the corresponding tensor factors to a one-dimensional space on which only the trivial representation can act. A Young diagram with some columns of maximum height corresponds to the same irrep as the Young diagram where these columns are eliminated, e.g. for maximum height $N = 4$:



(7.22)

Finally, the dual representation[7] corresponds to a dual Young diagram which is obtained by cutting out the original Young diagram from a rectangular block of maximum height N, e.g. for $N = 5$:



(7.23)

Note that some irreps, e.g. the adjoint, are self-dual. In the real algebra $\mathfrak{su}(N)$ they are thus real or quaternionic while all other irreps are complex.

Dimension. There is a simple algorithm for the dimension of the $\mathfrak{sl}(N)$ irreps given by the Young diagram. Here we fill the number N in the upper left corner of the diagram. The rows are filled with numbers increasing by 1 per box, and the columns are filled with numbers decreasing by 1 per box, for example for $N = 4$:

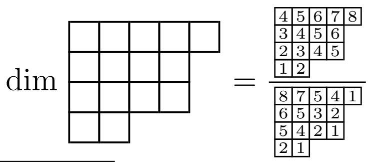
4	5	6	7	8
3	4	5	6	
2	3	4	5	
1	2			

(7.24)

The product of these numbers divided by the product of all hook lengths equals the dimension of the $\mathfrak{sl}(N)$ irrep:

$$\dim \rho = \frac{\displaystyle\prod_{j\in\text{boxes}(\mu)} \big(N + \text{column}(j) - \text{row}(j)\big)}{\displaystyle\prod_{j\in\text{boxes}(\mu)} \text{hook-length}(j,\mu)}. \tag{7.25}$$

For example, for $N = 4$ one obtains



$= 36.$ (7.26)

[7]The dual representation is the complex conjugate representation for $\mathfrak{su}(N)$.

Dynkin Labels. Finally, we want to determine the highest weight of the irrep corresponding to a Young diagram. We thus fill the Young diagram with numbers $k = 1, \ldots, N$ corresponding to the states $|k\rangle \in \mathbb{C}^N$. Note that the numbers in each column should all be distinct such that the state is not automatically destroyed by anti-symmetrisation. In order to maximise the weight[8] there should be as many 1's as possible. The remaining boxes should be filled with as many 2's as possible, and so on. Consequently, the k-th row should be filled with all equal numbers k:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 \\ \cline{1-2} \end{array} \tag{7.27}$$

The highest-weight state can be described schematically as a Young symmetriser acting on

$$|\mu\rangle \simeq \bigotimes_{k=1}^{N} |k\rangle^{\otimes l_k}, \qquad \mu = \sum_{k=1}^{N} l_k \lambda_k, \tag{7.28}$$

where l_k denotes the number of boxes on the k-th row. The weight of a state $|k\rangle$ expressed in the simple roots α_j, $j = 1, \ldots, r$, $r = N - 1$ is given by

$$\lambda_k = -\sum_{j=1}^{r} \frac{j}{N} \alpha_j + \sum_{j=k}^{r} \alpha_j. \tag{7.29}$$

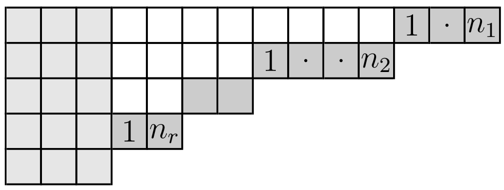
and the scalar product with the simple roots by

$$\langle \lambda_k, \alpha_j \rangle = \delta_{k,j} - \delta_{k-1,j}. \tag{7.30}$$

Therefore the Dynkin labels $[n_1, \ldots, n_r]$, of the highest-weight representation read

$$n_j = \langle \mu, \alpha_j \rangle = \sum_{k=1}^{N} l_k (\delta_{k,j} - \delta_{k-1,j}) = l_j - l_{j+1}. \tag{7.31}$$

We thus have a correspondence between the Dynkin labels and Young diagrams

$[n_1, \ldots, n_r] \equiv$  (7.32)

or alternatively

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & l_1 \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & l_2 \\ \cline{1-6} 1 & \cdot & \cdot & \cdot \\ \cline{1-4} 1 & l_N \\ \cline{1-2} \end{array} \equiv [l_1 - l_2, \ldots, l_{N-1} - l_N]. \tag{7.33}$$

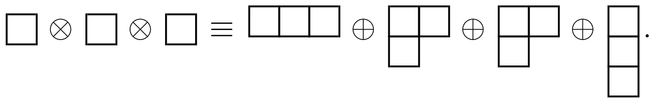
For example, one can now show that the two dimension formulas yield coincident number, but the general proof requires some non-trivial combinatorics.

[8] As before we assume the canonical ordering where the positive and negative generators of $\mathfrak{sl}(N)$ correspond to upper and lower triangular matrices.

Tensor Products. The formalism of Young diagrams for $\mathfrak{sl}(N)$ arose from the consideration of the m-fold tensor product of the defining representation. The above results phrased in terms of Young diagrams provide the decomposition

$$(\rho_{\text{def}})^{\otimes m} \equiv \bigoplus_{\mu \text{ with } m \text{ boxes}} (\rho_\mu)^{\oplus|\mu|}. \tag{7.34}$$

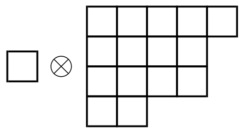
Note that the multiplicity of the irrep labelled by μ is given by the number $|\mu|$ of standard fillings of the Young tableau μ. For example



(7.35)

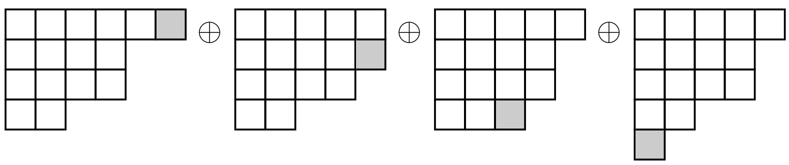
Note that this rule applies to all $\mathfrak{sl}(N)$, even for $\mathfrak{sl}(2)$ where it reads $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \equiv \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$. The final term must be left out because the total anti-symmetrisation of three two-dimensional spaces does not exist.

The formalism of Young diagrams is also very useful for decomposing tensor products of two arbitrary irreps. In particular, there is a simple algorithm for computing the tensor product of an irrep ρ with the defining representation:



(7.36)

The resulting diagrams should be a sum of diagrams with one additional box. All of these Young diagrams are obtained by adding a single box to the diagram of ρ. Since the resulting terms should be Young diagrams as well, one has to make sure to add the boxes only in certain allowed places, i.e. where the Young diagrams has a concave corner:



(7.37)

The tensor product decomposition is simply the sum of all permissible additions of boxes to the original diagram. There are analogous but more involved rules for computing the tensor product decomposition for two generic Young diagrams. Also for other tasks in group theory the Young tableaux can come in handy.

8 Classification of Simple Lie Algebras

In this chapter we want to classify all finite-dimensional simple Lie algebras and derive some of their properties and applications. We will again start with complex Lie algebras and later consider their various real forms.

8.1 Classification of Complex Algebras

We start with explaining how the classification of simple Lie algebras works and list some relevant features.

Chevalley–Serre Relations. We have already identified a useful set of generators of a complex simple Lie algebra $\mathfrak{g}$: These are the simple generators $(\mathrm{E}_j, \mathrm{F}_j, \mathrm{H}_j)$, $j = 1, \ldots, r$ along with non-simple positive and negative generators obtained as Lie brackets among the E_j and among the F_j, respectively. Their Lie brackets are reminiscent of r copies of the algebra $\mathfrak{sl}(2)$ with a suitable set of interactions between the factors

$$
\begin{aligned}
[\![\mathrm{H}_j, \mathrm{H}_k]\!] &= 0, & [\![\mathrm{E}_j, \mathrm{F}_k]\!] &= \delta_{jk}\mathrm{H}_k,\\
[\![\mathrm{H}_j, \mathrm{E}_k]\!] &= A_{jk}\mathrm{E}_k, & [\![\mathrm{H}_j, \mathrm{F}_k]\!] &= -A_{jk}\mathrm{F}_k.
\end{aligned}
\tag{8.1}
$$

The set of non-simple generators is constrained by the Serre relations ($j \neq k$)

$$
\operatorname{ad}(\mathrm{E}_j)^{1-A_{jk}}\mathrm{E}_k = 0, \qquad \operatorname{ad}(\mathrm{F}_j)^{1-A_{jk}}\mathrm{F}_k = 0.
\tag{8.2}
$$

The Cartan matrix A_{jk} is related to the roots as follows:

$$
A_{jk} := \alpha_k(\mathrm{H}_j) = \frac{2\langle\alpha_j, \alpha_k\rangle}{\langle\alpha_j, \alpha_j\rangle}\,.
\tag{8.3}
$$

In particular this implies that all diagonal elements equal 2,

$$
A_{kk} = 2.
\tag{8.4}
$$

The interactions between the $\mathfrak{sl}(2)$ factors can be understood as follows: The above relations for $j \neq k$ can be written as

$$
\begin{aligned}
\operatorname{ad}(\mathrm{H}_j)\mathrm{E}_k &= A_{jk}\mathrm{E}_k, & \operatorname{ad}(\mathrm{H}_j)\mathrm{F}_k &= -A_{jk}\mathrm{F}_k,\\
\operatorname{ad}(\mathrm{F}_j)\mathrm{E}_k &= 0, & \operatorname{ad}(\mathrm{E}_j)\mathrm{F}_k &= 0,\\
\operatorname{ad}(\mathrm{E}_j)^{1-A_{jk}}\mathrm{E}_k &= 0, & \operatorname{ad}(\mathrm{F}_j)^{1-A_{jk}}\mathrm{F}_k &= 0.
\end{aligned}
\tag{8.5}
$$

They express that E_k is the lowest-weight state under a representation of the $\mathfrak{sl}(2)$ spanned by $(\mathrm{E}_j, \mathrm{F}_j, \mathrm{H}_j)$; likewise F_k is the highest-weight state of an equivalent representation. From the constraints on the Cartan charges for a finite-dimensional representation of $\mathfrak{sl}(2)$ we know that A_{jk} must be a non-positive integer

$$A_{jk} \in \mathbb{Z}_0^-. \tag{8.6}$$

This ensures that $\mathrm{ad}(\mathrm{E}_j)^{-A_{jk}}\mathrm{E}_k$ is a highest-weight state within a lowest-weight representation and thus $\mathrm{ad}(\mathrm{E}_j)^{1-A_{jk}}\mathrm{E}_k$ must be zero; likewise for F_k.

For a finite-dimensional algebra, the scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*$ must be positive definite and likewise the Cartan matrix must be positive definite. One immediate corollary of the definition of A_{jk} is that

$$A_{jk} = 0 \quad \Longrightarrow \quad A_{kj} = 0. \tag{8.7}$$

Another corollary is that

$$A_{jk}A_{kj} = 4\cos^2\psi_{jk} < 4, \tag{8.8}$$

where ψ_{jk} denotes the angle between α_j and α_k as measured by the definite scalar product $\langle\cdot,\cdot\rangle$. Finally, by construction of A there is an invertible diagonal matrix D (given by $D_{jj} = \langle\alpha_j, \alpha_j\rangle$) such that DA is symmetric.

Putting these constraints together, every pair of off-diagonal elements of the Cartan matrix can only take one of 4 combinations

$$\{A_{jk}, A_{kj}\} \in \big\{\{0,0\}, \{-1,-1\}, \{-1,-2\}, \{-1,-3\}\big\}. \tag{8.9}$$

Dynkin Diagrams. Now that we know the properties of Cartan matrices for finite-dimensional simple Lie algebras, we can turn the logic around, and construct the latter algebras from scratch. One can show that the above algebra relations, where A_{jk} is a suitable Cartan matrix, describe a finite-dimensional semi-simple Lie algebra. This reduces the classification of finite-dimensional simple Lie algebras to the classification of matrices with the properties derived above.

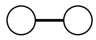
There is a useful representation of Cartan matrices in terms of so-called *Dynkin diagrams*:

- For each diagonal element $A_{jj} = 2$ of the Cartan matrix draw a white dot.[1] In total there are r dots where r is the rank of the Lie algebra.
- Two dots $j \neq k$ are connected by $0 \leq \max(|A_{jk}|, |A_{kj}|) \leq 3$ lines. In particular, two dots are not connected if $A_{jk} = A_{kj} = 0$.
- When $|A_{jk}| > |A_{kj}|$ draw an arrow head from dot k to dot j on top of the lines. No arrow is drawn if $A_{jk} = A_{kj} = 0$.[2]

[1]One may also associate the dots with the simple roots α_j.

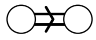
[2]One can also view the arrow from a long to a short root as an inequality sign between the lengths of the corresponding simple roots α_j.

For example, the Cartan matrix for $\mathfrak{sl}(3)$ is described by two dots connected by a line

$$A_{jk} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \equiv$$  . (8.10)

A useful feature of the Dynkin diagrams is that the ordering of simple roots plays no particular role, and one can identify a Lie algebra by a brief look at a diagram instead of the detailed table of matrix elements (most of which are typically 0).

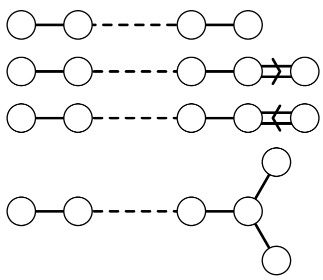
Dynkin diagrams can be characterised by two properties: A connected Dynkin diagram corresponds to a simple Lie algebra; a disconnected diagram then evidently corresponds to a semi-simple Lie algebra where each connection component corresponds to a simple factor. Here we shall mostly consider connected diagrams. Furthermore, a Lie algebra whose Dynkin diagram has only single lines is called *simply laced.* Here all roots have the same norm $\langle\alpha_j, \alpha_j\rangle$. An example of a simply laced Lie algebra is the above $\mathfrak{sl}(3)$. Conversely, a Lie algebra whose Dynkin diagram has multiple lines with arrows is called *non-simply laced.* Here the roots have different lengths. An example of a non-simply laced Lie algebra is $\mathfrak{so}(5)$ whose Cartan matrix is described by the Dynkin diagram

$$A_{jk} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \equiv$$  . (8.11)

Here, the long root α_1 corresponds to the dot on the left and the short root α_2 corresponds to the dot on the right.

Classical Lie Algebras. A complete enumeration of Dynkin diagrams for finite-dimensional Lie algebras yields four infinite families called $\mathfrak{a}_r$, $\mathfrak{b}_r$, $\mathfrak{c}_r$, $\mathfrak{d}_r$:

$\mathfrak{a}_r$: $\mathfrak{sl}(r+1)$
$\mathfrak{b}_r$: $\mathfrak{so}(2r+1)$
$\mathfrak{c}_r$: $\mathfrak{sp}(2r)$
$\mathfrak{d}_r$: $\mathfrak{so}(2r)$ (8.12)



These correspond to the three families of *classical Lie algebras* $\mathfrak{sl}/\mathfrak{su}$, $\mathfrak{so}$ and $\mathfrak{sp}$.[3] Here r describes the rank of the Lie algebras; their dimensions are given by

$$\begin{aligned} \dim\mathfrak{sl}(n) &= (n-1)(n+1), & \dim\mathfrak{a}_r &= r(r+2), \\ \dim\mathfrak{so}(n) &= \tfrac{1}{2}n(n-1), & \dim\mathfrak{d}_r &= r(2r-1), \\ & & \dim\mathfrak{b}_r &= r(2r+1), \\ \dim\mathfrak{sp}(2n) &= n(2n+1), & \dim\mathfrak{c}_r &= r(2r+1). \end{aligned} \tag{8.13}$$

[3]Note that there is a distinction between the orthogonal algebras for even and odd dimension. This can be related to the fact that one of the eigenvalues of odd-dimensional rotational matrices is always 1 while there is no such restriction for the even-dimensional ones.

Note that $\mathfrak{a}_r$ and $\mathfrak{d}_r$ are simply laced while $\mathfrak{b}_r$ and $\mathfrak{c}_r$ are non-simply laced.

The classical Lie algebras correspond to the matrix algebras

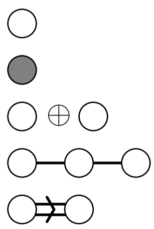
$$\begin{aligned}
\mathfrak{sl}(n) &= \bigl\{A \in \mathrm{End}(\mathbb{C}^n); \operatorname{tr} A = 0\bigr\},\\
\mathfrak{so}(n) &= \bigl\{A \in \mathrm{End}(\mathbb{C}^n); A + H^{-1}A^{\mathsf{T}}H = 0\bigr\},\\
\mathfrak{sp}(2n) &= \bigl\{A \in \mathrm{End}(\mathbb{C}^{2n}); A + E^{-1}A^{\mathsf{T}}E = 0\bigr\},
\end{aligned} \tag{8.14}$$

where H and E are invertible symmetric and anti-symmetric matrices, respectively. The concrete choice of H and E leads to equivalent algebras, and in a notation where each block corresponds to an $n \times n$ matrix the canonical choice is

$$H = \begin{pmatrix} 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}. \tag{8.15}$$

Small Rank. For small rank r, not all of the above sequences of diagrams make sense. Let us give some note-worthy cases at low rank:

$\mathfrak{a}_1 = \mathfrak{b}_1 = \mathfrak{c}_1$:
$\mathfrak{d}_1$:
$\mathfrak{d}_2$:
$\mathfrak{d}_3$:
$\mathfrak{b}_2 = \mathfrak{c}_2$:



(8.16)

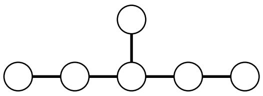
Here a shaded dot is meant to represent the one-dimensional abelian Lie algebra $\mathbb{C}$ which is usually not considered simple. These follow from identities among the classical Lie algebras at low rank which are related to the spinor representations of the orthogonal algebras

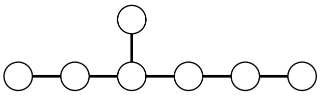
$$\begin{aligned}
\mathfrak{so}(2) &= \mathfrak{gl}(1), & \mathfrak{d}_1 &= \mathbb{C},\\
\mathfrak{so}(3) &= \mathfrak{sp}(2) = \mathfrak{sl}(2), & \mathfrak{b}_1 &= \mathfrak{c}_1 = \mathfrak{a}_1,\\
\mathfrak{so}(4) &= \mathfrak{sp}(2) \oplus \mathfrak{sp}(2), & \mathfrak{d}_2 &= \mathfrak{a}_1 \oplus \mathfrak{a}_1,\\
\mathfrak{so}(5) &= \mathfrak{sp}(4), & \mathfrak{b}_2 &= \mathfrak{c}_2,\\
\mathfrak{so}(6) &= \mathfrak{sl}(4), & \mathfrak{d}_3 &= \mathfrak{a}_3.
\end{aligned} \tag{8.17}$$

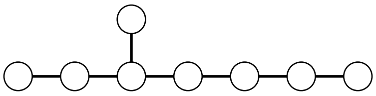
Note that the algebras $\mathfrak{d}_1 = \mathfrak{so}(2)$ and $\mathfrak{d}_2 = \mathfrak{so}(4)$ are not simple.[4]

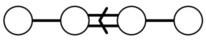
[4]With some imagination one can visualise how the Dynkin diagram $\mathfrak{d}_2$ arises at the lower end of the sequence $\mathfrak{d}_r$.

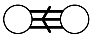
Exceptional Lie Algebras. In addition to the four classical Lie algebras, there are five *exceptional Lie algebras*:

$\mathfrak{e}_6$:  $\dim \mathfrak{e}_6 = 78 = 6 \cdot 13$

$\mathfrak{e}_7$:  $\dim \mathfrak{e}_7 = 133 = 7 \cdot 19$

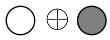
$\mathfrak{e}_8$:  $\dim \mathfrak{e}_8 = 248 = 8 \cdot 31$

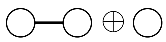
$\mathfrak{f}_4$:  $\dim \mathfrak{f}_4 = 52 = 4 \cdot 13$

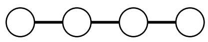
$\mathfrak{g}_2$:  $\dim \mathfrak{g}_2 = 14 = 2 \cdot 7$ (8.18)

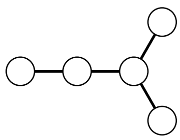
Their existence is often attributed to the existence of the division algebra of octonions. This fourth division algebra is non-associative and hence it does not make sense to construct arbitrary matrices with octonionic entries. This is possible only in some restricted cases, expressed by the exceptional algebras.

Note that the sequence of algebras $\mathfrak{e}_r$ can be extended to lower rank as particular combinations of the classical Lie algebras:[5]

$\mathfrak{e}_2 := \mathfrak{a}_1 \oplus \mathfrak{d}_1$:  $\mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$

$\mathfrak{e}_3 := \mathfrak{a}_2 \oplus \mathfrak{a}_1$:  $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2)$

$\mathfrak{e}_4 := \mathfrak{a}_4$:  $\mathfrak{sl}(5)$

$\mathfrak{e}_5 := \mathfrak{d}_5$:  $\mathfrak{so}(10)$ (8.19)

This finite sequence of exceptional algebras plays a role in various subjects of theoretical physics such as grand unified theories, supergravity and string theory.

[5]Again, with some imagination, one can see how the sequence is continued to at least $\mathfrak{e}_3$.

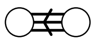
Casimir Operators. It is good to know the degrees of the higher Casimir invariants:[6] [7]

$$\begin{array}{l|l} \mathfrak{a}_r = \mathfrak{sl}(r+1) & 2, 3, \ldots, r+1 \\ \mathfrak{b}_r = \mathfrak{so}(2r+1) & 2, 4, 6, \ldots, 2r \\ \mathfrak{c}_r = \mathfrak{sp}(2r) & 2, 4, 6, \ldots, 2r \\ \mathfrak{d}_r = \mathfrak{so}(2r) & 2, 4, 6, \ldots, 2r-2, r \\ \mathfrak{e}_6 & 2, 5, 6, 8, 9, 12 \\ \mathfrak{e}_7 & 2, 6, 8, 10, 12, 14, 18 \\ \mathfrak{e}_8 & 2, 8, 12, 14, 18, 20, 24, 30 \\ \mathfrak{f}_4 & 2, 6, 8, 12 \\ \mathfrak{g}_2 & 2, 6 \end{array} \qquad (8.20)$$

8.2 The Exceptional Algebra $\mathfrak{g}_2$

Here we want to briefly discuss $\mathfrak{g}_2$ as one of the exceptional and non-simply laced algebras.

Roots. The Cartan matrix reads

$$A_{jk} = \frac{2\langle\alpha_j, \alpha_k\rangle}{\langle\alpha_j, \alpha_j\rangle} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \equiv$$  . (8.21)

This means that the root α_2 is $\sqrt{3}$ times as long as the root α_1 and that they are at an angle of 150°. The Serre relations read

$$\mathrm{ad}(\mathrm{E}_2)^2\mathrm{E}_1 = \mathrm{ad}(\mathrm{E}_1)^4\mathrm{E}_2 = 0. \qquad (8.22)$$

They imply that there are four non-simple positive generators of the form

$$\begin{aligned} \mathrm{E}_{12} &= \mathrm{ad}(\mathrm{E}_1)\mathrm{E}_2 = -\,\mathrm{ad}(\mathrm{E}_2)\mathrm{E}_1, \\ \mathrm{E}_{112} &= \mathrm{ad}(\mathrm{E}_1)\mathrm{E}_{12} = \mathrm{ad}(\mathrm{E}_1)^2\mathrm{E}_2, \\ \mathrm{E}_{1112} &= \mathrm{ad}(\mathrm{E}_1)\mathrm{E}_{112} = \mathrm{ad}(\mathrm{E}_1)^3\mathrm{E}_2, \\ \mathrm{E}_{21112} &= \mathrm{ad}(\mathrm{E}_2)\mathrm{E}_{1112} = \mathrm{ad}(\mathrm{E}_2)\,\mathrm{ad}(\mathrm{E}_1)^3\mathrm{E}_2. \end{aligned} \qquad (8.23)$$

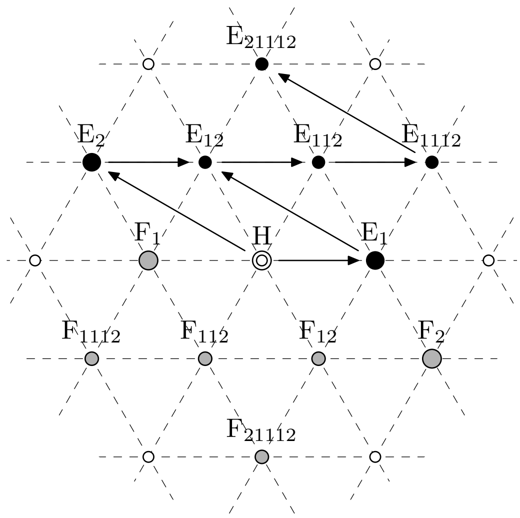
By means of the Jacobi identities and Serre relations one can show that all other combinations are trivial

$$\mathrm{ad}(\mathrm{E}_2)\mathrm{E}_{12} = \mathrm{ad}(\mathrm{E}_2)\mathrm{E}_{112} = \mathrm{ad}(\mathrm{E}_1)\mathrm{E}_{1112} = \mathrm{ad}(\mathrm{E}_j)\mathrm{E}_{21112} = 0. \qquad (8.24)$$

[6] For instance, the cubic Casimir invariant is related to chiral anomalies in four-dimensional quantum field theories. The table shows that these can appear only in connection to groups $\mathfrak{a}_r$, $r > 1$ as well as $\mathfrak{d}_3$ (which equals $\mathfrak{a}_3$). The higher Casimir invariants are related to anomalies in higher dimensions.

[7] The dimension of the Lie algebra equals twice the sum of the Casimir ranks minus r. It also equals r times the highest rank plus r.

The Lie algebra $\mathfrak{g}_2$ therefore has dimension 14. The distribution of roots is summarised in a diagram:



$$(8.25)$$

The dual basis is given by the vectors

$$\omega_1 = 2\alpha_1 + \alpha_2, \qquad \omega_2 = 3\alpha_1 + 2\alpha_2, \tag{8.26}$$

and one can convince oneself that the weight lattice Ω is spanned by the simple roots.

The dimensions of the smallest few irreps can be obtained from the Weyl dimension formula

$$[1,0] = \mathbf{7}, \qquad [0,1] = \mathrm{ad} = \mathbf{14}, \qquad [2,0] = \mathbf{27}, \qquad [1,1] = \mathbf{64}. \tag{8.27}$$

The decomposition of the smallest tensor products can be deduced as Diophantine equations

$$\begin{aligned} \mathbf{7} \otimes \mathbf{7} &= (\mathbf{27} \oplus \mathbf{1})_\vee \oplus (\mathbf{14} \oplus \mathbf{7})_\wedge, \\ \mathbf{7} \otimes \mathbf{14} &= \mathbf{64} \oplus \mathbf{27} \oplus \mathbf{7}. \end{aligned} \tag{8.28}$$

Octonions. The classical Lie algebras are related to the algebras of matrices (with certain additional properties). How can the exceptional algebras be interpreted?

For example, $\mathfrak{g}_2$ is known to be the automorphism algebra of the *octonions*. The octonions $\mathbb{O}$ form a non-associative non-commutative division algebra of real dimension 8. They are spanned by the real unit $\hat{e}_0 = 1$ and 7 imaginary units $\hat{e}_k$, $k = 1, \ldots 7$,

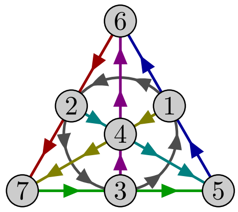
$$x = x_0 + \sum_{k=1}^{7} x_k \hat{e}_k \in \mathbb{O}. \tag{8.29}$$

As for the complex numbers and the quaternions, the real unit commutes with everything and squares to itself while the imaginary units square to minus the real unit

$$(\hat{e}_0 \cdot \hat{e}_j) = (\hat{e}_j \cdot \hat{e}_0) = \hat{e}_j, \qquad (\hat{e}_0 \cdot \hat{e}_0) = -(\hat{e}_j \cdot \hat{e}_j) = \hat{e}_0. \tag{8.30}$$

Furthermore, the imaginary units anti-commute according to the rule

$$(\hat{e}_j \cdot \hat{e}_k) = \pm \hat{e}_l, \qquad (\hat{e}_k \cdot \hat{e}_j) = \mp \hat{e}_l,$$



e.g.

$$(\hat{e}_2 \cdot \hat{e}_7) = +\hat{e}_6, \qquad (\hat{e}_7 \cdot \hat{e}_4) = -\hat{e}_1, \tag{8.31}$$

Here $\pm\hat{e}_l$ is determined by the diagram such that l lies on a common line with j, k; the sign is the parity of the permutation that orders the indices j, k, l according to the direction of the arrows along the line.[8]

The conjugate of an octonion is defined by

$$x^* = x_0 - \sum_{k=1}^{7} x_k \hat{e}_k, \tag{8.32}$$

the square norm as

$$|x|^2 := (x^* \cdot x) = (x \cdot x^*) = \sum_{j=0}^{7} x_j^2 \in \mathbb{R}_0^+, \tag{8.33}$$

and consequently the inverse takes the form $x^{-1} = x^*/|x|^2$.

The exceptional algebra $\mathfrak{g}_2$ has a 7-dimensional irrep which can act on the imaginary part of octonions. Combined with the trivial representation to act on the real part, it has a representation on the octonions

$$\rho_{\mathbf{oct}}(A)x := \sum_{j,k=1}^{7} \rho_{\mathbf{7}}(A)_{jk} x_j \hat{e}_k. \tag{8.34}$$

This representation is an isomorphism in the sense

$$(\rho_{\mathbf{oct}}(A)x \cdot y) + (x \cdot \rho_{\mathbf{oct}}(A)y) = \rho_{\mathbf{oct}}(A)(x \cdot y). \tag{8.35}$$

For an element g of the corresponding Lie group G_2 the isomorphism statement is $(\rho_{\mathbf{oct}}(g)x \cdot \rho_{\mathbf{oct}}(g)y) = \rho_{\mathbf{oct}}(g)(x \cdot y)$.

Subalgebras. Note that the isomorphisms of the imaginary octonions preserve the norm $|\cdot|$ by construction. Since the norm is spherically symmetric in the space $\mathbb{R}^7$, the algebra $\mathfrak{g}_2$ preserves the sphere $\mathrm{S}^6 \in \mathbb{R}^7$. This means that $\mathfrak{g}_2$ is a subalgebra of $\mathfrak{so}(7) = \mathfrak{b}_3$.

[8] For instance, this implies that any two $x, y \in \mathbb{O}$ generate an algebra of quaternions $\mathbb{H}$. A third element $z \in \mathbb{O}$ is needed to generate the whole of $\mathbb{O}$.

One may wonder how to characterise this subalgebra. For that we can look at the splitting of some irreps under $\mathfrak{so}(7) \to \mathfrak{g}_2$

$$
\begin{aligned}
\mathbf{7} &\to \mathbf{7}, \\
\mathbf{8} &\to \mathbf{7} \oplus \mathbf{1}, \\
\mathbf{21} &\to \mathbf{14} \oplus \mathbf{7}, \\
\mathbf{27} &\to \mathbf{27}, \\
\mathbf{48} &\to \mathbf{27} \oplus \mathbf{14} \oplus \mathbf{7}.
\end{aligned} \tag{8.36}
$$

In particular, the 8-dimensional spinor representation splits into a singlet and a 7-dimensional irrep. Furthermore, the adjoint splits into the adjoint and the same 7-dimensional irrep. This implies that $\mathfrak{g}_2$ is the stabiliser subalgebra of a (non-zero) spinor of $\mathfrak{so}(7)$.

Conversely one may ask about the biggest classical subalgebra of $\mathfrak{g}_2$. The regular triangular lattice of roots and weights suggests that this is $\mathfrak{sl}(3)$. One finds the following splitting of irreps for $\mathfrak{g}_2 \to \mathfrak{sl}(3)$

$$
\begin{aligned}
\mathbf{7} &\to \mathbf{3} \oplus \mathbf{3}^* \oplus \mathbf{1}, \\
\mathbf{14} &\to \mathbf{8} \oplus \mathbf{3} \oplus \mathbf{3}^*.
\end{aligned} \tag{8.37}
$$

Again this pattern of splittings tells that $\mathfrak{sl}(3)$ is the stabiliser of a non-zero element in the $\mathbf{7}$ of $\mathfrak{g}_2$.

8.3 Real Forms

In physics one is often interested in unitary representations, and those make sense only in real forms of the complex Lie algebra. A *real form* $\mathfrak{g}_\mathbb{R}$ of a complex Lie algebra $\mathfrak{g}_\mathbb{C}$ is specified by a complex conjugation on $\mathfrak{g}_\mathbb{C}$ which obeys

$$
[\![A^*, B^*]\!] = [\![A, B]\!]^*. \tag{8.38}
$$

The real slice $\mathfrak{g}_\mathbb{R} := \{A \in \mathfrak{g}_\mathbb{C}; A = A^*\}$ of the complex algebra is a real Lie algebra. Real forms are commonly classified by so-called Satake or Vogan diagrams which are decorations of Dynkin diagrams describing the complex conjugation. We will not introduce them here, but merely list the resulting real forms.

Classical Algebras. For the classical Lie algebras based on matrix algebras, the real forms follow a regular pattern. There are ten families of classical real forms listed along with their maximally compact subalgebras:

$$\begin{array}{ll}
\mathfrak{so}(p,n-p) & \mathfrak{so}(p)\oplus\mathfrak{so}(n-p) \\
\mathfrak{so}(n,\mathbb{C}) & \mathfrak{so}(n) \\
\mathfrak{so}(n,\mathbb{H}) & \mathfrak{su}(n)\oplus\mathbb{R} \\
\mathfrak{sl}(n,\mathbb{R}) & \mathfrak{so}(n) \\
\mathfrak{sl}(n,\mathbb{C}) & \mathfrak{su}(n) \\
\mathfrak{sl}(n,\mathbb{H}) & \mathfrak{sp}(n) \\
\mathfrak{su}(p,n-p) & \mathfrak{su}(p)\oplus\mathfrak{su}(n-p)\oplus\mathbb{R} \\
\mathfrak{sp}(2n,\mathbb{R}) & \mathfrak{su}(n)\oplus\mathbb{R} \\
\mathfrak{sp}(2n,\mathbb{C}) & \mathfrak{sp}(n) \\
\mathfrak{sp}(p,n-p) & \mathfrak{sp}(p)\oplus\mathfrak{sp}(n-p)
\end{array} \tag{8.39}$$

These algebras are defined analogously to their complex counterparts introduced before.[9]

Some comments are in order:

- The algebras are based on traceless ($\mathfrak{sl}$), anti-symmetric ($\mathfrak{so}$), anti-symplectic ($\mathfrak{sp}$) or anti-hermitian, traceless ($\mathfrak{su}$) matrices over the real ($\mathbb{R}$), complex ($\mathbb{C}$) or quaternionic ($\mathbb{H}$) numbers. The hermitian property only applies to the field of complex numbers.
- The default fields for $\mathfrak{so}$, $\mathfrak{su}$, $\mathfrak{sp}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, respectively. For these, the vector spaces are equipped with a metric of signature $(p, n-p)$ whose ordering does not matter. The definite signature $(n, 0)$ is abbreviated by (n).
- We use the convention that the number(s) indicate the dimension over the defining field. In other conventions the dimension of the complexified algebra is used. This leads to notational differences for matrices over the quaternions; here $(n,\mathbb{H})$ refers to $\mathbb{H}^n$, elsewhere it may refer to $\mathbb{H}^{n/2}\equiv\mathbb{C}^n$. Yet other conventions divide the dimension for $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{sp}(2n,\mathbb{C})$ by two.
- The algebras based on quaternions are frequently denoted alternatively as:

$$\begin{aligned}
\mathfrak{usp}(2p,2n-2p) &\to \mathfrak{sp}(p,n-p),\\
\mathfrak{so}^*(2n) &\to \mathfrak{so}(n,\mathbb{H}),\\
\mathfrak{su}^*(2n) &\to \mathfrak{sl}(n,\mathbb{H}).
\end{aligned} \tag{8.40}$$

Small Rank. For small rank, there are some equivalences between the various real forms. All the one-dimensional abelian Lie algebras are equal

$$\mathfrak{so}(2)=\mathfrak{so}(1,1)=\mathfrak{so}(1,\mathbb{H})=\mathfrak{u}(1)=\mathfrak{gl}(1,\mathbb{R})=\mathbb{R}. \tag{8.41}$$

[9] In order to define the quaternionic matrix algebras, note that for $q=w+x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ the transpose is defined by $q^{\mathsf{T}}:=w+x\hat{\imath}-y\hat{\jmath}+z\hat{k}$ and the trace by $\operatorname{tr}q:=w$. The canonical quaternionic symplectic metric is defined by $E=\operatorname{diag}(\pm1,\ldots,\pm1)\hat{\jmath}$.

For the non-abelian Lie algebras the equivalences are related to the spinor representations of the orthogonal algebras:

$$\begin{aligned}
\mathfrak{so}(3) &= \mathfrak{sp}(1) = \mathfrak{sl}(1,\mathbb{H}) = \mathfrak{su}(2),\\
\mathfrak{so}(2,1) &= \mathfrak{sp}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{su}(1,1),\\
\mathfrak{so}(4) &= \mathfrak{sp}(1) \oplus \mathfrak{sp}(1),\\
\mathfrak{so}(3,1) &= \mathfrak{sp}(2,\mathbb{C}),\\
\mathfrak{so}(2,2) &= \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R}),\\
\mathfrak{so}(2,\mathbb{H}) &= \mathfrak{sp}(1) \oplus \mathfrak{sp}(2,\mathbb{R}),\\
\mathfrak{so}(5) &= \mathfrak{sp}(2),\\
\mathfrak{so}(4,1) &= \mathfrak{sp}(1,1),\\
\mathfrak{so}(3,2) &= \mathfrak{sp}(4,\mathbb{R}),\\
\mathfrak{so}(6) &= \mathfrak{su}(4),\\
\mathfrak{so}(5,1) &= \mathfrak{sl}(2,\mathbb{H}),\\
\mathfrak{so}(4,2) &= \mathfrak{su}(2,2),\\
\mathfrak{so}(3,3) &= \mathfrak{sl}(4,\mathbb{R}),\\
\mathfrak{so}(3,\mathbb{H}) &= \mathfrak{su}(3,1),\\
\mathfrak{so}(6,2) &= \mathfrak{so}(4,\mathbb{H}).
\end{aligned} \tag{8.42}$$

For instance, one can convince oneself that the maximally compact subalgebras of the equivalent real forms coincide.

Standard Real Forms. There are always at least three real forms related to each complex simple Lie algebra $\mathfrak{g}$.

The *compact real form* is obtained by the conjugation

$$\mathrm{H}_k^* = -\mathrm{H}_k, \qquad \mathrm{E}_k^* = -\mathrm{F}_k, \qquad \mathrm{F}_k^* = -\mathrm{E}_k. \tag{8.43}$$

The algebraic relations are preserved by this conjugation. It has the distinguished property that the Killing form has (negative) definite signature. The associated Lie group is compact. The compact real forms of the complex algebras $\mathfrak{so}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{sp}(2n)$ are the real algebras $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n)$, respectively.

The *split real form* is the opposite of the compact real form. It is obtained by declaring all the Chevalley–Serre generators to be real

$$\mathrm{H}_k^* = \mathrm{H}_k, \qquad \mathrm{E}_k^* = \mathrm{E}_k, \qquad \mathrm{F}_k^* = \mathrm{F}_k. \tag{8.44}$$

The defining algebraic relations are real, and therefore this conjugation defines a proper real form of the algebra. The split real forms of the complex algebras $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sl}(n)$, $\mathfrak{sp}(2n)$ are the real algebras $\mathfrak{so}(n,n)$, $\mathfrak{so}(n,n+1)$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sp}(2n,\mathbb{R})$, respectively.

Finally, there is the *complex real form* of $\mathfrak{g} \oplus \mathfrak{g}$ which is largely equivalent to the complex Lie algebra $\mathfrak{g}$. Here, a Lie algebra over the complex numbers is interpreted as a Lie algebra over the real numbers with twice the dimension. Note that formally the complex real form is a real Lie algebra and thus different from the complex Lie algebra. For instance, the complex algebra has only complex representations whereas the real algebra can also have real or quaternionic representations. The compact real forms of the complex algebras $\mathfrak{so}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{sp}(2n)$ are the real algebras $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{su}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, respectively.

Exceptional Algebras. For the exceptional algebras $\mathfrak{g}_r$ there are several real forms denoted by $\mathfrak{g}_{r(p-q)}$ where (p, q) is the signature of the Killing form and where q is the dimension of the maximally compact subalgebra. We have seen above that there always exist the compact and the complex real forms; there is not much to say about these. For the split real forms $\mathfrak{g}_{r(r)}$ we specify the signature and the maximally compact subgroups:

$$\begin{array}{lll} \mathfrak{g}_{2(2)} & (8,6) & \mathfrak{su}(3) \\ \mathfrak{f}_{4(4)} & (28,24) & \mathfrak{so}(7) \oplus \mathfrak{su}(2) \\ \mathfrak{e}_{6(6)} & (42,36) & \mathfrak{sp}(4) \\ \mathfrak{e}_{7(7)} & (70,63) & \mathfrak{su}(8) \\ \mathfrak{e}_{8(8)} & (128,120) & \mathfrak{so}(16) \end{array} \tag{8.45}$$

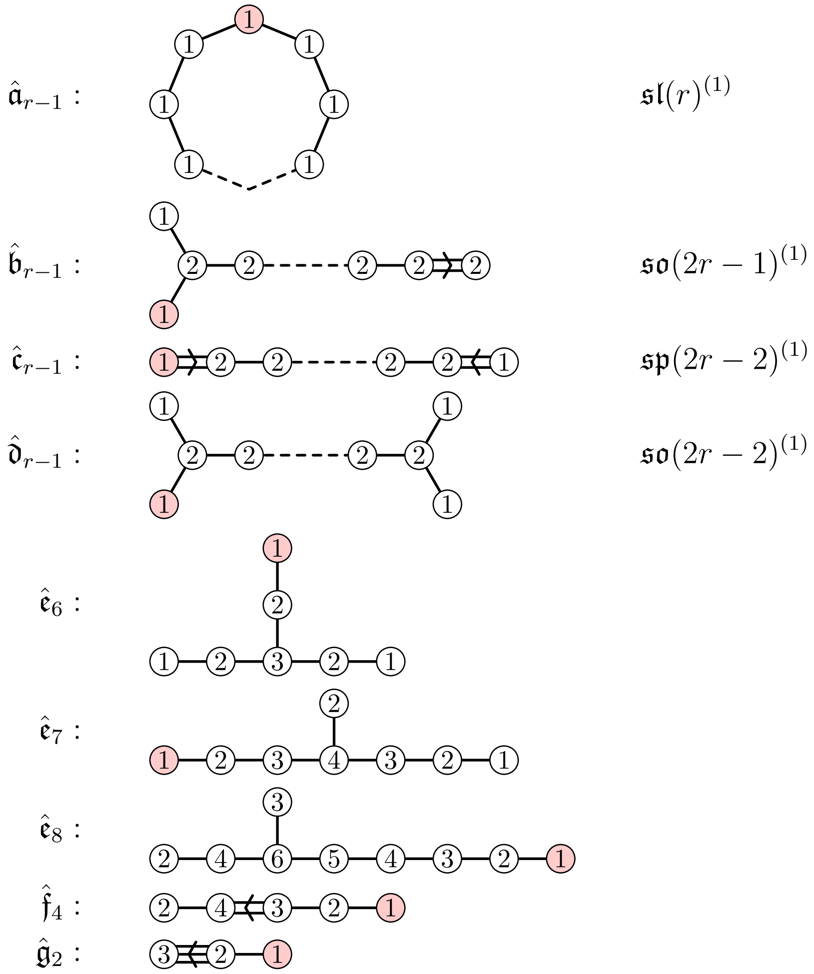
Furthermore, there are a few more real forms:

$$\begin{array}{lll} \mathfrak{f}_{4(-20)} & (16,36) & \mathfrak{so}(9) \\ \mathfrak{e}_{6(-26)} & (26,52) & \mathfrak{f}_4 \\ \mathfrak{e}_{6(-14)} & (32,46) & \mathfrak{so}(10) \oplus \mathbb{R} \\ \mathfrak{e}_{6(2)} & (40,38) & \mathfrak{su}(6) \oplus \mathfrak{su}(2) \\ \mathfrak{e}_{7(-25)} & (54,79) & \mathfrak{e}_{6(-78)} \oplus \mathbb{R} \\ \mathfrak{e}_{7(-5)} & (64,69) & \mathfrak{so}(12) \oplus \mathfrak{su}(2) \\ \mathfrak{e}_{8(-24)} & (112,136) & \mathfrak{e}_{7(-133)} \oplus \mathfrak{su}(2) \end{array} \tag{8.46}$$

8.4 Affine Algebras

The above formalism to classify simple finite-dimensional Lie algebras can be extended to construct some useful infinite-dimensional algebras. The construction uses the same algebra relations as above, however the Cartan matrices are now allowed to be positive semi-definite with one degenerate direction. This results in two classes of so-called *affine Kac–Moody algebras*.

Affine Algebras. The first class are the *untwisted affine algebras*. There is precisely one algebra $\hat{\mathfrak{g}} = \mathfrak{g}^{(1)}$ for each finite-dimensional simple Lie algebra $\mathfrak{g}$:[10]



(8.47)

Note that the diagram of $\hat{\mathfrak{g}}$ of rank r is obtained by adding one node to the Dynkin diagram of the corresponding finite-dimensional simple algebra $\mathfrak{g}$ of rank $r-1$. Moreover, this node $j=0$ is attached in accordance with the Dynkin labels of the adjoint representation of $\mathfrak{g}$. The numbers b_j indicated on the nodes[11] describe the kernel of the Cartan matrix

$$\sum_{j=0}^{r-1} A_{kj} b_j = 0. \tag{8.48}$$

The affine algebra $\hat{\mathfrak{g}}$ can be described conveniently in terms of the finite-dimensional algebra $\mathfrak{g}$: It is an extension of the loop algebra $\mathfrak{g}[x, x^{-1}]$ by a central element C and a derivation D

$$\hat{\mathfrak{g}} = \mathfrak{g}[x, x^{-1}] \oplus \mathbb{C}\mathrm{C} \oplus \mathbb{C}\mathrm{D}. \tag{8.49}$$

[10] The Dynkin diagrams of $\hat{\mathfrak{a}}_r$, $\hat{\mathfrak{d}}_r$ and $\hat{\mathfrak{e}}_r$ are composed from simple lines only. They make appearance in diverse topics in mathematics and theoretical physics, see Section 4.2, and they are called the *ADE-graphs*.

[11] The numbers b_j also determine the dimension of the underlying finite-dimensional Lie algebra as $\dim \mathfrak{g} = (r-1) + (r-1)\sum_j b_j$ where $r-1$ is the rank of $\mathfrak{g}$.

The *loop algebra* $\mathfrak{g}[x,x^{-1}] := \mathfrak{g}\otimes\mathbb{C}[x,x^{-1}]$ is the tensor product of a Lie algebra $\mathfrak{g}$ with Laurent polynomials in one (formal) variable x. Thus $\mathfrak{g}[x,x^{-1}]$ is spanned by the elements $\mathrm{J}_a\otimes x^n$, $n\in\mathbb{Z}$ where J_a, $a=1,\ldots,\dim\mathfrak{g}$ denotes a basis of $\mathfrak{g}$. The Lie brackets for $\hat{\mathfrak{g}}$ are given by

$$\begin{aligned}
[\![A\otimes x^m, B\otimes x^n]\!] &= [\![A,B]\!]\otimes x^{m+n} + m\delta_{m+n,0}\langle A,B\rangle \mathrm{C},\\
[\![\mathrm{D}, A\otimes x^m]\!] &= mA\otimes x^m,\\
[\![\mathrm{C}, A\otimes x^m]\!] = [\![\mathrm{C},\mathrm{D}]\!] &= 0,
\end{aligned}\tag{8.50}$$

Furthermore, there exist an invertible bilinear form[12]

$$\begin{aligned}
\langle A\otimes x^m, B\otimes x^n\rangle &= \delta_{m+n,0}\langle A,B\rangle,\\
\langle \mathrm{C},\mathrm{D}\rangle &= 1.
\end{aligned}\tag{8.51}$$

Note that the central charge C is related to the kernel of the transpose Cartan matrix

$$\mathrm{C} = \sum_{j=0}^{r-1} b_j^*\mathrm{H}_j \qquad\text{where}\qquad \sum_{j=0}^{r-1} b_j^* A_{jk} = 0.\tag{8.52}$$

This implies that $[\![\mathrm{C},\mathrm{E}_k]\!] = [\![\mathrm{C},\mathrm{F}_k]\!] = 0$. The derivation D is not accounted for by the Chevalley–Serre generators.[13]

Affine algebras have two types of representations of relevance in physics: The highest-weight representations are constructed as before using the Chevalley–Serre presentation of the algebra. The evaluation representation lifts a representation of the underlying finite-dimensional algebra $\mathfrak{g}$ to $\hat{\mathfrak{g}}$

$$\begin{aligned}
(A\otimes x^m)\big(|a\rangle\otimes|y\rangle\big) &= \big(A|a\rangle\big)\otimes\big(y^m|y\rangle\big),\\
D\big(|a\rangle\otimes|y\rangle\big) &= |a\rangle\otimes\big(-(y\partial/\partial y)|y\rangle\big),\\
C\big(|a\rangle\otimes|y\rangle\big) &= 0,
\end{aligned}\tag{8.53}$$

Twisted Affine Algebras. There also exist affine Dynkin diagrams which do not extend the ones of the finite-dimensional simple algebras. These take the form:

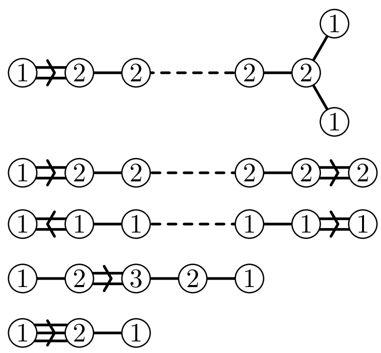
$\mathfrak{a}^{(2)}_{2r-3}$: $\mathfrak{sl}(2r-2)^{(2)}$

$\mathfrak{a}^{(2)}_{2r-2}$: $\mathfrak{sl}(2r-1)^{(2)}$

$\mathfrak{d}^{(2)}_{r}$: $\mathfrak{so}(2r)^{(2)}$

$\mathfrak{e}^{(2)}_{6}$:

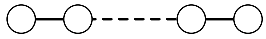
$\mathfrak{d}^{(3)}_{4}$: $\mathfrak{so}(8)^{(3)}$ (8.54)

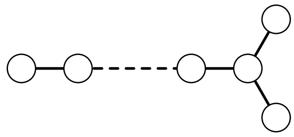


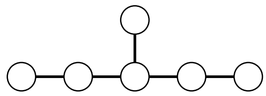
[12]The value assigned to $\langle\mathrm{C},\mathrm{D}\rangle$ is required to make the bilinear form non-singular and to define the quadratic Casimir invariant consistently.

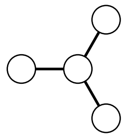
[13]The affine algebra is not simple. A simple algebra is obtained by dropping the derivation D and quotienting out the central charge C. The resulting algebra is the loop algebra $\mathfrak{g}[x,x^{-1}]$.

All of these algebras exist because some of the finite-dimensional Lie algebras possess outer automorphisms. These are related to symmetries of the Dynkin diagrams:[14]

$\mathfrak{a}_r$: 

$\mathfrak{d}_r$: 

$\mathfrak{e}_6$: 

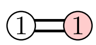
$\mathfrak{d}_4$:  (8.55)

Among these, the diagram $\mathfrak{d}_4$ is special because it has a three-fold symmetry as compared to the two-fold reflectional symmetries of the others.

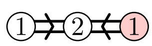
The above Dynkin diagrams correspond to *twisted affine algebras* $\mathfrak{g}_r^{(k)}$.[15] They can be considered as subalgebras of the above affine algebras $\mathfrak{g}_r^{(1)} := \hat{\mathfrak{g}}_r$ determined by the outer automorphism σ of order k. The elements $A \otimes x^n$ of the subalgebra are characterised by satisfying

$$\sigma A \otimes x^n = \mathrm{e}^{2\pi i n/k} A \otimes x^n. \tag{8.56}$$

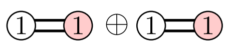
Small Rank. Again the sequences of classical affine algebras should be specified more clearly at low rank. The lower untwisted affine algebras are described by the following Dynkin diagrams:

$\hat{\mathfrak{a}}_1 = \hat{\mathfrak{b}}_1 = \hat{\mathfrak{c}}_1$: 

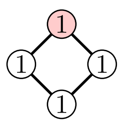
$\mathfrak{so}(3)^{(1)} = \mathfrak{sp}(2)^{(1)}$

$\hat{\mathfrak{b}}_2 = \hat{\mathfrak{c}}_2$: 

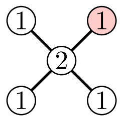
$\mathfrak{so}(5)^{(1)} = \mathfrak{sp}(4)^{(1)}$

$\hat{\mathfrak{d}}_2$: 

$\mathfrak{so}(4)^{(1)}$

$\hat{\mathfrak{d}}_3$: 

$\mathfrak{so}(6)^{(1)} = \mathfrak{sl}(4)^{(1)}$

$\hat{\mathfrak{d}}_4$: 

$\mathfrak{so}(8)^{(1)}$ (8.57)

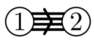
Here the double line without an arrow[16] describes a pair of equal off-diagonal elements $\{A_{jk}, A_{kj}\} = \{-2, -2\}$ of the Cartan matrix.

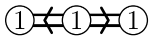
[14] The twisted affine Dynkin diagrams are related to foldings of the untwisted Dynkin diagrams under the symmetry.

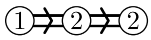
[15] The index r of $\mathfrak{g}_r^{(k)}$ is not directly related to the rank of the twisted affine algebra.

[16] Other conventions use double arrow or a quadruple line.

The small-rank twisted affine algebras are described by the following Dynkin diagrams:

$\mathfrak{a}_2^{(2)}$:  $\mathfrak{sl}(3)^{(2)}$

$\mathfrak{a}_3^{(2)} = \mathfrak{d}_3^{(2)}$:  $\mathfrak{sl}(4)^{(2)} = \mathfrak{so}(6)^{(2)}$

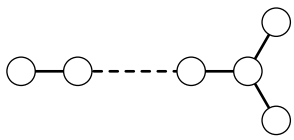
$\mathfrak{a}_4^{(2)}$:  $\mathfrak{sl}(5)^{(2)}$ (8.58)

Note that for affine algebras also a quadruple line with an arrow is permitted, which describes a pair of off-diagonal elements $\{A_{jk}, A_{kj}\} = \{-1, -4\}$ of the Cartan matrix.

8.5 Subalgebras

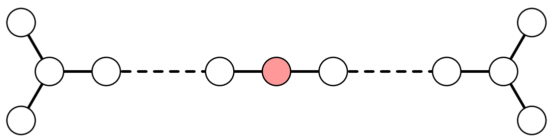
The Dynkin diagrams of the finite-dimensional algebras and their affine extensions can be used to construct many related algebras.

Construction. Often one is interested in finding subalgebras of a given algebra. For example, $\mathfrak{so}(p+q)$ has as subalgebras $\mathfrak{so}(p)$ and $\mathfrak{so}(q)$ as well as their direct sum $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$. How can this be seen from the Dynkin diagrams? Suppose $p, q > 2$ are even, then the relevant Dynkin diagrams are all of the form:



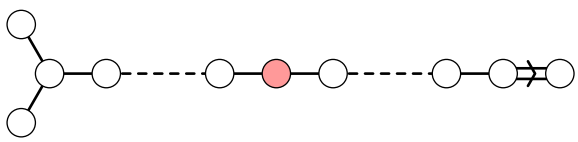
(8.59)

By eliminating some nodes one can obtain $\mathfrak{so}(p)$ and $\mathfrak{so}(q)$, but not their direct sum. However, the direct sum is contained in the affine Dynkin diagram of $\mathfrak{so}(p+q)^{(1)}$:



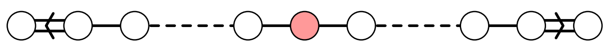
(8.60)

This trick also works if one of p, q is even and the other one is odd:



(8.61)

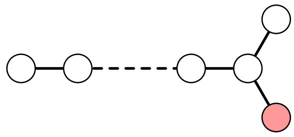
If both p, q are odd, however, the subalgebra $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ has lower rank than $\mathfrak{so}(p+q)$ and it cannot be obtained in this way. Here, the relevant Dynkin diagram is the twisted affine $\mathfrak{so}(p+q)^{(2)}$:



(8.62)

This method to obtain (large) subalgebras works for general simple Lie algebras. It is based on the fact that the affine Lie algebras contain infinitely many copies of a given simple Lie algebra under the projection $A \otimes x^n \to A$. By removing a node from the Dynkin diagram one obtains a subalgebra. When one node is removed from an affine Dynkin diagram, the resulting subalgebra is a direct sum of finite-dimensional simple algebras.

An alternative to obtain further subalgebras is to eliminate a node from a finite-dimensional Dynkin diagram. For example, $\mathfrak{so}(2n)$ has a $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathfrak{gl}(1)$ subalgebra given by the elimination:



(8.63)

The difference w.r.t. the above cases is that we keep the associated Cartan element H_k which yields the additional abelian component $\mathbb{C} = \mathfrak{gl}(1) = \mathfrak{so}(2)$. This is not an option when starting with an affine Dynkin diagram because H_k essentially corresponds to the central element C which is unrelated to the finite-dimensional algebra.

By using the above splitting method on the various Dynkin diagrams and iterating the procedure one obtains a wealth of subalgebras for any given simple Lie algebra. In particular, the rank is preserved (unless the twisted affine algebras are used). However, there are also many other subalgebras which do not follow from this procedure, e.g. $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ discussed above.

Grading. From the above construction of subalgebras one can further deduce a $\mathbb{Z}_n$ grading structure on the decomposition. This means that the Lie algebra $\mathfrak{g}$ decomposes

$$\mathfrak{g} = \bigoplus_{k=0}^{n-1} \mathfrak{g}_k \qquad \text{such that} \qquad [\![\mathfrak{g}_j, \mathfrak{g}_k]\!] \subset \mathfrak{g}_{j+k}. \tag{8.64}$$

The subalgebra is the component $\mathfrak{g}_0$ at grading 0. The label on the deleted node of the untwisted affine Dynkin diagram determines the order n of the grading. For a twisted Dynkin diagram $\mathfrak{g}^{(k)}$ the order of the grading is the product nk. When deleting a node from the finite Dynkin diagrams, the order is $2n$ where n is the label on the associated affine diagram.

Understanding the grading is useful for constructing coset models which are based on coset spaces G/H of a Lie group G and a Lie subgroup H. For instance, the $\mathbb{Z}_2$ cosets of Lie groups are called *symmetric spaces* and they have particularly nice properties.

Real Forms. The real form of the finite-dimensional Lie algebras can be obtained by considering their maximally compact subalgebras. These are given by the even part of a $\mathbb{Z}_2$ automorphism. Therefore there is a close correspondence between the real forms and the symmetric spaces, and the real forms can be classified via affine Dynkin diagrams:

- Remove a node of an untwisted affine Dynkin diagram with label 1. This corresponds to the compact real form because the remaining Dynkin diagram describes the original finite-dimensional Lie algebra.
- Remove a node of an untwisted affine Dynkin diagram with label 2. This corresponds to the real forms $\mathfrak{so}(2p, 2q)$, $\mathfrak{so}(2p, 2q-1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{8(-24)}$, $\mathfrak{e}_{8(8)}$, $\mathfrak{f}_{4(-20)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{g}_{2(2)}$.
- Remove two nodes of an untwisted affine Dynkin diagram with labels 1. This corresponds to the real forms whose maximally compact subalgebra contains a factor of $\mathbb{R}$, namely $\mathfrak{so}(p, 2)$, $\mathfrak{so}(n, \mathbb{H})$, $\mathfrak{su}(p, q)$, $\mathfrak{sp}(2n, \mathbb{R})$, $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(-25)}$.
- Remove a node of a $\mathbb{Z}_2$ twisted affine Dynkin diagram with label 1. This corresponds to the real forms whose maximally compact subalgebra has lower rank, namely $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{so}(2p-1, 2q-1)$, $\mathfrak{e}_{6(-26)}$, $\mathfrak{e}_{7(-25)}$.

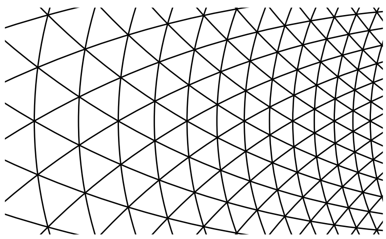
9 Conformal Symmetry

So far we have mainly discussed compact symmetry algebras with finite-dimensional representations. However, also the non-compact symmetry algebras play an important role in physics. In particular, in quantum field theory the unitary irreps of the Poincaré group can be used to classify the various kinds of particles. Here, we shall consider the somewhat bigger group of conformal symmetries, which are the transformations that preserve angles but not necessarily distances. Physical models with conformal symmetry are called conformal field theories. We shall see that the construction of suitable representations for such models is analogous to our previous constructions.

9.1 Conformal Field Theory

In the following, we will briefly introduce a few elements of conformal symmetry and conformal field theory.

Conformal Symmetry. Conformal transformations are the transformations of a space that preserve all angles between two intersecting smooth curves. We shall assume that the space is flat. In that case, the rotations and translations clearly preserve all angles. Furthermore, *scale transformations*, $x \mapsto \alpha x$, do preserve angles, but they evidently do not preserve distances. Finally, also *inversions about a point*, $x \mapsto x/x^2$, turn out to preserve angles. When combining inversions with translations, one finds the *conformal boosts* as additional continuous transformations.



(9.1)

Altogether the (continuous) conformal transformations in d spatial dimensions extend the euclidean transformations to the *conformal algebra* $\mathfrak{so}(d+1,1)$. For a d-dimensional spacetime, the conformal algebra is $\mathfrak{so}(d,2)$. We shall mainly consider the latter case here. Denote the generators of (Lorentz) rotations $\mathfrak{so}(d)$ or $\mathfrak{so}(d-1,1)$ by $\mathrm{M}_{\mu\nu} = -\mathrm{M}_{\nu\mu}$ and the generators of translations by P_μ. The additional generators of conformal symmetries are the *dila(ta)tion generator* D and the *conformal boost generators* K^μ.

The conformal action on the coordinates is specified by

$$\begin{aligned} \mathrm{P}_\mu x^\nu &= i\delta^\nu_\mu, & \mathrm{K}^\mu x^\nu &= i x^\mu x^\nu - \tfrac{i}{2}\eta^{\mu\nu}x^2, \\ \mathrm{D}x^\nu &= i x^\nu, & \mathrm{M}_{\mu\nu}x^\rho &= i\delta^\rho_\nu x_\mu - i\delta^\rho_\mu x_\nu. \end{aligned} \tag{9.2}$$

The corresponding conformal algebra therefore reads

$$\begin{aligned} [\![\mathrm{D},\mathrm{P}_\mu]\!] &= -i\mathrm{P}_\mu, \\ [\![\mathrm{D},\mathrm{K}^\mu]\!] &= +i\mathrm{K}^\mu, \\ [\![\mathrm{P}_\mu,\mathrm{K}^\nu]\!] &= -i\mathrm{M}_\mu{}^\nu + i\delta^\nu_\mu\mathrm{D}, \\ [\![\mathrm{D},\mathrm{M}_{\mu\nu}]\!] &= 0. \end{aligned} \tag{9.3}$$

Conformal Field Theory. A *conformal field theory* is a special kind of quantum field theory where the spacetime symmetries are given by the conformal group. Examples are given by classical electromagnetism as well as physical systems at a phase transition.[1]

Typically, the objects of interest in a conformal field theory are *local operators* $\mathcal{O}(x)$. These are some combinations of the fields of the theory evaluated at a particular point x in space(time). Often the local operators are not specified explicitly through particular combination of fields, but rather implicitly through their properties (spin, charges, quantum numbers). The observables related to these local operators are the n-point *correlation functions*

$$\big\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\big\rangle. \tag{9.4}$$

Conformal symmetry imposes constraints on the form of correlation functions. For example, translation symmetry evidently implies that two-point functions are functions of the difference of points $x_{12} := x_1 - x_2$. Moreover, rotational symmetry implies that only x_{12}^2 can appear. Finally, the scaling and conformal boost symmetries imply the very concrete form

$$\big\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\big\rangle = \frac{C}{(x_{12}^2)^{\Delta}}\,. \tag{9.5}$$

Here, $\mathcal{O}$ is assumed to be a scalar primary, Δ is its conformal dimension, and C is a normalisation constant. A *primary operator* is a local operator which is annihilated by the conformal boosts at the origin $x=0$ [2] and which has a definite *scaling dimension* Δ [3]

$$\rho_{\mathcal{O}}(\mathrm{K}_\mu)\mathcal{O}(0) = 0, \qquad \rho_{\mathcal{O}}(\mathrm{D})\mathcal{O}(0) = -i\Delta\mathcal{O}(0). \tag{9.6}$$

Furthermore, a scalar operator satisfies $\rho_{\mathcal{O}}(\mathrm{M}_{\mu\nu})\mathcal{O}(0) = 0$.

[1]When tuning the parameters of a model between two phases, its correlation functions become long-ranged and are described by conformal field theory.

[2]The point $x = 0$ is a fixed point of all conformal transformations except the translations.

[3]The origin is chosen as the reference point so that the conformal transformations on the coordinate x are as simple as possible.

Operators with spin transform in some non-trivial representation of $\mathfrak{so}(d)$ or $\mathfrak{so}(d-1,1)$ at the origin, and consequently they carry some vector (or spinor) indices. In n-point functions the vector indices of primary operators are typically contracted by the combination

$$I_{12}^{\mu\nu}(x_{12}) := \eta^{\mu\nu} - 2\,\frac{x_{12}^{\mu}x_{12}^{\nu}}{x_{12}^2}\,. \tag{9.7}$$

For example, a two-point function of vector primaries takes the form

$$\big\langle \mathcal{O}^{\mu}(x_1)\,\mathcal{O}^{\nu}(x_2)\big\rangle = \frac{C I^{\mu\nu}(x_{12})}{(x_{12}^2)^{\Delta}}\,. \tag{9.8}$$

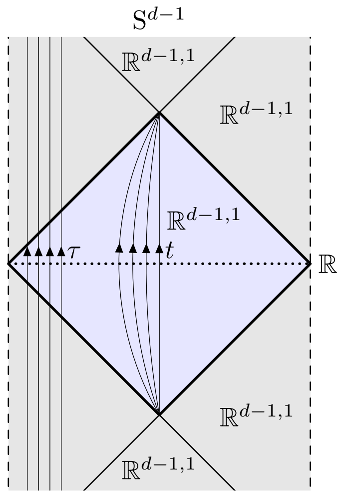
Most operators are not primary, but they are all related to some primary. A *descendant operator* is obtained from a primary operator by applying the translation generator

$$\mathcal{O}'_{\mu_1\ldots\mu_n}(x) = \rho_{\mathcal{O}}(\mathrm{P}_{\mu_1})\ldots\rho_{\mathcal{O}}(\mathrm{P}_{\mu_n})\mathcal{O}(x) = (-\mathring{\imath})^n\partial_{\mu_1}\ldots\partial_{\mu_n}\mathcal{O}(x). \tag{9.9}$$

Since descendants are partial derivatives of the primary, their correlation functions are all determined through the correlators of primaries. Therefore it largely suffices to focus attention to the primary operators; the descendants are encoded into the coordinate-dependence.

Global Aspects. The conformal group for spacetime has non-trivial topological features which are also relevant for the representation theory. In order to understand them, consider the following argument involving a finite conformal boost as the conjugation of a translation by an inversion. The inversion of a point in the distant past yields a point in the near future. A reasonably small translation can shift this point between near future and near past. A subsequent inversion maps this point to the distant future. This shows that even small conformal transformations relate the distant past and future.

Consequently, the corresponding regions of spacetime should be topologically nearby. In other words, spacetime appears to have *closed time-like curves*, a feature which is highly undesirable from the point of view of causality. Spacetime has a conformal topology of $\mathrm{S}^1 \times \mathrm{S}^{d-1}/\mathbb{Z}_2$ where the circle S^1 describes time. The resolution to the problem is to consider the *conformal completion* of spacetime. The topology of the completed spacetime is $\mathbb{R} \times \mathrm{S}^{d-1}$ where the time-like circle is unwound to a full time axis $\mathbb{R}$ as depicted in the following *Penrose diagram*[4]



(9.10)

The corresponding feature in group theory is that the (orthochronous proper) conformal group $\mathrm{SO}(d,2)^+$ has the maximal compact subgroup $\mathrm{SO}(d) \times \mathrm{SO}(2)$. The factor $\mathrm{SO}(2)$ describes time translations, and it has the topology of a circle S^1. This circle can be unwound, and one obtains a group $\widetilde{\mathrm{SO}}(d,2)$ which is an infinite covering of $\mathrm{SO}(d,2)^+$.[5] Now the group $\mathrm{SO}(d,2)^+$ acts as the conformal transformations on ordinary spacetime whereas the infinite cover $\widetilde{\mathrm{SO}}(d,2)$ acts on the conformal completion of spacetime. A relevant feature for representation theory will be that the fundamental group $\mathbb{Z}$ of the circle provides the central elements of $\widetilde{\mathrm{SO}}(d,2)$.

The inversions are also large conformal transformations. However, they correspond to reflections in the conformal completion of spacetime and thus they are not part of $\mathrm{SO}(d,2)^+$ or $\widetilde{\mathrm{SO}}(d,2)$. Here, we shall not discuss the discrete extensions like inversion, parity or time reversal.

9.2 Representations of $\mathfrak{sl}(2,\mathbb{R})$

Let us start with the simplest case in $d=1$ where the corresponding algebra has rank 1. The ordinary definition of conformal symmetry in one dimension does not actually make sense because there are no angles. Nevertheless the general conformal group in d dimensions can be specified to $d=1$ dimension where it

[4]Penrose diagrams display the global topology of a curved spacetime respecting angles (in particular, light-like directions are at 45°) but not distances.

[5]It is somewhat cleaner to describe this group at the level of the double covering spin group $\mathrm{Spin}(d,2)$: Its fundamental group is the fundamental group of the circle. The universal cover $\widetilde{\mathrm{Spin}}(d,2)$ is a double cover of $\widetilde{\mathrm{SO}}(d,2)$.

becomes the group $\mathrm{PSL}(2,\mathbb{R})$ [6] of real *Möbius transformations*[7]

$$x \mapsto \frac{ax+b}{cx+d}. \tag{9.11}$$

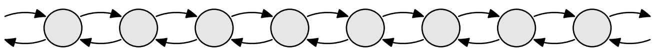
We already know the corresponding complexified Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, but the unitary representation theory of the split real form $\mathfrak{sl}(2,\mathbb{R})$ is much more diverse than the one of the compact real form $\mathfrak{su}(2)$.

Principal Series Representations. Gladly, the complete representation theory of $\mathfrak{sl}(2,\mathbb{C})$ derives from a single family of representations, the so-called *principal series representation*. Let us therefore discuss this representation in detail.

It acts on an infinite-dimensional vector space $\mathbb{V}^{\mathrm{ps}}$ spanned by the states $|m\rangle$ with $m \in \mathbb{Z}$. The representation of the generators $\mathrm{J}_z, \mathrm{J}_\pm$ is determined by two parameters γ, α

$$\begin{aligned}
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_+)|m\rangle &= \delta_m(m+\alpha+\gamma+\tfrac{1}{2})|m+1\rangle,\\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_z)|m\rangle &= (m+\alpha)|m\rangle,\\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_-)|m\rangle &= -\delta^{-1}_{m-1}(m+\alpha-\gamma-\tfrac{1}{2})|m-1\rangle.
\end{aligned} \tag{9.12}$$

The parameters δ_m determine the relative normalisation of the states $|m\rangle$ and $|m+1\rangle$, hence they do not count as parameters of the representation.[8]



(9.13)

A further equivalence of representations is given by shifting the labels of the states by some integer n and at the same time shifting α by n

$$\rho^{\mathrm{ps}}_{\gamma,\alpha} \equiv \rho^{\mathrm{ps}}_{\gamma,\alpha+n}, \qquad n \in \mathbb{Z}. \tag{9.14}$$

Finally, one can flip the sign of γ by multiplying δ_m by $(m+\alpha+\frac{1}{2}-\gamma)/(m+\alpha+\frac{1}{2}+\gamma)$. The resulting representation is also equivalent

$$\rho^{\mathrm{ps}}_{\gamma,\alpha} \equiv \rho^{\mathrm{ps}}_{-\gamma,\alpha}. \tag{9.15}$$

The two parameters can be understood as follows: The eigenvalue of the Casimir invariant on the representation is determined by the parameter γ

$$\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}^2)|m\rangle = (\gamma^2 - \tfrac{1}{4})|m\rangle. \tag{9.16}$$

[6] The group $\mathrm{PSL}(n)$ is $\mathrm{SL}(n)$ modulo its centre $\mathbb{Z}_n$. In particular, flipping the sign of all coefficients a, b, c, d does not change the map (but leaves the determination constraint unchanged).

[7] Composition of transformations is equivalent to multiplication of 2×2 matrices whose entries are a, b, c, d.

[8] Under some conditions, some of the δ_n can be zero or infinite. We shall exclude this case for the time being.

Conversely, the parameter α is related to the centre $\mathbb{Z}$ of the simply connected group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The centre is given by powers of the group element Z defined by

$$Z = \exp(2\pi\mathring{\imath}\mathrm{J}_z). \tag{9.17}$$

This element is central as can be seen from

$$\mathrm{Ad}(Z)\mathrm{J}_\pm = Z\mathrm{J}_\pm Z^{-1} = \mathrm{e}^{\pm 2\pi\mathring{\imath}}\,\mathrm{J}_\pm = \mathrm{J}_\pm. \tag{9.18}$$

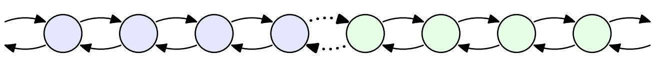
The representation of the centre is determined by the parameter α (mod 1)

$$\rho^{\mathrm{ps}}_{\gamma,\alpha}(Z)|m\rangle = \mathrm{e}^{2\pi\mathring{\imath}\alpha}\,|m\rangle. \tag{9.19}$$

The eigenvalue of the centre is important for selecting representations of the non-simply connected Lie groups $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$. In the former case $\alpha \in \frac{1}{2}\mathbb{Z}$ while the latter case further restricts $\alpha \in \mathbb{Z}$.

Reducibility. For generic parameters γ, α, the principal series representation is irreducible. For reducibility some of the coefficients $(m + \alpha \pm \gamma + \frac{1}{2})$ should be zero; without loss of generality let this be at the state $m = 0$. With a suitable choice of δ_m the representation is decomposable into a highest-weight and a lowest-weight representation[9]

$$\rho^{\mathrm{ps}}_{\pm(\alpha-1/2),\alpha+\mathbb{Z}} \equiv \rho^{\mathrm{hw}}_{\alpha-1} \oplus \rho^{\mathrm{lw}}_{\alpha}. \tag{9.20}$$


(9.21)

The lowest-weight representation is given by ($m \in \mathbb{Z}^+_0$)

$$\begin{aligned}
\rho^{\mathrm{lw}}_{\alpha}(\mathrm{J}_+)|m\rangle &= \delta_m(m + 2\alpha)|m+1\rangle,\\
\rho^{\mathrm{lw}}_{\alpha}(\mathrm{J}_z)|m\rangle &= (m+\alpha)|m\rangle,\\
\rho^{\mathrm{lw}}_{\alpha}(\mathrm{J}_-)|m\rangle &= -\delta^{-1}_{m-1} m|m-1\rangle.
\end{aligned} \tag{9.22}$$

Note that the equivalence between α and $\alpha + n$ does not hold here because the lowest-weight state $|0\rangle$ is now singled out. Furthermore, there is no equivalence between α and $1-\alpha$. The corresponding highest-weight representation based on the highest-weight state $|-1\rangle$ reads ($m \in \mathbb{Z}^-$)

$$\begin{aligned}
\rho^{\mathrm{hw}}_{\alpha}(\mathrm{J}_+)|m\rangle &= \delta_m(m+1)|m+1\rangle,\\
\rho^{\mathrm{hw}}_{\alpha}(\mathrm{J}_z)|m\rangle &= (m+1+\alpha)|m\rangle,\\
\rho^{\mathrm{hw}}_{\alpha}(\mathrm{J}_-)|m\rangle &= -\delta^{-1}_{m-1}(m+1+2\alpha)|m-1\rangle.
\end{aligned} \tag{9.23}$$

[9]The representation is decomposable if both $\delta_m(m+\alpha+\gamma+\frac{1}{2})$ for $m=-1$ and $-\delta^{-1}_{m-1}(m+\alpha-\gamma-\frac{1}{2})$ for $m=0$ approach zero while taking the limit for the parameters γ, α. If one of them remains finite, i.e. for $\delta_0 = 1$ or $\delta_0 = (m+\alpha-\gamma+\frac{1}{2})/(m+\alpha+\gamma+\frac{1}{2})$, the representation is merely reducible.

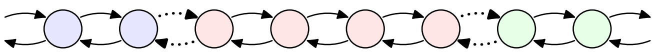
Note that the quadratic Casimir eigenvalues are given by

$$\rho^{\mathrm{lw}}_\alpha(\mathrm{J}^2) = \alpha(\alpha-1), \qquad \rho^{\mathrm{hw}}_\alpha(\mathrm{J}^2) = \alpha(\alpha+1). \tag{9.24}$$

Under yet more special conditions, all the representations can be split even further ($n \in \mathbb{Z}^+_0$)

$$\begin{aligned} \rho^{\mathrm{lw}}_{-n/2} &\equiv \rho^{\mathrm{fin}}_{n/2} \oplus \rho^{\mathrm{lw}}_{n/2+1}, \\ \rho^{\mathrm{hw}}_{n/2} &\equiv \rho^{\mathrm{hw}}_{-n/2-1} \oplus \rho^{\mathrm{fin}}_{n/2}, \\ \rho^{\mathrm{ps}}_{\pm(n+1)/2,-n/2+\mathbb{Z}} &\equiv \rho^{\mathrm{hw}}_{-n/2-1} \oplus \rho^{\mathrm{fin}}_{n/2} \oplus \rho^{\mathrm{lw}}_{n/2+1}. \end{aligned} \tag{9.25}$$

Here we find the $(n+1)$-dimensional representation $\rho^{\mathrm{fin}}_{n/2}$ with spin $s = n/2$.


(9.26)

Reality and Unitarity. Now we can consider the real form $\mathfrak{sl}(2,\mathbb{R})$ of the algebra and the constraints due to unitarity of the above three kinds of representations. The reality conditions can be imposed in different ways corresponding to the freedom of choice for a metric with a given signature. We chose them such that the above representations can admit a straight-forward unitary structure without an additional change of basis[10] [11]

$$\mathrm{J}^*_x = +\mathrm{J}_x, \qquad \mathrm{J}^*_y = +\mathrm{J}_y, \qquad \mathrm{J}^*_z = -\mathrm{J}_z. \tag{9.27}$$

With $\mathrm{J}_\pm := \mathrm{J}_x \pm \mathring{\imath}\mathrm{J}_y$ this implies $\mathrm{J}^*_\pm = \mathrm{J}_\mp$ and a unitary representation must obey

$$\rho(\mathrm{J}_z)^\dagger = \rho(\mathrm{J}_z), \qquad \rho(\mathrm{J}_\pm)^\dagger = -\rho(\mathrm{J}_\mp). \tag{9.28}$$

Let us first discuss the principal series representation. The first condition implies $\alpha \in \mathbb{R}$ and the equivalence $\alpha \equiv \alpha + \mathbb{Z}$ further allows us to restrict to the fundamental domain $-\frac{1}{2} < \alpha \leq \frac{1}{2}$. The second condition can be translated to the statement

$$|\delta_m|^2 = \frac{m+\alpha+\frac{1}{2}-\gamma}{m+\alpha+\frac{1}{2}+\gamma^*} \qquad \text{for all } m \in \mathbb{Z}. \tag{9.29}$$

The obvious solution is $\gamma \in \mathring{\imath}\mathbb{R}$ and δ_m some unspecified pure complex phase. There is also a less obvious solution with $\gamma \in \mathbb{R}$ where one has to make sure that $|\delta_m|^2 > 0$ for all $m \in \mathbb{Z}$. This holds provided that $|\gamma| < \frac{1}{2} - |\alpha|$ (with $|\alpha| \leq \frac{1}{2}$) and the representation is called *complementary series representation.*[12]

The above inequality (with solutions in $\mathbb{R}$ and $\mathring{\imath}\mathbb{R}$)

$$\gamma^2 < (\tfrac{1}{2} - |\alpha|)^2 \tag{9.30}$$

[10] Compared to the compact real form with $\mathrm{J}^*_k = -\mathrm{J}_k$ we flip two signs for $\mathfrak{so}(1,2)$.

[11] The equivalent choice $\mathrm{J}^*_x = +\mathrm{J}_x$, $\mathrm{J}^*_y = -\mathrm{J}_y$, $\mathrm{J}^*_z = +\mathrm{J}_z$ would more clearly reflect the real structure of $\mathfrak{sl}(2,\mathbb{R})$ formulated in terms of the real basis J_z and $\mathrm{J}_\pm$, but it would be harder to understand the unitarity of the above representations.

[12] It can be viewed as the analytic continuation of the principal series representation to the range $-\frac{1}{4} \leq \rho(\mathrm{J}^2) < -|\alpha|(1-|\alpha|)$ for the eigenvalue of the quadratic Casimir.

is called the *unitarity bound* for the principal series. One may wonder what happens at the unitarity bound? Quite generally, the representation becomes reducible, as can be verified with the above splitting formula. The reason for this behaviour is that the signature of the representation space changes when crossing the unitarity bound. At the unitarity bound the signature must therefore be indefinite. One can convince oneself that the null subspace admits a sub-representation.

For the lowest- and highest-weight representations also $\alpha \in \mathbb{R}$, but there is no longer an equivalence for shifts by integers. The unitarity condition for the lowest-weight representation reduces to

$$|\delta_m|^2 = \frac{m+1}{m+2\alpha} \qquad \text{for all } m \in \mathbb{Z}_0^+, \tag{9.31}$$

which implies that $\alpha > 0$. The corresponding condition for the highest-weight representation implies $\alpha < 0$. Note that at the unitarity bound the representation is again reducible and splits off a one-dimensional trivial representation.

All non-trivial finite-dimensional representations are not unitary for the real form $\mathfrak{sl}(2,\mathbb{R})$ in accordance with a general theorem for non-compact real forms.

9.3 Representations on Functions

In the following we translate the above representations to representations on spaces of functions and thus make contact to field theory.

Periodic Functions. The above infinite-dimensional representations naturally act on function spaces.[13]

A suitable space for the principal series irreps is given by periodic functions and their Fourier transform

$$f(\phi) = f(\phi + 2\pi) = \sum_{m=-\infty}^{\infty} f_m \, \mathrm{e}^{im\phi} . \tag{9.32}$$

The canonical square norm defined on this space reads

$$\|f\|^2 := \frac{1}{2\pi} \int \mathrm{d}\phi \, \big|f(\phi)\big|^2 = \sum_{m=-\infty}^{\infty} |f_m|^2. \tag{9.33}$$

Now we can map the function to the principal series space $\mathbb{V}^{\mathrm{ps}}$ by means of the Fourier coefficients f_m

$$|f\rangle = \sum_{m=-\infty}^{\infty} f_m |m\rangle. \tag{9.34}$$

[13]More precisely, one would typically restrict to square integrable functions in order to control issues of convergence and topology.

For the principal series representation we set $\delta_m = 1$ for convenience which is compatible with the unitarity conditions for $\gamma \in \mathring{\imath}\mathbb{R}$.[14] Then the representation reads

$$\begin{aligned}
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_z)|f\rangle &= \sum_{m=-\infty}^{\infty} (m+\alpha) f_m |m\rangle \\
&= \big|(-\mathring{\imath}\partial_\phi + \alpha) f\big\rangle, \\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_+)|f\rangle &= \sum_{m=-\infty}^{\infty} (m+\alpha+\gamma-\tfrac{1}{2}) f_{m-1} |m\rangle \\
&= \big|\mathrm{e}^{\mathring{\imath}\phi}(-\mathring{\imath}\partial_\phi + \alpha + \gamma + \tfrac{1}{2}) f\big\rangle, \\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_-)|f\rangle &= \sum_{m=-\infty}^{\infty} (-m-\alpha+\gamma+\tfrac{1}{2}) f_{m+1} |m\rangle \\
&= \big|\mathrm{e}^{-\mathring{\imath}\phi}(\mathring{\imath}\partial_\phi - \alpha + \gamma + \tfrac{1}{2}) f\big\rangle.
\end{aligned} \tag{9.35}$$

We can thus view the principal series representation as a representation on the function space $\mathbb{L}_2(\mathrm{S}^1)$ by differential operators and write

$$\begin{aligned}
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_z) &= -\mathring{\imath}\partial_\phi + \alpha, \\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_+) &= \mathrm{e}^{\mathring{\imath}\phi}(-\mathring{\imath}\partial_\phi + \alpha + \gamma + \tfrac{1}{2}), \\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{J}_-) &= \mathrm{e}^{-\mathring{\imath}\phi}(\mathring{\imath}\partial_\phi - \alpha + \gamma + \tfrac{1}{2}).
\end{aligned} \tag{9.36}$$

This representation is unitary for $\alpha \in \mathbb{R}$ and $\gamma \in \mathring{\imath}\mathbb{R}$ due to the hermitian conjugation property $\partial_\phi^\dagger = -\partial_\phi$ of the derivative operator with respect to the above square norm.

Functions of Time. Now we want to compare this representation to the conformal action in terms of the generators $\mathrm{H}, \mathrm{D}, \mathrm{K}$ acting on the time coordinate t as follows

$$\mathrm{H}t = \mathring{\imath}, \qquad \mathrm{D}t = \mathring{\imath}t, \qquad \mathrm{K}t = \tfrac{\mathring{\imath}}{2}t^2. \tag{9.37}$$

The $\mathfrak{sl}(2,\mathbb{R})$ algebra expressed in these (imaginary) generators reads

$$[\![\mathrm{D},\mathrm{H}]\!] = -\mathring{\imath}\mathrm{H}, \qquad [\![\mathrm{D},\mathrm{K}]\!] = +\mathring{\imath}\mathrm{K}, \qquad [\![\mathrm{H},\mathrm{K}]\!] = \mathring{\imath}\mathrm{D}. \tag{9.38}$$

In order to relate the representations we should match the generators with the J_k and the time t with the coordinate ϕ. The appropriate transformation is

$$t = \tan(\phi/2), \tag{9.39}$$

and it maps the complete time axis $\mathbb{R}$ to one period $-\pi < \phi < \pi$ of ϕ. The relationship between the generators reads

$$\mathrm{D} = \mathring{\imath}\mathrm{J}_x, \qquad \mathrm{H} = -(\mathrm{J}_z + \mathring{\imath}\mathrm{J}_y), \qquad \mathrm{K} = -\tfrac{1}{2}(\mathrm{J}_z - \mathring{\imath}\mathrm{J}_y). \tag{9.40}$$

[14] In order to make the complementary series representations with $\gamma \in \mathbb{R}$ unitary, one must choose a different (non-local) square norm for f.

The transformed representation on functions of time now reads

$$\begin{aligned}
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{H}) &= \mathring{\imath}\partial_t - 2\alpha\,\frac{1}{1+t^2} - 2\mathring{\imath}(\gamma+\tfrac{1}{2})\,\frac{t}{1+t^2}\,,\\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{D}) &= \mathring{\imath}t\partial_t - \alpha\,\frac{2t}{1+t^2} + \mathring{\imath}(\gamma+\tfrac{1}{2})\,\frac{1-t^2}{1+t^2}\,,\\
\rho^{\mathrm{ps}}_{\gamma,\alpha}(\mathrm{K}) &= \tfrac{\mathring{\imath}}{2}t^2\partial_t - \alpha\,\frac{t^2}{1+t^2} + \mathring{\imath}(\gamma+\tfrac{1}{2})\,\frac{t}{1+t^2}\,.
\end{aligned} \tag{9.41}$$

Note that this representation is unitary if we impose the following hermitian conjugation on the derivative operator

$$\partial_t^\dagger = -\partial_t + \frac{2t}{1+t^2}\,. \tag{9.42}$$

This relation follows from the transformation $\mathrm{d}\phi = 2\,\mathrm{d}t\,/(1+t^2)$ which implies the following square norm for functions of time[15]

$$\|f\|^2 = \int \frac{2\,\mathrm{d}t}{1+t^2}\,\big|f(t)\big|^2. \tag{9.43}$$

We can see that the conformal action on the time coordinate is the representation with $\gamma = -\frac{1}{2}$ and $\alpha = 0$. This representation is right on the unitarity bound where it splits into a pair of highest- and lowest-weight representations. These are unitary representations, but only for a square norm which is non-local in time.

Our findings in fact also explain how the above representation on periodic functions is relevant for conformal field theory: The periodic coordinate ϕ becomes global time τ when going to the conformal completion of time $\mathrm{S}^1 \to \mathbb{R}$. The representation on functions of global time reads

$$\begin{aligned}
\rho^{\mathrm{glob}}_{\gamma}(\mathrm{J}_z) &= -\mathring{\imath}\partial_\tau,\\
\rho^{\mathrm{glob}}_{\gamma}(\mathrm{J}_+) &= \mathrm{e}^{\mathring{\imath}\tau}\,(-\mathring{\imath}\partial_\tau + \gamma + \tfrac{1}{2}),\\
\rho^{\mathrm{glob}}_{\gamma}(\mathrm{J}_-) &= \mathrm{e}^{-\mathring{\imath}\tau}\,(\mathring{\imath}\partial_\tau + \gamma + \tfrac{1}{2}).
\end{aligned} \tag{9.44}$$

In terms of physics, $-\mathrm{J}_z$ clearly takes the role of the Hamiltonian for global time.

This representation is the direct integral of principal series representations $\rho^{\mathrm{glob}}_{\gamma,\alpha}$ over all $0 \leq \alpha < 1$. Namely, the representation of the central element $Z = \exp(2\pi\mathring{\imath}\mathrm{J}_z)$ acts as a shift over a whole period $\rho^{\mathrm{glob}}_{\gamma}(Z)f(\tau) = f(\tau+2\pi)$. Then then functions on the lattice $\tau + 2\pi\mathbb{Z}$ with $0 \leq \tau < 2\pi$ fixed can be can be decomposed according to the eigenvalues $\mathrm{e}^{\mathring{\imath}\alpha}$ of $\rho^{\mathrm{glob}}_{\gamma}(Z)$.

Local Operators. Let us find out the transformation properties of local operators. Define a state $|\mathit{\Delta}\rangle = \mathcal{O}(0)$ as some primary operator $\mathcal{O}$ at the origin $t = 0$. This state is characterised by the conditions

$$\rho(\mathrm{K})|\mathit{\Delta}\rangle = 0, \qquad \rho(\mathrm{D})|\mathit{\Delta}\rangle = -\mathring{\imath}\mathit{\Delta}|\mathit{\Delta}\rangle. \tag{9.45}$$

[15] The hermitian conjugate $\partial_t^\dagger$ follows from integration by parts between the functions $f(t)$ and $f(t)^*$, and the additional term is due to the measure factor.

It is a lowest-weight state with $\alpha = \varDelta$ if we were to identify the generators as $\mathrm{K} \to \frac{1}{2}\mathrm{J}_-$, $\mathrm{D} \to -\mathring{\imath}\mathrm{J}_z$, $\mathrm{H} \to \mathrm{J}_+$. These generators satisfy the algebra relations of $\mathfrak{sl}(2, \mathbb{C})$, but they have inconvenient reality conditions for the study of unitarity because complex conjugation does not interchange the ladder generators $\mathrm{K} \sim \mathrm{J}_-$ and $\mathrm{H} \sim \mathrm{J}_+$. For instance, we cannot claim this representation to be unitary for $\varDelta > 0$.

In order to study unitarity, let us first construct the complete representation on $\mathcal{O}(t)$. We reconstruct the local operator by shifting the state $|\varDelta\rangle$ to the appropriate time t

$$\mathcal{O}(t) = \exp\bigl(\mathring{\imath}t\rho(\mathrm{H})\bigr)|\varDelta\rangle. \tag{9.46}$$

By commuting the generators past the exponential we find the representation[16]

$$\begin{aligned}\rho(\mathrm{H})\mathcal{O}(t) &= -\mathring{\imath}\partial_t\mathcal{O}(t),\\ \rho(\mathrm{D})\mathcal{O}(t) &= -\mathring{\imath}t\partial_t\mathcal{O}(t) - \mathring{\imath}\varDelta\mathcal{O}(t),\\ \rho(\mathrm{K})\mathcal{O}(t) &= -\tfrac{\mathring{\imath}}{2}t^2\partial_t\mathcal{O}(t) - \mathring{\imath}\varDelta t\mathcal{O}(t).\end{aligned} \tag{9.47}$$

We should now find maximal-weight states w.r.t. the original generators

$$\mathrm{J}_z = -\tfrac{1}{2}\mathrm{H} - \mathrm{K}, \qquad \mathrm{J}_\pm = -\mathring{\imath}\mathrm{D} \mp (\tfrac{1}{2}\mathrm{H} - \mathrm{K}). \tag{9.48}$$

It turns out that a lowest-weight state is given by

$$|0\rangle = \int \mathrm{d}t\, (t + \mathring{\imath})^{2\varDelta-2}\mathcal{O}(t). \tag{9.49}$$

By (formally) applying integration by parts one can convince oneself that the following properties hold

$$\rho(\mathrm{J}_-)|0\rangle = 0, \qquad \rho(\mathrm{J}_z)|0\rangle = -(\varDelta - 1)|0\rangle. \tag{9.50}$$

Moreover, there is also a highest-weight state

$$|{-1}\rangle = \int \mathrm{d}t\, (t - \mathring{\imath})^{2\varDelta-2}\mathcal{O}(t), \tag{9.51}$$

which satisfies

$$\rho(\mathrm{J}_+)|{-1}\rangle = 0, \qquad \rho(\mathrm{J}_z)|{-1}\rangle = +(\varDelta - 1)|{-1}\rangle. \tag{9.52}$$

Altogether, we find that the local operator representation is a sum of two representations[17]

$$\rho \equiv \rho^{\mathrm{lw}}_{1-\varDelta} \oplus \rho^{\mathrm{hw}}_{\varDelta-1}. \tag{9.53}$$

This representation is in fact not unitary unless $\varDelta < 1$. Being non-unitary is a general feature of local operators in a (Lorentzian) conformal field theory.[18]

[16] Note that this is a representation on the operator $\mathcal{O}(t)$ (which happens to be a function of t). It is not a representation on the function $\mathcal{O}(t)$ (which happens to be an operator) even though it is expressed through derivatives acing on $\mathcal{O}(t)$. The difference between these two notions is important because it implies a different ordering of differential operators. In particular, this explains the opposite sign for the differential terms which is needed to satisfy the same algebra.

[17] According to the logic of Fourier transformations, the positive and negative frequency parts are separated into the highest- and lowest-weight representations, respectively.

[18] Perplexingly, the physically relevant local operators should satisfy the would-be unitarity conditions derived from the lowest-weight representation based on $|\varDelta\rangle$ in the above basis $\mathrm{H}, \mathrm{D}, \mathrm{K}$, i.e. $\varDelta > 0$.

9.4 Correlation Functions

Next we discuss some aspects of conformal correlators in general d dimensions in order to find out about the conformal properties of various objects.

Scalar Field. Consider first a scalar field $\Phi(x)$. The scalar is a primary field of conformal dimension Δ. The conformal correlator of two scalar fields reads

$$\big\langle \Phi(x_1)\,\Phi(x_2)\big\rangle = \frac{C}{(x_{12}^2)^{\Delta}}\,. \tag{9.54}$$

The scalar fields obey the massless Klein–Gordon equation $\partial_\mu\partial^\mu\Phi = 0$ (up to source terms or interactions), and thus the correlation function should do as well.[19] One finds

$$\big\langle \partial_\mu\partial^\mu\Phi(x_1)\,\Phi(x_2)\big\rangle = \partial_{1,\mu}\partial_1^\mu\,\frac{C}{(x_{12}^2)^{\Delta}} = \frac{2C\Delta(2\Delta+2-d)}{(x_{12}^2)^{\Delta+1}}\,. \tag{9.55}$$

On the one hand, the result implies that an elementary scalar field has scaling dimension $\Delta = \frac{1}{2}d - 1$ (which equals $\Delta = 1$ in $d = 4$ spacetime dimensions).[20] Thus the scaling dimension coincides with the usual mass dimension of the field. On the other hand, any scalar primary operator with $\Delta = \frac{1}{2}d - 1$ automatically obeys the Klein–Gordon equation, be it fundamental or not. Such a field carries substantially fewer degrees of freedom than an unconstrained field. We will discuss the implications of this observation on representation theory further below.

Field Strength. Next we consider the field strength $F_{\mu\nu}$ of electromagnetism. The latter is a conformal field theory and the field strength a conformal primary field,[21] but now with non-trivial spin. Based on the general construction of two-point functions, its correlator reads

$$\big\langle F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)\big\rangle = \frac{C(I_{\mu\rho}I_{\nu\sigma} - I_{\mu\sigma}I_{\nu\rho})}{(x_{12}^2)^{\Delta}}\,. \tag{9.56}$$

Now the homogeneous Maxwell equation $dF = 0$ within the correlation function yields

$$\big\langle dF_{\kappa\mu\nu}(x_1)F_{\rho\sigma}(x_2)\big\rangle = \frac{2(2-\Delta)C(x_{12,\kappa}\eta_{\mu\rho}\eta_{\nu\sigma} + \ldots)}{(x_{12}^2)^{\Delta+1}}\,. \tag{9.57}$$

The inhomogeneous Maxwell equation in the absence of sources reads $\partial^\mu F_{\mu\nu} = 0$, and within the correlation function one finds

$$\big\langle \partial^\mu F_{\mu\nu}(x_1)F_{\rho\sigma}(x_2)\big\rangle = \frac{2(\Delta-d+2)C(x_{12,\rho}I_{\nu\sigma} - x_{12,\sigma}I_{\nu\rho})}{(x_{12}^2)^{\Delta+1}}\,. \tag{9.58}$$

[19] We will not be interested in source terms which are localised to coincident points.

[20] Here we consider a free theory or the free limit of an interacting model. Furthermore, the fundamental fields are typically not observable on their own if they are charged under some local symmetry in which case their conformal dimension is not a meaningful quantity.

[21] In quantum field theory one would usually consider it as a descendant of the gauge potential A_μ. However, the latter is not gauge invariant and the correlator not uniquely determined. The one of the field strength is gauge-invariant and thus well-defined.

Altogether the equations imply $\Delta = 2$ and $d = 4$. This shows immediately that the electromagnetic field is conformal in four spacetime dimensions only.

Conserved Current. A third example is a conserved current of Noether's theorem. Within a conformal field theory this is a primary operator $\mathcal{J}^\mu$ with a vector index which satisfies the conservation equation $\partial_\mu \mathcal{J}^\mu$. The correlation function of a vector primary takes the form

$$\big\langle \mathcal{J}^\mu(x_1)\, \mathcal{J}^\nu(x_2)\big\rangle = \frac{C I^{\mu\nu}}{(x_{12}^2)^\Delta}\,, \tag{9.59}$$

and the correlator of the divergence of the current reads

$$\big\langle \partial_\mu \mathcal{J}^\mu(x_1)\, \mathcal{J}^\nu(x_2)\big\rangle = \frac{2(\Delta - d + 1) C x_{12}^\nu}{(x_{12}^2)^{\Delta+1}}\,. \tag{9.60}$$

This implies that a conserved current must have conformal dimension $\Delta = d - 1$ in accordance with its derivation from the Lagrangian density.

One current of central importance for any conformal field theory is the stress energy tensor $\mathcal{T}_{\mu\nu}$ with two symmetric traceless vector indices. It is the current responsible for conformal symmetry itself, and its conservation implies $\Delta = d$.

9.5 Representations of $\mathfrak{so}(4,2)$

Finally, we shall discuss the representation theory relevant to a conformal field theory in $d = 4$ spacetime dimensions. The conformal algebra is $\mathfrak{so}(4,2)$ which is equivalent to the unitary algebra $\mathfrak{su}(2,2)$. What makes the higher-dimensional conformal algebras interesting is that while their unitary representations are infinite-dimensional, they also consist of finite-dimensional components corresponding to the maximally compact subalgebra $\mathfrak{so}(4)$. As usual, we shall complexify the Lie algebras, e.g. the conformal algebra is $\mathfrak{so}(6,\mathbb{C}) = \mathfrak{sl}(4,\mathbb{C})$.

Local Operator Representations. The definition of a conformal primary operator $\mathcal{O}(x)$ is reminiscent of the definition of a highest-weight state[22]

$$\rho(\mathrm{K}_\mu)|\Delta, s_1, s_2\rangle = 0, \qquad \rho(\mathrm{D})\mathcal{O}(0) = -\mathring{\imath}\Delta|\Delta, s_1, s_2\rangle. \tag{9.61}$$

A minor difference between the concepts is the following: A primary local operator $\mathcal{O}_{\mu\ldots\rho}(x)$ with spin is usually considered to be the whole vector space of its $\mathfrak{so}(4)$ representation. Conversely, the highest-weight state is the highest-weight state of the $\mathfrak{so}(4)$ representation evaluated at the origin $\mathcal{O}_{+\ldots+}(0)$. The descendants of this

[22]Here we assume that the weight decreases with Δ such that the conformal boost K is a raising generator and the momentum P a lowering operator. We use this convention so that the finite-dimensional representations of the Lorentz subalgebra are described as usual by their highest weight rather than their lowest weight.

highest-weight state are the $\mathfrak{so}(4)$ descendants forming the whole multiplet $\mathcal{O}_{\mu\ldots\rho}(0)$ as well as the conformal descendants $\partial\ldots\partial\mathcal{O}(0)$ forming the Taylor expansion of the field $\mathcal{O}(x)$.

From the above considerations one can compute the $\mathfrak{sl}(4)$ Dynkin labels of a local operator representation

$$[p_1, r, p_2] \qquad \text{with} \qquad p_k = 2s_k \text{ and } r = -\varDelta - s_1 - s_2. \tag{9.62}$$

The Dynkin labels are a concise way to describe the local operator representation. The outer two labels are non-negative integers describing the spin, whereas the middle label is a typically negative real number describing the conformal dimension. For example $[0, -1, 0]$ describes a scalar field with dimension 1. The Dynkin labels are particularly useful when expressing the unitarity conditions of representations.

The local operator representation can be summarised by the character polynomial

$$\begin{aligned} P_{\Delta,s_1,s_2}(q) &= \frac{P_{s_1}(q_1)P_{s_2}(q_2)\, q_{\mathrm{D}}^{\varDelta}}{\prod_{m_1,m_2=\pm 1}\left(1 - q_{\mathrm{D}} q_1^{m_1} q_2^{m_2}\right)}\,, \\ P_s(q) &= \frac{q^{2s+1} - q^{-2s-1}}{q - q^{-1}}\,, \end{aligned} \tag{9.63}$$

where $q_1^{2m_1} q_2^{2m_2} q_{\mathrm{D}}^{\varDelta}$ describes a state with $\mathfrak{so}(4) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ weights (m_1, m_2) and conformal dimension $\varDelta$. Here the numerator describes the $\mathfrak{so}(4)$ representation of the primary operator and the denominator describes the descendants w.r.t. the 4 derivatives P_μ which can act any number of times.

Particular Representations. Above, we have discussed particular local operators and their correlation functions. Let us translate the results to representation theory of the conformal group. The (free) scalar field $\varPhi$ obeys the equation of motion

$$\rho(\mathrm{P}^\mu)\rho(\mathrm{P}_\mu)\varPhi = 0. \tag{9.64}$$

In other words, the descendant state $\rho(\mathrm{P}^2)\varPhi$ is absent from the local operator representation. We have discussed above that this is consistent only if the descendant is a highest-weight state itself so that the corresponding sub-representation can be projected out. Indeed, one finds that the full local operator representation $[0, -1, 0]$ has a sub-representation $[0, -3, 0]$. One can also convince oneself that the latter weight is in the orbit of the shifted Weyl group of the former weight. The character polynomial of the on-shell scalar fields reads

$$P_\varPhi = P_{[0,-1,0]} - P_{[0,-3,0]}. \tag{9.65}$$

The consideration for the other two examples is similar: The electromagnetic field strength corresponds splits into chiral and anti-chiral components with representations $[2, -3, 0]$ and $[0, -3, 2]$. Their Maxwell equations are operators in the representations $[1, -4, 1]$. Incidentally, this is the representation of the conserved Noether current (for the global gauge transformations). Its divergence is zero which means that the sub-representation $[0,-4,0]$ has been projected out. For the character polynomials one finds

$$\begin{aligned} P_F &= (P_{[2,-3,0]} - P_{[1,-4,1]} + P_{[0,-4,0]}) \\ &\quad + (P_{[0,-3,2]} - P_{[1,-4,1]} + P_{[0,-4,0]}), \\ P_{\mathcal{J}} &= P_{[1,-4,1]} - P_{[0,-4,0]}. \end{aligned} \tag{9.66}$$

This sequence of representations curiously continues to lower dimensions: The shifted Weyl orbit of the trivial weight $[0,0,0]$ contains virtually all the physical objects of electromagnetism

$$\begin{aligned} [0,0,0]\ (\mathit{\Delta}=0):&\quad \text{global gauge transformation,} \\ [1,-2,1]\ (\mathit{\Delta}=1):&\quad \text{gauge field / local gauge transformation,} \\ [2,-3,0]\ (\mathit{\Delta}=2):&\quad \text{chiral field strength,} \\ [0,-3,2]\ (\mathit{\Delta}=2):&\quad \text{anti-chiral field strength,} \\ [1,-4,1]\ (\mathit{\Delta}=3):&\quad \text{conserved electromagnetic current,} \\ [0,-4,0]\ (\mathit{\Delta}=4):&\quad \text{Lagrangian density.} \end{aligned} \tag{9.67}$$

The remaining 18 images under Weyl reflections serve as lowest-weight delimiters of the finite-dimensional representations of $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ for these 6 highest-weights.

Unitarity and Splitting. Finally, let us discuss unitarity of the above representations for the real form $\mathfrak{so}(4,2)=\mathfrak{su}(2,2)$. Unitary representations correspond to states of a quantum (field) theory on the space $\mathbb{R}\times\mathrm{S}^3$ which is the conformal completion of Minkowski space. The states' wave functions on the compact space S^3 decompose into a tower of spherical harmonics of S^3 which transform under the unitary finite-dimensional representations of $\mathfrak{so}(4)$.

Note that (as discussed earlier) local operators do not transform in unitary representation,[23] but their representations are nevertheless related to unitary representations. Effectively, the local operators transform in the very same representations of the complexified conformal algebra, but the reality conditions are not the ones needed for proper unitarity.[24]

Whether or not a given highest-weight representation is unitary mainly depends on the position of the images of the highest weight under the shifted Weyl reflections. The crucial point is that the squared norm of states changes sign roughly at the location of the images in $\mathfrak{h}^*$. There are two options to avoid a changing sign: Either the reflected weight is higher than the highest weight and therefore does not flip the sign within the highest-weight representation. Or the reflected weight is lower, but it is reachable from the highest weight by an integral combination of the negative roots. In the latter case, the representation is reducible and there is a

[23] For example, the momentum generator P_μ is a vector of $\mathfrak{so}(3,1)$ rather than the compact $\mathfrak{so}(4)$.

[24] Or something like this, it is complicated ...

chance that the unwanted signs are all contained in the sub-representation and can be projected out. The above words translate to constraints for the highest weight μ

$$\frac{2\langle\mu+\delta,\alpha\rangle}{\langle\alpha,\alpha\rangle}-1\in\mathbb{R}^-\cup\mathbb{Z}_0^+\qquad\text{for all }\alpha\in\Delta^+. \tag{9.68}$$

For the Dynkin labels $[p_1,r,p_2]$ of a $\mathfrak{su}(2,2)$ representation this condition implies that the six combinations

$$p_1,r,p_2,p_1+r+1,r+p_2+1,p_1+r+p_2+2 \tag{9.69}$$

are either negative real numbers or non-negative integers. More concretely, p_1 and p_2 should be non-negative integers because they determine the highest weights w.r.t. the compact subgroup $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$; conversely, r should be a non-positive real number

$$p_1,p_2\in\mathbb{Z}_0^+,\qquad r\in\mathbb{R}_0^-. \tag{9.70}$$

The concrete conditions for the other three combinations depend on whether some of the Dynkin labels are zero.

The unitarity conditions for unitary irreducible highest-weight representations of $\mathfrak{su}(2,2)$ can be summarised as [25]

$$\begin{aligned}
[p_1,r,p_2]:&\quad\text{unitary if }p_1,p_2\in\mathbb{Z}_*^+,\ r\leq -p_1-p_2-2,\\
[p,r,0]:&\quad\text{unitary if }p\in\mathbb{Z}_*^+,\ r\leq -2-p\text{ or }r=-1-p,\\
[0,r,p]:&\quad\text{unitary if }p\in\mathbb{Z}_*^+,\ r\leq -2-p\text{ or }r=-1-p,\\
[0,r,0]:&\quad\text{unitary if }r\leq -2\text{ or }r=-1\text{ or }r=0.
\end{aligned} \tag{9.71}$$

The solutions to these constraints yields the following physical objects along with their representation splitting (at the unitarity bounds):

- The trivial representation

$$[0,0,0]. \tag{9.72}$$

- The massless scalar particle ($\Delta=1$)

$$[0,-1,0]. \tag{9.73}$$

  The equation of motion transforms in the sub-representation $[0,-3,0]$.
- Massless particle with helicity $\pm p/2$, $p\in\mathbb{Z}_*^+$ ($\Delta=1+p/2$)

$$[p,-p-1,0]\text{ or }[0,-p-1,p],\ \text{respectively.} \tag{9.74}$$

  The equation of motion transforms in the sub-representation $[p-1,-p-2,1]$ or $[p-1,-p-2,1]$, respectively (which itself is reducible for $p>1$).
- Conserved current with spin $(p_1/2,p_2/2)$, $p_k\in\mathbb{Z}_*^+$ ($\Delta=2+p_1/2+p_2/2$)

$$[p_1,r,p_2],\qquad p_1+r+p_2+2=0. \tag{9.75}$$

  The conservation condition transforms in the sub-representation $[p_1-1,r,p_2-1]$.

[25] The unitarity conditions for non-compact real forms of other Lie algebras take a similar form.

- A generic local operator ($\varDelta \geq 2 + p_1/2 + p_2/2$)

$$[p_1, r, p_2], \qquad p_1 + r + p_2 + 2 < 0, \tag{9.76}$$

  or a non-conserved operator at the unitarity bound ($\varDelta = 2 + p/2$)

$$[p, r, 0] \text{ or } [0, r, p], \qquad p + r + 2 = 0. \tag{9.77}$$

  This representation has no sub-representations among the conformal descendants.

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