METRIC-AFFINE GAUGE THEORY OF GRAVITY: FIELD EQUATIONS, NOETHER IDENTITIES, WORLD SPINORS, AND BREAKING OF DILATION INVARIANCE

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Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance ★

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Abstract

In Einstein's gravitational theory, the spacetime is Riemannian, that is, it has vanishing torsion and vanishing nonmetricity (covariant derivative of the metric). In the gauging of the general affine group $A(4,R)$ and of its subgroup $GL(4,R)$ in four dimensions, energy-momentum and hypermomentum currents of matter are canonically coupled to the coframe and to the connection of a metric-affine spacetime with nonvanishing torsion and nonmetricity, respectively. Fermionic matter can be described in this framework by half-integer representations of the $\overline{SL}(4,R)$ covering subgroup. We set up a (first-order) Lagrangian formalism and build up the corresponding Noether machinery. For an arbitrary gauge Lagrangian, the three gauge field equations come out in a suggestive Yang–Mills like form. The conservation-type differential identities for energy-momentum and hypermomentum and the corresponding complexes and superpotentials are derived. Limiting cases such as the Einstein–Cartan theory are discussed. In particular we show, how the $A(4,R)$ may "break down" to the Poincaré (inhomogeneous Lorentz) group. In this context, we present explicit models for a symmetry breakdown in the cases of the Weyl (or homothetic) group, the $SL(4,R)$, or the $GL(4,R)$.

This article is dedicated to the memory of Rev. Dr. J. Dermott McCrea, OFM, who passed away on 21 May 1993. We lost a dear friend and an esteemed colleague whom we closely collaborated with since the early eighties.

1. Deformable spacetime structures in physics

1.1. Introduction: The need to go beyond Riemannian manifolds

A number of developments in physics in recent years have evoked the possibility that the treatment of spacetime might involve more than just the Riemannian spacetime V_4 of Einstein's general relativity:

(1) The vain effort so far to *quantize gravity* is, perhaps, the strongest piece of evidence for going beyond a geometry which is dominated by the classical distance concept.

(2) The generalization of the three-dimensional theory of elastic *continua with microstructure* to the four-dimensional spacetime of gravity suggests, in a rather convincing manner, physical interpretations for the newly emerging structures in post-Riemannian spacetime geometry.

(3) The description of hadron (or nuclear) matter in terms of *extended structures*: In particular, the quadrupole pulsation rates of that matter and, in a rest frame, their relation to representations of the volume-preserving three-dimensional linear group $SL(3,R)$ – with the rotation group $SO(3)$ as subgroup – have been established experimentally.

(4) The study of the *early universe* – in the light of the various theorems about a singular origin, the ideas about unification of the fundamental interactions (mostly involving additional dimensions, later compactified) and inflationary models with dilaton-induced Weyl covector

... and each of these developments necessitates the study of dynamical theories involving post-Riemannian geometries, whether in the context of local field theories or within the framework of string theories. We explain the interest in continua with microstructure, in extended structures, and the problematics of the early universe – as far as these are relevant as motivations for a relaxation of the Riemannian constraint in gravity – in the rest of this chapter, leaving the rather involved issue of quantum gravity to section 2.

The smallest departure from a V_4 would consist in admitting *torsion*, the field strength of local translations, arriving thereby at a Riemann–Cartan spacetime U_4 and, furthermore, *nonmetricity*, resulting in a *metric-affine* (L_4, g) spacetime [275]. In what follows, starting with section 3, we will deal with the geometry of spacetime, the Euler–Lagrange field equations, the Noether identities generalizing flat conservation laws, the conformal properties, and with a specific model of spontaneous symmetry breakdown – and all this in the framework of such metric-affine spacetimes. For reasons that will become clear in the sequel, we study in particular spacetime models arising from a Weyl/Yang–Mills-like gauge theory approach to gravity.

1.2. Spacetime as a continuum with microstructure

In Einstein's general relativity theory (GR), the linear connection of its Riemannian spacetime is (i) metric(-compatible), that is, the length and angle measurements are integrable, and (ii) symmetric. The symmetry of the Riemannian (or Levi-Civita) connection translates into the closure of infinitesimally small parallelograms, that is, in spaces with an asymmetric connection such parallelograms could carry a closure failure. Already the transition from the flat gravity-free Minkowski spacetime to the Riemannian spacetime in Einstein's theory can locally be understood as a *deformation* process. A strain tensor ε_{AB} in continuum mechanics [385] measures by its very definition $\varepsilon_{AB} := (\overset{\text{defor.}}{g_{AB}} - \overset{\text{undef.}}{g_{AB}})/2$ the change of the metric between the undeformed and the deformed state.

Thus, because of the pairing of stress and strain, it does not come as a surprize that in GR, according to Hilbert's definition, the *stress*-energy–momentum tensor couples the Lagrangian to the *metric*.

The lifting of the constraints of metric-compatibility and symmetry yields nonmetricity and torsion, respectively. The continuum under consideration, here classical spacetime, is thereby assumed to have a non-trivial microstructure, similar to that of a liquid crystal or a dislocated metal or the like. In particular, to drop the metricity condition, i.e. to allow for nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta} \neq 0$, and to "touch" thereby the lightcone, if parallelly displaced, is classically a step of unusual boldness, but may be unavoidable in quantum gravity. It is gratifying, though, to have the geometrical concepts of nonmetricity and torsion already arising in the (three-dimensional) continuum theory of lattice defects – and there they have concrete interpretations as densities of point defects and line defects (dislocations), respectively, cf. [373,375]. But even more, certain types of "hyperstress" are induced by these post-Riemannian structures: Double-stress without moment relates to nonmetricity, spin moment stress to torsion.[1]

Just as ordinary stress is the analogue of the (Hilbert) energy–momentum density, hyperstress finds its field-theoretical image in the density of hypermomentum which consists of

$$\text{spin current} \oplus \text{dilation current} \oplus \text{shear current}. \tag{1.2.1}$$

And these currents ought to couple to the corresponding post-Riemannian structures, a hypothesis which brought the metric-affine gravity theory under way in the first place [277,278].

According to Sakharov [584,585], gravitation represents a "metrical elasticity" of space which is brought about by quantum fluctuations of the vacuum. Here we pursue this analogy with continuum mechanics much further and introduce additional nonmetric and torsional degrees of freedom into spacetime, but, we believe, it is done in the same spirit.

1.3. Hadrons as extended structures – effective 'strong gravity'

With the discovery of a spatial spread for the hadrons – first in experiments measuring the electromagnetic form factors, then in the identification of the baryons with an SU(3) octet (rather than with the fundamental representation of the group, as in the Sakata model) and the conception of quarks as constituents – it became important to describe the dynamics and kinematics of quantum extended structures (extendons). The 1965 work dealt with three-dimensional vibrating and rotating "lumps" [167]. Then came dual models [699] and their reinterpretation as a quantum string [475,663], a one-dimensional extendon. It was later understood as an "effective" description of QCD flux tubes, extending between point-quarks [517].

Extendons can be deformed, and thus represent affine geometries in themselves. Hadron excitations show up as Regge trajectories, and the massive states fit $\overline{SL}(3,R)$ representations – as would indeed be expected from the pulsations of a (consider it as an approximation) fixed-volume three-extendon

[1] There exists an extended literature on continua with microstructure, see, for example, Ericksen [186], Jaunzemis [319], Mindlin [462], and Nye [527]. Kröner's articles [372–376,281] on lattice defects are particularly illuminating, since they relate differential geometric notions to distributions of lattice defects. His article on the lattice interpretation of nonmetricity [376] seems remarkable; however, no use of it has been made so far. The gauge-theoretical point of view is stressed by Kleinert [352]. The analogies between three-and four-dimensional continua with microstructure have been particularly worked out in [246,267,291,437]. In [274] the dislocation as a model for torsion and its similarity to spacetime structures in the Einstein–Cartan theory of gravity has been emphasized; for a recent development, see [678].

[167]. The quantum d-extendon involves covariance of a $(d+1)$ manifold (e.g. the world-sheet for the string), the extendon's time evolution. This resembles gravity, involves gauging geometrical groups, and often reproduces the same equations that were derived in the pursuit of quantum gravity.

It is thus not surprising that "effective strong-gravity" theories, in which the Planck length l is replaced by the Compton wavelength of the proton, were derived in the same context. One such example [292] with a confining Pöschl–Teller type potential [447] in the effective radial Schrödinger equation arose in the Poincaré gauge theory and its generalization to $SL(6,C)$ flavor models of Salam et al. [587,588,443]. By including the $SU(3)$ color group of QCD, one ends up with the $SL(6,C)^f \otimes SL(6,C)^c$ model of color geometrodynamics [445,447]. Another such treatment has used affine manifolds [490,498]. In these models, "low energy" means "hadron energies", i.e. 1–100 GeV. The slope of the trajectories is of the order of 1 GeV, as against the 10^{19} GeV of the theories we mentioned in section 1.1 above.

In a recent version of this approach, "chromogravity" [499,501,629,631] is derived from QCD itself, as an "effective" theory. A gravitation-like component is identified in the infrared limit of QCD, its contribution providing for color confinement, for the systematics of the excitations in the hadron spectrum (Regge sequences), and for the forces of longer-range responsible for the nuclear excitation spectrum. This QCD-generated graviton-like component is the analog of van der Waals forces in molecules, where a $J=2$ combination of two photons is exchanged between atoms; in QCD, a $J=2$-mediated zero-color component, plus all higher spin zero-color combinations of QCD gluons, make up this pseudo-gravitational component. The emergence of a $J=2$ contribution from a $J=1$ force (QCD) in higher orders is similar to the generation of the $J=2$ gravitational contribution in string theory, from closed strings – i.e. from the contraction of two open strings – an open string corresponding in the massless sector to a $J=1$-mediated force.

1.4. The early universe (cosmogony)

Already in the seventies, various theorems implied that, with a cosmology based on Riemannian geometry, the universe was forced either to have come out of a singularity – or, inevitably, to fall into one in the future. The simplest way of avoiding such a result is to assume that in the distant past – or the distant future – the geometry is not Riemannian.

In the late seventies and in the eighties, the same conclusion emerged from the new studies of the early universe connected with gauge unification theories[2] (GUT) [541] and their supersymmetric extensions – later replaced by unification and superunification as derived from the quantum superstring. In these theories the early universe has additional dimensions (and superdimensions). It is assumed that these extra dimensions spontaneously compactify, leaving internal symmetries as residual effects in the final four-dimensional spacetime, cf. [711]. The symmetries that we have identified phenomenologically include those of the $SU(3)\times SU(2)\times U(1)$ group of the standard model embedded within higher rank groups such as $E(6)$ or $SU(5)$. All of this implies geometries ranging from Kähler and Calabi–Yau to affine manifolds.

[2] Although the initials GUT were originally taken to mean Grand Unified Theories, it was later agreed (1979 HEPAP Conference) to read them as Gauge Unification Theories, in order to leave room, as might be needed, for 'grander' theories some day.

The eighties also ushered in *inflationary cosmology* [251,395,7], a new conception of the very early universe, now from the point of view of cosmology itself, rather than particle physics (though it does affect it too). In the more advanced "extended" models [382] one finds it necessary to abandon the Riemannian constraints [651], at the very least replacing Einstein's geometry by Weyl's. We deal with this situation in an example in section 6.

1.5. Organization of the paper and notation

In section 2 we take a tour d'horizon around quantum gravity. We mention the main open questions and unsolved problems.

In section 3 we show how, by starting with the affine group $A(n,R)$ and its Yang–Mills type gauging, we eventually arrive at a metric-affine geometry of spacetime, the structures and properties of which we explicate in the rest of this chapter. In particular, the potentials emerging from the affine connection are the coframe and the linear connection. The latter is decomposed into Riemannian and post-Riemannian pieces, and the interrelations of the Chern–Simons terms to the Bianchi identities are exhibited. The rules of exterior calculus we defer to appendix A and the irreducible decompositions of nonmetricity, torsion, curvature, and of the Bianchi identities to appendix B. All this is more or less traditional wisdom. However, we emphasize the post-Riemannian structures, such as the nonmetricity, the Weyl one-form, and the volume-preserving piece of the connection, within a coherent geometrical framework.

In section 4 the question is answered of how one can present especially *fermionic* matter in such a metric-affine spacetime. The results of this chapter are fairly new and have been found during the last 15 years or so by one of us (Y.N.) and his collaborators. World spinors are defined and their conformal properties studied. Technical details of the unitary irreducible representations of the $\overline{SA}(4,R)$ and the corresponding subgroups are collected in appendix C.

Having now a spacetime arena available and matter fields 'moving' therein, we can build up a Lagrangian of this gravitationally interacting matter system and an action function as well. This is done in section 5 in the conventional way. We postulate affine gauge invariance and switch on the Lagrange–Noether machinery. Besides the conventional canonical energy–momentum current, we define, generalizing the spin current, a hypermomentum current that is coupled to the linear connection, i.e. to the new gravitational potential of spacetime.

The Noether identities (5.2.10), (5.2.16) and the general form of the gravitational field equations (5.5.3), (5.5.4), (5.5.5) are derived. We discuss the Belinfante–Rosenfeld symmetrization of the energy–momentum current and study different limiting cases of the gravitational field equations by means of the Lagrange multiplier technique. Finally Astekhar type complex variables are generated by means of a metric-affine Chern–Simons term in the gauge Lagrangian. Whereas most of the material of this chapter appeared before, we claim some originality as to the *completeness* and the *rigor* of our presentation.

Up to including section 5, *no* gravitational gauge Lagrangian is specified explicitly. Thus we provided a 'kinematical' framework for metric-affine gauge gravity which has to be filled with physical life. This is done in section 6 in a preliminary way. Conformally invariant gravitational gauge Lagrangians, including dilaton fields, are studied and compared to alternative approaches in the literature. Various schemes of symmetry reduction from the linear to the Lorentz group are given explicitly. The physically relevant procedure is one of the main open problems. We believe, however, that the solution of exactly this problem is indispensable for future progress in gravity. Moreover, we discuss generalizations of recent inflationary models in our post-Riemannian framework.

In the list of literature all material relevant to our task is compiled as a service for the reader. Should we have overlooked some articles, we would like to ask the authors to let us know, possibly by email to hehl@thp.uni-koeln.de. We may want to supply this additional information in an Addendum.

In the body of the paper, special relativity (theory) will be abbreviated as SR, whereas the gravitational models used carry the following acronyms:

- GR = general relativity (theory), also called Einstein gravity (Riemannian spacetime V_4) [180].
- $\mathrm{GR}_{\parallel}$ = teleparallel (version of general relativity) theory (Weitzenböck spacetime W_4: Riemann–Cartan spacetime with vanishing (Cartan-)curvature and non-vanishing torsion), see [523,614].
- EC theory = Einstein–Cartan(–Sciama–Kibble) theory (of gravity); non-propagating torsion (Riemann–Cartan spacetime U_4: Metric and metric-compatible connection), see [683,275].
- PG (theory) = Poincaré gauge theory (of gravity); propagating torsion (Riemann–Cartan spacetime U_4), see [272].
- MAG = Metric-affine (gauge theory of) gravity (metric-affine spacetime (L_4, g): Independent $GL(4, R)$-connection and independent metric), see [288].
- We denote the covering of a certain group by an overline. We have, for instance, $SL(2, C) = \overline{SO}(1, 3)$. Sometimes we dispense with the overline for convenience provided it is clear from the context in any case.

2. Motivation: search for quantum gravity

2.1. No Einstein theory at quantum level perturbatively

It is now probably safe to state that GR, though fully validated "in the large", does not exist in the perturbative sense at the quantum level. This statement relies on the following two facts:

(i) Although the one-loop "vacuum" amplitude (i.e. gravitons interacting with gravitons, with no "matter" fields present) had been shown [306] to be finite, with the infinities cancelling through the unexpected action of a specific identity (Bach–Lanczos identity [21,384], cf. appendix A.3), no such "miracle" happens for the two-loop amplitude [232,233]. Explicit calculation shows it to be infinite.

(ii) In a renormalizable theory, such infinities are removed by appropriate counter-terms. However, the fact that in this case, the theory's coupling constant, i.e. Newton's constant, has dimensions $d_{\mathrm{N}} = -2$, leads to a need for an infinite number of different renormalization counterterms, one for each order of the perturbation expansion. It does not appear possible that this would represent a "manageable" renormalization procedure.

It is, of course, still possible that quantum gravity might be correctly represented by GR, but solely with non-perturbative realizations (cf. Ashtekar's new variables [15] within a Hamiltonian approach). The alternative would be to assume that the perturbation expansion does exist, but that it is Einstein's theory which is an incomplete description of gravity. It is because of the latter eventuality that the quest for quantum gravity adds an important motivation to go beyond Riemannian manifolds – the subject of this work. Noticing that the renormalizable paradigmatic *standard model* is entirely constituted of local gauge theories, and that GR itself is already a gauge theory at the classical level

[674] (as a matter of fact it had served as the model for the newer Weyl/Yang–Mills quantum gauge theories making up the standard model), we assume that quantum gravity is also a gauge theory. In this view, one should thus look for the "complete" gauge theory "G" of quantum gravity, existing fully at the quantum level. The Einstein theory (or Einstein–Cartan or any other theory reproducing the observational results of GR) would then just represent the "long-range" or "low-energy" component (and also limit) of "G". Since the quantum regime takes over at Planck mass $M_{\rm Planck}$, defined as the mass (or energy) at which the (quantum) Compton wave length is equal to the (gravitational) Schwarzschild radius, what is meant here by "low-energy" is "below Planck energies", i.e. below 10^{19} GeV. We are aware of the conceptual problems [309] which this perturbative extension to the Planck scale encounters with our usual macroscopic notions of space, time and causality.

2.2. *Dimensional analysis and renormalizability of Yang–Mills gauge theories*

Comparing with the successfully quantized Yang–Mills theories, we note that one key difficulty with GR as a relativistic quantum field theory resides in the abnormal dimensionality of its Lagrangian density. In Yang–Mills theories, the gauge potential is a Lie algebra-valued connection one-form and it reads [3]

$$A = A_i\,dx^i = A_i{}^K \lambda_K\,dx^i\,, \tag{2.2.1}$$

where the λ_K are the generators of the gauged Lie algebra. Its components A_i have the dimension $d_{\rm A} = -1$ of a spacetime derivative, since its very definition is related to its role as guaranteeing parallel-transport, as a piece of the gauge-covariant exterior derivative

$$D = d - A, \qquad D_i := e_i\rfloor D = \partial_i - A_i. \tag{2.2.2}$$

As a result, the components $F_{ij}{}^K$ of the field strength or "curvature" two-form $F = \frac{1}{2}F_{ij}{}^K\lambda_K\,dx^i\wedge dx^j$, a generalized curl of the potential or "connection" A, has $d_{\rm F} = -2$; the classical piece of the Yang–Mills Lagrangian density, which is *quadratic in the curvature*, thus has dimension $d_{\rm L} = -4$.

Having $d_{\rm L} = -4$ for the Lagrangian density is an important result, since it will directly yield a dimensionless expression for the Lagrangian four-form and consequently also for the action, after the integration over four-dimensional spacetime. This is as it should be; however, this also implies that *the coupling constant in such theories* (it appears as a coefficient of the squared curvature term) *is necessarily dimensionless*. As a result, we obtain two aspects essential to renormalizability:

(i) In the perturbative expansion, all powers of this coupling are then also dimensionless, and thus do not impose any constraints on the counter terms.

(ii) In calculating amplitudes, integrations, by definition, should be cut-off independent. This is generally identical to requiring scale invariance, i.e. no dependence on dimensional parameters – a requirement thus fulfilled in Yang–Mills theories, with their dimensionless couplings.

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[3] The index K runs from 1 to q, the dimension of the group manifold; the coordinate (or holonomic) indices are denoted by $i, j, k, \ldots = 0, 1, 2, 3$; the coordinate basis of our tangent space is e_i, and $\rfloor$ denotes the interior product. We use the notations of [272], where detailed definitions and derivations can be found. Incidentally, the use (à la Landau–Lifshitz [386]) of Latin indices for coordinates and Greek indices for frames is the opposite of what is generally adopted among particle physicists.

2.3. Dimensions in Einstein's GR and in Poincaré gravity; the gauge field of translations

This is not the situation in GR, with its Lagrangian, linear in the Lorentz curvature, i.e. with $d_R = -2$; the same is true in the simplest PG model [275], the EC theory, in which torsion is locally present but does not propagate. A partly similar situation occurs in generic PG [272], residing in a U_4 spacetime with propagating torsion (in addition to curvature as in GR). In PG, although the Lorentz group's squared curvatures do yield $d_{R^2} = -4$ terms, like a canonical Yang–Mills theory, one nevertheless has $d_{T^2} = -2$ contributions, this time from the torsion-square terms. This is due to the anomalous dimensionality $d_P = -1$ of the (inhomogeneous) translation generators in the Poincaré group (as against $d_\lambda = 0$ in the Yang–Mills gauge groups of the standard model and in the (homogeneous) Lorentz subgroup of the Poincaré group). In the connection (2.2.1) and in the covariant derivative (2.2.2), the λ_K is now replaced by a spacetime derivative, forcing the gauge-compensating mechanism to act multiplicatively with tetrad frame fields, rather than through a connection:

$$D_\alpha = \overset{\circ}{e}{}^i{}_\alpha D_i \,, \qquad D_\alpha := e_\alpha \rfloor D \,, \tag{2.3.1}$$

where $e_\alpha = \overset{\circ}{e}{}^i{}_\alpha \partial_i$ is the local Lorentz frame (the superscript $\circ$ stands in for *orthonormal* and anholonomic indices are denoted by $\alpha, \beta, \ldots = 0,1,2,3$).

The operator D_α thus replaces ∂_i and forms a semi-direct product with the $\overset{\circ}{L}_{\alpha\beta}$ of the $\overline{SO}(1,n-1) = \mathrm{spin}(1,n-1)$ [in four dimensions $SL(2,C)$] on the frames and involves the frame field as the (multiplicative mode) gauge field for translations. Alternatively, we can use the non-degenerate coframe (or one-form) $\vartheta^\beta = \overset{\circ}{e}{}_j{}^\beta \, dx^j$, which relates to the frame field via the relation

$$e_\alpha \rfloor \vartheta^\beta = \overset{\circ}{e}{}^i{}_\alpha \, \overset{\circ}{e}{}_i{}^\beta = \delta^\beta_\alpha \,. \tag{2.3.2}$$

Moreover, the coordinate components of the metric tensor field

$$g_{ij} = o_{\alpha\beta} \, \overset{\circ}{e}{}_i{}^\alpha \, \overset{\circ}{e}{}_j{}^\beta , \quad o_{\alpha\beta} = \mathrm{diag}(-1,1,1,1), \tag{2.3.3}$$

will enter the gravitational model, where $o_{\alpha\beta}$ is the local Minkowski metric. The components $\overset{\circ}{e}{}_j{}^\beta$, $\overset{\circ}{e}{}^i{}_\alpha$, g_{ij} are all dimensionless.

The gravitational "gauge field" of the translations is thus dimensionless. The translational field strength is the torsion,[4] the components of which thus have $d_T = -1$, as a curl of the $\overset{\circ}{e}{}_j{}^\beta$. Its square in the Lagrangian density then has $d_{T^2} = -2$. In PG and in the teleparallelism theory (in which one keeps only the torsion-square term, putting the curvature to zero), it is this $d_{T^2} = -2$ term which replaces Einstein's curvature scalar in describing macroscopic gravity – the distinction between these theories and Einstein's arising only in 5th order in the parametrized post-Newtonian (PPN) treatment [613,637,523].

To obtain a dimensionless action in GR or in PG, one is therefore forced to assign to the coupling constant (Newton's) a dimensionality $d_N = -2$, in order to make up for having only $d = -2$ from the fields and their derivatives. As mentioned above, this fact then interferes with the possibility of

[4] As we will discuss in section 3.2, the translational field strength of the affine (or the Poincaré) group does contain the torsion, but it also carries an additional piece, see (3.2.13).

developing a perturbation-theoretical renormalization procedure – in addition to introducing a scale (Planck mass) and thus indirectly making the amplitudes cut-off dependent.

Basically, the unconventional dimensionality is due to the fact that spacetime appears both as the base space and as a subspace of the gauged group manifold (spacetime being the manifold of the translations' parameters): In GR it allows gauge group indices of the curvature – due to the soldering of the Lorentz bundle to the base manifold – to be contracted with the frame indices of spacetime. In PG, it generates the anomalous dimensionality of the generators of translations. As a matter of fact, MacDowell and Mansouri [416] have shown that the Hilbert–Einstein Lagrangian can be obtained as an Inönü–Wigner contraction of a quadratic curvature Lagrangian gauging $SO(1,4)$; in this derivation, the Riemannian curvature scalar is a degenerate bilinear in which one factor is a contracted curvature whose connection has become the coframe with $d_\vartheta = 0$, and the $d_{\rm HE} = -2$ dimensionality of the Hilbert–Einstein Lagrangian density can thus again be blamed on the anomalous dimensionality of the translation generators – just as in PG.

2.4. The groups gauged in gravity

"...gravity is that field which corresponds to a gauge invariance with respect to displacement transformations" — *Richard Feynman (1963)*

Before returning to our dimensional analysis, in the search for a candidate "complete" gauge theory of gravity, with a $d_{\rm L} = -4$ Lagrangian density, we discuss the gauge groups relevant to gravity.

In Einsteinian gravity proper, there is only *one* such group, namely, the double covering $\overline{\rm Diff}(4, R)$ of the group of differentiable coordinate transformations or *covariance group* [461], acting on the (holonomic) spacetime indices $(i, j, k\ldots)$. Mostly, it is applied passively ("alias" transformations, only changing "names" amongst the curvilinear coordinates); sometimes, however, these locally dependent *orbital Poincaré transformations*, merging with the other diffeomorphisms, are interpreted actively. On the face of it, local diffeomorphisms can be considered as locally gauged translations; this view gains some additional justification from the fact that the gravitational field is coupled to the energy–momentum tensor density, i.e. to the translational current. However, in contrast to the Yang–Mills construction, the Lagrangian density involves only curvatures – the field strengths of the (homogeneous) Lorentz subgroup. In the presence of spinorial matter fields, when Lorentz labelled local frames have to be introduced, or in EC theory – in these cases there is in addition a local (homogeneous) Lorentz group acting on those frames as a separate gauge group. Here the compensating role of the gauge field is fulfilled by the connection (with anholonomic indices) $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ in a canonical Yang–Mills mode.

In the PG, the aim is for the full Poincaré group to be treated as a local gauge group. In a full Yang–Mills like mode [457], one starts from a generalized affine connection which contains the Lorentz and the true translational connection. The field strength contains the Lorentz curvature and a translational curvature. Only after a certain reduction, the translational connection and curvature are converted into coframe and torsion, respectively; see sections 3.1–3.5 for details.

Then, in order to preserve the action of the Lorentz group generators on the translations, the latter have to be defined by the frames, i.e. they should carry anholonomic indices. In PG, the Lorentz connection and the coframe are independent and we have the two modes of gauge compensation acting together. As we noted in our dimensional analysis, in the construction of a Yang–Mills-like quadratic Lagrangian for the Poincaré group, the squared torsion pieces are added to the squared Lorentz curvatures (torsion being the translational field-strength), providing for a Yang–Mills like dynamical local Poincaré gauge, aside from the action of $\overline{\mathrm{Diff}}(4,R)$.

However, it is partly possible (and physically more plausible) to unify the two local gauge groups – Poincaré on the frames and general covariance – and at the same time obtain a reinterpretion of the transformations wrought by $\overline{\mathrm{Diff}}(4,R)$, reproducing them by one and the same gauge group, "almost" in a Yang–Mills like mode. This is done by gauging a deformed and anholonomized version of the translations. More precisely, we have to take as translation generators D_α, i.e. the local tangent-space version ("anholonomic") of the Lorentz-covariant *parallel-displacements* [275,479,504], i.e. of the covariant exterior derivatives (2.3.1). Geometrically, these covariant parallel-displaceable frame-defined translations [275] (the "anholonomized general coordinate transformations" AGCT of [479], or "shift" and "laps" of the Hamiltonian formalism) are linked to Lie derivatives, see section 3.6, and their commutation relations involve "structure functions" instead of structure constants, i.e. they do not close:

$$[D_\alpha, D_\beta] = -T_{\alpha\beta}{}^{\gamma}(x)\, D_\gamma + R_{\alpha\beta}{}^{\gamma\delta}(x)\, \overset{\circ}{L}_{\gamma\delta}\,. \tag{2.4.1}$$

The structure functions are given by the components of torsion and curvature [275,488,504]. Effectively, the D_α generate the equivalent of the infinite-dimensional algebra of local translations, a topic we will discuss in section 3. Note that the commutation relations between the Lorentz generators $\overset{\circ}{L}_{\alpha\beta}$ are unmodified [275,488,498]; in the language of the "soft group manifold" approach [488,671,643], the Poincaré algebra undergoes a "spontaneous fibration" in which the Lorentz $SO(1,3)$ subgroup becomes the fiber of an on-mass-shell effective bundle – as a result of the equations of motion. The motions over the residual quotient base-manifold "Poincaré/Lorentz" after the "spontaneous selection" of the fiber, are still generated by the parallel-transport guaranteeing covariant derivatives. [5]

2.5. *Gravity theories with a 'dynamically truncated' gauging of the Poincaré group*

Let us now inspect the GR and PG gravitational theories from the point of view of a Yang–Mills like gauging of the Poincaré group. In both, the Hilbert–Einstein and the EC Lagrangian densities, it is only the Lorentz $SO(1,3)$ subgroup which is treated dynamically, i.e. with its curvature entering the Lagrangian. Thus, the connection generally plays a Yang–Mills like dynamical role. The translation generators – with their multiplicatively acting compensating fields (i.e. cancelling gradient terms in the variations) for local translations (i.e. diffeomorphisms) – are not independently gauged dynamically; indeed, the Lagrangian density does not contain a kinetic energy term for the tetrads or the metric. The opposite occurs in teleparallelism models, in which the curvature is constrained to zero and the physical degrees of freedom are solely carried by the coframe. In these theories, the connection is additionally constrained by the metricity condition

$$D_\alpha\, g_{\beta\gamma} = 0\,, \tag{2.5.1}$$

which guarantees the metric-compatibility of the connection.

[5] This quotient manifold is the spacetime for gravity, or the superspace in supergravity, where in addition to the above mentioned spacetime "AGCT", there are "local supersymmetry transformations" consisting in a similar set of "AGCT" modified translations, this time in the spinor sector of superspace.

The EC Lagrangian density is linear in the Lorentz curvature (which, as noted above, has the same dimensionality $d_R = -2$ as in the Yang-Mills case), so that only after counting the coupling constant $1/l^2$ it ends up having $d_{HE} = -4$:

$$V_{HE} = (1/2l^2)\det|\overset{\circ}{e}_k{}^\gamma|\, R_{ij}{}^{\alpha\beta}(\partial\Gamma, \Gamma)\, \overset{\circ}{e}{}^i{}_\beta \overset{\circ}{e}{}^j{}_\alpha\,. \tag{2.5.2}$$

This Lagrangian involves two fields, the coframe $\vartheta^\alpha = \overset{\circ}{e}_i{}^\alpha\, dx^i$ and the $so(1,3)$-valued connection $\Gamma^{\alpha\beta}$. There is only the derivative $d\,\Gamma^{\alpha\beta}$ appearing in the curvature, so that only one field propagates physically, i.e. has kinetic energy. In addition, one gets an algebraic equation (Sciama's [615,616] and Kibble's [344,345]) as a second equation of motion. It just reexpresses the torsion in terms of the spin current of the matter fields (or states that torsion is vanishing, if matter carries no spin). This is then a theory with local but non-propagating torsion. Simple ($N = 1$) supergravity can be derived from an EC Lagrangian plus the Lagrangian of a minimally coupled massless spin-3/2 Rarita-Schwinger field (spinor-valued one-form) [155,207,696], with the spin density of the Rarita-Schwinger gravitinos as sources of torsion.

Note that in PG theories in which a torsion-square term is present solely, as in teleparallelism theory [272,363,451], sometimes used as an alternative to the Hilbert-Einstein Lagrangian, or when it appears in addition to the curvature-squared term, as in Poincaré gauge theories [263,272], we have propagating torsion either instead of the metric graviton or in addition to it. However, although this approach may thus have an advantage in adding new true physical degrees of freedom, it leaves the dimensional issue unresolved as we already remarked, the squared torsion term in the Lagrangian having again $d_{T^2} = -2$ and thus interfering with renormalizability. There is one case in which this dimensionality issue can be overcome. However, as we shall see, there then arises a difficulty with unitarity.

2.6. *Unitarity violation in a gauge theory with quadratic Lorentz curvature*

The wrong dimensionality of the Hilbert-Einstein Lagrangian is not yet the full difficulty. Leaving out the torsion and staying in V_4 geometry, we could still have hoped to emulate the Yang-Mills case, through the use of a Lagrangian quadratic in the Lorentz curvature [302,348,654,728]. As we noted above, this will have $d = -4$ as in the Yang-Mills case; it does indeed lead to renormalizable theories, at least in vacuum [652]. Nevertheless, even using such Lagrangian densities, quadratic in the curvatures, does not resolve the problem. Taking *only* quadratic terms in the curvatures, there is no decent Newtonian limit. This then forces us to include the Hilbert-Einstein Lagrangian (to cover macroscopic gravity), in addition to the quadratic curvature terms, in the hope that the latter dominate in the high energy limit, which would make the theory renormalizable. This was indeed achieved by Stelle [652].

However, there then appear new difficulties, this time with unitarity. Catastrophic unitarity violations will occur in situations in which the Lorentz connection can be reexpressed in terms of the metric or the (co-)frame, i.e. when the metricity condition (2.5.1) holds, the connection then being given by the Christoffel formula.

Whenever the connection is given in terms of first derivatives ∂g of the metric or of the coframe, the Yang-Mills type gravity will be converted into a higher derivative theory, see section 5.8.3 for details. The curvature-squared terms will contain terms of the form

$$(\partial g)^4, \qquad (\partial^2 g)^2, \qquad (\partial^2 g)(\partial g)^2. \tag{2.6.1}$$

As a result, for a Lagrangian quadratic in the Lorentz curvature, the propagator, which is the inverted Fourier transform of the linearized Lagrangian, is quartic in the momenta. Such propagators automatically contain double poles, such as

$$\frac{1}{p^2(p^2-m^2)} = \frac{1}{m^2}\left(\frac{1}{p^2-m^2} - \frac{1}{p^2}\right), \tag{2.6.2}$$

so that one of the two poles has the wrong sign for its residue (leading to negative probabilities, i.e. ghosts, cf. [378] and references).

This may be remedied by adding new degrees of freedom, provided they do not reintroduce $d \neq -4$ terms. The connection $\Gamma^{\alpha\beta}$ should then still have $d = -1$ but it should become an independent field. One straightforward way of constructing precisely such a connection appears to lead to the abandonment of the metricity condition (2.5.1).

2.7. *Metric-affine gauge theories: gauging the $A(4, R)$ and relaxing the constraint of metricity*

In a general (L_4, g), the metric $g_{\alpha\beta}$, the coframe ϑ^α, and the linear connection $\Gamma_\alpha{}^\beta$, all are independent "potentials". In Table 1, we list the currents, potentials, field strengths, and Bianchi identities in such a framework.

Since a metric is given, the linear connection can, according to (3.10.12), be decomposed as follows:

$$\Gamma_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} - K_{\alpha\beta} + \tfrac{1}{2}Q_{\alpha\beta} + (e_{[\alpha}\rfloor Q_{\beta]\gamma})\vartheta^\gamma. \tag{2.7.1}$$

Here $\Gamma^{\{\}}_{\alpha\beta}$ is the Riemannian part of the connection depending solely on metric, frame, coframe, and their derivatives. The contortion one-form $K_{\alpha\beta} = -K_{\beta\alpha}$ is implicitly given in terms of the *torsion* two-form T^β by

$$T^\beta := D\vartheta^\beta = \vartheta^\alpha \wedge K_\alpha{}^\beta. \tag{2.7.2}$$

The *nonmetricity* one-form

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} \tag{2.7.3}$$

measures the deformation of length and angle standards during parallel-transport. In Riemann–Cartan spacetime U_4 it vanishes,

$$Q_{\alpha\beta} = 0, \tag{2.7.4}$$

Table 1
Currents of MAG and their associated gauge fields.

Current	Potential	Field strength	Bianchi identity
sym. en.-mom. $\sigma^{\alpha\beta}$	metric $g_{\alpha\beta}$	$Q_{\alpha\beta} = -Dg_{\alpha\beta}$	$DQ_{\alpha\beta} = 2R_{(\alpha\beta)}$
can. en.-mom. Σ_α	coframe ϑ^α	$T^\alpha = D\vartheta^\alpha$	$DT^\alpha = R_\mu{}^\alpha \wedge \vartheta^\mu$
hypermom. $\Delta^\alpha{}_\beta$	connection $\Gamma_\alpha{}^\beta$	$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\mu \wedge \Gamma_\mu{}^\beta$	$DR_\alpha{}^\beta = 0$

and, consequently, is not present in GR or EC theory. Moreover, if spinor fields – that would have induced torsion – are absent, Eq.(2.7.1) reduces just to the familiar Christoffel formula, in its anholonomic version.

Foregoing the metricity condition thus carries us over to metric-affine spacetimes (L_4, g), the physical picture should then be as follows:

(i) At very high energy ($E \geq M_{\text{Planck}}$), as in Yang–Mills gauge theories, gravity is described by $d_{\rm L} = -4$ quadratic Lagrangians in which the connection $\Gamma_\alpha{}^\beta$ is an independent field. The group to be gauged G has to include the translations and should contain an homogeneous subgroup

$$\overline{G} \supset SL(2, C)\,. \tag{2.7.5}$$

The theory should thus be both unitary (no metricity condition, thus leading to an independent connection and no p^{-4} propagators) and renormalizable ($d_{\rm L} = -4$ Lagrangian density).

(ii) At low energies ($E \leq M_{\text{Planck}}$), spontaneous breakdown should occur, $G' \Rightarrow SL(2, C)$ so that with an orthogonal (or pseudo-orthogonal) local invariance group the metricity condition becomes operational; yet this cannot be a "strong" statement about field operators, only a "weak" statement about matrix elements between low-energy states (l.e.s.):

$$\langle \text{l.e.s.}|Q_{\alpha\beta}|\text{l.e.s.}\rangle = 0. \tag{2.7.6}$$

As a result, the metric enters the game. The components of the Lie algebra-valued connection for that part of the group which does not survive as a residual local gauge group, in GR, EC, PG, or teleparallelism theory, i.e. in the low energy effective Lagrangian, all become massive ($M \sim M_{\text{Planck}}$).

Such models have recently been investigated [26,287,496,388,389,387], with $\overline{SL}(4, R) \supset SL(2, C)$ as the homogeneous part of the gauge group. This is denoted as the "affine" gauge approach, the affine group $\overline{A}(4, R) := R^4 \otimes \overline{GL}(4, R)$ consisting of the semi-direct product of the general linear group $\overline{GL}(4, R)$ and the Abelian group $T^4 = R^4$ of spacetime translations. For the representation of spinor fields we need their respective double-covering groups, see [495,628], denoted by an overline. We will discuss the role and structure of the double covering of the linear and diffeomorphism groups in our treatment of the matter fields in section 4 and in our construction of the spontaneous symmetry breakdown in section 6. Certain metric-affine gauge models have been proved to be renormalizable [388,389,387]; however, there is as yet no certainty with respect to the theory's unitarity.

We have thus sketched this particular logical chain that has led to the interest in metric-affine spacetimes, and in particular spacetimes that are generated by a local gauge.

A related, though different program [184] has centered on a Weyl-type gauge formalism; the obstacles to quantization do stem from the same basic dimensional analysis we have presented here.

2.8. *Supergravity and renormalizability?*

In all generality, $\overline{G}$ can be any group that contains $SL(2, C)$, as stated in (2.7.5). The most obvious possibilities are the Weyl, conformal, or linear groups and their respective supergroups. The Weyl or homothetic group (containing Lorentz transformations and dilations) is contained in both the 15-parameter group of conformal transformations and the general linear group $\overline{GL}(4, R)$. Conformal gauge theories have been investigated at the classical and the quantum level [184,185].

The inclusion of the Poincaré group in supersymmetry,[6] leading to supergravity [207,155], generated hopes of improved renormalizability. The bosonic homogeneous subgroup, however, is still $SL(2,C)$, or $SL(2,C) \times O(N)$ for N-extended supersymmetry. The enlargement happens in the system of translations, i.e. in the inhomogeneous quotient. For the flat case, this is

$$\text{super}\left(R^4 \otimes \overline{SO}(1,3)\right)/SL(2,C)\,. \qquad (2.8.1)$$

For the (curved space) gauged group, the spinorial local supersymmetry transformations are yet another example of a symmetry generated by Lie-derivatives as in (2.4.1).

Supergravity itself was developed within the same context of constructing a renormalizable "complete" gauge theory "G" of quantum gravity as explained in section 2.1. Its spacetime realization involves torsion and is thus a U_4 manifold, embedded in a superspace, the quotient (2.8.1). It has been shown, however, that though supergravity does induce finite results in many of the cases in which Einstein's theory gave infinities, it will not "cure" cases such as those discussed in [232,233]. Local supersymmetry in supergravity generalizes Einstein's theory so that a "vacuum" result in gravity (i.e. only gravitons, interacting with gravitons), becomes a vacuum result in supergravity (in which the graviton is part of a gauge multiplet, including at least one $J = 3/2$ spinorial field). Thus, after the discovery of supergravity, the finiteness of the one-loop vacuum amplitude in gravity [306], which had prior to that been shown [150,151] to fail in the presence of spins $0, \frac{1}{2}, 1$ matter fields, was now proved to hold in the one-loop "supergravity vacuum", which contains, aside from gravitons, the matter fields belonging to the graviton's supermultiplet. In extended supergravity with any $N \leq 8$, it was shown [331] that all vacuum amplitudes are finite, up to N loops. The graviton gauge supermultiplet in theories with $N \leq 4$ includes fields with spins $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, thus curing defects such as had been encountered by Deser and van Nieuwenhuizen [150,151]; but the same generalization caused by supersymmetry also causes the negative two-loops result in gravity [232,233] to reappear in $(N+1)$-loops in N-extended supergravity.

With only $SL(2,C)$ left as local gauge symmetry, only the graviton stays massless. The fields in the gauge supermultiplet acquire mass, through a Goldstone–Higgs mechanism. In supersymmetrized gauge unification theories (superGUTs), this mass may be related to the scale of the weak interaction, as part of a solution of the "hierarchy" problem: How to constrain the electroweak symmetry-breaking Higgs field so that its GUT- and gravity-induced radiative corrections do not bring its mass up to GUT (10^{15} GeV) or to Planck energy regions.[7]

The quantum gravity program that was based on supergravity has thus been useful, but cannot resolve the issue completely. It may prove necessary to the full program, but it is certainly insufficient. Nevertheless, one may speculate that, at the end, the *metric-affine program* may yet have to be reinforced through supersymmetrization. This would involve infinite-component spinor translation generators, a subject that has barely been "touched" to date [489].

[6] According to recent experiments [1], supersymmetric partners of the quarks and gluons of the standard model are excluded at a 90% confidence level below 126 GeV. Since the allowed energy region for these supersymmetric particles' masses (e.g. gluinos, squarks, etc..) is centered on 500 GeV and extends to 1 TeV, the final verdict will have to await the activation of the 8-10 TeV accelerator LHC at CERN (near Geneva), which will provide such center-of-mass values for the energy; see also the newly proposed uc of next footnote.

[7] The uc ("ultimate collider") up to Planck energies is described by Akahito [5] as an "experiment of the month".

2.9. Quantum superstring?

With William Thomson's idea of "vortex atoms" [675] coming of age in the shape of string and superstring theories, in recent years hopes for a *finite* theory of quantum gravity have centered on the quantum superstring (QSS) [612]. Although the perturbation expansion yields finite terms, the summations do involve infinities [248]. However, that would still be true in quantum electrodynamics (QED); in perturbative treatments in quantum field theory these infinities are assumed to arise because of non-perturbative solutions and are regarded as an indication of the latter's existence. Should we then consider the search for a theory of quantum gravity as having reached its goal and should we therefore cross it out as a motivation for the study of non-Riemannian gravitational theories? The basic assumption in the post-1984 treatment of the quantum superstring [238] "theory of everything" (TOE), an on-mass-shell S-matrix type theory, is that its truncation below Planck mass should go over smoothly into an off-mass-shell relativistic quantum (point) local field theory [8] (including a version of ten-dimensional supergravity, in one sector of the "heterotic string" [247], for instance); thus, even if the search were over, the same geometrical–gravitational question then relates to that truncated "low-energy" field theory and its gravitational sector. Moreover, it has been pointed out [105] that *consistency would then require the low-energy field theory to be finite by itself!* This then implies the existence of a finite or renormalizable relativistic quantum field theory of gravity.

As an S-matrix theory, the TOE should, for instance, provide the precise energy levels of positronium, including the (very small) gravitational contribution. QED gives us – to any order of the perturbation expansion – the electromagnetic component of these energy levels; the difference between the TOE and QED levels is then a finite contribution of quantum gravity, which should be calculable within that low energy quantum field theory of gravity.

Moreover, the success of the QSS as the *final* theory is not evident yet. In contrast to the psychological impact resulting from the apparent uniqueness of the original superstring, with only $E_8 \times E_8$ or $SO(32)$ as allowed internal symmetries, a very large number of superstring theories have subsequently been shown to exist, when reducing from the original 26 or 10 to our macroscopic four dimensions. These different theories have to be reinterpreted as different vacuum solutions of a meta-theory; the actual suggestion is to look for the non-perturbative solutions of a *string field theory* and to hope that some stability criteria will select the physical vacuum. Such string field theories have been constructed, but the search for a theory with a finite set of candidate vacua is still on, resembling at this stage the traditional search for a needle in a haysack.

String theory was originally developed in a flat embedding manifold (the "target space") with linearized gravity. To go beyond weak fields and to include the successful tests of GR, it had to be adapted to curved spacetime, while relating the spin-2 graviton in the string Hilbert space to the presence of macroscopic curvature. Einstein had originally realized that the symmetric energy–momentum tensor density is the source of a curvature field; quantum field theory had pointed to this field's particle realization as $J = 2$ gravitons, coupling to that energy–momentum tensor density. In the flat-embedding string the intermediate step is missing – and with it such effects as the precession of Mercury.

Two approaches have been followed, in dealing with the embedding of the string in a curved manifold. Both lines of work have consisted in constraining the embedding space so as to preserve

[8] *Point*-local as against *string*-local.

the features required by quantization: (a) Unitarity through critical dimensionality, i.e. the cancellation of the conformal anomaly – or of the Liouville field, breaking Weyl invariance of the world sheet description, in the quantized string; (b) tachyon cancellation through the preservation of the action of the super-Poincaré (flat supersymmetry) group; (c) the cancellation of the chiral anomaly, as obtained through the selection of the gauged internal symmetry group, e.g. $E_8 \times E_8$. The main line of work has taken a renormalization group approach, i.e. imposing the above results by requiring the cancellation of the radiative corrections to the critical dimensionality, etc. [203,97,98,413]. Einstein's vacuum equation then emerges as a result of such constraints; in first order, the theory involves in addition a scalar "dilaton" and an antisymmetric field. For strings which become "rigid" through the addition of extrinsic curvature terms [563] to the conventional Nambu–Goto action, a Weyl-invariant formulation was developed [398–401], relaxing the metricity condition (2.7.4), for instance. The alternative route [494] has consisted in developing a curved version of supersymmetry.

Two further developments in string theory are relevant to the quest for quantum gravity. Work on non-critical strings, i.e. strings that therefore do not have the Liouville field cancelling in the quantized version (i.e. a breakdown of Weyl invariance on the world sheet). The technique of *matrix models* [249] has yielded for the first time information about non-perturbative solutions to quantum gravity – but only in two dimensional quantum gravity, which therefore makes for inconclusive results as far as non-Riemannian aspects are concerned. The other line of work has consisted in the study of *extendons*, i.e. classical and quantum extended structures with more than one spatial dimension, e.g. a quantum membrane [484]. In these constructions, the main criterion for utilizability as a TOE or as a theory of quantum gravity is to ascertain whether or not the model has massless solutions fitting the graviton. The most favorable model, in this sense, appears to consist [55] of an embedding of the ten-dimensional superstring in a quantized membrane, through the addition of one new spatial dimension to both ten-dimensional supergravity – the underlying low energy field theory – and to the string and its world sheet. In the low energy field theory, this is set to yield eleven-dimensional supergravity.

Having said all this, we will now turn to the affine group and its gauging, leading eventually to a metric-affine geometric "arena" with fully liberated nonmetricity and torsion.

3. Geometry

3.1. Rigid affine group $A(n,R)$ and its Lie-algebra

In the flat n-dimensional affine space R^n, the rigid affine group $A(n,R) := R^n \rtimes GL(n,R)$ is the semidirect product of the group of n-dimensional *translations* and n-dimensional *general linear transformations*. This transformation group acts on an affine n-vector $x = \{x^\alpha\}$ according to

$$x \to x' = \Lambda x + \tau \tag{3.1.1}$$

where $\Lambda = \{\Lambda^\alpha{}_\beta\} \in GL(n,R)$ and $\tau = \{\tau^\alpha\} \in R^n$. [The range of the Greek indices is: $\alpha, \beta, \ldots = 0, 1, 2, \ldots, (n-1)$.] Thus it is a generalization of the Poincaré group $P := R^4 \rtimes SO(1,3)$, with the pseudo-orthogonal group $SO(1,n-1)$ being replaced by the general linear group $GL(n,R)$, cf. [347]. The semi-direct product property of both groups is reflected in the rather complicated law of group multiplication:

$$(\Lambda, \tau) \circ (\Lambda', \tau') = (\Lambda\Lambda', \Lambda\tau' + \tau). \tag{3.1.2}$$

Therefore in the following it is more convenient to work with a Möbius type representation [355,447] for which we take the same symbol $A(n,R)$: It is that subgroup of $GL(n+1,R)$ which leaves the n-dimensional hyperplane $\bar{\bar{R}}^n := \{\bar{\bar{x}} = \binom{x}{1} \in R^{n+1}\}$ invariant:

$$A(n,R) = \left\{ \begin{pmatrix} \Lambda & \tau \\ 0 & 1 \end{pmatrix} \in GL(n+1,R) \mid \Lambda \in GL(n,R),\ \tau \in R^n \right\}. \tag{3.1.3}$$

Thus, by an affine transformation, we obtain $\bar{\bar{x}}' = \left(\begin{smallmatrix} \Lambda & 0 \\ \tau & 1 \end{smallmatrix}\right)\binom{x}{1} = \binom{\Lambda x + \tau}{1}$, as is required for the action of the affine group on the flat affine space.

The Lie algebra $a(n,R)$ consists of the generators P_γ, representing n-dimensional translations, and the $L^\alpha{}_\beta$, which span the Lie algebra $gl(n,R)$ of n-dimensional linear transformations. Their commutation relations read:

$$[P_\alpha, P_\beta] = 0, \tag{3.1.4}$$

$$[L^\alpha{}_\beta, P_\gamma] = \delta^\alpha_\gamma P_\beta, \tag{3.1.5}$$

$$[L^\alpha{}_\beta, L^\gamma{}_\delta] = \delta^\alpha_\delta L^\gamma{}_\beta - \delta^\gamma_\beta L^\alpha{}_\delta. \tag{3.1.6}$$

Observe that the physical dimensions of these generators are $[L^\alpha{}_\beta] = \hbar$ and $[P_\alpha] = \hbar/\text{length}$. However, throughout this paper, natural units with $\hbar = c = 1$ are chosen.

For $n \geq 2$, the general linear group $GL(n,R)$ has two connected pieces, one with positive and the other one with negative determinant. The most important component is the *special* (or unimodular) linear group $SL_\circ(n,R)$ the elements of which are *continously connected* to the identity and, additionally, satisfy the condition $\det\Lambda = 1$. Guided by the four-dimensional spacetime, we distinguish, besides the group unit **1**, the operators $T, P, J := TP \in GL(n,R)$, which would correspond to time, space (parity), and total *reflections*, respectively. For $n \geq 3$, we can invariantly characterize them, after factoring out $SL_\circ(n,R)$-transformations, by the following properties:

$$\begin{aligned} T&: \quad \det T = -1, & \operatorname{tr} T &= n-2, \\ P&: \quad \det P = (-1)^{n-1}, & \operatorname{tr} P &= 2-n, \\ J&: \quad \det J = (-1)^n, & \operatorname{tr} J &= -n. \end{aligned} \tag{3.1.7}$$

The unconstrained determinant of $GL(n,R)$ contains, aside from the R^+ corresponding to the Abelian Lie subgroup of group elements $\Lambda \in GL(n,R)$ with positive determinant, the time reflection $T \in GL(n,R)$ with $\det T = -1$ by definition. For dimensions $n \geq 2$, we are thus lead to the isomorphic decomposition

$$GL(n,R) \approx [T \otimes SL(n,R)] \times R^+. \tag{3.1.8}$$

This decomposition induces the following splitting of the infinitesimal generators of the general linear group into the generators of traceless [9] linear transformations $\not{L}^\alpha{}_\beta$ and dilations [10] $\mathcal{D}$:

[9] Quite generally, for an arbitrary quantity $B_\alpha{}^\beta$ with two indices, we define its traceless part, the so-called "deviator" of continuum mechanics, according to $\not{B}_\alpha{}^\beta := B_\alpha{}^\beta - (1/n) B \delta^\beta_\alpha$, with $B := B_\gamma{}^\gamma$.

[10] Dilatations or dilations? "Since darkness dilates the pupils of our eyes and does not dilatate them, we see no reason for the extra 'ta' ". Cited from [642, p.37].

$$L^{\alpha}{}_{\beta} = \not{L}^{\alpha}{}_{\beta} + (1/n)\delta^{\alpha}_{\beta}\mathcal{D}, \quad \mathcal{D} := L^{\gamma}{}_{\gamma}. \tag{3.1.9}$$

For the dilation generator, we find the following commutation relations:

$$[\mathcal{D}, \mathcal{D}] = 0, \qquad [\mathcal{D}, P_{\alpha}] = P_{\alpha}, \qquad [\mathcal{D}, \not{L}^{\alpha}{}_{\beta}] = 0. \tag{3.1.10}$$

3.1.1. Additional structures in the presence of a metric

If the flat affine space carries a metric $g = g_{\alpha\beta}\, dx^{\alpha} \otimes dx^{\beta}$ with components $g_{\alpha\beta}$, one can lower indices and a finer splitting of the general linear group can be achieved:

$$L_{\alpha\beta} = \overset{\circ}{L}_{\alpha\beta} + \not{L}_{(\alpha\beta)} + (1/n) g_{\alpha\beta}\mathcal{D}. \tag{3.1.11}$$

Here $\overset{\circ}{L}_{\alpha\beta} := L_{[\alpha\beta]}$ are the generators of Lorentz rotations, $\not{L}_{(\alpha\beta)} = L_{(\alpha\beta)} - (1/n)\, g_{\alpha\beta} L^{\gamma}{}_{\gamma}$ represent shears, whereas $\mathcal{D} := L^{\gamma}{}_{\gamma}$ generates dilations. The first two pieces on the right-hand side of (3.1.11) correspond to the decomposition

$$sl(n, R) = \underline{k} \oplus \underline{n} = so(n) \oplus \underline{n} \tag{3.1.12}$$

of the simple real Lie algebra $sl(n, R)$ of $SL(n, R)$ into the maximal compact subalgebra $\underline{k}$ and its noncompact part $\underline{n}$. In our case this amounts to the well-known decomposition of a traceless square matrix into its skew-symmetric and its traceless symmetric parts.

As noted above, the $A(n, R)$ is an n-dimensional generalization of the Poincaré group $P := R^4 \rtimes SO(1, 3)$, with the pseudo-orthogonal group $SO(1, n-1)$ replaced by the general linear group $GL(n, R)$. This point of view, however, is no longer useful when one compares the two groups with respect to *algebraic deformations* of larger semi-simple geometrical transformation groups. Whereas the Poincaré group can be obtained by a Wigner–Inönü contraction from the de Sitter groups $SO(2, 3)$ or $SO(1, 4)$, cf. [234] for the corresponding gauge theories, the $A(n, R)$ cannot be derived by a group contraction, neither from $GL(n+1, R)$, nor from any other semi-simple group.[11]

3.2. Affine gauge approach

In a matrix representation analogous to (3.1.3), we can write the affine gauge group[12] as

[11] Contraction of $GL(n+1, R)$ or of the $SL(n+1, R)$, or of the related projective group $PGL(n, R) = SL(n+1, R)/Z$, with Z denoting the center, yields the *graded* affine group $A^{*}(n, R)$, with two Abelian subgroups instead of one. The origin of the two sets of "translations" can be exhibited by rewriting [355, p.132] the $sl(n+1, R)$ algebra, that generates these groups, as the graded algebra $a^{*} = R^{n} \oplus gl(n, R) \oplus R^{*n}$. Although this decomposition of $sl(n+1, R)$ seemingly looks like a generalization of the *conformal group*, with $so(1, n-1)$ replaced by $gl(n, R)$, one cannot identify the R^{n} and R^{*n} pieces with translations and special conformal transformations, respectively. This is related to the fact that it is impossible to find both $sl(n, R)$ and the conformal generators' algebra conf(n, R) as subalgebras of the same finite dimensional Lie algebra. Indeed, Ogievetsky [533] has proved the following important theorem: The algebraic closure of $sl(n, R)$ and conf(n, R), when these algebras are defined over the same n-dimensional manifold [non-linearly for conf(n, R)], is the infinite-dimensional algebra of analytical diffeomorphisms Diff(n, R): $[sl(n, R), \mathrm{conf}(n, R)] = \mathrm{diff}(n, R)$. A corollary to this theorem then states: There is no finite matrix embedding $sl(n, R)$ in its defining form simultaneously with the generators of special conformal transformations on the same manifold.

[12] In contrast to the structure group $G = A(n, R)$, the gauge group $\mathcal{G}$ consists of spacetime-dependent group elements $g(x)$ and is thus *infinite-dimensional*. This intuitive notion is made more precise in the fiber bundle approach [226,356,684,685], where one introduces first the *bundle of affine frames* $A(M) := P(M_n, A(n, R), \pi, \delta)$; here π denotes the projection to the

$$\mathcal{A}(n,R) = \left\{ \begin{pmatrix} \Lambda(x) & \tau(x) \\ 0 & 1 \end{pmatrix} \mid \Lambda(x) \in \mathcal{GL}(n,R), \quad \tau(x) \in \mathcal{T}(n,R) \right\}. \tag{3.2.1}$$

In accordance with a Yang-Mills type gauge approach, we introduce the *generalized affine* connection [356] (cf. also [213,214,525,564,694]):

$$\stackrel{\approx}{\Gamma} = \begin{pmatrix} \Gamma^{(L)} & \Gamma^{(T)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma^{(L)\beta}_{\alpha} L^{\alpha}{}_{\beta} & \Gamma^{(T)\alpha} P_{\alpha} \\ 0 & 0 \end{pmatrix}. \tag{3.2.2}$$

It is a one-form $\stackrel{\approx}{\Gamma} = \stackrel{\approx}{\Gamma}_i \, dx^i$ and transforms inhomogeneously under an affine gauge transformation:

$$\stackrel{\approx}{\Gamma} \stackrel{A^{-1}(x)}{\longrightarrow} \stackrel{\approx}{\Gamma}{}' = A^{-1}(x) \stackrel{\approx}{\Gamma} A(x) + A^{-1}(x) dA(x), \qquad A(x) \in \mathcal{A}(n,R). \tag{3.2.3}$$

Following Mack [417], we regard a transformation as an *active* [13] one, if it is formed with respect to the group element

$$A^{-1}(x) = \begin{pmatrix} \Lambda^{-1}(x) & -\Lambda^{-1}(x)\tau(x) \\ 0 & 1 \end{pmatrix}, \tag{3.2.4}$$

which is inverse to $A(x) \in \mathcal{A}(n,R)$.

The corresponding affine curvature is given by

$$\stackrel{\approx}{R} := d\stackrel{\approx}{\Gamma} + \stackrel{\approx}{\Gamma} \wedge \stackrel{\approx}{\Gamma} = \begin{pmatrix} R^{(L)} & R^{(T)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d\Gamma^{(L)} + \Gamma^{(L)} \wedge \Gamma^{(L)} & d\Gamma^{(T)} + \Gamma^{(L)} \wedge \Gamma^{(T)} \\ 0 & 0 \end{pmatrix}, \tag{3.2.5}$$

where the exterior product of Lie algebra-valued forms has to be evaluated with respect to the adjoint representation $Ad\, A(B) = [A, B]$. It transforms covariantly under the affine gauge group: [14]

$$\stackrel{\approx}{R} \stackrel{A^{-1}(x)}{\longrightarrow} \stackrel{\approx}{R}{}' = A^{-1}(x) \stackrel{\approx}{R} A(x). \tag{3.2.6}$$

The exterior covariant derivative $\stackrel{\approx}{D} := d + \stackrel{\approx}{\Gamma}\wedge$ acts on an affine p-form $\stackrel{\approx}{\Psi} = \binom{\Psi}{1}$ as follows

$$\stackrel{\approx}{D} \stackrel{\approx}{\Psi} = \begin{pmatrix} d\Psi + \Gamma^{(L)} \wedge \Psi + \Gamma^{(T)} \\ 0 \end{pmatrix} = \begin{pmatrix} D\Psi + \Gamma^{(T)} \\ 0 \end{pmatrix}. \tag{3.2.7}$$

base manifold and δ the (left or right) action of the structure group $A(n,R)$ on the bundle. Gauge transformations are vertical automorphisms of P. These are diffeomorphisms of the bundle P which preserve each fiber, i.e. act trivially on the base space. In general, the infinite-dimensional gauge group can be identified with $\mathcal{G} = C^{\infty}(P \times_{Ad} G)$, where the cross section in the associated bundle is abbreviated by C^{∞} and Ad denotes the adjoint representation with respect to G, cf. [73]. The affine gauge group $\mathcal{A}(n,R) := C^{\infty}(A(M) \times_{Ad} A(n,R))$ contains the group $\mathcal{GL}(n,R) := C^{\infty}(A(M) \times_{Ad} GL(n,R))$ of linear gauge transformations and the group $\mathcal{T}(n,R) := C^{\infty}(A(M) \times_{Ad} R^n)$ of local translations as subgroups. Due to its construction, the group $\mathcal{T}(n,R)$ is *locally* isomorphic to the group of active diffeomorphisms Diff(n,R) of the manifold [533,656]. We will not go any further into the bundle action of gauge groups, but refer the reader to [73,196,447,656] for details. In the more intricate case of the semidirect affine group, there may also exist a close interrelation of nonlinear gauge realizations [408,589,592,326] and induced representations [681,619,620,444], cf. appendix C.6.

[13] We regard, in accordance also with DeWitt, cf. [157], *active* diffeomorphisms or vertical automorphisms as those transformations which arise from a shift of a point in the manifold or in the fiber, respectively. Infinitesimally these actions can be expressed by Lie derivatives. Thus activ diffeomorphisms are equivalent to *local translations*, whereas vertical automorphisms yield (3.2.3) for the pull-back of the connection. On the other hand, passive transformations correspond to a mere relabeling of the coordinates, under which Cartan's exterior differential forms [103] are invariant by construction.

[14] Our matrix formalism, cf. [447,427] and references therein, is a spacetime generalization of the so-called *motor calculus* of von Mises [465,466].

Only by imposing the gauge $\Gamma^{(T)} = 0$, one would recover the covariant exterior derivative $D := d+\Gamma^{(L)}$ with respect to the linear connection $\Gamma^{(L)} = \Gamma_\alpha{}^\beta\, \rho(L^\alpha{}_\beta)$. Its action on (geometrical) fields depends on the representation type ρ to be specified later on. The Ricci identity takes the form

$$\tilde{\tilde{D}}\,\tilde{\tilde{D}}\,\tilde{\tilde{\Psi}} = \begin{pmatrix} DD\Psi + D\Gamma^{(T)} \\ 0 \end{pmatrix} = \tilde{\tilde{R}}\,\tilde{\tilde{\Psi}}.$$

After substitution of (3.2.2) and (3.2.4), the inhomogeneous transformation law (3.2.3) splits into

$$\Gamma^{(L)} \xrightarrow{\Lambda^{-1}(x)} \Gamma^{(L)\prime} = \Lambda^{-1}(x)\,\Gamma^{(L)}\Lambda(x) + \Lambda^{-1}(x)\, d\Lambda(x), \tag{3.2.8}$$

$$\Gamma^{(T)} \xrightarrow{\Lambda^{-1}(x)} \Gamma^{(T)\prime} = \Lambda^{-1}(x)\,\Gamma^{(T)} + \Lambda^{-1}(x)\, D\tau(x). \tag{3.2.9}$$

The local translations $\tau(x)$ automatically drop out in (3.2.8) due to the one-form structure of $\Gamma^{(T)}$. Thereby (3.2.8) acquires the conventional transformation rule (with the exterior derivative d) of a Yang–Mills-type connection for $\mathcal{GL}(n,R)$. Thus we can identify $\Gamma^{(L)} = \Gamma = \Gamma_\alpha{}^\beta\, L^\alpha{}_\beta$ with the linear connection. Due to the covariant exterior derivative term $D\tau(x) := d\tau(x) + \Gamma^{(L)}\,\tau(x)$ in (3.2.9), the translational part $\Gamma^{(T)}$ does not transform as a covector as is required for the coframe $\vartheta := \vartheta^\alpha P_\alpha$, i.e. a one-form with values in the Lie algebra of R^n.

However, it is suggestive to follow Trautman [683] and to introduce a vector (vector-valued zero-form) $\tilde{\tilde{\xi}} = \binom{\xi}{1} = \binom{\xi^\alpha P_\alpha}{1}$ which transforms as $\tilde{\tilde{\xi}}' = \Lambda^{-1}(x)\tilde{\tilde{\xi}}$, i.e. as

$$\xi \xrightarrow{\Lambda^{-1}(x)} \xi' = \Lambda^{-1}(x)\,(\xi - \tau(x)) \tag{3.2.10}$$

under an active affine gauge transformation. Then

$$\vartheta := \Gamma^{(T)} + D\xi \tag{3.2.11}$$

transforms [15] as a vector-valued one-form under the $\mathcal{A}(n,R)$, as required:

$$\vartheta \xrightarrow{\Lambda^{-1}(x)} \vartheta' = \Lambda^{-1}(x)\,\vartheta. \tag{3.2.12}$$

By differentiation of (3.2.11), the translational part of the affine curvature turns out to be

$$R^{(T)} := D\Gamma^{(T)} = (T^\alpha - R_\beta^{(L)\alpha}\xi^\beta)\,P_\alpha, \tag{3.2.13}$$

that is, the translational curvature contains, besides the torsion, a piece [16] induced by the linear curvature $R^{(L)}$. The vector field ξ is also found, perhaps in a more natural way, in the context of a group theoretical analysis of the gauged non-linear realizations of the translation group, see [408,340,341,326].

If $\Gamma^{(T)}$ vanished throughout the manifold, the vector field ξ would represent a n-dimensional version of Cartan's *generalized radius vector* [103]. The integrability condition is, in this instance, given by the vanishing of the translational part of the affine curvature (3.2.5), i.e. $R^{(T)} = 0$, which implies $T^\alpha = R_\beta^{(L)\alpha}\xi^\beta$. Incidentally, we correct hereby a sign error in [457].

[15] The canonical soldering one-form $\delta := \delta_\alpha^\beta \vartheta^\alpha \otimes e_\beta = \vartheta^\alpha \otimes e_\alpha$ would transform trivially under affine gauge transformation, cf. [617, p.342].

[16] One of us appreciates extended discussions which he had on this point with Luciano Mistura (Rome).

In a spacetime with a metric, a specific solution of the integrability condition would be $R^{(L)\alpha\beta} = R^{\alpha\beta} = a\vartheta^\alpha \wedge \vartheta^\beta$ and $T^\alpha = -ag_{\beta\gamma}\vartheta^\alpha \wedge \vartheta^\beta \xi^\gamma$, with the dimensionful constant a. For teleparallelism models with $R_\alpha^{(L)\beta} = 0$, the integrability condition yields a vanishing torsion.

3.3. Reduction to the metric-affine framework

In components, our key relation (3.2.11) takes the form

$$\Gamma_i^{(T)\alpha} = e_i{}^\alpha - D_i\xi^\alpha , \tag{3.3.1}$$

which, for $D_i\xi^\alpha = \delta_i^\alpha$, makes contact with the approach of Hayashi et al. [262–264], see also [471]. This condition identifies the coset space $A(n,R)/GL(n,R) \approx R^n$ with the *cotangent space* $T^*(M_n)$, cf. Niederle [513]. In a recent paper [243] on the Poincaré gauge approach, the ξ^α are kinematically interpreted as "Poincaré coordinates"; note that in Eq.(2.14) of that paper, vielbein and translational connection are identified opposite to our notation. Observe also that we do not have to put the "Poincaré coordinates" ξ^α to zero, in order to obtain the affine gauge transformation law (3.2.12) of the coframe. The reason is that the local translations are now "hidden" in the invariant transformation behavior of the exterior one-form ϑ under (passive) diffeomorphisms. Note also that in our approach in [457] we have shown of how to avoid a degenerate coframe. In contrast to Sexl and Urbantke [617, p.381], we do not need to break the affine gauge group kinematically via $D\tau(x) = 0$. There are attempts [593] to motivate the translational connection (3.3.1) from the theory of dislocations, whereas Hennig and Nitsch [300] provide an explanation in terms of jet bundles.

Since $\xi = \xi^\alpha P_\alpha$ acquires its values in the "orbit" (coset space) $A(n,R)/GL(n,R) \approx R^n$, it can be regarded as an affine vector field [17] (or "generalized Higgs field" according to Trautman [684]) which "hides" the action of the local translational "symmetry" $\mathcal{T}(n,R)$. We believe that the story of the ξ has not yet come to an end and that future developments on this point are possible. Probably one has to come up with an idea of how to construct an explicit symmetry breaking mechanism. For the time being, however, we require the condition [561]

$$D\xi = 0 \tag{3.3.2}$$

to hold. Then the translational connection $\Gamma^{(T)}$, together with the coframe ϑ, is soldered to the spacetime manifold, cf. [641], and the translational part of the affine gauge group is "spontaneously broken", cf. [540–542]. We may even postulate the stronger constraint of a "zero section" vector field $\xi = 0$. Then the generalized affine connection $\overset{\approx}{\Gamma}$ on the affine bundle $A(M)$ reduces to the *Cartan connection* [355]

$$\overset{=}{\Gamma} = \begin{pmatrix} \Gamma^{(L)} & \vartheta \\ 0 & 0 \end{pmatrix} \tag{3.3.3}$$

on the bundle $L(M)$ of linear frames. Due to (3.2.12), this is not anymore a connection in the usual sense.

[17] In the anti-deSitter gauge model of gravity of Stelle and West [653], the ξ^α parametrize the coset space $SO(2,3)/SO(1,3)$. The coframe ϑ^α and the Lorentz connection $\overset{\circ}{\Gamma}{}^{\alpha\beta} = -\overset{\circ}{\Gamma}{}^{\beta\alpha}$ can then be derived from the original $SO(2,3)$-connection via a nonlinear realization of that group involving the ξ-field. Such a Cartan connection arises not only from a reduction of (anti)-deSitter bundles [234], but also from conformal G-structures [298].

This affects also the measurability of a connection. Quantum interference measurements depend on the *non-integrable phase factor* $U(A,\gamma) = P\ \exp[(i/\hbar)\oint A]$, where $A = A_i^J\,\lambda_J\,dx^i$ is a Yang–Mills type connection. If the loop γ lies in a field-free region, i.e., one with Yang–Mills curvature $F = dA + A\wedge A = 0$, but encloses a region with "magnetic" flux $F \neq 0$, the potential A can still be measured (Aharanov–Bohm effect) via the amount of phase shift for closed loops. The same would hold true for a gravitationally induced phase factor $U(\overset{\approx}{\Gamma},\gamma) = P\ \exp[(i/\hbar)\oint(\Gamma^{(T)\alpha}P_\alpha + \Gamma_\alpha^{(L)\beta}L^\alpha{}_\beta)]$: For a field Φ carrying no $GL(4,R)$ excitations, i.e. no spin, no shear, and no dilation, we need a closed loop γ to detect the gravitational analog of the Aharonov–Bohm effect in a conical space, since outside the (rounded) apex of the cone there is $\Gamma^{(T)\alpha}\overset{*}{=}0$ locally. This analogy would break down, however, if one considered, instead of the true translational potential $\Gamma^{(T)\alpha}$, the coframe ϑ^α soldered to the spacetime manifold, cf. Anandan [10]. Because the coframe is non-degenerate by definition, it could be measured even by a non-closed loop, showing its essential classical character. Since the dimension of P_α is $2\pi\hbar/l$, the gravitational analog of Dirac's quantization condition would be $U(\overset{\approx}{\Gamma},\gamma) = (2\pi\hbar\mathcal{M}_n G/\hbar l c^2) = 2\pi n$, i.e. the mass would turn out to be a multiple $\mathcal{M}_n = nM_{\rm Planck}$ of the Planck mass.

Anyways, if additionally a metric is given, we recover the familiar metric-affine geometrical arena [18] with nonmetricity, torsion, and curvature, as is summarized in Table 1.

3.4. *Differentiable manifold M_n, frames, and coframes*

A geometrical arena which is large enough to eventually carry the Cartan connection (3.3.3) is an n-dimensional differentiable manifold M_n. At each point P of M_n, there is an n-dimensional vector space $T_P(M_n)$, the tangent vector space at P. We introduce a local vector basis, the *local frame* (or vielbein) e_α. Here we adhere to the following conventions: $\alpha,\beta,\ldots = 0,1,2,\ldots,(n-1)$ are anholonomic or frame indices, $i,j,k,\ldots = 0,1,2,\ldots,(n-1)$ are holonomic or coordinate indices, and $\wedge$ denotes the exterior product. In terms of the local coordinate basis $\partial_i := \partial/\partial x^i$, the frame e_α can be expanded according to

$$e_\alpha = e^i{}_\alpha\,\partial_i\,. \tag{3.4.1}$$

In order for e_α to serve as an anholonomic basis, the $e^i{}_\alpha$ are required to be non-degenerate, i.e. $\det e^i{}_\alpha \neq 0$. In the cotangent space $T_P^*(M_n)$ there exists a one-form basis or a *coframe*

$$\vartheta^\beta = e_j{}^\beta\,dx^j\,. \tag{3.4.2}$$

As a cobasis, the $e_j{}^\beta$ have to be non-degenerate as well. Moreover, the 'duality' of the frame e_α and the coframe with respect to the *interior product* $\rfloor$ requires for the frame and coframe coefficients that

$$e_\alpha\rfloor\vartheta^\beta = e^i{}_\alpha\,e_i{}^\beta = \delta^\beta_\alpha\,. \tag{3.4.3}$$

For the translational gauge potential $\Gamma^{(T)\alpha}$ this condition implies

$$e_\alpha\rfloor\Gamma^{(T)\beta} = \delta^\beta_\alpha - D_\alpha\xi^\beta = \delta^\beta_\alpha\,, \tag{3.4.4}$$

[18] Among the few textbooks which stress the importance of the interplay between affine and metric structures in the set-up of a theory of spacetime, we mention those of Kopczyński and Trautman [367] and of Ludwig [414], cf. Weyl [717] and Cartan [103].

since $D\xi^\alpha = 0$ for a Cartan connection. This is another manifestation of the 'soldering' of the frames to the manifold.

In an M_n, the frame e_α is determined only up to a local linear transformation (general frame *deformation*)

$$e_\alpha \to e'_\beta = \Lambda(x)_\mu{}^\nu \, \rho(L^\mu{}_\nu)_\beta{}^\gamma e_\gamma = \Lambda_\beta{}^\alpha(x)\, e_\alpha \,, \tag{3.4.5}$$

whereas the coframe, as part of the Cartan connection (3.3.3), transforms according to (3.2.12) or, more explicitly, as

$$\vartheta^\alpha \to \vartheta'^\beta = \Lambda^{-1}(x)_\mu{}^\nu \, \rho(L^\mu{}_\nu)_\alpha{}^\beta \vartheta^\alpha = \Lambda^{-1}(x)_\alpha{}^\beta \, \vartheta^\alpha \,. \tag{3.4.6}$$

The basis of $gl(n,R)$ acts on a geometrical object of representation type ρ via $\rho(L^\alpha{}_\beta)$. In the case of the frame (3.4.5), for instance, the corresponding representation is the Cartan–Weyl basis given by Kronecker deltas, i.e., $\rho(L^\mu{}_\nu)_\beta{}^\gamma = \delta^\mu_\beta \, \delta^\gamma_\nu$.

In the M_n we can use the operator of exterior derivation d and, of course, exterior $\wedge$ and interior $\rfloor$ multiplication. We call

$$C^\gamma := d\vartheta^\gamma = \tfrac{1}{2} C_{\alpha\beta}{}^\gamma \vartheta^\alpha \wedge \vartheta^\beta \tag{3.4.7}$$

the *anholonomity* two-form, which is not a gauge-covariant object. The components of the anholonomity C^γ can be obtained by repeated interior multiplication with the frame e_α:

$$C_{\alpha\beta}{}^\gamma = e_\beta \rfloor e_\alpha \rfloor C^\gamma = 2 e^i{}_\alpha e^j{}_\beta \, \partial_{[i} e_{j]}{}^\gamma \,. \tag{3.4.8}$$

In the 'holonomic gauge' where C^γ vanishes, we have a natural (or coordinate) coframe.

It has been noted already by Bergmann and Komar [53] that the apparent additional geometrical degrees of freedom, which are represented by the n^2 coefficients $e_i{}^\alpha(x)$ of the coframe one-form $\vartheta^\alpha = e_i{}^\alpha(x)\, dx^i$, can be absorbed by the action of the infinite-dimensional gauge group $\mathcal{G}L(n, R)$ of local linear transformations. An element of this group of automorphisms of the linear frame bundle may be locally represented by $\Lambda(x) = \Lambda_\alpha{}^\beta(x)\, L^\alpha{}_\beta$. The spacetime-dependent parameters $\Lambda_\alpha{}^\beta(x)$ consist of n-dimensional rotations (or Lorentz transformations), shear transformations, and scale transformations of the frames ("local Weyl transformations" [359]).

In an M_n we can define the *Lie derivative* of a scalar-valued p-form Ψ with respect to a vector field v as a rate of change of the p-form if transported along the vector. One finds (see [121]):

$$l_v \Psi := v \rfloor d\Psi + d(v \rfloor \Psi) \,. \tag{3.4.9}$$

The properties of this derivative are listed in (A.1.37). This expression is not gauge-covariant for tensor-valued forms and needs to be generalized.

3.5. Connection, exterior covariant derivative, torsion and curvature

In order to allow for a covariant differentiation of a tensor-valued form, we use the linear part of the Cartan *connection* (3.3.3) characterized by a one-form Γ which has values in the Lie algebra $gl(n,R)$ of the general linear group $GL(n,R)$. Hence Γ is expressible as

$$\Gamma = \Gamma_\alpha{}^\beta \, L^\alpha{}_\beta \,, \qquad \Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta \, dx^i \,, \tag{3.5.1}$$

where the $\Gamma_\alpha{}^\beta$'s are one-forms. A manifold M_n that is equipped with a linear connection Γ will be called [606] a *linearly connected manifold* L_n.

According to (3.2.8), the components of the linear connection transforms *inhomogeneously* under a local linear gauge transformation

$$\Gamma_\alpha{}^\beta \rightarrow \Gamma'^{\,\beta}_\alpha = \Lambda(x)_\alpha{}^\gamma \Gamma_\gamma{}^\delta \Lambda^{-1}(x)_\delta{}^\beta - \Lambda(x)_\alpha{}^\gamma d\Lambda^{-1}(x)_\gamma{}^\beta . \tag{3.5.2}$$

This is still a passive transformation, but with inverted factor ordering. The difference between two *different* connections behaves tensorial again:

$$\Delta\Gamma_\alpha{}^\beta := {}^{(1)}\Gamma_\alpha{}^\beta - {}^{(2)}\Gamma_\alpha{}^\beta , \qquad \Delta\Gamma'^{\,\beta}_\alpha = \Lambda(x)_\alpha{}^\gamma \Delta\Gamma_\gamma{}^\delta \Lambda^{-1}(x)_\delta{}^\beta . \tag{3.5.3}$$

If in (3.5.2) we pick a transformation which leads from a holonomic to an anholonomic frame, [19] that is, $\Lambda^{-1}(x)_i{}^\alpha = e_i{}^\alpha$, then we find

$$de_i{}^\alpha + \Gamma_\beta{}^\alpha e_i{}^\beta - \Gamma_i{}^j e_j{}^\alpha = 0 . \tag{3.5.4}$$

This is not a separate condition "$\mathcal{D}e_i{}^\alpha = 0$" on the tetrads, as often erroneously stated in the literature, but merely a relation which allows to compute the holonomic components $\Gamma_i{}^j$ in terms of the anholonomic ones $\Gamma_\beta{}^\alpha$, or vice versa.

For a tensor-valued p-form density of representation type ρ, the $GL(n,R)$*-covariant exterior* derivative is given by

$$D := d + \Gamma_\alpha{}^\beta\, \rho(L^\alpha{}_\beta) \wedge . \tag{3.5.5}$$

As an example, a vector-valued p-form Φ^α differentiates as

$$D\Phi^\alpha = d\Phi^\alpha + \Gamma_\beta{}^\alpha \Phi^\beta . \tag{3.5.6}$$

For a lower $GL(n,R)$-index, as in Ψ_α, a minus sign should be used instead. If we have a density, say $\hat{\Psi}_\alpha$, of anholonomic weight ω, then the trace $\Gamma := \Gamma_\alpha{}^\alpha$ of the connection appears, due to $\rho(L^\alpha{}_\beta)^\mu_\nu = -\delta^\alpha_\nu \delta^\mu_\beta - \omega \delta^\alpha_\beta \delta^\mu_\nu$, explicitly in the covariant derivative:

$$D\hat{\Psi}_\alpha = d\hat{\Psi}_\alpha - \Gamma_\alpha{}^\beta \hat{\Psi}_\beta - \omega\Gamma\hat{\Psi}_\alpha . \tag{3.5.7}$$

In general the weight ω is different from the dimension d_Ψ of the field Ψ.

Alternatively, one can derive this relation from the postulates that the Levi-Civita n-form density of (A.1.15) is covariantly constant and that the Leibniz rule is valid even if a density is involved. Note that the minus sign in front of the ω-term in (3.5.7) also occurs if a quantity with an upper Lie algebra index is differentiated, see [605]. By applying the exterior covariant derivatice twice, we arrive at the Ricci identity

$$DD = R_\alpha{}^\beta \rho(L^\alpha{}_\beta) \wedge , \tag{3.5.8}$$

which, if applied to a vector-valued zero-form, is often used for defining the curvature two-form $R_\alpha{}^\beta$, see (3.5.10).

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[19] This transformation provides a transition from an anholonomic to a holonomic cross section of the linear bundle.

The field strengths associated with the coframe and the linear connection are given by the *torsion* two-form

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \tfrac{1}{2} T_{\mu\nu}{}^\alpha\, \vartheta^\mu \wedge \vartheta^\nu, \tag{3.5.9}$$

and the linear part of the Lie-algebra-valued *curvature* two-form (3.2.5), respectively. From the commutation relation (3.1.5) and (3.1.6) of the affine algebra, we find for the components of the curvature:

$$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = \tfrac{1}{2} R_{\mu\nu\alpha}{}^\beta \vartheta^\mu \wedge \vartheta^\nu. \tag{3.5.10}$$

The left hand sides of (3.5.9) and (3.5.10) constitute Cartan's first and second structure equation, respectively.

A big boost for the application and in the visualization of torsion was Kazuo Kondo's 1952 discovery [360] (see also [373,352]) of the possibility of describing a quasi-continuous distribution of dislocations in three-dimensional crystals by means of a torsion field. From this example it is immediately clear that the torsion field can encompass singularities. For the curvature this sort of behavior is well-known from GR. There, a singular metric may induce a singularity in the curvature as, for example, in the vacuum Schwarzschild solution at the origin (vanishing radial coordinate). However, *torsion singularities* – being induced by a singular coframe and/or a singular linear connection – are qualitatively different therefrom and should be studied in their own right, as was first pointed out by Nester and Isenberg [511], see also Baker [30], Edelen [177], Garcia de Andrade [217], Tod [678], and Zhang and Chen [736].

We have demonstrated explicitly that torsion and curvature are merely parts of the generalized curvature of the Cartan connection discussed above. According to (3.4.6) and (3.5.2), torsion and curvature transform as

$$T^\alpha \to T'^\alpha = \Lambda^{-1}(x)_\beta{}^\alpha\, T^\beta, \tag{3.5.11}$$

$$R_\alpha{}^\beta \to R'_\alpha{}^\beta = \Lambda(x)_\alpha{}^\gamma \Lambda^{-1}(x)_\delta{}^\beta\, R_\gamma{}^\delta \tag{3.5.12}$$

under a linear gauge transformation.

Having now a prescribed connection at our disposal, we can, in generalizing the notion of the ordinary Lie derivative (3.4.9) of a scalar-valued form, come up with the notion of *gauge covariant Lie derivative* of a Lie algebra-valued form Ψ with respect to a vector v (see [504,506,510,674,704]):

$$\text{Ł}_v \Psi := v\rfloor D\Psi \;+\; D\left(v\rfloor\Psi\right). \tag{3.5.13}$$

In deriving the Noether identities in section 5.2, this gauge-covariant notion[20] will be of great help. Alternatively, Eq. (3.5.13) can be put into the form

$$\text{Ł}_v \Psi = l_v \Psi + (v\rfloor \Gamma_\alpha{}^\beta)\rho(L^\alpha{}_\beta)\Psi. \tag{3.5.14}$$

Later on in (3.11.9), we will employ the covariant exterior derivative $\widehat{D}$ with respect to the *transposed* connection (see [607,682])

$$\widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta. \tag{3.5.15}$$

[20] The 'ordinary' Lie derivative $\pounds_v$ of (A.1.38), which is already defined in an M_n, does not coincide with the Ł$_v$ of above. These two operators are interrelated by $\pounds_v \Psi = \text{Ł}_v \Psi - (\widehat{D}_\alpha v^\beta)\rho(L^\alpha{}_\beta)\Psi$, $\widehat{D}_\alpha := e_\alpha \rfloor \widehat{D}$.

Its somewhat unclear role becomes more transparent by the following observation: If applied to vector components v^α, the transposed derivative is identical to the gauge-covariant Lie derivative of the coframe with respect to the vector, i.e.

$$\text{Ł}_v\, \vartheta^\alpha \equiv \widehat{D}\, v^\alpha. \tag{3.5.16}$$

3.5.1. Geometric interpretation of torsion and curvature

In section 3.2 the affine connection $\overset{\approx}{\Gamma}$ and the affine curvature $\overset{\approx}{R}$ are defined in the standard Yang–Mills manner. If, by means of $\overset{\approx}{\Gamma}$, an affine vector $\overset{\approx}{v}$ is parallelly transported around a small closed loop, an affine transformation of $\overset{\approx}{v}$ is induced, the linear piece of which is determined by the linear curvature $R^{(L)}$ and the translational piece by the translational curvature $R^{(T)}$, see (3.2.13). In order to recover this sort of behavior on the level of the L_n, we have to introduce the so-called Cartan transport:

In a flat L_n – that is an L_n with $R_\alpha{}^\beta = 0$ – a *parallel* vector field $\zeta = \zeta^\alpha e_\alpha$ is one that satisfies the equation

$$D\zeta^\alpha = 0\,, \tag{3.5.17}$$

compare also (3.3.2). In an *arbitrary* linearly connected space L_n, the integrability condition for (3.5.17) is

$$DD\zeta^\alpha = R_\beta{}^\alpha \zeta^\beta = 0. \tag{3.5.18}$$

If the curvature tensor $R_\beta{}^\alpha$ does *not* vanish, we may still integrate (3.5.17) along a curve in order to get a vector which is parallelly transported along this curve. A standard calculation shows that if a vector ζ is parallelly transported around an *infinitesimal closed loop*, it is linearly deformed by an amount proportional to the curvature × area enclosed by the loop. An explicit statement of this result will be included in a more general result to be derived below, see (3.5.24). Note that the torsion plays no role here.

In order to find a context in which torsion does enter, let us consider an extension of the propagation law (3.5.17) along the lines of Cartan [103] (cf. [683]). Returning to the flat L_n, we may define a *radius vector* field (or position vector field) ξ as one that satisfies the equation

$$D\xi^\alpha = \vartheta^\alpha, \tag{3.5.19}$$

which corresponds to $\Gamma^{(T)\alpha} = 0$, compare (3.2.11). Let us refer to (3.5.19) as the equation of a *Cartan transport* ('rolling without sliding'). In terms of a Cartesian coordinate basis ∂_i with $\xi = \xi^i \partial_i$, Eq.(3.5.19) is simply $\partial_j \xi^i = \delta^i_j$ and hence $\xi^i = x^i + A^i$, with A^i as a constant vector, so that ξ^i is the radius (or position) vector of x^i with respect to an origin $x^i = -A^i$. In an L_n, the integrability condition for (3.5.19) is

$$DD\xi^\alpha - D\vartheta^\alpha = R_\beta{}^\alpha \xi^\beta - T^\alpha = 0. \tag{3.5.20}$$

Hence a necessary condition for the existence of global radius vector fields ξ is vanishing translational curvature; sufficient for this is zero torsion and zero linear curvature. Note that a *metric* is *not necessary* in the context of these considerations.

Suppose now that the torsion and curvature are non-zero and we integrate (3.5.19) around a closed infinitesimal curve C, with tangent vector v, beginning and ending at a point P having coordinates x^i. Let Q be any point on the curve with coordinates $x^i + y^i$ and Q' a neighboring point with coordinates $x^i + y^i + dy^i$. The equation of the curve is of the form $y^i = y^i(t)$ and the tangent vector field is $v = (dy^i/dt)\,\partial_i$. Then (3.5.19) becomes

$$\frac{d\xi^\alpha}{dt} = e_i{}^\alpha \frac{dy^i}{dt} - \Gamma_{i\beta}{}^\alpha \xi^\beta \frac{dy^i}{dt}. \tag{3.5.21}$$

Thus the change in ξ^α in the displacement from Q to Q' is

$$d\xi^\alpha = (e_i{}^\alpha\, dy^i - \Gamma_{i\beta}{}^\alpha \xi^\beta\, dy^i)|_{(x^i+y^i)}. \tag{3.5.22}$$

Following Cartan [103], we may interpret this equation by looking upon the tangent spaces at Q and Q' as affine spaces. Eq. (3.5.22) tells us that the mapping bringing the point ξ^α in the tangent space at Q to $\xi^\alpha + d\xi^\alpha$ at Q' consists of a soldered translation $e_i{}^\alpha\, dy^i$ and a linear deformation $-\Gamma_{i\beta}{}^\alpha\, dy^i\, \xi^\beta$.

If we now make a Taylor expansion about x^i of the functions $e_i{}^\alpha(x+y)$ and $\Gamma_{i\beta}{}^\alpha(x+y)$ and apply (3.5.22) to the infinitesimal displacement $x^i \to x^i + y^i$ to get $\xi^\beta(x+y) = \xi^\beta(x) + e_j{}^\beta y^j - \Gamma_{j\mu}{}^\beta(x)\xi^\mu(x)y^j$, then Eq. (3.5.22) becomes

$$d\xi^\alpha = d(e_i{}^\alpha y^i - \Gamma_{i\beta}{}^\alpha \xi^\beta y^i) + (T_{ij}{}^\alpha - R_{ij\beta}{}^\alpha \xi^\beta) y^{[i} dy^{j]}. \tag{3.5.23}$$

Hence, on integrating around C, it is found that the total change in ξ^α is

$$\Delta\xi^\alpha \simeq (T_{ij}{}^\alpha - R_{ij\beta}{}^\alpha \xi^\beta) \oint_C y^{[i}\, dy^{j]} = \tfrac{1}{2}(T_{ij}{}^\alpha - R_{ij\beta}{}^\alpha \xi^\beta) \int_S dy^i \wedge dy^j, \tag{3.5.24}$$

where S is the two-dimensional plane element enclosed by C. The $\simeq$ sign indicates that the surface S is so small that the components of curvature and torsion are constant in this area and can be taken in front of the surface integral. Thus, in going around the infinitesimal closed loop C, the vector ξ undergoes a translation and a linear deformation of the same order of magnitude as the area of S, the *translation* being determined by the *torsion* and the *linear deformation* by the *curvature*.

The Cartan transport may also be understood from the affine point of view. The affine version of (3.5.17) reads

$$\overset{\approx\approx}{D\xi^\alpha} = \begin{pmatrix} d\xi^\alpha + \Gamma^{(L)} \wedge \xi^\alpha + \Gamma^{(T)\alpha} \\ 0 \end{pmatrix} = \begin{pmatrix} D\xi^\alpha + \Gamma^{(T)\alpha} \\ 0 \end{pmatrix} = 0, \tag{3.5.25}$$

see (3.2.7). Parallel-transport along a tangent vector y of the Cartan circuit yields

$$\overset{\approx}{\mathrm{L}}_y \overset{\approx}{D} \overset{\approx}{\xi}{}^\alpha = y\rfloor \left(\overset{\approx}{D}\,\overset{\approx}{D}\,\overset{\approx}{\xi}{}^\alpha\right) + \overset{\approx}{D}\left(y\rfloor \overset{\approx}{D}\,\overset{\approx}{\xi}{}^\alpha\right) = y\rfloor \begin{pmatrix} DD\xi^\alpha + R^{(T)\alpha} \\ 0 \end{pmatrix} = 0. \tag{3.5.26}$$

Integration of the first one-form on a closed loop parametrized by y yields

$$\Delta\xi^\alpha = -\oint_C y\rfloor(DD\xi^\alpha) = \oint_C y\rfloor R^{(T)\alpha} = \int_S R^{(T)\alpha} \simeq \tfrac{1}{2}(T_{ij}{}^\alpha - R_{ij\beta}{}^\alpha\, \xi^\beta) \int_S dy^i \wedge dy^j, \tag{3.5.27}$$

see (3.2.13). This derivation is much nicer than the component approach starting with (3.5.19).

3.6. Affine gauge transformations versus active diffeomorphisms

The affine gauge transformations in (3.2.3) are finite transformations. If we expand them up to first order according to

$$\Lambda(x) = 1 + \omega_\alpha{}^\beta L^\alpha{}_\beta + \cdots, \tag{3.6.1}$$

$$\tau(x) = 0 + \varepsilon^\alpha P_\alpha + \cdots, \tag{3.6.2}$$

we obtain from (3.2.8) and (3.2.9), respectively,

$$\delta_{\Lambda^{-1}} \Gamma^{(L)} = (D\omega_\alpha{}^\beta) L^\alpha{}_\beta + \cdots, \tag{3.6.3}$$

$$\delta_{\Lambda^{-1}} \Gamma^{(T)} = (D\varepsilon_\alpha + \omega_\beta{}^\alpha \Gamma^{(T)\beta}) P_\alpha + \cdots. \tag{3.6.4}$$

Incidentally, for the "product" of Lie generators we use the Lie brackets of section 3.1, since we work in the adjoint representation. It is gratifying to note that the leading exterior covariant derivatives reveal, in particular, that the translational connection $\Gamma^{(T)}$ is really the "compensating" field for infinitesimal local translations ε in the Yang–Mills sense.

Let us compare this result with the "diffeomorphism" approach, which was orginally developed for the Poincaré subgroup of the $A(n,R)$: In essence, the translational part $\varepsilon = \varepsilon^\alpha P_\alpha$ of the transformation

$$\Pi = 1 + \varepsilon + \omega = 1 + \varepsilon^\alpha P_\alpha + \omega_\alpha{}^\beta L^\alpha{}_\beta, \tag{3.6.5}$$

is embedded as an n-parameter subgroup in the infinite-dimensional group of *active* diffeomorphisms of spacetime. In order to calculate the effect on the linear conection and the coframe, one has to consider the action [268] of the Lie derivative $\mathcal{L}_\varepsilon$ with respect to the vector field ε together with an infinitesimal frame rotation parametrized by ω. Since $\mathcal{L}_\varepsilon = l_\varepsilon$ holds for geometrical objects which are invariant under changes of the basis, a straightforward calculation yields

$$(\mathcal{L}_\varepsilon + \delta_\omega)\Gamma = \left[D(\omega_\alpha{}^\beta + \varepsilon\rfloor\Gamma_\alpha{}^\beta) + \varepsilon\rfloor R_\alpha{}^\beta\right] L^\alpha{}_\beta, \tag{3.6.6}$$

$$(\mathcal{L}_\varepsilon + \delta_\omega)\vartheta = \left[D\varepsilon^\alpha - (\omega_\beta{}^\alpha + \varepsilon\rfloor\Gamma_\beta{}^\alpha)\vartheta^\beta + \varepsilon\rfloor T^\alpha\right] P_\alpha. \tag{3.6.7}$$

The "annoying" linear connection terms in (3.6.6) and (3.6.7) can be dismissed by going over to the *parallel transport* version of the theory, as presented in [275,479], for example, in which, instead of P_α, the covariant derivative components $D_\alpha := e_\alpha\rfloor D$ are adopted as generators of local translations: Then the infinitesimal transformations read

$$\tilde{\Pi} = 1 + \varepsilon^\alpha D_\alpha + \omega_\alpha{}^\beta L^\alpha{}_\beta = \Pi - \varepsilon\rfloor\Gamma_\alpha{}^\beta L^\alpha{}_\beta. \tag{3.6.8}$$

Since this amounts to a redefinition $\tilde{\omega} := \omega - \varepsilon\rfloor\Gamma_\alpha{}^\beta L^\alpha{}_\beta$ of the parameters of the infinitesimal linear transformation, we can simply read off, from (3.6.6) and (3.6.7), the new results

$$(\mathcal{L}_\varepsilon + \delta_{\tilde{\omega}})\Gamma = \left[D\omega_\alpha{}^\beta + \varepsilon\rfloor R_\alpha{}^\beta\right] L^\alpha{}_\beta \tag{3.6.9}$$

$$(\mathcal{L}_\varepsilon + \delta_{\tilde{\omega}})\vartheta = \left[D\varepsilon^\alpha - \omega_\beta{}^\alpha\vartheta^\beta + \varepsilon\rfloor T^\alpha\right] P_\alpha. \tag{3.6.10}$$

In this *parallel transport* version, the leading covariant derivative pieces are the same as in the affine gauge approach. In particular, the "hidden" translational piece in the *affine* transformation (3.2.9) of the coframe gets thereby "uncovered" in (3.6.10). In the end, is it "...somewhat a matter of taste...", as

Nester [504] has put it, whether or not one prefers the parallel transport interpretation of translations over the affine gauge approach? One could argue that the Pauli-type curvature and torsion terms in the infinitesimal transformations (3.6.9) and (3.6.10) violate the spirit of the principle of *minimal coupling*, a cornerstone of a conventional Yang–Mills type gauge approach.

These terms also show up in the commutation relations for the operator D_α of parallel-transport, if applied to a zero-form. Let us start with the identity $\vartheta^\gamma \wedge (e_\gamma \rfloor \Psi) = p\Psi$ for any p-form Ψ. Then we get from

$$(p+1)DD\Psi = D \wedge \vartheta^\gamma \wedge (e_\gamma \rfloor D\Psi) = T^\gamma \wedge (e_\gamma \rfloor D\Psi) - \vartheta^\gamma \wedge D(e_\gamma \rfloor D\Psi) \tag{3.6.11}$$

and the Ricci identity (3.5.8), the relation

$$\vartheta^\alpha \wedge \vartheta^\beta \left(e_\alpha \rfloor D(e_\beta \rfloor D\Psi)\right) = -(p+1)T^\gamma \wedge (e_\gamma \rfloor D\Psi) + (p+1)^2 R_\mu{}^\nu \rho(L^\mu{}_\nu)\Psi. \tag{3.6.12}$$

If Ψ is a zero-form, Eq.(3.6.12) reduces to the commutation relation (see [275,479])

$$[\text{Ł}_{e_\alpha}, \text{Ł}_{e_\beta}] = [D_\alpha, D_\beta] = -T_{\alpha\beta}{}^\gamma D_\gamma + R_{\alpha\beta\mu}{}^\nu \rho(L^\mu{}_\nu), \tag{3.6.13}$$

cf. (2.4.1). Thus in a space with torsion and curvature the translations do not commute any longer, their Lie-algebra gets deformed and the former structure constants become x-dependent functions.

Such a *softening* [345] of the Lie algebra structure cannot be avoided in a diffeomorphism-type approach. Using the covariant derivatives (or Lie derivatives) has the advantage of being physically meaningful as a parallel transport, as explained in Ref. [272,275], once we put up a frame, and, in a corresponding first order approach, these 'non-minimal' structures do *not* touch the explicit form of the Lagrangian. However, they are algebraically less useful because (3.6.13) is not a Lie algebra any more.

3.7. Metric

For building up a macroscopical physical theory, we need a geometric structure which will enable us to measure lengths and angles. This is provided by the Riemannian metric g, a non-degenerate second order covariant *symmetric*[21] tensor field. In the L_n we introduce, besides the connection $\Gamma_\alpha{}^\beta$, an *independent* local metric structure by

$$g = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta, \quad g_{\alpha\beta} = g(e_\alpha, e_\beta). \tag{3.7.1}$$

A simple way to introduce a metric tensor field explicitly is to specify its $n(n+1)/2$ independent components $g_{ij} \equiv g(\partial_i, \partial_j)$ with respect to a given holonomic basis ∂_i:

$$g = g_{ij}\, dx^i \otimes dx^j, \quad g_{ij} = g(\partial_i, \partial_j) = g_{ji}. \tag{3.7.2}$$

The transition between the holonomic and anholonomic formulation is given by

$$g_{\alpha\beta} = e^i{}_\alpha e^j{}_\beta g_{ij}. \tag{3.7.3}$$

We assume that the metric is non-degenerate, i.e. $\det g_{\alpha\beta} \neq 0$.

[21] In contradistiction to Moffat's approach [467], we do not allow an *anti*symmetric part in the metric tensor, since it does not lend itself to a direct geometrical interpretation.

Let us consider, using standard results from linear algebra [241], the properties of the matrix $g_{\alpha\beta}$ at an arbitrary point P of the manifold: Because the matrix $g_{\alpha\beta}$ is symmetric, we can diagonalize it. Starting with a general frame field e_α with respect to which the metric components are given by (3.7.1), it is always possible to make a suitable local deformation $\Lambda_\alpha{}^\beta(x)$ to another frame $e'_\alpha = \Lambda_\alpha{}^\beta(x)\, e_\beta$ such that

$$g'_{\alpha\beta} = \Lambda_\alpha{}^\mu(x)\Lambda_\beta{}^\nu(x)\, g_{\mu\nu} = \mathrm{diag}\ (\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}) \tag{3.7.4}$$

is diagonal. We can normalize and rearrange the basis vectors e'_α in such a way that they yield

$$g(\overset{\circ}{e}_\alpha, \overset{\circ}{e}_\beta) = \overset{\circ}{g}_{\alpha\beta} = \mathrm{diag}\ (\underbrace{-1,\ldots,-1}_{\mathrm{Ind}(g)}, \underbrace{+1,\ldots,+1}_{n-\mathrm{Ind}(g)}) \quad =: o_{\alpha\beta}\,. \tag{3.7.5}$$

We call such a basis $\overset{\circ}{e}_\alpha$ a (pseudo)*orthonormal basis*. The natural number $\mathrm{Ind}(g)$ is the index [22] of the metric. According to Sylvester's law of inertia, the index is an invariant which is independent [23] of the particular orthonormal basis one arrived at. Only if we allow for degenerate metrics, see [174,586,183,147,143], then the index may change in different parts of the manifold.

The matrices $\overset{\circ}{\Lambda}(x) \in G(n,R)$, which 'rotate' one orthonormal basis into another one, are determined by the equation $\overset{\circ}{\Lambda}_\alpha{}^\mu(x)\overset{\circ}{\Lambda}_\beta{}^\nu(x)o_{\mu\nu} = o_{\alpha\beta}$. They build up the pseudo-orthogonal group $O(\mathrm{Ind}(g), n-\mathrm{Ind}(g))$.

If the metric is not positive (negative) definite, then there exists the 'lightcone'

$$LC_P := \{K \in T_P(M_n)\backslash\{0\} \mid g(K,K) = 0\}\,.$$

It uniquely determines the metric g in $T_P(M_n)$ up to a factor $\Omega \in R\backslash\{0\}$. If we act with an element of $O(\mathrm{Ind}(g), n-\mathrm{Ind}(g))$ on a vector, its length does not change. Accordingly, the $O(\mathrm{Ind}(g), n-\mathrm{Ind}(g))$ leaves the lightcone invariant.

So far we considered the properties of the metric on every $T_P(M_n)$. We now require the metric to be a smooth tensor field over M_n. It can then be shown that the index $\mathrm{Ind}(g)$ has to be constant [24] on M_n. If we restrict ourselves to a metric of *Minkowskian index*, that is, to $o_{\alpha\beta} = \mathrm{diag}(-1, 1, \cdots, 1)$, we find the $n(n-1)/2$-dimensional *Lorentz group* $O(1, n-1)$ as subgroup of the n^2-dimensional $GL(n,R)$. The "length square" of a vector $V = V^\alpha e_\alpha$ is defined by $V^2 := g(V,V) = V^\alpha V^\beta g(e_\alpha, e_\beta) = V^\alpha V^\beta g_{\alpha\beta}$. A vector V is called *timelike*, *lightlike (null)* or *spacelike*, according to whether $g(V,V) < 0$, $g(V,V) = 0$, or $g(V,V) > 0$, respectively.

The metric g induces an isomorphism $\phi : T_P(M_n) \longrightarrow T^*_P(M_n)$ by $\phi(V) \equiv g(V,\ \ )$ for each $V \in T_P(M_n)$. This isomorphism does not depend on a choice of a basis. Since the metric $g_{\alpha\beta}$ is non-degenerate, we can form the inverse metric $g^{\alpha\beta}$ such that $g_{\alpha\gamma}g^{\gamma\beta} = \delta^\beta_\alpha$. Now we can identify $T_P(M_n)$ with $T^*_P(M_n)$ or, technically speaking, we can raise and lower indices.

[22] More geometrically, we may define the index $\mathrm{Ind}(g)$ to be the maximal number of linearly independent timelike vectors $\in T_P(M_n)$.

[23] In order to change the signature in, for example, two dimensions from the Minkowskian form $\{o_{\alpha\beta}\} = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ to the Euclidian metric $\{\bar{g}_{\alpha\beta}\} = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, one has to transform $o_{\alpha\beta}$, according to (3.7.4), by a necessarily *complex* matrix. In our example, it is given by $\Lambda = \{\Lambda_\alpha{}^\beta\} = \left(\begin{smallmatrix}i\sqrt{1+\lambda^2} & \frac{i\lambda}{\sqrt{1+\lambda^2}}\\ \lambda & \end{smallmatrix}\right) \in GL(2,C)$, $\det\Lambda = i$.

[24] More exactly: constant on every connected part of M_n, see [259].

An L_n with an additional metric field will be called a metric-affine space (L_n, g). A connected four-dimensional oriented[25] differentiable manifold M_4 together with a linear *connection* $\Gamma_\alpha{}^\beta$ and a *metric* g of index 1 will be called a *SPACETIME* (L_4, g). Conventionally a spacetime is defined as an M_4 with a metric, see Sachs and Wu [582]. Our insistence on the independence of the linear connection basically results from the experience that one has made with gauge theories in which Lie-algebra valued connections play a prominent role as gauge potentials.

Owing to the existence of a metric, we can define the scalar density $\sqrt{|\det g_{\mu\nu}|}$ and, in view of (A.1.34), the g-volume element n-form[26]

$$\eta := \sqrt{|\det g_{\mu\nu}|}\vartheta^{\hat{1}} \wedge \cdots \wedge \vartheta^{\hat{n}} = \frac{1}{n!}\sqrt{|\det g_{\mu\nu}|}\,\epsilon_{\alpha_1\cdots\alpha_n}\vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_n} = {}^*1, \tag{3.7.6}$$

dual to the unit zero-form. The Hodge star * will be defined below. Picking a (pseudo) orthonormal positively oriented coframe $\overset{\circ}{\vartheta}{}^\alpha$, the g-volume element simplifies to

$$\eta = \overset{\circ}{\vartheta}{}^{\hat{1}} \wedge \cdots \wedge \overset{\circ}{\vartheta}{}^{\hat{n}}. \tag{3.7.7}$$

Having an n-form at our disposal, we can successively, as in (A.1.35), contract it by means of the frame e_α, thereby arriving at an $(n-1)$-form, an $(n-2)$-form, etc., until we terminate the series with a zero-form:

$$\begin{aligned}
\eta_{\alpha_1} &:= e_{\alpha_1} \rfloor \eta = \frac{1}{(n-1)!}\eta_{\alpha_1\alpha_2\cdots\alpha_n}\vartheta^{\alpha_2} \wedge \cdots \wedge \vartheta^{\alpha_n} = {}^*\vartheta_{\alpha_1}, \\
\eta_{\alpha_1\alpha_2} &:= e_{\alpha_2} \rfloor \eta_{\alpha_1} = \frac{1}{(n-2)!}\eta_{\alpha_1\alpha_2\alpha_3\cdots\alpha_n}\vartheta^{\alpha_3} \wedge \cdots \wedge \vartheta^{\alpha_n} = {}^*(\vartheta_{\alpha_1} \wedge \vartheta_{\alpha_2}), \\
&\vdots \\
\eta_{\alpha_1\cdots\alpha_n} &= e_{\alpha_n} \rfloor \cdots \rfloor e_{\alpha_1} \rfloor \eta = {}^*(\vartheta_{\alpha_1} \wedge \cdots \wedge \vartheta_{\alpha_n}).
\end{aligned} \tag{3.7.8}$$

The η-bases span the graded algebra of dual exterior forms on each cotangent space $T^*(M_n)$.

3.7.1. Hodge star

The Hodge star operator * maps a p-form into an $(n-p)$-form. Already in (3.7.6) and in (3.7.8) we specified how the star operator acts on zero-forms, one-forms, two-forms, etc.. Note that we had to lower the index of the coframe in (3.7.8) by means of the metric in order to achieve such a correspondence, that is, the star operator can only be defined if a (pseudo)Riemannian metric is at hand. Explicitly, the Hodge dual of a p-form Ψ is defined by

$$^*\Psi := \frac{1}{(n-p)!p!}\sqrt{|\det g_{\mu\nu}|}g^{\alpha_1\gamma_1}\cdots g^{\alpha_p\gamma_p}\epsilon_{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_{n-p}}\Psi_{\gamma_1\cdots\gamma_p}\vartheta^{\beta_1} \wedge \cdots \wedge \vartheta^{\beta_{n-p}}. \tag{3.7.9}$$

[25] Because of CP-violation in the Kaon system, space cannot be non-orientable, see Zel'dovich and Novikov [735].

[26] Following early attempts of Einstein, one may consider theories which are invariant merely under the special groups $SL(n,R)$ or SDiff(n,R) with determinant plus one. Then the determinant of the metric in (3.7.6) appears as an additional structure which is, at times, identified with the dilaton field σ according to $\sigma = \sqrt{\det g_{\mu\nu}}$ (as a result of symmetry breaking, cf. section 6 for details).

If f is a zero-form, Ψ and Φ p-forms, the Hodge star has the properties ${}^*(f\Psi) = f\,{}^*\Psi$ and ${}^*(\Psi + \Phi) = {}^*\Psi + {}^*\Phi$. These rules allow us to compute the Hodge dual of any p-form.

Possibly up to a sign, double application of the Hodge star to a p-form reproduces the original p-form,

$$^{**}\Phi = (-1)^{p(n-p)+\mathrm{Ind}(g)}\Phi, \tag{3.7.10}$$

where we used the normalization $\epsilon_{\alpha_1\cdots\alpha_n}\,\epsilon^{\alpha_1\cdots\alpha_n} = (-1)^{\mathrm{Ind}(g)}n!$. For Φ and Ψ of the *same* degree p, we find

$${}^*\Phi \wedge \Psi = {}^*\Psi \wedge \Phi, \tag{3.7.11}$$

and for a p-form Φ

$$\vartheta^\alpha \wedge (e_\alpha \rfloor \Phi) = p\Phi, \tag{3.7.12}$$

$${}^*(\Phi \wedge \vartheta_\alpha) = e_\alpha \rfloor {}^*\Phi. \tag{3.7.13}$$

3.8. Nonmetricity

In an (L_n, g), the field strengths are then given by the components of the *nonmetricity* one-form (relates to the relativistic mass *quadrupole* moment) [27]

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + 2\Gamma_{(\alpha|}{}^\gamma g_{\gamma|\beta)} = Q_{i\alpha\beta}\,dx^i, \tag{3.8.1}$$

the *torsion* two-form (3.5.9), and the *curvature* two-form (3.5.10). The Weyl one-form (note the conventional factor $1/n$)

$$Q := (1/n)Q_\gamma{}^\gamma = -(1/n)g^{\alpha\beta}Dg_{\alpha\beta}, \tag{3.8.2}$$

is one irreducible piece of the nonmetricity. Thus the traceless part of the nonmetricity reads

$$\not{Q}_{\alpha\beta} := Q_{\alpha\beta} - Qg_{\alpha\beta}. \tag{3.8.3}$$

For the complete irreducible decompositions of the field strengths (and of the Bianchi identities), we refer to Appendix B. The explicit expressions for their tensor components are given, e.g., in [24], [26]. Note that from (3.8.1) there result the contra- and covariant relations

$$Q^{\alpha\beta} = Dg^{\alpha\beta} \qquad \text{and} \qquad T_\alpha := g_{\alpha\beta}T^\beta = D\vartheta_\alpha + Q_{\alpha\beta} \wedge \vartheta^\beta. \tag{3.8.4}$$

The covariant derivatives of the η-bases read: [28]

$$D\eta_{\alpha_1} = -\tfrac{1}{2}nQ \wedge \eta_{\alpha_1} + T^\beta \wedge \eta_{\alpha_1\beta},$$
$$D\eta_{\alpha_1\alpha_2} = -\tfrac{1}{2}nQ \wedge \eta_{\alpha_1\alpha_2} + T^\beta \wedge \eta_{\alpha_1\alpha_2\beta},$$

[27] In the textbook of Landau–Lifshitz [386] it is proved that the nonmetricity of a spacetime manifold has to vanish. We leave it as an exercise to our readers to show that this "proof" consists in a *petitio principii*. Incidentally, also the proof in [386] of the vanishing of torsion of spacetime is of similar quality. With an analogous technique one could prove the vanishing of the curvature, too.

[28] This set of formulae has also been derived by Tresguerres, see [687–689].

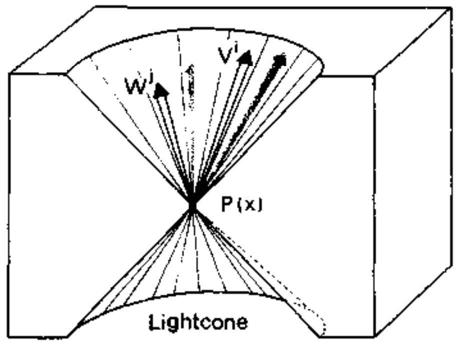


Fig. 1. Parallel displacement of vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ along a closed contour (not shown). The (shaded) null-vectors at the beginning may become, due to shear, both time-like at the end.

$$\vdots$$

$$D\eta_{\alpha_1\cdots\alpha_n} = -\tfrac{1}{2}nQ \wedge \eta_{\alpha_1\cdots\alpha_n}. \tag{3.8.5}$$

Clearly, the relation $D\eta \equiv 0$ for the n-form η is identically fulfilled. We remark that, due to the appearance of the nonmetricity, the (conformal) lightcone structure will, in general, not be preserved during parallel-transport. The parallel-displacement of vectors $\boldsymbol{V}$ and $\boldsymbol{W}$ along a curve with tangent vector $\boldsymbol{u}$ is given by the gauge covariant Lie derivative $\text{Ł}_u V^\alpha = u\rfloor DV^\alpha = 0$ and $\text{Ł}_u W^\alpha = 0$. Then, in a metric-affine spacetime, the scalar product $\langle V, W\rangle := g(V, W)$ is generally displaced as follows:

$$\text{Ł}_u \langle V, W\rangle = -u\rfloor(\not{Q}_{\alpha\beta}\, V^\alpha W^\beta + Q\langle V, W\rangle). \tag{3.8.6}$$

It shows that the Weyl one-form Q will leave the (conformal) light-cone structure intact, whereas the traceless $\not{Q}_{\alpha\beta}$, which corresponds to the shear in (3.1.11), deforms this structure, if transported along the vector $\boldsymbol{u}$ (cf. [174,260]). The local lightcone will not be touched therefrom. We will return to this issue in section 6 after our presentation of the symmetry-breaking mechanism and its induced geometry.

3.9. Bianchi identities and Chern–Simons terms

The field strengths nonmetricity, torsion, and curvature obey the following *Bianchi identities*:

$$\begin{aligned} &DQ_{\alpha\beta} \equiv 2R_{(\alpha}{}^\gamma g_{\beta)\gamma}, && \text{(0th)},\\ &DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma, && \text{(1st)},\\ &DR_\alpha{}^\beta \equiv 0, && \text{(2nd)}. \end{aligned} \tag{3.9.1}$$

This 0th identity comes into existence,[29] because the curvature two-form in an (L_n, g) enjoys no symmetry in the Lie algebra indices. In a Riemann–Cartan spacetime, however, where the curvature is antisymmetric in α and β, the 0th identity becomes trivial. In Riemannian spacetime, the 1st identity degenerates to the familiar symmetry condition $R_\gamma^{\{\}\alpha} \wedge \vartheta^\gamma = 0$ of the Riemannian curvature. The

[29] Its trace reads $dQ_\gamma{}^\gamma \equiv 2R_\gamma{}^\gamma$ or $dQ \equiv (2/n)R_\gamma{}^\gamma$.

irreducible decompositions of these Bianchi identities can be found in full detail in sections B.5 and B.6 of the appendix.

In metric-affine gravity for $n \geq 3$, there exist analogs of the Chern–Simons three-form of a non-Abelian gauge theory. However, let us first introduce the volume preserving connection

$$ {}^{\dagger}\Gamma_{\alpha}{}^{\beta} = \Gamma_{\alpha}{}^{\beta} - \tfrac{1}{2} Q \delta_{\alpha}^{\beta}, \tag{3.9.2} $$

which will be derived in section 3.12. The $GL(4,R)$ Chern–Simons term [116,117] reads

$$ \begin{aligned} C_{RR} &:= -\tfrac{1}{2}(\Gamma_{\alpha}{}^{\beta} \wedge R_{\beta}{}^{\alpha} + \tfrac{1}{3}\Gamma_{\alpha}{}^{\beta} \wedge \Gamma_{\beta}{}^{\gamma} \wedge \Gamma_{\gamma}{}^{\alpha}) \\ &= -\tfrac{1}{2}(\Gamma_{\alpha}{}^{\beta} \wedge d\Gamma_{\beta}{}^{\alpha} - \tfrac{2}{3}\Gamma_{\alpha}{}^{\beta} \wedge \Gamma_{\beta}{}^{\gamma} \wedge \Gamma_{\gamma}{}^{\alpha}) &(3.9.3) \\ &= {}^{\dagger}C_{RR} + \tfrac{1}{4} C_{trR\,trR}, &(3.9.4) \end{aligned} $$

see also Bardeen [35] and the earlier construction of Buchdahl [86] in the case of the Riemannian curvature, and that associated with *dilations* is

$$ C_{trR\,trR} := -\tfrac{1}{2}\Gamma \wedge R = -Q \wedge R - \tfrac{1}{4} d(R \ln|\det g_{\alpha\beta}|), \tag{3.9.5} $$

where $\Gamma = \Gamma_{\alpha}{}^{\alpha}$ and $R = d\Gamma_{\alpha}{}^{\alpha}$. As can be seen from (3.9.4), C_{RR} can be decomposed into an $SL(4,R)$-part ${}^{\dagger}C_{RR}$ and a dilation piece as given in (3.9.5). Hence (3.9.5) represents an independent Chern–Simons term for dilations.

These are the Chern–Simons three-forms proper which are independent of the metric.[30] Nevertheless, in the sense of our unified affine approach,[31] we may also expect Chern–Simons type terms attached to the translation. And indeed, a *translation* Chern–Simons type term (cf. [449,280] and references therein) can be found according to

$$ C_{TT} := \frac{1}{2l^2} g_{\alpha\beta} \vartheta^{\alpha} \wedge T^{\beta} = \frac{1}{2l^2}(\vartheta_{\alpha} \wedge d\vartheta^{\alpha} - \vartheta_{\alpha} \wedge \vartheta^{\beta} \wedge \Gamma_{\beta}{}^{\alpha}) = {}^{\dagger}C_{TT}. \tag{3.9.6} $$

With respect to the Clifford algebra-valued one- and two-forms $\gamma := \vartheta^{\alpha}\gamma_{\alpha}$, $\sigma := (i/2)\gamma \wedge \gamma$, and $D\gamma = T^{\alpha}\gamma_{\alpha}$, respectively, cf. [449,282], the translational Chern–Simons term can be rewritten as

$$ C_{TT} = \frac{1}{8l^2} \mathrm{Tr}(\gamma D\gamma) = \frac{i}{8l^2} \mathrm{Tr}(D\sigma). $$

[30] In odd dimensions $n = 2k-1$, the Chern–Simons term for the Lie algebra-valued linear connection (3.5.1) can be written in rather compact form as $C_R^{2k-1} = -(k/2)\int_0^1 dz\, \mathrm{Tr}\{\Gamma \wedge (z\, d\Gamma - z^2 \Gamma \wedge \Gamma)^{k-1}\}$, cf. [725].

[31] There may also be a Inönü–Wigner type contraction approach, where one does not use the Cartan connection (3.3.3), but instead starts off with a $sl(5,R)$-valued i.e. trace free connection on a four-dimensional manifold: $\hat{\Gamma} = \Gamma_A{}^B \cancel{L}^{*}{}^A{}_B = \Gamma_{\alpha}{}^{\beta} L^{\alpha}{}_{\beta} + \Gamma_4{}^{\beta} L^4{}_{\beta} + \Gamma_{\alpha}{}^4 L^{\alpha}{}_4 =: \Gamma_{\alpha}{}^{\beta} L^{\alpha}{}_{\beta} + (1/l)\vartheta^{\beta} L^4{}_{\beta} + (1/l)\theta_{\alpha} L^{\alpha}{}_4$, where $A, B, \ldots$ runs from 0 to 4. The $SL(5,R)$ Chern–Simons term reads $\hat{C} = -\frac{1}{2}(\Gamma_A{}^B \wedge d\Gamma_B{}^A - \frac{2}{3}\Gamma_A{}^B \wedge \Gamma_B{}^C \wedge \Gamma_C{}^A)$. Not surprisingly, it contains the Chern–Simons term (3.9.3) corresponding to $GL(4,R)$-gauge transformations. In order to isolate the remaining terms, we perform an expansion and find $\hat{C} = C_{RR} - (1/2l^2)[\vartheta^{\beta} \wedge d\theta_{\beta} + \theta_{\alpha} \wedge d\vartheta^{\alpha} - \frac{2}{3}(\Gamma_{\alpha}{}^{\beta} \wedge \theta_{\beta} \wedge \vartheta^{\alpha} + \theta_{\alpha} \wedge \vartheta^{\gamma} \wedge \Gamma_{\gamma}{}^{\alpha} + \vartheta^{\beta} \wedge \Gamma_{\beta}{}^{\gamma} \wedge \theta_{\gamma})] = C_{RR} - (1/2l^2)[\vartheta^{\beta} \wedge d\theta_{\beta} + \theta_{\alpha} \wedge d\vartheta^{\alpha} + 2\theta_{\alpha} \wedge \Gamma_{\beta}{}^{\alpha} \wedge \vartheta^{\beta}] = C_{RR} - (1/2l^2)[\vartheta^{\beta} \wedge D\theta_{\beta} + \theta_{\alpha} \wedge T^{\alpha}]$. After a Inönü–Wigner type group contraction, the one-forms ϑ^{β} and θ_{α} correspond to the R^4 and ${}^*R^4$ part of the graded affine algebra, respectively. This suggests to construct a metric simply via $g := \vartheta^{\alpha} \otimes \theta_{\alpha}$. The metric acquires the usual form, if we identify θ_{α}, after some symmetry reduction which eliminates the antisymmetric components of g, with the conventional coframe ϑ^{β} by means of $\theta_{\alpha} = g_{\alpha\beta}\vartheta^{\beta}$. Thus one may speculate that the metric has a *topological origin* which is induced by a $SL(5,R)$ Chern–Simons term. Since the term $\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge Q_{\alpha\beta}$, which arises in the course of this construction, vanishes identically, the $SL(5,R)$ Chern–Simons term decomposes finally into $\hat{C} = C_{RR} - 2C_{TT}$.

In any case, it is metric dependent, and it is unclear whether it has any topological meaning without taking recourse to dynamical identifications, cf. (6.7.8) and (6.7.10). Nevertheless, as we will see immediately, its exterior derivative behaves in the standard manner.

Namely, the exterior derivatives $B = dC$ of these three-forms are $GL(4,R)$ invariant and effectively constitute Lagrangians from which the three Bianchi identities (3.9.1) may be derived as Euler–Lagrange equations. [In order to obtain a non-trivial result from a boundary term, the structure equations (3.8.1), (3.5.9), and (3.5.10) have to be added by means of Lagrange multipliers, see [280] for details.] By differentiation and by using the volume-preserving connection we find, respectively,

$$B_{TT} = dC_{TT} = (1/2l^2)[g_{\alpha\beta}({}^{\dagger}T^{\alpha} \wedge {}^{\dagger}T^{\beta} + {}^{\dagger}R_{\gamma}{}^{\alpha} \wedge \vartheta^{\gamma} \wedge \vartheta^{\beta}) - {}^{\dagger}Q_{\alpha\beta} \wedge \vartheta^{\alpha} \wedge {}^{\dagger}T^{\beta}], \tag{3.9.7}$$

$$B_{RR} = dC_{RR} = -\tfrac{1}{2}R_{\alpha}{}^{\beta} \wedge R_{\beta}{}^{\alpha} = -\tfrac{1}{2}({}^{\dagger}R_{\alpha}{}^{\beta} \wedge {}^{\dagger}R_{\beta}{}^{\alpha} + \tfrac{1}{4}R \wedge R), \tag{3.9.8}$$

$$B_{trR\,trR} = dC_{trR\,trR} = -\tfrac{1}{2}R \wedge R. \tag{3.9.9}$$

Adding these boundary terms B to a dynamical Lagrangian does not affect the field equations, but may serve as generating functions for canonical transformations [449–451] in an Ashtekar type Hamiltonian formulation, cf. section 5.10.

By performing the variational derivative $\delta/\delta\Psi := \partial/\partial\Psi - (-1)^p d(\partial/\partial d\Psi)$ of (3.9.6) and (3.9.4), we find the beautiful formulae for the translation

$$\frac{\delta C_{TT}}{\delta g_{\alpha\beta}} = \frac{1}{2l^2}\vartheta^{(\alpha} \wedge {}^{\dagger}T^{\beta)}, \tag{3.9.10}$$

$$\frac{\delta C_{TT}}{\delta\vartheta^{\alpha}} = \frac{1}{l^2}(g_{\alpha\beta}\,{}^{\dagger}T^{\beta} + \tfrac{1}{2}\vartheta^{\beta} \wedge {}^{\dagger}Q_{\alpha\beta}), \tag{3.9.11}$$

$$\frac{\delta C_{TT}}{\delta\Gamma_{\alpha}{}^{\beta}} = \frac{1}{2l^2}g_{\beta\gamma}\vartheta^{\alpha} \wedge \vartheta^{\gamma}, \tag{3.9.12}$$

and for the linear deformation

$$\delta C_{RR}/\delta\Gamma_{\alpha}{}^{\beta} = -R_{\beta}{}^{\alpha}. \tag{3.9.13}$$

Again, Eq.(3.9.13) can be split into a volume-preserving and a volume-changing piece:

$$\frac{\delta^{\dagger}C_{RR}}{\delta^{\dagger}\Gamma_{\alpha}{}^{\beta}} = -{}^{\dagger}R_{\beta}{}^{\alpha}, \qquad \frac{\delta C_{trR\,trR}}{\delta\Gamma} = -R. \tag{3.9.14}$$

These equations show that the field strengths torsion/nonmetricity, curvature, and curvature trace are reproduced if we vary the corresponding Chern–Simons (type) term with respect to the appropriate potential. This fact can be exploited in setting up three-dimensional topological gravity models, cf. [29,392], that is, in three dimensions "we have a Lagrangian without having a Lagrangian". At the same time we recognize that, up to a sign, the constants in the definitions (3.9.3), (3.9.5), and (3.9.6) are reasonably chosen and that we were required to introduce a fundamental length l in (3.9.6) in order to guarantee for C_{TT}, C_{RR}, and $C_{trR\,trR}$ the same physical dimension.

3.10. Decomposing the linear connection into Riemannian and post-Riemannian pieces

Although torsion and nonmetricity are genuine field strengths, they can be reinterpreted as parts of the connection. The linear connection can be expressed in terms of metric, coframe, torsion, and nonmetricity.

To this end, we use the *anholonomity* two-form which was defined in (3.4.7)

$$C^\beta := d\vartheta^\beta = \tfrac{1}{2} C_{\mu\nu}{}^\beta \vartheta^\mu \wedge \vartheta^\nu , \qquad C_\alpha := g_{\alpha\beta} C^\beta , \tag{3.10.1}$$

and display the formula (3.8.1) for the nonmetricity:

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_\alpha{}^\gamma g_{\gamma\beta} + \Gamma_\beta{}^\gamma g_{\alpha\gamma} . \tag{3.10.2}$$

This relation provides an explicit expression for the symmetric part of the connection

$$\Gamma_{(\alpha\beta)} = \tfrac{1}{2}(dg_{\alpha\beta} + Q_{\alpha\beta}) . \tag{3.10.3}$$

Moreover, by means of (3.4.7), we can express the covariant torsion $T_\alpha := g_{\alpha\beta} T^\beta$ in terms of the anholonomity two-form C_α:

$$T_\alpha = g_{\alpha\beta}\, D\vartheta^\beta = g_{\alpha\beta}\, d\vartheta^\beta + g_{\alpha\beta} \Gamma_\gamma{}^\beta \wedge \vartheta^\gamma = C_\alpha + \Gamma_{\beta\alpha} \wedge \vartheta^\beta . \tag{3.10.4}$$

In analogy with (3.10.3), we put the piece with the connection on the left hand side of the equation:

$$\Gamma_{\beta\alpha} \wedge \vartheta^\beta = -C_\alpha + T_\alpha . \tag{3.10.5}$$

Eqs. (3.10.3) and (3.10.5) represent $[n^2(n+1)/2] + [n^2(n-1)/2] = n^3$ linear equations for the determination of the n^3 components of $\Gamma_{\alpha\beta}$ in terms of the variables $dg_{\alpha\beta}, Q_{\alpha\beta}, C_\alpha$, and T_α. Using the algebraic relation (A.1.32) for $\Phi_\alpha = T_\alpha - C_\alpha$ and $\psi_{\alpha\beta} = \Gamma_{\beta\alpha}$, these two sets of equations lead to the following expression for the connection:

$$\begin{aligned}
\Gamma_{\alpha\beta} = {} & \tfrac{1}{2} dg_{\alpha\beta} + (e_{[\alpha} \rfloor dg_{\beta]\gamma})\vartheta^\gamma + e_{[\alpha} \rfloor C_{\beta]} - \tfrac{1}{2}(e_\alpha \rfloor e_\beta \rfloor C_\gamma)\vartheta^\gamma && (V_n) \\
& - e_{[\alpha} \rfloor T_{\beta]} + \tfrac{1}{2}(e_\alpha \rfloor e_\beta \rfloor T_\gamma)\vartheta^\gamma && (U_n) \\
& + \tfrac{1}{2} Q_{\alpha\beta} + (e_{[\alpha} \rfloor Q_{\beta]\gamma})\vartheta^\gamma && (L_n, g) .
\end{aligned} \tag{3.10.6}$$

Note that $e_\alpha \rfloor e_\beta \rfloor = -e_\beta \rfloor e_\alpha \rfloor$, so that the symmetric and antisymmetric pieces of $\Gamma_{\alpha\beta}$ are clearly displayed in (3.10.6). With the Schouten braces of (A.1.24), Eq. (3.10.6) becomes

$$\Gamma_{\alpha\beta} = \tfrac{1}{2}[e_{\{\gamma} \rfloor dg_{\beta\alpha\}} + e_{\{\gamma} \rfloor e_\alpha \rfloor C_{\beta\}} + e_{\{\gamma} \rfloor Q_{\beta\alpha\}} - e_{\{\gamma} \rfloor e_\alpha \rfloor T_{\beta\}}] \vartheta^\gamma . \tag{3.10.7}$$

The components $(1/2) e_{\{\gamma} \rfloor dg_{\beta\alpha\}}$ correspond to the Christoffel symbol of the first kind. In components, Eq. (3.10.7) reads [606]:

$$\Gamma_{\gamma\alpha\beta} = \tfrac{1}{2}[\partial_{\{\gamma} g_{\beta\alpha\}} + C_{\{\gamma\beta\alpha\}} + Q_{\{\gamma\beta\alpha\}} - T_{\{\gamma\beta\alpha\}}] . \tag{3.10.8}$$

The first line of Eq. (3.10.6) represents the Riemannian piece of the connection. It will be abbreviated as (see also [435]):

$$\Gamma^{\{\}}_{\alpha\beta} := \tfrac{1}{2}\, dg_{\alpha\beta} + (e_{[\alpha} \rfloor dg_{\beta]\gamma})\vartheta^\gamma + e_{[\alpha} \rfloor C_{\beta]} - \tfrac{1}{2}(e_\alpha \rfloor e_\beta \rfloor C_\gamma)\vartheta^\gamma . \tag{3.10.9}$$

The additional pieces in (3.10.6) are of a tensorial nature. Let us introduce the *contortion* one-form $K_{\alpha\beta} = -K_{\beta\alpha}$ implicitly by

$$T^\alpha =: K^\alpha{}_\beta \wedge \vartheta^\beta, \tag{3.10.10}$$

or explicitly, cf. (A.1.23), by

$$K_{\alpha\beta} = e_{[\alpha}\rfloor T_{\beta]} - \tfrac{1}{2}(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\vartheta^\gamma = 2e_{[\alpha}\rfloor T_{\beta]} - \tfrac{1}{2}e_\alpha\rfloor e_\beta\rfloor(T_\gamma \wedge \vartheta^\gamma). \tag{3.10.11}$$

Observe that $(T_\gamma \wedge \vartheta^\gamma)$ in the last term is proportional to the irreducible axial piece of the torsion, see (B.2.7). Then our geometrical decomposition (3.10.6) of the linear connection can be summarized in

$$\Gamma_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} - K_{\alpha\beta} + \tfrac{1}{2}Q_{\alpha\beta} + (e_{[\alpha}\rfloor Q_{\beta]\gamma})\vartheta^\gamma. \tag{3.10.12}$$

Since the Weyl one-form Q is explicitly given by (3.8.2) in any (L_n, g), the following useful relation for the trace, i.e., the dilation part of the connection one-form can be derived which is itself neither a connection nor a $\mathcal{G}L(n,R)$-scalar:

$$\Gamma_\gamma{}^\gamma = \tfrac{1}{2}nQ + \tfrac{1}{2}g^{\alpha\beta}dg_{\alpha\beta} = \tfrac{1}{2}nQ + d\ln\sqrt{|\mathrm{det}g_{\alpha\beta}|}\,. \tag{3.10.13}$$

Not unexpectedly, the torsion piece drops out. Alternatively, using the covariant exterior derivative, this formula can be written as

$$\frac{D\sqrt{\det|g_{\alpha\beta}|}}{\sqrt{\det|g_{\alpha\beta}|}} = -\tfrac{1}{2}nQ. \tag{3.10.14}$$

An (L_n, g) with the constraints of a vanishing traceless nonmetricity $\not{Q}_{\alpha\beta} = 0$ and a vanishing torsion $T^\alpha = 0$ is called a W_n. In 1918, Weyl [716,719] used such a W_4 as a framework for spacetime in which he unsuccessfully attempted to construct a field theory unifying electromagnetism and gravitation. If $T^\alpha \neq 0$, we call such a spacetime a Y_n, see Fig. 2 and also (3.10.6) cum (3.8.3). An (L_n, g) with $Q_{\alpha\beta} = 0$ is called a Riemann–Cartan spacetime U_n. A U_n with $T^\alpha = 0$ corresponds to the Riemannian spacetime V_n of GR.

3.11. Deformation of a connection

Since any tensor-valued one-form A transforms homogenously with respect to linear gauge transformations, its subtraction from a connection can be regarded as a continuous *deformation*[32] within the space C of connections:

$$\Gamma_\alpha{}^\beta \quad \longrightarrow \quad \bar{\Gamma}_\alpha{}^\beta = \Gamma_\alpha{}^\beta + \varepsilon A_\alpha{}^\beta = \Gamma_\alpha{}^\beta + \varepsilon\left(\not{A}_\alpha{}^\beta + \frac{1}{n}\delta^\beta_\alpha A\right), \qquad \Gamma,\ \bar{\Gamma} \in C. \tag{3.11.1}$$

The trace part $A := A_\gamma{}^\gamma$ represents the *projective* piece of the deformation. In order to retain $\bar{\Gamma}_\alpha{}^\beta$ as a connection, $A_\alpha{}^\beta$ has to be tensor-valued. The deformation (3.11.1), which involves *no metric*, can

[32] These deformations may include the difference (3.5.3) of two gauge-equivalent connections as well as more general "gauge field copies" [149] as special cases. Our concept of a deformation seems to be a special case of a prolongation, see [22], where also $\overline{\vartheta}^\alpha = \vartheta^\alpha + \varepsilon^\alpha$ is considered.

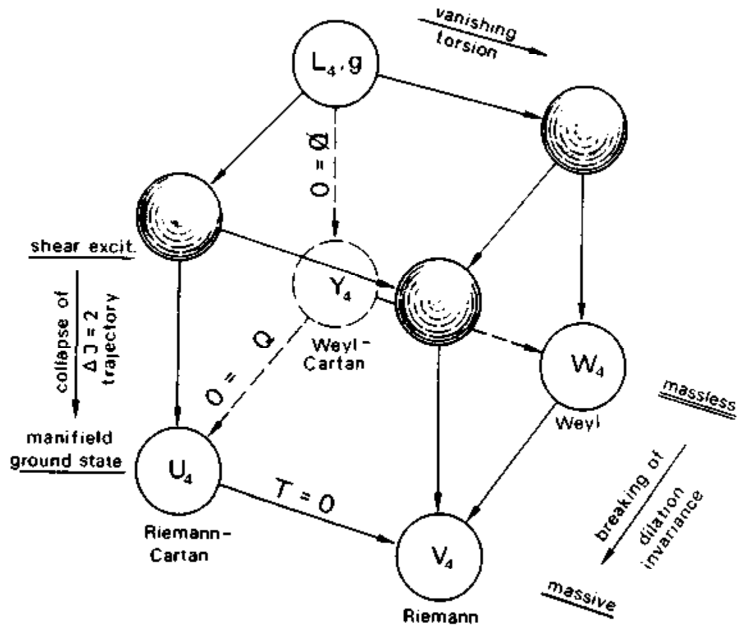


Fig. 2. MAGic cube: Classification of a metric-affine spacetime (L_4, g). $Q^{\frown}$ $\hat{=}$ tracefree nonmetricity, Q $\hat{=}$ Weyl one-form, and T $\hat{=}$ torsion.

be turned on and off via the continous parameter ε. In Ashetkar-like reformulations of GR and its teleparallelism equivalent [448,451], this parameter is allowed to become complex or, in particular, purely imaginary.

Any such deformation induces corresponding relations for nonmetricity, torsion, and curvature:

$$\bar{Q}_{\alpha\beta} = Q_{\alpha\beta} + 2\varepsilon A_{(\alpha\beta)}, \qquad \bar{Q} = Q + 2\varepsilon A_\gamma{}^\gamma / n, \tag{3.11.2}$$

$$\bar{T}^\alpha = T^\alpha + \varepsilon A_\mu{}^\alpha \wedge \vartheta^\mu, \tag{3.11.3}$$

$$\bar{R}_\alpha{}^\beta = R_\alpha{}^\beta + \varepsilon D A_\alpha{}^\beta - \varepsilon^2 A_\alpha{}^\mu \wedge A_\mu{}^\beta = R_\alpha{}^\beta + \varepsilon \bar{D} A_\alpha{}^\beta + \varepsilon^2 A_\alpha{}^\mu \wedge A_\mu{}^\beta, \tag{3.11.4}$$

$$\bar{R}_\gamma{}^\gamma = R_\gamma{}^\gamma + \varepsilon d A_\gamma{}^\gamma. \tag{3.11.5}$$

These decompositions can be transferred to the Bianchi identities (3.9.1) in a straightforward way. It requires, however, tedious calculations. It turns out that the Bianchi identities are *not* sensitive to a deformation of the connection at all.[33] Formally the same identities hold after passing to the deformed field strengths and the modified exterior covariant derivative $\bar{D}$ arising from the deformed connection:

$$\bar{D}\bar{Q}_{\alpha\beta} = 2\bar{R}_{(\alpha\beta)}, \qquad \bar{D}\bar{T}^\alpha = \bar{R}_\beta{}^\alpha \wedge \vartheta^\beta, \qquad \bar{D}\bar{R}_\alpha{}^\beta = 0. \tag{3.11.6}$$

The deformations of the Chern–Simons terms (3.9.6), (3.9.3), and (3.9.5) are, respectively, given by

[33] This excludes the possibility that Noether identities resulting from dilation invariance can ever be related to the Bianchi identities. In contrast, within the Einstein–Cartan theory, the energy-momentum and the angular momentum theorems can be formulated, via the field equations, in terms of the contracted Bianchi identities [286].

$$\bar{C}_{TT} = C_{TT} + (\varepsilon/2l^2)A_{[\alpha\beta]}\vartheta^\alpha \wedge \vartheta^\beta,$$
$$\bar{C}_{RR} = C_{RR} - \tfrac{1}{2}\varepsilon[A_\alpha{}^\beta \wedge d\Gamma_\beta{}^\alpha + \Gamma_\alpha{}^\beta \wedge (DA_\beta{}^\alpha - \varepsilon A_\beta{}^\gamma \wedge A_\gamma{}^\alpha)],$$
$$\bar{C}_{trR\,trR} = C_{trR\,trR} - \tfrac{1}{2}\varepsilon d(A \wedge \Gamma) - \varepsilon A \wedge R. \tag{3.11.7}$$

In the projective subcase $A_\alpha{}^\beta = \delta_\alpha^\beta P$ of the deformation (3.11.1), the curvature transforms in a particularly simple way:

$$\Gamma_\alpha{}^\beta \to \overset{\text{proj}}{\Gamma_\alpha{}^\beta} = \Gamma_\alpha{}^\beta + \varepsilon\,\delta_\alpha^\beta P \quad \Rightarrow \quad \overset{\text{proj.}}{R_\alpha{}^\beta} = R_\alpha{}^\beta + \varepsilon\,\delta_\alpha^\beta\,dP\,. \tag{3.11.8}$$

If a metric is available, the projective term can be written as $g_{\alpha\beta}\,dP$, that is, it contributes only to the symmetric part of the curvature. Thus $\overset{\text{proj.}}{R}_{[\alpha\beta]} = R_{[\alpha\beta]}$, a result which is important in GR and in the EC-theory of gravity.

The *transposed* connection (see [607], [682, paper IV])

$$\widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta \tag{3.11.9}$$

or, in components

$$\widehat{\Gamma}_{\gamma\alpha}{}^\beta = \Gamma_{\alpha\gamma}{}^\beta + C_{\alpha\gamma}{}^\beta, \tag{3.11.10}$$

with $C^\alpha = d\vartheta^\alpha$, is another example of a deformation in an L_n. In holonomic coordinates, and only then, we find, indeed, a transposition in the lower pair of indices. Moreover, in an (L_n, g), we can introduce the modified connection

$$\check{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta - \tfrac{1}{2}nQ\delta_\alpha^\beta \tag{3.11.11}$$

with the property that the corresponding covariant exterior derivative of the vector-valued g-volume form η_α vanishes:

$$\check{D}\eta_\alpha = 0\,. \tag{3.11.12}$$

A certain deformation yields, for example, the Christoffel connection $\bar{\Gamma}_\alpha{}^\beta = \Gamma^{\{\}\beta}_\alpha$. Then one should note, however, that the *deformed* nonmetricity and torsion vanish, i.e. $Q^{\{\}}_{\alpha\beta} = 0$ and $T^{\{\}\alpha} = 0$, so that the first two Bianchi identities degenerate to the familiar algebraic conditions $R^{\{\}}_{(\alpha\beta)} = 0$ and $R^{\{\}\alpha}_\beta \wedge \vartheta^\beta = 0$ of the Riemannian curvature $R^{\{\}\beta}_\alpha$. This deformation process can also be understood in the following way: In an (L_n, g) we have two classes of connections, the linear and the Riemannian one. Their difference ought to be a tensor-valued form, namely that corresponding to the contortion and nonmetricity pieces in (3.10.12).

In an (L_n, g), the most general *field redefinitions* of the basic variables, metric, coframe and linear connection, generated by some $(n-2)$-forms H_α and $H^\alpha{}_\beta$ and an n-form $m_{\alpha\beta}$, read

$$\overline{g}_{\alpha\beta} = g_{\alpha\beta} + e_{(\alpha|} \rfloor e_\gamma \rfloor {}^*H_{|\beta)}{}^\gamma + {}^*m_{\alpha\beta}, \tag{3.11.13}$$
$$\overline{\vartheta}^\alpha = \vartheta^\alpha + e_\beta \rfloor {}^*H^{\alpha\beta}, \tag{3.11.14}$$
$$\overline{\Gamma}_\alpha{}^\beta = \Gamma_\alpha{}^\beta + e_\alpha \rfloor {}^*H^\beta\,. \tag{3.11.15}$$

In a dynamical approach these m and the H will be gauge field momenta canonically conjugated to the metric, the coframe, and the connection, respectively. Due to the semidirect structure of the affine group, the gauge field momenta contribute just to the intertwined gauge potentials. The field redefinition (3.11.13) of the metric generalizes 't Hooft's ansatz [305], used in an attempt at perturbative renormalization of GR. For a general counterterm ΔV in the effective gauge Lagrangian, our geometrical variables become redefined according to the "intertwining relations" (3.11.13), (3.11.14), (3.11.15). In the four-dimensional Poincaré gauge theory, the Hodge star for gauge field momenta H can be dismissed, cf. [455]. However, in the coupling to matter, the field redefinitions may induce violations of the macroscopic principle of equivalence, cf. Brans [80]. Incidentally, our construction is more explicit than the rather formal field redefintion of Dixon [159] for non-Abelian gauge theories

$$\overline{A} = A + \delta G/\delta j, \tag{3.11.16}$$

where $j := \delta G/\delta A$ is the gauge current $(n-1)$-form of a generating n-form G.

3.12. Volume-preserving connection

An example of a deformation of a connection is provided by the construction of a *volume-preserving* connection ${}^{\dagger}\Gamma_\alpha{}^\beta$. By definition, it should leave the volume n-form η, or the associated η-basis, covariantly constant under parallel-transport:

$${}^{\dagger}D\eta_{\alpha_1\cdots\alpha_n} = 0 \Rightarrow {}^{\dagger}D\sqrt{|\det g_{\mu\nu}|} = d\sqrt{|\det g_{\mu\nu}|} - {}^{\dagger}\Gamma_\gamma{}^\gamma\sqrt{|\det g_{\mu\nu}|} = 0. \tag{3.12.1}$$

This merely determines the trace of our new connection. By means of a comparison with (3.10.14) we find the *standard volume-preserving* connection:

$${}^{\dagger}\Gamma_\alpha{}^\beta := \Gamma_\alpha{}^\beta - \tfrac{1}{2}Q\delta_\alpha^\beta, \qquad {}^{\dagger}\Gamma_\gamma{}^\gamma = \Gamma_\gamma{}^\gamma - \tfrac{1}{2}n\,Q = d\ln\sqrt{|\det g_{\alpha\beta}|}. \tag{3.12.2}$$

Consequently, the new connection is related, via $A = -(n/2)Q$, to the linear connection $\Gamma_\alpha{}^\beta$ by a projective transformation.

Note that a connection is not a gauge-invariant object. Instead, the vanishing of the contracted part of the curvature built from the connection ${}^{\dagger}\Gamma_\alpha{}^\beta$, i.e.,

$${}^{\dagger}R_\gamma{}^\gamma = 0, \tag{3.12.3}$$

is a necessary and sufficient condition that scalar densities and, in particular, the volume n-form η, are teleparallel under transport by means of this connection (cf. Schouten [606]). The indeterminacy of the volume-preserving connection of an (L_n, g) is related to the fact that, in an (L_n, g), DILCURV $R_\gamma{}^\gamma$ as such is *not* an irreducible piece of the curvature, but surfaces as an independent quantity only on the level of a Weyl–Cartan space Y_n. The vanishing of the DILCURV distinguishes a Riemannian V_n from a Weylian W_n and, as shown in Fig. 2, this is, mutatis mutandis, also true for a U_n and a Y_n, respectively.

The new volume-preserving nonmetricity

$${}^{\dagger}Q_{\alpha\beta} = Q_{\alpha\beta} - Qg_{\alpha\beta} =: \not{Q}_{\alpha\beta} \Rightarrow {}^{\dagger}Q = 0 \tag{3.12.4}$$

is traceless, and the new torsion and its trace read [34]

$$ {}^{\dagger}T^{\alpha} = T^{\alpha} - \tfrac{1}{2}Q \wedge \vartheta^{\alpha}, \qquad e_{\alpha}\rfloor {}^{\dagger}T^{\alpha} = e_{\alpha}\rfloor T^{\alpha} + \tfrac{1}{2}(n-1)Q. \tag{3.12.5}$$

For the curvature we have

$$ {}^{\dagger}R_{\alpha}{}^{\beta} = R_{\alpha}{}^{\beta} - \tfrac{1}{2}\delta^{\beta}_{\alpha}\, dQ\,, \tag{3.12.6}$$

where dQ denotes the dilation field strength derived from the one-form Q (Weyl covector). In a metric-affine spacetime (L_n, g), there arise two possible types of contractions of the curvature: One with respect to the frame indices (cf. Eddington [174, p.215]) and, after raising one index, a further contraction involving twice the interior product of the two-form:

$$ {}^{\dagger}R_{\gamma}{}^{\gamma} = R_{\gamma}{}^{\gamma} - \tfrac{1}{2}n\, dQ = 0, \qquad e_{\alpha}\rfloor e_{\beta}\rfloor ({}^{\dagger}R^{\alpha\beta} - R^{\alpha\beta}) = 0. \tag{3.12.7}$$

The last relation is valid for any projective transformation, cf. (3.11.8).

For an arbitrary tensor-valued p-form density of type $\binom{\mu}{\nu}$ of anholonomic weight ω, see (A.1.2), the exterior covariant derivative (3.5.5) splits into a volume-preserving piece and a dilation piece:

$$ D\hat{\Psi} = {}^{\dagger}D\hat{\Psi} + (\mu - \nu - n\omega)\tfrac{1}{2}Q\hat{\Psi}. \tag{3.12.8}$$

We will call

$$ \omega' := \mu - \nu - n\,\omega \tag{3.12.9}$$

the dilation weight of $\hat{\Psi}$, see (A.1.4). If we compare (3.12.8) with (3.12.4) and (3.12.5) for $Dg_{\alpha\beta}$ and $D\vartheta^{\alpha}$, respectively, we find $\omega'_g = -2$ and $\omega'_\vartheta = 1$.

3.13. Local scale transformations

Let us first turn to *local scale transformations* [35] in a *single* (L_n, g): Following partly Komar and Bergmann [358,359,53], such a scale transformation arises naturally from a local action (3.4.5) of the general linear group $GL(n,R) = [T \otimes SL(n,R)] \times R^{+}$ on the frames. Inasmuch as these gauge transformations do not change the metrical relations of spacetime, they may be regarded as *passive* transformations. A mere scale transformation [36] corresponds to the subcase

$$ \overset{+}{\Lambda}_{\alpha}{}^{\beta}(x) = \Omega(x)\delta^{\beta}_{\alpha} \tag{3.13.1}$$

which generates the volume-changing part of the $GL(n,R)$-gauge transformation. Then we have

$$ e_{\alpha} \rightarrow e'_{\alpha} = \overset{+}{\Lambda}_{\alpha}{}^{\gamma}(x) e_{\gamma} = \Omega e_{\alpha} \qquad (e'^{i}{}_{\alpha} = \Omega e^{i}{}_{\alpha}), \tag{3.13.2}$$

$$ \vartheta^{\alpha} \rightarrow \vartheta'^{\alpha} = \overset{+}{\Lambda}{}^{-1}{}_{\gamma}{}^{\alpha}(x)\vartheta^{\gamma} = \Omega^{-1}\,\vartheta^{\alpha} \qquad (e'^{\alpha}_{j} = \Omega^{-1} e_{j}{}^{\alpha}), \tag{3.13.3}$$

$$ g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \overset{+}{\Lambda}_{\alpha}{}^{\gamma}(x)\overset{+}{\Lambda}_{\beta}{}^{\delta}(x) g_{\gamma\delta} = \Omega^{2} g_{\alpha\beta}. \tag{3.13.4}$$

[34] A related result has been found by Bregman [82].

[35] They are also called dilations or homothetic transformations.

[36] With respect to the nomenclature, we follow the thesis of Kastrup [336] in which, for instance, 1 m = Ω cm, with $\Omega = 100$, is a transformation of the scale that leaves the length $l = (o_{\alpha\beta} v^{\alpha} v^{\beta})^{1/2}$ invariant.

By applying the inhomogeneous transformation law (3.5.2) for the linear connection, we find in the special case of local scale transformations:

$$\Gamma'^{\ \beta}_{\alpha} = \Gamma_{\alpha}{}^{\beta} + \delta^{\beta}_{\alpha}\, d \ln \Omega\,. \tag{3.13.5}$$

For later purposes we note also the local scale transformation of the trace of the connection, namely

$$\Gamma'^{\ \gamma}_{\gamma} = \Gamma_{\gamma}{}^{\gamma} + n\, d \ln \Omega\,. \tag{3.13.6}$$

These rules determine the local scale properties of the other geometric objects uniquely: The nonmetricity and the Weyl covector are invariant under local rescaling:

$$Q'^{\ \beta}_{\alpha} = Q_{\alpha}{}^{\beta}, \qquad ({}^{\dagger}Q)'^{\ \beta}_{\alpha} = {}^{\dagger}Q_{\alpha}{}^{\beta} \qquad Q' = Q. \tag{3.13.7}$$

This is what one would expect anyhow for tensor-valued forms with respect to a subgroup of local $GL(n,R)$-transformations. (In contradistinction to this local scale invariance, the Weyl covector will turn out to transform inhomogeneously under conformal changes.) For the translational field strength and the curvature we have

$$T'^{\alpha} = \Omega^{-1} T^{\alpha}, \qquad R'^{\ \beta}_{\alpha} = R_{\alpha}{}^{\beta}, \qquad ({}^{\dagger}R)'^{\ \beta}_{\alpha} = {}^{\dagger}R_{\alpha}{}^{\beta}. \tag{3.13.8}$$

It is straightforward to deduce the corresponding formulae for the various contractions of the curvature tensor.

3.14. Conformal changes in an (L_n, g)

It is a consequence of the $GL(n,R)$ gauge approach that local scale transformations are almost trivial for scalar- or tensor-valued forms. However, we may generalize local scale transformations (3.13.1) by admitting arbitrary exponents in the Ω factors. Thereby we arrive at a *conformal change* of the metric, which generalizes the original approach of Weyl [716]. Let us compare *two* metric-affine spacetimes (L_n, g) and $(L_n, \widetilde{g})$ which have conformally related metrics, but invariant $gl(n,R)$-valued curvatures. Since the second requirement does not fix the connection completely, but still allows it to be related via Einstein's λ-transformation $\Gamma_{\alpha}{}^{\beta} \to \Gamma_{\alpha}{}^{\beta} + \delta^{\beta}_{\alpha}\, d\lambda$ ([180, appendix 2], [52], see also Smalley [638]), we consider the *combined* transformations:

$$\widetilde{g}: \quad \begin{cases} \widetilde{g}_{\alpha\beta} &= \Omega^{L-2F} g_{\alpha\beta}\,, \qquad \widetilde{\vartheta}^{\alpha} = \Omega^{F} \vartheta^{\alpha}, \qquad \widetilde{e}_{\alpha} = \Omega^{-F} e_{\alpha}, \\ \widetilde{\Gamma}_{\alpha}{}^{\beta} &= \Gamma_{\alpha}{}^{\beta} - C \delta^{\beta}_{\alpha}\, d \ln \Omega \quad \Leftrightarrow \quad \widetilde{R}_{\alpha}{}^{\beta} = R_{\alpha}{}^{\beta}\,. \end{cases} \tag{3.14.1}$$

This puts a *conformal equivalence structure* on (L_n, g) in which g and $\widetilde{g} = \Omega^{L} g$ are conformally related. Due to

$$\widetilde{g} = \widetilde{\vartheta}^{\alpha} \otimes \widetilde{\vartheta}^{\beta}\, \widetilde{g}_{\alpha\beta} = (\widetilde{\vartheta})'^{\alpha} \otimes (\widetilde{\vartheta})'^{\beta}\, (\widetilde{g})'_{\alpha\beta} = \Omega^{L} g, \tag{3.14.2}$$

the conformal change of the anholonomic frame is determined only up to local $GL(n,R)$ transformations. Consequently, the weights L, F, and C for the conformal change of metric, coframe, and of the connection will, in general, be independent of each other.

Let us now consider the consequences of our combined conformal change (3.14.1): The curvature is invariant for a projective transformation which is, as in (3.14), constructed from an exact form, cf. (3.11.8):

$$\widetilde{R}_\alpha{}^\beta = R_\alpha{}^\beta, \qquad \widetilde{({}^\dagger R)}_\alpha{}^\beta = {}^\dagger R_\alpha{}^\beta. \tag{3.14.3}$$

For the nonmetricity and the Weyl covector, we find

$$\widetilde{Q}_{\alpha\beta} = -\widetilde{D}\widetilde{g}_{\alpha\beta} = \Omega^{L-2F}[Q_{\alpha\beta} - (L-2F+2C)g_{\alpha\beta}\, d\ln\Omega], \tag{3.14.4}$$

$$\widetilde{({}^\dagger Q)}_{\alpha\beta} = \Omega^{L-2F}\,{}^\dagger Q_{\alpha\beta}, \qquad \widetilde{Q} = Q - (L-2F+2C)\, d\ln\Omega, \tag{3.14.5}$$

respectively. For the translational field strength T^α and the torsion trace one-form $T := e_\alpha \rfloor T^\alpha$, we obtain

$$\widetilde{T}^\alpha = \Omega^F[T^\alpha + (F-C)\, d\ln\Omega \wedge \vartheta^\alpha] \tag{3.14.6}$$

$$\widetilde{({}^\dagger T)}^\alpha = \Omega^F[\,{}^\dagger T^\alpha + \tfrac{1}{2}L\, d\ln\Omega \wedge \vartheta^\alpha], \qquad \widetilde{T} = T + (C-F)(n-1)\, d\ln\Omega. \tag{3.14.7}$$

Note that the inhomogeneous term in $(3.14.7)_2$ parallels the corresponding term in $(3.14.5)_2$ for the Weyl covector [144,528]. As expected, the conformal change of the volume-preserving connection (3.12.2) is independent of C:

$$\widetilde{{}^\dagger\Gamma}_\alpha{}^\beta = {}^\dagger\Gamma_\alpha{}^\beta + \tfrac{1}{2}(L-2F)\delta_\alpha^\beta\, d\ln\Omega. \tag{3.14.8}$$

For the translational Chern–Simons term C_{TT} we find

$$\widetilde{C}_{TT} = \Omega^L\, C_{TT}\,. \tag{3.14.9}$$

The volume-preserving Chern–Simons term is conformally invariant

$$\widetilde{({}^\dagger C)}_{RR} = {}^\dagger C_{RR}, \tag{3.14.10}$$

whereas the corresponding dilation piece transforms as

$$\widetilde{C}_{tr\,R\,tr\,R} = C_{tr\,R\,tr\,R} + \tfrac{1}{2}nCR \wedge d\ln\Omega. \tag{3.14.11}$$

By adjusting the weights L, F, and C appropriately, we can recover all subcases earlier discussed in the literature. Formally, the local scale transformations are included in this scheme as a special case with $L=0$, $F=C=-1$.

The λ-transformation of the linear connection in (3.14.1) does not influence the conformal change of the Riemannian part (3.10.9) of the connection. In view of $\widetilde{\vartheta}_\alpha = \widetilde{g}_{\alpha\beta}\,\widetilde{\vartheta}^\beta = \Omega^{L-F}\vartheta_\alpha$, we find

$$\begin{aligned}\widetilde{\Gamma}^{\{\}}_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} &+ \tfrac{1}{2}(L-2F)\Omega^{L-2F}[g_{\alpha\beta} d\ln\Omega + 2(e_{[\alpha}\rfloor d\ln\Omega)g_{\beta]\gamma}\vartheta^\gamma] \\ &- \tfrac{1}{2}(L-F)\Omega^{L-2F}[(e_\alpha\rfloor e_\beta\rfloor(d\ln\Omega\wedge\vartheta_\gamma))\vartheta^\gamma - 2e_{[\alpha}\rfloor(d\ln\Omega\wedge\vartheta_{\beta]})].\end{aligned} \tag{3.14.12}$$

Moreover, from the transformation formula (3.14.1) of metric and coframe we find the relation

$$\widetilde{{}^*\Phi}^{(p)} = \Omega^{(n-2p)L/2}\, {}^*\widetilde{\Phi}^{(p)} \tag{3.14.13}$$

for the Hodge dual of a p-form. Thus for an exterior form in the "middle dimension" $p = n/2$ for n even, the Hodge operator is conformally invariant, cf. Dray et al. [170] for the four-dimensional case.

In order to further clearify our notion of conformal change, let us consider the space $\mathcal{M}$ of all (pseudo)Riemannian metrics g [50]. The infinite-dimensional *superspace* $\mathcal{S}$ is defined as the coset space $\mathcal{S} := \mathcal{M}/(\mathrm{Diff}(n,R))$ and, consequently, identifies all metrics g which are equivalent with respect to diffeomorphisms $\mathrm{Diff}(n,R)$. This infinite-dimensional group, which acts as transformation group on $\mathcal{M}$, can be enlarged to the group of *conformorphisms* $\widetilde{\mathrm{Diff}}(n,R) := C_+^\infty \otimes \mathrm{Diff}(n,R)$, where C_+^∞ denotes the Abelian group of all positive, infinitely differentiable functions Ω. The group $\widetilde{\mathrm{Diff}}(n,R)$ acts on $\mathcal{M}$ by pulling back *conformally eqivalent* metrics $\widetilde{g} = \sigma^*(\Omega^L g)$, where σ denotes a cross section. If $J = \{J_i{}^j\} \in \mathrm{Diff}(n,R)$ are the (passive) diffeomorphisms of appendix A.1, the left action of $(J,\Omega) \in \widetilde{\mathrm{Diff}}(n,R)$ on a metric $g \in \mathcal{M}$ is more explicitly given by $(J,\Omega)g(x) := J\Omega^L(x)g(x) = \Omega^L(Jx)g(Jx)$. In accordance with the pull-back notion, J acts directly on the coordinates x. Applying this rule twice, we obtain

$$
\begin{aligned}
&(J_1,\Omega_1)\circ(J_2,\Omega_2)g(x)\\
&\quad = J_1\Omega_1^L(x)\Omega_2^L(J_2x)g(J_2x) = \Omega_1^L(J_1x)\Omega_2^L(J_2J_1x)g(J_2J_1x) = \Omega_2^L(x')\Omega_1^L(J_2^{-1}x')g(x'),
\end{aligned}
$$

where $x' = J_2J_1x$. Thus the group multiplication [197] resulting from the composition of transformations, denoted by $\circ$, is determined by $(J_1,\Omega_1)\circ(J_2,\Omega_2) = (J_1\circ J_2, \Omega_2(\Omega_1\circ J_2^{-1}))$. This reveals that $\widetilde{\mathrm{Diff}}(n,R)$ is, under composition, the *semidirect product* of $\mathrm{Diff}(n,R)$ with the Abelian group C_+^∞.

In the *conformal superspace* $\widetilde{\mathcal{S}} := \mathcal{M}/\widetilde{\mathrm{Diff}}(n,R) \subset \mathcal{S}$, conformally equivalent metrics are identified, see also Swift [664]. It should be pointed out that the $(n+1)(n+2)/2$-parameter *isotropy group* $C(M_n,g) \subset \mathrm{Diff}(n,R)$ of *conformal transformations of spacetime* such that $\widetilde{g} = g$ can be regarded as an isometry within the conformal superspace $\widetilde{\mathcal{S}}$ ([442], see Giulini [225] for the homotopy groups of $\mathcal{S}$).

3.15. Orthonormal, holonomic, conformal, and pure gauges

There exist certain gauge conditions on the frame field for which the description of our geometrical arena simplifies considerably.

The most obvious gauge condition is the *orthonormal gauge*. We choose the frame to be orthonormal, i.e.

$$e^i{}_\alpha \stackrel{*}{=} \overset{\circ}{e}{}^i{}_\alpha. \tag{3.15.1}$$

The 'star equal' sign indicates a specific gauge that has been introduced. Because of (3.7.3) and (3.7.5), we then have the 'metrical' subsidiary condition

$$g_{\alpha\beta} \stackrel{*}{=} g_{ij}\overset{\circ}{e}{}^i{}_\alpha \overset{\circ}{e}{}^j{}_\beta = o_{\alpha\beta} \tag{3.15.2}$$

(which resembles those known from nonlinear σ models [556]). Geometrically, this gauge condition is intimately connected with the celebrated 'reduction theorem' for linear frame bundles [356, p.88].

A much more stringent condition is the *holonomic gauge* [406] which one imposes ordinarily in GR:

$$C^{\alpha} := d\vartheta^{\alpha} \stackrel{*}{=} 0 \quad \text{or} \quad e^{i}{}_{\alpha} \stackrel{*}{=} \delta^{i}_{\alpha}. \tag{3.15.3}$$

Then the frame is a *natural* frame and looses its independent degrees of freedom. No linear gauge transformation of the vielbeins are permitted anymore. What remains is the action of the diffeomorphism group and, along with it, a differential identity for energy-momentum. With metric and holonomic connection as remaining geometrical variables, we can derive two field equations (ZEROTH and SECOND of section 5.5) and one Noether identity from diffeomorphism invariance. The 2nd Noether identity, however, would then be concealed since the holonomic hypermomentum identity would be implicitly contained in the momentum identity. This gauge should not be confused with the *harmonic* or Hilbert gauge $d\eta_{\alpha} = d^{*}\vartheta_{\alpha} \stackrel{*}{=} 0$ which appears, for instance, in the formulation of the Cauchy problem in GR.

In the framework of two conformally related spacetimes (L_n, g) and $(L_n, \widetilde{g})$, we may pick in each spacetime a natural frame field, that is, we impose the *conformal holonomic gauge*

$$C_{\alpha} = d\vartheta_{\alpha} = d\widetilde{\vartheta}_{\alpha} \stackrel{*}{=} 0 \quad \Rightarrow \quad L = F. \tag{3.15.4}$$

Then, according to (3.14.12), the Riemannian piece of the linear connection reduces, to

$$\widetilde{\Gamma}^{\{\}}_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} - \tfrac{1}{2} L \Omega^{-L} [g_{\alpha\beta}\, d\ln\Omega + 2(e_{[\alpha}\rfloor d\ln\Omega) g_{\beta]\gamma}\vartheta^{\gamma}]. \tag{3.15.5}$$

Since, according to (3.14.1), the contravariant components of the metric change as $g^{kl} \rightarrow \widetilde{g}^{kl} = \Omega^{L} g^{kl}$, we recover for the holonomic components, i.e., the Christoffel symbol, the familiar transformation law under a conformal change of metric:

$$\widetilde{\Gamma}^{\{\}k}_{ij} = \Gamma^{\{\}k}_{ij} - \tfrac{1}{2} L \{\delta^{k}_{i}\partial_{j} + \delta^{k}_{j}\partial_{i} - g_{ij}\partial^{k}\} \ln\Omega. \tag{3.15.6}$$

In order to study conformal properties in a Riemann–Cartan space U_n or in the purely holonomical framework (cf. [287]) of the Riemannian spacetime (V_n), one has to fix again a 'gauge'. In order to achieve the passage from the natural (L_n, g) to an U_n (or even to a V_4), let us require, besides vanishing nonmetricity (and torsion), the *conformal gauge* (cf. Lindström et al. [398]) in which the *conformally invariant* metric density is identified with the Minkowski metric:

$$\hat{g}^{\alpha\beta} := |\det g_{\mu\nu}|^{1/n} g^{\alpha\beta} \stackrel{*}{=} o^{\alpha\beta}, \qquad \hat{g}_{\alpha\beta} := |\det g_{\mu\nu}|^{-1/n} g_{\alpha\beta} \stackrel{*}{=} o_{\alpha\beta}. \tag{3.15.7}$$

This gauge is qualified by the fact that the local Minkowski metric becomes conformally invariant, i.e.

$$\widetilde{o}^{\alpha\beta} = \Omega^{-nL/n}\Omega^{L} o^{\alpha\beta} = o^{\alpha\beta}, \qquad \widetilde{o}_{\alpha\beta} = o_{\alpha\beta}. \tag{3.15.8}$$

Incidentally, in the context of SDiff(n, R)-invariant models (diffeomorphisms with determinant one), metric densities of the type $g_{\alpha\beta}/|\det g_{\mu\nu}|$ have been considered [89].

In the parlance of Yang–Mills theories, one speaks of a *pure gauge* connection when its field strength vanishes. Take a flat and uncontorted Minkowski spacetime. According to (3.10.9), in curvilinear coordinates and arbitrary frames, its connection reads

$$\Gamma^{\{\}}_{\alpha\beta} = \tfrac{1}{2}\, dg_{\alpha\beta} + (e_{[\alpha}\rfloor dg_{\beta]\gamma})\vartheta^{\gamma} + e_{[\alpha}\rfloor C_{\beta]} - \tfrac{1}{2}(e_{\alpha}\rfloor e_{\beta}\rfloor C_{\gamma})\vartheta^{\gamma}. \tag{3.15.9}$$

It has to fulfill the constraint

$$R^{\{\}}_{\alpha\beta} = 0. \tag{3.15.10}$$

Eqs.(3.15.9) and (3.15.10) represent a pure gauge. Suppose we pick on top of that an orthonormal gauge. Then (3.15.9) specializes to

$$\Gamma^{\{\}}_{\alpha\beta} \stackrel{*}{=} e_{[\alpha}\rfloor C_{\beta]} - \tfrac{1}{2}(e_\alpha\rfloor e_\beta\rfloor C_\gamma)\vartheta^\gamma. \tag{3.15.11}$$

In these orthonormal frames, the anholonomity C_β looks like a torsion $-T_\beta$, as a glance at (3.10.6) will tell. In the orthonormal pure gauge, a torsion can locally be *simulated* by an anholonomity. In the sense of the equivalence principle, this supports the existence of torsion as long as one has reason to believe in the fundamental meaning of a local $SO(1, n-1)$.

By the same token, in a holonomic pure gauge we have

$$\Gamma^{\{\}}_{\alpha\beta} \stackrel{*}{=} \tfrac{1}{2}\,dg_{\alpha\beta} + (e_{[\alpha}\rfloor dg_{\beta]\gamma})\vartheta^\gamma\,. \tag{3.15.12}$$

A comparison with (3.10.6) shows that a nonmetricity $Q_{\alpha\beta}$, in this holonomic gauge, can be simulated by a non-vanishing $dg_{\alpha\beta}$. Thus by means of a non-orthonormal frame locally a nonmetricity of the pure gauge type can be generated.

4. Matter fields: manifields and world spinors

4.1. Existence of a double covering of the special linear group

For a long time, progress in affine gauge models of gravity was delayed by an error. Throughout the community of researchers in GR, there was a prevailing belief that the linear groups possess no non-trivial universal covering,[37] i.e. that it is impossible to construct spinor states, i.e. states carrying two-valued unitary representations of (in the Euclidean case) the $SO(n)$ compact subgroup, or in Minkowski spacetime, of the *spin*, i.e. the mathematical "stability subgroup" (for massive states this is the subgroup $SO(n-1)$ of spatial rotations) and thus also of the linear groups themselves. For the same reason, it was thought that it is impossible to construct spinor *fields* transforming linearly under $GL(n,R)$ or $SL(n,R)$. As a result, it was also thought that there could be no "world" spinors, i.e. that $\mathrm{Diff}(n,R)$ too has no *universal* (double for $n \geq 3$) *covering group*. Statements of that nature exist (and unfortunately continue to appear) in almost every textbook in GR and should be disregarded (see the examples cited in Ref.7 of [495]; we refer the reader to that article for proofs of the algebraic and topological theorems in the following discussion). The existence of a universal covering of the entire group inclusion

$$SL(n,R) \subset GL(n,R) \subset A(n,R) \subset \mathrm{Diff}(n,R) \tag{4.1.1}$$

was first pointed out in [476,477] and proved in [478]. This result has only recently been incorporated in a textbook on spinors [94] and noted in a corrective footnote in another text [617].[38]

[37] An exception is Post [565] who was aware of the double covering of the linear group; Bargmann [36] had even constructed the unitary representations of the (infinite) universal covering of $SL(2,R)$, but this remained unknown in the GR community, where the supposed inexistence of such representations for $SL(2,R)$ was quoted as a mathematical argument in physical discussions.

[38] Although no such structural error appears in the mathematical literature, very little attention was given to the representations of the double-covering of the $SL(n,R)$ for $n \geq 3$, prior to [648], in which the physically motivated [623] is referred to; see also [664], paper II, using [628].

The misunderstanding was due to the fact that representations [702] of the universal covering groups of the $GL(n,R)$ exist only as infinite matrices (this is no worse than in the case of the Heisenberg algebra, for which the same statement is true). The inexistence of *finite* representations was misconstrued as implying the inexistence of *any* representations.

Let us remember that any semisimple Lie group G can be decomposed according to the Iwasawa decomposition, which reads

$$G = K \times A \times N, \tag{4.1.2}$$

where K, A, N are analytic subgroups of G: K is the maximal compact subgroup, A is an Abelian subgroup, and N is a nilpotent subgroup, i.e. one with "triangular" matrices, whose non-vanishing matrix elements are all on one side of the diagonal. Since A and N are simply connected [296, p.234] and, by definition, are such as to be trivially deformable to the identity, the first homotopy group π_1 of the Lie group G is the same as that of its maximal compact subgroup K. In other words, the topology of a non-compact Lie group G is that of its maximal compact subgroup K.

If, as happens for the special linear groups, K is the group $SO(n) \subset SL(n,R)$, then we know that there is a covering group for K, namely $\overline{K} = \mathrm{Spin}(n)$; in any case in which K itself is not yet simply connected, we can replace it by its universal covering group $\overline{K}$, and, in the wake of it, we shall have thus defined and constructed the universal covering of G, namely $\overline{G} = \overline{K} \times A \times N$.

In general, the connectivity properties of the connected component $SO_\circ(n)$ of the orthogonal group is marked by the first homotopy groups [296, p.346]

$$\pi_1(SO_\circ(n)) = \begin{cases} Z_2 & \text{for} \quad n = 2k+1 \\ Z_2 \oplus Z_2 & \text{for} \quad n = 4k \\ Z_4 & \text{for} \quad n = 2(2k+1), \end{cases} \tag{4.1.3}$$

where Z_k is the group of integers modulo k, with respect to arithmetic addition, or the group of the k complex k^{th} roots of the identity. These homotopy groups determine the types of spinors existing in each case [94] – namely Weyl, Majorana, or Weyl–Majorana – and the structure of the related supersymmetry.

Thus, to define and construct the universal covering groups $\overline{SL}(n,R)$, remembering that for the special linear group $SL(n,R)$ we have the Iwasawa decomposition, $SL(n,R) = SO(n) \times A \times N$, we have

$$\overline{SL}(n,R) := \mathrm{Spin}(n) \times A \times N, \tag{4.1.4}$$

and applying the relationship between $GL(n,R)$ and $SL(n,R)$ as described in equation (3.1.8), this extends to

$$\overline{GL}(n,R) = [T \otimes \overline{SL}(n,R)] \times R^+, \tag{4.1.5}$$

$$\overline{A}(n,R) = R^n \otimes \overline{GL}(n,R). \tag{4.1.6}$$

In a playful spirit, with $\overline{SO}(n) = \mathrm{Spin}(n)$, the mathematicians have adopted the notation $\overline{O}(n) =: \mathrm{Pin}(n)$. Thus, for the covering group $\overline{GL}(n,R)$, the maximal compact subgroup is denoted by $\overline{O}(n) = \mathrm{Pin}(n)$.

The infinite-dimensional group $\mathrm{Diff}(n,R)$ is Einstein's covariance group, realizing the passive symmetry under general coordinate transformations, which is trivially represented in our exterior form notation. According to Stewart's theorem [657], in the neighborhood of the identity, it can be decomposed [39]

$$\mathrm{Diff}_{o}(n,R) \approx GL(n,R) \times H \times R^{n}\,, \tag{4.1.7}$$

where the subgroup H is contractible to a point. As a result, $O(n)$ is the deformation retract of $\mathrm{Diff}(n,R)$, and there exists a double covering group

$$\overline{\mathrm{Diff}}_{o}(n,R) \approx \overline{GL}(n,R) \times H \times R^{n}\,, \tag{4.1.8}$$

i.e. we shall be able to construct *"world" spinors*, in addition to the possibility of having *anholonomic spinors* under a $GL(4,R)$ or $A(4,R)$ gauge group acting on local frames. We shall name the latter *"affine frame spinors"*, generalizing the conventional finite Dirac (Lorentz) frame spinors as introduced by Tetrode, Wigner, Weyl, as well as by Fock and Ivanenko (see, for example, [447] for references). Affine frame spinors undergo, in addition to local Lorentz-rotations, deformations which have the *shear* current as conjugate Noether current.

It is not proven that $\overline{\mathrm{Diff}}(n,R)$, as given by (4.1.8), represents the full universal covering group, when considering transformations which are not close to the identity. It is possible that much richer structure exists globally (cf. [461]), especially in four dimensions, where an infinity of *exotic* R^4 has been demonstrated to exist [208,162], no two of which are globally diffeomorphic to each other and to the standard one [81], even though they are globally topologically equivalent. However, Eq.(4.1.8) is all that we shall need in the following.

4.2. The deunitarizing automorphism $\mathcal{A}$: covariance and equivalence as algebraic constraints

For the construction of our so-called *manifield* representations, a decomposition involving the $(n-1)$-dimensional subgroup $SL(n-1,R) \subset SL(n,R)$ is instrumental. For $u, v, w = 1,\dots,n-1$ let us define

$$\mathcal{S} := \mathcal{L}^{u}{}_{u} = -\mathcal{L}^{\hat{0}}{}_{\hat{0}}, \tag{4.2.1}$$

$$\overset{(\pm)}{B}{}_{u} := \tfrac{1}{2}(\mathcal{L}^{\hat{0}}{}_{u} \mp \mathcal{L}^{u}{}_{\hat{0}}). \tag{4.2.2}$$

With respect to these boost-like generators, $\mathcal{S}$ acts as

$$[\mathcal{S}, \overset{(\pm)}{B}{}_{u}] = \overset{(\mp)}{B}{}_{u}\,, \tag{4.2.3}$$

which will give rise to an *inner automorphism*.

Since we are distinguishing between the time index $\hat{0}$ – we assume $\mathrm{Ind}(g) = 1$, i.e. the signature $(-+++)$ – and the space indices denoted by $u, v\dots$, we can now identify more directly the following generators:

[39] Note that *active* $\mathrm{Diff}(n,R)$ is isomorphic to the infinite-dimensional group of local translations, i.e. $\mathrm{Diff}(n,R) \approx \mathcal{T}^{n} := C^{\infty}(A(M) \times_{Ad} R^{n})$, where $A(M)$ is the affine bundle on which the translations R^{n} act via the adjoint representation [656,447] with respect to $GL(n,R)$.

$$
L^{\alpha}{}_{\beta}:\begin{cases}
J_{\cdots u}=\frac{1}{2}\epsilon_{\hat{0}\cdots uvw}\mathcal{E}^{vw} & \text{angular momentum (compact)}\\
\overset{(+)}{B}_{u}=-\mathcal{E}_{(\hat{0}u)} & \text{time–space shears (compact)}\\
\mathcal{E}_{(uv)} & \text{spatial shears (non-compact)}\\
\overset{(-)}{B}_{u}=-\mathcal{E}_{[\hat{0}u]} & \text{Lorentz boost (non-compact)}\\
\mathcal{S}=-\mathcal{E}^{\hat{0}}{}_{\hat{0}} & \text{time shear (non-compact).}
\end{cases}
\tag{4.2.4}
$$

The important subalgebras and the relevant generators are

- $so(n-1)$: the spatial rotations, generated by the $J_{\cdots u}$,
- $sl(n-1)$: the $(n-1)$-dimensional volume-preserving algebra of the $J_{\cdots u}$ and $\mathcal{E}_{(uv)}$,
- R^{+}: a one-dimensional Abelian algebra generated by $\mathcal{S}$,
- $so(n)$: the maximal compact subalgebra, generated by $J_{\cdots u}$ and $\overset{(+)}{B}_{u}$,
- $so(1,n-1)$: the n-dimensional Lorentz algebra, generated by $J_{\cdots u}$ and $\overset{(-)}{B}_{u}$.

The following *First Manifield Theorem*, proved in [628], is an immediate consequence of the commutation relations: For any $sl(n,R)$, $n>3$, there exists an inner automorphism

$$
\mathcal{A}=\exp(\tfrac{1}{4}i\pi\mathcal{S}),\tag{4.2.5}
$$

which leaves the subalgebra $[R^{+}\times sl(n-1,R)]$ invariant and transforms

$$
\mathcal{A}J_{\cdots u}\mathcal{A}^{-1}=J_{\cdots u},\qquad \mathcal{A}\mathcal{E}_{uv}\,\mathcal{A}^{-1}=\mathcal{E}_{uv},\qquad \mathcal{A}\mathcal{S}\mathcal{A}^{-1}=\mathcal{S},\qquad \mathcal{A}\mathcal{D}\mathcal{A}^{-1}=\mathcal{D},\tag{4.2.6}
$$

$$
\mathcal{A}\overset{(+)}{B}_{u}\mathcal{A}^{-1}=i\overset{(-)}{B}_{u},\qquad \mathcal{A}\overset{(-)}{B}_{u}\mathcal{A}^{-1}=i\overset{(+)}{B}_{u},\tag{4.2.7}
$$

i.e. it replaces the compact $\overset{(+)}{B}_{u}$ by $i\overset{(-)}{B}_{u}$, i.e. an imaginary (and thereby formally compactified) Lorentz-boost, at the same time replacing the noncompact $\overset{(-)}{B}_{u}$ by an imaginary (and thereby decompactified) time–space shear $i\overset{(+)}{B}_{u}$.

The automorphism $\mathcal{A}$ is the key to the construction of physically fitting infinite-component fields, our *manifields* from the infinite unitary irreducible ("bandor") representations of $SL(n,R)$, including the two-valued spinorial ones (cf. appendix C.3). The fields have to be non-unitary, like ordinary tensors [these are constructed as finite and thus non-unitary representations of $SL(n,R)$] or Lorentz spinors [finite and thus non-unitary representations of $SO(1,n-1)$].

To understand the interplay between the homogeneous and inhomogeneous groups, it is important to remember that the algebraic foundations of *relativistic quantum field theory* (RQFT) stand on Einstein's two *principles* of *covariance and equivalence*, a fact which is generally glossed over in textbooks in RQFT, when these are not group-oriented or specifically related to GR.

The principle of covariance requires the physical fields to appear in expressions invariant under the group Diff(n,R). In order to realize this principle and a smooth transition to curved spacetime, fields (differential forms) are therefore constructed as representations of the homogeneous, holonomic, linear group $SL(n,R)_{H}\subset$ Diff(n,R); in Minkowski space, the symmetric spin two tensor field $\phi_{ij}(x)$, for example, includes all ten components of the relevant finite irreducible representation of $SL(4,R)$, reducing under the orthogonal subgroup $SO(4)$ (or under the pseudo-orthogonal $SO(1,3)$) as $\mathbf{10}=\mathbf{9}+\mathbf{1}$, i.e. $(\mathbf{1},\mathbf{1})\oplus(\mathbf{0},\mathbf{0})$, when seggregating the trace. Note that in holonomic constructions, the linear group plays a specific auxiliary role: As the irreducible linear subgroup of the diffeomorphism group, it provides a standard "realization" of that group, with the quotient $\mathrm{Diff}(n,R)/SL(n,R)$ being represented non-linearly over the matrices of the linear subgroup.[40] For Lorentz spinor fields, *with a finite number of components*, there is no linear embedding in $SL(n,R)$, and the non-linear action of the diffeomorphisms is transmitted through the local frames; in this case, the linear subgroup action is that of the Lorentz group $\overline{SO}(1,n-1) \subset \mathrm{Diff}(n,R)$, acting on the (tangent) frame indices, and the non-linear realization of the quotient group is given by the frame fields themselves.

At the same time, the principle of equivalence is embodied in the *affine* properties of the tangent space. It requires the physics of GR and PG to contain the transition to a frame which now carries the gravitational degrees of freedom and the kinematics is that of special relativity (SR), i.e. that of the Poincaré group, now acting in Minkowski spacetime. Reciprocally, in flat spacetime and RQFT, this kinematics is such as to provide, here too, a smooth transition. It is realized through the structure of the particle Hilbert space: It carries the representations of the *inhomogeneous* Lorentz group, i.e., the Poincaré group. When returning to GR and PG, this action, though hidden by spacetime curvature and torsion, is nevertheless omnipresent in the anholonomic frames, as constrained by the equivalence principle, cf. sections 3.2 and 3.3.

Note, however, that in metric-affine gravity (MAG), we have to consider a two-step transition. The action on the frames (in curved spacetime) is that of the affine group $A(n,R)$; this cannot be applied as a symmetry to flat spacetime, when extinguishing the gravitational field. The complete treatment [496] thus requires the intermediate stage, e.g. the long-range or low-energy regime, still within curved spacetime, of section 6.5, corresponding then to the Riemannian case, as in Einsteinian GR.

RQFT requires a smooth overlap between the algebraic realizations of the two principles, as examplified by the following case:

The constraint of non-unitarity of the field representations is imposed by the physical requirements, stemming from the above: With the physical $g_{\alpha\beta} \stackrel{*}{=} o_{\alpha\beta}$ identification of the metric g, the Lorentz boosts become non-compact generators. Should we use unitary representations, the Lorentz subgroup would be represented by the unitary infinite-dimensional representations of Gel'fand and Naimark [218,219]. These are representations that excite the spin degree of freedom: Acting with $\overset{(-)}{B}_{u}$ on a state of spin j, we get states with spins $j-1$ or $j+1$, ad infinitum. After all, these are representations of the (homogeneous) Lorentz group, i.e. a group that does not include translations and, consequently, the momenta, and can only excite that which it contains, namely spins.

Such representations for the fields, however, do not give the correct overlap with the representations of the particle Hilbert space, which are those of the Poincaré group, the inhomogeneous group that does induce the momenta. In practice, indeed, the spin-exciting representations for the boosts do not fit our understanding of the known particles or states corresponding to the known fundamental fields (quarks and leptons, W and Z bosons, etc.) and their particle realizations on a Hilbert space. Observationally, we find that the Lorentz boosts act to accelerate a particle, i.e. to modify its momentum and its kinetic energy, without changing its spin, which is in fact one of the permanent characteristics identifying a particle: It is the quantum number describing the representation of the

[40] This is the reason for the appearance of the $SL(4,R)$ matrices in the covariant derivative, even though Riemannian gravity is not the Yang–Mills gauge theory of that group [156].

stability subgroup, $\overline{SO}(3)$ for massive states, over which one induces, following Wigner and Mackey, the unitary representations of the Poincaré group, cf. [38].

For ordinary tensors and for Lorentz spinors, the representation is finite and thus non-unitary. The Lorentz boosts' intrinsic action is realized by antihermitian operators. When derived from a Noether theorem, they just cancel, because the physically acceptable Lagrangian four-form L itself is guaranteed to be hermitian (or made hermitian by taking $\frac{1}{2}(L + \text{h.c.})$, i.e. adding to it the hermitian conjugate). The surviving piece of the boost is then its orbital part, such as $tP_z - zP_t$. It is unitary because it acts in the infinite-dimensional Hilbert space of momenta; its effect is, indeed, to change the momentum and not the spin.

To achieve the same result when using infinite-dimensional "bandor" type representations, whether spinorial or tensorial, we apply the deunitarizing automorphism $\mathcal{A}$ in the following manner:

(1) We first apply $\mathcal{A}$ to the generator algebra $sl(n,R)$ and get an $sl(n,R)_{\mathcal{A}}$ algebra in which the Lorentz boosts are given by $i\overset{(+)}{B}_{u}$ and the time–space shears by $i\overset{(-)}{B}_{u}$.

(2) We construct unitary irreducible representations for this algebra $sl(n,R)_{\mathcal{A}}$: the $i\overset{(-)}{B}_{u}$ generators close with the $J_{\dots u}$ on the compact $so(n)_{\mathcal{A}}$ and will be represented by an infinite sum of finite unitary representations, those corresponding to the maximal compact subalgebra, onto which one induces the entire $sl(n,R)_{\mathcal{A}}$ unirrep. The noncompact $so(1,n-1)_{\mathcal{A}}$, formed by the $i\overset{(+)}{B}_{u}$ with the $J_{\dots u}$, will be represented by infinite unirreps à la Gel'fand and Naimark.

(3) After the construction of this unirrep of $sl(n,R)_{\mathcal{A}}$, we apply $\mathcal{A}^{-1}$. As a result, we regain the original identification of the generators of $sl(n,R)$, as defined by taking the Minkowskian signature for $g_{\alpha\beta}$; the boosts $\overset{(-)}{B}_{u}$ are given by finite and non-unitary (they are multiplied by $1/i$) representations of $so(n)$, precisely as it is in ordinary tensor and spinor fields; the time–space shears $\overset{(+)}{B}_{u}$ are represented by non-unitary (because of the same $1/i$) infinite Gel'fand–Naimark representations. They too, like the boosts, will have their intrinsic action cancelled for a hermitianized Lagrangian and will therefore only act orbitally. The construction is thus given by the sequence

$$L \to \mathcal{A}^{-1}[U(\mathcal{A}L\mathcal{A}^{-1})]\mathcal{A}, \tag{4.2.8}$$

where U denotes the unitary irreducible representation.

Returning to our considerations as based upon covariance and equivalence, we note that in the above construction, we have managed to produce a consistent overlap of the two principles' application: The selection of the representations of the covariance subgroup $\overline{SL}(4,R)$ has been fitted to the constraints imposed by equivalence, namely having a Poincaré group particle Hilbert space.

4.3. *Manifields and some of their applications*

In flat spacetime, structureless particles are thus represented by induced unitary representations of the Poincaré group $P = R^4 \otimes SO_o(1,3)$ or, in the case of spinor fields, by its simply connected covering group $\overline{P} = R^4 \otimes SL(2,C)$. In the presence of gravity with its curved spacetime, this group then functions *"vertically"*, i.e. as a structure group, acting on the local frames (*"anholonomic tangent group"*).

The study of the phenomenological systematics of Hilbert space representations shows that such a description in terms of the Poincaré group (with spin and mass squares as invariants) is sufficient for the leptons, even though even in this case there is a need for other "internal" algebraically defined quantum numbers, to explain interactions (such as "weak" isospin and hypercharge) and phenomenological conservation laws (the various types of "lepton numbers", chiralities, etc.). For the various gauge fields, we require in addition the quantum numbers corresponding to the adjoint representation of the relevant gauge groups. On the other hand, for hadrons and nuclei – states in Hilbert space which do not coincide with the simplest materialization of a fundamental field – aside from *"internal"* symmetries, such as *flavor*-isospin $SU(2)$ and unitary symmetry $SU(3)$, there is an extremely rich struture of correlated dynamical excitations (tens of thousands of states in nuclei, tens in hadrons) which have been shown to emerge from the dynamics, like the spectrum of states of a hydrogen atom or of a harmonic oscillator.

For the hadrons, the basic phenomenological description, to begin with, was in terms of the S-matrix poles along a *Regge trajectory* in the analytically continued complex angular momentum plane. It was then shown that an appropriate algebraic description could be provided by the action of a *spectrum generating group* (SGG) [166,167,39,68], emerging from the dynamics and correlating an infinite sequence of representations of the Lorentz group (i.e. *energy levels*). SR and the O'Raifeartaigh/Coleman–Mandula theorems [539,126] do not allow an embedding of the Poincaré (or, for massless systems, of the conformal) group in larger symmetry groups (except for supersymmetry [253]) but the SGGs are not symmetries of the S-matrix, they only correlate a sequence of dynamically related Poincaré group representations.[41]

The most obvious choice [166,167] for the description of Regge sequences appeared to involve the infinite-dimensional unitary *bandor* representations of $SL(3,R)$, with an $SO(3)$ subgroup defined as the difference between the total angular momentum of a hadron and the "quark spin" contained in $SU(6)$, the latter being given by current algebra [220]. With the post-1974 picture, as given by the *standard model* of the physics of particles and fields, such sequences represent the system of bound and resonant states, either of three quarks (the baryons), of quark–antiquark structures (the mesons), or of three antiquarks (the antibaryons).

At the time, there appeared, however, to be one difficulty with the use of $SL(n,R)$ systematics, either as a SGG for the hadrons in Minkowski space, or as an anholonomic local tangent group for metric-affine gravity – in which the "vertical" group acting on the frames is enlarged by embedding the Lorentz group $SO_o(1,3)$ in $SL(4,R) \subset GL(4,R)$ (or of the Poincaré group in $SA(4,R) \subset A(4,R)$). Since $SL(4,R)$ or the affine group $SA(4,R) := R^4 \otimes SL(4,R)$, respectively, were (erroneously) considered to possess no double covering, this seemed to restrict the applications of both the hadronic $SL(3,R)$ in flat spacetime and of metric-affine gravity to bosonic matter. The issue was settled, first for $\overline{SL}(3,R)$ [324,708], and then for all $\overline{SL}(n,R)$, when the existence of the covering group was demonstrated [478] and when Ne'eman et al. could show that $\overline{SL}(4,R)$ possesses *"bandor" type* unitary infinite-dimensional representations.

For the hadrons, although there exist several other candidate SGGs [68], note that $\overline{SL}(4,R)$ *is unique, in providing the only possible field representation ("manifields")*, thus allowing for a *covariant*

[41] Dothan [165] has shown that the SGG can be considered as coordinate-dependent (time-dependent, in a non-relativistic context) symmetries of the S-matrix, which, on the other hand, do not commute with the Hamiltonian, their generators thus raising and lowering the energy levels, somewhat like Lorentz boosts.

description, within the context of phenomenological field theory. The same description in terms of manifields fits the needs of metric-affine gravity. Moreover, the manifield description of the hadrons lends itself in the usual way to the replacement of flat Minkowski space by curved spacetime, in the presence of gravity, whatever the nature of the manifold (Riemannian torsionless V_4, torsion containing U_4, metric-affine (L_4, g), etc.).

A similar need exists in the description of nuclear excitation systematics, except that the nuclear structure methodology makes much less use of field methods. The application of these ideas in that domain therefore exploits the Hilbert space representation structure and the action of transition operators between them, rather than manifields. Moreover, it is often practical in nuclei to truncate the infinite bandors and replace them by finite unitary representations of a compact real form with the same algebra, i.e. replace $SU(2) \subset \overline{SL(3,R)}$ by $SU(2) \subset SU(3)$ [68].

In sections C.1, C.2 and C.3 of the appendix we list the unitary (infinite-dimensional) irreducible representations of $\overline{SL}(n,R)$ for $2 \leq n \leq 4$, which we also apply, using the automorphism $\mathcal{A}$, as non-unitary field representations. In appendix C.4 we also present the unitary representations of $\overline{SA}(n,R)$ in a particle Hilbert space and explain the possible induced representations, see also appendix C.6. For spinor matter fields and for infinite tensor fields – in other words, for *manifields*, assuming they create massive particles, the Hilbert space will support unitary infinite-dimensional representations of $\overline{SA}(n,R)$, actually given by the little group's representations (in Wigner's nomenclature). These will be unitary irreducible representations of $\overline{SL}(n-1,R)$, unmodified by $\mathcal{A}$, or "bandors": Bands of states with mounting spins; in the cases of $\overline{SL}(3,R)$ as a stability subgroup of $\overline{SA}(4,R)$, in the simplest representations the spin intervals are restricted to $\Delta J = 2$; otherwise we encounter $\Delta J = 1$ or 2.

Returning to the Minkowski space dynamics of hadrons and nuclei, we note that the bandor or manifield algebraic structure suggests that the mounting levels can be excited by the action of a tensor field such as gravity – or of an effective tensor field originating elsewhere, but with properties resembling gravity ("strong gravity" [587,588,445,570]). Several different approaches [448,499,205,41,415] have pointed to the emergence of such an effective tensor field in QCD, thereby explaining the latter's geometrical features, [42] as expressed in color confinement. The approach also fits the fact that in *string* theory, while the open string reduces under truncation (doing away with the Planck mass sector) to a Yang–Mills theory (such as QCD), the closed string reduces in the same truncation to gravity. Indeed, the closed string is a contraction of two open strings (summed over the color index) and appears in the quantum corrections of the open string – in just the same manner as happens in QCD for *chromogravity* or *pseudogravity* in the models discussed by the above authors, with the tensor field appearing as a bilinear in the Yang–Mills field (either the potential or the field strengths, depending on the model).

Note that in the Ne'eman–Šijački approach, the geometric effects are limited to the *infrared* region [499,501,500], whereas the other approaches appear at this point to cover the entire energy spectrum – although still indirectly linked with the IR region through the fact that the description does not involve the color index. Several effects in hadron [629] and nuclear [630] physics can be attributed to such a *"chromogravity"*. Among the features explained by chromogravity in hadrons are the observed Regge trajectories [166], corresponding to the unitary irreducible representations of the little group $\overline{SL}(3,R)$, thus fitting the relativistic treatment based on $\overline{SA}(4,R)$. The key question with respect to

[42] For the $SU(2)$–Yang–Mills case, as mapped to a Riemann–Cartan space, see [458].

the spectrum is to know whether or not there are kinematical constraints on the "Regge trajectories", i.e. the spin versus mass square correlation, within a given representation. In the past, such constraints have either made it extremely hard to construct a physical representation (e.g. the "angular condition" for local current algebra [220]) or shown the field representation to have properties which clash with experiment (e.g. the Majorana equation). This is the main reason for our being interested in the Casimir invariants of $SA(n,R)$ and for including them in appendix C.4.

The most important result we have in this context is that for the representations used in our manifields (world spinors and infinitensors), i.e. the representations of class IIA, cf. appendix C.4, the Casimir invariant $C(4)$ vanishes, and there are *no constraints*. Representations of class IIB 2b, for instance, have a non-vanishing Casimir – but we do not use them in physics at this stage.

In the corresponding excitation bands in nuclei, cf. [708,579], the $SL(3,R)$ has been enlarged to $Sp(3,R)$. The theory has also provided for the first time a derivation of the $j = 0,2$ ground state of the successful *interacting boson model* ("IBM" [13]) in nuclei. In hadrons, aside from the direct mathematical demonstration of the emergence in QCD (in the infrared region) of a system of pseudo-diffeomorphisms[43] for which the "chromogravity" effective tensor field acts as a Riemannian metric, the notion is heuristically justifiable by the geometric, bag-like nature of color confinement. The energy levels represent the deformation–pulsation frequencies of these compound systems. Note that $SL(4,R)$ is then the invariance group of a bag-like extended structure evolving in spacetime (just as an evolving string is $SL(2,R)$ invariant [600]) and the little group $SL(3,R)$ represents the invariance of the three-volume of the color-confining bag, throughout its vibrations–pulsations. The theory is also unique in providing a derivation of the observed linear relation in hadron Regge trajectories between the spins and squared masses[44] [499,501].

In gravity, aside from providing for matter fields affine and metric-affine theories, the new structures can be used for phenomenological studies and calculations in Einsteinian gravity, when protons or neutrons, for instance, are involved. In a fundamental Lagrangian, we would have quark fields and QCD gluons. From these it would be almost impossible to calculate precise effects on their bound states, the protons and neutrons. In the past, the method used was to treat the protons and neutrons as phenomenological Dirac fields; this, however, is an incorrect prescription, since nucleons are not Dirac particles. Not only do they have large anomalous magnetic moments, but in addition, they possess a large sequence of excited states for which there is no description in the Dirac equation. The correct covariant formalism is that described in this chapter.

Returning now to our algebraic constructs, we note that the simplest $\overline{SL}(4,R)$ manifields are the multiplicity-free ones (appendix C.3). In particular, the two conjugate representations in the discrete series

$$\mathcal{D}^{\text{disc.}}_{SL(4,R)}(\tfrac{1}{2},0) \oplus \mathcal{D}^{\text{disc.}}_{SL(4,R)}(0,\tfrac{1}{2}) \tag{4.3.1}$$

[43] Recently it has been proved [500] that the exchange of n-gluon sets with the color index contracted (example: $d_{abc}\,B_i{}^a B_j{}^b\, d^c{}_{de}\, B_k{}^d B_l{}^e$ etc..., with $a,b,\ldots$ $SU(3)$ indices and $i,j,\ldots$ indices for spacetime) generates a representation of the n-dimensional Ogievetsky algebra.

[44] This relation also emerges in a different context – perhaps related through gravity/chromogravity parallelism – namely for the Kerr–Newman metric in GR [342,503], as pointed out by Salam [587], [also in a personal communication to Y.N.] and made more exact in [443,445,447] by employing the Christodoulou–Ruffini notion of the irreducible mass of a black hole. Chromogravity and the other treatments of QCD as a geometrical theory will be reviewed in greater detail in a coming review of the applications of QCD to the understanding of the hadron spectrum [501].

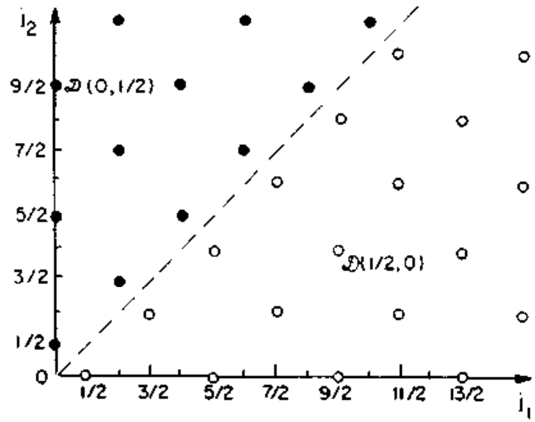


Fig. 3. The basic spinor manifield $\mathcal{D}^{\text{disc.}}_{SL(4,R)}(\frac{1}{2},0) \oplus \mathcal{D}^{\text{disc.}}_{SL(4,R)}(0,\frac{1}{2})$, extending the Minkowski space Dirac spinor field of the same 'name' into $SL(4,R)$. This reducible $SL(4,R)$ representation is the direct sum of two infinite, $\mathcal{A}$-deunitarized irreducible representations, denoted by hollow and full circles respectively, cf. [477,481,628].

contain, as representations of the little group $SL(3,R)$, sequences of $\mathcal{D}^{\text{ladd.}}_{SL(3,R)}(\underline{j} = \frac{1}{2}; \sigma_2 = 0)$, the special unirrep of (C.2.4). This unirrep reduces to an infinite sum of $SU(2)$ representations $(\underline{j}, \underline{j} + 2, \ldots)$

$$j = \frac{1}{2} \oplus \frac{5}{2} \oplus \frac{9}{2} \oplus \cdots . \tag{4.3.2}$$

The $\mathcal{D}^{\text{disc.}}_{SL(4,R)}(\frac{1}{2},0) \oplus \mathcal{D}^{\text{disc.}}_{SL(4,R)}(0,\frac{1}{2})$ is the direct sum of two finite, chiral, and non-unitary representations of the Lorentz group, related by parity. In the Dirac case, this describes the Lorentz behavior of the Dirac field; for the $SL(4,R)$ representation, this is the behavior of the lowest Lorentz submultiplet. Thus it contains a Dirac-like spinor at the lowest level. In a region where the shears and dilations are switched off, with no tensor field to relate the various $\Delta j = 2$ levels, this ground state decouples and becomes a true Dirac spinor field.

Other important manifields appearing in the ladder series in (C.3.10) are

$$\mathcal{D}^{\text{ladd.}}_{SL(4,R)}(\underline{j} = 0, e_2), \qquad e_2 \in R, m = n = j, \quad j \cong \underline{j} \,(\text{mod}\, 1), \tag{4.3.3}$$

those yielding bosonic bands corresponding to the $SL(3,R)$ bandors in (C.2.4)

$$\mathcal{D}^{\text{ladd.}}_{SL(3,R)}(\underline{j} = 0; \sigma_2), \qquad \sigma_2 \in R, \; \{j^n\} = \{\underline{j}, \underline{j} + 2, \cdots\}, \quad \text{i.e.} \quad j = 0 \oplus 2 \oplus 4 \oplus \cdots, \tag{4.3.4}$$

$$\mathcal{D}^{\text{ladd.}}_{SL(3,R)}(\underline{j} = 1; \sigma_2), \qquad \sigma_2 \in R, \; \{j^n\} = \{\underline{j}, \underline{j} + 2, \cdots\}, \quad \text{i.e.} \quad j = 1 \oplus 3 \oplus 5 \oplus \cdots, \tag{4.3.5}$$

and the manifield

$$\mathcal{D}^{\text{ladd.}}_{SL(4,R)}(\underline{j} = \tfrac{1}{2}, e_2), \qquad e_2 \in R, \; m = n = j, \quad j \cong \underline{j} \,(\text{mod}\, 1), \tag{4.3.6}$$

yielding the same $SL(3,R)$ bandors as (4.3.4) and (4.3.5).

These infinite-dimensional representations of $\overline{SL}(4,R)$ are realized by infinite component spinor fields, i.e. *spinorial manifields;* taking first the case in which the action of the group is defined on the frames, i.e. representing the anholonomic $\overline{SL}(4,R)_A$, the manifield is $\Psi^{\Xi}(x)$, where the anholonomic (upper case Greek) indices $\Xi, \Pi, \Theta = 1,2,\ldots,\infty$ run over the countable infinity of the components of the manifield representations, namely, in the usual case, the quantum numbers of the representations of the $\mathrm{Spin}(1,3) = \overline{SO}(1,3)$ – finite and non-unitary, by construction – isomorphic to the (finite) unitary representations of $\overline{SO}(4,R)$. For example, we may have a sequence $(0,0),(1,1),(2,2),\ldots$, i.e. single-valued representations, in the case of the *infinitensor* (4.3.3), or the sequence $(1/2,0),(3/2,1),(5/2,2)\ldots$, double-valued representations, for one of the two "ladders" in the spinorial manifield in (4.3.1).

For these spinor representations, the group-structure can be stepwise enlarged, e.g. according to a subgroup chain, beginning with the Lorentz group,

$$SL(2,C) \subset \overline{P} = R^4 \otimes SL(2,C) \subset \overline{SA}(4,R) \subset \overline{\mathrm{Diff}}(4,R). \tag{4.3.7}$$

This is, however, an inclusion in the abstract. To realize, for instance, the covariance group, we have to specify $\overline{SL}(4,R)_H$, the *holonomic* special linear subgroup of $\overline{\mathrm{Diff}}(4,R)$; in addition, we shall have to correct for the action of the covariance group on the coordinates in the argument of the manifield. We shall study one such construction in the next section.

If we lift the Lie generators $L^i{}_j$ of the linear connection $\Gamma_i{}^j$ to the manifield representation, denoting it by $\rho(L^i{}_j) = \{(L^i{}_j)_P{}^Q\}$ (for this holonomic case we use upper case Latin indices), we can introduce the exterior covariant derivative

$$D\Psi^P(x) = (\delta^P_Q d + \Gamma_i{}^j (L^i{}_j)_Q{}^P)\Psi^Q(x). \tag{4.3.8}$$

The manifield representation is irreducible, because the shear generators (still in the holonomic case) $\not{E}_{(ij)}$, i.e., the generators of the coset space $\overline{SL}(n,R)/\overline{SO}(n)$, connect all these various substates over the $|\Delta j| = 2$ intervals. Physically, it is the gravitational field (or the effective field of "*chromogravity*", in the flat spacetime physics of the hadrons) which connects these substates, since they are coupled through the covariant derivative by means of the matrices $\{(L^i{}_j)_Q{}^P\}$. Note that to the extent that we are treating (in MAG) the case of an anholonomic action of $\overline{SL}(4,R)_A$ on the frames, a similar covariant derivative can be defined, with the indices P,Q replaced by Ξ,Π; the manifield (infinitesimal) variation, in that case, does not directly involve the variation $x^i \to x^i + \xi^i(x)$.

4.4. *World spinors and world infinitensors, new concepts in field theory*

The relationship between $\overline{SL}(4,R)_A$ and $\overline{SL}(4,R)_H$ is somewhat less straightforward than the one existing between the Lorentz groups in the anholonomic and holonomic physical realizations. We have mentioned previously that SR does not allow a symmetry larger than the Poincaré group. Thus, the frames over which the anholonomic group is acting are not the orthonormal frames as in GR. We shall however have occasion to use the anholonomic linear group and its double-covering in flat space; in such situations, the manifield breaks down and reduces to an infinite sum of Minkowski spacetime fields. Phenomenologically, $\overline{SL}(4,R)$ is then used as a SGG [68].

There are, however, other physical situations in which the frame manifield remains irreducible – e.g. in models of quantum gravity in which the high-energy (above Planck mass) regime is non-Riemannian. In either case, we can introduce a local *manibein* frame [493], by lifting the usual frame fields $e^i{}_\alpha(x)$ – (4×4) field-valued matrices, relating here $SL(4,R)_A$ to $SL(4,R)_H$ in their

$(1/2, 1/2)$ four-vector representations [45] – via the manifield representation, to the infinite-dimensional matrix $E^P{}_{II}(x)$. As a result, we can define holonomic *world spinors*,

$$\Psi^P(x) = E^P{}_{II}(x)\Psi^{II}(x), \qquad E^P{}_{II}(x) := E(e^i{}_\alpha)^P{}_{II}, \tag{4.4.1}$$

where $\Psi^{II}(x)$ is now a spinorial manifield under the action of $\overline{SL}(4,R)_H$; with the action of the covariance group double-covering $\overline{\mathrm{Diff}}(4,R)$ following the usual prescription of a non-linear realization over its linear subgroup, we have thereby obtained a field which will carry faithful representations of that covariance group's double-covering, fitting the "*world*" *spinor* description. Before dealing in detail with the transformation properties of both world spinors and affine-frame spinor manifields, we note that for a general world tensor field $\Phi^{\alpha\cdots\beta}_{\gamma\cdots\zeta}$, there are two ways of describing the transformations under $SL(4,R)$: (a) As a direct product of p covariant and q contravariant four-vector "fundamental" representations, or, (b) after contractions over indices of complementary types, and symmetrization and antisymmetrization over indices of the same type, the (now irreducible) tensor carries an *irreducible* representation of $SL(4,R)$. There is one single type of manifield, constructed as a ladder representation (see appendix C), which can be considered as a limiting case of some types of ordinary tensors – namely the totally symmetric contravariant $\Phi^{\alpha\cdots\beta}$ and its covariant analog – when the number of indices goes to infinity (for a spinor, Φ itself is a Lorentz spinor). In that case, option (a) is available. In the general case, however, we have to use option (b), i.e. we deal with the field as the carrier of an irreducible (non-unitary) representation. For an ordinary tensor field we have the infinitesimal variation,

$$\delta\phi^m(x) = -i\,\partial_i\xi^j(x)\,[\rho(L^i{}_j)]^m{}_n\phi^n((\sigma^i{}_j)^{-1}x) + \xi^i\partial_i\phi^m(x), \tag{4.4.2}$$

where $\xi^j(x) = \delta x^j$, the indices m,n denote components of the field representation, $\rho(L)$ is the appropriate matrix representation and $\sigma(L^{-1})$ is the (4×4) matrix representation of the inverted action of that generator. The $SL(4,R)_H$ matrix is multiplied by a parameter which is the relevant component of the gradient of x^i, resulting from the Jacobian determinant [156] and producing the non-linear action of $\mathrm{Diff}(4,R)/SL(4,R)_H$ quotient.

The infinitesimal variation of the world spinor manifield under the action of the diffeomorphism $x^i \to x^{i'} = x^i + \xi^i$ will be very similar, except that the action on the spinorial manifield will be realized through the infinite matrices of $\overline{SL}(4,R)_H$,

$$\delta\Psi^M(x) = -i\,\partial_i\xi^j(x)\,[\rho(\overline{L}^i{}_j)]^M{}_N\Psi^N((\sigma^i{}_j)^{-1}x) + \xi^i\partial_i\Psi^M(x). \tag{4.4.3}$$

We note that the generators of $\overline{SL}(4,R)_H$ are given, in terms of the anholonomic ones, by

$$(\overline{L}^i{}_j)^M{}_N = E^M{}_\Xi(x)\,(\overline{L}^\alpha{}_\beta)^\Xi{}_{II}H_N{}^{II}(x), \tag{4.4.4}$$

where $H = E^{-1}$ is the inverted manibein, lifted from the one-form basis.

The "infinitesimal" parallel-transport of a manifield under a one-parameter subgroup of $\mathcal{T}^n \approx \mathrm{Diff}(n,R)$ with generator ξ is provided by the $GL(n,R)$-gauge covariant Lie derivative

$$\text{Ł}_\xi\Psi^{II}(x) = \xi\rfloor D\Psi^{II}(x) + D\xi\rfloor\Psi^{II}(x)\,. \tag{4.4.5}$$

[45] We denote the finite unirreps of $SL(4,R)$ by the largest unirrep of its compact $SO(4)$ subgroup, or by the non-unitary unirrep of $SO(1,3)$ in the Minkowski case.

Holonomically, i.e., for world spinor $\Psi^M(x)$, Eq. (4.4.5) simplifies to the usual covariant Lie derivative $l_\xi = \xi \rfloor d + d\xi \rfloor$.

Fermionic holonomic manifields (with half-integer Lorentz subgroup spin) are *world spinors*, to be clearly distinguished from *bosonic* ones. The bosonic ones contain the same information as ordinary tensor fields, in as much as they belong to Diff(n, R). They are thus better known as *infinitensors*. The fermionic world spinors, on the other hand, display the action of the covering group $\overline{\text{Diff}}(n, R)$ *faithfully* and are therefore more general.

In order to introduce covariant differentiation for world spinors, the linear connection (3.5.1) has to be lifted to the manifield and then written holonomically. Similarly as in (3.5.2), we find the following inhomogeneous relation

$$\Gamma'^{\ \beta}_{\alpha}(L^\alpha{}_\beta)_{II}{}^{\Theta} = E^P{}_{II}\ \Gamma_j{}^k(L^j{}_k)_P{}^T\ H_T{}^\Theta - E^P{}_{II} dH_P{}^\Theta \qquad (4.4.6)$$

between the anholonomic (Greek indices) and the holonomic (Latin indices) connection for a manifield with respect to a manibein transformation.

In the minimal coupling prescription, the coupling to the connection, is then achieved by means of the covariant manifield derivative $(D_i := \partial_i \rfloor D)$)

$$D_i \Psi^M(x) = [\delta^M_N \partial_i + \Gamma_{ij}{}^k (L^j{}_k)_N{}^M]\, \Psi^N(x)\ . \qquad (4.4.7)$$

Then, in our first order formalism, the world spinor matter Lagrangian reads

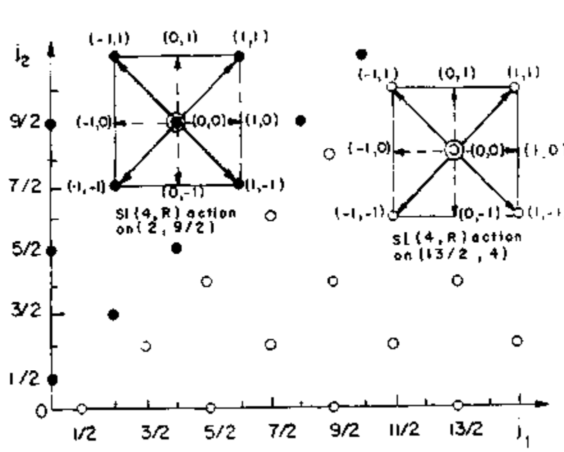
$$L_{\text{mat}} = L_{\text{mat}}(H_L{}^\Lambda(x), \Psi^M(x), D_i\Psi^M(x)). \qquad (4.4.8)$$

The applications of our new concepts – the world infinitensors and spinors, especially the latter – range over a number of areas, which we touched upon in section 4.3. The above mentioned gauge coupling to gravity provides the key for applications in gravity itself and in theories involving a phenomenological or effective field resembling gravity, aside from providing for matter fields in affine and metric-affine theories. We mentioned phenomenological studies and calculations in Einsteinian gravity, when protons or neutrons, for instance, are involved. Nucleons should be treated as components of a phenomenological world spinor, coupled to gravity through the connection as in (4.4.7). Falling into a black hole, for instance, they might well get their sequences of resonances excited by the strong gravitational field of the hole. In the case of *chromogravity* (QCD induced), the dynamics provide mass formulae for the Regge sequences, for instance.

4.5. Manifield equations

For the rest of section 4, we will assume the dimension of the differential manifold to be $n = 4$, that is, we will consider spacetime. Two key theorems constrain the construction of manifield equations [99], the *First Manifield Theorem* of section 4.2 and the
Second Manifield Theorem: In $\overline{SL}(4, R)$ *or* $\overline{\text{Diff}}(4, R)$ *covariant equations, the manifield cannot correspond to a multiplicity-free representation.*

This can be readily seen in Fig. 4: In a curved space and covariant equation of motion, the 'γ-type matrices' X_α span a four-vector representation $(1/2, 1/2)$ under $sl(4, R)$, i.e. $[\mathcal{L}^\alpha{}_\beta, X_\alpha] = \delta^\alpha_\gamma X^\beta$, cf. (3.1.5). Taking the commutator between states connected by the X_α, we find that for the left hand side not to vanish, $\mathcal{L}^\alpha{}_\beta$ has to act (non-trivially) as a $(0, 0)$ transition.



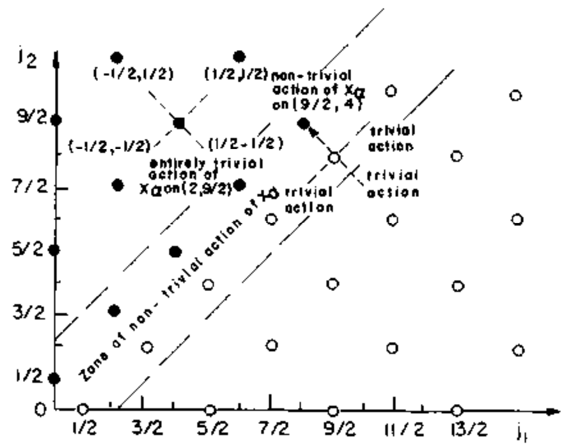


Fig. 4. The action of the $(1,1)$ 'symmetric' algebraic generators spanning the quotient $sl(4,R)/so(1,3)$ – and the second algebraic theorem for world spinor dynamics, the curved space multiplicity theorem [99]. The $(|\Delta j_1|, |\Delta j_2|) = (1,1)$ ninefold action of the shears spans a 'Union Jack' – eight arrows and the central point itself, namely (in the helicity representation), $(1,1), (1,0), (1,-1), (0,1), (0,0), (0,-1), (-1,1), (-1,0), (-1,-1)$ representing the raising and lowering action on the $so(1,3)$ (Lorentz) subgroup representations. Of these, naturally, only $(0,0)$ preserves a state's eigenvalues. Multiplicity-free representations, by definition, do not leave a state unchanged and always connect it to higher or lower states; here, the Lorentz submultiplets are connected by the 'St.Andrew' $\times$-like transitions to higher or lower Lorentz submultiplets. The $(0,0)$ transitions' action is thus trivial here; this is also true of the 'St.George' ('+'-like) transitions, acting trivially, in both the 'ladder' and in these specific 'discrete'-type representations.

Fig. 5. The action of the infinite (constant) 'γ-matrices' X_α.

An $sl(4,R)$ generator, such as $\mathcal{S} = -\not{L}^{\hat{0}}{}_{\hat{0}}$, maps the infinite matrices X_α through $[\mathcal{S}, X_\alpha] = -\delta^{\hat{0}}{}_\alpha X_{\hat{0}}$, since X_α is a four-vector under $sl(4,R)$. Since $\mathcal{S}$ is a spin zero object (as a time-time component of the shears), this requires $\mathcal{S}$ to couple a state to itself, so as to get the same bra state on both sides of these commutation relations. However, in a multiplicity-free representation, the generators take you only up or down, by definition.

Thus equations for "holonomic" or world spinors, being $SL(4,R)_H$ invariant, exist only for manifields which are not multiplicity-free. This is an inconvenience in any approach in which one tries to use the simplest manifields. Mickelsson [440] first wrote down such a covariant equation. The Mickelsson equation, however, for vanishing gravitational field (i.e. in the flat limit), remains invariant under global $\overline{SL}(4,R)$ and thus does not reduce to simple $SL(2,C)$-invariance. This can be remedied through a mechanism of spontaneous symmetry breakdown with a Goldstone–Higgs field or manifield. We shall discuss such a model in section 6.4. The

Third Manifield Theorem: In the absence of the gravitational field, a Lorentz invariant equation decouples the states of $\mathcal{D}^{\text{disc.}}_{SL(4,R)}(1/2,0) \oplus \mathcal{D}^{\text{disc.}}_{SL(4,R)}(0,1/2)$ *outside of the main diagonal.*

This can be seen in Figs.3 and 5: the infinite γ-like matrices X_i are still a Lorentz four-vector, i.e. a $(1/2,1/2)$ representation. Such operators cannot connect a state to any other state, except between both sides of the main diagonal "alley".

The constraints imposed by these two theorems notwithstanding, there is a simple method [99] in which one can still use the simplest (multiplicity-free) manifields, which we listed in section 4.3, and can construct a holonomic equation, i.e. one describing a world spinor. We first construct a

Lorentz-invariant (i.e. flat space) equation, e.g. for the manifield $\mathcal{D}^{\rm disc.}_{\overline{SL}(4,R)}(1/2,0) \oplus \mathcal{D}^{\rm disc.}_{\overline{SL}(4,R)}(0,1/2)$ (while applying the deunitarizing automorphism of section 4.2),

$$\left[i\, o^{ik} X_i \partial_k - m(p)\right] \Psi = 0. \tag{4.5.1}$$

The anholonomic X_α are constructed in the following way [493]: first we embed $\overline{SL}(4,R)$ in $\overline{SL}(5,R)$ and then select a pair of parity conjugate principal series representations which are contained in the $SL(4,R)$ reduction of our spinorial representations. Let the generators of $\overline{SL}(5,R)$ be $\overline{L}_A{}^B$, $A,B = 0,\cdots,4$. We define

$$X_\alpha = \overline{L}_{[4\alpha]}, \qquad \alpha = 0,1,2,3, \tag{4.5.2}$$

which yields an $SL(4,R)$ four-vector. Note that the X_α operators constructed in this way yield, upon commutation, the $SL(2,C)$ generators, generalizing a property of Dirac's γ-matrices. The holonomic matrices in (4.5.1) are then given by $X_i = e_i{}^\alpha X_\alpha$.

In the presence of gauge transformations (or of curvature), the partial derivative of the world spinor $\Psi^M(x)$ is replaced by the covariant exterior derivative (4.4.7), in which $(L^j{}_k)_N{}^M$ is an infinite $\overline{SL}(4,R)$ matrix. This matrix will connect states outside of the diagonal alley, with $\Delta J = 0,1,2$. Note that even though $X_i = e_i{}^\alpha X_\alpha$ will now contain a tetrad field, it is still a four-vector under the holonomic $SL(4,R)_H \subset \overline{\rm Diff}(4,R)$. In addition the Minkowski metric has, of course, to be replaced by the relevant metric.

The third manifield theorem, for world spinor dynamics, the "flat space disconnection" theorem: The $(|\Delta j_1|,|\Delta j_2|) = (1/2,1/2)$ action of the X_α, namely, in the helicity representation, the four possible transitions $(1/2,1/2)$, $(1/2,-1/2)$, $(-1/2,1/2)$, $(-1/2,-1/2)$, is represented in two cases: Acting anywhere within the $\mathcal{D}^{\rm disc.}_{\overline{SL}(4,R)}(1/2,0) \oplus \mathcal{D}^{\rm disc.}_{\overline{SL}(4,R)}(0,1/2)$ representation, e.g. on the $(2,9/2)$ submultiplets, it yields nothing (entirely trivial action). However, when acting within the shaded zone, a non-trivial action becomes possible, connecting submultiplets belonging to the two different irreducible components. As a result, in flat space, the states lying outside of the shaded zone disconnect, in a Dirac-like Lorentz invariant equation [481]. In the presence of a gravitational field, the covariant derivative in (4.3.8) replaces the ordinary derivative in the Dirac-type equation. It involves the connection, contracted with an infinite representation of the $\not{L}^\alpha{}_\beta$, with the 'Union Jack' action described in Fig.4 – and all states become connected.

In the following section, we shall see that the above special-relativistic anholonomic equation is transformed into a covariant equation for a world spinor by means of the application of infinite frames (manibeins).

4.6. Manibeins

In accordance with the principle of equivalence, we construct first the equation for a spinor manifield anholonomically, i.e. in a local frame, and then holonomize it [99].

We start with a manifield, which is an $\mathcal{A}$-deunitarized representation of $SL(4,R)$, but we impose Lorentz invariance only, as for an ordinary finite spinor. We thus have a direct sum of an infinite sequence of spins 1/2, 3/2, etc.. If we use (4.3.1), i.e., a multiplicity-free representation

$$\mathcal{D}^{\rm disc.}_{\overline{SL}(4,R)}(\tfrac{1}{2},0) \oplus \mathcal{D}^{\rm disc.}_{\overline{SL}(4,R)}(0,\tfrac{1}{2}), \tag{4.6.1}$$

then the γ-matrices are just the infinite direct sum of corresponding Bargmann–Wigner γ's for the special-relativistic equation at each level. In curved space, the "minimal coupling" prescription, i.e. replacement in the Lagrangian of the partial derivative by a covariant derivative, will connect the levels within $\Delta J = 2$.

To go over to a world spinor field, we use "manibeins", i.e. tetrad-like infinite[46] components $E^P{}_\Pi(x) := E(e^i{}_\alpha)^P{}_\Pi$ of the local frame, cf. [493]. We denote by $\Psi^P(x)$ the P-component of the holonomic manifield, carrying a realization of $\mathrm{Diff}(4, R)$. According to the second theorem of above, this is no more a multiplicity-free representation. This is not surprizing, considering that the manibein $E^P{}_\Pi(x)$ represents the translational part of the gravitational field.

In the local (anholonomic) frame, the components $\Psi^\Pi(x)$ of the discrete series representation of above correspond to its reduction over the $\mathcal{A}$-deunitarized (Lorentz) subgroup $\overline{SO}(4)$. We have

$$\Psi^P(x) = E^P{}_\Pi(x)\Psi^\Pi(x), \qquad P, \Pi = 1, \ldots, \infty. \tag{4.6.2}$$

The $E^P{}_\Pi(x)$ and their inverses, the components $H_P{}^\Pi(x)$ of the coframe, are thus infinite matrices with the infinitesimal transformation properties

$$(\mathcal{L}_\varepsilon + \delta_{\tilde{\omega}})E^P{}_\Pi(x) = -\omega_\alpha{}^\beta(x)(\mathcal{E}^\alpha{}_\beta)_\Pi{}^\Theta E^P{}_\Theta(x) + (D_i\varepsilon^j + \varepsilon^k T_{ki}{}^j)(\mathcal{E}^i{}_j)_N{}^P E^N{}_\Pi(x), \tag{4.6.3}$$

where ε is the generator of a one-parameter subgroup of $\overline{\mathrm{Diff}}(4, R)$ and $\mathcal{E}^\alpha{}_\beta$ the generators and $\omega_\alpha{}^\beta$ the infinitesimal parameters of the $SL(4, R)$, cf. (3.6.10).

The conventional transition between holonomic and anholonomic indices for tensors is mediated by the vielbein coefficients $e_i{}^\alpha$ such that $g_{ij} = e_i{}^\alpha e_j{}^\beta g_{\alpha\beta}$. Denoting by X^0 the constant γ^0-like matrix in the X^i set, we can form the Dirac-type adjoint $\overline{\Psi} := \Psi^\dagger X^0$ and find similarly from the scalar product of world spinors,

$$\overline{\Psi}(x)\Psi(x) := \Psi^{\dagger M}(x)H_M{}^\Xi H_N{}^\Pi(X^0)_{\Xi\Pi}\Psi^N(x) =: \Psi^{\dagger M}(x)G_{MN}(x)\Psi^N(x), \tag{4.6.4}$$

where the symmetric infinite-component tensor $G_{MN}(x)$ is a functional of the gravitational field realizing the metric g_{ij} on the world spinor components. After spontaneous symmetry breakdown to Riemann–Cartan or even to Riemannian spacetime, as discussed in section 6.4, we expect to get, in accordance with the principle of equivalence, a "weak" equation for "low energy states":

$$\langle \text{l.e.s.}|D_i G_{MN}(x)|\text{l.e.s.}\rangle \cong 0. \tag{4.6.5}$$

In writing the Ψ-Lagrangian, we thus have to use coframes. In this case, the coframes have their components' labels ranging over a countable infinity. Such coframes are matrix-valued translational potentials, relating $\overline{GL}(4, R)_A$ to $\overline{GL}(4, R)_H$ [99]. For bosonic manifields, the double covering is collapsed and the coframe can then be given [490,491] in terms of the conventional coframes ϑ^β by

$$\theta^\Xi = C^\Xi{}_\beta\vartheta^\beta = H_J{}^\Xi G^J{}_i\,dx^i \quad \begin{cases} H_J{}^\Xi = C^\Xi{}_\beta e_j{}^\beta G^j{}_J, \\ E^I{}_\Pi = G^I{}_i e^i{}_\alpha C^\alpha{}_\Pi, \end{cases} \tag{4.6.6}$$

where the $C^\Xi{}_\beta$ and $G^J{}_i$ are rectangular $n \times \infty$ transition matrices of $SL(2, C)$ and $SL(4, R)$, respectively, connecting the $\underline{4}$ (i-index) and $(1/2, 1/2)$ (β-index) to the infinite-dimensional representations

[46] Instead of tetrads we now have "$\aleph_0$-ads".

(Ξ and J indices, respectively). The $C^{\Xi}{}_{\beta}$ consist of a reduced infinite sum of rectangular matrices. They relate, within one single $\mathcal{A}$-deunitarized representation of $\overline{SL}(4,R)_A$, the A, B labels of the finite (non-unitary) representations of $SL(2,C)$ – replacing here the $\overline{SO}(4)$ compact subgroup representations in the $\mathcal{A}$-deunitarized representation of $\overline{SO}(4,R)_A$ itself – to the four-dimensional α, β indices of the local Lorentz group, also saturating a four-dimensional representation of $SL(4,R)_A$. The $G^J{}_i$ relate the four-dimensional i, j indices of $\overline{SL}(4,R)_H$ to the infinite-dimensional I, J indices of the $\mathcal{A}$-deunitarized representation of that group.

Having followed the transition from the anholonomic to the holonomic equation, we can understand how the manibein frames transform a representation of the discrete series, which is multiplicity-free, into a representation with multiplicity, presumably a reducible sum, following a Clebsch–Gordan expansion.

However, if we are interested in a holonomic equation, we can, following Mickelsson [440], also start out directly with an appropriate non-multiplicity-free pair of parity-conjugate representations.

4.7. Conformal properties of world spinor densities

In a metric-affine spacetime, the exterior covariant derivative for the manifield takes the general form of (4.3.8). In the context of Lagrangians, the manifield as a $SL(4,R)$ representation has to be lifted to a $GL(4,R)$ representation and thus to be regarded as a density $\hat{\Psi}$ of appropriate dimension d_Ψ. Thus, following Schouten [604], it is more natural to consider (world) spinor *densities*, cf. section A.1.1, of a suitable *dimension* d_Ψ. Then (4.3.8) converts, according to the prescription (3.5.7), into

$$D\hat{\Psi}^{\Xi}(x) = [D\hat{\Psi}^{\Xi}(x)]_{\text{manif.}} - \omega_\Psi \Gamma \delta^{\Xi}_{\Pi}\, \hat{\Psi}^{\Pi}(x). \tag{4.7.1}$$

Here the first piece on the right hand side denotes the covariant exterior manifield derivative as if the Ψ did not carry a hat.

We adopt the convention that a (mani-)field density $\hat{\Psi}(x)$ transforms under a conformal change $\tilde{g} = \Omega^L g$ of the underlying metric structure, cf. section 3.14, according to

$$\hat{\Psi}(x) \to \tilde{\hat{\Psi}}(x) = \Omega(x)^{d_\Psi L/2}\hat{\Psi}(x) \qquad \begin{cases} d_\eta := n & \text{volume density } \eta, \\ d_0 := -\frac{1}{2}(n-2) & \text{scalar field } \sigma, \\ d_{1/2} := -\frac{1}{2}(n-1) & \text{spinor field } \psi. \end{cases} \tag{4.7.2}$$

In $n = 4$ dimensions, the dimension of a scalar field is given by $d_0 = -1$, whereas $d_{1/2} = -3/2$ holds for a conventional Dirac spinor. For the weight $L = 2$ of the metric, this corresponds to the *canonical* (physical) dimensions which, for a scalar field, e.g., is (length)$^{-1}$. For a world spinor density we are assuming that it transforms as

$$\tilde{\hat{\Psi}}^{\Xi}(x) = \Omega^{-(n-1)L/4}\hat{\Psi}^{\Xi}(x), \tag{4.7.3}$$

which, for $L = 2$ and $n = 4$, is in conformity with the canonical dimension $d = -3/2$ of conventional Dirac fields, cf. [419,733]. Then the trace of the connection acts as a compensating potential in the manifield derivative (4.7.1) for densities, which suffices to ensure conformal covariance. In a (L_n, g), according to (3.14.1), the exterior covariant derivative (4.7.1) of a world spinor density (cf. [726] for the Riemannian case) transforms under conformal changes as follows (cf. also [145,515,528]):

$$\widetilde{D}\widetilde{\hat{\Psi}}^{\Xi}(x) = \Omega^{-(n-1)L/4}\{D + [(n\omega_{\Psi} - 1)C - \tfrac{1}{4}(n-1)L]d\ln\Omega\}\hat{\Psi}^{\Xi}(x). \tag{4.7.4}$$

Consequently, conformal covariance of the exterior $GL(n,R)$-gauge covariant derivative can be achieved:

$$\widetilde{D}\widetilde{\hat{\Psi}}^{\Xi}(x) = \Omega^{-(n-1)L/4}\, D\hat{\Psi}^{\Xi}(x), \quad \omega_{\Psi} = (1/n)[1 + \tfrac{1}{4}(n-1)L/C]. \tag{4.7.5}$$

Let us consider a Dirac-type Lagrangian L_{Ψ} and take into account the definition

$$X(x)^{\Xi}{}_{\Pi} := (X_{\alpha}\vartheta^{\alpha})^{\Xi}{}_{\Pi} \quad \Rightarrow \quad \widetilde{X}(x)^{\Xi}{}_{\Pi} = \Omega^{F}X(x)^{\Xi}{}_{\Pi} \tag{4.7.6}$$

of the vielbein-deformed γ-type matrices X_{α}, together with the conformal property (3.14.1) of the coframe ϑ^{α}. Moreover, we employ the transformation formula (3.14.13) for the Hodge dual of a p-form. Taken together, this implies that the Dirac-type hermitian Lagrangian

$$L_{\hat{\Psi}} = \tfrac{1}{2}i[\overline{\hat{\Psi}_{\Xi}}(x)X(x)^{\Xi}{}_{\Pi} \wedge {}^{*}D\hat{\Psi}^{\Pi}(x) - \overline{D\hat{\Psi}_{\Xi}}(x) \wedge {}^{*}X(x)^{\Xi}{}_{\Pi}\,\hat{\Psi}^{\Pi}(x)] \tag{4.7.7}$$

for massless world spinor densities is conformally invariant in any dimension,

$$\widetilde{L}_{\Psi} = \Omega^{F-L/2}L_{\Psi} = L_{\Psi}, \tag{4.7.8}$$

provided the conformal weight of the coframe is given by $F = L/2$, which, for $L = 2$, is the canonical value.

Compared to a conventional Dirac field carrying only spin, the hypothetical manifields, in addition, supply a shear and a dilational current, which will become the source of a symmetric connection field.

These world spinors cannot be equipped with a rest mass[47] in a conformally invariant manner. However, we may introduce a Yukawa-type coupling to the dilaton field σ such that the resulting interaction Lagrangian is conformally invariant in $n = 4$ dimensions:

$$L_{\sigma\Psi} = -\sigma\overline{\hat{\Psi}^{\Xi}}(x)\,\hat{\Psi}_{\Xi}(x)\eta, \qquad \widetilde{L}_{\sigma\Psi} = \Omega^{-(n-4)L/4}L_{\sigma\Psi}. \tag{4.7.9}$$

The manifield equation, resulting from the combined Lagrangian (4.7.7) and (4.7.9) by variation with respect to $\hat{\Psi}$, reads (cf. [447,282, p.114] for the Riemann–Cartan case):

$$i[X \wedge {}^{*}D + \tfrac{1}{2}(D\,{}^{*}X) + i\sigma\eta]\hat{\Psi} = 0. \tag{4.7.10}$$

A constant mass term $m \approx \langle\sigma\rangle$ for world spinors can be induced as a result of the breaking of the local scale symmetry by means of a non-trivial vacuum expectation value $\langle\sigma\rangle = \sqrt{\chi}/l_0$ of the dilaton field σ, see sections 6.4 and 6.5.

In a model of Audretsch et al. [18], which was devised in order to understand the relation between a Weylian and a Riemannian spacetime, a condition $Dm(x) = 0$ for a so-called mass function $m(x)$ is necessary and sufficient for the existence of geodesic particle trajectories in the WKB-limit. The manifield mass of our model, however, will be induced by a breaking of the dilation symmetry, see section 6.4. Thereby the corresponding condition $D\sigma = 0$ should surface as a requirement for the

[47] To us, a mildly spacetime dependent mass $m = m(x)$ à la Hayashi et al. [261] and Audretsch et al. [18] seems to be an inappropriate device, since, according to Coleman [124], "...scale transformations ... do not change numerical parameters – that is to say, they stay within a given physical theory".

groundstate. As it turns out, this implies the vanishing of the dilation current and, accordingly, the emergence of a preferred pseudo-Riemannian structure as well.

5. Lagrange–Noether machinery

5.1. The external currents of a matter field

The external currents of a matter field are those currents which are related to spacetime symmetries. On a fundamental level, we adopt the view that tangible matter is described in terms of infinite-dimensional spinor[48] or tensor representations of $SL(4,R)$, the *manifields* Ψ of the preceeding section.

In a first order formalism we assume that the material Lagrangian n-form for these manifields depends most generally on Ψ, $d\Psi$, and the potentials $g_{\alpha\beta}$, ϑ^α, $\Gamma_\alpha{}^\beta$. According to the *minimal coupling* prescription, derivatives of these potentials are not permitted. We usually adhere to this principle. However, *Pauli type terms* such as $R_\alpha{}^\beta \bar{\Psi} L^\alpha{}_\beta {}^*(\sigma\Psi)$, where $\sigma := (i/2) X_\mu X_\nu \vartheta^\mu \wedge \vartheta^\nu$, and the Jordan–Brans–Dicke type term $|\Phi|\, R^{\alpha\beta} \wedge \eta_{\alpha\beta}$ may occur in phenomenological models or in the context of symmetry breaking. Therefore, we develop our Lagrangian formalism in sufficient generality in order to cope with such models by including in the Lagrangian also the derivatives $dg_{\alpha\beta}$, $d\vartheta^\alpha$, and $d\Gamma_\alpha{}^\beta$ of the gravitational potentials:

$$L = L(g_{\alpha\beta}, dg_{\alpha\beta}, \vartheta^\alpha, d\vartheta^\alpha, \Gamma_\alpha{}^\beta, d\Gamma_\alpha{}^\beta, \Psi, d\Psi)\,. \tag{5.1.1}$$

As a further bonus, we can then also read off the Noether identities for the gravitational gauge fields in $n \geq 2$ dimensions.

The requirement of invariance under the local affine group $\mathcal{A}(n,R)$, as we will discuss at the beginning of section 5.2 below, means that the Lagrangian L should be invariant under both, linear gauge transformations of the frame and (active) diffeomorphisms. Invariance under frame transformations leads immediately to the result that L depends on $\Gamma_\alpha{}^\beta$ only via the exterior *covariant* derivative D (or via the nonmetricity $Q_{\alpha\beta}$, torsion T^α, and curvature $R_\alpha{}^\beta$ in Pauli type terms). To see this, one simply has to note that we can always choose the coframe field ϑ^α such that $\Gamma_\alpha{}^\beta \stackrel{*}{=} 0$ at a given event P, in which case

$$L(g_{\alpha\beta}, dg_{\alpha\beta}, \vartheta^\alpha, d\vartheta^\alpha, \Gamma_\alpha{}^\beta, d\Gamma_\alpha{}^\beta, \Psi, d\Psi) \stackrel{*}{=} L(g_{\alpha\beta}, Q_{\alpha\beta}, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta, \Psi, D\Psi) \tag{5.1.2}$$

at P. Now, since the right-hand side of (5.1.2) is a scalar-valued n-form constructed from tensorial and spinorial manifields, it is invariant under linear frame transformations. Also the left-hand side is similarly invariant by hypothesis. It follows that, at an event P, Eq. (5.1.2) holds in general for an arbitrary frame field. Applying the same argument at every event P of spacetime, the result (5.1.2) follows quite generally. Here the $GL(n,R)$-gauge covariant exterior derivative is given by

$$D\Psi^\Xi(x) = [\delta^\Xi_\Pi d + \Gamma_\alpha{}^\beta (L^\alpha{}_\beta)_\Pi{}^\Xi]\Psi^\Pi(x)\,. \tag{5.1.3}$$

[48] In treating the matter Lagrangian L, the covering groups of $GL(n,R)$, $SL(n,R)$, and $A(n,R)$ or $SA(n,R)$ occur and thus should be overlined, since the matter fields are spinor manifields. Only for the gauge Lagrangian V, it can remain non-overlined. For convenience, we often drop the overline, if it is clear from the context anyways.

where $(L^\alpha{}_\beta)_\Pi{}^\Xi$ denotes a lifting of the $GL(n,R)$ generators to the manifield representation.

Independent variations of $g_{\alpha\beta}$, $Q_{\alpha\beta}$, ϑ^α, T^α, $R_\alpha{}^\beta$, Ψ, and $D\Psi$ yield

$$\delta L = \delta g_{\alpha\beta}\frac{\partial L}{\partial g_{\alpha\beta}} + \delta Q_{\alpha\beta}\wedge\frac{\partial L}{\partial Q_{\alpha\beta}} + \delta\vartheta^\alpha\wedge\frac{\partial L}{\partial\vartheta^\alpha} + \delta T^\alpha\wedge\frac{\partial L}{\partial T^\alpha} + \delta R_\alpha{}^\beta\wedge\frac{\partial L}{\partial R_\alpha{}^\beta}$$
$$+\delta\Psi\wedge\frac{\partial L}{\partial\Psi} + \delta(D\Psi)\wedge\frac{\partial L}{\partial(D\Psi)}, \tag{5.1.4}$$

where the partial derivatives are implicitly *defined* by (5.1.4). Note that for a tensor-valued p-form $\Phi_{\alpha\beta}$ which is symmetric or antisymmetric in α and β, expressions like $\delta\Phi_{\alpha\beta}\wedge(\partial L/\partial\Phi_{\alpha\beta})$ are only formal. In order to avoid counting the nondiagonal components twice in the variation procedure, a *strict ordering* of the indices is necessary. This is to be understood in the first two terms of (5.1.4). Since we assume that the variation δ and the exterior derivative d commute, i.e. $[\delta, d] = 0$, we can transform the variations with respect to $Q_{\alpha\beta}$, T^α, and $R_\alpha{}^\beta$ defined by (3.8.1), (3.5.9), and (3.5.10), via "partial integration", into variations with respect to the original variables $g_{\alpha\beta}$, ϑ^α, and $\Gamma_\alpha{}^\beta$. We find

$$\delta L = \tfrac{1}{2}\delta g_{\alpha\beta}\sigma^{\alpha\beta} + \delta\vartheta^\alpha\wedge\Sigma_\alpha + \delta\Gamma_\alpha{}^\beta\wedge\Delta^\alpha{}_\beta + \delta\Psi\wedge(\delta L/\delta\Psi) \tag{5.1.5}$$
$$+ d\left(-\delta g_{\alpha\beta}\wedge\frac{\partial L}{\partial Q_{\alpha\beta}} + \delta\vartheta^\alpha\wedge\frac{\partial L}{\partial T^\alpha} + \delta\Gamma_\alpha{}^\beta\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} + \delta\Psi\wedge\frac{\partial L}{\partial D\Psi}\right),$$

where the variational derivative

$$\frac{\delta L}{\delta\Psi} = \frac{\partial L}{\partial\Psi} - (-1)^p D\frac{\partial L}{\partial(D\Psi)}, \tag{5.1.6}$$

for a gauge-invariant Lagrangian L, becomes identically the $GL(n,R)$-covariant *variational derivative* of L with respect to the p-form Ψ. The *matter currents* in (5.1.5) are given by[49]

$$\sigma^{\alpha\beta} := 2\frac{\delta L}{\delta g_{\alpha\beta}} = 2\frac{\partial L}{\partial g_{\alpha\beta}} + 2D\frac{\partial L}{\partial Q_{\alpha\beta}}, \tag{5.1.7}$$

$$\Sigma_\alpha := \frac{\delta L}{\delta\vartheta^\alpha} = \frac{\partial L}{\partial\vartheta^\alpha} + D\frac{\partial L}{\partial T^\alpha}, \tag{5.1.8}$$

$$\Delta^\alpha{}_\beta := \frac{\delta L}{\delta\Gamma_\alpha{}^\beta} = (L^\alpha{}_\beta\Psi)\wedge\frac{\partial L}{\partial(D\Psi)} + 2g_{\beta\gamma}\frac{\partial L}{\partial Q_{\alpha\gamma}} + \vartheta^\alpha\wedge\frac{\partial L}{\partial T^\beta} + D\frac{\partial L}{\partial R_\alpha{}^\beta}. \tag{5.1.9}$$

Since $D\Gamma_\alpha{}^\beta$ is not defined, the variational derivative $\delta L/\delta\Gamma_\alpha{}^\beta$ in (5.1.9) cannot be understood according to the covariant version of (5.1.6). Rather, it is just an abbreviation of the right-hand side of (5.1.9), which is the outcome of the transition from (5.1.4) to (5.1.5). The last term on the right hand side of (5.1.5) is an exact form which does not contribute to the action integral because of the

[49] In some textbooks, see Thirring and his associates [674,704], the matter currents are defined with the Hodge star introduced on the left-hand sides of these equations, such as ${}^*\tilde\Sigma_\alpha := \cdots$. Then the currents are one-forms by definition. The price one has to pay for this "convenience" is that the matter currents $\tilde\Sigma_\alpha = {}^*(\cdots)$ etc., via the Hodge-star, become *explicitly* contaminated with the gravitational potential $g_{\alpha\beta}$. There is, however, no need for such a convention. We try to keep matter and gravitation cleanly separated in order to get insight into their structural interdependence.

usual assumption that $\delta g_{\alpha\beta} = 0$, $\delta\vartheta^\alpha = 0$, $\delta\Gamma_\alpha{}^\beta = 0$, and $\delta\Psi = 0$ on the boundary ∂M of the domain M of integration.

A classical convective model of the matter currents (5.1.7), (5.1.8), (5.1.9) has been developed by Obukhov and Tresguerres [532] following the pattern of the Weyssenhoff ansatz.

5.1.1. Energy-momentum

The n-form $\sigma^{\alpha\beta}$ and the $(n-1)$-form Σ_α are the *metrical* (Hilbert) and the *canonical* (Noether) *energy-momentum currents*, respectively. These currents occur, though in a more restricted sense, also in GR. In Lagrangian field theory, a current is fundamentally an $(n-1)$-form (some quantity spread over a hypersurface) and not an n-form such as the metric energy-momentum "current" given by (5.1.7). This suggests that we consider the equivalent $(n-1)$-form

$$\sigma_\beta := e_\alpha\rfloor\sigma^\alpha{}_\beta, \quad e_\beta\rfloor\sigma^\beta \equiv 0. \tag{5.1.10}$$

Later, when we will have the Noether theorems at our disposal, we will relate the symmetric σ_α, via a Belinfante–Rosenfeld type formula, to the canonical current Σ_α.

From the canonical energy-momentum current we can extract its *trace* $\vartheta^\alpha\wedge\Sigma_\alpha$ with one independent component [50] according to

$$\not{\Sigma}_\alpha := \Sigma_\alpha - (1/n)e_\alpha\rfloor(\vartheta^\gamma\wedge\Sigma_\gamma) \tag{5.1.11}$$

such that $\not{\Sigma}_\alpha$ is traceless:

$$\vartheta^\alpha\wedge\not{\Sigma}_\alpha = 0. \tag{5.1.12}$$

The *antisymmetric* piece $\vartheta_{[\alpha}\wedge\Sigma_{\beta]}$ has $n(n-1)/2$ independent components, exactly as the $(n-2)$-form

$$\Sigma := g^{\alpha\beta}e_\alpha\rfloor\Sigma_\beta = e_\alpha\rfloor\Sigma^\alpha. \tag{5.1.13}$$

Via some contractions, we find

$$\vartheta_{[\alpha}\wedge\Sigma_{\beta]} = \tfrac{1}{2}\vartheta_\alpha\wedge\vartheta_\beta\wedge\Sigma. \tag{5.1.14}$$

Consequently, the irreducible decomposition of the canonical energy-momentum $(n-1)$-form Σ_α into a symmetric tracefree, trace, and antisymmetric piece reads

$$\Sigma_\alpha = \overset{\frown}{\not{\Sigma}}_\alpha + (1/n)e_\alpha\rfloor(\vartheta^\gamma\wedge\Sigma_\gamma) + \tfrac{1}{2}\vartheta_\alpha\wedge\Sigma, \tag{5.1.15}$$

where this equation can be understood as defining the symmetric tracefree piece $\overset{\frown}{\not{\Sigma}}_\alpha$ with its $(n-1)(n+2)/2$ components. For the symmetric piece

$$\overset{\frown}{\Sigma}_\alpha = \Sigma_\alpha - \tfrac{1}{2}\vartheta_\alpha\wedge\Sigma, \tag{5.1.16}$$

[50] It is called the trace because in terms of the components $\mathcal{T}^\beta{}_\alpha$ of the conventionally defined energy–momentum tensor density, with $\Sigma_\alpha = \mathcal{T}^\beta{}_\alpha\epsilon_\beta$, we have $\vartheta^\alpha\wedge\Sigma_\alpha = \vartheta^\alpha\wedge\mathcal{T}^\beta{}_\alpha\epsilon_\beta = \mathcal{T}^\alpha{}_\alpha\epsilon$. Here ϵ is the (metric-independent) Levi-Civita four-form density and $\epsilon_\alpha = e_\alpha\rfloor\epsilon$, see (A.1.18) and (A.1.19). Analogously, the expression $g_{\gamma[\alpha}\vartheta^\gamma\wedge\Sigma_{\beta]} = \vartheta_{[\alpha}\wedge\Sigma_{\beta]} = \mathcal{T}_{[\alpha\beta]}\epsilon$ represents the antisymmetric piece.

we find

$$e_\alpha \rfloor \widehat{\Sigma}^\alpha = 0, \qquad \vartheta^\alpha \wedge \widehat{\Sigma}_\alpha = 0, \qquad e_\alpha \rfloor \widehat{\Sigma}^\alpha = 0. \tag{5.1.17}$$

Moreover, in analogy to (5.1.14), we have

$$\vartheta_{(\alpha} \wedge \Sigma_{\beta)} = \vartheta_{(\alpha} \wedge \widehat{\Sigma}_{\beta)} + (1/n) g_{\alpha\beta} (\vartheta^\gamma \wedge \Sigma_\gamma). \tag{5.1.18}$$

5.1.2. Hypermomentum

A rather new concept [277,278] is that of the *hypermomentum current* which is given by the $(n-1)$-form $\Delta^\alpha{}_\beta$. According to the direct product structure $GL(n,R) = [T \otimes SL(n,R)] \times R^+$ of the general linear group, its trace

$$\Delta := \Delta^\gamma{}_\gamma = \delta L / \delta \Gamma_\gamma{}^\gamma, \tag{5.1.19}$$

the *dilation current*, can be split off, leaving over the traceless hypermomentum current

$$\not{\Delta}^\alpha{}_\beta := \Delta^\alpha{}_\beta - (1/n) \delta^\alpha_\beta \Delta. \tag{5.1.20}$$

This separation is somewhat reminiscent of the decomposition (3.12.2) of the connection into volume-changing and volume-preserving pieces.

In an (L_n, g), a metric is available which allows us to lower the index α and to split (5.1.9) into a symmetric and an antisymmetric piece. Then we arrive at the decomposition

$$\begin{aligned} \Delta_{\alpha\beta} \quad &= \tau_{\alpha\beta} \qquad\quad + (1/n) g_{\alpha\beta} \Delta \quad + \widehat{\not{\Delta}}_{\alpha\beta} \\ &\sim \text{spin current} \oplus \text{dilation current} \oplus \text{shear current}, \end{aligned} \tag{5.1.21}$$

where

$$\tau_{\alpha\beta} := \Delta_{[\alpha\beta]} = \vartheta_{[\alpha} \wedge \mu_{\beta]} \tag{5.1.22}$$

is the (dynamical) *spin current* and

$$\widehat{\not{\Delta}}_{\alpha\beta} := \Delta_{(\alpha\beta)} - (1/n) g_{\alpha\beta} \Delta \tag{5.1.23}$$

the symmetric and tracefree *shear current*. Since $\tau_{\alpha\beta} = -\tau_{\beta\alpha}$ is an $(n-1)$-form, it can be equivalently expressed in terms of a vector-valued $(n-2)$-form μ_α, as displayed in (5.1.22). The explicit form of the spin energy potential[51] $(n-2)$-form μ_α, according to (A.1.26), reads

$$\mu_\alpha = -2 e_\beta \rfloor \tau_\alpha{}^\beta + \tfrac{1}{2} \vartheta_\alpha \wedge (e_\beta \rfloor e_\gamma \rfloor \tau^{\beta\gamma}). \tag{5.1.24}$$

The symmetric piece of (5.1.21), namely

$$\widehat{\Delta}_{\alpha\beta} := \Delta_{(\alpha\beta)} = (1/n) g_{\alpha\beta} \Delta + \widehat{\not{\Delta}}_{\alpha\beta}, \tag{5.1.25}$$

we will call the *strain current*.

[51] In comparison to [366], our conventions with respect to μ differ by a factor of -2. Incidentically, in some earlier articles [454,288,280] we have erroneously claimed that the trace $e_\alpha \rfloor \mu^\alpha$ of the spin energy potential vanishes.

Observe that the decomposition (5.1.21) is only irreducible with respect to the Lie-algebra indices α and β. In contrast to the decomposition (5.1.21), which will turn out to be useful in section 5.3, a further decomposition of the components $\Delta_{\mu_1\cdots\mu_{n-1}\alpha\beta}$ of $\Delta_{\alpha\beta} = (1/(n-1)!)\Delta_{\mu_1\cdots\mu_{n-1}\alpha\beta}\vartheta^{\mu_1}\wedge\cdots\wedge\vartheta^{\mu_{n-1}}$, in analogy to section B.6, has found no natural application so far.

The dynamical spin $\tau_{\alpha\beta}$ is a source term which, besides the energy–momentum current Σ_α, is crucial in the Poincaré gauge theory of gravity [272,273,434,447], whereas the dilation current Δ is an essential ingredient of conformal models of gravity based on a Weyl geometry [287,288] and familiar from canonical field theory [124,404]. For gravity with its Planck scale, only the shear $\widehat{\Delta}_{\alpha\beta}$ seems to be more remote from *direct* physical experience [52]. In the manifield approach of Ne'eman et al. [284,480,490,491], it manifests itself indirectly by the occurrence of states within the infinite-dimensional representations which are lying along Regge trajectories. The shear current has an analogue in the three-dimensional continua with microstructure, see [246,267,462].

The field equation for the manifields Ψ is given by the familiar Euler–Lagrange equation

$$\delta L/\delta\Psi = 0. \tag{5.1.26}$$

If it is assumed to be fulfilled in the derivation of identities, we call them *weak* identities in the following ("on shell" in the parlance of particle physicists).

5.2. Noether identities for material energy–momentum and hypermomentum

According to the Noether theorem, the conservation identities of the matter system result from the postulated invariance of L under a local symmetry group, here the group of local affine transformations. Actually, this is only true "weakly", i.e., provided the Euler–Lagrange equation (5.1.26) for the manifield is satisfied. A direct gauging of the affine group $A(n,R) := R^n \rtimes GL(n,R)$ would yield a Yang–Mills type "internal" interpretation of local spacetime symmetries, as explained in section 3.2, and would lead to one (combined) Noether identity within a "motor calculus". Here we follow the more conventional route and consider the reduced or "parallel-transport" version of affine gauge transformations, cf. sections 3.3 and 3.6, in which the infinite-dimensional group $\mathcal{T} := C^\infty(A(M)\times_{Ad} R^n)$ of local translations is replaced by the group of Diff(n,R) of diffeomorphisms on the spacetime manifold M_n itself. Then invariance of L under the group of local affine transformations means that L is invariant under (i) diffeomorphisms on M_n, and (ii) $\mathcal{G}L(n,R)$ deformations of the frame field according to (3.4.5). First we consider the invariance of L under ...

5.2.1. Diffeomorphisms

Let ξ be the generator of an arbitrary one-parameter group $\mathcal{T}_t$ of diffeomorphisms. In order to obtain a *covariant* Noether identity from invariance of L under a one-parameter group of *local* translations $\mathcal{T}_t \subset \mathcal{T} \approx$ Diff(n,R), it would be sufficient to use the conventional Lie derivative $l_\xi := \xi\rfloor d + d\xi\rfloor$ on M with respect to an arbitrary vector field ξ (cf. [363]). Since our Lagrangian L is also required to

[52] On a much higher length scale, however, back in 1965 already Dothan et al. [166] showed that the shears represent (orbitally) the *time-derivatives of "gravitational" quadrupoles*, cf. [291]. For a fixed volume, e.g. a nucleus, we obtain the very physical picture of pulsation changing it from looking like a pear to looking like a cigar (these are the "deformed nuclei", cf. [579]). The time-derivatives mean that we are considering the *pulsation rates* for these quadrupoles, i.e., for the various bulges or departures from spherical shape.

be a *scalar* under $GL(n,R)$-gauge transformations, we can equivalently replace l_ξ by the $GL(n,R)$-covariant Lie derivative $\text{Ł}_\xi := \xi\rfloor D + D\xi\rfloor$ (cf. [479,488,504,506,510,674]). Then we will obtain directly a $GL(n,R)$*-gauge covariant* Noether identity by substituting Ł_ξ into (5.1.4):

$$\begin{aligned} l_\xi L = \text{Ł}_\xi L = (\text{Ł}_\xi g_{\alpha\beta})\frac{\partial L}{\partial g_{\alpha\beta}} &+ (\text{Ł}_\xi Q_{\alpha\beta})\wedge\frac{\partial L}{\partial Q_{\alpha\beta}} \\ &+ (\text{Ł}_\xi\vartheta^\alpha)\wedge\frac{\partial L}{\partial\vartheta^\alpha} + (\text{Ł}_\xi T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\text{Ł}_\xi R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} \\ &+ (\text{Ł}_\xi\Psi)\wedge\frac{\partial L}{\partial\Psi} + (\text{Ł}_\xi D\Psi)\wedge\frac{\partial L}{\partial D\Psi}. \end{aligned} \tag{5.2.1}$$

Recall that $\xi\rfloor$, which formally acts analogously to a derivative of degree -1, obeys a Leibniz rule. Since the Lagrangian L is an n-form, its Lie derivative reduces to $\text{Ł}_\xi L = D\xi\rfloor L$. After expanding the Lie derivatives and performing some "partial integrations", we equivalently get

$$\begin{aligned} D\xi\rfloor L = D\Bigg(&(\xi\rfloor Q_{\alpha\beta})\frac{\partial L}{\partial Q_{\alpha\beta}} + (\xi\rfloor\vartheta^\alpha)\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} \\ &+ (\xi\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} + (\xi\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi}\Bigg) \\ &- (\xi\rfloor Q_{\alpha\beta})\frac{\partial L}{\partial g_{\alpha\beta}} - (\xi\rfloor Q_{\alpha\beta})D\frac{\partial L}{\partial Q_{\alpha\beta}} + (\xi\rfloor R_\beta{}^\gamma)\wedge 2g_{\gamma\alpha}\frac{\partial L}{\partial Q_{\alpha\beta}} \\ &- (\xi\rfloor\vartheta^\alpha)D\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor\vartheta^\alpha)R_\alpha{}^\beta\wedge\frac{\partial L}{\partial T^\beta} + (\xi\rfloor T^\alpha)\wedge D\frac{\partial L}{\partial T^\alpha} \\ &+ (\xi\rfloor R_\beta{}^\gamma)\wedge\vartheta^\beta\wedge\frac{\partial L}{\partial T^\gamma} + (\xi\rfloor R_\beta{}^\gamma)\wedge D\frac{\partial L}{\partial R_\beta{}^\gamma} \\ &+ (\xi\rfloor R_\beta{}^\gamma)\wedge L^\beta{}_\gamma\Psi\wedge\frac{\partial L}{\partial D\Psi} + (\xi\rfloor D\Psi)\wedge\frac{\delta L}{\delta\Psi} + (-1)^p(\xi\rfloor\Psi)\wedge D\frac{\delta L}{\delta\Psi}. \end{aligned} \tag{5.2.2}$$

By collecting those terms which form variational derivatives, we obtain, following Kopczyński [363,364],

$$A + dB = 0, \tag{5.2.3}$$

where

$$\begin{aligned} A := &-(\xi\rfloor Q_{\alpha\beta})\frac{\delta L}{\delta g_{\alpha\beta}} - (\xi\rfloor\vartheta^\alpha)D\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor R_\beta{}^\gamma)\wedge\frac{\delta L}{\delta\Gamma_\beta{}^\gamma} \\ &+ (\xi\rfloor D\Psi)\wedge\frac{\delta L}{\delta\Psi} + (-1)^p(\xi\rfloor\Psi)\wedge D\frac{\delta L}{\delta\Psi}, \\ B := \xi\rfloor L - \Big(&(\xi\rfloor Q_{\alpha\beta})\frac{\partial L}{\partial Q_{\alpha\beta}} + (\xi\rfloor\vartheta^\alpha)\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} \\ &+ (\xi\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} + (\xi\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi}\Big). \end{aligned} \tag{5.2.4}$$

The functions A and B have the form

$$A = \xi^\alpha A_\alpha, \qquad B = \xi^\alpha B_\alpha. \tag{5.2.5}$$

and hence, by (5.2.3),

$$\xi^\alpha (A_\alpha + dB_\alpha) + d\xi^\alpha \wedge B_\alpha = 0, \tag{5.2.6}$$

where *both* ξ^α *and* $d\xi^\alpha$ are *pointwise arbitrary*. Hence we can conclude that both B_α and A_α vanish so that

$$A = 0, \qquad B = 0. \tag{5.2.7}$$

From $B = 0$ we can read off the identity

$$\begin{aligned}\xi\rfloor L = (\xi\rfloor Q_{\alpha\beta})\frac{\partial L}{\partial Q_{\alpha\beta}} + (\xi\rfloor\vartheta^\alpha)\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} \\ + (\xi\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} + (\xi\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi}.\end{aligned} \tag{5.2.8}$$

After replacing the vector field by the vector basis, $\xi \to e_\alpha$, Eq. (5.2.8) yields directly the explicit form of the *canonical energy–momentum current*

$$\begin{aligned}\Sigma_\alpha = e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi)\wedge\frac{\partial L}{\partial D\Psi} - (e_\alpha\rfloor\Psi)\wedge\frac{\partial L}{\partial\Psi} \\ - (e_\alpha\rfloor Q_{\beta\gamma})\frac{\partial L}{\partial Q_{\beta\gamma}} - (e_\alpha\rfloor T^\beta)\wedge\frac{\partial L}{\partial T^\beta} + D\frac{\partial L}{\partial T^\alpha} - (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\frac{\partial L}{\partial R_\beta{}^\gamma}.\end{aligned} \tag{5.2.9}$$

The first line in (5.2.9) represents the result known in the context of *special* relativistic classical field theory. In the Maxwellian case, for example, the Ψ stands for the electromagnetic potential one-form $A = A_i\, dx^i$, with the field strength two-form $F = DA = dA$. Then, by (5.2.9), we find directly Minkowski's $U(1)$-gauge invariant canonical energy–momentum current of the Maxwell field. [53] The last Ψ-dependent term in the first line of (5.2.9) vanishes for a zero-form, as is exemplified by the Dirac field; however, for a Proca field, for example, we get a contribution. Furthermore, our formalism is general enough to account also for a Rarita–Schwinger type spinor-valued one-form field $\hat{\Psi}$. The second line in (5.2.9) accounts for possible Pauli terms as well as for Lagrange multiplier terms in variations with constraints and is absent in the case of minimal coupling.

From $A = 0$, we can read off the *first Noether identity*

$$\begin{aligned}D\Sigma_\alpha &\equiv (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\Delta^\beta{}_\gamma - \tfrac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})\sigma^{\beta\gamma} + W_\alpha \\ &\cong (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\Delta^\beta{}_\gamma - \tfrac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})\sigma^{\beta\gamma}, \qquad \text{(1st)}\end{aligned} \tag{5.2.10}$$

where

$$W_\alpha := (e_\alpha\rfloor D\Psi)\frac{\delta L}{\delta\Psi} + (-1)^p(e_\alpha\rfloor\Psi)\wedge D\frac{\delta L}{\delta\Psi}. \tag{5.2.11}$$

[53] The definition of the canonical energy–momentum tensor of the Maxwell field à la Landau–Lifshitz [386] does *not* yield a gauge invariant quantity, since they pick the *components* A_α of A, that is covector-valued zero-forms, as their field variables. Because of $A = A_\alpha\vartheta^\alpha$, the relation $F = DA = dA = DA_\alpha\wedge\vartheta^\alpha + A_\alpha T^\alpha$, see [364], links the gauge invariant field strength $F = DA = dA$, which is assumed to occur in the Lagrangian, see (5.1.2), to the gauge dependent field DA_α. Therefore the "canonical" Landau–Lifshitz tensor has to be fixed up by an ad hoc procedure.

Our first result in (5.2.10) is given in the *strong* form, where no field equation is invoked.[54]

In the differential identity (5.2.10) for the canonical energy–momentum current there occur, on the right-hand side, Lorentz-type forces of the general structure field strength $\times$ current. The translational force[55] (à la Peach–Koehler, see [281,437]) and the Mathisson–Papapetrou type force are already known from the Poincaré gauge theory [272,275]. The contribution from the nonmetricity is a new feature of the gauge approach to local spacetime deformations; it arises because of the non-vanishing strain piece of the connection. The Lorentz-type contribution (5.2.11) from the matter field Ψ itself is, in quantum field theory, known as the functional differential operator W_α of the Ward-identity for translation invariance, cf. [122].

On the other hand, in a new formulation of the ^{3}He superfluid with a Lagrangian $L = L(\rho, s)$, depending on the particle current $(n-1)$-form ρ and the entropy $(n-1)$-form s, the *material* Lorentz-type forces in the Noether identity (5.2.10) are given by

$$\begin{aligned} W_\alpha = (-1)^{n-1}\left(e_\alpha\rfloor\frac{\delta L}{\delta\rho}\right)d\rho - \rho\wedge e_\alpha\rfloor\left(D\frac{\delta L}{\delta\rho}\right) \\ +(-1)^{n-1}\left(e_\alpha\rfloor\frac{\delta L}{\delta s}\right)ds - s\wedge e_\alpha\rfloor\left(D\frac{\delta L}{\delta s}\right). \end{aligned} \tag{5.2.12}$$

By putting these two forces successively to zero, we recover the force n-form constraints of Carter and Khalatnikov [104]. Accordingly, the keeping of these off-shell terms in (5.2.10) yields a better understanding of classical as well as quantum-field-theoretical results.

5.2.2. *Linear transformations*

The invariance of L with respect to local linear transformations (3.4.5) of the frames gives rise to a further identity: Under an infinitesimal $GL(n,R)$-transformation

$$\omega_\alpha{}^\beta(x) := \Lambda_\alpha{}^\beta(x) - \delta_\alpha^\beta, \tag{5.2.13}$$

the geometrical objects and the matter fields vary according to

$$\delta g_{\alpha\beta} = 2\omega_{(\alpha\beta)}, \qquad \delta\vartheta^\alpha = -\omega_\beta{}^\alpha\vartheta^\beta, \qquad \delta\Gamma_\alpha{}^\beta = D\omega_\alpha{}^\beta, \qquad \delta\Psi = -\omega_\alpha{}^\beta L^\alpha{}_\beta\Psi. \tag{5.2.14}$$

If we insert (5.2.14) into (5.1.5), we obtain

$$\delta L = -\omega_\alpha{}^\beta\left(-g_{\beta\gamma}\,\sigma^{\alpha\gamma} + \vartheta^\alpha\wedge\Sigma_\beta + D\Delta^\alpha{}_\beta + (L^\alpha{}_\beta\Psi)\wedge\frac{\delta L}{\delta\Psi}\right)$$

[54] If the Bianchi identities (3.9.1) were not assumed, the term $(e_\alpha\rfloor B'^\beta)\wedge(\partial L/\partial T^\beta) + (e_\alpha\rfloor B_\beta{}''^\gamma)\wedge(\partial L/\partial R_\beta{}^\gamma)$ would, in addition, occur on the right-hand side of (5.2.10), cf. [280]. The *weak* identity, which is denoted by $\cong$, holds only provided the matter field equation $\delta L/\delta\Psi = 0$ is satisfied, cf. [303].

[55] The difference of the left-hand-side of (5.2.10) and the translational force, by means of the transposed connection (3.11.9), can be written in the compact form $D\Sigma_\alpha - (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta \equiv \widehat{D}\,\Sigma_\alpha$. For the special case of a *Riemann–Cartan space* U_n – that is, if the nonmetricity $Q_{\alpha\beta}$ vanishes – we find by using the contortion one-form $K_{\alpha\beta}$, see (3.10.11): $\widehat{D}\,\Sigma_\alpha \overset{U_n}{=} D^{\{\}}\Sigma_\alpha + (e_\alpha\rfloor K^{\beta\gamma})\vartheta_{[\beta}\wedge\Sigma_{\gamma]} = D^{\{\}}\Sigma_\alpha + [T_\alpha + \frac{1}{2}e_\alpha\rfloor(\vartheta^\beta\wedge T_\beta)]\wedge\Sigma$. Sometimes this formula (see [456]) is very useful: Should the energy–momentum current be *symmetric*, then, in a U_n, merely the Christoffel derivative is left over [439], in spite of the possible presence of torsion.

$$+ d\left(\omega_\alpha{}^\beta(\Delta^\alpha{}_\beta - (L^\alpha{}_\beta\Psi)\wedge\frac{\partial L}{\partial(D\Psi)} - 2g_{\beta\gamma}\frac{\partial L}{\partial Q_{\alpha\gamma}} - \vartheta^\alpha\wedge\frac{\partial L}{\partial T^\beta} - D\frac{\partial L}{\partial R_\alpha{}^\beta})\right). \tag{5.2.15}$$

The boundary term vanishes by virtue of the explicit expression (5.1.9) for the hypermomentum current $\Delta^\alpha{}_\beta$. Then, from the arbitrariness of $\omega_\alpha{}^\beta$, there follows the *second Noether identity*

$$D\Delta^\alpha{}_\beta + \vartheta^\alpha\wedge\Sigma_\beta - g_{\beta\gamma}\,\sigma^{\alpha\gamma} \equiv -(L^\alpha{}_\beta\Psi)\wedge(\delta L/\delta\Psi)\cong 0. \qquad \text{(2nd)} \tag{5.2.16}$$

Again, the weak Noether identity holds provided the matter field equation (5.1.26) is satisfied.[56]

In Yang–Mills theory with the internal indices of the gauge potential A suppressed, the Noether identity takes the form

$$DJ \equiv \Psi\wedge(\delta L/\delta\Psi), \tag{5.2.17}$$

where the internal current of a field theory with optional Pauli terms reads

$$J := \frac{\delta L}{\delta A} = \Psi\wedge\frac{\partial L}{\partial D\Psi} + D\frac{\partial L}{\partial F}. \tag{5.2.18}$$

Needless to say that this structure is, for the $GL(n,R)$, also displayed in (5.2.16) and (5.1.9), respectively. In the gravitational case, however, *additional* terms generated by the translational group occur.

A relation between the metrical ("Hilbert") and the canonical ("Noether") energy–momentum currents $\sigma^{\alpha\beta}$ and Σ_α, respectively, which generalizes that of Belinfante and Rosenfeld [46,47,578], can rather straightforwardly be derived in a metric-affine framework: To this end, recall the definition (5.1.10) of the "auxiliary" energy–momentum current σ_β. Then, the original second Noether identity (5.2.16), simply by interior multiplication, supplies us with the prototype of the *Belinfante–Rosenfeld relation* in a metric-affine spacetime:

$$\sigma_\beta \cong \Sigma_\beta + e_\alpha\rfloor D\Delta^\alpha{}_\beta \quad \text{with } \vartheta_{[\alpha}\wedge\sigma_{\beta]}\equiv 0. \tag{5.2.19}$$

The first formula can be understood as a symmetrization of an otherwise asymmetric energy–momentum current (see, for example, [271,366]). We will come back to this question in section 5.6.

5.3. Decomposition of the Noether identities

The dilational part of the *second* Noether identity can be easily extracted from (5.2.16) by sheer contraction:

$$d\Delta + \vartheta^\alpha\wedge\Sigma_\alpha - \sigma^\alpha{}_\alpha \equiv -(L^\gamma{}_\gamma\Psi)\wedge(\delta L/\delta\Psi)\cong 0. \tag{5.3.1}$$

Note that we have here $D\Delta = d\Delta$, since Δ is a scalar-valued $(n-1)$-form (with vanishing weight). Only the trace piece $\vartheta^\alpha\wedge\Sigma_\alpha$ in the decomposition (5.1.15) of the energy–momentum current

[56] If we had not assumed the structure equations (3.8.1), (3.5.9), (3.5.10) in the transition from (5.1.4) to the the variational expression (5.1.5), we would, in addition, get the term $2S_{\beta\gamma}\wedge(\partial L/\partial Q_{\alpha\gamma}) - S'^\alpha\wedge(\partial L/\partial T^\beta) - S''^\alpha_\gamma\wedge(\partial L/\partial R_\gamma{}^\beta) + S''^\gamma_\beta\wedge(\partial L/\partial R_\alpha{}^\gamma)$ on the right-hand side of (5.2.16).

contributes to this dilational identity. The approximate *Bjorken scaling*[57] discovered in hadron deep-inelastic electron scattering off nucleons is such an example of the *conservation of the dilation current* in flat spacetime, cf. also [277,283].

In order to perform the (anti-)symmetrization of the second Noether identity, we have to use the metric and to lower an index. We obtain

$$D\Delta_{\alpha\beta} + \vartheta_\alpha \wedge \Sigma_\beta + Q_{\alpha\gamma} \wedge \Delta^\gamma{}_\beta - \sigma_{\alpha\beta} \equiv -(L_{\alpha\beta}\Psi) \wedge (\delta L/\delta\Psi) \cong 0. \tag{5.3.2}$$

Having also decomposed the nonmetricity into its trace and tracefree pieces according to (3.12.4), a straightforward antisymmetrization of (5.3.2), together with (5.1.21), yields the metric-affine angular momentum identity

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} + \not{Q}_{[\alpha|\gamma} \wedge \tau^\gamma{}_{|\beta]} + \not{Q}_{[\alpha|\gamma} \wedge \overset{\frown}{\not\Delta}{}^\gamma{}_{|\beta]} + Q \wedge \tau_{\alpha\beta} \cong 0, \tag{5.3.3}$$

in which the first two terms (spin and orbital part) are already known from a Riemann–Cartan spacetime [273]. As could have been expected, the dilation current is not involved in this total angular momentum balance, whereas the shear current contributes a new term to the orbital angular momentum. The symmetric part of (5.3.2) reads

$$\begin{aligned} &D\Delta_{(\alpha\beta)} + \vartheta_{(\alpha} \wedge \Sigma_{\beta)} - \sigma_{\alpha\beta} \\ &\quad + \not{Q}_{(\alpha|\gamma} \wedge \overset{\frown}{\not\Delta}{}^\gamma{}_{|\beta)} + Q \wedge \overset{\frown}{\not\Delta}{}_{\alpha\beta} + \frac{1}{n}\not{Q}_{\alpha\beta} \wedge \Delta + \frac{1}{n} g_{\alpha\beta} Q \wedge \Delta + \not{Q}_{(\alpha|\gamma} \wedge \tau^\gamma{}_{|\beta)} \cong 0. \end{aligned} \tag{5.3.4}$$

Observe that in this balance of the total shear plus dilation current in metric-affine spacetime there occurs a correction term carrying the spin current.

Since (5.3.1) does not depend on the Abelian part of the connection, i.e. the Weyl one-form Q, we may apply the definition (3.12.2) of the volume-preserving connection to (5.2.16) and find

$${}^\dagger D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - \sigma^\alpha{}_\beta \cong 0. \tag{5.3.5}$$

If we use (5.3.1) to project out the trace of (5.3.5), then the tracefree intrinsic hypermomentum current

$$\not\Delta^\alpha{}_\beta := \Delta^\alpha{}_\beta - (1/n)\delta^\alpha_\beta \Delta^\gamma{}_\gamma = \overset{\frown}{\not\Delta}{}^\alpha{}_\beta + \tau^\alpha{}_\beta \tag{5.3.6}$$

obeys the identity

$${}^\dagger D\not\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - \frac{1}{n}\delta^\alpha_\beta \vartheta^\gamma \wedge \Sigma_\gamma - \left(\sigma^\alpha{}_\beta - \frac{1}{n}\delta^\alpha_\beta \sigma^\gamma{}_\gamma\right) \cong 0. \tag{5.3.7}$$

We lower the index α in (5.3.7) and find, see (5.1.11),

$${}^\dagger D\not\Delta_{\alpha\beta} + {}^\dagger Q_{\mu\alpha} \wedge \not\Delta^\mu{}_\beta + \vartheta_\alpha \wedge \Sigma_\beta - \not\sigma_{\alpha\beta} \cong 0. \tag{5.3.8}$$

[57] The other issue, which motivated Coleman [123,124] (and before that Feynman and Huggins) to consider scaling laws, is *renormalizability*. As is mentioned already in section 2, for a field theory to be renormalizable, the amplitude has to be cut-off independent, which implies scale invariance. This necessarily occurs at the energy of the cut-off, which could be at any level (even at Planck mass).

Here we introduced the tracefree piece $\not\sigma_{\alpha\beta} := \sigma_{\alpha\beta} - g_{\alpha\beta}\sigma^{\gamma}{}_{\gamma}/n$ of the metrical energy–momentum $\sigma_{\alpha\beta}$. The decomposition of (5.3.8) into its symmetric tracefree and its antisymmetric pieces yields the remarkably compact formulae

$$ {}^{\dagger}D\widehat{\not\Delta}_{\alpha\beta} + {}^{\dagger}Q_{\mu(\alpha} \wedge \not\Delta^{\mu}{}_{\beta)} + \vartheta_{(\alpha} \wedge \widehat{\not\Sigma}_{\beta)} - \not\sigma_{\alpha\beta} \cong 0, \tag{5.3.9} $$

$$ {}^{\dagger}D\tau_{\alpha\beta} + {}^{\dagger}Q_{\mu[\alpha} \wedge \not\Delta^{\mu}{}_{\beta]} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} \cong 0. \tag{5.3.10} $$

Accordingly, Eq. (5.3.1) represents the law for the *dilation* current, Eq. (5.3.9) that for the *shear* current, whereas (5.3.10) is the general version of the *angular momentum* law.

From the dilational part (5.3.1) of the second Noether identity we find, alternatively to (5.1.15), the weakly equivalent decomposition of the energy–momentum current (see [268,289,365,366])

$$ \Sigma_{\alpha} \cong \widehat{\not\Sigma}_{\alpha} + (1/n)e_{\alpha}\rfloor(\sigma^{\gamma}{}_{\gamma} - d\Delta) + \tfrac{1}{2}\vartheta_{\alpha} \wedge \Sigma, \tag{5.3.11} $$

which subsequently will be instrumental for the derivation of an *improved energy–momentum* current which is required to have a "soft", i.e. derivative-free, trace for scalar fields [96,713].

Let us now turn to the *first* Noether identity (5.2.10). If we use (5.3.1) and express the first Noether identity in terms of the volume-preserving connection (3.12.2), we get

$$ \begin{aligned} {}^{\dagger}D\Sigma_{\alpha} \cong\, & (e_{\alpha}\rfloor{}^{\dagger}T^{\beta}) \wedge \Sigma_{\beta} + (e_{\alpha}\rfloor{}^{\dagger}R_{\beta}{}^{\gamma}) \wedge \not\Delta^{\beta}{}_{\gamma} - \tfrac{1}{2}(e_{\alpha}\rfloor{}^{\dagger}Q_{\beta\gamma})\not\sigma^{\beta\gamma} \\ & + \tfrac{1}{2}[(e_{\alpha}\rfloor dQ) \wedge \Delta - (e_{\alpha}\rfloor Q)d\Delta]. \end{aligned} \tag{5.3.12} $$

The terms in the last line represent the explicit Weyl-pieces.

From the first Noether identity, we will also derive, for the sake of completeness, relations for the covariant exterior derivatives of the antisymmetric and the trace pieces of the energy–momentum current, respectively:

$$ \begin{aligned} D\Sigma \cong\, & (e_{\alpha}\rfloor T^{\beta}) \wedge (e^{\alpha}\rfloor\Sigma_{\beta}) + (e_{\alpha}\rfloor R_{\beta}{}^{\gamma}) \wedge (e^{\alpha}\rfloor\Delta^{\beta}{}_{\gamma}) \\ & + \tfrac{1}{2}(e_{\alpha}\rfloor Q_{\beta\gamma})\ (e^{\alpha}\rfloor\sigma^{\beta\gamma}) - e_{\alpha}\rfloor(Q^{\alpha\beta} \wedge \Sigma_{\beta}) + \text{Ł}_{e_{\alpha}}\Sigma^{\alpha}, \end{aligned} \tag{5.3.13} $$

$$ D[e_{\alpha}\rfloor(\vartheta^{\beta} \wedge \Sigma_{\beta})] \cong \text{Ł}_{e_{\alpha}}(\vartheta^{\beta} \wedge \Sigma_{\beta}). \tag{5.3.14} $$

5.4. Gauge field momenta and Noether identities for the gauge Lagrangian

In a gauge approach to gravity such as in the Poincaré gauge theory [272] or in supergravity [206,207,155], the total Lagrangian L_{tot} is given by

$$ L_{\text{tot}} = V + L, \tag{5.4.1} $$

where L is the material Lagrangian dealt with in sections 4.1 and 5.1, while V is the gauge field Lagrangian. We assume that the n-form V depends on the potentials $g_{\alpha\beta}$, ϑ^{α}, $\Gamma_{\alpha}{}^{\beta}$ and their first derivatives, $dg_{\alpha\beta}$, $d\vartheta^{\alpha}$ and $d\Gamma_{\alpha}{}^{\beta}$. By an argument similar to the one used in section 5.1, it may be shown that invariance of V under tetrad deformations requires V to be of the form

$$ V = V(g_{\alpha\beta}, Q_{\alpha\beta}, \vartheta^{\alpha}, T^{\alpha}, R_{\alpha}{}^{\beta}) \,. \tag{5.4.2} $$

Consequently, we can use the results of section 5.2 and transfer them to the gauge Lagrangian simply by replacing L by V and by dropping all Ψ-dependent terms in the outcome. For an easier bookkeeping we condense our notation and introduce, according to the conventional canonical prescription, the following *gauge field momenta*: The $(n-1)$-form

$$M^{\alpha\beta} := 2\frac{\partial V}{\partial dg_{\alpha\beta}} = -2\frac{\partial V}{\partial Q_{\alpha\beta}} \tag{5.4.3}$$

and the $(n-2)$-forms [58]

$$H_\alpha := -\frac{\partial V}{\partial d\vartheta^\alpha} = -\frac{\partial V}{\partial T^\alpha}, \qquad H^\alpha{}_\beta := -\frac{\partial V}{\partial d\Gamma_\alpha{}^\beta} = -\frac{\partial V}{\partial R_\alpha{}^\beta}. \tag{5.4.4}$$

Moreover, the *metrical* energy-momentum n-form

$$m^{\alpha\beta} := 2\partial V/\partial g_{\alpha\beta}, \tag{5.4.5}$$

the *canonical* energy-momentum $(n-1)$-form

$$E_\alpha := \partial V/\partial \vartheta^\alpha \tag{5.4.6}$$

and the *hypermomentum* $(n-1)$-form

$$E^\alpha{}_\beta := \partial V/\partial \Gamma_\alpha{}^\beta = -\vartheta^\alpha \wedge H_\beta - g_{\beta\gamma}M^{\alpha\gamma} \tag{5.4.7}$$

for the gauge fields themselves will also occur. If we apply the variational principle with respect to the *independent variables* $g_{\alpha\beta}, \vartheta^\alpha$, and $\Gamma_\alpha{}^\beta$ and compare it with (5.1.7)–(5.1.9), the following relations are obtained:

$$2\delta V/\delta g_{\alpha\beta} = -DM^{\alpha\beta} + m^{\alpha\beta}, \tag{5.4.8}$$

$$\delta V/\delta\vartheta^\alpha = -DH_\alpha + E_\alpha, \tag{5.4.9}$$

$$\delta V/\delta\Gamma_\alpha{}^\beta = -DH^\alpha{}_\beta + E^\alpha{}_\beta. \tag{5.4.10}$$

The Noether procedure may be applied to the gravitational Lagrangian (5.4.2) in precisely the same way as it has been applied to the material Lagrangian in section 5.2.

(i) Diffeomorphism invariance yields the explicit structure of the canonical energy-momentum $(n-1)$-form

$$E_\alpha = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma + \tfrac{1}{2}(e_\alpha \rfloor Q_{\beta\gamma})M^{\beta\gamma} \tag{5.4.11}$$

of the gauge fields (cf. (5.2.9) for the material case), which implies for its trace piece

$$\vartheta^\alpha \wedge E_\alpha = nV + 2T^\beta \wedge H_\beta + 2R_\beta{}^\gamma \wedge H^\beta{}_\gamma + \tfrac{1}{2}Q_{\beta\gamma} \wedge M^{\beta\gamma}. \tag{5.4.12}$$

Furthermore we find the *first Noether identity*

$$D\frac{\delta V}{\delta\vartheta^\alpha} \equiv (e_\alpha \rfloor T^\beta) \wedge \frac{\delta V}{\delta\vartheta^\beta} + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge \frac{\delta V}{\delta\Gamma_\beta{}^\gamma} - (e_\alpha \rfloor Q_{\beta\gamma})\frac{\delta V}{\delta g_{\beta\gamma}}, \qquad \text{(1st)} \tag{5.4.13}$$

which completely parallels the identity (5.2.10) for the matter Lagrangian.

[58] In Maxwell's theory the gauge field momentum $H = -\partial V_{\rm Max}/\partial F$, with the field strength $F := dA$ is also called "excitation" field.

(ii) Invariance with respect to (infinitesimal) $\mathcal{G}L(n,R)$-transformations yields the *second Noether identity*,

$$D\frac{\delta V}{\delta \Gamma_\alpha{}^\beta} + \vartheta^\alpha \wedge \frac{\delta V}{\delta \vartheta^\beta} - 2g_{\beta\gamma}\frac{\delta V}{\delta g_{\alpha\gamma}} \equiv 0. \qquad \text{(2nd)} \tag{5.4.14}$$

Observe that the Noether identities for the gravitational gauge fields are both *strong* identities, since no field equation is involved in their derivation. By inserting (5.4.8)–(5.4.10) into the second Noether identity (5.4.14), we obtain the following explicit expression for the metrical gauge energy–momentum current:

$$m^\alpha{}_\beta = \vartheta^\alpha \wedge E_\beta + Q_{\beta\gamma} \wedge M^{\alpha\gamma} - T^\alpha \wedge H_\beta - R_\gamma{}^\alpha \wedge H^\gamma{}_\beta + R_\beta{}^\gamma \wedge H^\alpha{}_\gamma. \tag{5.4.15}$$

Consequently, its trace is given by

$$\begin{aligned} m^\alpha{}_\alpha &= \vartheta^\alpha \wedge E_\alpha + Q_{\alpha\beta} \wedge M^{\alpha\beta} - T^\alpha \wedge H_\alpha \\ &= nV + \tfrac{3}{2}Q_{\beta\gamma} \wedge M^{\beta\gamma} + T^\beta \wedge H_\beta + 2R_\beta{}^\gamma \wedge H^\beta{}_\gamma. \end{aligned} \tag{5.4.16}$$

In conformally invariant four-dimensional Lagrangians, this identity explains why quadratic Lagrangians in the curvature are admitted, but not those quadratic in nonmetricity or torsion. For the special choice $L = 2F - 2C$ and/or $F = C$ of the conformal weights L, F, C, as given in section 3.14, *quartic* Lagrangians in nonmetricity and/or torsions could, however, be viable from this point of view.

5.5. Metric-affine field equations

Now we are in the position to formulate the action principle in complete generality: The total action of gravitational gauge fields and minimally coupled matter fields reads

$$W = \int [V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) + L(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi)]. \tag{5.5.1}$$

The a priori *independent variables* for the application of the variational principle are $\Psi, g_{\alpha\beta}, \vartheta^\alpha$, and $\Gamma_\alpha{}^\beta$. Their independent variation yields, by means of (5.4.8)–(5.4.10) and the definitions (5.4.3)–(5.4.7) and (5.1.6)–(5.1.9), the *Yang–Mills type gauge field equations* of metric-affine gravity:

$$\delta L/\delta\Psi = 0, \qquad \text{(MATTER)} \tag{5.5.2}$$

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}, \qquad \text{(ZEROTH)} \tag{5.5.3}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \qquad \text{(FIRST)} \tag{5.5.4}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \Delta^\alpha{}_\beta. \qquad \text{(SECOND)} \tag{5.5.5}$$

Already the gauge covariant exterior derivatives D of the gauge field momenta are of Yang–Mills type.[59] Due to the universality of the gravitational interaction, there arise additional *self-couplings* which involve the currents of the metrical energy–momentum $m^{\alpha\beta}$, the canonical energy–momentum

[59] Recall the inhomogeneous Yang–Mills equation $-D(\partial V_{YM}/\partial F) = DH = J$.

E_α, or the hypermomentum $E^\alpha{}_\beta$ of the gauge fields, respectively. They, together with the corresponding material currents $\sigma^{\alpha\beta}$, Σ_α, and $\Delta^\alpha{}_\beta$, act as sources of the gauge field potentials.

This dynamical framework is very general. It contains the field equations of GR and those of the Einstein-Cartan theory [273] as special, but dynamically degenerate cases. The Poincaré gauge theory [272] results by requiring, by means of a Lagrange multipliers, the connection to be metric compatible, see section 5.8.

As soon as an explicit gauge Lagrangian V is specified, all we have to do is *to partially differentiate* this Lagrangian with respect to the field strengths $Q_{\alpha\beta}$, T^α, and $R_\alpha{}^\beta$, respectively. Thereby we find the gauge field momenta in (5.4.3) and (5.4.4). If we substitute those into (5.4.7), (5.4.11), and (5.4.15) and, afterwards, into the field equations (5.5.3)-(5.5.5), then we can display the field equations in their explicit form. Our framework allows to investigate different gauge Lagrangians in a straightforward way. Note, in particular, that we do not need to vary the Hodge star, a computation which would complicate things appreciably.[60] The explicit introduction of the gauge field momenta as operationally meaningful quantities in their own rights – together with the temporary suspension of the relations between the momenta and the field strengths – this is our trick, taken from the Kottler-Cartan-van Dantzig representation of electrodynamics (see [282]), which does the job.

Compared to the earlier work on this subject, cf. [406,407], including our own work [279,277,273], in which only two field equations occur, we have obtained a system of *three* gauge field equations for the gravitational potentials [287,288]. This can be traced back to the assumption that the coframe field ϑ^α is *not* assumed to be *orthonormal*, as one usually does. This allows for more flexibility in the process of the eventual solving of the field equations.

Instead of the geometrical variables $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta)$, used in (5.5.1), one can also turn to the set $(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha)$ or $(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, K^{\alpha\beta})$, see [730] and [275,284,285], although this distracts from our Yang-Mills type approach.

The second Noether identities (5.3.2) and (5.4.14) reveal that one of the field equations, ZEROTH or the symmetric part of FIRST, i.e.

$$\vartheta_{(\alpha} \wedge DH_{\beta)} - \vartheta_{(\alpha} \wedge E_{\beta)} = \vartheta_{(\alpha} \wedge \Sigma_{\beta)}, \tag{5.5.6}$$

can be deduced from these combined second Noether identities, provided the remaining two field equations, i.e. FIRST or ZEROTH, both together with SECOND, are satisfied. Since the matter field equation (5.5.2) is a prerequisite for the validity of the differential Noether identity, we obtain the important result that one of the first two metric-affine field equations is "weakly" *redundant*.

Not unexpectedly, SECOND does *not* follow from the other field equations, merely the weaker conditions

$$D(\Delta^\alpha{}_\beta + \delta V/\delta\Gamma_\alpha{}^\beta) \equiv 0, \tag{5.5.7}$$

$$(e_\alpha \rfloor R_\beta{}^\gamma) \wedge (\Delta^\beta{}_\gamma + \delta V/\delta\Gamma_\beta{}^\gamma) \equiv 0 \tag{5.5.8}$$

can be derived under these premises.

[60] This is only correct *cum grano salis*. In calculating $m^{\alpha\beta}$ explicitly according to (5.4.5), we have to determine the partial derivative with respect to $g_{\alpha\beta}$ of an expression which contains a Hodge star $*$ in general. However, provided FIRST and SECOND are fulfilled, ZEROTH is redundant, see below, and we can dismiss it as independent field equation.

5.6. Belinfante–Rosenfeld relation via change of variables

In order to exploit further properties of the Belinfante–Rosenfeld type symmetrization relation (5.2.19), we consider the minimally coupled matter Lagrangian

$$L = L(g_{\alpha\beta}, \vartheta^{\alpha}, \Psi, D\Psi). \tag{5.6.1}$$

Then the terms $\partial L/\partial Q_{\alpha\beta}$, $\partial L/\partial T^{\alpha}$, and $\partial L/\partial R_{\alpha}{}^{\beta}$ in (5.1.7)–(5.1.9) vanish.

Next we want to go over from $(g_{\alpha\beta}, \Gamma_{\alpha}{}^{\beta})$ to an equivalent set of geometrical variables by expressing the n^3 components of the connection $\Gamma_{\alpha}{}^{\beta}$, occuring in the covariant derivative D, in terms of the $n^2(n-1)/2$ components of the torsion T^{α} and the $n^2(n+1)/2$ components of the nonmetricity $Q_{\alpha\beta}$, compare (3.10.7):

$$\Gamma_{\alpha}{}^{\beta} = \Gamma_{\alpha}{}^{\beta}(g_{\mu\nu}, dg_{\mu\nu}, \vartheta^{\mu}, d\vartheta^{\mu}, T^{\mu}, Q_{\mu\nu}). \tag{5.6.2}$$

Then we have $(g_{\alpha\beta}, \Gamma_{\alpha}{}^{\beta}) \to (g_{\alpha\beta}, Q_{\alpha\beta}, T^{\alpha})$. Because this *change of variables* involves implicitly also derivatives of the metric and the coframe, it is not simply a point transformation of the Lagrangian formalism. Formally, we thereby leave the minimal coupling prescription. However, we can apply our general formalism of sections 5.1 and 5.2 and, as a bonus, we obtain momenta which are conjugate to the tensor-valued forms $Q_{\alpha\beta}$ and T^{α}, which have a more direct physical interpretation.

In order to perform this change of independent variables in an consistent manner within the variational procedure, we supplement the matter Lagrangian (5.6.1), applying *Lagrange multipliers*, by the structure relations $S_{\alpha\beta} := Q_{\alpha\beta} + dg_{\alpha\beta} - \Gamma_{\alpha}{}^{\gamma} g_{\gamma\beta} - \Gamma_{\beta}{}^{\gamma} g_{\alpha\gamma}$ and $S'^{\alpha} := T^{\alpha} - d\vartheta^{\alpha} - \Gamma_{\beta}{}^{\alpha} \wedge \vartheta^{\beta}$ which, for $S_{\alpha\beta} = 0$ and $S'^{\alpha} = 0$, yield the definitions (3.8.1) and (3.5.9) of nonmetricity and torsion, respectively. Thus (cf. [285,346])

$$\begin{aligned}\hat{L} &= \hat{L}(g_{\alpha\beta}, dg_{\alpha\beta}, \vartheta^{\alpha}, d\vartheta^{\alpha}, \Gamma_{\alpha}{}^{\beta}, T^{\alpha}, Q_{\alpha\beta}, \Psi, d\Psi, \Xi^{\alpha\beta}, \mu_{\alpha}) \\ &= L(g_{\alpha\beta}, \vartheta^{\alpha}, \Psi, D\Psi) + \tfrac{1}{2} S_{\alpha\beta} \wedge \Xi^{\alpha\beta} + S'^{\alpha} \wedge \mu_{\alpha},\end{aligned} \tag{5.6.3}$$

in which the $(n-1)$-form $\Xi^{\alpha\beta} = \Xi^{(\alpha\beta)}$ and the $(n-2)$-form μ_{α} feature as Lagrange multipliers. The variation of the original Lagrangian L is given by (5.1.4). From the variations of the structure equations for nonmetricity and torsion we obtain (cf. [280]):

$$\delta S_{\alpha\beta} = \delta Q_{\alpha\beta} + D(\delta g_{\alpha\beta}) - g_{\beta\gamma}\delta\Gamma_{\alpha}{}^{\gamma} - g_{\alpha\gamma}\delta\Gamma_{\beta}{}^{\gamma} \tag{5.6.4}$$

$$\delta S'^{\alpha} = \delta T^{\alpha} - D(\delta\vartheta^{\alpha}) - \delta\Gamma_{\beta}{}^{\alpha} \wedge \vartheta^{\beta}. \tag{5.6.5}$$

After shifting exterior derivatives to boundary terms, the variation of the equivalent Lagrangian $\hat{L}$ yields the result:

$$\begin{aligned}\delta\hat{L} = {}& \tfrac{1}{2}\delta g_{\alpha\beta}(\sigma^{\alpha\beta} - D\Xi^{\alpha\beta}) + \delta\vartheta^{\alpha} \wedge (\Sigma_{\alpha} - D\mu_{\alpha}) + \delta\Gamma_{\alpha}{}^{\beta} \wedge (\Delta^{\alpha}{}_{\beta} - \vartheta^{\alpha} \wedge \mu_{\beta} - \Xi^{\alpha}{}_{\beta}) \\ & + \delta T^{\alpha} \wedge \mu_{\alpha} + \tfrac{1}{2}\delta Q_{\alpha\beta} \wedge \Xi^{\alpha\beta} + \delta\Psi \wedge \frac{\delta L}{\delta\Psi} + \tfrac{1}{2} S_{\alpha\beta} \wedge \delta\Xi^{\alpha\beta} + S'^{\alpha} \wedge \delta\mu_{\alpha} \\ & + d\left(\delta\Psi \wedge \frac{\partial L}{\partial D\Psi} - \delta\vartheta^{\alpha} \wedge \mu_{\alpha} + \tfrac{1}{2}\delta g_{\alpha\beta} \wedge \Xi^{\alpha\beta}\right).\end{aligned} \tag{5.6.6}$$

The "constraints" $S_{\alpha\beta} = 0$ and $S'^{\alpha} = 0$, arising from the variations $\delta\mu_{\alpha}$ and $\delta\Xi^{\alpha\beta}$ of the multipliers, are nothing but the geometrical definitions (3.8.1) and (3.5.9) of nonmetricity and torsion. Now the

Lagrange multipliers $\Xi^{\alpha\beta}$ and μ_α acquire the status of truely dynamical field momenta which are canonically conjugated to nonmetricity $Q_{\alpha\beta} \neq 0$ and torsion $T^\alpha \neq 0$. Moreover, the new *relocalized* currents are, in accordance (5.1.7)–(5.1.9), given by

$$\hat{\sigma}^{\alpha\beta} = \sigma^{\alpha\beta} - D\Xi^{\alpha\beta}, \tag{5.6.7}$$

$$\hat{\Sigma}_\alpha = \Sigma_\alpha - D\mu_\alpha, \tag{5.6.8}$$

$$\hat{\Delta}^\alpha{}_\beta = \Delta^\alpha{}_\beta - \vartheta^\alpha \wedge \mu_\beta - \Xi^\alpha{}_\beta. \tag{5.6.9}$$

Let us derive from (5.6.6) the Noether identity which results from a frame transformation. Under an infinitesimal $\mathcal{GL}(n,R)$-transformation (5.2.13), we find, in addition to (5.2.14), the response of nonmetricity and torsion

$$\delta Q_{\alpha\beta} = 2\omega_{(\alpha|\gamma}Q_{\beta)}{}^\gamma, \qquad \delta T^\alpha = -\omega_\beta{}^\alpha T^\beta. \tag{5.6.10}$$

Then, similarly as in section 5.2, we get the *modified* second Noether identity

$$D\hat{\Delta}^\alpha{}_\beta + \vartheta^\alpha \wedge \hat{\Sigma}_\beta - g_{\beta\gamma}\hat{\sigma}^{\alpha\gamma} + T^\alpha \wedge \mu_\beta - Q^{\alpha\gamma} \wedge \Xi_{\gamma\beta} \cong 0. \tag{5.6.11}$$

It is again crucial to note that this identity holds *weakly* ("on shell"), i.e., provided the matter field equation is fulfilled.

Up to now, all variables in (5.6.6) are on an equal footing. Let us now perform the change of independent variables according to (5.6.2). Then the relocalized matter Lagrangian $\hat{L}$ does *not* depend on the connection $\Gamma_\alpha{}^\beta$ anymore. Consequently, the relocalized hypermomentum (5.6.9), conjugate to the connection, has to vanish, and we can read off from (5.6.6) that

$$\hat{\Delta}^\alpha{}_\beta = 0 \quad \Leftrightarrow \quad \Delta^\alpha{}_\beta = \vartheta^\alpha \wedge \mu_\beta + \Xi^\alpha{}_\beta. \tag{5.6.12}$$

The $n^2(n-1)/2$ components of its antisymmetric piece read

$$\tau^{\alpha\beta} := \Delta^{[\alpha}{}_\gamma g^{\beta]\gamma} = \vartheta^{[\alpha} \wedge \mu_\gamma g^{\beta]\gamma} = \vartheta^{[\alpha} \wedge \mu^{\beta]}. \tag{5.6.13}$$

If we compare this result with (5.1.22), we recognize that we can identify the Lagrange multiplier μ_α, defined in (5.6.3), with the spin energy potential (5.1.24).

The metrical energy–momentum $(n-1)$-form σ_α, corresponding to the set of variables (5.6.2), can be derived by substituting (5.6.12) directly into (5.2.19):

$$\sigma_\beta = \Sigma_\beta - D\mu_\beta + e_\alpha \rfloor (T^\alpha \wedge \mu_\beta) + g_{\beta\gamma} e_\alpha \rfloor D\Xi^{\alpha\gamma} - e_\alpha \rfloor (Q_{\beta\gamma} \wedge \Xi^{\alpha\gamma}). \tag{5.6.14}$$

This relation represents the central result of this section.

The U_4-limit can be recovered by adding, similarly as later on in (5.8.1), the term $(1/2)Q_{\alpha\beta}\wedge\mu^{\alpha\beta}$ to the relocalized Lagrangian (5.6.3). Then we find $Q_{\alpha\beta} = 0$ as a constraint and the canonical conjugate strain current $\Xi^{\alpha\beta} + \mu^{\alpha\beta}$ replaces $\Xi^{\alpha\beta}$ in (5.6.14). Thereby we recover the Belinfante–Rosenfeld relation

$$\sigma_\beta = \Sigma_\beta - D\mu_\beta + e_\alpha \rfloor (T^\alpha \wedge \mu_\beta) \tag{5.6.15}$$

of the *Poincaré gauge theory* [268,366,456]. The *Einsteinian* V_4-limit

$$\sigma_\beta = \Sigma_\beta - D^{\{\}}\mu_\beta, \tag{5.6.16}$$

which is the historical Belinfante–Rosenfeld formula [46,47,578], can be obtained in a similar manner.

The decomposition of the hypermomentum current $\Delta^\alpha{}_\beta$ in (5.6.12) can be understood as an alternative to the decomposition (5.1.21). As displayed in the second line of (5.6.6), the spin energy potential μ_α couples to torsion T^α and the strain type current $\Xi^{\alpha\beta}$ to the nonmetricity $Q_{\alpha\beta}$:

$$\delta\hat{L} = \cdots + \delta T^\alpha \wedge \mu_\alpha + \tfrac{1}{2}\delta Q_{\alpha\beta} \wedge \Xi^{\alpha\beta} + \ldots . \tag{5.6.17}$$

The quantities torsion and nonmetricity both have an obvious geometrical interpretation. Thus they lend support to the fundamental importance of the "potential" μ_α and of the current $\Xi^{\alpha\beta}$. In contrast to $\Xi^{\alpha\beta}$, the strain current $\widehat{\Delta}^{(\alpha\beta)} = \Delta^{(\alpha}{}_\gamma g^{\beta)\gamma}$ does not directly couple to a simple geometrical quantity. Both currents are interrelated according to

$$\widehat{\Delta}^{(\alpha\beta)} = \vartheta^{(\alpha} \wedge \mu^{\beta)} + \Xi^{\alpha\beta}. \tag{5.6.18}$$

5.7. *Energy–momentum and hypermomentum complexes*

From the field equations (5.5.3)–(5.5.5) one can readily construct energy–momentum and hypermomentum complexes in which the gauge field momenta feature as "superpotentials":

$$\check{m}^{\alpha\beta} := \sigma^{\alpha\beta} + m^{\alpha\beta} - 2\Gamma_\gamma{}^{(\alpha} \wedge M^{\gamma|\beta)} \simeq dM^{\alpha\beta}, \tag{5.7.1}$$

$$\check{E}_\alpha := \Sigma_\alpha + E_\alpha + \Gamma_\alpha{}^\beta \wedge H_\beta \simeq dH_\alpha, \tag{5.7.2}$$

$$\check{E}^\alpha{}_\beta := \Delta^\alpha{}_\beta + E^\alpha{}_\beta - \Gamma_\gamma{}^\alpha \wedge H^\gamma{}_\beta + \Gamma_\beta{}^\gamma \wedge H^\alpha_\gamma \simeq dH^\alpha{}_\beta. \tag{5.7.3}$$

The complex $\check{m}^{\alpha\beta}$ is an n-form and, as such, not really a current. By similar arguments as above, it is redundant. Consequently, we can concentrate on the canonical energy–momentum complex $\check{E}_\alpha$ and the hypermomentum complex $\check{E}^\alpha{}_\beta$, which are both $(n-1)$-forms. Both complexes consist of a tensor part, the sum of the corresponding material and gauge currents, and a gauge part of the type *connection* $\wedge$ *field momentum*, that is, they are not gauge-covariant with respect to $\mathcal{G}L(n,R)$. If the gauge field equations (5.5.4) and (5.5.5) are fulfilled, we get "weakly" (now in the sense that the gauge field equations are satisfied) the right hand sides of (5.7.2) and (5.7.3). Then the complexes are exact forms and the gauge field momenta $M^{\alpha\beta}$, H_α, and $H^\alpha{}_\beta$ feature as "superpotentials". Hence we find the local conservation laws

$$d\check{E}_\alpha \simeq ddH_\alpha = 0, \qquad d\check{E}^\alpha{}_\beta \simeq ddH^\alpha{}_\beta = 0. \tag{5.7.4}$$

Therefore these complexes are *locally* conserved $(n-1)$-forms which, upon integration over a spacelike hypersurface, provide the corresponding "charges" of the metric-affine gauge theory.

For metric-affine gravity, Eqs. (5.7.2) and (5.7.3) were given already in [23,26] and rewritten in exterior form notation in [609,450], for example. In the proper teleparallelism model of section 5.9, the "superpotential" H_α becomes the Freud complex of GR, cf. [468,450,707].

If the geometry admits a global Killing symmetry, there exist conserved currents in metric-affine gravity, cf. [268]. More precisely, we require [524] $\xi = \xi^\alpha e_\alpha$ to be a Killing vector field for metric *and* connection,

$$\pounds_\xi g = \left(\text{Ł}_\xi g_{\alpha\beta} + 2g_{\gamma(\alpha}e_{\beta)}\rfloor \text{Ł}_\xi \vartheta^\gamma\right)\vartheta^\alpha \otimes \vartheta^\beta = 0, \qquad \pounds_\xi \Gamma_\alpha{}^\beta = 0. \tag{5.7.5}$$

According to (A.1.38), (3.5.14), and (3.5.16), these conditions can be recast into the form

$$g_{\gamma(\alpha}e_{\beta)}\rfloor\widehat{D}\xi^\gamma - \tfrac{1}{2}\xi\rfloor Q_{\alpha\beta} = 0, \qquad D(e_\alpha\rfloor\widehat{D}\xi^\beta) + \xi\rfloor R_\alpha{}^\beta = 0. \tag{5.7.6}$$

Note that the first equation of (5.7.6) can be written alternatively, in terms of the Riemannian derivative,[61] as $e_{(\alpha}\rfloor D^{\{\}}\xi_{\beta)} = 0$.

Let us define the current $(n-1)$-form

$$\varepsilon_{\mathrm{MA}} := \xi^\alpha \Sigma_\alpha + (e_\beta\rfloor \widehat{D}\xi^\gamma)\Delta^\beta{}_\gamma. \tag{5.7.7}$$

We compute its exterior covariant derivative, substitute the two Noether identities (5.2.10) and (5.2.16), and reshuffle the emerging expressions:

$$\begin{aligned} d\varepsilon_{\mathrm{MA}} &= (D\xi^\alpha)\wedge\Sigma_\alpha + \xi^\alpha D\Sigma_\alpha + D(e_\beta\rfloor\widehat{D}\xi^\gamma)\wedge\Delta^\beta{}_\gamma + (e_\beta\rfloor\widehat{D}\xi^\gamma)D\Delta^\beta{}_\gamma \\ &\cong (D\xi^\alpha)\wedge\Sigma_\alpha + (\xi\rfloor T^\beta)\wedge\Sigma_\beta + (\xi\rfloor R_\beta{}^\gamma)\wedge\Delta^\beta{}_\gamma - \tfrac{1}{2}(\xi\rfloor Q_{\beta\gamma})\sigma^{\beta\gamma} \\ &\quad + D(e_\beta\rfloor\widehat{D}\xi^\gamma)\wedge\Delta^\beta{}_\gamma + (e_\beta\rfloor\widehat{D}\xi^\gamma)(\sigma^\beta{}_\gamma - \vartheta^\beta\wedge\Sigma_\gamma) \\ &= [\widehat{D}\xi^\alpha - \vartheta^\beta(e_\beta\rfloor\widehat{D}\xi^\alpha)]\wedge\Sigma_\alpha + [(De_\beta\rfloor\widehat{D}\xi^\gamma) + \xi\rfloor R_\beta{}^\gamma]\wedge\Delta^\beta{}_\gamma \\ &\quad + (g_{\alpha(\beta}e_{\gamma)}\rfloor\widehat{D}\xi^\alpha - \tfrac{1}{2}\xi\rfloor Q_{\beta\gamma})\sigma^{\beta\gamma}. \end{aligned} \tag{5.7.8}$$

While transforming the exterior derivative into the gauge covariant derivative, we assumed that ε has zero weight, which is in accordance with the zero weight for the Lagrangian and a usual vector field ξ. Moreover, we recognize that the expression in the first square bracket vanishes identically because of the relation $p\psi = \vartheta^\alpha\wedge(e_\alpha\rfloor\psi)$, which is valid for any p-form. In view of the generalized Killing equations (5.7.6), also the other expressions in the square brackets vanish. Thus, the current (5.7.7) is weakly conserved

$$d\varepsilon_{\mathrm{MA}} \cong 0. \tag{5.7.9}$$

For the Riemann–Cartan spacetime of the EC theory a similar result has been obtained by Trautman [682] and, for the linearized case, by Tod [677]. The corresponding current reads

$$\varepsilon_{\mathrm{RC}} := \xi^\alpha\Sigma_\alpha + (e_\beta\rfloor\widehat{D}\xi^\gamma)\tau^\beta{}_\gamma, \tag{5.7.10}$$

where the spin current is defined according to $\tau^{\beta\gamma} := \Delta^{[\beta\gamma]}$. In Audretsch et al. [19], this current was used to construct a Hamiltonian for the Dirac field.

Provided a timelike Killing vector field exists, we obtain, via (5.7.9), a globally conserved energy $\int_S \varepsilon_{\mathrm{MA}}$. Our deduction of this expression follows the pattern laid out in GR, but generalizes it to a metric-affine spacetime. Some steps of this deduction resemble Penrose's *local mass construction* [552], except that we refrain from using spinor or twistor methods at this stage.

5.7.1. Conserved dilation and proper conformal currents

If the metric-affine spacetime admits even a conformal symmetry, an important generalization of (5.7.7) can be constructed as follows: Let $\xi = \xi^\alpha e_\alpha$ be a conformal Killing vector field such that the Lie derivative of the metric g and the connection $\Gamma_\alpha{}^\beta$ read[62]

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[61] For the second equation, but only in the restricted case of a Riemann–Cartan space, we find $\xi\rfloor R_\alpha{}^\beta + D(e_\alpha\rfloor\widehat{D}\xi^\beta) \equiv \xi\rfloor R^{\{\}\beta}_\alpha + D^{\{\}}(e_\alpha\rfloor D^{\{\}}\xi^\beta) - \mathrm{L}_\xi K_\alpha{}^\beta$.

[62] Eq. $(5.7.11)_2$ implies the vanishing of the tracefree part of $(5.7.6)_2$. The same would hold, too, by requiring $(5.7.11)_2$ for the volume-preserving connection instead.

$$\pounds_\xi g = \omega g, \qquad \pounds_\xi \Gamma_\alpha{}^\beta = \tfrac{1}{2}\delta_\alpha^\beta d\omega. \tag{5.7.11}$$

The same algebra as that leading to (5.7.6) yields

$$g_{\gamma(\alpha} e_{\beta)} \rfloor \widehat{D}\xi^\gamma - \tfrac{1}{2}\xi\rfloor Q_{\alpha\beta} = \tfrac{1}{2} g_{\alpha\beta}\omega, \qquad D(e_\alpha \rfloor \widehat{D}\,\xi^\beta) + \xi\rfloor R_\alpha{}^\beta = \tfrac{1}{2}\delta_\alpha^\beta\, d\omega, \tag{5.7.12}$$

compare with (5.7.6), which we recover for $\omega = 0$. It follows from $(5.7.11)_1$ that ξ generates a transformation, parametrized by ω, of the spacetime manifold such that the metric undergoes the special [442] conformal change $g \to \tilde{g} = e^{L\omega} g$. For a given geometry, the scalar function $\omega = \omega(x)$ is determined by the trace of $(5.7.12)_1$, i.e. by

$$\omega = (2/n) e_\gamma \rfloor \widehat{D}\xi^\gamma - \xi\rfloor Q. \tag{5.7.13}$$

Thus, in metric-affine spacetime the conformal Killing equation for the metric reads

$$g_{\gamma(\alpha}\, e_{\beta)} \rfloor \widehat{D}\xi^\gamma - (1/n) g_{\alpha\beta} e_\gamma \rfloor \widehat{D}\xi^\gamma = \tfrac{1}{2}\xi\rfloor \not{Q}_{\alpha\beta}, \tag{5.7.14}$$

where $\not{Q}_{\alpha\beta} := Q_{\alpha\beta} - g_{\alpha\beta} Q$ is the tracefree part of the nonmetricity.

Going through a similar analysis as in the context of the metric-affine current (5.7.7), we obtain, see [268] for details, the *conformal current*

$$\varepsilon_C = \xi^\alpha \not{\Sigma}_\alpha + (e_\beta \rfloor \widehat{D}\xi^\gamma) \not{\Delta}^\beta{}_\gamma + (1/n)\xi\rfloor(\vartheta^\alpha \wedge \Sigma_\alpha) + \tfrac{1}{2}(\xi\rfloor Q)\Delta. \tag{5.7.15}$$

For conformally invariant gauge theories, such as the Maxwell or the Yang–Mills vacuum theory, the *trace* $\vartheta^\alpha \wedge \Sigma_\alpha$ of the energy–momentum current *vanishes* and (5.7.15) provides the conserved quantity

$$d\varepsilon_C \cong \tfrac{1}{2}\omega(\vartheta^\alpha \wedge \Sigma_\alpha) = 0. \tag{5.7.16}$$

Thus we have found generalizations of the well-known dilation and proper conformal currents in Minkowski spacetime [317] to a metric-affine spacetime. Such a spacetime provides the most natural gravitational background for these currents.

Imposing conformal invariance on a metric-affine spacetime is, however, an extremely strong condition. According to the Ogievetsky theorem [533], a metric-affine spacetime, which admits a conformal symmetry, will have its frames locally invariant (in the *active* operational sense) under the group of analytical diffeomorphisms, cf. [486] for further details. This result overlaps with the fact that we have included in our affine gauge approach local translations, i.e. active diffeomorphisms, except that whereas the latter are only infinitesimal (their generators do not form a Lie algebra anyhow), the Ogievetsky transformations can be integrated to finite diffeomorphisms, without involving an infinite-component connection, since the algebraic generators are multiplied by constant parameters solely: the local dependence has already been taken care of through the generating Taylor expansion itself.

The emergence of an explicit infinite-dimensional Lie algebra may make it possible to treat conformal fields in four dimensions similarly to what is done in the special case of two dimensions. In $n = 2$, there is an infinite-dimensional conformal algebra which is isomorphic to the algebra of analytic two-dimensional diffeomorphisms [330]. In two dimensions, this feature constrains the fields and leads to the highly restrictive "fusion" rules [330], which have recently put two-dimensional conformal field theory into the focus of interest of statistical mechanics and string theory. Note that the Ogievetsky algebra in four dimensions should possess a quantum extension with central charges as in the case of its two-dimensional analog, the Virasoro algebra. Neither this extension nor the representation theory have been investigated to date.

5.7.2. Noether identities from conformal changes in an (L_n, g)

The conformal equivalence structure introduced in section 3.14 compares *two* different metric-affine spacetimes (L_n, g) and $(L_n, \tilde{g})$. Therefore it is not a local symmetry in the strict sense. However, in its infinitesimal version we can regard it as a Noether symmetry and derive the corresponding Noether identities. This was done in the recent work of Obukhov [530] which we follow closely.

We start from a conformal change of the metric in its combined, most general form (3.14.1) and expand the arbitrary conformal function Ω according to

$$\Omega = \exp\omega \simeq 1 + \omega. \tag{5.7.17}$$

For the variation of the basic fields, this implies[63]

$$\delta g_{\alpha\beta} = (l - 2F)\,\omega g_{\alpha\beta}, \qquad \delta\vartheta^\alpha = F\omega\vartheta^\alpha, \qquad \delta\Gamma_\alpha{}^\beta = -C\delta^\beta_\alpha d\omega, \qquad \delta\Psi = I\omega\Psi. \tag{5.7.18}$$

Here I is a matrix necessarily diagonal, the elements of which describe the conformal weights of the individual components.

The variation of a Langrangian which is invariant under such infinitesimal conformal changes reads

$$\begin{aligned}\delta L = {} & \omega[(\tfrac{1}{2}l - F)\sigma^\alpha{}_\alpha + F\vartheta^\alpha \wedge \Sigma_\alpha + Cd\Delta + I\Psi \wedge \delta L/\delta\Psi] \\ & - d\left[\omega\left(C\Delta - I\Psi \wedge \frac{\partial L}{\partial(D\Psi)} + (l-2F)g_{\alpha\beta}\frac{\partial L}{\partial Q_{\alpha\beta}} - F\vartheta^\alpha \wedge \frac{\partial L}{\partial T^\alpha} - C\delta^\beta_\alpha D\frac{\partial L}{\partial R_\alpha{}^\beta}\right)\right].\end{aligned} \tag{5.7.19}$$

Let us proceed, similarly as in the derivation of the $GL(n, R)$ Noether identities in section 5.2.2, and assume that ω and $d\omega$ are pointwise arbitrary ($A + dB$ scheme). Then, from $dB = 0$, we obtain the following identity for the dilation current

$$C\Delta = I\Psi \wedge \frac{\partial L}{\partial(D\Psi)} - (l-2F)g_{\alpha\beta}\frac{\partial L}{\partial Q_{\alpha\beta}} + F\vartheta^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + C\delta^\beta_\alpha D\frac{\partial L}{\partial R_\alpha{}^\beta}, \tag{5.7.20}$$

whereas $A = 0$ yields the strong *conformal Noether identity*

$$Cd\Delta + F\vartheta^\alpha \wedge \Sigma_\alpha + (\tfrac{1}{2}l - F)\sigma^\alpha{}_\alpha \equiv -\Psi \wedge \delta L/\delta\Psi \cong 0. \tag{5.7.21}$$

After integrating out the matter field Ψ in the effective Lagrangian, the left hand side of (5.7.21) is proportional to the *conformal anomaly*, see Deser and Schwimmer [153,632] for the Riemannian case. The conformal changes (3.14.1), for $l = 0$ and $F = C = -1$, contain the local scale transformations (3.13.2)–(3.13.4). Thus, for this choice of conformal weights, we recover the Noether identity (5.3.1) for the dilational current.

[63] The 'weight' factor L, defined in (3.14.1), will in this subsection be denoted by l in order to prevent a mixing up of this factor with the Lagrangian.

Let us now require, as we will do in more detail in section 6, a theory to be invariant under both, the $GL(n,R)$-gauge group and the group of conformorphisms. Then we can insert the explicit form (5.1.19) cum (5.1.9) of the dilation current into (5.7.20) and find quite generally

$$(l-2F+2C)g_{\alpha\beta}\frac{\partial L}{\partial Q_{\alpha\beta}}+(C-F)\vartheta^\alpha\wedge\frac{\partial L}{\partial T^\alpha}=(I-CL^\alpha{}_\alpha)\Psi\wedge\frac{\partial L}{\partial(D\Psi)}. \qquad (5.7.22)$$

This identity provides us with a constraint for possible Pauli-type terms in a *matter* Lagrangian which depends on the Weyl covector Q and the trace or the axial piece of the torsion, cf. section B.2. The dilational Noether identity (5.3.1) and the conformal version (5.7.21) yields the "weak" relations

$$l\sigma^\alpha{}_\alpha \simeq -2(C-F)\,d\Delta\,, \qquad l\vartheta^\alpha\wedge\Sigma_\alpha \simeq -(l-2F+2C)\,d\Delta. \qquad (5.7.23)$$

For pure *gravitational* Lagrangian (without the manifield Ψ), the identity (5.7.22) implies that terms depending on the Weyl covector and/or the vector or axial vector pieces of the torsion are proportional to each other or are even not permitted, unless $l = 2F + 2C$ and/or $C = F$. The geometrical meaning of these special choices of conformal weights is rather clear from inspecting the conformal transformations (3.14.5) and (3.14.7). In these specific instances, the Weyl covector and/or the torsion trace one-form $T = e_\alpha\rfloor T^\alpha$ become conformally invariant and are, consequently, qualified as generic concommitants of a conformally and $GL(n,R)$-gauge invariant Lagrangian n-form V. Then, according to (5.7.23), the trace of the metrical and/or the canonical energy–momentum will vanish.

5.8. Variational principle with constraints

In order to make contact with gravitational gauge theories on spacetimes with *restricted* geometrical degrees of freedom in a dynamically consistent manner, we use the method of *Lagrange multipliers* and will (successively) enforce the constraints of vanishing nonmetricity, torsion, or even curvature, cf. [361,272,276,285,346,364].

A word of caution seems to be in order here. We will find, e.g., that PG can be consistently derived from such a variation principle with constraints and fits nicely into our metric-affine scheme. However, the constraint of vanishing nonmetricity is ad hoc, provided matter generates a non-vanishing strain current $\widehat{\Delta}_{\alpha\beta}$, see (5.1.25). In such a situation, which is assumed to occur at very high energies, the vanishing of the nonmetricity, which couples to the strain current, must not be postulated. Instead, the complete metric-affine framework with the Lagrangian (5.5.1) should then be applied.

Our method of specializing the general metric-affine field equations to more restricted spacetimes can also be inverted. If we had started from a Riemannian spacetime and had incorporated the constraint $T^\alpha = 0$ and $Q_{\alpha\beta} = 0$ via Lagrange multipliers as in (5.8.14) or (5.8.1), the passage to an (L_n, g) could have been achieved via the relaxation of constraints as described in [285]. In quantum gravity, we expect anyways that these constraints hold *at most* as vacuum expectation values $\langle 0|\hat{T}^\alpha|0\rangle = 0$, $\langle 0|\hat{Q}_{\alpha\beta}|0\rangle = 0$ for the corresponding operators, whereas the vacuum expectation values of the torsion and nonmetricity operators squared may very likely pick up nonvanishing values due to quantum fluctuations [731].

5.8.1. Vanishing nonmetricity: Poincaré gauge gravity

Let us first enforce the vanishing of the nonmetricity which results in PG: To this end we consider, instead of (5.5.1), the total Lagrangian

$$L_{\{Q=0\}} = V + L + \tfrac{1}{2}Q_{\alpha\beta} \wedge \mu^{\alpha\beta}, \tag{5.8.1}$$

where the Lagrangian multiplier $\mu^{\alpha\beta} = \mu^{\beta\alpha}$ is a symmetric $(n-1)$-form. By varying with respect to $g_{\alpha\beta}$, ϑ^α, $\Gamma_\alpha{}^\beta$, and the Lagrange multiplier, we obtain the modified gravitational gauge field equations:

$$DM^{\alpha\beta} - m^{\alpha\beta} - D\mu^{\alpha\beta} = \sigma^{\alpha\beta}, \tag{5.8.2}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{5.8.3}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta - g_{\beta\gamma}\mu^{\alpha\gamma} = \Delta^\alpha{}_\beta, \tag{5.8.4}$$

$$Q_{\alpha\beta} = 0. \tag{5.8.5}$$

Since the nonmetricity vanishes, we can freely raise and lower indices. We resolve the symmetric part of the second field equation (5.8.4) with respect to the Lagrange multiplier,

$$\mu^{\alpha\beta} = DH^{(\alpha\beta)} - E^{(\alpha\beta)} - \Delta^{(\alpha\beta)}, \tag{5.8.6}$$

and insert this expression into the zeroth field equation. This results in

$$DM^{\alpha\beta} - m^{\alpha\beta} - DDH^{(\alpha\beta)} + DE^{(\alpha\beta)} + D\Delta^{(\alpha\beta)} = \sigma^{\alpha\beta}. \tag{5.8.7}$$

By inserting the definition (5.4.7) of $E^\alpha{}_\beta$, we get

$$-m^{\alpha\beta} - DDH^{(\alpha\beta)} - D[\vartheta^{(\alpha} \wedge H^{\beta)}] = \sigma^{\alpha\beta} - D\Delta^{(\alpha\beta)}, \tag{5.8.8}$$

or, together with the first field equation (5.8.3) and the constraint (5.8.5),

$$\begin{aligned} &-m^{\alpha\beta} - R^{(\alpha}{}_\gamma \wedge H^{\gamma|\beta)} + R_\gamma{}^{(\alpha} \wedge H^{\gamma|\beta)} - T^{(\alpha} \wedge H^{\beta)} + \vartheta^{(\alpha} \wedge E^{\beta)} \\ &\quad = \sigma^{\alpha\beta} - D\Delta^{(\alpha\beta)} - \vartheta^{(\alpha} \wedge \Sigma^{\beta)} \cong 0. \end{aligned} \tag{5.8.9}$$

According to (5.8.5) and to the symmetric part (5.3.4) of the second Noether identity for matter, this reduces weakly to the explicit relation (5.4.15) for the metrical energy-momentum current of the gauge fields. Thus in a Riemann–Cartan spacetime, the zeroth field equation drops out altogether, and we are left with the first field equation

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{5.8.10}$$

and the antisymmetric part

$$DH_{[\alpha\beta]} - E_{[\alpha\beta]} = DH_{[\alpha\beta]} + \vartheta_{[\alpha} \wedge H_{\beta]} = \tau_{\alpha\beta}, \tag{5.8.11}$$

of the second field equation. This finding is also consistent with our previous results on the redundancy of the zeroth field equation.

Within the framework of PG, the subtle interplay between gauge field equations and Noether identities shows up again. As we saw, the field equation (5.5.3), which is symmetric in α and β, degenerates to a non-propagating one and can be regarded as one of the redundant equations.

Moreover, one can deduce from the antisymmetric part of (5.4.15) and the second Noether identity (5.3.3) for $Q_{\alpha\beta}=0$ that the *antisymmetric* part of the FIRST field equation of PG

$$
\begin{aligned}
&-D(\vartheta_{[\alpha}\wedge H_{\beta]}) + T_{[\alpha}\wedge H_{\beta]} - \vartheta_{[\alpha}\wedge E_{\beta]} \\
&\quad = DDH_{[\alpha\beta]} + T_{[\alpha}\wedge H_{\beta]} - \vartheta_{[\alpha}\wedge E_{\beta]} - D\tau_{\alpha\beta} = \vartheta_{[\alpha}\wedge \Sigma_{\beta]}, \qquad (\mathrm{FIRST}_{[\ ]}) \qquad (5.8.12)
\end{aligned}
$$

is redundant as well. This redundancy [275] has proved to be instrumental in the reduction of the quadratic Poincaré gauge field equations to an effectively Einsteinian system by means of a modified double duality ansatz, cf. [446,22]. It also simplifies the formulation of the Cauchy problem [158] of this model.

At low energies (in terms of the Planck energy), when the strain current cannot be excited, the matter Lagrangian is restricted to be locally Poincaré invariant instead of being invariant under the full affine group. In other word, if PG is supposed to apply, the strain current should vanish,

$$\widehat{\Delta}_{\alpha\beta} = \Delta_{(\alpha\beta)} = 0. \qquad (5.8.13)$$

Actually the constraint (5.8.5) of vanishing nonmetricity allows to impose (5.8.13) a posterori, because of the redundancy of (5.8.9). Poincaré gauge covariance is left over and the considerations of this section 5 can easily be redone under these auspicies. Clearly, the connection one-form is no longer an independent variable, rather it is constrained to be metric compatible. Obviously, the field equations [272] turn out to be (5.8.10) and (5.8.11), and from the decomposed second Noether identity only the antisymmetric piece is left over. Moreover, in analogy to (5.2.19), it is possible to derive, in the framework of PG, the generalized Belinfante–Rosenfeld formula (5.6.15), cf. [366].

We will see in section 6.5, in which manner the condition (5.8.5) and (5.8.13) can be derive from a dynamical model of symmetry breaking.

5.8.2. Vanishing torsion

As a second example, which contains generalizations of Einstein's GR to a (L_n, g) with *symmetric connection*, we enforce the vanishing of torsion and consider the toy Lagrangian

$$L_{\{T=0\}} = V + L + T^\alpha \wedge \lambda_\alpha. \qquad (5.8.14)$$

Here the Lagrange multiplier λ_α is a $(n-2)$-form. Then the modified gravitational gauge field equations read:

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}, \qquad (5.8.15)$$

$$DH_\alpha - E_\alpha - D\lambda_\alpha = \Sigma_\alpha, \qquad (5.8.16)$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta - \vartheta^\alpha \wedge \lambda_\beta = \Delta^\alpha{}_\beta, \qquad (5.8.17)$$

$$T^\alpha = 0. \qquad (5.8.18)$$

The antisymmetric part of (5.8.17), with (5.4.7) inserted, i.e.

$$DH_{[\alpha\beta]} + Q_{[\alpha|\gamma}\wedge H^\gamma{}_{|\beta]} + \vartheta_{[\alpha}\wedge H_{\beta]} - \vartheta_{[\alpha}\wedge \lambda_{\beta]} = \tau_{\alpha\beta}, \qquad (5.8.19)$$

has $n^2(n-1)/2$ components and thus completely determines the Langrange multiplier λ_α. For its resolution with respect to λ_α, we employ the algebraic identity (A.1.26) and obtain

$$\lambda_\alpha = H_\alpha - 2e^\beta\rfloor(DH_{[\alpha\beta]} + Q_{[\alpha|\gamma} \wedge H^\gamma{}_{|\beta]}) + \tfrac{1}{2}\vartheta_\alpha e^\gamma\rfloor e^\delta\rfloor(DH_{[\gamma\delta]} + Q_{[\gamma|\epsilon} \wedge H^\epsilon{}_{|\delta]}) - \mu_\alpha, \tag{5.8.20}$$

where μ_α is again the spin energy potential (5.1.24). Then the first field equation (5.8.16) reduces to

$$\begin{aligned} &-E_\alpha + 2D[e^\beta\rfloor(DH_{[\alpha\beta]} + Q_{[\alpha|\gamma} \wedge H^\gamma{}_{|\beta]}) - \tfrac{1}{4}\vartheta_\alpha e^\gamma\rfloor e^\delta\rfloor(DH_{[\gamma\delta]} + Q_{[\gamma|\epsilon} \wedge H^\epsilon{}_{|\delta]})] \\ &\quad = \Sigma_\alpha - D\mu_\alpha. \end{aligned} \tag{5.8.21}$$

The new energy–momentum current

$$\hat{\Sigma}_\alpha := \Sigma_\alpha - D\mu_\alpha \tag{5.8.22}$$

contains, due to the constraint (5.8.18) and the decomposed Noether identity (5.3.3), in general the antisymmetric piece

$$\begin{aligned} \vartheta_{[\alpha} \wedge \hat{\Sigma}_{\beta]} &= \vartheta_{[\alpha} \wedge \Sigma_{\beta]} + D\tau_{\alpha\beta} \\ &= -\not{Q}_{[\alpha|\gamma} \wedge \tau^\gamma{}_{|\beta]} - \not{Q}_{[\alpha|\gamma} \wedge \widehat{\not{\Delta}}^\gamma{}_{|\beta]} - Q \wedge \tau_{\alpha\beta}. \end{aligned} \tag{5.8.23}$$

5.8.3. Riemannian spacetime and general relativity

However, if additionally the nonmetricity is required to vanish, we will recover from (5.8.22) the *Belinfante–Rosenfeld symmetrized* energy–momentum current [46,47,578] in Riemannian spacetime, cf. [268,364,456]. There is a corresponding expression for the gravitational gauge fields: If we consider topological gravity in $n = 3$ dimensions and use in the Lagrangian, as a supplement to the Hilbert–Einstein term, the Chern–Simons three-form for the curvature, the Cotton tensor surfaces in the first field equation, cf. [452,29]. This demonstrates that the constraint of vanishing torsion is not as innocent as is usually surmized; in particular, it does not only change the variables but also the order of differentiation from one to two. Moreover, solving the constraint $d\vartheta^\alpha + \Gamma_\beta^{\{\}\alpha} \wedge \vartheta^\beta = 0$ for the connection is possible only for a non-degenerate coframe [429].

More precisely, the Riemannian case is obtained from the Lagrangian

$$L_{\{Q=0,T=0\}} = V + L + \tfrac{1}{2}Q_{\alpha\beta} \wedge \mu^{\alpha\beta} + T^\alpha \wedge \lambda_\alpha. \tag{5.8.24}$$

We can go through both previous reduction procedures and eventually obtain as the only field equation

$$-E_\alpha + 2D\left(e^\beta\rfloor DH_{[\alpha\beta]} - \tfrac{1}{4}\vartheta_\alpha e^\gamma\rfloor e^\delta\rfloor DH_{[\gamma\delta]}\right) = \Sigma_\alpha - D\mu_\alpha, \tag{5.8.25}$$

together with the constraints

$$Q_{\alpha\beta} = 0, \qquad T^\alpha = 0. \tag{5.8.26}$$

Incidentically, Eq. (5.8.25) applies also to the higher-derivative models [335,601] entertained for singularity-free cosmologies. Einsteinian GR is obtained from (5.8.25) by means of the Hilbert–Einstein Lagrangian

$$V_{\mathrm{HE}} = -\frac{1}{2l^{n-2}} R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} \quad \Longrightarrow \quad H_\alpha = 0, \qquad H_{[\alpha\beta]} = \frac{1}{2l^{n-2}}\eta_{\alpha\beta}. \tag{5.8.27}$$

Since $D\eta_{\alpha\beta} = 0$ for vanishing torsion, we find, by substituting the identification (5.4.11),

$$-e_\alpha \rfloor V_{\mathrm{HE}} - \frac{1}{2l^{n-2}}(e_\alpha \rfloor R^{\{\}\beta\gamma}) \wedge \eta_{\beta\gamma} = \Sigma_\alpha - D^{\{\}}\mu_\alpha \tag{5.8.28}$$

or

$$\frac{1}{2l^{n-2}} R^{\{\}\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = \Sigma_\alpha - D^{\{\}}\mu_\alpha. \tag{5.8.29}$$

The current on the right hand side of the *Einstein equation* is the momentum current of (5.6.16).

5.9. Teleparallelism

As our last example, we consider the case of vanishing curvature which is a constraint in (extended) *teleparallelism* models. We start with the total Lagrangian

$$L_{\{R=0\}} = V + L + R_\alpha{}^\beta \wedge \lambda^\alpha{}_\beta, \tag{5.9.1}$$

where the Lagrange multiplier $\lambda^\alpha{}_\beta$ is a $gl(n, R)$-valued $(n-2)$-form. Then the modified gravitational gauge field equations read:

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}, \tag{5.9.2}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{5.9.3}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta - D\lambda^\alpha{}_\beta = \Delta^\alpha{}_\beta, \tag{5.9.4}$$

$$R_\alpha{}^\beta = 0. \tag{5.9.5}$$

In this case the resolution with respect to the Lagrange multiplier is not unique due to the occurrence of the exterior covariant derivative in front of $\lambda^\alpha{}_\beta$ in (5.9.4).

A further differentiation merely leads to the trivial identity:

$$\begin{aligned} &-R^\alpha{}_\gamma \wedge \lambda^\gamma{}_\beta + R^\gamma{}_\beta \wedge \lambda^\alpha{}_\gamma \\ &\quad = D\Delta^\alpha{}_\beta + \vartheta^\alpha \wedge \Sigma_\beta - g_{\beta\gamma}\sigma^{\alpha\gamma} - T^\alpha \wedge H_\beta + \vartheta^\alpha \wedge E_\beta + Q_{\beta\gamma} \wedge M^{\alpha\gamma} \\ &\qquad - R_\gamma{}^\alpha \wedge H^\gamma{}_\beta + R_\beta{}^\gamma \wedge H^\alpha{}_\gamma - g_{\beta\gamma} m^{\alpha\gamma} \\ &\quad \cong 0. \end{aligned} \tag{5.9.6}$$

Observe that the left hand side is trivial due to the vanishing of the curvature, whereas the right hand side reproduces the second Noether identity (5.2.16) together with the explicit formula (5.4.15) of the metrical energy-momentum current.

Teleparallelism models in the limit of vanishing nonmetricity (Riemann-Cartan spacetime) can be described by the total Lagrangian

$$L_{\{Q=0,R=0\}} = V + L + \tfrac{1}{2} Q_{\alpha\beta} \wedge \mu^{\alpha\beta} + R_\alpha{}^\beta \wedge \lambda^\alpha{}_\beta. \tag{5.9.7}$$

The field equations turn out to be

$$DM^{\alpha\beta} - m^{\alpha\beta} - D\mu^{\alpha\beta} = \sigma^{\alpha\beta}, \tag{5.9.8}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \tag{5.9.9}$$

$$DH^{\alpha}{}_{\beta} - E^{\alpha}{}_{\beta} - g_{\beta\gamma}\mu^{\alpha\gamma} - D\lambda^{\alpha}{}_{\beta} = \Delta^{\alpha}{}_{\beta}, \tag{5.9.10}$$

$$Q_{\alpha}{}^{\beta} = 0, \qquad R_{\alpha}{}^{\beta} = 0. \tag{5.9.11}$$

The symmetric part of (5.9.10) can be resolved with respect to $\mu^{\alpha\beta}$ with the result of

$$\mu^{\alpha\beta} = DH^{(\alpha\beta)} - E^{(\alpha\beta)} - D\lambda^{(\alpha\beta)} - \Delta^{(\alpha\beta)}. \tag{5.9.12}$$

If we substitute this into (5.9.8) and use the constraints (5.9.11), we find, similarly as in (5.8.9), that the zeroth field equation drops out completely:

$$-m^{\alpha\beta} - T^{(\alpha} \wedge H^{\beta)} + \vartheta^{(\alpha} \wedge E^{\beta)} = \sigma^{\alpha\beta} - D\Delta^{(\alpha\beta)} - \vartheta^{(\alpha} \wedge \Sigma^{\beta)} \cong 0. \tag{5.9.13}$$

The remaining equations are

$$DH_{\alpha} - E_{\alpha} = \Sigma_{\alpha}, \tag{5.9.14}$$

$$DH_{[\alpha\beta]} - E_{[\alpha\beta]} - D\lambda_{[\alpha\beta]} = \tau_{\alpha\beta}. \tag{5.9.15}$$

The *proper* teleparallelism model in Riemann–Cartan spacetime is singled out by its effective equivalence to Einstein's GR. In order to prove this equivalence, one contracts the curvature $\overline{R}_{\alpha}{}^{\beta}$ built, according to (3.11.4), from a deformed connection $\overline{\Gamma}_{\alpha}{}^{\beta} = \Gamma_{\alpha}{}^{\beta} + A_{\alpha}{}^{\beta}$, by $\eta_{\alpha\beta}$. For $n \geq 2$, this provides us with the following identity:

$$\begin{aligned}\overline{R}^{\alpha\beta} \wedge \eta_{\alpha\beta} \equiv R^{\alpha\beta} \wedge \eta_{\alpha\beta} - A^{\alpha\mu} \wedge A_{\mu}{}^{\beta} \wedge \eta_{\alpha\beta} - Q^{\alpha\mu} \wedge A_{\mu}{}^{\beta} \wedge \eta_{\alpha\beta} \\ - 2A^{\alpha\beta} \wedge Q \wedge \eta_{\alpha\beta} + A^{\alpha\beta} \wedge T^{\gamma} \wedge \eta_{\alpha\beta\gamma} + d(A^{\alpha\beta} \wedge \eta_{\alpha\beta}).\end{aligned} \tag{5.9.16}$$

Incidentally, the term proportional to $\eta_{\alpha\beta\gamma}$ would vanish in $n = 2$ dimensions. Moreover, if the deformed connection is identified with the Riemannian piece $\Gamma^{\{\}}_{\alpha\beta}$, we find from (3.10.12) that the deformation one-form takes the form

$$A_{\alpha\beta} = K_{\alpha\beta} - \tfrac{1}{2}Q_{\alpha\beta} - (e_{[\alpha}\rfloor Q_{\beta]\gamma})\vartheta^{\gamma}. \tag{5.9.17}$$

Hence, for vanishing nonmetricity, we recover the geometric identity

$$\begin{aligned}R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} &\equiv R^{\alpha\beta} \wedge \eta_{\alpha\beta} - K^{\alpha\mu} \wedge K_{\mu}{}^{\beta} \wedge \eta_{\alpha\beta} + K^{\alpha\beta} \wedge T^{\gamma} \wedge \eta_{\alpha\beta\gamma} + d(K^{\alpha\beta} \wedge \eta_{\alpha\beta}) \\ &= R^{\alpha\beta} \wedge \eta_{\alpha\beta} + T^{\alpha} \wedge {}^{*}(-{}^{(1)}T_{\alpha} + 2{}^{(2)}T_{\alpha} + \tfrac{1}{2}{}^{(3)}T_{\alpha}) + 2d(\vartheta^{\alpha} \wedge {}^{*}T_{\alpha})\end{aligned} \tag{5.9.18}$$

for the Hilbert–Einstein Lagrangian (multiplied by the dimensionful factor $-2l^{n-2}$). For the teleparallelism condition $R^{\alpha\beta} = 0$, the first term on the right-hand side drops out. Thus we arrive effectively at the *teleparallelism* Lagrangian (see [272,292,293,439,451,580,613,614,672,673])

$$V_{\parallel} = V_{\mathrm{T}^2} + R_{\alpha}{}^{\beta} \wedge \lambda^{\alpha}{}_{\beta}, \qquad V_{\mathrm{T}^2} := -\frac{1}{2l^{n-2}} T^{\alpha} \wedge {}^{*}(-{}^{(1)}T_{\alpha} + 2{}^{(2)}T_{\alpha} + \tfrac{1}{2}{}^{(3)}T_{\alpha}). \tag{5.9.19}$$

Via (5.9.19) and (5.9.18) the resulting field equation (5.9.3) can be shown to be equivalent to (5.8.28) or (5.8.29), see [451]. Élie Cartan showed that the Einstein equation in the teleparallel version $\mathrm{GR}_{\parallel}$ of general relativity forms an *involutive system*, cf. [661,662,535,536,670]. In the framework of $\mathrm{GR}_{\parallel}$ the Cauchy problem and the coupling to non-scalar matter were thought to be problematic [524,363,269]. Recently Maluf [425,426] was able to show that, by fixing a suitable gauge, the Hamiltonian and the Cauchy problem are well-behaved in vacuo.

5.10. Ashtekar type complex field momenta generated by Chern–Simons terms

In four dimensions, we were able to determine the Chern–Simons terms of metric-affine spacetime in section 3.9. Multiplying each term with a dimensionless coupling constant θ_i, the corresponding *boundary term* reads [280]:

$$\begin{aligned} dC_{MAG} &= \theta_1 dC_{TT} + \theta_2 dC_{RR} + \theta_3 dC_{trR\, trR} \\ &= \frac{\theta_1}{2l^2}[g_{\alpha\beta}(T^\alpha \wedge T^\beta + R_\gamma{}^\alpha \wedge \vartheta^\gamma \wedge \vartheta^\beta) - Q_{\alpha\beta} \wedge \vartheta^\alpha \wedge T^\beta] \\ &\quad - \tfrac{1}{2}\theta_2 R_\alpha{}^\beta \wedge R_\beta{}^\alpha - \tfrac{1}{2}\theta_3 R \wedge R. \end{aligned} \tag{5.10.1}$$

If we decompose the connection, according to (3.12.2), into a volume-preserving and a volume-changing part, we find

$$\begin{aligned} dC_{MAG} &= \frac{\theta_1}{2l^2}[g_{\alpha\beta}({}^\dagger T^\alpha \wedge {}^\dagger T^\beta + {}^\dagger R_\gamma{}^\alpha \wedge \vartheta^\gamma \wedge \vartheta^\beta) - {}^\dagger Q_{\alpha\beta} \wedge \vartheta^\alpha \wedge {}^\dagger T^\beta] \\ &\quad - \tfrac{1}{2}\theta_2 {}^\dagger R_\alpha{}^\beta \wedge {}^\dagger R_\beta{}^\alpha - \tfrac{1}{8}(\theta_2 + 4\theta_3) R \wedge R. \end{aligned} \tag{5.10.2}$$

Observe already here that the generic Lagrangian of the metric-affine gauge theory is expected to have the same overall structure as (5.10.2): Merely one Hodge star should be distributed appropriately to each term in (5.10.2), and, in addition, there could occur different irreducible pieces of the field strengths multiplied with suitable dimensionless coupling constants.

Then we may supplement the original gravitational gauge Lagrangian V with the boundary term idC_{MAG}, where i denotes the imaginary unit, $i^2 = -1$. Subsequently we consider the complexified Lagrangian

$$\overset{(\pm)}{V} = V \pm i dC_{MAG}. \tag{5.10.3}$$

The purely imaginary character of the additional piece is necessary if we want to preserce CP, i.e. the combined charge and parity transformation, as an exact symmetry in gravitational interactions, cf. [390,451].

Clearly, the new Lagrangians $\overset{(\pm)}{V}$ yield field equations which are equivalent to the old ones, since we have merely added a boundary term to the original Lagrangian. However, the new field momenta [280], which are canonically conjugate to *real* nonmetricity, torsion, and curvature, become necessarily *complex*:

$$\overset{(\pm)}{M}{}^{\alpha\beta} := -2\frac{\partial \overset{(\pm)}{V}}{\partial Q_{\alpha\beta}} = M^{\alpha\beta} \pm i\frac{\theta_1}{l^2}\vartheta^{(\alpha} \wedge T^{\beta)}, \tag{5.10.4}$$

$$\overset{(\pm)}{\Pi}{}_\alpha := -\frac{\partial \overset{(\pm)}{V}}{\partial T^\alpha} = H_\alpha \mp i\frac{\theta_1}{l^2}T_\alpha, \tag{5.10.5}$$

$$\overset{(\pm)}{\Pi}{}^\alpha{}_\beta := -\frac{\partial \overset{(\pm)}{V}}{\partial R_\alpha{}^\beta} = H^\alpha{}_\beta \mp i\theta_2 R_\beta{}^\alpha \mp i\theta_3 \delta^\alpha_\beta R. \tag{5.10.6}$$

In the special case of PG, with the normalization $\theta_i = 1$, these field momenta read

$$\overset{(\pm)}{\Pi}_{\alpha} = H_{\alpha} \mp (i/l^2)T_{\alpha}, \tag{5.10.7}$$

$$\overset{(\pm)}{\Pi}_{\alpha\beta} = H_{\alpha\beta} \mp iR_{\alpha\beta}. \tag{5.10.8}$$

In particular, the case of a self-dual curvature can be understood in terms of the (complex) SO(3,C) formalism of Brans and others, see [78,79].

Following the general lead of Ashtekar [15], it has been shown [449,707] that this *change of variables* yields already on the Lagrangian level a very efficient Yang–Mills type formulation of the teleparallelism equivalent $GR_{\|}$ of Einstein's theory:

$$\overset{(\pm)}{V}_{\|} = \mp(i/4)l^2\, \overset{(\pm)}{\Pi}{}^{\alpha} \wedge \overset{(\pm)}{\Pi}_{\alpha}. \tag{5.10.9}$$

Applying a $(3+1)$-decomposition, the gravitational Hamiltonian for $GR_{\|}$ becomes polynomial in the Ashtekar type variables $\overset{(\pm)}{\underline{A}}_{\alpha} := \underline{{}^{*}\overset{(\pm)}{\Pi}}_{\alpha}$, given by the three-dual of the tangential field momenta (5.10.7). In this Hamiltonian formalism, the normal part $\mathcal{F} = n \rfloor C_{MAG}$ with respect to a timelike vector field n turns out to be the true *generating function* for an Ashtekar type pair of new variables and their canonically conjugated momenta [15]. This approach has been generalized [450,451] to the Poincaré gauge theory.[64]

6. Dynamics

6.1. Conformal gravity: quadratic model Lagrangians

Let us consider a *primordial* world which is invariant under arbitrary local $GL(4,R)$-deformations of the frame. In particular this implies that each single piece of our Lagrangian has to be invariant with respect to a local (positive) scale transformation (R^{+} gauge transformation). Moreover, we require, with Weyl, an *additional* invariance[65] of our primordial Lagrangian with respect to conformal changes of the underlying metric structure. In section 3.14 we discussed such conformorphisms which extend the diffeomorphisms by C_{+}^{∞}, the Abelian group of all positive, infinitely differentiable functions Ω. This Abelian part we have already applied *actively* in the conformal Noether procedure.

According to the covariance under conformorphisms, the Lagrangian, which specifies the dynamics, should be independent of any dimensional coupling constant. Furthermore, for a Yang–Mills-type description, we want the field equations to be *linear* in the second derivatives of the gauge potential. Then, at most quadratic terms in curvature, torsion, and nonmetricity are admitted.[66] At this stage it

[64] The $GL(4,R)$-gauge approach of Floreanini and Percacci [199,200] is different from ours: Their $GL(4,R)$ does *not* contain the physical Lorentz group. The latter one comes rather "in addition" to the "internal" $GL(4,R)$; in other words, their $GL(4,R)$ does commute with the Lorentz group.

[65] This is roughly the situation at the onset of the big bang when no particles are frozen out which, with their masses and other dimensionful properties, could provide a length scale.

[66] If we required the theory to be also supersymmetric, superconformal curvature-square Lagrangians would arise in the study of the "low"-energy limit of superstring models in $n = 10$ dimensions, see Bergshoeff, Salam, and Sezgin [54]. Moreover, in the string expansion, besides the graviton, a massless scalar field, the dilaton σ, will be the most relevant field in the bosonic sector [55], see also [167].

is not necessary to restrict ourselves to a Weyl spacetime, rather we are able to unveil our point of view in the general metric-affine framework.

According to the irreducible decompositions of appendix B, see also [626,287], the field momentum linear in the curvature can be expanded in terms of the irreducible curvature pieces as follows:

$$\overline{H}^{\alpha}{}_{\beta} = -\frac{1}{\kappa}\,{}^{*}\left(\sum_{N=1}^{11} b_{(N)}\, g^{\alpha\gamma} g_{\beta\delta}\, {}^{(N)}R_{\gamma}{}^{\delta}\right). \tag{6.1.1}$$

The number of irreducible pieces depends on the dimension n of the spacetime: For $n > 3$ we have eleven pieces , but ten for $n = 3$, and three for $n = 2$.

Since for $n \geq 3$ there are three irreducible torsion pieces, as in the restricted case of the Poincaré gauge field theory [272], we obtain for the translational gauge field momentum linear in the torsion the expansion

$$\overline{H}_{\alpha} = -\,{}^{*}\left(\sum_{M=1}^{3} a_{(M)}\, g_{\alpha\gamma}\, {}^{(M)}T^{\gamma}\right). \tag{6.1.2}$$

Let us exhibit, for arbitrary dimensions n, the conformal properties of these gauge field momenta in the case of an (L_n, g). Then the curvature, together with its irreducible pieces, is, contrary to, e.g., its behavior in a purely Riemannian space, conformally invariant, compare (3.14.3). The formula for conformal transformations for the Hodge dual of a p-form is given in (3.14.13). Consequently, we have

$$\widetilde{\overline{H}}^{\alpha}{}_{\beta} = \Omega^{(n-4)L/2}\, \overline{H}^{\alpha}{}_{\beta}\,. \tag{6.1.3}$$

Apparently, this is invariant in $n = 4$ dimensions, and we can construct therefrom the quadratic curvature Lagrangian which is likewise invariant in four dimensions:

$$V_R := -\tfrac{1}{2} R_{\alpha}{}^{\beta} \wedge \overline{H}^{\alpha}{}_{\beta}, \qquad \widetilde{V}_R = \Omega^{(n-4)L/2}\, V_R\,. \tag{6.1.4}$$

In contradistinction, the classical Hilbert-Einstein term as well as the Einstein-Cartan Lagrangian $V_{\rm EC}$ are not invariant under conformal changes, except for the trivial case with $n = 2$. They involve the Hodge dual, the metric, and the coframe for their construction:

$$V_{\rm EC} := -\tfrac{1}{2} g^{\alpha\gamma} R_{\alpha}{}^{\beta} \wedge \eta_{\gamma\beta} = -\tfrac{1}{2} g_{\beta\gamma}\, R_{\alpha}{}^{\beta} \wedge {}^{*}(\vartheta^{\alpha} \wedge \vartheta^{\gamma}), \qquad \widetilde{V}_{\rm EC} = \Omega^{(n-2)L/2}\, V_{\rm EC}\,. \tag{6.1.5}$$

Let us turn to possible contributions from the contortion of the world: According to (3.14.6), (3.14.7), the axial and the tensor torsion two-forms defined in (B.2.7) and (B.2.8), respectively, transform in a conformally covariant manner, whereas this is true for the trace torsion only for the choice $C = F$. Under this condition, the translational momentum (6.1.2) inherits, with respect to conformal changes of the metric, the modified transformation law

$$\widetilde{\overline{H}}_{\alpha} = \Omega^{(n-2)L/2-F}\, \overline{H}_{\alpha}\,, \quad F = C. \tag{6.1.6}$$

Consequently, the most general quadratic torsion Lagrangian transforms as

$$V_T := -\tfrac{1}{2} T^{\alpha} \wedge \overline{H}_{\alpha}\,, \qquad \widetilde{V}_T = \Omega^{(n-2)L/2}\, V_T\,. \tag{6.1.7}$$

Even for the choice above of $F = C$, it would only be invariant in $n = 2$ dimensions.[67] Therefore, as long as the dimensionality $n = 4$ of our macroscopic world is not reduced in the high-energy region, a quadratic torsion Lagrangian would not be admissible as a conformally invariant piece.

Let us compare the Lagrangians considered so far with those which would be obtained by distributing one Hodge star in the general boundary term (5.10.1) of an (L_n, g). We then observe that the mixed nonmetricity term[68]

$$V_{QT} := \sum_{K=1}^{3} c_{(K)}\ {}^{(K)\dagger}Q_{\alpha\beta} \wedge \vartheta^{\alpha} \wedge \overline{H}_{\gamma}\, g^{\gamma\beta} \tag{6.1.8}$$

is missing. From the conformal properties (3.14.4), (3.14.6) of the gauge fields involved, we find

$$\widetilde{V}_{QT} = \Omega^{(n-2)L/2}\, V_{QT}\ . \tag{6.1.9}$$

Analogously to the torsion-squared Lagrangian (6.1.7), this piece is conformally invariant only for $n = 2$.

It has been pointed out earlier that the ZEROTH field equation is redundant *on shell*. Nevertheless, it is instructive to exhibit the corresponding conformally invariant contributions to quadratic Lagrangians. Again in the metrical gauge field momentum only the irreducible pieces of volume-preserving[69] nonmetricity are allowed to occur

$$^{\dagger}\overline{M}^{\alpha\beta} = -\,{}^{*}\left(\sum_{K=1}^{3} q_{(K)} g^{\alpha\gamma} g^{\beta\delta}\ {}^{(K)\dagger}Q_{\gamma\delta}\right). \tag{6.1.10}$$

However, the corresponding quadratic nonmetricity Lagrangian transforms as

$$V_Q := -\tfrac{1}{4}{}^{\dagger}Q_{\alpha\beta} \wedge {}^{\dagger}\overline{M}^{\alpha\beta}, \qquad \widetilde{V}_Q = \Omega^{(n-2)L/2} V_Q. \tag{6.1.11}$$

Accordingly it has to be excluded in the four-dimensional world.

Summing up, in four dimensions, the most general conformally invariant "gravitational" Lagrangian, under the premises of quasi-linearity of the gauge field momenta, is of the Yang–Mills type, i.e., it involves only the curvature, but explicitly neither the torsion nor the nonmetricity. Neither a Hilbert–Einstein type term is admissible, nor a "cosmological" term $\Lambda\eta$ proportional to the volume density η of the world, which would transform as $\tilde{\eta} = \Omega^{nL/2}\eta$.

[67] In view of the invariance of (6.1.7) in two dimensions, such a term is a possible candidate for the bosonic part of the superstring action (see [338,339,237]). For $n = 2$ torsion is irreducible such that (6.1.7) is equivalent to a term quadratic in the trace T of the torsion (TRATOR). On the other hand, the Hilbert–Einstein term is trivial in two dimensions since it degenerates to an exact form $-(1/2)R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} = d[-(1/2)(\Gamma^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta})]$. This is due to $D\eta_{\alpha\beta} = 0$ and the Abelian nature of the $SO(2)$ part $\Gamma^{[\alpha\beta]} = (-1)^{\mathrm{Ind}(g)}\eta^{\alpha\beta}\Gamma^{*}$ of the connection. However, if one supplements the Hilbert–Einstein Lagrangian by a boundary term $d(\vartheta^{\alpha} \wedge {}^{*}T_{\alpha})$, one recovers the Yang–Mills term (5.9.19) that is quadratic in the torsion. It turned out that not only this constrained model but also the general PG theory is completely integrable in two dimensions (cf. [453] and refs. given). A constrained $GL(2,R)$ gauge model is analyzed in [350].

[68] We drop the contributions from the Weyl covector piece ${}^{(4)}Q_{\alpha\beta} = Qg_{\alpha\beta}$ since, for $L \neq 2F - 2C$, the Weyl one-form Q transform inhomogeneously under conformal changes of the metric, cf. (3.14.1) and (3.14.5). This excludes also the term $Q \wedge {}^{*}T$ which, due to an intriguing coupling of the Weyl covector to the torsion trace $T = e_{\alpha}\rfloor T^{\alpha}$, could possibly be responsible for a symmetry breaking from Weyl to Riemann–Cartan spacetimes, cf. section 8 of [280].

[69] The quadratic Weyl covector piece $Q \wedge {}^{*}Q$, which is related to a boundary term via $D\,{}^{*}Q - Q \wedge {}^{*}Q = d\,{}^{*}Q$, in general does not transform homogeneously under conformal changes.

Our conformally invariant quadratic curvature Lagrangian V_R contains volume-preserving and volume-changing parts. In order to exhibit this on par with the corresponding decomposition (3.12.6) of the curvature, we split the deformational gauge field momentum as follows:

$$H^\alpha{}_\beta = {}^{\dagger}H^\alpha{}_\beta + (1/n)\, \delta^\alpha_\beta\, H\,. \tag{6.1.12}$$

The trace part of $H^\alpha{}_\beta$ is, up to proportionality, nothing but the field momentum canonically conjugate to the Weyl one-form:

$$H := H^\alpha{}_\alpha = -\frac{\partial V_R}{\partial R_\alpha{}^\alpha} = -\frac{2}{n}\frac{\partial V_R}{\partial dQ}\,. \tag{6.1.13}$$

Then the conformal Lagrangian (6.1.4) may as well be rewritten as

$$V_R = -\tfrac{1}{2}({}^{\dagger}R_\alpha{}^\beta \wedge {}^{\dagger}H^\alpha{}_\beta + \tfrac{1}{2} dQ \wedge H)\,, \tag{6.1.14}$$

which clearly displays the separation into a volume-preserving and a volume-changing dynamics. This is also reflected in the following splitting of the second field equation which may be deduced from (6.1.12) by independent variations of the volume-preserving connection and of the Weyl vector or, simply, by decomposing (5.5.5) into its tracefree and trace part, respectively:

$$ {}^{\dagger}D\, {}^{\dagger}H^\alpha{}_\beta - {}^{\dagger}E^\alpha{}_\beta = {}^{\dagger}\Delta^\alpha{}_\beta\,, \tag{6.1.15}$$

$$DH + \vartheta^\alpha \wedge H_\alpha + M^\beta{}_\beta = DH = \Delta\,. \tag{6.1.16}$$

Since $H_\alpha = 0$ and $M^{\alpha\beta} = 0$ for (6.1.14), the volume-changing part of our theory is determined, not unexpectedly, by an inhomogeneous equation which formally resembles the Maxwell equation. Observe that the Weyl one-form has become a truly dynamical, i.e., propagating degree of freedom, even for a vanishing dilation current Δ. Incidentically, Buchmüller and Dragon [90] take the vanishing of DILCURV, i.e. the tensor relation ${}^{\dagger}R_\gamma{}^\gamma = 0$, as the gauge fixing condition for volume-preserving coordinate transformations. In this condition, an overlap between holonomic $GL(n,R)_H \subset Diff(n,R)$ and the anholonomic $GL(n,R)$ occurs.

In our primordial world, in which no explicit torsion terms are present in the Lagrangian, the first field equation is purely algebraic in the curvature, i.e., non-dynamical:

$$-E_\alpha = \Sigma_\alpha\,. \tag{6.1.17}$$

According to (5.4.12) and to (6.1.4) with (6.1.1), the trace of the deformational energy-momentum reads:

$$\vartheta^\alpha \wedge E_\alpha = nV + 2R_\alpha{}^\beta \wedge \overline{H}^\alpha{}_\beta = nV - 2R_\alpha{}^\beta \wedge \frac{\partial V}{\partial R_\alpha{}^\beta} = (n-4)\,V\,. \tag{6.1.18}$$

Observe that this relation is contained as a special case in the general formula (5.4.16). Its material counterpart is given by (5.7.23).

As a consequence of (6.1.18), for $n = 4$, as one would expect for a conformally invariant Yang–Mills type theory, the energy–momentum trace $\vartheta^\alpha \wedge \Sigma_\alpha$ vanishes[70] and only matter with vanishing

[70] For the Lagrangian $V = \kappa\, \underbrace{R_\alpha{}^\beta \wedge R_\beta{}^\gamma \wedge \cdots \wedge R_\gamma{}^\alpha}_{n/2 \text{ factors}}$, this is also true in n dimensions, provided n is even.

trace can consistently couple to $GL(4, R)$ gauge fields. In Riemannian spacetime, the quadratic Weyl curvature piece in (6.1.14) is frequently considered as a model of conformal gravity, see [329,447] and refs. therein. Recently, by calculating the independent deformations of conformally self-dual gravitational instantons [for Euclidean signature, i.e. for $\mathrm{Ind}(g) = 0$], Perry and Teo [560] found a topological invariant which resembles the first Donaldson invariant [162], [163] in the Yang–Mills case.

6.2. *Coupling to the dilaton*

So far our world is ruled from the beginning solely by primordial, conformally invariant deformational gauge fields ("conformal gravity"). Following Isham et al. [310], e.g., we may immerse, in addition, a primordial scalar field, the so-called *dilaton field* $\sigma(x)$, into this world. [71] The introduction of scalar fields into gravity has already a long history, see [310,311,577,211,431]; in our approach, in contrast to that of Jordan, Brans, and Dicke [322,77], the crucial feature is the onset of a possible symmetry breaking, following, for example, Englert et al. [184,185], Minkowski [464], Smolin [639,640], Gregorash and Papini [239,240], Zee [733,734], Nieh [515], Šijački [625], and Germán [221], see also the related work [145,528]. More recently, Flato and Raczka [198], as well as Cheng [111,112], cf. [34,59], have suggested to couple gravity via the isodoublet (complex) Higgs field ϕ^J $(J = 1, 2)$ to the Weinberg–Salam model such that the dilaton field σ is included as one of the four scalar degrees of freedom.

Since the Higgs field of the standard model couples very strongly to fermion matter – it makes the mass of the electron, muon, quarks, etc. – it cannot be related to the dilaton as Flato and Raczka [198] surmised. Had it been a Jordan–Brans–Dicke field, it would be 10^{40} times stronger than gravity. Also, the present thinking about the dilaton in stringy and other inflation is that the scalar field was active in the first 10^{-43} seconds and then became so weakly coupled as to obey the upper bounds on Brans–Dicke couplings – whereas the Higgs is expected to be produced at the LHC (Large Hadron Collider) because it is strongly coupled to all massive matter.

On the classical level, we can assume that the scalar field carries canonical dimensions, i.e. $(\text{length})^{-1}$ in $n = 4$ dimensions. Therefore, with respect to a conformal change (3.14.1) with weight L of the underlying metric structure, the scalar field is really a density which transforms [693] according to

$$\sigma(x) \to \tilde{\sigma}(x) = \Omega(x)^{Ld_\sigma/2}\,\sigma(x) = \Omega(x)^{-(n-2)L/4}\,\sigma(x)\,. \tag{6.2.1}$$

In order to account for possible propagating modes of this scalar field, we have to construct a conformally invariant term for the kinetic part of the dilaton Lagrangian. Since the dilaton carries the dimension $d_\sigma = (2-n)/2$, cf. (4.7.2), the gauge-covariant exterior derivative reduces to the usual exterior derivative amended by a connection trace term $\Gamma = \Gamma_\alpha{}^\alpha$ as a compensating "potential". In an (L_n, g), the conformally covariant derivative of a scalar field is then given by

[71] According to Adler [2], e.g. the dilaton is not elementary, but should rather be regarded as a fermion condensate, i.e. $\sigma \doteq (\overline{\Psi}\Psi)^{(n-2)/2(n-1)}$. In the literature, there are various parametrizations of the dilaton field in use: According to Coleman [124], the prescription to render a theory dilation or scale invariant is to replace each mass scale μ by a field $\hat{\sigma}$, the dilaton, such that the dilaton couples in a universal way to these mass terms via $\overline{\mu} = \mu\, e^{\hat{\sigma}/f}$, where f is the dilaton decay constant [87]. In other conventions, the dilaton is related via $\sigma(x) = \frac{1}{2}[(n+2)/n]^{1/2} \ln \Omega(x)$ to the conformal part Ω of the metric $g_{\alpha\beta} = \Omega^{1/n}\, s_{\alpha\beta}$.

$$D\sigma = (d - \omega_\sigma \Gamma)\,\sigma\,. \tag{6.2.2}$$

For $\omega_\sigma = -(Ld_\sigma/2nC)$, the conformal covariance of $D\sigma$ follows from (6.2.1) and the transformation formula (3.14.1) for the trace of the connection:

$$\begin{aligned}\widetilde{D}\widetilde{\sigma} &= (d - \omega_\sigma \Gamma)\,\Omega^{-(n-2)L/4}\sigma \\ &= \Omega^{-(n-2)L/4}[-\tfrac{1}{4}(n-2)L\,d\ln\Omega + d - \omega_\sigma\Gamma + nC\omega_\sigma\,d\ln\Omega]\sigma \\ &= \Omega^{-(n-2)L/4}\,D\sigma\,. \end{aligned}\tag{6.2.3}$$

Note that, contrary to the description of Nieh [515], in our approach the constant in front of the trace connection one-form is fixed by the dimension d_σ of the scalar field relative to that of the metric, cf. (6.2.1) and (3.12.8). Thus an (L_n, g) provides a rather natural framework to accommodate conformal changes of metrics; in particular, there seems to be no need to define anew a "conformally covariant" derivative (cf. Bregman [82], Smolin [639]). Observe also that the conformal variation [318,622] of a scalar field is, in our formalism, simply obtained by the gauge-covariant Lie derivative, cf. (A.1.38):

$$\delta^{(c)}\sigma = \pounds_\xi\sigma = \mathrm{Ł}_\xi\sigma - \frac{d_\sigma}{n}\left(e_\alpha\rfloor\,\widehat{D}\,\xi^\alpha\right)\sigma = \xi\rfloor D\sigma + \frac{n-2}{2n}\left(e_\alpha\rfloor\,\widehat{D}\,\xi^\alpha\right)\sigma\,. \tag{6.2.4}$$

For a p-form $\alpha^{(p)}$, the $GL(n,R)$ gauge-invariant d'Alembertian operator reads

$$\Box\alpha^{(p)} := (-1)^{pn+s}\left[{}^*D\,{}^*D + (-1)^n D\,{}^*D\,{}^*\right]\alpha^{(p)}\,. \tag{6.2.5}$$

(In a Riemannian space, a related operator for an arbitrary tensor is explicitely constructed by Yano [729, p.67].) For a zero form, i.e. a scalar field σ, the second part in the definition (6.2.5) would lead to an $(n+1)$-form and, consequently, drops out. Then the gauge-invariant d'Alembertian can be rewritten as

$$\Box\sigma = (-1)^s\,{}^*D\,{}^*D\sigma = \overline{\Box}\sigma - (-1)^s\omega_\sigma\,{}^*(d\,{}^*\Gamma - \omega_\sigma\,\Gamma\wedge{}^*\Gamma)\,\sigma, \tag{6.2.6}$$

where $\overline{\Box} := (-1)^s\,{}^*d\,{}^*d$ denotes the usual d'Alembertian operator in curved spacetime. Due to (6.2.3), the operator $\Box$ is also conformally invariant. In a Riemannian spacetime, the known conformally invariant wave operator (cf.[442,726], e.g.) reads:

$$\Box\sigma = \overline{\Box}\sigma + \frac{d_\sigma}{(n-1)}\,(e_\alpha\rfloor e_\beta\rfloor R^{\{\}\alpha\beta})\,\sigma\,. \tag{6.2.7}$$

In view of (6.2.3) and (3.14.13), our procedure implies that the kinetic part of the dilaton Lagrangian is conformally invariant in any dimension:

$$L_\Box := -\tfrac{1}{2}(-1)^{\mathrm{Ind}(g)}\,D\sigma\wedge{}^*D\sigma, \qquad \widetilde{L}_\Box = \Omega^{(Ld_\sigma+(n-2)L/2)}\,L_\Box = L_\Box\,. \tag{6.2.8}$$

By variation of (6.2.8) with respect to σ, we obtain the conformally invariant scalar wave operator (6.2.6).

The dilaton Lagrangian with a polynomial self-interaction n-form $U(|\sigma|)$ is given by

$$L_\sigma = L_\Box + U(|\sigma|)\,\eta. \tag{6.2.9}$$

By variation of (6.2.9) with respect to σ, we obtain the scalar wave equation

$$\frac{\delta L_\sigma}{\delta\sigma} = (-1)^{\mathrm{Ind}(g)}\, D\,{}^*D\sigma + \frac{\partial U(|\sigma|)}{\partial\sigma}\,\eta = 0$$
$$\Leftrightarrow\quad {}^*\left(\frac{\delta L_\sigma}{\delta\sigma}\right) = \Box\,\sigma + (-1)^{\mathrm{Ind}(g)}\,\frac{\partial U(|\sigma|)}{\partial\sigma} = 0. \tag{6.2.10}$$

6.3. Improved energy–momentum current

The trace of the dilaton's energy–momentum current, i.e.

$$\vartheta^\alpha \wedge \Sigma_\alpha(\sigma) = (n-2)\, L_\Box + nU(|\sigma|)\,\eta \tag{6.3.1}$$

does not vanish. Therefore, we need to improve Σ_α in this respect. As an intermediate step, we could proceed from the identity

$$\sigma_\beta \equiv \Sigma_\beta + e_\alpha \rfloor D\Delta^\alpha{}_\beta + e_\alpha\rfloor\left(\rho(L^\alpha{}_\beta)\,\Psi\wedge\frac{\delta L}{\delta\Psi}\right) \tag{6.3.2}$$

for the metrical energy–momentum $(n-1)$-form defined by (5.1.10), which follows from the strong Noether identity (5.2.16). For the dilaton field carrying canonical dimensions $d_\sigma = (2-n)/2$, only the trace of the hypermomentum current and of the linear generators contribute:

$$\Delta^\alpha{}_\beta = \frac{1}{n}\,\delta^\alpha_\beta\,\Delta\,, \qquad \rho(L^\alpha{}_\beta)\sigma = \frac{1}{n}\,\delta^\alpha_\beta\,\mathcal{D}\sigma = \frac{1}{n}\,\delta^\alpha_\beta\, d_\sigma\sigma\,. \tag{6.3.3}$$

Consequently, Eq. (6.3.2) simplifies to

$$\sigma_\alpha \equiv \Sigma_\alpha + \frac{1}{n} e_\alpha\rfloor D\Delta + \frac{d_\sigma}{n}\, e_\alpha\rfloor\left(\sigma\wedge\frac{\delta L_\sigma}{\delta\sigma}\right)\,. \tag{6.3.4}$$

Since the kinetic part of the dilaton Lagrangian (6.2.8) explicitly depends on the connection trace one-form $\Gamma_\gamma{}^\gamma$, the scalar field does also provide an intrinsic dilation current. According to (5.1.19), the latter is dynamically defined by

$$\Delta := \Delta^\gamma{}_\gamma = \frac{\delta L}{\delta\Gamma_\gamma{}^\gamma} = -(-1)^{\mathrm{Ind}(g)}\,\frac{2-n}{2n}\,\sigma\,{}^*D\sigma\,, \qquad \widetilde{\Delta} = \Delta\,. \tag{6.3.5}$$

The trace of the metrical energy–momentum current σ_α does not vanish either; but for $n \geq 3$ it depends also on kinetic terms (which would become "hard" in the momentum representation) as follows

$$\begin{aligned}\vartheta^\alpha\wedge\sigma_\alpha &= \vartheta^\alpha\wedge\Sigma_\alpha(\sigma) + D\Delta + d_\sigma\left(\sigma\wedge\frac{\delta L_\sigma}{\delta\sigma}\right)\\ &= \frac{(n-1)(n-2)}{n} L_\Box + (-1)^{\mathrm{Ind}(g)}\,\frac{(n-1)(n-2)}{2n}\,\sigma D{}^*D\sigma\\ &\quad + \left(nU(|\sigma|) - \frac{n-2}{2}\,\sigma\,\frac{\partial U(|\sigma|)}{\partial\sigma}\right)\eta.\end{aligned} \tag{6.3.6}$$

In our formalism we may define a "new improved" energy–momentum current for scalar fields by

$$\widetilde{\sigma}_\alpha := \sigma_\alpha + \frac{(n-1)}{n}\, e_\alpha\rfloor D\Delta. \tag{6.3.7}$$

For its trace we find the "strong" relation

$$\vartheta^{\alpha} \wedge \widetilde{\sigma}_{\alpha} = \vartheta^{\alpha} \wedge \sigma_{\alpha} + (n-1) D\Delta = \left(nU(|\sigma|) - \frac{n-2}{2} \sigma \frac{\partial U(|\sigma|)}{\partial \sigma} \right) \eta. \tag{6.3.8}$$

Compared to (6.3.1) and (6.3.6), kinetic terms such as $L_{\square}$ are now absent in (6.3.8). In contradistinction to our ealier "weak" result in [268], this is now a *strong* improvement, because no field equations have been employed in our derivation; cf. Kraus and Sibold [368] for the related case of flat spacetime.

Moreover, due to Euler's theorem for homogeneous functions, the $\sigma^{2n/(n-2)}$ piece in the potential $U(|\sigma|)$ drops out. For a polynomial potential $U(|\sigma|)$ of degree $p \leq 2n/(n-2)$, the operator dimensionality is then smaller than n (for $n \geq 3$). Therefore, the new trace is indeed "soft" in a momentum representation in the sense of Jackiw ([317, p.213]; cf. Kopczyński et al. [365]). Note that a pure $\sigma^{2n/(n-2)}$ model is, in flat spacetime, known to be renormalizable according to the criteria of power counting [693]. Moreover, the left hand side of (6.3.8) is related to dilation anomalies which are measured by the deviation of the effective self-interaction potential U from its conformally invariant form, compare [715, Eq. (3.4)].

It is a further consequence of (6.3.5) that a necessary and sufficient condition for a vanishing dilation current is the covariant constancy of the dilaton field, provided it is non-zero:

$$D\sigma = 0 \Longleftrightarrow \Delta = 0. \tag{6.3.9}$$

In the wake of a symmetry breaking, to which we will turn in the next section, there occur "mixed" terms involving, besides σ, also the curvature scalar. Due to its projective invariance, cf. (3.11.8), such a term does not contribute to the dilation current.

By construction, the kinetic part (6.2.8) of the dilaton Lagrangian is independent of the tracefree, that is, volume-preserving nonmetricity (3.12.4). Thus the dilaton field σ does *not* contribute to the shear current, and we may infer from the field equation (6.1.15) that the *ground state* of our metric-affine world is undeformed by material shear so far. Only for such violent spacetime fluctuations which break up the metrical continuum into a possibly non-causal "spacetime foam" and create "baby universes" and "wormholes" ([125], cf. [441]), the shear degrees of freedom of the gravitational gauge fields could get excited by self-interaction, yielding, due to the essentially dynamical character of (6.1.15), propagating modes.

6.4. Breaking of the dilation symmetry: induced Einsteinian gravity

Local scale invariance of fundamental non-gravitational interactions is valid only approximately in the high energy limit of Bjorken scaling. For gravity we expect the same at the onset of the big bang (or at extremely high energies). After a very short time lag, the Weyl group of local scale invariance would have broken down to the Poincaré group. In order to model this symmetry breaking in our post-Riemannian framework, further dynamical ingredients have to be added to our highly symmetrical, but for these reasons rather unphysical, Yang-Mills like gravitational world.

As in Goldstone's model field theory [231], nonlinear terms in the dilaton field σ provide the essential means to achieve this. Let us consider the nonlinear self-interaction potential [72]

$$U(|\sigma|) = \frac{1}{4!}\lambda_\sigma \sigma^4 \eta, \qquad U(|\tilde{\sigma}|) = \Omega^{-(n-4)L/2} U(|\sigma|). \tag{6.4.1}$$

In four dimensions, we find that the (renormalizable) σ^4-term of the completed dilaton Lagrangian is conformally invariant.

On the dynamical part of the geometrical gauge fields, the curvature scalar term (6.1.5) and the explicit torsion-squared terms (6.1.7) had been prohibited by the requirement of conformal invariance. With the advent of the dilaton field, this is no longer true. In fact, the "mixed" n-form [73]

$$V_{\sigma\Gamma} = \sigma^2 \left(\frac{1}{\chi} V_{\rm EC} + V_{\rm T} + V_{\rm Q} \right), \tag{6.4.2}$$

which represents a "contact"-type interaction of the deformational and scalar degrees of freedom, aquires invariance with respect to conformal changes of metric in any dimensions:

$$\widetilde{V}_{\sigma\Gamma} = V_{\sigma\Gamma} \,. \tag{6.4.3}$$

Therefore it should be included in our canon of at most quadratic Lagrangians, although such contact type coupling to the dilaton is perturbatively non-renormalizable [90].

In the derivation of the Noether identities for the total Lagrangian, such contact terms of the Brans–Dicke type have formally been accounted for by the generalized definitions (5.1.7)–(5.1.9) of the matter currents. In effect, these terms just provide us with the following additional field momenta of translations and deformations:

$$M^{\alpha\beta}(\sigma\Gamma) = \sigma^2 M^{\alpha\beta}, \qquad H_\alpha(\sigma\Gamma) = \sigma^2 H_\alpha, \qquad H^\alpha{}_\beta(\sigma\Gamma) = (1/2\chi)\sigma^2 g^{\alpha\gamma}\eta_{\gamma\beta}\,. \tag{6.4.4}$$

The degenerate form of the $GL(n,R)$ gauge field momentum is pertinent to the Hilbert–Einstein or Einstein–Cartan type Lagrangian.

The complete Lagrangian for the so-called dilaton field, including these mixed terms, eventually assumes the form

$$L_\sigma = L_\Box + U(|\sigma|) + V_{\sigma\Gamma}. \tag{6.4.5}$$

An extremum (preferable a minimum) of this gravitationally coupled nonlinear scalar field Lagrangian necessarily occurs for

$$\frac{\delta L_\sigma}{\delta\sigma} = \Box\,\sigma + 2\left(\frac{1}{\chi} V_{\rm EC} + V_T + V_Q\right)\sigma + \frac{\partial U(|\sigma|)}{\partial\sigma} = 0, \tag{6.4.6}$$

provided the scalar field equation is satisfied. Due to the presence of the connection trace one-form in the derivative D and in $\Box$, the specification of the ground state configuration is more involved than

[72] Although this term is usually introduced ad hoc in a purely Riemannian geometry, it may itself be the result of a conformally rescaled and then "frozen in" gravity, cf. [44].

[73] In (6.4.2), due to its conformal invariance, we could also include the term $\sigma^2 V_{\rm QT}$ with the novel mixing of nonmetricity and torsion.

in an uncoupled model. Similarly as in the Abelian Higgs model, cf., e.g., [114,566], we may seek configurations for which

$$\sigma \simeq \langle\sigma\rangle = v_\sigma, \qquad \frac{\partial U(|\sigma|)}{\partial\sigma} = 0, \qquad D\sigma \simeq 0 \tag{6.4.7}$$

hold asymptotically. According to (6.3.9), the dilaton current vanishes too, i.e. $\Delta \simeq 0$.

For a model of massless scalar electrodynamics, Coleman and E. Weinberg [127,709] have shown that radiative corrections *induce* a spontaneous symmetry breakdown of the $U(1)$ gauge symmetry. In the first loop approximation, there arises an *effective* self-interacting potential, which has a minimum away from the origin:

$$\frac{\lambda_\sigma}{4!}\sigma^4 \overset{\text{radiative corr.}}{\Longrightarrow} V_{\text{eff.}} = \frac{3\omega_\sigma^4}{64\pi^2}\sigma^4\left(\ln\frac{\sigma^2}{\langle\sigma^2\rangle} - \frac{1}{2}\right). \tag{6.4.8}$$

It is tempting to adopt the hypothesis that a similar mechanism applies to the dilatons of our gravitational model also in a non-flat background, in particular, since there is a relation [328] between chiral and scale invariance for extended models.

However, a *dimensional transmutation* [124] appears to be a general feature of such an induced symmetry breaking: The dimensionless parameter λ_σ is transmuting into a dimensional one, i.e. $\langle\sigma\rangle$, which breaks conformal invariance *explicitly*. By comparison with the macroscopic world in which Newton's gravitational constant G_N is inherent, the physical scale is then necessarily determined by

$$\langle\sigma\rangle = {}_{(-)}^{+}\sqrt{\chi}/l, \tag{6.4.9}$$

where $l = \sqrt{8\pi G_N \hbar/c^3}$ denotes the Planck length. In this setting, Newton's gravitational constant in Einstein's GR is a result of such a symmetry breaking (cf. [185,464]) of conformal gravity.

Let us consider the "induced" gravitational world in the vicinity of this ground state. In this crude approximation, we obtain

$$L_\sigma \simeq L'_{\text{EC}} = \frac{1}{l^2}\left(V_{\text{EC}} + \chi V_{\text{T}} + \chi V_{\text{Q}} - \Lambda_{\text{ind.}}\right), \quad \Lambda_{\text{ind.}} = \frac{3\omega_\sigma^4\chi^2}{128\pi^2 l^2}. \tag{6.4.10}$$

In this order of approximation, the kinetic part of the scalar field vanishes. Thus we end up with an Einstein-Cartan Lagrangian plus possible explicit torsion terms of teleparallelism type theories and optional nonmetricity terms. In the conformal gauge Lagrangian (6.1.14), only the volume-preserving part survives, due to $dQ \simeq 0$, which follows from (6.4.7). Having started from a gauge theory of local $GL(n,R)$-deformations, we find that the gravitational vacuum structure of spacetime gets broken down from an (L_n, g) to a Riemann-Cartan spacetime U_n with its associated Poincaré group. The *induced* "cosmological" constant $\Lambda_{\text{ind.}}$ is really of microscopic origin and is notoriously large, cf. [712].

A constant mass term $m = \langle\sigma\rangle$ for world spinors is induced as a further result of the breaking of the local scale symmetry through the non-trivial vacuum expectation value of the dilaton field σ. In view of the small length scale induced by (6.4.9), the rest mass m of world spinors obeys the relation

$$8\pi G_N m^2/(\hbar c) \sim \chi. \tag{6.4.11}$$

By adjusting the coupling constant χ in the Einstein–Cartan part of the Lagrangian, the manifield mass m can be shifted to a value far below the superheavy Planck mass M_{Planck}.

If we had admitted the term $\sigma^2 Q \wedge {}^*Q$ in our gauge Lagrangian, a superheavy Weyl vector boson would occur after symmetry breaking [644]. So far, dilation or Weyl invariance has not offered a solution of the *cosmological constant problem* [712,129], i.e. its vanishing in the present epoch.

6.5. Triggering spontaneous symmetry breakdown of $\overline{SL}(4,R)$

The discussion of renormalizability, which we touched upon in section 2.2, made it clear that the very-high-energy (VHE) regime (i.e. higher than Planck energies) of our theory of quantum gravity has to be ruled, instead of the Brans–Dicke type terms (6.4.2) of the previous section, rather by $d_{R^2} = -4$ terms and that the gauge field appearing in these terms should represent physically independent degrees of freedom. This has forced us to enlarge the (anholonomic) gauge group in its homogeneous part beyond $SL(2,C)$. In this section, we are going to discuss a model in which a spontaneous symmetry breakdown (SSB) occurs in the volume-preserving part $\overline{SL}(4,R)_A \subset \overline{GL}(4,R)_A$. The breakdown occurs either directly or indirectly. The indirect process is induced by the same mechanism we described in section 6.4, which triggered the breakdown of local scale invariance, i.e. of the R^+ part in $\overline{GL}(4,R) = (T \otimes \overline{SL}(4,R)) \times R^+$.

Continuing in our quest for renormalizability, we use the Yang–Mills model for our gauge theory. Our $d = -4$ VHE Lagrangian should be quadratic in the $\overline{GL}(4,R)_A$ curvatures. According to (6.1.4) and (6.1.1), the most general quadratic Lagrangian four-form reads:

$$V_{\text{sym.}} = V_R = -\frac{1}{2\kappa}\sum_{N=1}^{11} b_{(N)} g^{\alpha\gamma} g_{\beta\delta}\, {}^{(N)}R_\alpha{}^\beta \wedge {}^*({}^{(N)}R_\gamma{}^\delta). \tag{6.5.1}$$

This Lagrangian would encompass the "SKY" (Stephenson–Kilmister–Yang) [74] Lagrangian [654], were it not for the addition of spontaneous $\overline{GL}(4,R)_A$-symmetry breaking terms. The conformal gravity Lagrangian has non-vanishing nonmetricity $Q_{\alpha\beta} = -Dg_{\alpha\beta} \neq 0$ contributed by the $\overline{SL}(4,R)_A/SL(2,C)$ components of the connection, as can be seen from (3.10.12).

In fact, the Schwarzschild–Einstein–Newton component and the related macroscopic horizons will be provided by the $d_R = -2$ terms corresponding to the SSB of $\overline{GL}(4,R)_A$ and of its $\overline{SL}(4,R)_A$ subgroup in the low energy region, underneath the Planck energy, i.e. at distances much larger than the Planck length $l = 10^{-33}$ cm.

The terms that will dominate the low-energy region will effectively *generate* a *vanishing nonmetricity* and reproduce the Hilbert–Einstein Lagrangian (or equivalent macrocopic Lagrangians). It will correspond to the "Higgs sector" in a Weinberg–Salam type model: Although the $-\mu^2\phi^2$ term that generates the SSB in those theories has precisely the same dimensionality as the Einstein and/or

[74] It will be remembered that criticism of the SKY Lagrangian [654,348,728] had centered on its unsuitability for a theory of gravity in the large, since it has no Birkhoff theorem, i.e. the exterior Schwarzschild solution is not the unique spherically symmetric vacuum solution. Moreover, there is no decent Newtonian limit because of the surfacing (from the Riemannian connection) of 3*rd* derivatives of the metric (Baekler et al. [25]). However, the physical requirements are different in our case: The Lagrangian (6.5.1) of the renormalizable Yang–Mills type dominates the VHE regime, but needs to be amended by symmetry breaking terms in order to cope with the macroscopic, low energy region. Thus, the Newtonian limit is reproduced by other (dimension 2) terms.

the $(torsion)^2$-terms in gravity, explicit mass terms are *not* favored in our conformal approach to gravity. In our approach, the effective Hilbert–Einstein Lagrangian $\sigma^2 V_E$ will have the same structure as the effective W mass term $\langle\Phi^2\rangle W^2$ of SSB in Yang–Mills–Higgs theories.

The low energy (broken $\overline{GL}(4,R)_A$ gauge symmetry) region preserves a subgroup of $\overline{GL}(4,R)_A$, and this is precisely the Lorentz group, thus ensuring that the long-range component corresponds to a Riemannian geometry and has Einsteinian features.

The symmetry breakdown could occur "spontaneously" through the assignment of a non-vanishing vacuum expectation value,

$$\langle\Phi_{00}\rangle \neq 0 \tag{6.5.2}$$

for the $(0,0)$-component of a manifield Φ, i.e. an infinite-component field behaving as an "$\mathcal{A}$-deunitarized" representation of $\overline{SL}(4,R)_A$, cf. section 4.

We deal with manifields in the context of the matter fields in MAG. Here, however, Φ is a bosonic manifield, behaving as an $\mathcal{A}$-deunitarized representation of $\overline{GL}(4,R)_A$. Indeed, to treat the breakdown of the Yang–Mills-like gauge symmetry appropriately, the Higgs field has to transform as a non-trivial representation of that group, with a component that has the Lorentz group (on the frames) as its stability subgroup; this is Φ_{00}, a component behaving as the $(0,0)$, i.e. scalar representation of the Lorentz deunitarized subgroup, and it acquires a non-vanishing vacuum expectation value.

In other words, Φ is reduced over the "apparent" compact subgroup $SO(4)$, which is physically just the A-transform of the Lorentz group, the $\overline{SO}(1,3)_A = SL(2,C)_A$ subgroup of $\overline{GL}(4,R)_A$. It has finite non-unitary representations, just as in finite tensors (in which the $GL(4,R)$ representation itself is "naturally" non-unitary, and so are those of the $SL(2,C)$ subgroup).

In writing the Φ-Lagrangian, we thus have to use coframes. In this case, the coframes have their component's labels ranging over a countable infinity. Such coframes are gravitational-field valued matrices, relating $\overline{GL}(4,R)_A$ to $\overline{GL}(4,R)_H$, cf. [99]. For bosonic manifields, the double covering is collapsed and the coframe can then be given in terms of the conventional coframes by

$$\theta^B = C^B{}_\beta \vartheta^\beta = H_J{}^B G^J{}_i\, dx^i \qquad \begin{cases} H_J{}^B = C^B{}_\beta e_j{}^\beta G^j{}_J \\ E^I{}_A = G^I{}_i e^i{}_\alpha C^\alpha{}_A\,, \end{cases} \tag{6.5.3}$$

where the $C^B{}_\beta$ and $G^J{}_i$ are transition matrices of $SL(2,C)$ and $SL(4,R)$, respectively. The $C^B{}_\beta$ consist of a reduced infinite sum of rectangular matrices that relate, within one single $\mathcal{A}$-deunitarized representation of $\overline{SL}(4,R)_A$, the A, B labels of the finite (non-unitary) representations of $SL(2,C)$ – replacing here the $\overline{SO}(4)$ compact subgroup representations in the $\mathcal{A}$-deunitarized representation of $\overline{SO}(4,R)_A$ itself – to the four-dimensional α, β indices of the local Lorentz group, also saturating a four-dimensional representation of $SL(4,R)_A$. The $G^J{}_i$ relate the four-dimensional i, j indices of $\overline{SL}(4,R)_H$ to the infinite-dimensional I, J indices of the $\mathcal{A}$-deunitarized representation of that group.

We take the following conventional Lagrangian for the Φ manifield, together with the dilaton field σ that breaks $\overline{GL}(4,R)_A$ but is invariant under the traceless $\overline{SL}(4,R)_A$ and under conformal changes:

$$L_{\rm Higgs} = -\tfrac{1}{2}(-1)^{{\rm Ind}(g)}\, D\Phi \wedge {}^*D\Phi - \tfrac{1}{2}(-1)^{{\rm Ind}(g)}\, D\sigma \wedge {}^*D\sigma - U(\Phi,\sigma)\eta + L_{\rm Yu}, \tag{6.5.4}$$

where

$$U(\Phi,\sigma) = \tfrac{1}{4}[\lambda_\Phi(\Phi^\dagger\Phi)^2 + 2\lambda(\Phi^\dagger\Phi)\sigma^2 + \lambda_\sigma\sigma^4] \tag{6.5.5}$$

represents a self-interaction, and

$$L_{\rm Yu} = (\mu \bar{\Psi} \Phi \Psi + \nu \sigma \bar{\Psi} \Psi) \eta \tag{6.5.6}$$

a Yukawa coupling to the fermions. In the presence of the dilaton, we do not have to assume the nonvanishing vacuum expectation value for Φ_{00} as a separate ansatz. Rather the λ term in (6.5.5), representing an interaction between Φ^2 and σ^2, when, as in section 6.4,

$$\langle \sigma \rangle = v_\sigma \neq 0, \tag{6.5.7}$$

will provide an effective negative mass-squared term for Φ^2. Writing

$$\sigma(x) = v_\sigma + \sigma_{\rm Higgs}(x), \tag{6.5.8}$$

we find, cf. (6.4.9),

$$l = \sqrt{\chi}/v_\sigma, \qquad G_N = \chi/8\pi {v_\sigma}^2. \tag{6.5.9}$$

The $\Phi^2\sigma^2$ term contributes to the $U(\Phi, \sigma)$-potential, which, up to the constant term $\lambda_\sigma v_\sigma^4$, becomes

$$U(\Phi^2) = \tfrac{1}{4}[2\lambda {v_\sigma}^2 \Phi^2 + \lambda_\Phi (\Phi^2)^2]\eta, \tag{6.5.10}$$

where $\Phi^2 = \Phi^\dagger \Phi := \sum_B \Phi_B^* \Phi^B$, $B := (j_1, j_2)$. A sufficient condition for a minimum is

$$\left(\lambda v_\sigma^2 + \lambda_\Phi \sum_B \Phi_B^* \Phi^B\right) \Phi^C = 0,$$

$$D\Phi^C = [\delta_B^C (d - \omega_\Phi \Gamma) + \Gamma_{(\alpha\beta)} (\mathcal{E}^{(\alpha\beta)})_B{}^C] \Phi^B \simeq 0, \tag{6.5.11}$$

so that for $\lambda < 0$ we get a SSB of $\overline{SL}(4, R)$. Applying a suitable $\overline{SL}(4, R)/\overline{SO}(1, 3)$ transformation, within an irreducible subspace, i.e. after a choice of a gauge, we have

$$\sum_B \Phi_B^* \Phi^B = \Phi_0^* \Phi^0, \tag{6.5.12}$$

thus obtaining

$$\Phi^B(x) = v'_\Phi \delta_0^B + \Phi^B_{\rm Higgs}(x) \tag{6.5.13}$$

$$v'_\Phi = \sqrt{-\lambda/\lambda_\Phi}\, v_\sigma. \tag{6.5.14}$$

The spinorial manifield Ψ acquires a mass

$$M(\Psi) = \mu v'_\Phi + \nu v_\sigma \tag{6.5.15}$$

also of the order of the Planck mass (up to the coupling constants μ and ν).

Returning to the Φ manifield, above its lowest level $(0,0)$ we find the three representations $(2,0)$, $(1,1)$, $(0,2)$. The $(2,0)$ and $(0,2)$ cannot be reached from $(0,0)$ by a single application of the $sl(4,R)/sl(2,C)$ generators $\mathcal{E}_{(\alpha\beta)}$, while

$$|(1,1)\rangle = \mathcal{E}_{(\alpha\beta)}\, |(0,0)\rangle, \tag{6.5.16}$$

i.e. the $(1,1)$ components have the quantum numbers of the $sl(4,R)/sl(2,C)$ generators and supply longitudinal components to the shear connections. These components of the connection (contributing via (3.10.3) to the nonmetricity) are thus massive.

Performing a gauge transformation

$$\Lambda_{(\alpha\beta)} = \exp(\not{E}^{(\alpha\beta)}\, \omega_{(\alpha\beta)}), \quad \omega_{(\alpha\beta)} \sim (1,1), \tag{6.5.17}$$

we find for the infinitesimally gauge-equivalent connection,

$$\bar{\Gamma}_{(\alpha\beta)} \cong \Gamma_{(\alpha\beta)} - d\omega_{(\alpha\beta)}, \tag{6.5.18}$$

cf. (3.6.9) for the precise relation. The mass matrix of the shear part of the connection itself is given by

$$\langle \not{E}^{(\alpha\beta)}\, v_\Phi | v_\Phi \not{E}^{(\gamma\delta)} \rangle \tag{6.5.19}$$

with the $(\not{E}^{(\alpha\beta)})_B{}^C$ matrix elements corresponding to the transitions $(0,0) \to (1,1) \to (0,0)$. Using the 3-$j$ coefficients [625,496] of the $sl(4,R)$, we obtain

$$M^2(\Gamma_{(\alpha\beta)}) = \frac{1}{4}(16 + e_2{}^2) \begin{bmatrix} 0 & 1 & 1 \\ 0 & -a & a \end{bmatrix}^2 \begin{bmatrix} 0 & 1 & 1 \\ 0 & -b & b \end{bmatrix}^2 v_\Phi'^2, \tag{6.5.20}$$

with $\alpha, \beta = 1, 0, -1$ (the spherical basis).

The mass of the dilational part $\bar{\Gamma}^\gamma{}_\gamma$ of the affine connection turns out to be

$$M^2(\bar{\Gamma}^\gamma{}_\gamma) = v_\Phi'^2. \tag{6.5.21}$$

The Lorentz connection does not acquire mass and for the Higgs manifield we find

$$M^2(\Phi^B_{\text{Higgs}}) = -4\lambda v_\Phi'^2, \quad \lambda < 0, \tag{6.5.22}$$

where the $(j_1, j_2) \neq (1,1)$ have become the longitudinal components of the shear part of the connection.

This model has been studied further, first by constructing the appropriate BRST equations [388] and then proving that it is renormalizable [387,389]. Its renormalizability does not derive from $(1/p^4)$-propagators as in the case of quadratic Riemannian Lagrangians [652], but is rather akin to renormalizable Yang–Mills theories, whose proof of renormalizability is followed in [387,389], except for the complications induced by the diffeomorphism gauge. What remains unanswered is the question of unitarity. One would have to prove that in higher orders, no effective dipole-ghost terms would ever emerge, etc.

6.6. *Extended inflation*

The inflationary model, see Linde [396] and Guth [252] for recent overviews, is a modification of the standard big bang model which is aiming at providing answers to such cosmological issues as: (1) large scale uniformity, (2) flatness [near to the critical density], (3) absence of magnetic GUT monopoles, (4) almost scale-invariant spectrum of the microwave background as seen by COBE, etc..

The standard as well as the inflationary model is based on the isotropic Robertson-Walker metric [75]

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) \tag{6.6.1}$$

of Einstein's equation with effective cosmological constant $\Lambda_{\rm eff.}$ induced by the "false" vacuum density. (An open, flat, or closed universe is characterized by $k = -1, 0, +1$, respectively.) For the metric (6.6.1), Einstein's equation reduces to the Friedmann equation

$$\dot{a}^2 = -k + \tfrac{1}{3}a^2(l^2\rho + \Lambda_{\rm eff.}) \tag{6.6.2}$$

together with the Raychaudhuri equation of energy conservation.

The inflationary period lasts only for a small fraction of a second and provides really a model for the "bang" itself. After this period, the de Sitter type exponential inflation with

$$a = a_0 \exp[(\Lambda_{\rm eff.}/3)^{1/2} t], \qquad k = 0, \quad \rho = 0$$

is supposed to join smoothly onto the standard model based on the Friedmann solution ($k = 0$, $\Lambda_{\rm eff.} = 0$) of Einstein's equation ("graceful exit" problem). The possible effects of a primordial anisotropy on inflation have also been discussed in the literature [301].

What is important in our context is that almost all mechanisms for inflation depend on a dynamical scalar field that has a sufficiently long flat region in its potential to allow for the de Sitter phase. The interaction with Einsteinian gravity is of the Jordan-Brans-Dicke type [77] and, therefore, resembles the dilaton model of section 6.4. Moreover, in the extended inflationary model of La and Steinhardt [382] and in the "eternal" modification of Linde [397], cf. [301,558], the graceful exit problem is attacked by introducing two scalar fields with a specific selfinteracting potential $U(\phi, \sigma)$ similar to our manifield potential (6.5.5) of SSB: After singling out and identifying one component of the bosonic manifield Φ with the Brans-Dicke scalar ϕ, the structure of the inflationary Lagrangian is

$$\begin{aligned} L_{\rm infl} &= V_{\phi R} + L_\phi + L_\sigma \\ &= -(1/2\chi)\phi^2 R^{\alpha\beta}\wedge\eta_{\alpha\beta} + \tfrac{1}{2}D\phi\wedge{}^*D\phi + \tfrac{1}{2}D\sigma\wedge{}^*D\sigma - U(\phi,\sigma)\eta . \end{aligned} \tag{6.6.3}$$

The dilaton is the one which can be in practice the Brans-Dicke field. The inflaton is generally the GUT Higgs field, and it is generally not coupled to gravity in the Brans-Dicke fashion. In extended inflation, the dilaton-Brans-Dicke field is coupled to the curvature scalar and causes a change in the effective value of Newton's constant; this in turn changes the Hubble "constant" which is proportional to the square-root of G, and this helps the transition from false to true vacuum. If these two scalar fields were related via $\sigma = \frac{1}{2}[(n+2)/n]^{1/2}\ln\phi$, as in [120], the dilaton could play, at the same time, the role of the "inflaton".

For $R^{\alpha\beta} = R^{\{\}\alpha\beta}$, such Hilbert-type Lagrangians coupled to a scalar field ϕ have been analyzed in various models before, see [44,45,139], [239,638] and references given.

[75] In such a geometry, GR coupled to a scalar field allows [183] also a smooth *change of the signature* which may be important in quantum cosmology.

Extensions of inflationary models to Riemann-Cartan spacetime [190,28,6,576] as well as to Weyl spacetimes [333] have been discussed in the literature.[76] Therefore, it seems to be appropriate to expose the common geometrical features of these models. To this end, we vary (6.6.3) with respect to ϕ, σ, $g_{\alpha\beta}$, ϑ^α, and $\Gamma_\alpha{}^\beta$. Since $M^{\alpha\beta}$ and H_α in (6.4.4) vanish, the following field equations are obtained:

$$\frac{\delta L_\phi}{\delta\phi} - \frac{1}{\chi}\phi R_\alpha{}^\beta \wedge \eta^\alpha{}_\beta = 0, \qquad \frac{\delta L_\sigma}{\delta\phi} = 0, \qquad \text{(matter)} \tag{6.6.4}$$

$$-(1/\chi)\phi^2(g^{\alpha\beta}R_\gamma{}^\delta \wedge \eta^\gamma{}_\delta + R^{(\alpha|\gamma} \wedge \eta_\gamma{}^{|\beta)}) = \sigma^{\alpha\beta}, \qquad \text{(0th)} \tag{6.6.5}$$

$$(1/2\chi)\phi^2 R_\beta{}^\gamma \wedge \eta^\beta{}_{\gamma\alpha} = \Sigma_\alpha, \qquad \text{(1st)} \tag{6.6.6}$$

$$(1/2\chi)D(\phi^2\eta^\alpha{}_\beta) = 0, \quad \Delta = 0. \qquad \text{(2nd)} \tag{6.6.7}$$

From the 2nd Noether theorem we know that (0th) is redundant, provided (1st) and (2nd) are fulfilled. Moreover, we see that $\phi\ \delta L_\phi/\delta\phi = \vartheta^\alpha \wedge \Sigma_\alpha$, which is the strong dilation identity (5.3.1) for $d_\phi = -1$, cf. [712]. Hence we only have to solve $\delta L_\sigma/\delta\sigma = 0$ and the (1st) and (2nd) field equations. In vacuum, none of the gauge field equations determines the scalar field ϕ. The situation changes, however, if we consider the full dilaton type Lagrangian (6.6.3).

According to (3.8.5), the second field equation, which is a result of the vanishing of the dilaton spin, reads explicitly:

$$Q^{\mu\alpha} \wedge \eta_{\mu\beta} - 2Q \wedge \eta^\alpha{}_\beta + T^\gamma \wedge \eta^\alpha{}_{\beta\gamma} = -2\eta^\alpha{}_\beta \wedge \frac{D\phi}{\phi}. \tag{6.6.8}$$

In a Weyl spacetime, the torsion and the volume-preserving nonmetricity ${}^{\dagger}Q_{\alpha\beta}$, cf. (3.12.4), both vanish. Then we can resolve (6.6.8) with respect to the Weyl covector:

$$Q = 2\frac{D\phi}{\phi} \quad \Rightarrow \quad Q = \frac{2}{1+4\omega_\phi}(d\ln\phi - \omega_\phi d\ln\sqrt{|\det g_{\alpha\beta}|}). \tag{6.6.9}$$

Incidentally, putting the torsion to zero already in the Lagrangian, would lead to the same result. According to (3.10.12), we find a Weyl spacetime with the connection [638], cf. [432]:

$$\Gamma_{\alpha\beta} = \Gamma^{\{\}}_{\alpha\beta} + (1/\phi)(g_{\alpha\beta}D\phi + \vartheta_\beta D_\alpha\phi - \vartheta_\alpha D_\beta\phi). \tag{6.6.10}$$

After applying a conformal change of the metric according to (3.14.5) with the identification $\Omega = \phi$ and $L - 2F + 2C = 2$, the new Weyl one-form $\widetilde{Q}$ will vanish.

Alternatively, if we put nonmetricity to zero and work in a Riemann-Cartan spacetime [221], then (6.6.8) determines the torsion as

$$T^\alpha = \vartheta^\alpha \wedge \frac{D\phi}{\phi} \Longleftrightarrow K_{\alpha\beta} = \frac{1}{\phi}\vartheta_{[\alpha}\, D_{\beta]}\phi. \tag{6.6.11}$$

Now we perform the conformal change (3.14.1) with $F = 1$, $C = 0$, and $\Omega = \phi$ of the orthonormal one-form basis:

$$\vartheta^\alpha \longrightarrow \widetilde{\vartheta}^\alpha = \phi\vartheta^\alpha. \tag{6.6.12}$$

[76] In the paper [190] there are minor algebraic slips (see Assad and Letelier [16]) which, however, do not touch the main conclusions of the paper.

According to (3.14.6) and (3.14.3) we obtain for the rescaled torsion

$$\widetilde{T}^\alpha = \phi\left(T^\alpha - \vartheta^\alpha \wedge \frac{D\phi}{\phi}\right) = 0, \tag{6.6.13}$$

whereas the curvature remains invariant. Therefore, in both cases nonmetricity and torsion are almost of the pure gauge type, cf. also [56], and can locally made to vanish by a conformal change of metric and/or coframe. Globally, however, such solutions may still have crucial imprints on cosmology and galaxy formation, see the next section.

For the full dynamically coupled Brans–Dicke type model (6.6.3), the energy-momentum three-form

$$\Sigma(\phi)_\alpha = \tfrac{1}{2}[D\phi \wedge (e_\alpha \rfloor {}^*D\phi) + (e_\alpha \rfloor D\phi) \wedge {}^*D\phi] - U(\phi)\eta_\alpha \tag{6.6.14}$$

of the scalar field ϕ, for instance, has to be taken into account. Then, in contrast to the procedures of Smalley and Germán, the scalar field ϕ is determined dynamically by the field equation (6.6.6), see also [287].

In order to symplify the dynamics, we may reparametrize the Brans–Dicke scalar ϕ according to $\phi^2 = \chi(1 + |\xi|\varphi^2)/l^{n-2}$ and apply a conformal change of the metric and of the scalar field ϕ following (6.2.1). Since the covariant exterior derivative transforms according to (6.2.3), this implies for the reparametrized field φ the conformal relations

$$\widetilde{g} = \Omega^L g = (1 + |\xi|\varphi^2)^{2/(n-2)} g, \qquad \widetilde{D}\widetilde{\varphi} = \Omega^{(2-n)L/4} D\varphi. \tag{6.6.15}$$

Then, using (6.1.5) and (3.14.3), the inflationary Lagrangian (6.6.3) for $\sigma = 0$ can be rewritten in terms of the conformally related structures as

$$L_{\rm infl} = -\frac{1}{2l^{n-2}} \widetilde{R}_\alpha{}^\beta \wedge \widetilde{\eta}_\alpha{}^\beta + \tfrac{1}{2} \widetilde{D}\widetilde{\varphi} \wedge \widetilde{{}^*} \widetilde{D}\widetilde{\varphi} - W(\varphi)\widetilde{\eta}, \tag{6.6.16}$$

where the new selfinteraction is implicitly given by $W(\varphi) = U(\varphi)(1 + |\xi|\varphi^2)^{-n/(n-2)}$. Hence the transformations (6.6.15) map the original Brans–Dicke type Lagrangian (6.6.3) into the Einstein–Klein–Gordon Lagrangian (6.6.16). The same is true for the corresponding field equations. For a suitable quartic Higgs type potential $U(\varphi)$, we simply obtain $W(\varphi) = \Lambda_{\rm eff.}$, i.e. a minimally coupled Einstein–Klein–Gordon Lagrangian with cosmological term $\Lambda_{\rm eff.}$, and vice versa. In Riemannian spacetime, the transformations (6.6.15) still hold and, due to $D\varphi = d\varphi$, we recover the (restricted) *Wagoner–Bekenstein–Starobinsky transformation*, see [44,703,335], generalized here to n dimensions:

$$\widetilde{D}\widetilde{\varphi} =: d\theta \quad \Rightarrow \quad \theta = \int \Omega^{(2-n)L/4}\, d\varphi = (1/\sqrt{|\xi|})\, \mathrm{Arsinh}(\sqrt{|\xi|}\,\varphi). \tag{6.6.17}$$

If we put $\varphi = C/r$, we obtain a similar structure as in the exact scalar solution of Baekler et al. [28]. According to our preliminary analysis, such solutions *bifurcate* with respect to the vanishing and the non-vanishing of the parameter $\Lambda_{\rm eff.}$.

Albanese and de Ritis [6] used this procedure for the derivation of a de Sitter type solution of an Einstein–Cartan Lagrangian coupled to a scalar field with asymptotic constant scalar field and damped torsion, cf.[668]. The inflationary solution of Kao [333] starts from (6.6.3) in a Weyl spacetime and breaks the symmetry via the pure gauge solution (6.6.9) to a (Riemannian) de Sitter spacetime.

An exponential potential [731], such as $W(\theta) \sim \exp\theta$, may, for instance, result from extra dimensions after a Kaluza–Klein reduction of a higher-dimensional Einstein–Cartan Lagrangian with torsion, cf. [447, p.146] and [222]. For such an exponential dependence there exists an exact solution [575] with an asymptotic expansion factor

$$\begin{aligned} a(t) &\sim t^{4/(n-1)} \quad \text{for small } t \\ &\sim t^{2/(n-1)} \quad \text{for large } t. \end{aligned} \tag{6.6.18}$$

Thus, for $1 < n \leq 4$, we obtain a *power-law* inflation. The $t^{2/3}$ behavior indicates a "graceful exit" to a matter-dominated universe. Potentials which induce inflation are classified by Barrow [37]. For a (conformally) flat universe there exists now a formal solution in terms of the Hubble parameter as a new time coordinate, from which the inflaton potential $W(\varphi)$ with an almost flat or scale-invariant COBE spectrum can be reconstructed, cf. [611,381,460]. There exists also attempts [688,689] (see also Minkevich [463]) to derive cosmological solutions in the quadratic MAG. After the inflationary period, the previously existent shear and dilation currents die out, and the final fate of the model consists in a rudimentary Poincaré gauge invariance. In such a scenario the transition from metric-affine gravity to Poincaré gauge gravity is achieved by exploiting the similarity between the Weyl vector solution (6.6.9) and the vector torsion solution (6.6.11) via the ansatz $T^\alpha = (1/2)Q\wedge\vartheta^\alpha$. After the inflationary period, the condensation of such primordial scalar fields may give rise to absolutely stable *boson stars* [459,379,380,610] as possible contributions to the hypothetical *dark matter*.

6.7. Cosmic strings with non-trivial Weyl vector or with torsion?

The breaking of local scale (or dilation) invariance is intimately connected with a quantum-theoretical groundstate configuration which violates parity and has asymptotically the structure of a Weyl vector vortex (6.4.7), i.e.,

$$\sigma \simeq \langle\sigma\rangle, \qquad D\sigma \simeq 0. \tag{6.7.1}$$

After the expansion of the universe, these relics of the dilaton field may give rise to "cosmic strings" (see, e.g., the instructive review of Straumann [659]), which at times are advocated for the explanation of apparent double quasars. In the outside region of this stringlike vortex configuration, which comes about by "freezing in" classical Riemannian gravity, we find from the requirement (6.4.7) of a covariant constant dilaton field, because of (6.2.2), that

$$\Gamma \simeq \frac{1}{\omega_\sigma} d\ln\sigma \quad \Rightarrow \quad Q \simeq \frac{2}{n}\left(\frac{1}{\omega_\sigma} d\ln\sigma - d\ln\sqrt{\det|g_{\alpha\beta}|}\right). \tag{6.7.2}$$

This is the analog of the Meissner–Ochsenfeld effect, or the Nielsen–Olesen vortex solution [518] in a Weyl spacetime, cf. [287], the corresponding dilational field strength dQ is expelled from the "superconducting" gravitational "ether", inasmuch as the trace of the connection is a pure gauge field.

However, for cosmological solutions on spacetime manifolds which are not simply connected (i.e., which have non-trivial Betti numbers) a non-trivial winding number[77]

[77] A non-trivial winding number for the trace part of the torsion, in an Y_4, has been taken into consideration by Gregorash and Papini [240]. Moreover, the torsion kinks of PG theories with scalar field coupling may be of interest in this respect [28].

$$N_Q := \frac{n\omega_\sigma}{4\pi}\int_{c_2} dQ = \frac{n\omega_\sigma}{4\pi}\oint_{\partial c_2\to 0} Q = \frac{1}{2\pi}\oint_{\partial c_2\to 0}\left(d\ln\sigma - \omega_\sigma\, d\ln\sqrt{\det|g_{\alpha\beta}|}\right) \tag{6.7.3}$$

may arise (c_2 denotes a two-dimensional space-like hypersurface, i.e. a two-cycle with boundary ∂c_2). For example, this is the case in the *conical* metric of the Vilenkin string [701] for which $d\ln(\det|g_{\alpha\beta}|)^{1/2} = d\ln r$ except for the singularity at the location $r = 0$ of the string. Moreover, a further contribution would occur, if σ is not single-valued for a loop ∂c_2 enclosing the "singularity" line of the string. This is typically the case for a functional behavior of the scalar field such as

$$\sigma = e^{\pm \arcsin(|x|)}. \tag{6.7.4}$$

Other examples of manifolds with a non-trivial global topology are "wormholes"

$$M^n = \mathbb{R}\times S^1 \times S^{n-2}, \tag{6.7.5}$$

and a "torus universe"

$$M^T = \mathbb{R}\times \underbrace{S^1\times\cdots\times S^1}_{(n-1)\text{ factors}}. \tag{6.7.6}$$

The idea to consider the gravitational analogue of the Meissner effect and to allow Abrikosov vortices in spacetime, has been developed by Hanson and Regge [257]. However, their example of a "torsion vortex" constructed via a conformal change of the metric should apply to a Weyl geometry, rather than to a Riemann–Cartan spacetime.

The pure gauge solution (6.6.9) of the conformally invariant Brans–Dicke type model (6.6.3) formally had the structure (6.7.2) which could yield a non-trivial "Weyl charge" N_Q.

When we solve the same model alternatively for the torsion, we obtain (6.6.11). Although the torsion is of the pure gauge type, i.e.

$$T^\alpha = \vartheta^\alpha \wedge \frac{D\phi}{\phi}, \tag{6.7.7}$$

the translational Chern–Simons term does not yield a topological charge, but vanishes globally:

$$C_{\mathrm{TT}} = \frac{1}{2l^2}\vartheta^\alpha\wedge T_\alpha = \frac{1}{2l^2}\vartheta^\alpha\wedge\vartheta_\alpha\wedge\frac{D\phi}{\phi} \equiv 0. \tag{6.7.8}$$

However, the torsion one-form

$$T := e_\alpha \rfloor T^\alpha = (n-1)\frac{D\phi}{\phi} = (n-1)\left(d\ln\phi - \omega_\phi d\ln\sqrt{\det|g_{\alpha\beta}|}\right) \tag{6.7.9}$$

would lead to the topological "torsion charge"

$$N_T := \frac{1}{2\pi(n-1)}\oint_{\partial c_2\to 0} T = \frac{1}{2\pi}\oint_{\partial c_2\to 0}\left(d\ln\phi - \omega_\phi d\ln\sqrt{\det|g_{\alpha\beta}|}\right) = N_Q, \tag{6.7.10}$$

which becomes identical to the "Weyl charge" for $\phi = \sigma$. Torsion solutions of the type (6.7.9) are considered [30] in the conical background of the Vilenkin cosmic string, see also [722] and [678] and, in [568], some speculative aspects of charge quantization in the Weinberg–Salam model coupled to the Weyl spacetime are discussed.

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Appendix A. Differential geometric formalism

Some differential geometric formalism which we are using in the body of our article is collected here. There exists a detailed mathematical literature on these structures. We found Choquet-Bruhat et al. [121], Loomis and Sternberg [402], Schouten [606], and Trautman [686] particularly useful. In theoretical physics, Sexl and Urbantke [617] and Thirring [674] are highly informative and stimulating, see also the 'evergreen' [607] of Schrödinger.

A.1. Exterior calculus on the 'bare' manifold M_n

We assume a connected n-dimensional differential manifold M_n as the underlying structure. A vector basis e_α of its tangent space $T_P(M_n)$ is dual to the one-form basis ϑ^β of the cotangent space $T^*_P(M_n)$. On the manifold there acts the group of diffeomorphisms $\mathrm{Diff}(n, R)$ and in the (co)tangent space the general linear group $GL(n, R)$. The geometric objects will be characterized by their transformation behavior under these two groups.

A.1.1. Geometric objects

Group representations with infinite many components have been separately discussed in section 4, those with a finite number are listed here. Under a non-degenerate, differentiable coordinate transformation $x^i \to x'^i(x^j)$ with $J_{i'}{}^j := \partial x^j/\partial x'^i$ (passive diffeomorphism) and under the $GL(n, R)$-frame transformation $e_\alpha \to e'_\alpha = \Lambda_\alpha{}^\beta\, e_\beta$ (deformation) with the respective Jacobians

$$J := \det\left(\frac{\partial x^j}{\partial x'^i}\right) = \det(J_i{}^j), \qquad \Lambda := \det(\Lambda_\alpha{}^\beta), \tag{A.1.1}$$

the components $\Psi_{i_1\cdots i_p}{}^{k\,\alpha_1\cdots\alpha_\mu}{}_{\beta_1\cdots\beta_\nu}$ of a p-form/vector density transform as

$$\begin{aligned}\Psi'_{i_1\cdots i_p}{}^{k\,\alpha_1\cdots\alpha_\mu}{}_{\beta_1\cdots\beta_\nu} = {} & (\operatorname{sgn} J)^P\,(\operatorname{sgn}\Lambda)^\pi\,|J|^w\,|\Lambda|^\omega \frac{\partial x^{j_1}}{\partial x'^{i_1}}\cdots\frac{\partial x^{j_p}}{\partial x'^{i_p}}\,\frac{\partial x'^k}{\partial x^l} \\ & \times (\Lambda^{-1})_{\gamma_1}{}^{\alpha_1}\cdots(\Lambda^{-1})_{\gamma_\mu}{}^{\alpha_\mu}\,\Lambda_{\beta_1}{}^{\delta_1}\cdots\Lambda_{\beta_\nu}{}^{\delta_\nu}\,\Psi_{j_1\cdots j_p}{}^{l\,\gamma_1\cdots\gamma_\mu}{}_{\delta_1\cdots\delta_\nu}.\end{aligned} \tag{A.1.2}$$

We assume that Ψ is totally antisymmetric in its lower coordinate indices: $\Psi_{i_1\cdots i_p}\cdots \equiv \Psi_{[i_1\cdots i_p]}\cdots$. Furthermore, we have $P, \pi \in \{0,1\}$ and $\omega, w \in R$. If $P = 1$ and/or $\pi = 1$, we speak of a holonomically and/or an anholonomically odd parity of Ψ. If $w \neq 0$ and/or $\omega \neq 0$, we call Ψ a holonomic and/or an anholonomic density of weight w and/or ω, respectively. We shall be primarily concerned with the following type of quantities:

(i) Tensor-valued p-forms[78] of representation type $\rho = \binom{\mu}{\nu}$ (with $k=0$ and $P=\pi=\omega=w=0$).
(ii) Vectors (with $p=\mu=\nu=0$ and $P=\pi=\omega=w=0$).
(iii) Scalar densities of anholonomic weight ω (with $p=k=\mu=\nu=0$ and $P=\pi=w=0$).
(iv) Connection one-forms with values in the adjoint representation of $GL(n,R)$. In this case an inhomogeneous term has to be added in (A.1.2), exactly as in (3.2.8).
(v) Manifields which have infinitly many components and thus are *not* covered by (A.1.2). They will be introduced in section 4.

The transformation formula (A.1.2) is conventionally used in the literature, see [606,692]. However, group theoretically one would approach the transformation of geometrical objects from a slightly different angle. The group of diffeomorphisms $\mathrm{Diff}(n,R)$ and the linear group $GL(n,R)$ are decomposable groups, i.e., they split into the *direct product* of the Abelian subgroups $R' = \{J\}$ and $R = \{\Lambda\}$, respectively, and the special groups with determinant plus one:

$$\begin{aligned} J_i{}^j &= J\,\check{J}_i{}^j \in \mathrm{Diff}(n,R) = R' \times -\mathrm{SDiff}(n,R), \\ \Lambda_\alpha{}^\beta &= \Lambda\check{\Lambda}_\alpha{}^\beta \in GL(n,R) = R\times SL(n,R).\end{aligned} \tag{A.1.3}$$

In view of the formula $\det M = e^{trM}$, we distinguish the elements of the special groups with a 'check' from those of the general groups. Therefore (A.1.2) can be understood as a product representation, i.e., as a tensor representation of the special groups times a representation of R^+ with arbitrary weight times the sign of the relevant determinants:

$$\begin{aligned}\Psi'_{i_1\cdots i_p}{}^{k\,\alpha_1\cdots\alpha_\mu}{}_{\beta_1\cdots\beta_\nu} = {} & (\operatorname{sgn} J)^{P+p}\,(\operatorname{sgn}\Lambda)^{\pi+\mu-\nu}\,|J|^{w+p}\,|\Lambda|^{\omega+\mu-\nu} \\ & \times \check{J}_{i_1}{}^{j_1}\cdots\check{J}_{i_p}{}^{j_p}\,(\check{J}^{-1})_l{}^k\,(\check{\Lambda}^{-1})_{\gamma_1}{}^{\alpha_1}\cdots(\check{\Lambda}^{-1})_{\gamma_\mu}{}^{\alpha_\mu} \\ & \times \check{\Lambda}_{\beta_1}{}^{\delta_1}\cdots\check{\Lambda}_{\beta_\nu}{}^{\delta_\nu}\,\Psi_{j_1\cdots j_p}{}^{l\,\gamma_1\cdots\gamma_\mu}{}_{\delta_1\cdots\delta_\nu}.\end{aligned} \tag{A.1.4}$$

Observe that thereby the weights and parities, as defined in (A.1.2), get shifted. Clearly, both points of view are possible, but the latter one is more natural if seen from the theory of group representations.

[78] For capital $P = 1$ we have odd (or twisted) differential forms à la de Rham [572], see also [69,95]. They are necessary for integration on nonorientable manifolds, which is not possible with ordinary (even) forms with $P = 0$. Note that the de Rham p-currents Ψ are in general distribution-valued, i.e. defined only for test $(n-p)$-forms Ψ of compact support via the integral $\rho(\Psi) := \int_{M_n} \rho\wedge\Psi$.

A.1.2. Exterior multiplication

With the *p-forms*, for $p = 0, \cdots, n$, we can build up the graded exterior algebra of forms. A p-form Ψ can be expanded with respect to its anholonomic components according to

$$\Psi = \frac{1}{p!}\Psi_{\beta_1\cdots\beta_p}\,\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_p}\,. \tag{A.1.5}$$

The exterior multiplication $\wedge$ has the following properties:

$$\begin{aligned}
&(\mathrm{i}) && (\Phi+\Psi)\wedge\Pi = \Phi\wedge\Pi+\Psi\wedge\Pi,\\
&(\mathrm{ii}) && (a\Phi)\wedge\Pi = \Phi\wedge(a\Pi) = a(\Phi\wedge\Pi),\\
&(\mathrm{iii}) && (\Phi\wedge\Pi)\wedge\Xi = \Phi\wedge(\Pi\wedge\Xi),\\
&(\mathrm{iv}) && \Phi\wedge\Pi = (-1)^{pq}(\Pi\wedge\Phi),
\end{aligned} \tag{A.1.6}$$

where Φ and Ψ are p-forms, Π is a q-form, Ξ an r-form, and a a factor.

A.1.3. Interior multiplication

If, in addition, *vectors* $u, v, w\cdots$ are available, we can define a (metric independent!) interior multiplication $\rfloor$ of a vector with a p-form. For a zero-form f we have $u\rfloor f = 0$. The further properties of $\rfloor$ are:

$$\begin{aligned}
&(\mathrm{i}) && v\rfloor(\Phi+\Psi) = v\rfloor\Phi + v\rfloor\Psi,\\
&(\mathrm{ii}) && (v+u)\rfloor\Phi = v\rfloor\Phi + u\rfloor\Phi,\\
&(\mathrm{iii}) && (av)\rfloor\Phi = a(v\rfloor\Phi),\\
&(\mathrm{iv}) && v\rfloor u\rfloor\Phi = -u\rfloor v\rfloor\Phi,\\
&(\mathrm{v}) && v\rfloor(\Phi\wedge\Pi) = (v\rfloor\Phi)\wedge\Pi + (-1)^p\Phi\wedge(v\rfloor\Pi).
\end{aligned} \tag{A.1.7}$$

where Φ and Ψ are p-forms, Π is a q-form, v and u vectors, and a a factor.

The 'duality' of frame and coframe implies

$$e_\alpha\rfloor\vartheta^\beta = \vartheta^\beta(e_\alpha) = \delta^\beta_\alpha. \tag{A.1.8}$$

Hence, for

$$\Psi = \frac{1}{p!}\Psi_{\alpha_1\cdots\alpha_p}\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_p}, \tag{A.1.9}$$

the application of the interior product with the properties listed above yields

$$e_\beta\rfloor\Psi = \frac{1}{(p-1)!}\Psi_{\beta\alpha_2\cdots\alpha_p}\vartheta^{\alpha_2}\wedge\cdots\wedge\vartheta^{\alpha_p}. \tag{A.1.10}$$

A.1.4. Volume elements and orientation

The nonzero elements of an arbitrary n-form

$$\overline{\eta} = \frac{1}{n!}\overline{\eta}_{\alpha_1\cdots\alpha_n}\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_n} = \overline{\eta}_{\hat{1}\cdots\hat{n}}\vartheta^{\hat{1}}\wedge\cdots\wedge\vartheta^{\hat{n}} \tag{A.1.11}$$

are called volume elements. We put a hat on numbers which represent *an*holononic indices. A linear transformation (3.4.5) of the frame yields

$$\vartheta'^{\hat{1}} \wedge \cdots \wedge \vartheta'^{\hat{n}} = \Lambda^{-1} \vartheta^{\hat{1}} \wedge \cdots \wedge \vartheta^{\hat{n}}. \tag{A.1.12}$$

Since $\overline{\eta}$ in (A.1.11) remains invariant, its components transform according to

$$\overline{\eta}'_{\hat{1}\cdots\hat{n}} = \Lambda \overline{\eta}_{\hat{1}\cdots\hat{n}}. \tag{A.1.13}$$

The manifold is orientable, if there exists a nowhere vanishing n-form. An orientation is defined by choosing a definite volume element $\overline{\eta}$. If a basis e_α is given with a cobasis ϑ^β, then $\overline{\eta}$ is expressible as

$$\overline{\eta} = f\, \vartheta^{\hat{1}} \wedge \cdots \wedge \vartheta^{\hat{n}}. \tag{A.1.14}$$

The basis is said to be positively oriented if $f > 0$.

A.1.5. Levi-Civita n-form density

The Levi-Civita n-form density ϵ with weight $\omega = -1$ and odd π-parity transforms as follows:

$$\epsilon' = (1/\Lambda)\epsilon = (\operatorname{sgn}\Lambda)^1 |\Lambda|^{-1} \epsilon. \tag{A.1.15}$$

In terms of its components we have

$$\epsilon' = \frac{1}{n!} \epsilon'_{\alpha_1\cdots\alpha_n} \vartheta'^{\alpha_1} \wedge \cdots \wedge \vartheta'^{\alpha_n} = \epsilon'_{\hat{1}\cdots\hat{n}} \vartheta'^{\hat{1}} \wedge \cdots \wedge \vartheta'^{\hat{n}} \tag{A.1.16}$$

and an analogous relation for the unprimed components. We substitute these decompositions into (A.1.15) and take care of (A.1.12), then the components of ϵ turn out to be invariant

$$\epsilon'_{\hat{1}\cdots\hat{n}} = \epsilon_{\hat{1}\cdots\hat{n}}. \tag{A.1.17}$$

In other words, the transformation law (A.1.15) of the Levi-Civita density is prescribed such that its components do *not* change under frame transformations. This property of the Levi-Civita density is only shared by the Kronecker δ_α^β, a zero-form of type $\binom{1}{1}$. We normalize (A.1.17) according to

$$\epsilon_{\hat{1}\cdots\hat{n}} = +1 \qquad (\text{in spacetime} \quad \epsilon_{\hat{0}\cdots(\hat{n}-\hat{1})} = +1\,). \tag{A.1.18}$$

By executing successively the interior product on ϵ, another representation of the bases is induced which span the graded algebra of exterior form densities on each $T^*(M_n)$:

$$\begin{aligned}
\epsilon_{\alpha_1} &:= e_{\alpha_1} \rfloor \epsilon = \frac{1}{(n-1)!} \epsilon_{\alpha_1\alpha_2\cdots\alpha_n} \vartheta^{\alpha_2} \wedge \cdots \wedge \vartheta^{\alpha_n}, \\
\epsilon_{\alpha_1\alpha_2} &:= e_{\alpha_2} \rfloor \epsilon_{\alpha_1} = \frac{1}{(n-2)!} \epsilon_{\alpha_1\alpha_2\alpha_3\cdots\alpha_n} \vartheta^{\alpha_3} \wedge \cdots \wedge \vartheta^{\alpha_n}, \\
&\vdots \\
\epsilon_{\alpha_1\cdots\alpha_n} &= e_{\alpha_n} \rfloor \cdots \rfloor e_{\alpha_1} \rfloor \epsilon.
\end{aligned} \tag{A.1.19}$$

This so-called ϵ-basis can be used to define a metric independent duality operation.

The exterior product of the coframe with the ϵ-basis satisfy the following relations:

$$\begin{aligned}
&\vartheta^\beta \wedge \epsilon_{\alpha_1} = \delta^\beta_{\alpha_1}\epsilon, \\
&\vartheta^\beta \wedge \epsilon_{\alpha_1\alpha_2} = -\delta^\beta_{\alpha_1}\epsilon_{\alpha_2} + \delta^\beta_{\alpha_2}\epsilon_{\alpha_1}, \\
&\vartheta^\beta \wedge \epsilon_{\alpha_1\alpha_2\alpha_3} = \delta^\beta_{\alpha_1}\epsilon_{\alpha_2\alpha_3} - \delta^\beta_{\alpha_2}\epsilon_{\alpha_1\alpha_3} + \delta^\beta_{\alpha_3}\epsilon_{\alpha_1\alpha_2}, \\
&\vdots \\
&\vartheta^\beta \wedge \epsilon_{\alpha_1\cdots\alpha_n} = (-1)^{n-1}\delta^\beta_{\alpha_1}\epsilon_{\alpha_2\cdots\alpha_n} + (-1)^{n-2}\delta^\beta_{\alpha_2}\epsilon_{\alpha_1\alpha_3\cdots\alpha_n}\cdots + \delta^\beta_{\alpha_n}\epsilon_{\alpha_1\cdots\alpha_{n-1}}.
\end{aligned} \tag{A.1.20}$$

A.1.6. Equivalence of a two-form Φ_α and an antisymmetric one-form $\Pi_{\alpha\beta}$

We expand these forms with respect to their holonomic components:

$$\Phi_\alpha = \tfrac{1}{2}\Phi_{ij\alpha}\, dx^i \wedge dx^j, \qquad \Pi_{\alpha\beta} = \Pi_{i\alpha\beta}\, dx^i. \tag{A.1.21}$$

Since the number $n^2(n-1)/2$ of independent components of Φ_α and $\Pi_{\alpha\beta} = -\Pi_{\beta\alpha}$ is equal, we can make the ansatz

$$\Phi_\alpha = \Pi_{\alpha\beta} \wedge \vartheta^\beta. \tag{A.1.22}$$

By repeated contraction of Φ_α by means of the frame e_β, we find the reciprocal of this formula:

$$\Pi_{\alpha\beta} = e_{[\alpha}\rfloor\Phi_{\beta]} - \tfrac{1}{2}(e_\alpha\rfloor e_\beta\rfloor\Phi_\gamma)\vartheta^\gamma = \tfrac{1}{2}(e_{\{\gamma}\rfloor e_\alpha\rfloor\Phi_{\beta\}})\vartheta^\gamma. \tag{A.1.23}$$

Here we introduced, as convenient abbrevation, the Schouten braces according to

$$\{\alpha\beta\gamma\} := \alpha\beta\gamma - \beta\gamma\alpha + \gamma\alpha\beta. \tag{A.1.24}$$

Torsion $T_\alpha = g_{\alpha\beta}T^\beta$ and contortion $K_{\alpha\beta}$ are an example of such equivalent forms.

A.1.7. Equivalence of an $(n-2)$-form μ^α and an antisymmetric $(n-1)$-form $\tau^{\alpha\beta}$

The arguments with respect to the number of independent components of the last subsection translate, mutatis mutandis, to the forms μ^α and $\tau^{\alpha\beta} = -\tau^{\beta\alpha}$. With

$$\tau^{\alpha\beta} = \vartheta^{[\alpha} \wedge \mu^{\beta]} = -\tau^{\beta\alpha}, \tag{A.1.25}$$

we can interrelate both forms. Again we contract twice with the frame and find

$$\mu^\alpha = -2e_\beta\rfloor\tau^{\alpha\beta} + \tfrac{1}{2}\vartheta^\alpha \wedge (e_\beta\rfloor e_\gamma\rfloor\tau^{\beta\gamma}). \tag{A.1.26}$$

The spin current and the spin-energy potential are an example of such two equivalent quantities.

A.1.8. Expressing a one-form $\Psi_{\alpha\beta}$ in terms of a two-form Φ_α

Let us consider a one-form $\Psi_{\alpha\beta}$ which is related to a vector-valued two-form Φ_α via

$$\Psi_{\alpha\beta} \wedge \vartheta^\beta = \Phi_\alpha. \tag{A.1.27}$$

We will resolve this equation with respect to $\Psi_{\alpha\beta}$.

Note that $\Psi_{\alpha\beta}$, in contrast to $\Pi_{\alpha\beta}$ in (A.1.22), carries no symmetries. Then from counting the components it is immediately clear that $\Psi_{[\alpha\beta]}$ has $n^2(n-1)/2$ components, exactly the same number as the two-form Φ_α. Since $\Psi_{(\alpha\beta)} \neq 0$ in general, this piece of Ψ will enter the formulae in its own right.

By repeated interior multiplication of (A.1.27) with e_α we find as an intermediate step

$$e_\delta \rfloor \Psi_{\alpha\gamma} - e_\gamma \rfloor \Psi_{\alpha\delta} = e_\gamma \rfloor e_\delta \rfloor \Phi_\alpha . \tag{A.1.28}$$

In order to obtain a partial solution for the antisymmetric part of $\Psi_{\alpha\beta}$, we apply again the Schouten braces of (A.1.24). Thereby we find

$$\begin{aligned}\Psi_{[\alpha\beta]} &= e_{[\alpha} \rfloor \Phi_{\beta]} - \tfrac{1}{2}(e_\alpha \rfloor e_\beta \rfloor \Phi_\gamma)\vartheta^\gamma + (e_\beta \rfloor \Psi_{(\alpha\gamma)} - e_\alpha \rfloor \Psi_{(\beta\gamma)})\vartheta^\gamma \\ &= \tfrac{1}{2}(e_{\{\gamma} \rfloor e_\alpha \rfloor \Phi_{\beta\}})\vartheta^\gamma + (e_\beta \rfloor \Psi_{(\alpha\gamma)} - e_\alpha \rfloor \Psi_{(\beta\gamma)})\vartheta^\gamma .\end{aligned} \tag{A.1.29}$$

By adding to (A.1.29) the symmetric part of $\Psi_{\alpha\beta}$ in the equivalent form

$$\Psi_{(\alpha\beta)} = (e_\gamma \rfloor \Psi_{(\alpha\beta)})\vartheta^\gamma , \tag{A.1.30}$$

we finally arrive, for $n > 1$, at the general formula

$$\Psi_{\alpha\beta} = \tfrac{1}{2}(e_{\{\gamma} \rfloor e_\alpha \rfloor \Phi_{\beta\}})\vartheta^\gamma + (e_{\{\gamma} \rfloor \Psi_{(\alpha\beta)\}})\vartheta^\gamma = \tfrac{1}{2} e_{\{\gamma} \rfloor \Big(e_\alpha \rfloor \Phi_{\beta\}} + 2\Psi_{(\alpha\beta)\}} \Big)\vartheta^\gamma . \tag{A.1.31}$$

For computer algebra programs [609,435,650], however, it is more time saving to evaluate the interior products as far as possible. Then we have the alternative formula

$$\Psi_{\alpha\beta} = e_{[\alpha} \rfloor \Phi_{\beta]} - \tfrac{1}{2}(e_\alpha \rfloor e_\beta \rfloor \Phi_\gamma)\vartheta^\gamma + \Psi_{(\alpha\beta)} + (e_\beta \rfloor \Psi_{(\alpha\gamma)} - e_\alpha \rfloor \Psi_{(\beta\gamma)})\vartheta^\gamma . \tag{A.1.32}$$

A.1.9. A scalar density simulates the determinant of the metric

Clearly, if we prescribe a scalar density field with the weight $\omega = +1$ and odd π-parity, then it transforms as

$$\sigma' = (\operatorname{sgn} \Lambda)^1 |\Lambda|^1 \sigma , \tag{A.1.33}$$

and, in view of (A.1.15), the field

$$\tilde{\epsilon} := \epsilon\sigma \tag{A.1.34}$$

is a 'pure' n-form. Then, in analogy to (A.1.19), we can construct the 'pure' p-form basis

$$\begin{aligned}\tilde{\epsilon}_{\alpha_1} &:= e_{\alpha_1} \rfloor \tilde{\epsilon} = \frac{1}{(n-1)!}\tilde{\epsilon}_{\alpha_1\alpha_2\cdots\alpha_n}\vartheta^{\alpha_2} \wedge \cdots \wedge \vartheta^{\alpha_n} , \\ \tilde{\epsilon}_{\alpha_1\alpha_2} &:= e_{\alpha_2} \rfloor \tilde{\epsilon}_{\alpha_1} = \frac{1}{(n-2)!}\tilde{\epsilon}_{\alpha_1\alpha_2\alpha_3\cdots\alpha_n}\vartheta^{\alpha_3} \wedge \cdots \wedge \vartheta^{\alpha_n} , \\ &\ \vdots \\ \tilde{\epsilon}_{\alpha_1\cdots\alpha_n} &= e_{\alpha_n} \rfloor \cdots \rfloor e_{\alpha_1} \rfloor \tilde{\epsilon} .\end{aligned} \tag{A.1.35}$$

Thus by means of a suitably chosen scalar density field – in our applications in section 6 it is the dilaton field, see also [712,88,169,101] – we can devise the $\tilde{\epsilon}$-basis and the corresponding duality operator without need of a metric, cf. [258]. And again we find formulae analogous to (A.1.20).

A.1.10. Exterior derivative

The exterior derivative maps a p-form into a $(p+1)$-form. It has the following properties:

$$\begin{aligned} &\text{(i)} && d(\Psi+\Phi)=d\Psi+d\Phi, \\ &\text{(ii)} && d(\Psi\wedge\Pi)=d\Psi\wedge\Pi+(-1)^p\Psi\wedge d\Pi, \\ &\text{(iii)} && df(v)=v(f) \\ &\text{(iv)} && d(d\Psi)=0. \end{aligned} \tag{A.1.36}$$

Here Ψ and Φ are p-forms, Π a q-form, f a zero-form, and v a vector.

A.1.11. Lie derivative

The *Lie derivative* l_v of a scalar-valued p-form Φ with respect to a vector field v had been defined in (3.4.9). If v and u are vectors, Ψ and Φ p-forms, and Π a q-form, then the properties of the Lie derivative read as follows (cf. [265]):

$$\begin{aligned} &\text{(i)} && l_v(\Psi+\Phi)=l_v\Psi+l_v\Phi, \\ &\text{(ii)} && l_v(\Psi\wedge\Pi)=(l_v\Psi)\wedge\Pi+\Psi\wedge(l_v\Pi), \\ &\text{(iii)} && l_v(d\Psi)=d(l_v\Psi), \\ &\text{(iv)} && l_v\, l_u\Psi=l_u\, l_v\Psi+l_{[v,u]}\Psi, \\ &\text{(v)} && u\rfloor l_v\Psi=l_v(u\rfloor\Psi)-[v,u]\rfloor\Psi. \end{aligned} \tag{A.1.37}$$

For a tensor-valued form Π – we suppress the indices – one can take the Lie derivative operator $\pounds_v$ of Ricci calculus [601] as a lead. Then one finds:

$$\pounds_v\Pi=l_v\Pi-(e_\beta\rfloor l_v\vartheta^\alpha)\,\rho(L^\alpha{}_\beta)\Pi\,. \tag{A.1.38}$$

This 'ordinary' Lie derivative we only use in the context of Killing symmetries, see (5.7.5) and (5.7.11). However, the gauge covariant Lie derivative of (3.5.13) has a much broader field of application.

A.2. Derivatives of the bases in a linearly connected manifold

If we determine the covariant exterior derivative of the ϵ-basis (A.1.19), we do need a linear connection $\Gamma_\alpha{}^\beta$ but no metric:

$$D\epsilon_{\alpha_1}=T^\beta\wedge\epsilon_{\alpha_1\beta},\qquad D\epsilon_{\alpha_1\alpha_2}=T^\beta\wedge\epsilon_{\alpha_1\alpha_2\beta},\qquad \dots,\qquad D\epsilon_{\alpha_1\cdots\alpha_n}=0. \tag{A.2.1}$$

For the scalar-modified $\tilde\epsilon$-basis the computations run on the same track,

$$\begin{gathered} D\tilde\epsilon_{\alpha_1}=\frac{D\sigma}{\sigma}\wedge\tilde\epsilon_{\alpha_1}+T^\beta\wedge\tilde\epsilon_{\alpha_1\beta},\qquad D\tilde\epsilon_{\alpha_1\alpha_2}=\frac{D\sigma}{\sigma}\wedge\tilde\epsilon_{\alpha_1\alpha_2}+T^\beta\wedge\tilde\epsilon_{\alpha_1\alpha_2\beta}, \\ \dots,\qquad D\tilde\epsilon_{\alpha_1\cdots\alpha_n}=\frac{D\sigma}{\sigma}\wedge\tilde\epsilon_{\alpha_1\cdots\alpha_n}, \end{gathered} \tag{A.2.2}$$

the advantage being that this basis is composed of pure forms. Eq. (A.2.2) should be compared with (3.10.14). Then, if a metric is given, it becomes evident that $\sigma=(\det|g_{\alpha\beta}|)^{1/2}$ and (A.2.2) thereby specializes to (3.8.5).

A.3. Euler four-form and generalized Bach–Lanczos identity

If we allow, in a metric-affine spacetime, the Lagrangian to involve also the dual with respect to the Lie algebra, we can extend the list of admissible boundary forms in section 3.9. In a four-dimensional Riemannian manifold, an important example is the integrand of the Gauss–Bonnet integral $\int R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta} \sim \chi_{\rm Euler}$. Since its integration over a closed manifold yields the *Euler number* as a topological characteristic, the integrand is also known as the Euler four-form. In a metric-affine spacetime the corresponding four-form [79] reads

$$B_{RR^{(\star)}} = -\tfrac{1}{2} R_\alpha{}^\beta \wedge R^{(\star)\alpha}_\beta = \tfrac{1}{4} \eta_\beta{}^\alpha{}_\delta{}^\gamma R_\alpha{}^\beta \wedge R_\gamma{}^\delta. \tag{A.3.1}$$

The Lie dual $(\star)$ interchanges the Lie-algebra indices according to

$$\Gamma_\alpha^{(\star)\beta} := \tfrac{1}{2} \eta_\alpha{}^\beta{}_\delta{}^\gamma \Gamma_\gamma{}^\delta, \quad \eta_\alpha{}^\beta{}_\delta{}^\gamma = g^{\beta\mu} g^{\gamma\nu} \eta_{\alpha\mu\delta\nu}.$$

In (A.3.1), because of the action of the star, only the antisymmetric piece $R_{[\alpha\beta]}$ is left over of the curvature, whereas the symmetric piece $R_{(\alpha\beta)}$, which is so characteristic for the (L_4, g), drops out. In the framework of the Riemann–Cartan spacetime and using orthonormal frames, we have $B_{RR^{(\star)}} = dC_{RR^{(\star)}}$ with

$$C_{RR^{(\star)}} = \tfrac{1}{4} \eta^\alpha{}_\beta{}^\gamma{}_\delta (R_\alpha{}^\beta \wedge \Gamma_\gamma{}^\delta + \tfrac{1}{3} \Gamma_\alpha{}^\beta \wedge \Gamma_\gamma{}^\varepsilon \wedge \Gamma_\varepsilon{}^\delta). \tag{A.3.2}$$

If we perform the variational derivative of $B_{RR^{(\star)}}$ with respect to $\Gamma_\alpha{}^\beta$ and $g_{\alpha\beta}$ (there is no explicit dependence on ϑ^α) in the context of the general metric-affine geometry, we obtain

$$\delta B_{RR^{(\star)}} / \delta\vartheta^\alpha = 0, \tag{A.3.3}$$

$$\delta B_{RR^{(\star)}} / \delta g_{\alpha\beta} = \tfrac{1}{2} g^{\alpha\beta} R^{\mu\nu} \wedge R^{(\star)}_{\mu\nu} - 2 R^{(\alpha|\gamma} \wedge R^{(\star)|\beta)}{}_\gamma, \tag{A.3.4}$$

$$\delta B_{RR^{(\star)}} / \delta\Gamma_\alpha{}^\beta = \tfrac{1}{2} \eta_{\mu\beta\gamma\delta} [(Q^{\alpha\mu} - 2Q\, g^{\alpha\mu}) \wedge R^{\gamma\delta} + g^{\alpha\mu} Q^{\gamma\lambda} \wedge R_\lambda{}^\delta]. \tag{A.3.5}$$

Invariance of $B_{RR^{(\star)}}$ under linear transformations of the frame leads to the *2nd Noether identity* [280]

$$g_{\beta\gamma} \frac{\delta B_{RR^{(\star)}}}{\delta g_{\alpha\gamma}} = D \left(\frac{\delta B_{RR^{(\star)}}}{\delta \Gamma_\alpha{}^\beta} \right), \tag{A.3.6}$$

which we write out fully as

$$\begin{aligned} & g^{\alpha\beta} R^{\mu\nu} \wedge R^{(\star)}_{\mu\nu} - 4 R^{(\alpha|\gamma} \wedge R^{(\star)|\beta)}{}_\gamma \\ & \quad = \eta_\mu{}^\beta{}_{\gamma\delta} [2 R^{(\alpha\mu)} \wedge R^{\gamma\delta} + g^{\alpha\mu} (2 R^{(\gamma\rho)} \wedge R_\rho{}^\delta - R \wedge R^{\gamma\delta})], \end{aligned} \tag{A.3.7}$$

where $R := R_\gamma{}^\gamma$.

This is the *Bach–Lanczos identity* of GR generalized to a *metric-affine* spacetime. In a Riemann–Cartan spacetime and in the Riemannian spacetime of GR the right-hand side of (A.3.7) vanishes and we obtain the familiar form of the Bach–Lanczos identity [21,384]. A derivation of this identity for GR by means of variations was given earlier by Ray [569], who used a similar Lagrangian but a much more involved argumentation (see also [20] and [429]).

[79] In five dimensions, the term $B_{RR^{(\star)}} = \frac{1}{4} \eta^A{}_B{}^C{}_D R_A{}^B \wedge R_C{}^D$, with $A, B, \cdots = 0, \cdots, 4$ is the simplest Lagrangian in the $SO(3,2)$ gauge model of gravity, see [234] and references given.

Appendix B. Irreducible decompositions

The irreducible decompositions of nonmetricity, torsion, curvature and Bianchi identities invariant under both, the general linear and the pseudo-orthogonal group, are presented here. For the decompositions with respect to the (pseudo)orthogonal group, see also [436].

B.1. Irreducible decomposition of nonmetricity

Since the nonmetricity is already a metric-dependent object, we can raise one index and split the nonmetricity $Q_{\alpha\beta}$ into its tracefree and trace parts, as is already done in (3.12.4),

$$Q_{\alpha\beta} = \not{Q}_{\alpha\beta} + Q\, g_{\alpha\beta}, \tag{B.1.1}$$

where $\not{Q}_\gamma{}^\gamma = g^{\gamma\beta}\, Q_{\gamma\beta} = 0$. The Weyl covector $Q = (1/n)\, Q_\alpha{}^\alpha$ cannot be reduced any further. Applying the Young diagram method to the components $\not{Q}_{\gamma\alpha\beta} = e_\gamma \rfloor \not{Q}_{\alpha\beta}$ together with forming traces yields three further irreducible pieces, provided $n > 2$. We may express the resulting decomposition in terms of exterior forms as follows: Let us define

$$\Lambda_\alpha := e^\beta \rfloor \not{Q}_{\alpha\beta}, \qquad \Lambda := \Lambda_\alpha\, \vartheta^\alpha,$$

$$\Theta_\alpha := {}^*(\not{Q}_{\alpha\beta} \wedge \vartheta^\beta), \qquad \Theta := \vartheta^\alpha \wedge \Theta_\alpha, \qquad \Omega_\alpha := \Theta_\alpha - \frac{1}{n-1} e_\alpha \rfloor \Theta. \tag{B.1.2}$$

Then the irreducible decomposition of $Q_{\alpha\beta}$ invariant under the (pseudo)orthogonal group is given by [80]

$$\begin{aligned} Q_{\alpha\beta} &= {}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}, \\ &= \text{TRINOM} + \text{BINOM} + \text{VECNOM} + \text{CONOM}, \end{aligned} \tag{B.1.3}$$

$$\tfrac{1}{2}n^2(n+1) = \tfrac{1}{6}n(n-1)(n+4) + \tfrac{1}{3}n(n^2-4) + n + n, \tag{B.1.4}$$

where

$${}^{(2)}Q_{\alpha\beta} = (-1)^{n-1+\mathrm{Ind}(g)}\, \tfrac{2}{3}\, {}^*(\vartheta_{(\alpha} \wedge \Omega_{\beta)}), \tag{B.1.5}$$

$${}^{(3)}Q_{\alpha\beta} = \frac{2n}{(n-1)(n+2)} \left(\vartheta_{(\beta}\, \Lambda_{\alpha)} - \frac{1}{n} g_{\alpha\beta} \Lambda \right), \tag{B.1.6}$$

$${}^{(4)}Q_{\alpha\beta} = g_{\alpha\beta} Q, \tag{B.1.7}$$

$${}^{(1)}Q_{\alpha\beta} = Q_{\alpha\beta} - {}^{(2)}Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} - {}^{(4)}Q_{\alpha\beta}, \tag{B.1.8}$$

and $(-1)^{\mathrm{Ind}(g)} = (\det g_{\alpha\beta}) / |\det g_{\alpha\beta}|$. Note that, for any p-form Φ, double application of the Hodge star operator yields

$${}^{**}\Phi = (-1)^{p(n-p)+\mathrm{Ind}(g)}\, \Phi. \tag{B.1.9}$$

[80] In four dimensions, the components ${}^{(1)}Q_{\gamma\alpha\beta}$ have a spin content of maximal three ("TRI") etc.. These names are used in our computer programs [433,435,609,650,294] which automatize the irreducible decompositions inter alia.

Table 2
Irreducible decomposition of nonmetricity: Number of independent components in four, three, and two dimensions.

n	$Q_{\alpha\beta}$	TRINOM	BINOM	VECNOM	CONOM
4	40	16	16	4	4
3	18	7	5	3	3
2	6	2	0	2	2

There are two types of vanishing traces

$$ {}^{(1)}Q_\alpha{}^\alpha = {}^{(2)}Q_\alpha{}^\alpha = {}^{(3)}Q_\alpha{}^\alpha = 0, \qquad e_\beta \rfloor {}^{(1)}Q^{\alpha\beta} = e_\beta \rfloor {}^{(2)}Q^{\alpha\beta} = 0, \tag{B.1.10}$$

whereas the Young symmetries of those parts for which both traces vanish read

$$ \vartheta_\beta \wedge {}^{(1)}Q^{\alpha\beta} = 0, \qquad e_{(\alpha} \rfloor {}^{(2)}Q_{\beta\gamma)} = 0. \tag{B.1.11}$$

If we had started by applying the Young diagram procedure directly to the components $Q_{\gamma\alpha\beta} = e_\gamma \rfloor Q_{\alpha\beta}$, two of the four irreducible pieces obtained would not have been canonical. They would have been arbitrary combinations of ${}^{(3)}Q_{\alpha\beta}$ and ${}^{(4)}Q_{\alpha\beta}$ above, while the other two pieces would have been ${}^{(1)}Q_{\alpha\beta}$ and ${}^{(2)}Q_{\alpha\beta}$ as in (B.1.8) and (B.1.5), see [278]. The initial splitting (B.1.1) with respect to the indices on the nonmetricity *one-form* has ensured that the four irreducible subspaces which we have obtained are uniquely defined (cf. also [684]).

The four irreducible components defined by (B.1.5)–(B.1.8) have an interesting "orthogonality" property. Suppose that for any two rth order tensor-valued p-forms $A_{\alpha_1\ldots\alpha_r}$ and $B_{\alpha_1\ldots\alpha_r}$ $(0 \le r \le n)$ we define a (pseudo)*scalar product* by

$$ A \cdot B := {}^*(A_{\alpha_1\ldots\alpha_r} \wedge {}^*B^{\alpha_1\ldots\alpha_r}) \, . \tag{B.1.12}$$

It is clear that the scalar product so defined is symmetric, $A \cdot B = B \cdot A$, and it may be verified that

$$ {}^{(i)}Q \cdot {}^{(j)}Q \neq 0 \quad \text{for} \quad i = j; \qquad {}^{(i)}Q \cdot {}^{(j)}Q = 0 \quad \text{for} \quad i \neq j, \tag{B.1.13}$$

so that

$$ Q \cdot Q = \sum_{i=1}^{4} {}^{(i)}Q \cdot {}^{(i)}Q. \tag{B.1.14}$$

In two dimensions $\Omega_\alpha = 0$ and hence ${}^{(2)}Q_{\alpha\beta}$ drops out leaving three irreducible pieces. This is also clear from (B.1.5). Thus

$$ Q_{\alpha\beta} = {}^{(1)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}, \quad n = 2. \tag{B.1.15}$$

B.2. Irreducible decomposition of torsion

For $n > 2$, the torsion in an L_n has two irreducible pieces invariant under the general linear group,

$$ T^\alpha = {}^{(I)}T^\alpha + {}^{(II)}T^\alpha, \tag{B.2.1}$$

$$ \tfrac{1}{2}n^2(n-1) = \tfrac{1}{2}(n-2)n(n+1) + n, \tag{B.2.2}$$

Table 3
Irreducible decomposition of torsion: Number of independent components in four, three, and two dimensions.

n	T^α	TENTOR	TRATOR	AXITOR
4	24	16	4	4
3	9	5	3	1
2	2	0	2	0

where

$$^{(II)}T^\alpha = \frac{1}{n-1}\,\vartheta^\alpha \wedge T, \qquad ^{(I)}T^\alpha = T^\alpha - {}^{(II)}T^\alpha, \tag{B.2.3}$$

and $T := e_\alpha \rfloor T^\alpha$ is the trace torsion one-form. In two dimensions, T^α is irreducible and of a pure vector type, cf. appendix B of [453]:

$$T^\alpha = {}^{(II)}T^\alpha. \tag{B.2.4}$$

In an (L_n, g) the torsion has the same algebraic structure, and hence the same irreducible decomposition with respect to the (pseudo)orthogonal group, as in a Riemann–Cartan spacetime (see [103,260,389,705,278] for the case of four dimensions). For $n = 2$, the torsion T^α is already irreducible and for $n > 2$ we may write its irreducible components as follows:

$$\begin{aligned} T^\alpha &= {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha \\ &= \text{TENTOR} + \text{TRATOR} + \text{AXITOR}, \\ \tfrac{1}{2}n^2(n-1) &= \tfrac{1}{3}n(n^2-4) + n + \tfrac{1}{6}n(n-1)(n-2), \end{aligned} \tag{B.2.5}$$

where

$$^{(2)}T^\alpha = \frac{1}{n-1}\,\vartheta^\alpha \wedge (e_\beta \rfloor T^\beta)\,, \tag{B.2.6}$$

$$^{(3)}T^\alpha = (-1)^s\,\tfrac{1}{3}\,{}^*\{\vartheta^\alpha \wedge {}^*(T^\beta \wedge \vartheta_\beta)\}\,, \tag{B.2.7}$$

$$^{(1)}T^\alpha = T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{B.2.8}$$

Again here, the irreducible pieces are canonical. They satisfy

$$\begin{aligned} &^{(1)}T^\alpha \wedge \vartheta_\alpha = 0, \qquad e_\alpha \rfloor {}^{(1)}T^\alpha = 0\,, \\ &^{(2)}T^\alpha \wedge \vartheta_\alpha = 0, \qquad e_\alpha \rfloor {}^{(3)}T^\alpha = 0\,. \end{aligned} \tag{B.2.9}$$

In four dimensions the Hodge dual ${}^*T^\alpha$ of the torsion is also a two-form and a parallel decomposition of it is given by

$$^*T^\alpha = {}^{(1)*}T^\alpha + {}^{(2)*}T^\alpha + {}^{(3)*}T^\alpha\,, \tag{B.2.10}$$

where ${}^{(i)*}T^\alpha$ $(i = 1, 2, 3)$ are defined by substituting ${}^*T^\alpha$ for T^α in (B.2.6)–(B.2.8). These are related to the irreducible pieces ${}^{(i)}T^\alpha$ of T^α by the relations [81]

[81] These relations, together with the corresponding ones (B.4.35) for the antisymmetric part of the curvature, were first pointed out by Hecht [265].

$$ {}^{(1)*}T^{\alpha} = {}^{*\,(1)}T^{\alpha}, \qquad {}^{(2)*}T^{\alpha} = {}^{*\,(3)}T^{\alpha}, \qquad {}^{(3)*}T^{\alpha} = {}^{*\,(2)}T^{\alpha}. \tag{B.2.11} $$

For other dimensions, the most straightforward way to write the irreducible pieces of the Hodge dual is

$$ {}^{*}T^{\alpha} = \sum_{i=1}^{3} {}^{(i)\,*}T^{\alpha}, \tag{B.2.12} $$

with ${}^{(i)\,*}T^{\alpha} = {}^{*\,(i)}T^{\alpha}$ $(i = 1, 2, 3)$, a numbering which could equally have been used in four dimensions.

It may be verified that, with the scalar product as defined by (B.1.12), ${}^{(i)}T^{\alpha}$ $(i = 1, 2, 3)$ satisfy an "orthogonality" property similar to that possessed by the nonmetricity,

$$ {}^{(i)}T \cdot {}^{(j)}T \neq 0 \quad \text{for} \quad i = j; \qquad {}^{(i)}T \cdot {}^{(j)}T = 0 \quad \text{for} \quad i \neq j, \tag{B.2.13} $$

and hence

$$ T \cdot T = \sum_{i=1}^{3} {}^{(i)}T \cdot {}^{(i)}T. \tag{B.2.14} $$

In four dimensions we can define another (pseudo)scalar product $\langle A, B\rangle$ for any two tensor-valued two-forms $A_{\alpha_1 \ldots \alpha_r}$ and $B_{\alpha_1 \ldots \alpha_r}$ by

$$ \langle A, B\rangle := {}^{*}(A_{\alpha_1 \ldots \alpha_r} \wedge B^{\alpha_1 \ldots \alpha_r}) . \tag{B.2.15} $$

It follows from these definitions that it has the following properties:

$$ \langle {}^{*}A, B\rangle = \langle A, {}^{*}B\rangle = A \cdot B, \tag{B.2.16} $$

$$ {}^{*}A \cdot B = \langle {}^{*}A, {}^{*}B\rangle, \qquad A \cdot {}^{*}B = (-1)^{p+\mathrm{Ind}(g)}\langle A, B\rangle, $$
$$ {}^{*}A \cdot {}^{*}B = (-1)^{p+\mathrm{Ind}(g)} A \cdot B. \tag{B.2.17} $$

The "orthogonality" relations for the irreducible torsion components with respect to this additional scalar product take the form

$$ \begin{cases} \langle {}^{(1)}T, {}^{(1)}T\rangle \neq 0, \\ \langle {}^{(2)}T, {}^{(3)}T\rangle = \langle {}^{(3)}T, {}^{(2)}T\rangle \neq 0, \\ \langle (i)T, {}^{(j)}T\rangle = 0 \text{ otherwise}, \end{cases} \tag{B.2.18} $$

so that

$$ \langle T, T\rangle = \langle {}^{(1)}T, {}^{(1)}T\rangle + 2\langle {}^{(2)}T, {}^{(3)}T\rangle. \tag{B.2.19} $$

It should be noted however that the scalar product (B.2.15) is invariant only under (pseudo)orthogonal transformations of determinant +1.

B.3. Irreducible decomposition of curvature in an L_n

As a first step in deriving the irreducible decomposition of the curvature in an L_n, invariant under the general linear group, we can split the curvature two-form $R_\alpha{}^\beta$ into its trace and trace-free parts.

$$R_\alpha{}^\beta = \not{R}_\alpha{}^\beta + (1/n)\delta_\alpha^\beta R, \quad R = R_\gamma{}^\gamma. \tag{B.3.1}$$

Let

$$U_\alpha := e_\gamma \rfloor \not{R}_\alpha{}^\gamma, \qquad U_\alpha{}^\beta := \vartheta^\beta \wedge U_\alpha, \qquad U := U_\gamma{}^\gamma, \tag{B.3.2}$$

$$V^\beta := \vartheta^\gamma \wedge \not{R}_\gamma{}^\beta, \qquad V_\alpha{}^\beta := e_\alpha \rfloor V^\beta, \qquad V := V_\gamma{}^\gamma. \tag{B.3.3}$$

be given. Then, the curvature decomposes as

$$R_\alpha{}^\beta = A_\alpha{}^\beta + B_\alpha{}^\beta + C_\alpha{}^\beta + D_\alpha{}^\beta + E_\alpha{}^\beta, \tag{B.3.4}$$

$$\begin{aligned} \tfrac{1}{2}n^3(n-1) = {} & \tfrac{1}{3}n^2(n^2-4) + \tfrac{1}{6}n(n^2-1)(n-3) \\ & + \tfrac{1}{2}n(n+1) + \tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n-1), \end{aligned} \tag{B.3.5}$$

where

$$B_\alpha{}^\beta = \tfrac{1}{3}V_\alpha{}^\beta - \frac{1}{3(n-2)} e_\alpha \rfloor (V \wedge \vartheta^\beta), \tag{B.3.6}$$

$$C_\alpha{}^\beta = \frac{1}{n-1}\left[U_\alpha{}^\beta - \tfrac{1}{2}\vartheta^\beta \wedge (e_\alpha \rfloor U)\right], \qquad \text{(RICSYM)} \tag{B.3.7}$$

$$D_\alpha{}^\beta = \frac{1}{(n+1)}\left(\frac{1}{2}\vartheta^\beta \wedge (e_\alpha \rfloor U) + \frac{1}{n-2} e_\alpha \rfloor (V \wedge \vartheta^\beta)\right), \qquad \text{(RICANTI)} \tag{B.3.8}$$

$$E_\alpha{}^\beta = (1/n)\delta_\alpha^\beta R, \qquad \text{(DILCURV)} \tag{B.3.9}$$

$$A_\alpha{}^\beta = R_\alpha{}^\beta - B_\alpha{}^\beta - C_\alpha{}^\beta - D_\alpha{}^\beta - E_\alpha{}^\beta. \tag{B.3.10}$$

The symmetries and trace properties are

$$\vartheta^\alpha \wedge A_\alpha{}^\beta = 0, \qquad e_\beta \rfloor A_\alpha{}^\beta = 0, \qquad A_\alpha{}^\alpha = 0, \tag{B.3.11}$$

$$B_\alpha{}^\beta - \tfrac{1}{3} e_\alpha \rfloor (\vartheta^\gamma \wedge B_\gamma{}^\beta) = 0, \qquad B_\alpha{}^\alpha = 0, \tag{B.3.12}$$

$$\vartheta^\alpha \wedge C_\alpha{}^\beta = 0. \tag{B.3.13}$$

In the case $n = 2$, we have $A_\alpha{}^\beta = B_\alpha{}^\beta = D_\alpha{}^\beta = 0$ and

$$R_\alpha{}^\beta = C_\alpha{}^\beta + E_\alpha{}^\beta \quad \text{for} \quad n = 2, \tag{B.3.14}$$

$$4 = 3 + 1.$$

When $n = 3$ we have $B_\alpha{}^\beta = 0$ and

$$R_\alpha{}^\beta = A_\alpha{}^\beta + C_\alpha{}^d + D_\alpha{}^\beta + E_\alpha{}^\beta, \tag{B.3.15}$$

$$27 = 15 + 6 + 3 + 3.$$

In four dimensions, we can define a symmetric (pseudo)scalar product for any $gl(4,R)$-valued two-forms $F_\alpha{}^\beta$ and $G_\alpha{}^\beta$, taking values in the one-dimensional space of four-forms, by

$$F \bullet G := F_\alpha{}^\beta \wedge G_\beta{}^\alpha . \tag{B.3.16}$$

It is found that the products of the two-forms A, B, C, D, E defined by (B.3.6)–(B.3.10) are all zero except for $A \bullet A$, $B \bullet C$, $D \bullet D$ and $E \bullet E$, so that

$$R \bullet R = A \bullet A + 2B \bullet C + D \bullet D + E \bullet E . \tag{B.3.17}$$

B.4. Irreducible decomposition of curvature in an (L_n, g)

If we now pass on to an (L_n, g), we may lower the second index on the curvature two-form and consider the irreducible decomposition of $R_{\alpha\beta}$ under the (pseudo)orthogonal group. The first step is to separate it into its antisymmetric and symmetric parts,

$$R_{\alpha\beta} = W_{\alpha\beta} + Z_{\alpha\beta}, \tag{B.4.1}$$

$$W_{\alpha\beta} = R_{[\alpha\beta]}, \qquad Z_{\alpha\beta} = R_{(\alpha\beta)} . \tag{B.4.2}$$

The irreducible decomposition of $W_{\alpha\beta}$ is the same as for the curvature of a Riemann–Cartan spacetime (see [278] for four dimensions). In two dimensions $W_{\alpha\beta}$ cannot be reduced any further, but for $n > 2$ we get

$$\begin{aligned} W^{\alpha\beta} &= {}^{(1)}W^{\alpha\beta} + {}^{(2)}W^{\alpha\beta} + {}^{(3)}W^{\alpha\beta} + {}^{(4)}W^{\alpha\beta} + {}^{(5)}W^{\alpha\beta} + {}^{(6)}W^{\alpha\beta} \\ &= \mathrm{WEYL} + \mathrm{PAIRCOM} + \mathrm{PSCALAR} + \mathrm{RICSYMF} + \mathrm{RICANTI} + \mathrm{SCALAR}, \end{aligned} \tag{B.4.3}$$

$$\begin{aligned} \tfrac{1}{4}n^2(n-1)^2 = {} & \tfrac{1}{12}(n+2)(n+1)n(n-3) + \tfrac{1}{8}(n+2)n(n-1)(n-3) \\ & + \tfrac{1}{24}n(n-1)(n-2)(n-3) + \tfrac{1}{2}(n+2)(n-1) + \tfrac{1}{2}n(n-1) + 1, \end{aligned} \tag{B.4.4}$$

where

$${}^{(2)}W_{\alpha\beta} = (-1)^s\, {}^*(\vartheta_{[\alpha} \wedge \Psi_{\beta]}), \tag{B.4.5}$$

$${}^{(3)}W_{\alpha\beta} = (-1)^s \tfrac{1}{12}\, {}^*(X \wedge \vartheta_\alpha \wedge \vartheta_\beta), \tag{B.4.6}$$

$${}^{(4)}W_{\alpha\beta} = -\frac{2}{n-2}\, \vartheta_{[\alpha} \wedge \Phi_{\beta]}, \tag{B.4.7}$$

$${}^{(5)}W_{\alpha\beta} = -\frac{1}{n-2}\, \vartheta_{[\alpha} \wedge e_{\beta]} \rfloor (\vartheta^\alpha \wedge W_\alpha), \tag{B.4.8}$$

$${}^{(6)}W_{\alpha\beta} = -\frac{1}{n(n-1)} W\, \vartheta_\alpha \wedge \vartheta_\beta, \tag{B.4.9}$$

$${}^{(1)}W_{\alpha\beta} = W_{\alpha\beta} - \sum_{n=2}^{6} {}^{(n)}W_{\alpha\beta}, \tag{B.4.10}$$

with

Table 4
Irreducible decomposition of the curvature in four dimensions.

96 L_4	96 (L_4, g)	42 Y_4	36 U_4	20 V_4
	10 ${}^{(1)}W$ (WEYL)	10 ${}^{(1)}W$	10 ${}^{(1)}W$	10 ${}^{(1)}W$
	9 ${}^{(2)}W$ (PAIRCOM)	9 ${}^{(2)}W$	9 ${}^{(2)}W$	–
	1 ${}^{(3)}W$ (PSCALAR)	1 ${}^{(3)}W$	1 ${}^{(3)}W$	–
10 C	9 ${}^{(4)}W$ (RICSYMF)	9 ${}^{(4)}W$	9 ${}^{(4)}W$	9 ${}^{(4)}W$
6 D	6 ${}^{(5)}W$ (RICANTI)	6 ${}^{(5)}W$	6 ${}^{(5)}W$	–
(10 C)→	1 ${}^{(6)}W$ (SCALAR)	1 ${}^{(6)}W$	1 ${}^{(6)}W$	1 ${}^{(6)}W$
	30 ${}^{(1)}Z$	–	–	–
	9 ${}^{(2)}Z$	–	–	–
	6 ${}^{(3)}Z$	–	–	–
6 E	6 ${}^{(4)}Z$ (DILCURV)	6 ${}^{(4)}Z$	–	–
	9 ${}^{(5)}Z$	–	–	–

$$W^\alpha := e_\beta \rfloor W^{\alpha\beta}, \qquad W := e_\alpha \rfloor W^\alpha, \qquad X^\alpha :=^* (W^{\beta\alpha} \wedge \vartheta_\beta), \qquad X := e_\alpha \rfloor X^\alpha, \tag{B.4.11}$$

and

$$\Psi_\alpha := X_\alpha - \frac{1}{4}\vartheta_\alpha \wedge X - \frac{1}{n-2} e_\alpha \rfloor (\vartheta^\beta \wedge X_\beta),$$
$$\Phi_\alpha := W_\alpha - \frac{1}{n} W \vartheta_\alpha - \frac{1}{2} e_\alpha \rfloor (\vartheta^\beta \wedge W_\beta). \tag{B.4.12}$$

The trace and symmetry properties of of the various irreducible components are summarized by

$$e^\beta \rfloor {}^{(i)}W_{\alpha\beta} = 0, \quad i = 1,2,3; \qquad e^\alpha \rfloor e^\beta \rfloor {}^{(i)}W_{\alpha\beta} = 0, \quad i = 1,2,3,4,5, \tag{B.4.13}$$
$$\vartheta^\alpha \wedge (e^\beta \rfloor {}^{(i)}W_{\alpha\beta}) = 0, \quad i = 1,2,3,4,6, \tag{B.4.14}$$
$$\vartheta^\beta \wedge {}^{(i)}W_{\alpha\beta} = 0 \quad i = 1,4,6; \qquad \vartheta^\alpha \wedge \vartheta^\beta \wedge {}^{(i)}W_{\alpha\beta} = 0, \quad i = 1,2,4,5,6. \tag{B.4.15}$$

As indicated by (B.4.4), ${}^{(1)}W_{\alpha\beta}$, ${}^{(2)}W_{\alpha\beta}$, and ${}^{(3)}W_{\alpha\beta}$ are all vanishing when $n = 3$. In two dimensions, the single irreducible component is

$$W_{\alpha\beta} = {}^{(6)}W_{\alpha\beta}, \quad n = 2. \tag{B.4.16}$$

Turning to the symmetric part $Z_{\alpha\beta}$ of the curvature two-form (cf. [684,287]), we first of all split it into a tracefree and a trace part,

$$Z_{\alpha\beta} = \not{Z}_{\alpha\beta} + (1/n) g_{\alpha\beta} Z, \quad Z = Z_\gamma{}^\gamma, \tag{B.4.17}$$

so that $\not{Z}_\gamma{}^\gamma = 0$. In two dimensions no further reduction is possible. For $n > 2$, Z is still irreducible but $\not{Z}_{\alpha\beta}$ may be decomposed into further irreducible pieces. The full irreducible decomposition for $Z_{\alpha\beta}$ may be displayed as follows: Let

$$\not{Z}_\alpha := e^\beta \rfloor \not{Z}_{\alpha\beta}, \qquad \Delta := \frac{1}{(n-2)} (\vartheta^\alpha \wedge \not{Z}_\alpha), \qquad Y_\alpha := {}^*(\not{Z}_{\alpha\beta} \wedge \vartheta^\beta), \tag{B.4.18}$$

$$\Xi_\alpha := Z_\alpha - \tfrac{1}{2}e_\alpha \rfloor (\vartheta^\gamma \wedge Z_\gamma), \qquad \Upsilon_\alpha := Y_\alpha - \frac{1}{(n-2)}\, e_\alpha \rfloor (\vartheta^\gamma \wedge Y_\gamma) \tag{B.4.19}$$

be given. Note that Ξ_α and Υ_α satisfy

$$e_\alpha \rfloor \Xi^\alpha = 0, \qquad \Xi_\alpha \wedge \vartheta^\alpha = 0, \tag{B.4.20}$$

$$e_\alpha \rfloor \Upsilon^\alpha = 0, \qquad \Upsilon_\alpha \wedge \vartheta^\alpha = 0. \tag{B.4.21}$$

The irreducible decomposition of $Z_{\alpha\beta}$ may then be written as

$$Z_{\alpha\beta} = {}^{(1)}Z_{\alpha\beta} + {}^{(2)}Z_{\alpha\beta} + {}^{(3)}Z_{\alpha\beta} + {}^{(4)}Z_{\alpha\beta} + {}^{(5)}Z_{\alpha\beta}, \tag{B.4.22}$$

$$\begin{aligned}\tfrac{1}{4}n^2(n^2-1) &= \tfrac{1}{8}(n-2)(n+4)(n^2-1) + \tfrac{1}{8}(n+2)n(n-1)(n-3)\\ &\quad + \tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n-1) + \tfrac{1}{2}(n+2)(n-1),\end{aligned} \tag{B.4.23}$$

where

$${}^{(2)}Z_{\alpha\beta} = \tfrac{1}{2}(-1)^s\, {}^*\{\vartheta_{(\alpha} \wedge \Upsilon_{\beta)}\}, \tag{B.4.24}$$

$${}^{(3)}Z_{\alpha\beta} = \frac{1}{(n+2)}\{n\vartheta_{(\alpha} \wedge (e_{\beta)} \rfloor \Delta) - 2g_{\alpha\beta}\, \Delta\}, \tag{B.4.25}$$

$${}^{(4)}Z_{\alpha\beta} = (1/n)\, g_{\alpha\beta} Z, \tag{B.4.26}$$

$${}^{(5)}Z_{\alpha\beta} = (2/n)\, \vartheta_{(\alpha} \wedge \Xi_{\beta)}, \tag{B.4.27}$$

$${}^{(1)}Z_{\alpha\beta} = Z_{\alpha\beta} - {}^{(2)}Z_{\alpha\beta} - {}^{(3)}Z_{\alpha\beta} - {}^{(4)}Z_{\alpha\beta} - {}^{(5)}Z_{\alpha\beta}. \tag{B.4.28}$$

The Young symmetries of the various irreducible pieces are displayed by

$$\vartheta^\beta \wedge {}^{(1)}Z_{\alpha\beta} = \vartheta^\beta \wedge {}^{(5)}Z_{\alpha\beta} = 0, \qquad {}^{(2)}Z_{\alpha\beta} - \tfrac{1}{2}e_{(\alpha} \rfloor \{\vartheta^\gamma \wedge {}^{(2)}Z_{\beta)\gamma}\} = 0, \tag{B.4.29}$$

while the trace properties are

$$e_\alpha \rfloor {}^{(1)}Z^{\alpha\beta} = e_\alpha \rfloor {}^{(2)}Z^{\alpha\beta} = 0, \tag{B.4.30}$$

$${}^{(1)}Z_\alpha{}^\alpha = {}^{(2)}Z_\alpha{}^\alpha = {}^{(3)}Z_\alpha{}^\alpha = {}^{(5)}Z_\alpha{}^\alpha = 0. \tag{B.4.31}$$

By (B.4.23), the second irreducible piece ${}^{(2)}Z_{\alpha\beta}$ vanishes for $n = 3$. The two irreducible components in the case of two dimensions are given by

$$Z_{\alpha\beta} = {}^{(4)}Z_{\alpha\beta} + {}^{(5)}Z_{\alpha\beta}, \quad n = 2. \tag{B.4.32}$$

With regard to the uniqueness of the decomposition, a remark similar to that made in the case of the nonmetricity applies here. If we simply apply the Young diagram procedure to the components $Z_{\mu\nu\alpha\beta}$ of $Z_{\alpha\beta}$ and take traces, three of the five irreducible pieces obtained are ${}^{(1)}Z_{\alpha\beta}$, ${}^{(2)}Z_{\alpha\beta}$, and ${}^{(5)}Z_{\alpha\beta}$ as above, but the remaining two pieces are arbitrary combinations of the two irreducible subspaces involved in ${}^{(3)}Z_{\alpha\beta}$ and ${}^{(4)}Z_{\alpha\beta}$ above and hence are not canonical (see [103,287]). Here, however, the initial decomposition (B.4.17) with respect to the indices on the *two-form* $Z_{\alpha\beta}$ has led to a unique canonical set of irreducible pieces. [82]

[82] The irreducible decomposition of Šijački [626], invariant under $GL(4,R)$, starts with two abstract tensors of type $\binom{0}{3}$ and $\binom{0}{4}$ which carry the symmetries of torsion and curvature, respectively, see his table on page 394. However, his results are are not comparable to ours, except for the $SO(1,3)$-subcase.

The orthogonality property noted in the cases of the nonmetricity and torsion holds also for the curvature. With respect to the scalar product (B.1.12) the 11 irreducible pieces of the curvature are mutually orthogonal, i.e.

$$\begin{cases} {}^{(i)}W \cdot {}^{(j)}W = 0 & \text{for } i \neq j, \\ {}^{(i)}Z \cdot {}^{(j)}Z = 0 & \text{for } i \neq j, \\ {}^{(i)}W \cdot {}^{(j)}Z = 0 & \text{for } i = 1 \cdots 6, \quad j = 1 \cdots 5. \end{cases} \tag{B.4.33}$$

Hence

$$R \cdot R = \sum_{i=1}^{6} {}^{(i)}W \cdot {}^{(i)}W \; + \; \sum_{j=1}^{5} {}^{(j)}Z \cdot {}^{(j)}Z. \tag{B.4.34}$$

In analogy with the equations (B.2.11) for the torsion, we note the following relations (omitting the indices) which hold in four dimensions

$$\begin{aligned} &{}^{*(1)}W = {}^{(1)*}W, \qquad &&{}^{*(2)}W = {}^{(4)*}W, \qquad &&{}^{*(4)}W = {}^{(2)*}W, \\ &{}^{*(3)}W = {}^{(6)*}W, \qquad &&{}^{*(6)}W = {}^{(3)*}W, \qquad &&{}^{*(5)}W = {}^{(5)*}W, \end{aligned} \tag{B.4.35}$$

and

$$\begin{aligned} &{}^{*(1)}Z = {}^{(1)*}Z, \qquad &&{}^{*(3)}Z = {}^{(3)*}Z, \qquad &&{}^{*(4)}Z = {}^{(4)*}Z, \\ &{}^{*(2)}Z = {}^{(5)*}Z, \qquad &&{}^{*(5)}Z = {}^{(2)*}Z. && \end{aligned} \tag{B.4.36}$$

Similarly as in the case of the torsion, there is an additional (pseudo)scalar product in four dimensions defined by (B.2.15). For this product the "orthogonality" properties are summed up by the equation

$$\begin{aligned} \langle R, R\rangle = {} & \langle {}^{(1)}W, {}^{(1)}W\rangle + 2\langle {}^{(2)}W, {}^{(4)}W\rangle + 2\langle {}^{(3)}W, {}^{(6)}W\rangle + \langle {}^{(5)}W, {}^{(5)}W\rangle \\ & + \langle {}^{(1)}Z, {}^{(1)}Z\rangle + 2\langle {}^{(2)}Z, {}^{(5)}Z\rangle + \langle {}^{(3)}Z, {}^{(3)}Z\rangle + \langle {}^{(4)}Z, {}^{(4)}Z\rangle. \end{aligned} \tag{B.4.37}$$

B.5. Irreducible decompositions of the zeroth and first Bianchi identities

If we treat the 0th Bianchi identity

$$B^{(0)}_{\alpha\beta} \equiv D\,Q_{\alpha\beta} - 2R_{(\alpha\beta)} \equiv 0, \tag{B.5.1}$$

as a 2nd order symmetric tensor-valued two-form, it splits under the general linear group into two irreducible pieces,

$$B^{(0)}_{\alpha\beta} = {}^{(I)}B^{(0)}_{\alpha\beta} + {}^{(II)}B^{(0)}_{\alpha\beta}, \tag{B.5.2}$$

$$\tfrac{1}{4}n^2(n^2-1) = \tfrac{1}{8}n(n^2-1)(n+2) + \tfrac{1}{8}n(n^2-1)(n-2), \tag{B.5.3}$$

with

$${}^{(II)}B^{(0)}_{\alpha\beta} = \tfrac{1}{2}e_{(\alpha}\rfloor(B^{(0)}_{\beta)\gamma}\wedge\vartheta^\gamma), \qquad {}^{(I)}B^{(0)}_{\alpha\beta} = B^{(0)}_{\alpha\beta} - {}^{(II)}B^{(0)}_{\alpha\beta}, \tag{B.5.4}$$

provided $n > 2$. ${}^{(I)}B^{(0)}_{\alpha\beta}$ satisfies the symmetries

$$ {}^{(I)}B^{(0)}_{\alpha\beta} \wedge \vartheta^{\beta} = 0 \tag{B.5.5}$$

and is the only piece that survives in two dimensions.

Since the 0th Bianchi identity is a second order symmetric tensor-valued two-form, which, by construction, is already metric dependent, its irreducible decomposition invariant under the (pseudo)orthogonal group is the same as the corresponding decomposition for the symmetric part of the curvature given in the previous section.

The *1st Bianchi identity* is

$$ B'^{\alpha} \equiv 0, \tag{B.5.6}$$

$$ B'^{\alpha} = DT^{\alpha} - R_{\beta}{}^{\alpha} \wedge \vartheta^{\beta}, \tag{B.5.7}$$

B'^{α} is a vector-valued three-form and hence, in two dimensions it is zero, while in three dimensions, it is irreducible, with respect to both, the general linear and the (pseudo)orthogonal group. In an L_n ($n > 3$) its irreducible decomposition invariant under $GL(n, R)$ is given by

$$ B'^{\alpha} = {}^{(I)}B'^{\alpha} + {}^{(II)}B'^{\alpha}, \tag{B.5.8}$$

$$ \tfrac{1}{6}n^2(n-1)(n-2) = \tfrac{1}{6}n(n^2-1)(n-3) + \tfrac{1}{2}n(n-1), \tag{B.5.9}$$

where

$$ {}^{(II)}B'^{\alpha} = \frac{1}{n-2}\vartheta^{\alpha} \wedge (e_{\gamma} \rfloor B'^{\gamma}), \qquad {}^{(I)}B'^{\alpha} = B'^{\alpha} - {}^{(II)}B'^{\alpha}. \tag{B.5.10}$$

${}^{(I)}B'^{\alpha}$ clearly satisfies

$$ e_{\alpha} \rfloor {}^{(I)}B'^{\alpha} = 0. \tag{B.5.11}$$

In an (L_n, g) the irreducible components with respect to the (pseudo)orthogonal group

$$ B'^{\alpha} = {}^{(1)}B'^{\alpha} + {}^{(2)}B'^{\alpha} + {}^{(3)}B'^{\alpha}, \tag{B.5.12}$$

$$ \tfrac{1}{6}n^2(n-1)(n-2) $$
$$ = \tfrac{1}{8}(n+2)n(n-1)(n-3) + \tfrac{1}{2}n(n-1) + \tfrac{1}{24}n(n-1)(n-2)(n-3), \tag{B.5.13}$$

with

$$ {}^{(2)}B'^{\alpha} = \frac{1}{(n-2)}(e_{\beta} \rfloor B'^{\beta}) \wedge \vartheta^{\alpha}, \qquad {}^{(3)}B'^{\alpha} = \tfrac{1}{4}e_{\alpha} \rfloor (\vartheta^{\beta} \wedge B'_{\beta}), \tag{B.5.14}$$

$$ {}^{(1)}B'^{\alpha} = B'^{\alpha} - {}^{(2)}B'^{\alpha} - {}^{(3)}B'^{\alpha}. \tag{B.5.15}$$

The symmetries and trace properties are given by

$$ \vartheta^{\alpha} \wedge {}^{(1)}B'_{\alpha} = \vartheta^{\alpha} \wedge {}^{(2)}B'_{\alpha} = 0, \qquad e_{\alpha} \rfloor {}^{(1)}B'^{\alpha} = e_{\alpha} \rfloor {}^{(3)}B'^{\alpha} = 0. \tag{B.5.16}$$

One may also verify that ${}^{(i)}B'^{\alpha}$ ($i = 1, 2, 3$) are mutually orthogonal with respect to the scalar product (B.1.12). In three dimensions we have ${}^{(1)}B'^{\alpha} = {}^{(3)}B'^{\alpha} = 0$ and, consequently,

$$ B'^{\alpha} = {}^{(2)}B'^{\alpha}, \quad n = 3. \tag{B.5.17}$$

B.6. Irreducible decomposition of the second Bianchi identity

The left-hand side of the *2nd Bianchi identity*

$$B_{\alpha}^{\prime\prime\beta} \equiv 0, \tag{B.6.1}$$

$$B_{\alpha}^{\prime\prime\beta} = DR_{\alpha}{}^{\beta} \tag{B.6.2}$$

is a second order tensor-valued three-form, so we need only to consider dimensions greater than two.

With respect to the general linear group, and for $n > 3$, we obtain five irreducible pieces as follows: Let

$$V^{\beta} := \vartheta^{\gamma} \wedge \not{B}_{\gamma}{}^{\beta}, \qquad V_{\alpha}{}^{\beta} := e_{\alpha} \rfloor V^{\beta}, \qquad V := V_{\alpha}{}^{\alpha}, \tag{B.6.3}$$

be given, where

$$\not{B}_{\alpha}^{\prime\prime\beta} = B_{\alpha}^{\prime\prime\beta} - (1/n)\delta_{\alpha}^{\beta} B_{\gamma}^{\prime\prime\gamma}. \tag{B.6.4}$$

Furthermore, let us define

$$C_{\alpha}^{\prime\prime\beta} := \not{B}_{\alpha}^{\prime\prime\beta} - \tfrac{1}{4} V_{\alpha}{}^{\beta}, \tag{B.6.5}$$

$$X_{\alpha} = e_{\gamma} \rfloor C_{\alpha}^{\prime\prime\gamma}, \qquad X_{\alpha}{}^{\beta} = \vartheta^{\beta} \wedge X_{\alpha}, \qquad X = X_{\alpha}{}^{\alpha}. \tag{B.6.6}$$

Then

$$B_{\alpha}^{\prime\prime\beta} = {}^{(I)}B_{\alpha}^{\prime\prime\beta} + {}^{(II)}B_{\alpha}^{\prime\prime\beta} + {}^{(III)}B_{\alpha}^{\prime\prime\beta} + {}^{(IV)}B_{\alpha}^{\prime\prime\beta} + {}^{(V)}B_{\alpha}^{\prime\prime\beta}, \tag{B.6.7}$$

$$\begin{aligned} &\tfrac{1}{6} n^3 (n-1)(n-2) \\ &\quad = \tfrac{1}{8} n^2 (n+2)(n-1)(n-3) + \tfrac{1}{24} n(n^2-1)(n-2)(n-4) \\ &\qquad + \tfrac{1}{6} n(n-1)(n-2) + \tfrac{1}{3} n(n^2-1) + \tfrac{1}{6} n(n-1)(n-2), \end{aligned} \tag{B.6.8}$$

where

$${}^{(II)}B_{\alpha}^{\prime\prime\beta} = \frac{1}{4}\left(V_{\alpha}{}^{\beta} - \frac{1}{n-3} e_{\alpha} (\vartheta^{\beta} \wedge V) \right), \tag{B.6.9}$$

$${}^{(III)}B_{\alpha}^{\prime\prime\beta} = \frac{4}{(n+1)(n-3)} (\tfrac{1}{3} n \vartheta^{\beta} \wedge (e_{\alpha} \rfloor X) - \delta_{\alpha}^{\beta} X), \tag{B.6.10}$$

$${}^{(IV)}B_{\alpha}^{\prime\prime\beta} = \frac{1}{n-2} \vartheta^{\beta} \wedge (X_{\alpha} - \tfrac{1}{3} e_{\alpha} \rfloor X), \tag{B.6.11}$$

$${}^{(V)}B_{\alpha}^{\prime\prime\beta} = (1/n)\delta_{\alpha}^{\beta} B_{\gamma}^{\prime\prime\gamma}, \tag{B.6.12}$$

$${}^{(I)}B_{\alpha}^{\prime\prime\beta} = B^{\prime\prime} - {}^{(II)}B_{\alpha}^{\prime\prime\beta} - {}^{(III)}B_{\alpha}^{\prime\prime} - {}^{(IV)}B_{\alpha}^{\prime\prime\beta} - {}^{(V)}B_{\alpha}^{\prime\prime\beta}. \tag{B.6.13}$$

It may be verified that these satisfy the following properties:

$${}^{(K)}B_{\alpha}^{\prime\prime\alpha} = 0, \qquad K = I, II, III, IV, \tag{B.6.14}$$

$$\vartheta^{\beta} \wedge {}^{(K)}B_{\alpha}^{\prime\prime\beta} = 0, \qquad K = I, IV, \tag{B.6.15}$$

$$e_{\alpha} \rfloor (\vartheta^{\beta} \wedge {}^{(K)}B_{\alpha}^{\prime\prime\beta}) = 0, \qquad K = I, II, IV, \tag{B.6.16}$$

$$e_\alpha \rfloor {}^{(K)}B''^{\alpha}_{\beta} = 0, \qquad K = I, II, \tag{B.6.17}$$

$$\vartheta^\beta \wedge (e_\alpha \rfloor {}^{(K)}B''^{\alpha}_{\beta}) = 0, \qquad K = I, II, IV. \tag{B.6.18}$$

Note, however, that (B.6.16) is a consequence of (B.6.14) and (B.6.18).

In four dimensions

$${}^{(II)}B''^{\beta}_{\alpha} = 0, \quad n = 4 \tag{B.6.19}$$

leaving four irreducible pieces. In three dimensions the number of irreducible components is two,

$$B''^{\beta}_{\alpha} = \not{B}''^{\beta}_{\alpha} + {}^{(V)}B''^{\beta}_{\alpha}, \tag{B.6.20}$$

or equivalently

$$B''^{\beta}_{\alpha} = {}^{(IV)}B''^{\beta}_{\alpha} + {}^{(V)}B''^{\beta}_{\alpha}, \tag{B.6.21}$$

$$9 = 8 + 1. \tag{B.6.22}$$

As a first step in deriving its irreducible decomposition, we split it into its antisymmetric and symmetric parts, $A_{\alpha\beta} = B''_{[\alpha\beta]}$ and $S_{\alpha\beta} = B''_{(\alpha\beta)}$:

$$B''^{\alpha\beta} = A^{\alpha\beta} + S^{\alpha\beta}, \tag{B.6.23}$$

$$\tfrac{1}{6}n^3(n-1)(n-2) = \tfrac{1}{12}n^2(n-1)^2(n-2) + \tfrac{1}{12}n^2(n^2-1)(n-2).$$

In three dimensions $A_{\alpha\beta}$ cannot be reduced any further.

For the case of dimension $n > 3$ the irreducible components may be exhibited as follows (cf. [287] for $n = 4$): Let

$$A_\alpha = A_{\alpha\beta} \wedge \vartheta^\beta, \qquad A = A_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta, \qquad A_{\alpha\beta\gamma\delta} = e_\alpha \rfloor e_\beta \rfloor A_{\gamma\delta}. \tag{B.6.24}$$

Then $A_{\alpha\beta}$ has three irreducible pieces invariant under $GL(n,R)$:

$$A_{\alpha\beta} = {}^{(I)}A_{\alpha\beta} + {}^{(II)}A_{\alpha\beta} + {}^{(III)}A_{\alpha\beta}, \tag{B.6.25}$$

$$\tfrac{1}{12}n^2(n-1)^2(n-2) = \tfrac{1}{120}n(n-1)(n-2)(n-3)(n-4) + \tfrac{1}{30}n(n^2-1)(n-2)(n-3) + \tfrac{1}{30}n^2(n^2-1)(n-2),$$

where

$$\begin{aligned} {}^{(I)}A_{\alpha\beta} &= -\tfrac{1}{20}e_\alpha \rfloor e_\beta \rfloor A, \\ {}^{(II)}A_{\alpha\beta} &= \tfrac{2}{15}(e_{[\alpha} \rfloor A_{\beta]} + A_{\alpha\beta\gamma\delta} \wedge \vartheta^\gamma \wedge \vartheta^\delta + 2A_{\alpha\beta}), \\ {}^{(III)}A_{\alpha\beta} &= A_{\alpha\beta} - {}^{(I)}A_{\alpha\beta} - {}^{(II)}A_{\alpha\beta}. \end{aligned} \tag{B.6.26}$$

To get the irreducible components under (pseudo)orthogonal transformations we take traces. All traces on ${}^{(I)}A_{\alpha\beta}$ are zero, and there is just one independent trace on each of ${}^{(II)}A_{\alpha\beta}$ and ${}^{(III)}A_{\alpha\beta}$. Let

$$\tau_\alpha = e^\gamma \rfloor {}^{(III)}A_{\alpha\gamma} \tag{B.6.27}$$

be a vector-valued two-form of the same algebraic structure as the torsion, but with vanishing axial piece, i.e.$\tau_\alpha \wedge \vartheta^\alpha = 0$. Hence

$$\tau_\alpha = {}^{(1)}\tau_\alpha + {}^{(2)}\tau_\alpha, \tag{B.6.28}$$

$${}^{(2)}\tau_\alpha = \frac{1}{(n-1)}\,\vartheta_\alpha \wedge (e_\beta \rfloor \tau^\beta), \qquad {}^{(1)}\tau_\alpha = \tau_\alpha - {}^{(2)}\tau_\alpha. \tag{B.6.29}$$

On the other hand, ${}^{(II)}A_{\alpha\beta}$ has got only an axial piece and is therefore not reducible any further. We therefore obtain

$$A_{\alpha\beta} = {}^{(1)}A_{\alpha\beta} + {}^{(2)}A_{\alpha\beta} + {}^{(3)}A_{\alpha\beta} + {}^{(4)}A_{\alpha\beta} + {}^{(5)}A_{\alpha\beta} + {}^{(6)}A_{\alpha\beta}, \tag{B.6.30}$$

$$\begin{aligned}\tfrac{1}{12}n^2(n-1)^2(n-2) = {} & \tfrac{1}{24}n(n+2)(n^2-1)(n-4) + \tfrac{1}{3}n(n^2-4) + n \\ & + \tfrac{1}{30}n(n-1)(n^2-4)(n-4) + \tfrac{1}{6}n(n-1)(n-2) \\ & + \tfrac{1}{120}n(n-1)(n-2)(n-3)(n-4),\end{aligned} \tag{B.6.31}$$

with

$${}^{(2)}A_{\alpha\beta} = -\frac{2}{(n-3)}\vartheta_{[\alpha} \wedge {}^{(1)}\tau_{\beta]}, \tag{B.6.32}$$

$${}^{(3)}A_{\alpha\beta} = -\frac{1}{(n-2)}\vartheta_{[\alpha} \wedge {}^{(2)}\tau_{\beta]}, \tag{B.6.33}$$

$${}^{(4)}A_{\alpha\beta} = {}^{(II)}A_{\alpha\beta} - {}^{(5)}A_{\alpha\beta}, \tag{B.6.34}$$

$${}^{(5)}A_{\alpha\beta} = -\frac{2}{(n-3)}\vartheta_{[\alpha} \wedge \{e^\gamma \rfloor {}^{(II)}A_{\beta]\gamma}\}, \tag{B.6.35}$$

$${}^{(6)}A_{\alpha\beta} = {}^{(I)}A_{\alpha\beta}, \tag{B.6.36}$$

$${}^{(1)}A_{\alpha\beta} = A_{\alpha\beta} - {}^{(2)}A_{\alpha\beta} - {}^{(3)}A_{\alpha\beta} - {}^{(4)}A_{\alpha\beta} - {}^{(5)}A_{\alpha\beta} - {}^{(6)}A_{\alpha\beta}. \tag{B.6.37}$$

We note the following properties:

$${}^{(i)}A_{\alpha\beta} \wedge \vartheta^\beta = 0, \quad i = 1,2,3, \tag{B.6.38}$$

$$e_\beta \rfloor {}^{(i)}A_{\alpha\beta} = 0, \quad i = 1,4,6, \tag{B.6.39}$$

$$e_\alpha \rfloor e_\beta \rfloor {}^{(i)}A_{\alpha\beta} = 0, \quad i = 1,2,4,5,6, \tag{B.6.40}$$

$${}^{(i)}A_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta = 0, \quad i = 1,2,3,4,5. \tag{B.6.41}$$

It is clear from (B.6.31) that ${}^{(1)}A_{\alpha\beta}$, ${}^{(4)}A_{\alpha\beta}$ and ${}^{(6)}A_{\alpha\beta}$ vanish for $n = 4$. In three dimensions, the single irreducible piece is given by

$$A_{\alpha\beta} = {}^{(3)}A_{\alpha\beta}, \quad n = 3. \tag{B.6.42}$$

To find the irreducible pieces of the symmetric part $S_{\alpha\beta}$ of the 2nd Bianchi identity (cf. [684] for $n = 4$), we first of all split the three-form $S_{\alpha\beta}$ into its tracefree and trace parts:

$$S_{\alpha\beta} = \mathcal{S}_{\alpha\beta} + (1/n)g_{\alpha\beta}S, \quad S = S_\gamma{}^\gamma. \tag{B.6.43}$$

In three dimensions these are the only irreducible components. For dimension $n > 3$ the second term in (B.6.43) is clearly irreducible and, under $GL(n, R)$, the first term $\mathcal{S}_{\alpha\beta}$ has two irreducible pieces

$$ {}^{(I)}\,\mathcal{S}_{\alpha\beta} = \tfrac{2}{5} e_{(\alpha} \rfloor U_{\beta)}, \qquad {}^{(II)}\,\mathcal{S}_{\alpha\beta} = \mathcal{S}_{\alpha\beta} - {}^{(I)}\,\mathcal{S}_{\alpha\beta}, \tag{B.6.44} $$

where

$$ U_\alpha = \vartheta^\gamma \wedge \mathcal{S}_{\gamma\alpha}. \tag{B.6.45} $$

On taking traces each of these splits into two so that, when we take S of (B.6.43) into account, the eventual result for $S_{\alpha\beta}$, and for $n > 3$, is

$$ S_{\alpha\beta} = {}^{(1)}S_{\alpha\beta} + {}^{(2)}S_{\alpha\beta} + {}^{(3)}S_{\alpha\beta} + {}^{(4)}S_{\alpha\beta} + {}^{(5)}S_{\alpha\beta}, \tag{B.6.46} $$

$$ \begin{aligned} &\tfrac{1}{12} n^2 (n^2-1)(n-2) \\ &\quad = \tfrac{1}{20} n(n+1)(n+4)(n-2)(n-3) + \tfrac{1}{30} n(n-1)(n^2-4)(n-4) \\ &\qquad + \tfrac{1}{6} n(n-1)(n-2) + \tfrac{1}{3} n(n^2-4) + \tfrac{1}{6} n(n-1)(n-2), \end{aligned} \tag{B.6.47} $$

where

$$ {}^{(2)}S_{\alpha\beta} = \frac{2}{5}\left(e_{(\alpha} \rfloor U_{\beta)} + \frac{1}{(n-3)} \{ g_{\alpha\beta} \Xi - \vartheta_{(\alpha} \wedge (e_{\beta)} \rfloor \Xi) \} \right), \tag{B.6.48} $$

$$ {}^{(3)}S_{\alpha\beta} = \frac{2}{(n+2)(n-3)} [-g_{\alpha\beta} \Xi + \tfrac{1}{3} n \vartheta_{(\alpha} \wedge (e_{\beta)} \rfloor \Xi)], \tag{B.6.49} $$

$$ {}^{(4)}S_{\alpha\beta} = \frac{2}{(n-1)} \vartheta_{(\alpha} \wedge P_{\beta)}, \tag{B.6.50} $$

$$ {}^{(5)}S_{\alpha\beta} = (1/n) g_{\alpha\beta} S, \tag{B.6.51} $$

$$ {}^{(1)}S_{\alpha\beta} = S_{\alpha\beta} - {}^{(2)}S_{\alpha\beta} - {}^{(3)}S_{\alpha\beta} - {}^{(4)}S_{\alpha\beta} - {}^{(5)}S_{\alpha\beta}, \tag{B.6.52} $$

and

$$ \Xi_\alpha = e^\gamma \rfloor \mathcal{S}_{\alpha\gamma}, \qquad \Xi = \Xi_\alpha \wedge \vartheta^\alpha, \qquad P_\alpha = \Xi_\alpha - \tfrac{1}{3} e_\alpha \rfloor \Xi. \tag{B.6.53} $$

These satisfy the following properties,

$$ {}^{(i)}S_{\alpha\beta} \wedge \vartheta^\beta = 0, \quad i = 1, 4, \tag{B.6.54} $$

$$ e^\beta \rfloor {}^{(i)}S_{\alpha\beta} = 0, \quad i = 1, 2, \tag{B.6.55} $$

$$ {}^{(i)}S_\alpha{}^\alpha = 0, \quad i = 1, 2, 3, 4. \tag{B.6.56} $$

According to (B.6.47), the term ${}^{(2)}S_{\alpha\beta}$ drops out in four dimensions. In three dimensions $U_\alpha = 0$, $\Xi = 0$ and the irreducible piece $\mathcal{S}_{\alpha\beta}$ may be written as

$$ \mathcal{S}_{\alpha\beta} = {}^{(4)}S_{\alpha\beta}, \quad n = 3, \tag{B.6.57} $$

where the right-hand side is that of (B.6.50), with $P_\alpha = \Xi_\alpha$.

To sum up, the left-hand side of the 2nd Bianchi identity in an n-dimensional metric-affine spacetime ($n > 3$) may be written in terms of its 11 irreducible parts as follows:

$$ \begin{aligned} B''^{\alpha\beta} = {} & {}^{(1)}A^{\alpha\beta} + {}^{(2)}A^{\alpha\beta} + {}^{(3)}A^{\alpha\beta} + {}^{(4)}A^{\alpha\beta} + {}^{(5)}A^{\alpha\beta} + {}^{(6)}A^{\alpha\beta} \\ & + {}^{(1)}S^{\alpha\beta} + {}^{(2)}S^{\alpha\beta} + {}^{(3)}S^{\alpha\beta} + {}^{(4)}S^{\alpha\beta} + {}^{(5)}S^{\alpha\beta}. \end{aligned} \tag{B.6.58} $$

Here ${}^{(n)}A^{\alpha\beta}$ $(n = 1,\dots,6)$ are defined by (B.6.32)–(B.6.37), while ${}^{(m)}S^{\alpha\beta}$ $(m = 1,\dots,5)$ are as given by (B.6.48)–(B.6.52). The number of independent components in each of the irreducible pieces is as indicated in (B.6.31) and (B.6.47) and from these it is clear that in four dimensions the number of irreducible parts reduces to seven. In three dimensions there are three irreducible pieces,

$$B''_{\alpha\beta} = {}^{(3)}A_{\alpha\beta} + {}^{(4)}S_{\alpha\beta} + {}^{(5)}S_{\alpha\beta} \quad \text{for} \quad n = 3. \tag{B.6.59}$$

Finally one may verify that

$$\begin{cases} {}^{(i)}A^{\alpha\beta} \wedge {}^{*(j)}A_{\alpha\beta} = 0, \text{ for } i \neq j, \\ {}^{(i)}A^{\alpha\beta} \wedge {}^{*(j)}S_{\alpha\beta} = 0, \text{ for } i = 1\dots 6,\ j = 1\cdots 5, \\ {}^{(i)}S^{\alpha\beta} \wedge {}^{*(j)}S_{\alpha\beta} = 0, \text{ for } i \neq j \end{cases} \tag{B.6.60}$$

so that, once again, the irreducible pieces are mutually orthogonal with respect to the scalar product (B.1.12).

Appendix C. Group representations

C.1. Unirreps of the $\overline{SL}(2,R)$

The *un*itary infinite-dimensional *ir*reducible *rep*resentations (*unirreps*) of $\overline{SL}(2,R)$ were constructed and catalogued by Bargmann [36]. The two-dimensional linear group $SL(2,R)$ is special insofar as it has infinite coverings: The maximal compact subgroup $SO(2)$ is, regarded as a manifold, isomorphic to the circle S^1, which is infinitely often covered by the line.

Bargmann listed four classes of representations, defined by τ, the eigenvalue of the quadratic $SL(2,R)$ Casimir operator

$$C_2 := \frac{1}{2}\mathcal{L}^{\alpha}{}_{\beta}\mathcal{L}^{\beta}{}_{\alpha} = \left(\overset{(-)}{B}\right)^2 + S^2 - \left(\overset{(+)}{B}\right)^2 = -\tau^2\,, \tag{C.1.1}$$

and by m, the (helicity-like) eigenvalue of the normal subalgebra generated by J_u:

(i) *Principal series* $\mathcal{D}^{\text{princ.}}_{SL(2,R)}(\underline{m},\tau)$:

$$\begin{aligned} &1/4 \le \tau, \qquad \tau \in R; \\ &\underline{m} = 0, \quad \{m\} = \{0, \pm1, \pm2, \pm3, \dots\}, \quad \text{i.e.} \quad \{m\} = Z, \\ &\text{or} \\ &\underline{m} = 1/2, \quad \{m\} = \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots\right\}, \quad \text{i.e.} \quad \{m\} = Z/2, \end{aligned} \tag{C.1.2}$$

(ii) *Supplementary series* $\mathcal{D}^{\text{suppl.}}_{SL(2,R)}(\underline{m},\tau)$:

$$\begin{aligned} &0 < \tau < 1/4, \\ &\underline{m} = 0, \quad \{m\} = \{0, \pm1, \pm2, \pm3, \dots\}, \quad \text{i.e.} \quad \{m\} = Z, \\ &\text{or} \\ &\underline{m} = 1/2, \quad \{m\} = \left\{\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots\right\}, \quad \text{i.e.} \quad \{m\} = Z/2, \end{aligned} \tag{C.1.3}$$

(iii) *Discrete series (mounting)* $\mathcal{D}^{\text{disc.}}_{SL(2,R)}(\underline{m})$:

$$\tau = \underline{m}\,(1-\underline{m});\quad \{\underline{m}\} = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots\right\},$$
$$\{m\} = \{\underline{m}, \underline{m}+1, \underline{m}+2, \dots\}, \tag{C.1.4}$$

(iv) *Discrete series (descending)* $\mathcal{D}^{\text{disc.}}_{SL(2,R)}(\overline{m})$:

$$\tau = -\overline{m}\,(1+\overline{m});\quad \{\overline{m}\} = \left\{-\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \dots\right\},$$
$$\{m\} = \{\dots, (\overline{m}-2), \overline{m}-1, \overline{m}\}. \tag{C.1.5}$$

In these formulae, $\underline{m}$ and $\overline{m}$ are the minimal or maximal eigenvalues of $\overset{(-)}{B}$ in a representation in which m is mounting or descending, respectively. The names of the various series refer to the values of τ.

Consider the $SL(n,R)$. Preserving for $n=2$ the identification $g_{\alpha\beta} = o_{\alpha\beta}$, but regarding the algebra of the $SL(2,R)$ as acting on a two-dimensional Minkowski space, we could represent the corresponding generators (4.2.4) in terms of the Pauli matrices (yielding real values in the exponentiation $e^{\mathcal{K}}$):

$$\overset{(-)}{B} \to \tfrac{1}{2}\sigma_3,\qquad \overset{(+)}{B} \to \tfrac{1}{2}i\sigma_2,\qquad S \to \tfrac{1}{2}\sigma_1, \tag{C.1.6}$$

with the $\overset{(+)}{B}$ as the only compact generator.

Note that besides being a real form of $SU(1,1)$, the group $SL(2,R)$ is in itself the double-covering group of $SO(1,2)$. As a result, for instance, $\overline{SL}(2,R)$ is the quadruple covering of $SO(1,2)$. Bargmann's point of departure [83] is derived from the $SO(1,2)$-realization, rather than the $SL(2,R)$. Following Bargmann and making our physical identifications at the $SO(1,2)$ level, i.e. on a three-dimensional Minkowski space, we find, going directly over to the spinor representation [i.e. $SL(2,R)$ as double-covering of $SO(1,2)$]:

$$J \to \tfrac{1}{2}i\sigma_2,\qquad \overset{(-)}{B}_1 \to \tfrac{1}{2}\sigma_3,\qquad \overset{(-)}{B}_2 \to \tfrac{1}{2}\sigma_1. \tag{C.1.7}$$

The representations with half-integer m in the supplementary series are thus two-valued in $SO(1,2)$ and *single-valued* in $SL(2,R)$, but they are not faithful representations of the double covering of that group itself. The representations of the double-covering group $\overline{SL}(2,R)$, displaying the structure

$$\overline{SL}(2,R)/Z_2 = SL(2,R)\,, \tag{C.1.8}$$

are of class (ii), i.e. are $\mathcal{D}^{\text{suppl.}}_{SL(2,R)}(h,\tau)$, with $h = 1/4$. More generally, the representations of the multiple covering have $h = 1/(2c)$, where c denotes the order of the covering.

[83] Although Bargmann had constructed these representations of the multiple covering in 1947, these results were not assimilated by the physics community, who continued to assume the inexistence of a covering group of $GL(n,R)$ even for $n=2$. The fact that $SL(2,R)$ is itself a double covering of $SO(1,2)$ added to the confusion and strengthened the impression, throughout 1928–1977, that linear groups have no double covering (see the discussion of $n=2$ in [477], including the examples of Ref. 11 in that article).

C.2. Unirreps of the $\overline{SL}(3,R)$

There are four series of unirreps [84] of $\overline{SL}(3,R)$. Denoting by j the angular momentum eigenvalue corresponding to the maximal compact subgroup $\overline{SO}(3) = SU(2)$, and k an additional quantum number of this rank two group, by $\underline{j}$ and $\underline{k}$ the minimal values of J and $\overset{(+)}{B} \equiv K$ in a unirrep, and n the multiplicity, we have

(i) *Principal series* $\mathcal{D}^{\text{princ.}}_{SL(3,R)}(\underline{j},\underline{k};\sigma_2,\delta_2)$:

$$\begin{aligned}
&\sigma_2,\delta_2 \in R; \quad \underline{k} = 0, 1/2, 1;\\
&\underline{k} = 0 \to \underline{j} = 0, \quad \{j^n\} = \{0, 2^2, 3, 4^3, 5^2, 6^4, 7^3, \ldots\},\\
&\underline{k} = \frac{1}{2} \to \underline{j} = \frac{1}{2}, \quad \{j^n\} = \left\{\frac{1}{2}, \left(\frac{3}{2}\right)^2, \left(\frac{5}{2}\right)^3, \ldots\right\},\\
&\underline{k} = 0, 1 \to \underline{j} = 1, \quad \{j^n\} = \{1, 2, 3^2, 4^2, 5^3, 6^3, 7^4, \ldots\},
\end{aligned} \tag{C.2.1}$$

(ii) *Supplementary series* $\mathcal{D}^{\text{suppl.}}_{SL(3,R)}(\underline{j},\underline{k};\sigma_2,\delta_1)$:

$$\begin{aligned}
&\sigma_2 \in R, \quad 0 < |\delta_1| \le 1/2, \quad \underline{k} = 1/2 \quad \text{or} \quad 0 < |\delta_1| < 1, \quad \underline{k} = 0, \quad \underline{j} = 0, 1;\\
&\underline{k} = 0 \to \underline{j} = 0, \quad \{j^n\} = \{0, 2^2, 3, 4^3, 5^2, 6^4, 7^3, \ldots\},\\
&\underline{k} = 1/2 \to \underline{j} = 1/2, \quad \{j^n\} = \left\{\frac{1}{2}, \left(\frac{3}{2}\right)^2, \left(\frac{5}{2}\right)^3, \ldots\right\},\\
&\underline{k} = 0, 1 \to \underline{j} = 1, \quad \{j^n\} = \{1, 2, 3^2, 4^2, 5^3, 6^3, 7^4, \ldots\},
\end{aligned} \tag{C.2.2}$$

(iii) *Discrete series* $\mathcal{D}^{\text{disc.}}_{SL(3,R)}(\underline{j};\sigma_2)$:

$$\begin{aligned}
&\sigma_2 \in R, \quad \underline{j} = \underline{k}, \quad \underline{j} = 3/2, 2, 5/2, 3, \ldots,\\
&\{j^n\} = \{\underline{j}, \underline{j}+1, (\underline{j}+2)^2, (\underline{j}+3)^2, (\underline{j}+4)^3, (\underline{j}+5)^3, \ldots\},
\end{aligned} \tag{C.2.3}$$

(iv) *Ladder series* $\mathcal{D}^{\text{ladd.}}_{SL(3,R)}(\underline{j};\sigma_2)$:

$$\begin{aligned}
&\underline{j} = 0, 1 \;\to\; \sigma_2 \in R, \quad \underline{j} = 1/2 \;\to\; \sigma_2 = 0;\\
&\{j^n\} = \left\{\underline{j}, \underline{j}+2, \underline{j}+4, \underline{j}+6, \underline{j}+8, \ldots\right\}, \quad \text{i.e.} \quad \Delta j = 2.
\end{aligned} \tag{C.2.4}$$

In terms of the unirreps of the maximal compact subgroup, the last sequence consists of those multiplicity-free representations which are known as the "ladder representations".

In addition, there are two discrete quantum numbers ϵ and ϵ', in the principal, the supplementary, and the discrete series. For the principal series, they are $\epsilon = \pm 1$ and $\epsilon' = \pm 1$. In the supplementary

[84] After many years in which the double covering was not mentioned and the study and classification had dealt with $SL(3,R)$ proper only (by Gel'fand and Graev, G. Rosen, Dothan, Gell-Mann, and Ne'eman, etc. – see [166]), the double covering was discovered by Joseph and Ne'eman and reconfirmed by several groups (see [324], referred to in [58]). This established also the existence of spinor representations; see also [533]. Šijački published a complete classification [623]. The complete unitary complement (i.e. listing of representations) of $SL(3,R)$ was published in [648], referring to [623] for those of the double covering.

series, they are $\epsilon = +1$ (only) whereas $\epsilon' = \pm 1$. For the discrete series, they are again both $\epsilon = \pm 1$ and $\epsilon' = \pm 1$. They do not exist in the ladder series. These quantum numbers are related to the occurence of even or odd values of k and j.

C.3. Unirreps of the $\overline{SL}(4, R)$

Since $n = 4$ is where our main interest lies, we remind the reader of the results of our construction of field representations, using the inner automorphism $\mathcal{A}$. Our otherwise unitary representations should be reinterpreted as nonunitary, after the reidentification of the $\overline{SO}(1,3)$ and $\overline{SO}(4)$ subgroups according to (4.2.7). In fact, conventional finite representations of $SL(4, R)$ (ordinary tensor fields) also reduce into direct sums of representations of the compact subgroup $SO(4)_{\mathcal{A}}$. We use the corresponding index for the $\mathcal{A}$-transformed subgroups, after the mapping (4.2.7). Physically, this $SO(4)_{\mathcal{A}}$ represents nonunitary finite-dimensional representations of the physical Lorentz group $\overline{SO}(1,3)$, rather than unitary finite-dimensional representations of the true compact subgroup $SO(4)$ that would have been generated by the J_u and $\overset{(+)}{B}_u$ of (4.2.4). That $SO(4)$ is represented in our bandor field representations by nonunitary infinite-dimensional representations of $\overline{SO}(1,3)$.

In the systematics of finite $SL(4, R)$ representations, a symmetric tensor $Z_{\alpha\beta}$, for example, is a $\underline{10}$ under $SL(4, R)$; it reduces (in the Riemannian case) to $\underline{9} + \underline{1}$ under $SO(4)_{\mathcal{A}}$; and it then reduces further into $\underline{9} \to \underline{5} + \underline{3} + \underline{1}$ and $\underline{1}$ under J_u, i.e. $SO(3) \subset SO(4)_{\mathcal{A}}$. The same picture now holds for our infinite-dimensional representations.

It is useful to gain a better understanding of the structure of the double covering $\overline{SL}(4, R)$ by studying its center, and those of $SL(4, R)$ and $SO(3,3)$. Indeed, as in the case of $SL(2, R)$, the group $SL(4, R)$ is itself also a covering group, $SL(4, R) = \overline{SO}(3,3)$. The group $\overline{SL}(4, R)$ is thus the quadruple covering of $SO(3,3)$. The compact subgroup is

$$\overline{SO}(4)_{\mathcal{A}} \cong \overline{SO}(3) \times \overline{SO}(3) = SU(2) \times SU(2). \tag{C.3.1}$$

For each $SO(3)$ subgroup, we can also write

$$SO(3) = [SU(2)/Z_2]. \tag{C.3.2}$$

In $SO(3,3)$, the compact subgroup is $SO(3) \times SO(3)$, and the center is trivial. In $SL(4, R)$, the compact subgroup is

$$SO(4) = [(SU(2) \times SU(2))/Z_2^d], \tag{C.3.3}$$

where Z_2^d is the *diagonal discrete subgroup* whose representations are given by

$$Z_2^d : \{1, (-1)^{2j_1} = (-1)^{2j_2}\}, \tag{C.3.4}$$

and where j_1 and j_2 are the Casimir labels of the two $SU(2)$ representations.

In $\overline{SL}(4, R)$, the center consists of both Z_2 centers, one in each $SU(2)$,

$$Z_2 \times Z_2 : \{1, (-1)^{2j_1}\} \times \{1, (-1)^{2j_2}\}. \tag{C.3.5}$$

The structure is best described by the exact sequences:

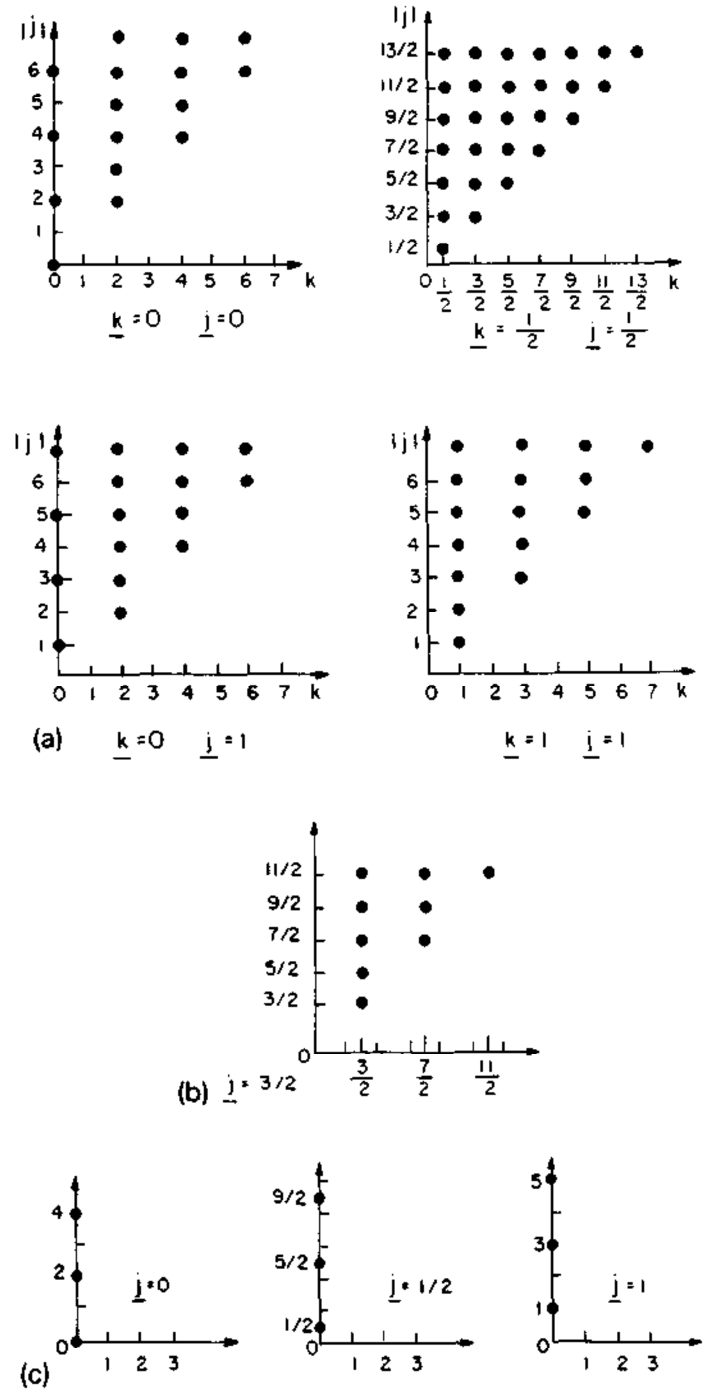

Fig. 6. Sample unirreps of $\overline{SL}(3,R)$: (a) Principal and supplementary series with the continuous parameters σ_2, δ_2, and δ_1 suppressed, see Eqs. (C.2.1) and (C.2.2). (b) Discrete series with the continuous parameter σ_2 suppressed. We have the same pattern for $\underline{j} = 2, 5/2, \ldots$, see Eq. (C.2.3). (c) Ladder series with the continuous parameter σ_2 either suppressed, or $\sigma_2 = 0$ for the spinorial case $\underline{j} = 1/2$, see Eq. (C.2.4).

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z_2^d & \rightarrow & Z_2\times Z_2 & \rightarrow & Z_2 & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \rightarrow & Z_2^d & \rightarrow & \overline{SL}(4,R) & \rightarrow & SL(4,R) & \rightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & SO(3,3) & & SO(3,3) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{C.3.6}$$

The $\overline{SL}(4,R)$ unirreps reduce into infinite sums of finite-dimensional $\overline{SO}(4)_{\mathcal{A}}$ representations. The multiplicity-free ones, i.e. those with one single representation of $\overline{SO}(4)_{\mathcal{A}}$ for each type appearing in the reduction of an $\overline{SL}(4,R)_{\mathcal{A}}$ representation, were constructed and finalized in [628]. We list first the multiplicity-free set

(i) *Principal series* $\mathcal{D}^{\text{princ.}}_{SL(4,R)}(\underline{m},\underline{n};e_2)$:

$$\begin{aligned}
&\underline{m}=\{0,1\},\quad \underline{n}=0;\qquad e_1=0,\ e_2\in R;\\
&\{(m,n)\}: m+n\cong \underline{m}\ (\mathrm{mod}\ 2).
\end{aligned} \tag{C.3.7}$$

(ii) *Supplementary series* $\mathcal{D}^{\text{suppl.}}_{SL(4,R)}(0,0;e_1)$:

$$\begin{aligned}
&0\le |e_1|\le 1,\quad e_2=0;\\
&\{(m,n)\}: m+n\cong 0\ (\mathrm{mod}\ 2).
\end{aligned} \tag{C.3.8}$$

(iii) *Discrete series* $\mathcal{D}^{\text{disc.}}_{SL(4,R)}((\underline{j},0)\ \text{or}\ (0,\underline{j}))$:

$$\begin{aligned}
&e_1=1-\underline{j},\quad e_2=0;\\
&\{(m,n)\}: m+n\cong \underline{j}\ (\mathrm{mod}\ 2),\quad |m-n|\ge \underline{j};\quad \underline{j}=1/2,\ 1,\ 3/2,
\end{aligned} \tag{C.3.9}$$

(iv) *Ladder series* $\mathcal{D}^{\text{ladd.}}_{SL(4,R)}(\underline{j},e_2)$:

$$\begin{aligned}
&\underline{j}=0,\ 1/2,\quad e_1=0,\quad e_2\in R,\\
&m=n=j;\qquad j\cong \underline{j}\ (\mathrm{mod}\ 1)\quad \text{for}\ \underline{j}=1/2;\qquad j\cong \underline{j}\ (\mathrm{mod}\ 2)\quad \text{for}\ \underline{j}=0.
\end{aligned} \tag{C.3.10}$$

The enumeration and classification of $\overline{SL}(4,R)$ unirreps have been recently completed [627]. First, let us make up a list of three types of quantum numbers of $\overline{SL}(4,R)$:

$$\begin{aligned}
&A: e_1=0;\quad e_2\in R,\\
&B_1: d_1=0;\quad d_2\in R,\\
&B_2: d_1=m+n,\quad d_2=0;\quad m+n=1/2,1,3/2,\dots,\\
&B_3: 0\le d_1\le 1,\quad d_2=0;\quad m+n=0,\pm 2,\pm 4,\dots,\\
&B_4: 0\le d_1\le 1/2,\quad d_2=0;\quad m+n\cong 1/2\ \text{or}\ 3/2\ \ (\mathrm{mod}\ 2),\\
&C_1: c_1=0;\quad c_2\in R,\\
&C_2: c_1=m+n,\quad c_2=0;\quad m-n=1/2,1,3/2,\dots,
\end{aligned}$$

$$C_3 : 0 \leq c_1 \leq 1, \quad c_2 = 0; \quad m - n = 0, \pm 2, \pm 4, \ldots,$$
$$C_4 : 0 \leq c_1 \leq 1/2, \quad c_2 = 0; \quad m - n \cong 1/2 \text{ or } 3/2 \pmod 2). \tag{C.3.11}$$

There are 16 series (or subseries) of $\overline{SL}(4,R)$ unirreps (or nonunitary $\mathcal{A}$-transformed representations) having nontrivial multiplicities, given in the above by the set

$$(A, B_r, C_s): \qquad \{r, s = 1, 2, 3, 4\}; \qquad j_1 \geq |m|, \quad j_2 \geq |n| \tag{C.3.12}$$

for all choices of r and s.

The eigenvalues of the second-order Casimir operator of the $\overline{SL}(4,R)$ are given by

$$C_2 := -\mathcal{E}^\alpha{}_\beta \mathcal{E}^\beta{}_\alpha = 4 - \tfrac{1}{4}\,(e_1 + ie_2)^2, \tag{C.3.13}$$

cf. [628], where, however, the notation is different: our $\mathcal{E}^\alpha{}_\beta = iQ^\alpha{}_\beta$ of [628].

C.4. Casimir invariants of the $SA(n,R)$

The fundamental particles in Minkowski spacetime correspond to the unitary irreducible representations of the Poincaré group. For $P_+^\uparrow$ and massive states, the eigenvalues of the *two* Casimir operators classify these representations uniquely. According to the Wigner classification [724], fundamental particles in Minkowski spacetime are classified by mass and spin. For massless particles, we need, in addition, the eigenvalue helicity of the Casimir operator of the Weyl group. Do similar results hold in theories which are invariant under the $GA(n,R)$ or $SA(n,R)$ groups, repectively?

Casimir invariants of real low-dimensional Lie algebras were evaluated by Patera et al. [546], and the invariant of $SA(2,R)$ was listed as $A_{5,40}$. The next basic advance in the study of the Casimir invariants of the affine and related groups followed the work of Sternberg [655]. Rais [567], Perroud [559], as well as Demichev and Nelipa [142] finally demonstrated that the $SA(n,R)$ have a single such operator (whereas the $GA(n,R)$ have none) which, using the Cartan–Weyl basis of the related $gl(n,R)$, i.e.

$$(L^\alpha{}_\beta)_\delta{}^\gamma = \delta^\alpha_\delta \delta^\gamma_\beta, \tag{C.4.1}$$

is given by

$$C(n) = \mathrm{Sym}\left[\epsilon^{\alpha_0 \cdots \alpha_{n-1}} P_{\alpha_0} (L^{\beta_0}{}_{\alpha_1} P_{\beta_0})(L^{\gamma_0}{}_{\alpha_2} L^{\delta_1}{}_{\gamma_0} P_{\delta_1}) \cdots (L^{\kappa_0}{}_{\alpha_{n-1}} L^{\kappa_1}{}_{\kappa_0} \cdots L^{\kappa_{n-2}}{}_{\kappa_{n-3}} P_{\kappa_{n-2}})\right], \tag{C.4.2}$$

where Sym denotes the symmetrization of all generators. Eq. (C.4.2) is equivalent to the determinant

$$C(n) = \det \begin{pmatrix} P_{\hat 0} & L^\beta{}_{\hat 0} P_\beta & L^\gamma{}_{\hat 0} L^\delta{}_\gamma P_\delta & \cdots \\ P_{\hat 1} & L^\beta{}_{\hat 1} P_\beta & L^\gamma{}_{\hat 1} L^\delta{}_\gamma P_\delta & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ P_{\hat n - \hat 1} & L^\beta{}_{\hat n - \hat 1} P_\beta & L^\gamma{}_{\hat n - \hat 1} L^\delta{}_\gamma P_\delta & \cdots \end{pmatrix}, \tag{C.4.3}$$

or, in shorthand notation, to

$$C(n) = \det\left(P, LP, (L)^2 P, \ldots, (L)^{n-1} P\right), \tag{C.4.4}$$

thus involving powers of the basis L in the Lie algebra $gl(n,R)$ ranging from 0 to $n-1$. Hence the invariant $C(n)$ is a polynomial of degree n in the translations p and of degree $n(n-1)/2$ in the $sl(n,R)$ generators; altogether it is thus of degree $n(n+1)/2$. Although written in terms of the Cartan–Weyl basis (C.4.1), the formula (C.4.2) automatically takes care of the tracelessness of the $sl(n,R)$ generators $\not{L}^{\alpha}{}_{\beta} = L^{\alpha}{}_{\beta} - \delta^{\alpha}_{\beta}\mathcal{D}/4$. Therefore the Casimir operator can also be represented as

$$\not{\mathcal{C}}\,(n) = \det\left(P, \not{L}\,P, (\not{L})^2 P, \cdots, (\not{L})^{n-1} P\right) . \tag{C.4.5}$$

This can explicitly be demonstrated by inserting the decomposition (3.1.9) into (C.4.2) and employing successively the commutation relations (3.1.10) and (3.1.4), respectively:

$$\begin{aligned} C(n) = {} & \not{\mathcal{C}}\,(n) + \mathrm{Sym}\left[\epsilon^{\alpha_0\cdots\alpha_{n-1}} P_{\alpha_0}\left(\frac{1}{n}\delta^{\beta_0}_{\alpha_1}\mathcal{D}P_{\beta_0}\right)\right. \\ & \times\left(\not{L}^{\gamma_0}{}_{\alpha_2}\not{L}^{\delta_1}{}_{\gamma_0}P_{\delta_1} + \delta^{\gamma_0}_{\alpha_2}\frac{1}{n}\mathcal{D}\not{L}^{\delta_1}{}_{\gamma_0}P_{\delta_1} + \not{L}^{\gamma_0}{}_{\alpha_2}\frac{1}{n}\delta^{\delta_1}_{\gamma_0}\mathcal{D}P_{\delta_1} + \delta^{\gamma_0}_{\alpha_2}\frac{1}{n}\mathcal{D}\frac{1}{n}\delta^{\delta_1}_{\gamma_0}\mathcal{D}P_{\delta_1}\right) \\ & \left.\cdots + P_{\alpha_0}(\not{L}^{\beta_0}{}_{\alpha_1}P_{\beta_0})\left(\frac{2}{n}\mathcal{D}\not{L}^{\delta_1}{}_{\alpha_2}P_{\delta_1} + \frac{1}{n^2}\mathcal{D}\mathcal{D}P_{\alpha_2}\right)\cdots\right] \\ = {} & \not{\mathcal{C}}\,(n) + \mathrm{Sym}\left[\epsilon^{\alpha_0\cdots\alpha_{n-1}}\frac{1}{2n}[P_{\alpha_0}, P_{\alpha_1}](\mathcal{D}+1)\left(\frac{2}{n}\mathcal{D}\not{L}^{\delta_1}{}_{\alpha_2}P_{\delta_1} + \frac{1}{n^2}\mathcal{D}\mathcal{D}P_{\alpha_2}\right)\right. \\ & \left.\cdots + \frac{1}{n}P_{\alpha_0}\left([\not{L}^{\beta_0}{}_{\alpha_1}P_{\beta_0}, \not{L}^{\beta_0}{}_{\alpha_2}P_{\beta_0}] + \frac{1}{2n}\mathcal{D}\mathcal{D}[\not{L}^{\beta_0}{}_{\alpha_1}, P_{\alpha_2}]P_{\beta_0}\right)\cdots\right] \\ = {} & \not{\mathcal{C}}\,(n). \end{aligned} \tag{C.4.6}$$

Observe that the last but one term, after applying the commutation relation (3.1.5), yields $[P_{\alpha_1}, P_{\alpha_2}] = 0$. Proceeding further by induction, higher order terms can likewise be shown to vanish.

Let us also sketch the proof that $C(n)$ is a Casimir operator of the special affine group $SA(n,R) = R^n \otimes SL(n,R)$: The determinant is built up from the n vectors $\{P, LP, (L)^2P, \ldots, (L)^{n-1}P\}$. The first of these vectors, on commutation with the generator P_α of translations, becomes linearly dependent on the second one (see [393] for details, also Eq. (6.11) of [391]). In order to conclude the proof, we recall that the determinant gives the volume of a vector space. Since $SL(n,R)$-transformations are *volume-preserving*, the operator $C(n)$ is also invariant under the rigid $SL(n,R)$.

For the construction of the eigenvalues of $C(n)$, let us first consider the simplest non-trivial dimension[85], namely that for $n=2$. Let $\Phi(x) = \Phi(t,x)$ be a field which transforms according to:

$$\Phi'(x) := D(\Lambda)\,\Phi[\Lambda^{-1}(x-a)] . \tag{C.4.7}$$

(i) In the *scalar* representation $D(\Lambda) = 1$ we obtain for the generators

$$\Lambda^1{}_0 = -x\partial_t\,, \qquad \Lambda^1{}_1 - \Lambda^0{}_0 = t\partial_t - x\partial_x\,, \qquad \Lambda^0{}_1 = -t\partial_x\,, \tag{C.4.8}$$

$$P_0 = -\partial_t\,, \qquad P_1 = -\partial_x\,. \tag{C.4.9}$$

Inserting this into (C.4.2) yields $C(2) = 0$.

(ii) For the *vector* representation $D(\Lambda) = \Lambda$ we have the generators

[85] These examples were provided by Jürgen Lemke (Cologne).

$$\Lambda^1{}_0 = -x\partial_t + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Lambda^1{}_1 - \Lambda^0{}_0 = t\partial_t - x\partial_x + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Lambda^0{}_1 = -t\partial_x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{C.4.10}$$

$$P_0 = -\partial_t, \qquad P_1 = -\partial_x. \tag{C.4.11}$$

Thus we obtain for the Casimir operator

$$C(2) = \begin{pmatrix} -\partial_t\partial_x & -\partial_x^2 \\ \partial_t^2 & \partial_t\partial_x \end{pmatrix} = \begin{pmatrix} -\partial_x \\ \partial_t \end{pmatrix} (\partial_t, \partial_x). \tag{C.4.12}$$

The eigenvalues λ are given by

$$\det\left[\begin{pmatrix} -P_tP_x & -P_x^2 \\ P_t^2 & P_tP_x \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda^2 = 0, \tag{C.4.13}$$

that is, the unique solution is $\lambda = 0$.

Analogously to the Poincaré group, the orbital parts of the generators do not contribute the the Casimir operator. Already for $n = 3$, one better uses computer algebra [650] in order to calculate the eigenvalues of the full Casimir operator for a given representation.

C.5. Classification of the unirreps of the $\overline{SA}(4, R)$

We now go over to the particle aspect and follow for the representation theory of $\overline{SA}(4, R)$ Wigner's classical treatment [724] of the Poincaré group's Hilbert space and projective representations (for quantum mechanics). The induction ($\leftarrow$) is done over the stability subgroup (Wigner's little group), whereas the inclusion sequence is denoted by $\subset$. For the special affine group

$$\overline{SA}(n, R) = R^n \otimes\!\!\!\!\!\!\subset \overline{SL}(n, R), \tag{C.5.1}$$

there is an affine subgroup inherent, in which the new 'translations' in $SA(n-1, R)$ correspond to the generators $\not{E}^{\,0}{}_u$ in $SA(n, R)$, cf. section 4.2. At some stage, for $k = 0, \ldots, n-2$, there occur two possibilities:

(i) The $\not{E}^0{}_u\,(n-k)$ are trivialized

$$\not{E}^0{}_u\,(n-k)|\text{phys.}\rangle = 0, \tag{C.5.2}$$

and the representation is induced over the infinite unitary linear representations of $\overline{SL}(n-k, R)$. This happens, for instance, in matter manifields, i.e. their particle states will correspond to 'Regge trajectories' described by unitary infinite-dimensional representations of $\overline{SL}(3, R)$, see Ref. [166].

(ii) If the $SL(n-k, R)$ representation is not linear and infinite, the $SA(n-k, R)$ is induced over the finite unitary linear representations of $\overline{SO}(n-k)$. Example: The gauge field of $\overline{SA}(n, R)$ will end up being induced over the transverse $\overline{SO}(n-2)$, with the *CPT* conjugate representation added.

The group $SA(n, R)$, acting on the space of momenta, has two orbits:

$$\text{Orb}_1 = \{0\}, \qquad \text{Orb}_2 = R^n - \{0\}. \tag{C.5.3}$$

For the null orbit, i.e. when we select states for which all n components of the momenta vanish, the Casimir operator vanishes, since it is a homogeneous symmetric polynomial of degree n in the momenta. In the second orbit – which, incidentally, is invariant under the entire $GL(n,R)$ – for small values of the momenta, the invariance of the Casimir operator implies that the eigenvalues of the $SL(n,R)$ homogeneous operators must grow fast.

These two orbits provide for a classification [393] of the unitary irreducible representations of $\overline{SA}(4,R)$. We have a hierarchy of stability subgroups over which the unirreps are constructed as *induced* representations. The four-vector $\overset{\circ}{p}$ either vanishes, $\overset{\circ}{p}=0$ (case I) and $C(4)=0$, or it does not, $\overset{\circ}{p}\neq 0$ (case II) and $C(4)\sim(\overset{\circ}{p})^4=m^4$, where m denotes the particle's mass.

Case I: Physically, it is useful to think of this case as the very-low frequency limit of a massless particle including its Regge excitations. The little group is $\overline{SL}(4,R)$. The unirreps of this group have been classified [628,627]. They are rather unphysical in that the Lorentz subgroup will appear in unitary infinite representations, the unirreps of Gel'fand and Yaglom [219]. These contain all spins, and the action of the Lorentz boost on a state with spin j connects it with those of spin $j+1$ and $j-1$. Such 'particles' are thus not characterized by definite spins, as phenomenologically required. Note that we do not encounter this difficulty with the fields and manifields, since these are constructed [628] with the *deunitarizing automorphism* $\mathcal{A}$. In a non-unitary and finite representation, the Lorentz boosts stay anti-Hermitian and cancel.

Case II: The little group is $\overline{SA}(3,R)'$. This affine group consists of the semi-direct product of the spatial $\overline{SL}(3,R)$ with a 'fake' set of three 'translation' momenta p', in fact representing contributions of the spatial shears to the 0 direction. We now have two subcases:

- **Case II A:** All three components $p'=0$. The effective little group is then $\overline{SL}(3,R)$. The unirreps are induced over this subgroup, they can be reduced to infinite discrete sums of spins, fitting the hadron situation and also providing an interesting model for primordial fermion fields (in fact manifields). This picture has been studied in [492,493,498]. It fits all applications mentioned in section 4. Note that $C(3')=0$, and, as a result, $C(4)=0$ as well, since the multiplier of $(\overset{\circ}{p})^4$ is precisely the $C(3')$ Casimir invariant of the stability subgroup defined by $\overset{\circ}{p}$.
- **Case II B:** The fake momenta satisfy $p'\neq 0$. We can select a frame in which only p'^{0} does not vanish, a fake energy-like component. $C(3')\sim(p'^{0})^3=(m')^3$, m' a mass-like eigenvalue. The new little group is $\overline{SA}(2,R)''$. Again, the 'translations' are fake momenta p''. We can have two subcases:
  - **Case II B1:** All components of $p''=0$ and $C(2'')=0$. In that case, we get again both $C(3")=0$ and $C(4)=0$. The effective little group is $\overline{SL}(2,R)$ (i.e. the double-covering, in an infinitely covered group). The unirreps have been classified by Bargmann [36] and are useful in a variety of physical contexts.
  - **Case II B2:** $p''\neq 0$, $C(2'')\sim(p'')^2=(m'')^2$. The little group is $SA(1,R)$, with one fake momentum p'''. Again we have two possibilities:
    - **Case II B2a:** $p'''=0$, $C(1''')=0$. This is a scalar representation. As a result, $C(2'')=C(3')=C(4)=0$.
    - **Case II B2b:** $p'''\neq 0$, $C(1''')=q=m'''$. Note that here $C(2'')=(m'')^2m'''$, $C(3')=(m')^3(m'')^2m'''$ and $C(4)=m^4(m')^3(m'')^2m'''$.

To summarize, we have five classes of representations: I, II A, II B1, II B2a, II B2b; which are illustrated in the following diagram:

$$
\begin{array}{lcll}
\overline{SA}(4,R) & \supset & \overline{SL}(4,R) & : \quad \mathrm{I} \\
\quad\uparrow & & & \\
\overline{SA}(3,R)' & \supset & \overline{SL}(3,R) & : \quad \mathrm{IIA} \\
\quad\uparrow & & & \\
\overline{SA}(2,R)'' & \supset & \overline{SL}(2,R) & : \quad \mathrm{IIB1} \\
\quad\uparrow & & & \\
SA(1,R) & & & : \quad \mathrm{IIB2a}\,,\ \mathrm{IIB2b}
\end{array}
\tag{C.5.4}
$$

Moreover,

$$
C(4)=0, \quad \text{for} \quad \mathrm{I},\ \mathrm{II\,A},\ \mathrm{II\,B1},\ \mathrm{II\,B2a}\,; \qquad C(4)=m^4\,(m')^3\,(m'')^2\,m''' \quad \text{for} \quad \mathrm{II\,B2b}\,.
\tag{C.5.5}
$$

At first sight, the Casimir invariant appears to constrain the masses and spins in a wrong manner, as in the Majorana [421] infinite-dimensional equation: the higher the spin, the lower the mass; this is the opposite of what we observe in hadron phenomenology and of what is assumed in the Chew–Frautschi plot for a Regge trajectory. However, considering that in the general case (including the most useful case II A) the invariant vanishes, the value of m^4 stays unconstrained in all but case II B2b. Instead, constraints on the value of the masses may be derived dynamically [499,501]. It is remarkable that an evaluation based on the 'pseudo-gravity' approximation for QCD in the infrared region does reproduce the linear correlation between m^2 and the spin j.

C.6. *Induced representations of the $\overline{SA}(n,R)$*

We have seen in section 4 that the manifields are constructed out of the linear representations of the (homogeneous) $SL(4,R)$ – just like our conventional tensor fields (except for the involvement of the double-covering, for world-spinors). However, just as the Minkowski space particles' Hilbert space spectrum is given by unitary representations of the (inhomogeneous) Poincaré group $R^n \rtimes \overline{SO}(1,n-1)$, so is the particle Hilbert space in a metric-affine geometry given by unitary representations of $SA(n,R) = R^n \rtimes \overline{SL}(n,R)$, i.e. by those of $SA(4,R)$ in our simplest physical situation. *These representations have to be Wigner–Mackey induced representations, the induction being over the various stability subgroups*, as they appear in the classification we have given in the previous section.

We start with the somewhat unphysical – but mathematically edifying – case I in our classification, namely situations in which the little group is the full homogeneous subgroup, i.e. $SL(4,R)$. This fits precisely the five-dimensional Möbius representation-space of section 3.1, when we point the translation parameter vector into the "fifth" dimension. Following Trautman [681], the notion of G-vector bundles enables us to derive for this case the formula

$$
\mathcal{D}(A)\Psi(\bar{\bar{x}}) := \mathcal{D}(A_{\bar{\bar{x}}}^{-1}\,A\,A_{A^{-1}\bar{\bar{x}}},\overset{\circ}{\bar{\bar{x}}})\,\Psi(A^{-1}\,\bar{\bar{x}})\,.
\tag{C.6.1}
$$

According to section 3.1, the group element $\underline{A} := \left(\begin{smallmatrix} \Lambda & \tau \\ 0 & 1 \end{smallmatrix}\right)$ of the affine group acts on a vector $\bar{\bar{x}} := \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) \in R^{n+1}$ of the n-dimensional hyperplane $\bar{\bar{R}}^n$. The representation depends on the representative vector $\overset{\circ}{\bar{\bar{x}}}$ of the chosen coset space ("group orbit"). For the "zero section" vector $\overset{\circ}{\bar{\bar{x}}} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, the stability group ('little group' in the sense of Wigner) is the $SL(n,R)$. By definition, the group element $A_{\bar{\bar{x}}}$

transforms this zero section vector into an arbitrary one according to $A_{\bar{\bar{x}}} \overset{\circ}{\bar{\bar{x}}} = \bar{\bar{x}}$. Such an element is, up to an $SL(n,R)$ similarity transformation, given by

$$\tilde{A}_{\bar{\bar{x}}} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad \tilde{A}_{\bar{\bar{x}}}^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}. \tag{C.6.2}$$

Moreover, we need the group element with the transformation behavior

$$A_{A^{-1}\bar{\bar{x}}} \overset{\circ}{\bar{\bar{x}}} = A^{-1}\bar{\bar{x}} = \begin{pmatrix} \Lambda^{-1}(x-\tau) \\ 1 \end{pmatrix}. \tag{C.6.3}$$

This condition can be satisfied, for instance, by

$$\tilde{A}_{A^{-1}\bar{\bar{x}}} = \begin{pmatrix} 1 & \Lambda^{-1}(x-\tau) \\ 0 & 1 \end{pmatrix}. \tag{C.6.4}$$

Thus the argument of the inducing representation $\mathcal{D}$ is given by

$$\tilde{A}_{\bar{\bar{x}}}^{-1} A\, \tilde{A}_{A^{-1}\bar{\bar{x}}} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \Lambda^{-1}(x-\tau) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.6.5}$$

Since this is really an element of the little group $SL(n,R)$ for the orbit chosen, we can write the induced representations in coordinate space simply as [cf. (C.4.7)]:

$$\mathcal{D}(\Lambda,\tau)\Psi(x) = \mathcal{D}(\Lambda)\Psi[\Lambda^{-1}(x-\tau)]. \tag{C.6.6}$$

Abandoning our mathematical Möbius space, we now return to spacetime with $n = 4$ – and for the physical particles' Hilbert space representations, to its Fourier transform – namely, momentum space. As explained in the previous section, the physical criteria point to case II A, with $\overline{SL}(3,R)$ as the residual *effective* little group (after putting $p' = 0$ for the eigenvalues of the full stability subgroup $\overline{SA}(3,R)'$). Note that the Fourier transform itself requires an $SL(4,R)$-invariant measure $d\mu(p)$ in $\Psi(x) = \int d\mu(p)\, \exp(-ip\cdot x)\, \Psi(p)$.

In the following, we apply the induced representation $\mathcal{D}(\overline{\Lambda},\tau)$ on momentum states $\omega^{\{S\}}(p,M)$, where $\{S\}$ denotes the unirreps of $\overline{SL}(3,R)$, i.e. the $(\underline{j},\underline{k},\sigma_2,\delta_2)$ of section C.2. By $\{M\}$ we labeled the quantum numbers *inside* an $SL(3,R)$ unirreps, i.e. the spins j and helicities k. Let $\gamma(p)$, an element of the coset space, denote the special linear transformation which boosts, for a non-zero mass m, the momentum p to its rest frame $\overset{\circ}{p} = (m,0,0,0)$. This boost is generated by the operators $\overset{(+)}{B}_{\mu\nu}$, $\overset{(-)}{B}_{\mu\nu}$ and $\mathcal{S}$ of (4.2.4). From the Fourier transform $\Psi(p)$ one gets the Wigner function $\omega(p)$ via $\omega(p) := \mathcal{D}(\gamma(p))\Psi(p)$. Then the induced representation acting on Wigner functions takes the form

$$\mathcal{D}(\overline{\Lambda},\tau)\omega^{\{S\}}(p,M) = e^{ip\cdot\tau} \sum_{M'} \mathcal{D}^{\{S\}}_{M'M}\left(\gamma(p)\overline{\Lambda}\gamma^{-1}(\Lambda^{-1}p),\, {}^{\circ}p\right) \omega^{\{S\}}(p\Lambda, M'). \tag{C.6.7}$$

In the case of the Poincaré group, we would have $p\Lambda = \Lambda^{-1}p$ in the argument of the right hand side due to the Hermitian property of Lorentz transformations. Then we recovered exactly Eq. (16.2) of Niederer and O'Raifeartaigh [512].

If, instead of working with Wigner functions, we work with ket vectors, i.e. with the conventional hadron Hilbert space of particle physics, Eq. (C.6.7) is replaced by the formula (B1) of [498], cf.

[626]. There, the "states" are Hilbert space Dirac ket vectors, labeled by the quantum numbers of the little group. For the Poincaré case, this would be the semi-direct product $T_3 \otimes SU(2)$ and the state is denoted $|p,\sigma\rangle$. The Poincaré group element is represented by $\mathcal{D}(\overline{\Lambda},\tau) = \mathcal{D}(1,\tau)\ \mathcal{D}(\overline{\Lambda},0)$. Applied to the state, the Lorentz element $\mathcal{D}(\overline{\Lambda},0)$ acts first, resulting in $|\Lambda p\rangle$ where Λ is the $SO(1,3)$ correspondent to $\overline{\Lambda}$ (which is in the covering group $SL(2,C)$). The Wigner rotation thus acts on this state, which yields the exponential $\exp(i\tau\cdot\Lambda p)$, as in [498]. The Wigner functions $\omega(p)$ of [512] do not represent the kets; instead, they are the *matrix elements* $\langle p|\omega\rangle$, with the momentum p labeling the *dual* (bra) Hilbert space. The action of $\mathcal{D}(1,\tau)\mathcal{D}(\overline{\Lambda},0)$ is actually *to the left*! The translation is thus applied to $\langle p|$, becoming $exp(i\tau\cdot p)$, which is then further multiplied by the Wigner rotation representation. As a result, the ordering of the decomposed operations is also inverted: The $\overline{\Lambda}$, $\gamma(p)$, $\gamma^{-1}(\Lambda^{-1}p)$ in (C.6.7) are in Eq. (B1) of [498] replaced by A, L_p, L_{Ap}, respectively.

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