# Properties of HYMNs in Examples of <br> Four-Color, Five-Color, and Six-Color Adinkras 

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#### Abstract

The mathematical concept of a "Banchoff index" associated with discrete Morse functions for oriented triangular meshes has been shown to correspond to the height assignments of nodes in adinkras. In recent work there has been introduced the concept of "Banchoff matrices" leading to HYMNs - height yielding matrix numbers. HYMNs map the shape of an adinkra to a set of eigenvalues derived from Banchoff matrices. In the context of some examples of four-color, minimal five-color, and minimal six-color adinkras, properties of the HYMNs are explored.


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## 1 Introduction

Graphs have demonstrated an unexpected power to "clear out the mathematical underbrush" encountered by theoretical physicists. Feynman Diagrams are a spectacular example of this. While it is difficult to recall, there was a time when Feynman Diagrams were not held in high regard. This all changed rapidly when a sufficient number of physicists employed their power to efficiently calculate quantum corrections to physical processes.

While we make no similar claims about breadth of possible impacts from the developments which began with the recognition of the existence of the $\mathcal{G} \mathcal{R}(\mathrm{d}, \mathcal{N})$ ("Garden Algebras") [1,2]) as a foundation of supersymmetric representation theory and their evolution into the introduction of adinkras [3], we do hold that adinkras provide similarly important tools within the domain of the representation theory of supersymmetrical theories. One hint about this involves the pathways [4, $5,6]$ that adinkras have opened from supersymmetrical theories, including field theories, to errorcorrection codes.

There is a substantial and rapidly growing literature on the relation of quantum error-correction codes [7] to the very structure of space-time (e. g. [8]) itself. Over and above this particular focus, there is the related similar discussion underway regarding quantum entanglement and space-time. Literally, it is bona fide query to ask, "Is space-time a quantum error-correction code?" However, from the perspective of field theory, the fields themselves are primary dynamical entities. This suggests the similar question, "Are there relations between fields that describe physical reality and quantum error-correction codes?"

To our knowledge, the works in $[4,5,6]$ are the singular ones that hint at such a linkage. To be clear, there have been no claims that these works provide for a role for quantum error-correction codes. The implication of the works in $[4,5,6]$ is all irreducible supersymmetric field theory representations in four or dimensions involve classical error-corrections codes. However, it also true that the $\mathbf{L}$-matrices and their inverses the $\mathbf{R}$-matrices that arise from adinkras are generalizations of Pauli matrices. Looking at many discussions of quantum error-correction codes, the Pauli matrices play an important role.

Among the most primitive of adinkras are those which have the structure of nodes only occurring at two distinct height. These are called "valise adinkras" with all bosonic nodes at the same value of height and all fermion nodes at the same (but different from the bosonic one) height. The fact that the nodes representing functions can be differentiated or integrated is reflected in a change of nodal height in adinkras [9]. In the works of [10] - [16], various prescriptions have been presented for ways to map the graphical representations into numerical data. While some of these involve the use of eigenvalues for this purpose, we recently used a modified formulation [17] (called Height Yielding Matrix Numbers - HYMNs) also using eigenvalues. The "HYMNs" approach encode the heights of various nodes into diagonal matrices. This approach called, "dressing," was introduced in work by Toppan et. al. [18,19,20,21,22]. In the discussion to follow we explore properties of the HYMNs through examples of four-color, five-color, and six-color adinkras.

Works by mathematicians [23,24] have connected adinkras to algebraic geometry and in particular Riemann surfaces. The concept of discrete Morse functions for oriented triangular meshes on Riemann surfaces was introduced into the mathematical literature some time ago [25]. Thus, nodal
heights in adinkras are associated with values of these Morse functions. The latter are piecewiselinear over the meshes constructed by linear extension across edges and faces of the plaquettes associated with the meshes. The bottom line is the height assignment of any node in an adinkra corresponds to the integer value of the Morse function and the height assignment can be "Banchoff index" of the node. The fact that adinkra contain many nodes leads to matrices of these indices that we call "Banchoff Matrices." The Height Yielding Matrix Numbers - HYMNs - are the eigenvalue of these matrices.

HYMNs provide to an intrinsic definition of the shape of an adinkra. In a vaguely reminiscent way of how the Riemann curvature tensor provides a definition of intrinsic curvature for hyper surfaces, HYMNs provide an intrinsic way to define the shape of any adinkra. In our conclusions section, a further discussion of the importance of this will be covered more extensively.

## 2 HYMNs: Height Yielding Matrix Numbers

Here, we generalize the notion of HYMNs [17] from those with only bosonic nodes lifted to also include lifting of fermion nodes. Supersymmetric transformation laws encoded by valise adinkras can be described succinctly by the equations

$$
\begin{equation*}
\mathrm{D}_{\mathrm{I}} \Phi_{i}=i\left(\mathrm{~L}_{\mathrm{I}}\right)_{i \hat{k}} \Psi_{\hat{k}} \quad, \quad \mathrm{D}_{\mathrm{I}} \Psi_{\hat{k}}=\left(\mathrm{R}_{\mathrm{r}}\right)_{\hat{k} i} \partial_{0} \Phi_{i}, \tag{2.1}
\end{equation*}
$$

and the $N \mathbf{L}_{\mathrm{I}}$ and $N \mathbf{R}_{\mathrm{I}}$ matrices satisfy the algebra of general, real matrices describing $N$ supersymmetries between $d$ bosons and $d$ fermions, the so-called Garden Algebra:

$$
\begin{equation*}
\mathbf{L}_{\mathrm{I}} \mathbf{R}_{\mathrm{J}}+\mathbf{L}_{\mathrm{J}} \mathbf{R}_{\mathrm{I}}=2 \delta_{I J} \mathbf{I}_{d} \quad, \quad \mathbf{R}_{\mathrm{I}} \mathbf{L}_{\mathrm{J}}+\mathbf{R}_{\mathrm{J}} \mathbf{L}_{\mathrm{I}}=2 \delta_{I J} \mathbf{I}_{d} \tag{2.2}
\end{equation*}
$$

with $\mathbf{I}_{d}$ the $d \times d$ identity matrix. These relations imply that off-diagonal $N$ real $2 d \times 2 d$ matrices $\widehat{\gamma}_{\mathrm{I}}$ constructed from the $\mathbf{L}_{\mathrm{I}}$ and $\mathbf{R}_{\mathrm{I}}$ matrices form a Euclidean Clifford Algebra.

We define the node lifting operator $\boldsymbol{M}(m, w)$ that acts on an arbitrary number of $d$ fields as

$$
\begin{align*}
\boldsymbol{M}(m, w) & \equiv\left(\begin{array}{ccccc}
m^{p_{1}} & 0 & 0 & \ldots & 0 \\
0 & m^{p_{1}} & 0 & \ldots & 0 \\
0 & 0 & m^{p_{3}} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \ldots & m^{p_{\mathrm{d}}}
\end{array}\right) \quad,  \tag{2.3}\\
w & \equiv p_{1} 2^{0}+p_{2} 2^{1}+p_{3} 2^{2}+\cdots+p_{\mathrm{d}} 2^{\mathrm{d}-1} \quad, \quad \text { with } p_{i}=0,1 \tag{2.4}
\end{align*}
$$

where the word parameter $w$ is as in [26]. We define lifted bosons $\Phi\left(m_{B}, w_{B}\right)$ and lifted fermions $\Psi\left(m_{F}, w_{F}\right)$ as

$$
\begin{align*}
\Phi\left(m_{B}, w_{B}\right) & =\boldsymbol{M}\left(m_{B}, w_{B}\right) \Phi  \tag{2.5}\\
\Psi\left(\mu_{F}, w_{F}\right) & =\boldsymbol{M}\left(\mu_{F}, w_{F}\right) \Psi \tag{2.6}
\end{align*}
$$

In [17], only lifting bosonic nodes was considered. As $\boldsymbol{M}(1, w)=\boldsymbol{M}(m, 0)=\mathbf{I}_{d}$, lifting only bosonic nodes amounts to setting either $m_{F}=1$ or $w_{F}=0$ in the following.

Multiplying Eqs. (2.1) by node lifting matrices $\boldsymbol{M}\left(m_{B}, w_{B}\right)$ and $\boldsymbol{M}\left(\mu_{F}, w_{F}\right)$ and inserting factors of $\mathbf{I}_{d}=\boldsymbol{M}\left(m_{F}^{-1}, w_{F}\right) \boldsymbol{M}\left(m_{F}, w_{F}\right)$ and $\mathbf{I}_{d}=\boldsymbol{M}\left(\mu_{B}^{-1}, w_{B}\right) \boldsymbol{M}\left(\mu_{B}, w_{B}\right)$ results in the transformation laws

$$
\begin{gather*}
\mathrm{D}_{\mathrm{I}} \Phi\left(m_{B}, w_{B}\right)=i \mathbf{L}_{\mathrm{I}}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \Psi\left(m_{F}, w_{F}\right),  \tag{2.7}\\
\mathrm{D}_{\mathrm{I}} \Psi\left(\mu_{F}, w_{F}\right)=\mathbf{R}_{\mathrm{I}}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \partial_{0} \Phi\left(\mu_{B}, w_{B}\right), \tag{2.8}
\end{gather*}
$$

where

$$
\begin{align*}
\mathbf{L}_{\mathrm{I}}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) & =\boldsymbol{M}\left(m_{B}, w_{B}\right) \mathbf{L}_{\mathrm{I}} \boldsymbol{M}\left(m_{F}^{-1}, w_{F}\right)  \tag{2.9}\\
\mathbf{R}_{\mathrm{I}}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) & =\boldsymbol{M}\left(\mu_{F}, w_{F}\right) \mathbf{R}_{\mathrm{r}} \boldsymbol{M}\left(\mu_{B}^{-1}, w_{B}\right) \tag{2.10}
\end{align*}
$$

The way we have written these equations, $\mu$ is not specific to fermions and $m$ is not specfic to bosons. Rather, $\mu$ is specific to the R-matrices and $m$ to the $\mathbf{L}$-matrices. See Eqs. (2.7) and (2.8).

Based on equation (2.9) and (2.10), it's clear that the final $\mathbf{L}$ and $\mathbf{R}$ matrices after lifting both bosons and fermions do not depend on the order of lifting operations. Since two diagonal matrices commute with each other and matrix multiplication is associative. Namely, there is only one unique set of L and R matrices corresponding to a specified adinkra, no matter whether it is valise or non-valise.

The redefined matrices $\mathbf{L}_{\mathrm{I}}(m, w)$ and $\mathbf{R}_{\mathrm{I}}(\mu, w)$ satisfy the $G R(d, N)$ algebra in the $\mu_{F} \rightarrow m_{F}$ and $\mu_{B} \rightarrow m_{B}$ limit:

$$
\begin{align*}
& \mathbf{L}_{\mathrm{I}}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \mathbf{R}_{\mathrm{J}}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right)+\mathbf{L}_{\mathrm{J}}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \mathbf{R}_{\mathrm{I}}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \\
& =\boldsymbol{M}\left(m_{B}, w_{B}\right)\left[\mathbf{L}_{\mathrm{I}} \boldsymbol{M}\left(\mu_{F} / m_{F}, w_{F}\right) \mathbf{R}_{\mathrm{J}}+\mathbf{L}_{\mathrm{J}} \boldsymbol{M}\left(\mu_{F} / m_{F}, w_{F}\right) \mathbf{R}_{\mathrm{I}}\right] \boldsymbol{M}\left(\mu_{B}^{-1}, w_{B}\right) \\
& \rightarrow \boldsymbol{M}\left(m_{B}, w_{B}\right)\left[\mathbf{L}_{\mathrm{I}} \mathbf{R}_{\mathrm{J}}+\mathbf{L}_{\mathrm{J}} \mathbf{R}_{\mathrm{I}}\right] \boldsymbol{M}\left(\mu_{B}^{-1}, w_{B}\right)=2 \delta_{\mathrm{IJ}} \boldsymbol{M}\left(m_{B} / \mu_{B}, w_{B}\right), \text { for } \mu_{F} \rightarrow m_{F} \\
& \rightarrow 2 \delta_{\mathrm{IJ}} \mathbf{I}_{d}, \quad \text { for } \mu_{B} \rightarrow m_{B} \tag{2.11}
\end{align*}
$$

The same results hold in the $\mu_{B} \rightarrow m_{B}$ and $\mu_{F} \rightarrow m_{F}$ limit for the $\mathbf{R}_{\mathbf{I}} \mathbf{L}_{\mathrm{J}}+\mathbf{R}_{\mathrm{J}} \mathbf{L}_{\mathrm{I}}$ algebra

$$
\begin{align*}
& \mathrm{R}_{\mathrm{I}}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \mathrm{L}_{\mathrm{J}}\left(m_{B}, m_{F}, w_{B}, w_{F}\right)+\mathrm{R}_{\mathrm{J}}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \mathrm{L}_{\mathrm{I}}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \\
& \rightarrow 2 \delta_{\mathrm{IJ}} \mathbf{I}_{d}, \quad \text { for } \mu_{B} \rightarrow m_{B} \quad, \quad \text { and } \quad \mu_{F} \rightarrow m_{F} \tag{2.12}
\end{align*}
$$

The $m / \mu$ ratio will be prevalent in the rest of our calculations, so we define

$$
\begin{equation*}
\rho_{B}=m_{B} / \mu_{B} \quad, \quad \rho_{F}=\mu_{F} / m_{F} . \tag{2.13}
\end{equation*}
$$

For $N=4$, lifted Banchoff $\boldsymbol{B}$-matrices are defined as:

$$
\begin{align*}
& \boldsymbol{B}_{L}\left(\rho_{B}, \rho_{F}, w_{B}, w_{F}\right)= \\
& \mathbf{L}_{4}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \mathbf{R}_{3}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \mathbf{L}_{2}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \mathbf{R}_{1}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right)  \tag{2.14}\\
& \boldsymbol{B}_{R}\left(\rho_{B}, \rho_{F}, w_{B}, w_{F}\right)= \\
& \mathbf{R}_{4}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \mathbf{L}_{3}\left(m_{B}, m_{F}, w_{B}, w_{F}\right) \mathbf{R}_{2}\left(\mu_{B}, \mu_{F}, w_{B}, w_{F}\right) \mathbf{L}_{1}\left(m_{B}, m_{F}, w_{B}, w_{F}\right)
\end{align*}
$$

We will define Banchoff matrices for $N=5$ and $N=6$ in the following sections.
In the remainder of this work, we will demonstrate, in the context of specific example, some properties of these constructions. Our examples are chosen from systems of four color, five color, and then six color adinkras.

## 3 GR(4,4) Calculations

Since adinkras can be obtained from the dimensional reduction of supersymmetric theories from higher dimensions, there are multiple higher dimensional starting points. Thus, models with the same number of independent supercharges under such reduction lead to the same adinkras. In particular, $4 \mathrm{D}, \mathcal{N}=1$ [27], $2 \mathrm{D}, \mathcal{N}=(2,2)[28], 2 \mathrm{D}, \mathcal{N}=(4,0)$ [29] and $1 \mathrm{D}, \mathcal{N}=4$ supersymmetrical theories [30] must produce adinkras that lie in the set of the 36,864 adinkras associated with the Coxeter Group $B C_{4}$.

If the starting point is chosen in the domain of $4 \mathrm{D}, \mathcal{N}=1$ supersymmetrical theories, among the most familiar models involve the chiral, vector and tensor supermultiplets. The explicit forms of the $\mathbf{L}$-matrices and $\mathbf{R}$-matrices for the $4 \mathrm{D}, \mathcal{N}=1$ supermultiplets can be found in the work seen in [27]. This is our starting point discussed below. In the following sections, we not only present HYMNs for various supermultiplets, but also HYMNs if we drop all dashings in the associated adinkra.

### 3.1 Chiral Supermultiplet:



Figure 1: Chiral Supermultiplet Adinkra
In Fig. 1, there is shown the adinkra for chiral supermultiplet.

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}\right\}$ | $\left\{-1,-1,-\rho_{B}^{2},-\rho_{B}^{2}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\left\{\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}\right\}$ | $\left\{1,1, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ |

Table 1: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the adinkra in Fig 1

### 3.2 Vector Supermultiplet:

In Fig. 2, there is shown the adinkra for vector supermultiplet.


Figure 2: Vector Supermultiplet Adinkra

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{-1,-1,-\rho_{B},-\rho_{B}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ |

Table 2: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the adinkra in Fig 2

### 3.3 Tensor Supermultiplet:

In Fig. 3, there is shown the adinkra for tensor supermultiplet.


Figure 3: Tensor Supermultiplet Adinkra

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\{-1,-1,-1,-1\}$ | $\{1,1,1,1\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\{1,1,1,1\}$ | $\{1,1,1,1\}$ |

Table 3: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the adinkra in Fig 3

### 3.4 Four Supermultiples: SM-I, SM-II, SM-III, SM-IV

In this subsection, the valise adinkras associated with the domain of $2 \mathrm{D}, \mathcal{N}=(4,0)$ supersymmetric models are considered below. The explicit forms of the $\mathbf{L}$-matrices and $\mathbf{R}$-matrices for the
$2 \mathrm{D}, \mathcal{N}=(4,0)$ and $1 \mathrm{D}, \mathcal{N}=4$ supermultiplets can be found in the work seen in [30]. Thus in Fig. 4 through 7, there are shown the adinkras for supermultiplets SM-I to SM-IV.


Figure 4: Adinkra Diagram for SM-I


Figure 6: Adinkra Diagram for SM-III


Figure 5: Adinkra Diagram for SM-II


Figure 7: Adinkra Diagram for SM-IV

These four adinkras, as well as all 36,864 valise $\operatorname{GR}(4,4)$ adinkras share the same eigenvalues. Eigenvalues for $\boldsymbol{B}_{L}$ matrix are: where

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{\chi_{\mathrm{o}}, \chi_{\mathrm{o}}, \chi_{\mathrm{o}}, \chi_{\mathrm{o}}\right\}$ | $\left\{-\chi_{\mathrm{o}},-\chi_{\mathrm{o}},-\chi_{\mathrm{o}},-\chi_{\mathrm{o}}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\{1,1,1,1\}$ | $\{1,1,1,1\}$ |

Table 4: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the 36,864 valise adinkras associated with $(4,0)$ SUSY

$$
\chi_{o}(\mathcal{R})= \begin{cases}1 & \text { when } \mathcal{R}=\text { SM-I, SM-IV }  \tag{3.1}\\ -1 & \text { when } \mathcal{R}=\text { SM-II, SM-III }\end{cases}
$$

which is true for all the valise $\mathrm{GR}(4,4)$ adinkras.

### 3.5 The Collapse of the Variant Representations

There are ten off-shell $4 \mathrm{D}, \mathcal{N}=1$ off-shell supermultiplets. For completeness (as well as correcting some sign errors in previous publications) the complete description of the component fields as well as SUSY transformation laws are given in Appendix A.

The point is to note that any theory that has a description in terms of $4 \mathrm{D}, \mathcal{N}=1$ off-shell supermultiplet can be reduced to a 1D, $N=4$ off-shell supermultiplet which may be transformed into a valise adinkra by bring all the bosonic fields to the same level in the corresponding adinkra.

We begin by briefly giving the names and component fields listed in each of these supermultiplets. We have for all those supermultiplets related to the chiral supermultiplet the following:
(1.) Chiral Supermultiplet: (A, B, $\left.\psi_{\mathrm{a}}, \mathrm{F}, \mathrm{G}\right)$
(2.) Hodge - Dual \#1 Chiral Supermultiplet: (A, B, $\left.\psi_{\mathrm{a}}, \mathrm{f}_{\mu \nu \rho}, \mathrm{G}\right)$
(3.) Hodge - Dual \#2 Chiral Supermultiplet: $\left(\mathrm{A}, \mathrm{B}, \psi_{\mathrm{a}}, \mathrm{F}, \mathrm{g}_{\mu \nu \rho}\right)$
(4.) Hodge - Dual \#3 Chiral Supermultiplet: (A, B, $\left.\psi_{\mathrm{a}}, \mathrm{f}_{\mu \nu \rho}, \mathrm{g}_{\mu \nu \rho}\right)$

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ | $\left\{-1,-1,-\rho_{B},-\rho_{B}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ |

Table 5: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for Hodge-Dual \#1 and HodgeDual \#2 Chiral Supermultiplet

The eigenvalues shown in tables \# 5 and \# 6 result with the four dimensional supermultiplet is case into the form of a 1 D valise supermultiplet.

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\{1,1,1,1\}$ | $\{-1,-1,-1,-1\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\{1,1,1,1\}$ | $\{1,1,1,1\}$ |

Table 6: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for Hodge-Dual \#3 Chiral Supermultiplet

It should be noted that performing a parity exchange on the component bosonic fields of the original chiral supermultiplet returns the same field content. The reason for this is that the bosonic fields in the chiral-like supermultiplets all come in opposite parity pairs. So performing a parity transformation on the bosons only "swaps" members of the parings. This is not so for the remaining supermultiplets.

We have for all those supermultiplets related to the vector supermultiplet the following
(1.) Vector Supermultiplet: $\left(\mathrm{A}_{\mu}, \lambda_{\mathrm{b}}, \mathrm{d}\right)$
(2.) Axial - Vector Supermultiplet: $\left(U_{\mu}, \widetilde{\lambda}_{b}, \widetilde{d}\right)$
(3.) Hodge - Dual Vector Supermultiplet: $\left(\mathrm{A}_{\mu}, \lambda_{\mathrm{b}}, \mathrm{d}_{\mu \nu \rho}\right)$
(4.) Hodge - Dual Axial - Vector Supermultiplet : $\left(\widetilde{\mathrm{U}}_{\mu}, \widetilde{\lambda}_{\mathrm{b}}, \widetilde{\mathrm{d}}_{\mu \nu \rho}\right)$

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{-1,-1,-\rho_{B},-\rho_{B}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}\right\}$ |

Table 7: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for Axial-Vector Supermultiplet

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\{-1,-1,-1,-1\}$ | $\{1,1,1,1\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\{1,1,1,1\}$ | $\{1,1,1,1\}$ |

Table 8: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for Hodge-Dual Vector Supermultiplet, Hodge-Dual Axial-Vector Supermultiplet, and Axial-Tensor Supermultiplet

Finally, we come to the tensor supermultiplet and its parity exhanged version. Since this supermultiplet does not possess any auxiliary fields, there is not a possibility to perform a Hodge-type duality. So the only possible transformation is to perform a parity exchange on the bosons. This leads to:
(1.) Tensor Supermultiplet: $\left(\varphi, \mathrm{B}_{\mu \nu}, \chi_{\mathrm{a}}\right)$
(2.) Axial - Tensor Supermultiplet : $\left(\widetilde{\varphi}, \mathrm{C}_{\mu \nu}, \widetilde{\chi}_{\mathrm{a}}\right)$

These studies suggest that two simple functions of the eigenvalues that are important for the introduction of a class-based structure for these diagrams would be the trace and determinant. The trace and determinant have the properties that they are invariant under any permutation of the eigenvalues. Actually, since there are two sets of eigenvalues (one set for $\boldsymbol{B}_{L}$ and one for $\boldsymbol{B}_{R}$ ). we are able to analyze the trace and determinant for each. These results are shown in appendix B.

## 4 An Example of 4-color Non-minimal Adinkra

In this chapter, the discussion will turn to an exploration of the response of the HYMNs to changes of shape of 4 -color adinkras which possess 8 -closed and 8 -open nodes as seen in Fig. 8. These correspond to reducible superfields. The $4 \mathrm{D}, \mathcal{N}=1$ real scalar superfield was shown in [31] to be depicted by the adinkra in Fig. 8 similar to how the $4 \mathrm{D}, \mathcal{N}=1$ chiral, vector, and tensor multiplets were shown in [27] to be depicted by the adinkras in Figs. 1, 2, and 3.


Figure 8: An Example of 4-color Non-minimal Adinkra
Recall that the definition of Banchoff $\boldsymbol{B}$-matrices is:

$$
\begin{align*}
\boldsymbol{B}_{L} & =\mathbf{L}_{4} \mathbf{R}_{3} \mathbf{L}_{2} \mathbf{R}_{1}  \tag{4.1}\\
\boldsymbol{B}_{R} & =\mathbf{R}_{4} \mathbf{L}_{3} \mathbf{R}_{2} \mathbf{L}_{1} \tag{4.2}
\end{align*}
$$

For the adinkra in Figure 8, the corresponding L-matrices are:

$$
\mathbf{L}_{1}=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4.3}\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right) \quad, \quad \mathbf{L}_{2}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{array}\right),
$$

$$
\mathbf{L}_{3}=\left(\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0  \tag{4.4}\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right), \mathbf{L}_{4}=\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right)
$$

In this case, we not only lift bosons but also lift fermions. We define the boson lifting matrix $M\left(m_{B}, w_{B}\right)$ and fermion lifting matrix $M\left(m_{F}, w_{F}\right)$ as in Eqs. (2.5) and (2.6), respectively. By analyzing the HYMNs, we get results listed in Table 9. Eigenvalues are not sensitive to dashings.

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F}\right.$, | $\left\{-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F}\right.$, |
| $\left.\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\}$ | $\left.\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\left\{-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F}\right.$, | $\left\{-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F}\right.$, |
| $\left.\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\}$ | $\left.\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\}$ |

Table 9: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the adinkra in Fig 8
Consider the $\boldsymbol{B}^{2}$ matrix, which is

$$
\boldsymbol{B}^{2}=\left[\begin{array}{cc}
\boldsymbol{B}_{L}^{2} & 0  \tag{4.5}\\
0 & \boldsymbol{B}_{R}^{2}
\end{array}\right]
$$

Eigenvalues for $\boldsymbol{B}_{L}^{2}$ matrix (which is diagonal) are

$$
\begin{equation*}
\left\{\rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}\right\} \tag{4.6}
\end{equation*}
$$

Eigenvalues for $\boldsymbol{B}_{R}^{2}$ matrix (which is diagonal) are

$$
\begin{equation*}
\left\{\rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}, \rho_{B}^{4} \rho_{F}^{2}\right\} \tag{4.7}
\end{equation*}
$$

Then we can find:

$$
\begin{align*}
\operatorname{det}\left(\boldsymbol{B}_{L}^{2}\right) & =\rho_{B}^{32} \rho_{F}^{16}  \tag{4.8}\\
\operatorname{det}\left(\boldsymbol{B}_{R}^{2}\right) & =\rho_{B}^{32} \rho_{F}^{16} \tag{4.9}
\end{align*}
$$

which are consistent with our conjecture:

$$
\begin{equation*}
\operatorname{det}\left(\boldsymbol{B}_{L}^{2}\right)=\operatorname{det}\left(\boldsymbol{B}_{R}^{2}\right)=\rho_{B}^{N \times(\# \text { of bosons lifted })} \rho_{F}^{N \times(\# \text { of fermions lifted })} \tag{4.10}
\end{equation*}
$$

### 4.1 Step-by-step Results



Figure 9: An Example of 4-color Non-minimal Adinkra

Look at Figure 9, start the analysis from the valise adinkra in which all bosons live in level-1 and all fermions live in level-2. In order to get the dimond adinkra as Figure 9 shown, there are three steps to do the transformations.

Note that lifting or lowering nodes would not influence dashing properties, so in all following steps, eigenvalues after dropping all dashing are the same as with dashing.

In the following tables, we use $<\mid>$ notation to denote the "shape" of adinkras we met in intermediate steps. The number sitting next to $<$ is the number of bosons living in level-1. | is to divide adjacent levels. Thus the second number is the number of fermions living in level-2, and so on. So $<8 \mid 8>$ is the valise adinkra, our starting point, and the one in Figure 9 is $<1|4| 6|4| 1>$. Figure 10 shows the $<1|4| 7 \mid 4>$ type adinkra.

Step 1: Raise Boson 2-8 to Level-3. See Table 10. All values in the last column of Table 10 satisfy our conjecture (4.10).

Step 2: Raise Fermions 5-8 to Level-4. See Table 11. All values in the last column of Table 11 satisfy our conjecture (4.10).

Step 3: Raise Boson 8 to Level-5 and we get the final results as Table 9 and Equation (4.6) to (4.9).

|  | $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues | $\operatorname{det}\left(\boldsymbol{B}_{L / R}^{2}\right)$ |
| :---: | :---: | :---: | :---: |
| $<8 \mid 8>$ | $\begin{gathered} \hline-1,-1,-1,-1, \\ 1,1,1,1\} \end{gathered}$ | $\begin{gathered} \hline-1,-1,-1,-1, \\ 1,1,1,1\} \end{gathered}$ | $\rho_{B}^{4 \times 0}$ |
| $<7\|8\| 1>$ | $\begin{gathered} \{-1,-1,1,1, \\ \left.-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}\right\} \end{gathered}$ | $\begin{gathered} \{-1,-1,1,1, \\ \left.-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}\right\} \end{gathered}$ | $\rho_{B}^{4 \times 1}$ |
| $<6\|8\| 2>$ | $\begin{gathered} \left\{-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2}\right. \\ \left.\rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}\right\} \end{gathered}$ | $\begin{gathered} \{-1,-1,1,1, \\ \left.-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B}\right\} \end{gathered}$ | $\rho_{B}^{4 \times 2}$ |
| $<5\|8\| 3>$ | $\begin{gathered} \left\{-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}\right. \\ \left.-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B}\right\} \end{gathered}$ | $\begin{gathered} \left\{-\rho_{B}^{1 / 2},-\rho_{B}^{1 / 2}, \rho_{B}^{1 / 2}, \rho_{B}^{1 / 2},\right. \\ \left.-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B}\right\} \end{gathered}$ | $\rho_{B}^{4 \times 3}$ |
| $<4\|8\| 4>$ | $\begin{gathered} \left\{-\rho_{B},-\rho_{B},-\rho_{B},-\rho_{B},\right. \\ \left.\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}\right\} \end{gathered}$ | $\begin{gathered} \left\{-\rho_{B},-\rho_{B},-\rho_{B},-\rho_{B},\right. \\ \left.\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}\right\} \end{gathered}$ | $\rho_{B}^{4 \times 4}$ |
| $<3\|8\| 5>$ | $\begin{gathered} \left\{-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B},\right. \\ \left.-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}\right\} \\ \hline \end{gathered}$ | $\begin{gathered} \left\{-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B},\right. \\ \left.-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}\right\} \\ \hline \end{gathered}$ | $\rho_{B}^{4 \times 5}$ |
| $<2\|8\| 6>$ | $\begin{gathered} \left\{-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2},\right\} \\ \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2} \end{gathered}$ | $\begin{aligned} & \left\{-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B},\right. \\ & \left.-\rho_{B}^{2},-\rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\} \end{aligned}$ | $\rho_{B}^{4 \times 6}$ |
| $<1\|8\| 7>$ | $\begin{gathered} \left\{-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2},\right. \\ \left.-\rho_{B}^{2},-\rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\} \end{gathered}$ | $\begin{gathered} \left\{-\rho_{B}^{3 / 2},-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2}, \rho_{B}^{3 / 2},\right. \\ \left.-\rho_{B}^{2},-\rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\} \end{gathered}$ | $\rho_{B}^{4 \times 7}$ |

Table 10: Eigenvalues of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ matrices and determinants of $\boldsymbol{B}_{L}^{2}$ and $\boldsymbol{B}_{R}^{2}$ matrices in Step 1

|  | $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues | $\operatorname{det}\left(\boldsymbol{B}_{L / R}^{2}\right)$ |
| :---: | :---: | :---: |
| $<1\|7\| 7\|1\rangle$ | $\begin{aligned} & \boldsymbol{B}_{L}:\left\{-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2},-\rho_{B}^{2}, \rho_{B}^{2},-\rho_{B}^{3 / 2} \rho_{F}^{1 / 2}, \rho_{B}^{3 / 2} \rho_{F}^{1 / 2},-\rho_{B}^{2} \rho_{F}^{1 / 2}, \rho_{B}^{2} \rho_{F}^{1 / 2}\right\} \\ & \boldsymbol{B}_{R}:\left\{-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2},-\rho_{B}^{2}, \rho_{B}^{2},-\rho_{B}^{3 / 2} \rho_{F}^{1 / 2}, \rho_{B}^{3 / 2} \rho_{F}^{1 / 2},-\rho_{B}^{2} \rho_{F}^{1 / 2}, \rho_{B}^{2} \rho_{F}^{1 / 2}\right\} \\ & \hline \end{aligned}$ | $\rho_{B}^{4 \times 7} \rho_{F}^{4 \times 1}$ |
| $<1\|6\| 7 \mid 2>$ | $\begin{gathered} \boldsymbol{B}_{L}:\left\{-\rho_{B}^{3 / 2}, \rho_{B}^{3 / 2},-\rho_{B}^{2}, \rho_{B}^{2},-\rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\} \\ \boldsymbol{B}_{R}:\left\{-\rho_{B}^{3 / 2} \rho_{F}^{1 / 2},-\rho_{B}^{3 / 2} \rho_{F}^{1 / 2}, \rho_{B}^{3 / 2} \rho_{F}^{1 / 2}, \rho_{B}^{3 / 2} \rho_{F}^{1 / 2},-\rho_{B}^{2} \rho_{F}^{1 / 2}, \rho_{B}^{2} \rho_{F}^{1 / 2},-\rho_{B}^{2} \rho_{F}^{1 / 2}, \rho_{B}^{2} \rho_{F}^{1 / 2}\right\} \end{gathered}$ | $\rho_{B}^{4 \times 7} \rho_{F}^{4 \times 2}$ |
| $<1\|5\| 7 \mid 3>$ | $\begin{aligned} & \boldsymbol{B}_{L}:\left\{-\rho_{B}^{3 / 2} \rho_{F}^{1 / 2}, \rho_{B}^{3 / 2} \rho_{F}^{1 / 2},-\rho_{B}^{2} \rho_{F}^{1 / 2}, \rho_{B}^{2} \rho_{F}^{1 / 2},-\rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\} \\ & \boldsymbol{B}_{R}:\left\{-\rho_{B}^{3 / 2} \rho_{F}^{1 / 2}, \rho_{B}^{3 / 2} \rho_{F}^{1 / 2},-\rho_{B}^{2} \rho_{F}^{1 / 2}, \rho_{B}^{2} \rho_{F}^{1 / 2},-\rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\} \end{aligned}$ | $\rho_{B}^{4 \times 7} \rho_{F}^{4 \times 3}$ |
| $<1\|4\| 7 \mid 4>$ | $\begin{aligned} & \boldsymbol{B}_{L}:\left\{-\rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\} \\ & \boldsymbol{B}_{R}:\left\{-\rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F}, \rho_{B}^{3 / 2} \rho_{F},-\rho_{B}^{2} \rho_{F},-\rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}, \rho_{B}^{2} \rho_{F}\right\} \end{aligned}$ | $\rho_{B}^{4 \times 7} \rho_{F}^{4 \times 4}$ |

Table 11: Eigenvalues of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ matrices and determinants of $\boldsymbol{B}_{L}^{2}$ and $\boldsymbol{B}_{R}^{2}$ matrices in Step 2


Figure 10: An Example of 4-color $\langle 1| 4|7| 4>$ Non-minimal Adinkra

## 5 Two Five-color Adinkras

At this point, let us emphasize a matter of some importance that occurs whenever adinkras with an odd number of colors is under consideration. From (2.14), it follows that the index structure of the $\boldsymbol{B}_{R}$ and $\boldsymbol{B}_{L}$ matrices is very different. Explicitly expressions describing the matrix entries for each, take the forms

$$
\begin{equation*}
\boldsymbol{B}_{L}=\left(B_{L}\right)_{j k} \quad, \quad \boldsymbol{B}_{R}=\left(B_{R}\right)_{\hat{j} \hat{k}} \tag{5.1}
\end{equation*}
$$

if $N$ is even but also

$$
\begin{equation*}
\boldsymbol{B}_{L}=\left(B_{L}\right)_{j \hat{k}} \quad, \quad \boldsymbol{B}_{R}=\left(B_{R}\right)_{\hat{j} k} \tag{5.2}
\end{equation*}
$$

if $N$ is odd. This observation has some powerful implications for the multiplications of the the $\boldsymbol{B}_{R}$ and $\boldsymbol{B}_{L}$ matrices:
(a.) if $N$ is even, only eigenvalues of $\left(\boldsymbol{B}_{R}\right)^{p}$ and $\left(\boldsymbol{B}_{L}\right)^{q}$ for any real numbers $p$ and $q$ have well defined mathematical meanings, and
(b.) if $N$ is odd, only eigenvalues of $\left(\boldsymbol{B}_{R} \boldsymbol{B}_{L}\right)^{p}$ and $\left(\boldsymbol{B}_{L} \boldsymbol{B}_{R}\right)^{q}$ for any real numbers $p$ and $q$ have well defined mathematical meanings.

In Fig. 11, there are shown two five-color adinkras.

and


Figure 11: Two Five-color Adinkras

Since in this case, $N=5$ is odd, we have to study the eigenvalues of Banchoff $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ and $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ matrices. For each of these, we respectively find

$$
\begin{align*}
& \boldsymbol{B}_{R} \boldsymbol{B}_{L}=\mathbf{R}_{5} \mathbf{L}_{4} \mathbf{R}_{3} \mathbf{L}_{2} \mathbf{R}_{1} \mathbf{L}_{5} \mathbf{R}_{4} \mathbf{L}_{3} \mathbf{R}_{2} \mathbf{L}_{1}  \tag{5.3}\\
& \boldsymbol{B}_{L} \boldsymbol{B}_{R}=\mathbf{L}_{5} \mathbf{R}_{4} \mathbf{L}_{3} \mathbf{R}_{2} \mathbf{L}_{1} \mathbf{R}_{5} \mathbf{L}_{4} \mathbf{R}_{3} \mathbf{L}_{2} \mathbf{R}_{1}
\end{align*}
$$

For the first adinkra, the L-matrices are:

$$
\begin{align*}
& \mathbf{L}_{1}=\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right) \quad, \quad \mathbf{L}_{2}=\left(\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{array}\right),  \tag{5.4}\\
& \mathbf{L}_{3}=\left(\begin{array}{cccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right), \mathbf{L}_{4}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)  \tag{5.5}\\
& \mathbf{L}_{5}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right) \tag{5.6}
\end{align*}
$$

By analyzing the HYMNs, we obtain the results listed in Table 12.

| $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ eigenvalues | $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ eigenvalues |
| :---: | :---: |
| $\left\{1,1, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ |
| $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) |
| $\left\{1,1, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ | $\left\{1,1, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ |

Table 12: Results of $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ and $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ eigenvalues with/without dashings for the first adinkra in Figure 11

For the second adinkra, the $\mathbf{L}$-matrices are:

$$
\begin{align*}
& \mathbf{L}_{1}=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right) \quad, \quad \mathbf{L}_{2}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right)  \tag{5.7}\\
& \mathbf{L}_{3}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right), \mathbf{L}_{4}=\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)  \tag{5.8}\\
& \mathbf{L}_{5}=\left(\begin{array}{cccccccc}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right) . \tag{5.9}
\end{align*}
$$

By analyzing the HYMNs for this case we find the results listed in Table 13.

| $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ eigenvalues | $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ eigenvalues |
| :---: | :---: |
| $\left\{\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ | $\left\{\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ |
| $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) |
| $\left\{\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ | $\left\{\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}^{2}, \rho_{B}^{2}\right\}$ |

Table 13: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the second adinkra in Figure 11

## 6 Two Six-color Adinkras

In this chapter, we are going to study HYMNs for six-color adinkras. In Figure 12, there are shown two six-color adinkras.

and


Figure 12: Two Six-color Adinkras

Since in this case, $N=6$ is even, we define Banchoff $\boldsymbol{B}$-matrices in the way similar to $N=4$ case.

$$
\begin{align*}
\boldsymbol{B}_{L} & =\mathbf{L}_{6} \mathbf{R}_{5} \mathbf{L}_{4} \mathbf{R}_{3} \mathbf{L}_{2} \mathbf{R}_{1},  \tag{6.1}\\
\boldsymbol{B}_{R} & =\mathbf{R}_{6} \mathbf{L}_{5} \mathbf{R}_{4} \mathbf{L}_{3} \mathbf{R}_{2} \mathbf{L}_{1} \tag{6.2}
\end{align*}
$$

For the first adinkra, the $\mathbf{L}$-matrices are:

$$
\begin{align*}
& \mathbf{L}_{1}=\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \mathbf{L}_{2}=\left(\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{array}\right),  \tag{6.3}\\
& \mathbf{L}_{3}=\left(\begin{array}{cccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{array}\right), \mathbf{L}_{4}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right),  \tag{6.4}\\
& \mathbf{L}_{5}=\left(\begin{array}{ccccccccccc} 
\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right), \mathbf{L}_{6}=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0
\end{array}\right) . \tag{6.5}
\end{align*}
$$

By analyzing the HYMNs, we get results listed in Table 14.

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\left\{-i \sqrt{\rho_{B}},-i \sqrt{\rho_{B}}, i \sqrt{\rho_{B}}, i \sqrt{\rho_{B}}\right.$, | $\left\{-i \sqrt{\rho_{B}},-i \sqrt{\rho_{B}}, i \sqrt{\rho_{B}}, i \sqrt{\rho_{B}}\right.$, |
| $\left.-i \rho_{B},-i \rho_{B}, i \rho_{B}, i \rho_{B}\right\}$ | $\left.-i \rho_{B},-i \rho_{B}, i \rho_{B}, i \rho_{B}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\left\{-\sqrt{\rho_{B}},-\sqrt{\rho_{B}}, \sqrt{\rho_{B}}, \sqrt{\rho_{B}}\right.$ | $\left\{-\sqrt{\rho_{B}},-\sqrt{\rho_{B}}, \sqrt{\rho_{B}}, \sqrt{\rho_{B}}\right.$, |
| $\left.-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B}\right\}$ | $\left.-\rho_{B},-\rho_{B}, \rho_{B}, \rho_{B}\right\}$ |

Table 14: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the first adinkra in Fig 12

For the second adinkra, the $\mathbf{L}$-matrices are:

$$
\begin{align*}
& \mathbf{L}_{1}=\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right), \mathbf{L}_{2}=\left(\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right)  \tag{6.6}\\
& \mathbf{L}_{3}=\left(\begin{array}{llllllll}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right), \mathbf{L}_{4}=\left(\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right)  \tag{6.7}\\
& \mathbf{L}_{5}=\left(\begin{array}{lllllll}
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \mathbf{L}_{6}=\left(\begin{array}{cccccc}
0
\end{array}\right) \tag{6.8}
\end{align*}
$$

By analyzing the HYMNs, we get results listed in Table 15.

| $\boldsymbol{B}_{L}$ eigenvalues | $\boldsymbol{B}_{R}$ eigenvalues |
| :---: | :---: |
| $\{-i,-i, i, i$, | $\left\{-i, i,-i \rho_{B},-i \rho_{B}\right.$, |
| $\left.-i \rho_{B}, i \rho_{B},-i \rho_{B}^{2}, i \rho_{B}^{2}\right\}$ | $\left.-i \rho_{B}, i \rho_{B}, i \rho_{B}, i \rho_{B}\right\}$ |
| $\boldsymbol{B}_{L}$ eigenvalues (dropping all dashings) | $\boldsymbol{B}_{R}$ eigenvalues (dropping all dashings) |
| $\{-1,-1,1,1$, | $\left\{-1,1,-\rho_{B},-\rho_{B}\right.$, |
| $\left.-\rho_{B}, \rho_{B},-\rho_{B}^{2}, \rho_{B}^{2}\right\}$ | $\left.-\rho_{B}, \rho_{B}, \rho_{B}, \rho_{B}\right\}$ |

Table 15: Results of $\boldsymbol{B}_{L}$ and $\boldsymbol{B}_{R}$ eigenvalues with/without dashings for the second adinkra in Fig 12

## 7 Conclusion

In this paper, we have extended our previous discussions about using HYMNs (height-yielding matrix numbers) which are the eigenvalues [14] of functions of the adjacency matrices associated with the $\mathbf{L}$-matrics and $\mathbf{R}$-matrices derived from adinkras. The traces and determinants of the Banchoff matrices defined in this paper yield polynomials that are sensitive of the "shapes" of the adinkras in all cases examined. Further study will be required to support the current speculation that these polynomials split adinkras into equivalent classes. Even more intriguing is the possibility that these polynomials could play an important role in the concept of "SUSY Holography," as introduced in the work of [49]. These topics will be the subject of future explorations.
> "The object of pure Physics is the unfolding of the laws of the intelligible world; the object of pure Mathematics that of unfolding the laws of human intelligence."

- J. J. Sylvester


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## A Transformation Laws of $4 \mathrm{D}, \mathcal{N}=1$ Minimal Supermultiplets

Although a lot of literature about Adinkras has presented transformation laws of ten minimal off-shell $4 \mathrm{D}, \mathcal{N}=1$ supermultiplets, for example chapter three in [17], there are some chronic typos. For clarity, we include the corrected transformation laws in this appendix. Note that the convention is $\epsilon^{0123}=+1$ and $\eta^{\mu \nu}=\eta_{\mu \nu}=\operatorname{diag}(-1,1,1,1)$.

Chiral Supermultiplet: $\left(A, B, \psi_{a}, F, G\right)$

$$
\begin{align*}
\mathrm{D}_{a} A & =\psi_{a} \\
\mathrm{D}_{a} \psi_{b} & =i\left(\gamma^{\mu}\right)_{a b} \partial_{\mu} A-\left(\gamma^{5} \gamma^{\mu}\right)_{a b} \partial_{\mu} B-i\left(\gamma^{5}\right)_{a}^{b} \psi_{b}  \tag{A.1}\\
\mathrm{D}_{a} F & =\left(\gamma_{a b}^{\mu}\right)_{a}^{b} \partial_{\mu} \psi_{b} \quad, \quad \mathrm{D}_{a} G=i\left(\gamma^{5} \gamma^{\mu}\right)_{a}^{b} \partial_{\mu} \psi_{b}
\end{align*}
$$

Hodge - Dual \#1 Chiral Supermultiplet : $\left(A, B, \psi_{a}, \mathrm{f}_{\mu \nu \rho}, G\right)$

$$
\begin{align*}
\mathrm{D}_{a} A & =\psi_{a} \\
\mathrm{D}_{a} \psi_{b} & =i\left(\gamma^{\mu}\right)_{a b} \partial_{\mu} A-\left(\gamma^{5} \gamma^{\mu}\right)_{a b} \partial_{\mu} B-i\left(\gamma^{5}\right)_{a}^{b} \psi_{b}  \tag{A.2}\\
\mathrm{D}_{a} \mathrm{f}_{\mu \nu \rho} & =-\left(\gamma^{\sigma}\right)_{a}{ }^{b} \epsilon_{\sigma \mu \nu \rho} \psi_{b}\left(\epsilon^{\sigma \mu \nu \rho} \partial_{\sigma} \mathrm{f}_{\mu \nu \rho}\right)+\quad \mathrm{D}_{a} G=i\left(\gamma^{5} \gamma^{\mu}\right)_{a}{ }^{b} \partial_{\mu} \psi_{a b} G
\end{align*}
$$

Hodge - Dual \#2 Chiral Supermultiplet: $\left(A, B, \psi_{a}, F, \mathrm{~g}_{\mu \nu \rho}\right)$

$$
\begin{align*}
\mathrm{D}_{a} A=\psi_{a} & , \quad \mathrm{D}_{a} B=i\left(\gamma^{5}\right)_{a}^{b} \psi_{b} \\
\mathrm{D}_{a} \psi_{b} & =i\left(\gamma^{\mu}\right)_{a b} \partial_{\mu} A-\left(\gamma^{5} \gamma^{\mu}\right)_{a b} \partial_{\mu} B-i C_{a b} F+\frac{1}{3!}\left(\gamma^{5}\right)_{a b}\left(\epsilon^{\sigma \mu \nu \rho} \partial_{\sigma} \mathrm{g}_{\mu \nu \rho}\right)  \tag{A.3}\\
\mathrm{D}_{a} F & =\left(\gamma^{\mu}\right)_{a}^{b} \partial_{\mu} \psi_{b} \quad, \quad \mathrm{D}_{a} \mathrm{~g}_{\mu \nu \rho}=-\left(\gamma^{5} \gamma^{\sigma}\right)_{a}^{b} \epsilon_{\sigma \mu \nu \rho} \psi_{b}
\end{align*}
$$

Hodge - Dual \#3 Chiral Supermultiplet: $\left(A, B, \psi_{a}, \mathrm{f}_{\mu \nu \rho}, \mathrm{g}_{\mu \nu \rho}\right)$

$$
\begin{align*}
\mathrm{D}_{a} A= & \psi_{a} \quad, \quad \mathrm{D}_{a} B=i\left(\gamma^{5}\right)_{a}^{b} \psi_{b} \\
\mathrm{D}_{a} \psi_{b}= & i\left(\gamma^{\mu}\right)_{a b} \partial_{\mu} A-\left(\gamma^{5} \gamma^{\mu}\right)_{a b} \partial_{\mu} B \\
& -i \frac{1}{3!} C_{a b}\left(\epsilon^{\sigma \mu \nu \rho} \partial_{\sigma} \mathrm{f}_{\mu \nu \rho}\right)+\frac{1}{3!}\left(\gamma^{5}\right)_{a b}\left(\epsilon^{\sigma \mu \nu \rho} \partial_{\sigma} \mathrm{g}_{\mu \nu \rho}\right)  \tag{A.4}\\
\mathrm{D}_{a} \mathrm{f}_{\mu \nu \rho}= & -\left(\gamma^{\sigma}\right)_{a}{ }^{b} \epsilon_{\sigma \mu \nu \rho} \psi_{b} \quad, \quad \mathrm{D}_{a} \mathrm{~g}_{\mu \nu \rho}=-\left(\gamma^{5} \gamma^{\sigma}\right)_{a}{ }^{b} \epsilon_{\sigma \mu \nu \rho} \psi_{b}
\end{align*}
$$

For the Hodge - Dual \#1 Chiral Supermultiplet one should perform the replacement of the auxiliary fields according to

$$
\begin{equation*}
\int d t F \rightarrow \mathrm{f}_{123} \quad, \quad G \rightarrow G \tag{A.5}
\end{equation*}
$$

where $f_{123}$ is the purely spatial component of the Lorentz 3-form $f_{\mu \nu \rho}$.
For the Hodge - Dual \#2 Chiral Supermultiplet one should perform the replacement of the auxiliary fields according to

$$
\begin{equation*}
F \rightarrow F \quad, \quad \int d t G \rightarrow \mathrm{~g}_{123} \tag{A.6}
\end{equation*}
$$

where $g_{123}$ is the purely spatial component of the Lorentz 3-form $g_{\mu \nu \rho}$.
For the Hodge - Dual \#3 Chiral Supermultiplet one should perform the replacement of the auxiliary fields according to

$$
\begin{equation*}
\int d t F \rightarrow \mathrm{f}_{123} \quad, \quad \int d t G \rightarrow \mathrm{~g}_{123} \tag{A.7}
\end{equation*}
$$

where $f_{123}$ is the purely spatial component of the Lorentz 3 -form $f_{\mu \nu \rho}$ and $g_{123}$ is the purely spatial component of the Lorentz 3-form $\mathrm{g}_{\mu \nu \rho}$.

Vector Supermultiplet: $\left(A_{\mu}, \lambda_{b}, \mathrm{~d}\right)$

$$
\begin{align*}
\mathrm{D}_{a} A_{\mu} & =\left(\gamma_{\mu}\right)_{a}^{b} \lambda_{b}, \\
\mathrm{D}_{a} \lambda_{b} & =-i \frac{1}{4}\left(\left[\gamma^{\mu}, \gamma^{\nu}\right]\right)_{a b}\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)+\left(\gamma^{5}\right)_{a b} \mathrm{~d}  \tag{A.8}\\
\mathrm{D}_{a} \mathrm{~d} & =i\left(\gamma^{5} \gamma^{\mu}\right)_{a}^{b} \partial_{\mu} \lambda_{b},
\end{align*}
$$

Axial-Vector Supermultiplet : $\left(U_{\mu}, \widetilde{\lambda}_{b}, \widetilde{\mathrm{~d}}\right)$

$$
\begin{align*}
\mathrm{D}_{a} U_{\mu} & =i\left(\gamma^{5} \gamma_{\mu}\right)_{a}{ }^{b} \widetilde{\lambda}_{b} \\
\mathrm{D}_{a} \widetilde{\lambda}_{b} & =\frac{1}{4}\left(\gamma^{5}\left[\gamma^{\mu}, \gamma^{\nu}\right]\right)_{a b}\left(\partial_{\mu} U_{\nu}-\partial_{\nu} U_{\mu}\right)+i C_{a b} \widetilde{\mathrm{~d}}  \tag{A.9}\\
\mathrm{D}_{a} \widetilde{\mathrm{~d}} & =-\left(\gamma^{\mu}\right)_{a}^{b} \partial_{\mu} \widetilde{\lambda}_{b},
\end{align*}
$$

Hodge - Dual Vector Supermultiplet: $\left(A_{\mu}, \lambda_{b}, \mathrm{~d}_{\mu \nu \rho}\right)$

$$
\begin{align*}
\mathrm{D}_{a} A_{\mu} & =\left(\gamma_{\mu}\right)_{a}^{b} \lambda_{b} \\
\mathrm{D}_{a} \lambda_{b} & =-i \frac{1}{4}\left(\left[\gamma^{\mu}, \gamma^{\nu}\right]\right)_{a b}\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)+\frac{1}{3!}\left(\gamma^{5}\right)_{a b}\left(\epsilon^{\sigma \mu \nu \rho} \partial_{\sigma} \mathrm{d}_{\mu \nu \rho}\right) \\
\mathrm{D}_{a} \mathrm{~d}_{\mu \nu \rho} & =-i\left(\gamma^{5} \gamma^{\sigma}\right)_{a}^{b} \epsilon_{\sigma \mu \nu \rho} \lambda_{b} \tag{A.10}
\end{align*}
$$

Hodge - Dual Axial - Vector Supermultiplet : $\left(A_{\mu}, \widetilde{\lambda}_{b}, \widetilde{\mathrm{~d}}_{\mu \nu \rho}\right)$

$$
\begin{align*}
\mathrm{D}_{a} U_{\mu} & =i\left(\gamma^{5} \gamma_{\mu}\right)_{a}{ }^{b} \widetilde{\lambda}_{b} \\
\mathrm{D}_{a} \widetilde{\lambda}_{b} & =\frac{1}{4}\left(\gamma^{5}\left[\gamma^{\mu}, \gamma^{\nu}\right]\right)_{a b}\left(\partial_{\mu} U_{\nu}-\partial_{\nu} U_{\mu}\right)+i \frac{1}{3!} C_{a b}\left(\epsilon^{\sigma \mu \nu \rho} \partial_{\sigma} \widetilde{\mathrm{d}}_{\mu \nu \rho}\right)  \tag{A.11}\\
\mathrm{D}_{a} \widetilde{\mathrm{~d}}_{\mu \nu \rho} & =\left(\gamma^{\sigma}\right)_{a}{ }^{b} \epsilon_{\sigma \mu \nu \rho} \widetilde{\lambda}_{b}
\end{align*}
$$

For Axial - Vector Supermultiplet one should perform the replacement of the fermionic fields according to

$$
\begin{equation*}
\lambda_{b} \rightarrow-i\left(\gamma^{5}\right)_{b}^{c} \widetilde{\lambda}_{c} \tag{A.12}
\end{equation*}
$$

and the bosonic fields according to

$$
\begin{equation*}
A_{\mu} \rightarrow U_{\mu} \quad, \quad \mathrm{d} \rightarrow \widetilde{\mathrm{~d}} \tag{A.13}
\end{equation*}
$$

For Hodge - Dual Vector Supermultiplet one should perform the replacement of the the auxiliary field according to

$$
\begin{equation*}
\int d t \mathrm{~d} \rightarrow \mathrm{~d}_{123} \tag{A.14}
\end{equation*}
$$

where $d_{123}$ is the purely spatial component of the Lorentz 3 -form $d_{\mu \nu \rho}$. Note that the last line in Equation (3.9) in [17] has typos and the corrected one is (A.10).

For Hodge - Dual Axial - Vector Supermultiplet one should perform the replacement of the auxiliary field according to

$$
\begin{equation*}
\int d t \tilde{\mathrm{~d}} \rightarrow \widetilde{\mathrm{~d}}_{123} \tag{A.15}
\end{equation*}
$$

where $\widetilde{\mathrm{d}}_{123}$ is the purely spatial component of the Lorentz 3 -form $\widetilde{\mathrm{d}}_{\mu \nu \rho}$. Note that the last line in Equation (3.10) in [17] has typos and the corrected one is (A.11).

Tensor Supermultiplet: $\left(\varphi, B_{\mu \nu}, \chi_{a}\right)$

$$
\begin{align*}
\mathrm{D}_{a} \varphi & =\chi_{a} \quad, \quad \mathrm{D}_{a} B_{\mu \nu}=-\frac{1}{4}\left(\left[\gamma_{\mu}, \gamma_{\nu}\right]\right)_{a}^{b} \chi_{b}  \tag{A.16}\\
\mathrm{D}_{a} \chi_{b} & =i\left(\gamma^{\mu}\right)_{a b} \partial_{\mu} \varphi-\left(\gamma^{5} \gamma^{\mu}\right)_{a b} \epsilon_{\mu}^{\rho \sigma \tau} \partial_{\rho} B_{\sigma \tau}
\end{align*}
$$

Axial - Tensor Supermultiplet: $\left(\widetilde{\varphi}, C_{\mu \nu}, \widetilde{\chi}_{a}\right)$

$$
\begin{align*}
\mathrm{D}_{a} \widetilde{\varphi} & =i\left(\gamma^{5}\right)_{a}{ }^{b} \widetilde{\chi}_{b} \quad, \quad \mathrm{D}_{a} C_{\mu \nu}=-i \frac{1}{4}\left(\gamma^{5}\left[\gamma_{\mu}, \gamma_{\nu}\right]\right)_{a}{ }^{b} \widetilde{\chi}_{b},  \tag{A.17}\\
\mathrm{D}_{a} \widetilde{\chi}_{b} & =-\left(\gamma^{5} \gamma^{\mu}\right)_{a b} \partial_{\mu} \widetilde{\varphi}-i\left(\gamma^{\mu}\right)_{a b} \epsilon_{\mu}{ }^{\rho \sigma \tau} \partial_{\rho} C_{\sigma \tau},
\end{align*}
$$

For Axial - Tensor Supermultiplet one should perform the replacement of the fermionic fields according to

$$
\begin{equation*}
\chi_{a} \rightarrow i\left(\gamma^{5}\right)_{a}^{b} \widetilde{\chi}_{b} \tag{A.18}
\end{equation*}
$$

and the bosonic fields according to

$$
\begin{equation*}
\varphi \rightarrow \widetilde{\varphi} \quad, \quad B_{\mu \nu} \rightarrow C_{\mu \nu} \tag{A.19}
\end{equation*}
$$

Note that the first line in Equation (3.12) in [17] has typos and the corrected one is (A.17).

## B Traces and Determinants

As mentioned in the body of the paper, the traces and determinants of the various Banchoff matrices presented are an important calculation as to the classification of the HYMNs. These are listed below.

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | $4 \rho_{B}$ | $-2\left(1+\rho_{B}^{2}\right)$ | $4 \rho_{B}$ | $2\left(1+\rho_{B}^{2}\right)$ |
| determinant | $\rho_{B}^{4}$ | $\rho_{B}^{4}$ | $\rho_{B}^{4}$ | $\rho_{B}^{4}$ |

Table 16: Results of trace and determinant eigenvalues with/without dashings for the adinkra in Fig 1

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | $-2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ |
| determinant | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ |

Table 17: Traces and determinants with/without dashings for the adinkra in Fig 2

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | -4 | 4 | 4 | 4 |
| determinant | 1 | 1 | 1 | 1 |

Table 18: Traces and determinants with/without dashings for the adinkra in Fig 3

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | $4 \chi_{\mathrm{o}}$ | $-4 \chi_{\mathrm{o}}$ | 4 | 4 |
| determinant | 1 | 1 | 1 | 1 |

Table 19: Traces and determinants with/without dashings for the 36, 864 valise adinkras associated with $(4,0)$ SUSY
where

$$
\chi_{o}(\mathcal{R})= \begin{cases}1 & \text { when } \mathcal{R}=\text { SM-I, SM-IV }  \tag{B.1}\\ -1 & \text { when } \mathcal{R}=\text { SM-II, SM-III }\end{cases}
$$

which is true for all the valise $\operatorname{GR}(4,4)$ adinkras.

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | $2\left(1+\rho_{B}^{2}\right)$ | $-2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ |
| determinant | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ |

Table 20: Traces and determinants with/without dashings for Hodge-Dual \#1 and HodgeDual \#2 Chiral Supermultiplet

For the adinkra described in Chapter 4, since as we already seen that eigenvalues are not sensitive to dashings, we will only show $\operatorname{Tr}\left(\boldsymbol{B}_{L}\right), \operatorname{Tr}\left(\boldsymbol{B}_{R}\right), \operatorname{Det}\left(\boldsymbol{B}_{L}\right)$, and $\operatorname{Det}\left(\boldsymbol{B}_{R}\right)$ in Table 24.

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | 4 | -4 | 4 | 4 |
| determinant | 1 | 1 | 1 | 1 |

Table 21: Traces and determinants with/without dashings for Hodge-Dual \#3 Chiral Supermultiplet

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | $-2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}^{2}\right)$ |
| determinant | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ |

Table 22: Traces and determinants with/without dashings for Axial-Vector Supermultiplet

|  | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | -4 | 4 | 4 | 4 |
| determinant | 1 | 1 | 1 | 1 |

Table 23: Traces and determinants with/without dashings for Hodge-Dual Vector Supermultiplet, Hodge-Dual Axial-Vector Supermultiplet, and Axial-Tensor Supermultiplet

|  | $\operatorname{Tr}\left(\boldsymbol{B}_{L}\right)$ | $\operatorname{Tr}\left(\boldsymbol{B}_{R}\right)$ | $\operatorname{Det}\left(\boldsymbol{B}_{L}\right)$ | $\operatorname{Det}\left(\boldsymbol{B}_{R}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| $<8 \mid 8>$ | 0 | 0 | 1 | 1 |
| $<7\|8\| 1>$ | 0 | 0 | $\rho_{B}^{2}$ | $\rho_{B}^{2}$ |
| $<6\|8\| 2>$ | 0 | 0 | $\rho_{B}^{4}$ | $\rho_{B}^{4}$ |
| $<5\|8\| 3>$ | 0 | 0 | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ |
| $<4\|8\| 4>$ | 0 | 0 | $\rho_{B}^{8}$ | $\rho_{B}^{8}$ |
| $<3\|8\| 5>$ | 0 | 0 | $\rho_{B}^{10}$ | $\rho_{B}^{10}$ |
| $<2\|8\| 6>$ | 0 | 0 | $\rho_{B}^{12}$ | $\rho_{B}^{12}$ |
| $<1\|8\| 7>$ | 0 | 0 | $\rho_{B}^{14}$ | $\rho_{B}^{14}$ |
| $<1\|7\| 7 \mid 1>$ | 0 | 0 | $\rho_{B}^{14} \rho_{F}^{2}$ | $\rho_{B}^{14} \rho_{F}^{2}$ |
| $<1\|6\| 7 \mid 2>$ | 0 | 0 | $\rho_{B}^{14} \rho_{F}^{4}$ | $\rho_{B}^{14} \rho_{F}^{4}$ |
| $<1\|5\| 7 \mid 3>$ | 0 | 0 | $\rho_{B}^{14} \rho_{F}^{6}$ | $\rho_{B}^{14} \rho_{F}^{6}$ |
| $<1\|4\| 7 \mid 4>$ | 0 | 0 | $\rho_{B}^{14} \rho_{F}^{8}$ | $\rho_{B}^{14} \rho_{F}^{8}$ |
| $<1\|4\| 6\|4\| 1>$ | 0 | 0 | $\rho_{B}^{16} \rho_{F}^{8}$ | $\rho_{B}^{16} \rho_{F}^{8}$ |

Table 24: Traces and determinants for adinkras described in Chapter 4

In summary, for the 4-color nonminimal adinkras described in Chapter 4, we have general equations

$$
\begin{align*}
& \operatorname{Tr}\left(\boldsymbol{B}_{L}\right)=\operatorname{Tr}\left(\boldsymbol{B}_{R}\right)=0 \\
& \operatorname{Det}\left(\boldsymbol{B}_{L}\right)=\operatorname{Det}\left(\boldsymbol{B}_{R}\right)=\rho_{B}^{2 \times \# \text { of bosons lifted }} \rho_{F}^{2 \times \# \text { of fermions lifted }} \tag{B.2}
\end{align*}
$$

For two five-color adinkras described in Chapter 5, since eigenvalues are not sensitive to dashings as well, we will only show results with dashings.

| 1st Adinkra | $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ | $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ |
| :---: | :---: | :---: |
| trace | $2\left(1+\rho_{B}+2 \rho_{B}^{2}\right)$ | $2\left(1+\rho_{B}+2 \rho_{B}^{2}\right)$ |
| determinant | $\rho_{B}^{10}$ | $\rho_{B}^{10}$ |
| 2nd Adinkra | $\boldsymbol{B}_{L} \boldsymbol{B}_{R}$ | $\boldsymbol{B}_{R} \boldsymbol{B}_{L}$ |
| trace | $2\left(3 \rho_{B}+\rho_{B}^{2}\right)$ | $2\left(3 \rho_{B}+\rho_{B}^{2}\right)$ |
| determinant | $\rho_{B}^{10}$ | $\rho_{B}^{10}$ |

Table 25: Traces and determinants for the adinkras in Fig 11

For two six-color adinkras described in Chapter 6, we have Table 26.

| 1st Adinkra | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| trace | 0 | 0 | 0 | 0 |
| determinant | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ |
| 2nd Adinkra | $\boldsymbol{B}_{L}$ | $\boldsymbol{B}_{R}$ | $\left\|\boldsymbol{B}_{L}\right\|$ | $\left\|\boldsymbol{B}_{R}\right\|$ |
| trace | 0 | 0 | 0 | 0 |
| determinant | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ | $\rho_{B}^{6}$ |

Table 26: Traces and determinants with/without dashings for the adinkras in Fig 12

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