Course on

Beyond

the

Standard Model

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1995

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Chapter 1

The Standard Model

After three decades of great cumulative theoretical and experimental effort, today we seem to possess the theory of both strong and electroweak interactions. This is the so-called *Standard Model*, based on the invariance under the symmetry group

$$G_{st} = SU(2)_L \times U(1)_Y \times SU(3)_C \tag{1.1}$$

More correctly phrased, it is based on the partial spontaneous symmetry breaking of G_{st}.

The basic constituents of matter, the elementary fermions, i.e. the quarks and leptons, have the following transformation properties under G_{st} (where for the time being we concentrate only in the first generation of fermions)

Quarks :

	$\begin{pmatrix} u \\ d \end{pmatrix}^i_L$	u^i_R	d^i_R
$SU(3)_C$	3_C	3_C	3_C
$SU(2)_L$	2_L	$\bar{1}_L$	1_L
$U(1)_Y$	$1/3$	$4/3$	$-2/3$

(where $i = 1, 2, 3 \equiv Nc$)

Leptons :

$$\begin{array}{llll} & \begin{pmatrix} \nu \\ e \end{pmatrix}_L & e_R & \not{\nu}_R \,(\text{no } \ \nu_{\rm R}) \\ & & & \\ SU(3)_C & 1_C & 1_C & \\ & & & \\ SU(2)_L & 2_L & 1_L & \\ & & & \\ U(1)_Y & -1 & -2 & \end{array}$$

In the above we have used the formula for the electromagnetic charge

$$Q_{em} = T_{3L} + \frac{Y}{2} \qquad\qquad (\,photon \ = \ color\ blind\,) \tag{1.2}$$

with Q_{em}

$$\begin{array}{llcll} u: & 2/3 & \qquad\qquad & e: & -1 \\ d: & -1/3 & & \nu: & 0 \end{array}$$

An often used notation is also

$$\begin{array}{ccc} \begin{pmatrix} u \\ d \end{pmatrix}_L^i & : & (3_C,\, 2_L,\, 1/3) \\ & & \\ u_R & : & (3_C,\, 1_L,\, 4/3) \\ & & \\ d_R & : & (3_C,\, 1_L,\, -2/3) \\ & & \\ ------ & & -------------- \\ \begin{pmatrix} \nu \\ e \end{pmatrix}_L & : & (1_C,\, 2_L,\, -1) \\ & & \\ e_R & : & (1_C,\, 1_L,\, -2) \end{array}$$

The Lagrangian for the above theory (ignore $SU(3)_C$ for the time being), more precisely for the $SU(2) \times U(1)$ electroweak model, can be written as:

$$\mathcal{L} = \mathcal{L}_{g.bosons} + \mathcal{L}_{fermions} \tag{1.3}$$

where

$$\mathcal{L}_{g.bosons} = -\frac{1}{4}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} - \frac{1}{4}B_{\mu\nu} \cdot B^{\mu\nu} \tag{1.4}$$

with

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_n u \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu \tag{1.5}$$

or

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon_{ijk} A^j_\mu A^k_\nu \tag{1.6}$$

being the $SU(2)_L$ field strength, and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial - \nu B_\mu \tag{1.7}$$

the corresponding field strength for the $U(1)_Y$ gauge boson.

The symmetry invariance holds under the following transformations

$$\frac{\vec{\tau}\cdot\vec{A}_\mu}{2} \longrightarrow U(\theta)\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U(\theta)^{-1} - \frac{i}{g}\left[\partial_\mu U(\theta)\right]U(\theta)^{-1} \tag{1.8}$$

where

$$U(\theta) = e^{-i\frac{\vec{\tau}}{2}\cdot\vec{\theta}(x)} \qquad SU(2) \tag{1.9}$$

which for small θ becomes

$$A^{i\prime}_\mu = A^i_\mu - \underset{\substack{\uparrow \\ \text{a “gauge” transformation}}}{\frac{1}{g}\partial_\mu\theta^i} + \quad \underset{\substack{\uparrow \\ \text{an } SU(2) \text{ global rotation}}}{\epsilon_{ijk}\theta^j A^k_\mu} \tag{1.10}$$

$\Longrightarrow$ Thus a vector $\vec{A}_\mu$ of “photons” (gauge bosons).

Under the same transformations $\vec{F}_{\mu\nu}$ is a vector

$$\frac{\vec{\tau}}{2}\cdot\vec{F}'_{\mu\nu} = U(\theta)\frac{\vec{\tau}}{2}\cdot\vec{F}_{\mu\nu}U(\theta)^{-1} \tag{1.11}$$

or for small θ

$$F^{i\ \prime}_{\mu\nu} = F^i_{\mu\nu} + g\epsilon_{ijk}\theta^j F^k_{\mu\nu} \tag{1.12}$$

and so $\vec{F}^2_{\mu\nu}$ is a group invariant.

Similarly,

$$B'_\mu = B_\mu - \frac{1}{g'}\partial_\mu\theta \tag{1.13}$$

$$B'_{\mu\nu} = B_{\mu\nu} \tag{1.14}$$

the usual $U(1)$ “photon”.

Next, the fermionic Lagrangian has the usual covariant Dirac form

$$\mathcal{L}_f = \sum_f i\bar{f}\gamma_\mu D^\mu f \tag{1.15}$$

with

$$D_\mu = \partial_\mu - ig\underset{\underset{SU(2)\text{ gen.}}{\uparrow}}{\frac{\vec{\tau}}{2}} \cdot \vec{A}_\mu - ig'\underset{\underset{U(1)\text{ gen.}}{\uparrow}}{\frac{Y}{2}} B_\mu \tag{1.16}$$

to be read as:

$$D_\mu \begin{pmatrix} u \\ d \end{pmatrix}_L = \partial_\mu \begin{pmatrix} u \\ d \end{pmatrix}_L - \left(ig\frac{\vec{\tau}}{2} \cdot \vec{A}_\mu + ig'\frac{1}{6}B_\mu \right) \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{1.17}$$

$$D_\mu u_R = \partial_\mu u_R - ig'\frac{2}{3}B_\mu u_R \tag{1.18}$$

etc.

Invariant under

$U(1)_Y$

$$f \longrightarrow e^{-i\theta Y/2} f$$

$SU(2)_L$

$$\begin{array}{ccc} \begin{pmatrix} u \\ d \end{pmatrix}_L & \longrightarrow & e^{-i\frac{\vec{\tau}}{2}\cdot\vec{\theta}} \begin{pmatrix} u \\ d \end{pmatrix}_L \\ u_R & \longrightarrow & u_R \\ d_R & \longrightarrow & d_R \end{array}$$

etc.

As it is: $\left.\begin{array}{l}\text{4 massless gauge fields} \\ \text{all fermions massless}\end{array}\right\} \Leftarrow$ symmetry.

Thus we must *break the symmetry* and must break it *spontaneously* to ensure renormalizability.

As you know by now, this is achieved through the doublet of Higgs scalars

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{1.19}$$

with $Y = 1$ or $\Phi = (1_C,\ 2_L,\ 1)$.

Once again, by fixing the quantum numbers of Φ its interactions are determined

$$\mathcal{L}_\Phi = \frac{1}{2}(D_\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi) + \mathcal{L}_Y \tag{1.20}$$

where the potential has the well-known form

$$V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2 \tag{1.21}$$

and $\mathcal{L}_Y$ are the Yukawa couplings of the Higgs doublet to the fermions.

$$\mathcal{L}_Y = h_d(\bar{u} \quad \bar{d})_L \Phi d_R + h_U(\bar{u} \quad \bar{d})_L i\tau_2 \Phi^* u_R + h_\nu(\bar{\nu} \quad \bar{e})_L \Phi e_R + h.c. \tag{1.22}$$

Also, the covariant derivative has the form

$$D_\mu \Phi = \partial_\mu \Phi - ig\frac{\vec{\tau}}{2} \cdot \vec{A}_\mu \Phi - i\frac{g'}{2} B_\mu \Phi \tag{1.23}$$

Before one can study the physics of the theory, one must find the vacuum, i.e. determine the lowest energy state. This will follow from $V(\Phi)$, since as you notice we have a freedom in the sign of the μ^2 term (recall that $\lambda > 0$ in order to

ensure that there is a minimum of energy, i.e. that the potential is bounded from below). An innocently looking choice $\mu^2 > 0$ provides all the difference, the potential takes the form

Obviously, at the minimum

$$< \Phi^\dagger \Phi >= v^2 \equiv \frac{\mu^2}{\lambda} \tag{1.24}$$

which by an $SU(2)$ transformation (show it) can be cast in the form

$$< \Phi^\dagger \Phi >= \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{1.25}$$

with

$$\begin{aligned} T_i < \Phi > &= \frac{\tau_i}{2} \neq 0 \\ Y < \Phi > &= 0 \end{aligned} \tag{1.26}$$

but

$$Q_{em} < \Phi > \equiv (T_3 + \frac{Y}{2}) < \Phi > = 0 \tag{1.27}$$

Thus, the symmetry is broken, schematically

$$SU(2)_L \times U(1)_Y \underset{< \Phi > \neq 0}{\longrightarrow} U(1)_{em}$$

The consequences are dramatic:

1.0.1 a. Particle spectrum

From $\frac{1}{2}(D_\mu \Phi)^\dagger (D_\mu \Phi)$, one finds in the vacuum $< \Phi > = \begin{pmatrix} 0 \\ v \end{pmatrix}$

$$\frac{1}{2} < (D_\mu \Phi)^\dagger (D_\mu \Phi) > = \frac{1}{2} \frac{g^2 v^2}{4} \left[(A_1^2 + A_2^2) + \frac{1}{\cos^2 \theta_W} (\cos \theta_W A_3 - \sin \theta_W B)^2 \right] \tag{1.28}$$

where $\tan \theta_W \equiv g'/g$.

This, in the language of eigenstates of definite charge

$$\begin{aligned} W^\pm &\equiv \frac{1}{\sqrt{2}}(A_1 \mp A_2) \\ Z &\equiv \cos \theta_W A_3 - \sin \theta_W B \\ A &\equiv \sin \theta_W A_3 + \cos \theta_W B \end{aligned} \tag{1.29}$$

$$\tag{1.30}$$

One concludes

$$M_W = \frac{g}{2} v \ \ ; \quad M_Z = \frac{M_W}{\cos \theta_W} \ \ ; \quad M_A = 0 \tag{1.31}$$

Similarly, from the Yukawa interactions (1.22), in the vacuum the fermions become massive

$$m_d = h_d v \ \ ; \quad m_u = h_u v \ \ ; \quad m_e = h_e v \tag{1.32}$$

except for the neutrino which in the absence of the ν_R field remain massless. However, the masses are completely arbitrary, due to the arbitrariness of the couplings h_i.

As you know, by a local $SU(2)_L$ group transformation we can always bring the Higgs doublet to the form (show it)

$$\Phi = \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} \tag{1.33}$$

implying only one physical Higgs scalar η, the real and neutral spin-0 particle. From (1.21) its mass is

$$m_\eta^2 = 2\lambda v^2 \tag{1.34}$$

All we know about λ is unfortunately only its sign, this leaves m_η undetermined.

Of course, you know that we did not lose the other 3 members of Φ; they just got traded for the extra 3 longitudinal degrees of freedom of the massive W^+, W^- and Z.

1.0.2 b. Interactions

b.1 Gauge Currents

From $\sum_f i\bar{f}\gamma^\mu D_\mu f$ it is easy to derive the charged and neutral weak and electromagnetic interactions

$$\begin{aligned}
\mathcal{L}_W &= \frac{g}{\sqrt{2}}\bar{u}_L\gamma^\mu d_L W_\mu^+ + h.c. \\
\mathcal{L}_A &= e\sum_f \bar{f}\gamma^\mu Q_{em} f A_\mu \\
\mathcal{L}_Z &= \frac{g}{\cos\theta_W}\sum_f \bar{f}\left(T_3 L - Q\sin^2\theta_W\right)\gamma^\mu f Z_\mu
\end{aligned} \tag{1.35}$$

where

$$e \equiv g\sin\theta_W \tag{1.36}$$

$$L \equiv \frac{1+\gamma_5}{2} \tag{1.37}$$

The fact that $M_A = 0$ and the form of $\mathcal{L}_A$ allow us to identify A_μ as a photon, whereas $W_\mu^\pm$ and Z_μ mediate the weak interaction. Since $M_Z\cos\theta_W = M_W$, the theory is completely characterized by only two parameters: M_W and (say) $\sin^2\theta_W$. The great success of the standard model lies in the fact that all the charged and neutral current data and the measurements of M_W and M_Z agree spectacularly with the theory for

$$M_W \simeq 80GeV \quad ; \quad \sin^2\theta_W \simeq 0.23 \tag{1.38}$$

Notice that the prediction $M_Z = M_W/\cos\theta_W = 90GeV$ follows from the spontaneous symmetry breaking. Had you given the mass to W and Z by hand, not only the theory would not be renormalizable, but the masses would have been uncorrelated.

1.0.3 b.2 Yukawa Couplings

Besides coupling to $W_\mu^\pm$ and Z_μ, the Higgs also has interactions with the fermions. It is easy to show from(1.22) and (1.31) that

$$\mathcal{L}_Y = \frac{g}{2M_W}\eta \sum_f m_f \bar{f} f \tag{1.39}$$

You can see why it was and still is hard to find the Higgs. Not only that its mass is arbitrary, but also the couplings to the fermions are suppressed by m_f/M_W . For example, for $f = e$, the electron, this suppression is about 10^{-5} ! Similarly, for down and up quarks, with $m_d \simeq m_u \simeq 10MeV$, we do not do much better.

May you keep in mind that the experiment only provides a lower bound:

$$(m_\eta)_{exp} \geq 60GeV \tag{1.40}$$

And the search for the Higgs scalar must be considered *the* most central task of weak interaction physics.

The $SU(2)_L \times U(1)$ Electro-Weak Theory

Summary of Main formulas

Fermions
$$\begin{cases} \left\{ \begin{matrix} u \\ d \end{matrix} \right)^i_L & u^i_R & d^i_R \\ \\ \begin{pmatrix} \nu \\ e \end{pmatrix}_L & e_R & \end{cases}$$

Higgs $\quad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$

Charge $\quad Q_{em} = T_3 + \frac{Y}{2}$

Lagrangian

$$\mathcal{L} = \mathcal{L}_{gb} + \mathcal{L}_f + \mathcal{L}_\Phi + \mathcal{L}_Y \tag{1.41}$$

$$\begin{aligned} \mathcal{L}_{gb} &= -\frac{1}{4}\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu} - \frac{1}{4}B_{\mu\nu} \cdot B^{\mu\nu} \\ \mathcal{L}_f &= \sum_f i\bar{f}\gamma_\mu D^\mu f \\ \mathcal{L}_\Phi &= \frac{1}{2}(D_\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi) + \mathcal{L}_Y \\ \mathcal{L}_Y &= h_d(\bar{u}\ \ \bar{d})_L\Phi d_R + h_U(\bar{u}\ \ \bar{d})_L i\tau_2\Phi^* u_R + h_\nu(\bar{\nu}\ \ \bar{e})_L\Phi e_R + h.c. \end{aligned}$$

with $V(\Phi) = -\frac{\mu^2}{2}\Phi^\dagger\Phi + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2$

From $V(\Phi) \quad \Rightarrow \quad <\Phi>= \begin{pmatrix} 0 \\ v \end{pmatrix}$

Physical Gauge Fields

$$W^{\mu}\pm \equiv \tfrac{1}{\sqrt{2}}(A_1^{\mu} \mp A_2^{\mu}) \qquad M_W^2 = \tfrac{g^2}{4}v^2$$

$$Z^{\mu} \equiv \cos\theta_W A_3^{\mu} - \sin\theta_W B^{\mu} \qquad M_Z = \tfrac{M_W}{\cos\theta_W};\ \tan\theta_W \equiv g'/g$$

$$A^{\mu} \equiv \sin\theta_W A_3^{\mu} + \cos\theta_W B^{\mu} \qquad M_A = 0$$

Currents

$$\begin{aligned} J_W^{\mu} &= \frac{g}{\sqrt{2}}\bar{u}_L\ \gamma^{\mu}\ d_L \\ J_A^{\mu} &\equiv J_{em} = e\sum_f \bar{f}\ \gamma^{\mu}\ Q_{em}\ f \\ J_W^{\mu} &= \frac{g}{\cos\theta_W}(J_3\ L - \sin^2\theta_W\ J_{em}) \end{aligned}$$

Physical Higgs

$$\Phi = \begin{pmatrix} 0 \\ v+\eta \end{pmatrix} \quad ; \quad m_{\eta}^2 = 2\lambda v^2 \tag{1.42}$$

$$\mathcal{L}_y = \frac{g}{2M_W}\eta\sum_f m_f\ \bar{f}\ f \tag{1.43}$$

On the positive notice:

- the standard model is a remarkably successful phenomenological theory, since with only two independent parameters M_W and $\sin\theta_W$ it correctly describes all the electroweak processes.
- it has an extremely economical Higgs sector which can account for all the particle masses.
- It is easy to see that both baryon and lepton number are automatically conserved (at least in perturbation theory).

Questions one may pose

- where is the Higgs ?
- could we unify the different gauge couplings g_s, g and g' ? Can we predict $\sin^2\theta_w$?
- can we relate the quark and lepton charges ?
- could we explain the origin of parity and time reversal breaking ? Why are there broken only by the weak interaction ?
- could we predict the quark mass spectrum ? Also the mixing angles in the case of more generations.

As far as the first question is concerned, have patience. In what follows we shall attempt to offer some possible answers to the others.

Chapter 2

Unification: The Basic Concepts

The energy dependence of the $U(1)_{em}$ and $SU(3)_c$ coupling constants discussed in Lecture 2 can be generalized to any group G. The result, which we only quote here, is

$$\frac{1}{\alpha_g(E_2)} = \frac{1}{\alpha_g(E_1)} - \frac{1}{2\pi} b_G \ln \frac{E_2}{E_1} \tag{2.1}$$

where E is to be interpreted as $\sqrt{(-q^2)}$ (for $-q^2 \gg m^2$, where m is the mass of the contributing particle) and

$$b_G \equiv \frac{11}{3} T_{GB}(R) - \frac{4}{3} T_F(R) - \frac{1}{3} T_H(R) \tag{2.2}$$

with

$$T(R)\delta_{ij} = Tr\,(T_i T_j) \tag{2.3}$$

for the representation R; and GB, F anf H refer to the gauge boson, fermionic and Higgs scalar contributions, respectively. Keep in mind please that only the particles with a mass smaller or equal to E_1 will contribute (the particles with $M \geq E_2$ decouple as a result of the renormalizability of the theory). Thus if there are particles with $E_1 \leq m \leq E_2$, one should break the above formula in the regions $E \leq m$ and $E \geq m$ separately, and M will contribute only in the latter region.

Normally gauge bosons are in the adjoint representation of the group G, in which case we denote $T_{GB}(adjoint) = C_2(G)$. For the $SU(N)$ groups which are our primary interest (at least for the time being), we choose for the fundamental representation

$$T(vector) = T(N) = \frac{1}{2} \tag{2.4}$$

You can then show that for the adjoint representation

$$T(adjoint) = T(N^2 - 1) = C_2\left(SU(N)\right) = N \tag{2.5}$$

(we will come back to this).

Let us see now what it implies for the standard model group $G_{st} = SU(3) \times SU(2) \times U(1)_Y$.

STRONG INTERACTIONS: $SU(3)_c$

- GB**:** The gluons belong to the adjoint representation, the octet of $SU(3)$. Now, $T(8) = C_2\left(SU(3)\right) = 3$, which you can easily calculate by decomposing the adjoint representation of $SU(3)$ under $SU(2) \subset SU(3)$, as $8 = 3 + 2 + 2 + 1$.

- F**:** For one generation of fermions, only u_α and d_α contribute. Then

$$T_F(1\ gen) = T_F(3) + T_F(3) = \frac{1}{2} + \frac{1}{2} = 1 \tag{2.6}$$

  A noteworthy remark: if you work with Weyl fermions u_L, u_R, d_L and d_R you must include the factor of $\frac{1}{2}$ for each helicity state, which follows from $TrL(R) = \frac{1}{2}$, where $L(R) = (1 \pm \gamma_5)/2$. Then again

$$T_F(1\ gen) = 4\ \cdot \frac{1}{2}\ \cdot \frac{1}{2} = 1 \tag{2.7}$$

- H**:** There are no colored scalars in G_{st}; therefore $T_H = 0$

We have finally

$$b_3 = \frac{33}{3} - \frac{4}{3} n_g \tag{2.8}$$

which is in full agreement with (63) (n_g is the number of generations). Of course, formula (2.8) is valid above the scale of the heaviest fermion; i.e. only for $E > m_t$, m_t being the mas of the top quark, $m_t \simeq 180 GeV$.

WEAK INTERACTIONS: $SU(2)_L$

- GB**:** Now, $T(3) = C_2\left(SU(3)\right) = 2$
- F**:** The fermions contributing are only the four left-handed doublets (keep in mind that quarks come in colors!), and thus

$$T_F(1\ gen) = 4\,\frac{1}{2}\,\frac{1}{2} = 1 \tag{2.9}$$

  where the last factor of $\frac{1}{2}$ is just due to helicity.
- H**:** We have the standard Higgs doublet, thus $T_H = \frac{1}{2}$.

Then

$$b_2 = \frac{22}{3} - \frac{4}{3}n_g - \frac{1}{6} \tag{2.10}$$

Electromagnetic interactions: $U(1)_{em}$

- GB**:** We have now $C_2\left(U(1)\right) = 0$, but the $W^{\pm}$ bosons carry charge, thus

$$T_{GB} = \frac{11}{3}(Q^2_{W^+} + Q^2_{W^-}) = \frac{22}{3} \tag{2.11}$$

- F**:** For one generation of fermions

$$T_F(1\ gen) = \sum Q^2_{em} = (Q^2_u + Q^2_d)3 + Q^2_e = \frac{5}{3} + 1 = \frac{8}{3} \tag{2.12}$$

- H**:** We have a charged Higgs ϕ^+ in the doublet Φ, and so

$$T_H = Q^2_{\phi^+} = 1 \tag{2.13}$$

Thus

$$b_{em} = \frac{22}{3} - \frac{8}{3}\,\frac{4}{3}n_g - \frac{1}{3} \tag{2.14}$$

Notice the factor $\frac{8}{3}$ discrepancy in the fermionic contribution compared to $SU(2)_L$ and $SU(3)_c$. Why is it there ?

2.1 a. Unification: running of couplings

Imagine that above some energy M_X, you succeed to unify strong and electro-weak interactions, i.e. assume $b_2(E > M_x) = b_3(E > M_x)$. Obviously, no new fermions are needed to achieve that, since $b_2(F) = b_3(F)$ already, but the same is not true of the gauge boson and Higgs contributions.

- *GB*: Imagine a set of new gauge bosons, denoted by X. Obviously, $b_2 = b_3$ implies from $\Delta b_2 = \frac{11}{3} T_2(X)$ and $\Delta b_3 = \frac{11}{3} T_3(X)$

$$T_2(X) = T_3(X) + 1 \tag{2.15}$$

1) One possible (the simplest) solution is

$$T_3(X) = 0 \quad ; \quad T_2(X) = 1 \tag{2.16}$$

implying additional two $SU(2)$ doublets of gauge bosons ($1 = 1/2 + 1/2$). This would correspond to an octet of gauge bosons (if we also add a singlet), or the group $SU(3)_L$. We shall see below that it is impossible to construct a theory with the usual fermions based on $SU(3)_L \times SU(3)_c$.

2) Try then $T_3(X) = \frac{1}{2}$ (*a color triplet*)
But then $T_2 = 0$, and so (2.15) does not work. Thus we need at least $T_3(X) = 1$ or two color triplets X^α and Y^α, but now $T_2(X) = 3/2$, and if we also include the antiparticles $\bar{X}^\alpha$ and $\bar{Y}^\alpha$ we get

$$T_3(X, Y + \bar{Y}, \bar{X}) = 2 \qquad T_2(X, Y + \bar{Y}, \bar{X}) = 3 \tag{2.17}$$

which is OK. Notice that we predict 12 new gauge bosons $\begin{pmatrix} X \\ Y \end{pmatrix}_\alpha$, $\begin{pmatrix} \bar{Y} \\ \bar{X} \end{pmatrix}^\alpha$, where $\alpha = red,\ green,\ blue$, which with the "old ones": 8 gluons and 4 electro-weak bosons, gives altogether 24. Notice also that the group $SU(5)$ has 24 generators, or 24 gauge bosons.

- **H:** since $T_2(\Phi) = 1/2$, $T_3(\Phi) = 0$, we need a new triplet of scalars h_α with $T_3(h) = 1/2$, $T_2(h) = 0$.

In short, we have now (at least) 5 scalar fields. Notice that the fundamental representation of $SU(5)$ is 5-dimensional.

Keep in mind that we expect the new particles to have the mass of order M_X.

In summary, we have for $E > M_X$

$$b_2 = b_3 = \frac{55}{3} - \frac{4}{3}n_g - \frac{1}{6} \tag{2.18}$$

Notice in addition that $C_2\,(SU(5)) = 5$, and $55 = 5 \cdot 11$. Everything we have computed so far points in the direction of the unifying group $SU(5)$.

2.2 b. Unification: physics

Before moving to $SU(5)$, let us see if the smaller group $SU(3)_L \times SU(3)_c$ could work. It does not take long to see that $SU(3)_L$ requires new fermions, since the charge operator Q_{em} must now be in $SU(3)_L$, or $TrQ_{em} = 0$. In order to be as conservative as possible and for the sake of simplicity, we shall try (at least for the time being) to look for theories with only the usual content of fermions; i.e. with three (or more) generations of fermions.

Furthermore we need the fermions of a given representation to have the same helicity, which forces us to work with the fields

$$\begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad , \quad \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{2.19}$$

$$(e^c)_L \quad , \quad (u^c)_L \quad , \quad (d^c)_L \tag{2.20}$$

where

$$(\psi_c)_L \equiv C(\bar{\psi})^T_R \tag{2.21}$$

The quantum numbers of the antiparticles are

	$SU(3)_c$	$SU(2)_L$	$Y/2$
$(e^C)_L$	1_c	1_L	1
$(u^C)_L$	3^*_c	1_L	$-\frac{2}{3}$
$(d^C)_L$	3^*_c	1_L	$+\frac{1}{3}$

Try to put the quarks in a 3-dimensional representation of $SU(3)_L$. Obviously it would have the form

$$3_L = \begin{pmatrix} u \\ d \\ \mathcal{D} \end{pmatrix}_L \tag{2.22}$$

where from $TrQ_{em} = 0$ we get $Q_D = -1/3$ and thus $\mathcal{D} \neq d^c$; hence we have introduced a new quark $\mathcal{D}$ against our rules.

On top of that, in order to have different quark and lepton charges we would need a large representation for leptons or many new fields.

Chapter 3

SU(5): A Prototype GUT

Let us summarize the results of Lecture 3. In order to have a unified theory above some energy scale M_X, with n_g generations of fermions (and no new fermions), we need at least:

- 12 new gauge bosons: an $SU(2)_L$ doublet, color triplet $\begin{pmatrix} X \\ Y \end{pmatrix}_\alpha$ and their antiparticles;
- a color triplet; $SU(2)_L$ singlet of Higgs scalars h_α.

Thus we seem to need at least an $SU(5)$ symmetry. In what follows we shall pursue seriously the idea of $SU(5)$ unification. We shall construct a realistic grand unified theory (GUT) of both strong and electro-weak interactions, and we shall confront it with experiment. We will also show (later) that it is the minimal such theory.

Our task at this point, if not simple, is at least straightforward. We need

1. to put the fermions in the irreducible representation(s) of $SU(5)$ (we will do it generation by generation as in the standard model);
2. to show that the extra 12 gauge bosons have the quantum numbers of X and Y bosons (and their antiparticles)
3. and finally, to show that the $SU(5)$ symmetry can be broken spontaneously down to G_{st} and furthermore down $SU(3)_c \times U(1)_{em}$.

Before the systematic treatment of $SU(5)$, let us attempt to extract as much information as possible from general considerations. Recall only that $SU(5)$ is a group of rank four; it can be generated by 24 (see below) traceless 5×5, matrices which is a reflection of the fact that the $SU(5)$ is a unitary transformation $UU^\dagger = 1$ with $detU = 1$.

It is obvious that we should try to put the electro-weak doublet Φ and the new color triplet h_α in the 5-dimensional fundamental representation

$$5_H = \Phi = \begin{pmatrix} h^r \\ h^g \\ h^b \\ \phi^+ \\ \phi^0 \end{pmatrix} \quad \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} SU(3)_c \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} SU(2)_L \end{array} \tag{3.1}$$

where in the obvious notation the $SU(3)_c$ symmetry is acting on the first 3 components and the $SU(2)_L$ on the last two.

Now, the electromagnetic charge $Q_{em} = T_{3W} + Y/2$ (we call T_{3W} the weak isospin) must be a part of $SU(5)$ and so $TrQ_{em} = 0$. Taking into account that color commutes with $U(1)_{em}$, we predict $Q_h = -1/3$, or in other words

$$Q_{em}(5) = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix} \tag{3.2}$$

for any 5-dimensional representation. From (3.1) we can determine T_{3W}

$$T_{3W}(5) = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & +\frac{1}{2} & \\ & & & & -\frac{1}{2} \end{pmatrix} \equiv \frac{\lambda_{11}}{2} \tag{3.3}$$

where λ_{11} is one of the 24 matrices λ_a generating $SU(5)$ transformations, and so

$$\frac{Y}{2}(5) = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & +\frac{1}{2} & \\ & & & & -\frac{1}{2} \end{pmatrix} \equiv \sqrt{\frac{5}{3}}\frac{\lambda_{12}}{2} \tag{3.4}$$

Here the factor $\sqrt{\frac{5}{3}}$ is needed to ensure that the generator $\lambda_{12}/2$ is normalized as $\lambda_{11}/2$, and for that matter, any of the $SU(5)$ generators

$$Tr\frac{\lambda_a}{2}\frac{\lambda_b}{2} = \frac{1}{2}\delta_{ab} \tag{3.5}$$

$a, b = 1...24$.

But since in $SU(5)$ we have only one coupling constant g ($g_c = g_W = g$), we get from

$$g' \frac{Y}{2} = g \frac{\lambda_{12}}{2} \tag{3.6}$$

an important prediction for the weak mixing angle

$$\begin{aligned} \tan \theta_W &\equiv g'/g = \sqrt{\frac{3}{5}} \qquad \text{or} \\ \sin^2 \theta_W &= \frac{3}{8} \end{aligned} \tag{3.7}$$

Of course (3.7) is valid at M_X, and thus we shall need to run it down to M_W to see if it agrees with experiment.

3.1 Fermions

We have 15 Weyl fields in each generation as shown in (2.19,2.20). It is then only natural to try to put them in a 15-dimensional representation of $SU(5)$. As you know by symmetrizing and antisymmetrizing

$$5 \otimes 5 = 15_s + 10_{as} \tag{3.8}$$

Since $5 = (3_c\,,\, 1_L) + (1_c\,,\, 2_L)$ (in an obvious notation), since $(3_c \otimes 3_c)_s = 6_c$, and since quarks come only in color triplets, we must abandon the idea of 15_S. What about 5 and 10_{as} ?

We know the quantum numbers of 5 from (3.1) and (3.2), implying uniquely

$$5_F \equiv \psi = \begin{pmatrix} d^r \\ d^g \\ d^b \\ e^+ \\ -\nu^C \end{pmatrix}_R \tag{3.9}$$

(recall that $(f^C)_R \equiv C \bar{f}_L$).

Now, from $\psi \longrightarrow U\psi$ under $SU(5)$, the 10-dimensional representation χ must transform as

$$\chi \longrightarrow U \, \chi \, U^\dagger \tag{3.10}$$

This is enough to give the quantum numbers of the particles in 10

$$\chi = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_b^C & -u_g^C & -u^r & -d^r \\ -u_b^C & 0 & u_r^C & -u^g & -d^g \\ u_g^C & -u_r^C & 0 & -u^b & -d^b \\ u^r & u^g & u^b & 0 & e^+ \\ d^r & d^g & d^b & -e^+ & 0 \end{bmatrix}_L \tag{3.11}$$

The factor $1/\sqrt{2}$ will be clear later. Notice that in (3.9), a minus sign convention fo the ν^C field is to ensure that $\begin{pmatrix} e^+ \\ -\nu^C \end{pmatrix}_R$ and $\begin{pmatrix} e \\ \nu \end{pmatrix}_L$ transform identically, and in (3.11) the signs are the property of χ being antisymmetric. To reproduce (3.11) you may notice that from (3.10), a typical generator of $SU(5)$ will act on Ξ as

$$T_a \chi = \frac{\lambda_a}{2}\chi + \chi \frac{\lambda_a^T}{2} \tag{3.12}$$

which for a diagonal matrix such as Q_{em} would imply

$$(Q_{em}\chi)_{ij} = (Q_i + Q_j)\chi_{ij} \tag{3.13}$$

where Q_i are the elements of Q_{em} for the 5-representation, formula (3.2). We will often in the future work with χ and $\bar{5}_F$

$$\psi^C = \begin{pmatrix} d_r^C \\ d_g^C \\ d_b^C \\ e \\ -\nu \end{pmatrix}_L \tag{3.14}$$

In any case, it is clear that the fermions of each generation fit correctly into $\bar{5}+10$. Most importantly, a unified theory such as $SU(5)$ explains charge quantization, i.e. it relates the quark and lepton charges. From (3.14)

$$Q(d^C) = -\frac{1}{3}Q(e) = \frac{1}{3} \tag{3.15}$$

and then from (3.11) we see that $Q(u) = Q(d) + 1 = 2/3$.

Furthermore, from (3.14) we see that $(d^C)_L \equiv C\bar{d}_R^T$ must be a singlet under $SU(2)_L$. In other words the group theoretic structure *implies* the V-A structure of the weak currents, i.e. the fact that the right-handed fermions are $SU(2)_L$ singlets. These are two important successes of our desire to unify $SU(3)_c$ and $SU(2)_L$; before, these facts were put in by hand.

3.2 Gauge Bosons

In order to proceed with the systematic study of the structure and interactions in $SU(5)$ we need to find the physical gauge bosons. For this we need the form of the generators of $SU(5)$ for the 5-dimensional representation; i.e. the 24 matrices λ_a. It is easy to write down the 12 of these that correspond to rotations in the $SU(3)_c$ and $SU(2)_L \times U(1)$ space.

$\mathbf{SU(3)_c}$

$$\lambda_{1\ldots8} = \begin{pmatrix} & & & 0 & 0 \\ & \lambda^c_{1\ldots8} & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{O}} \\ 0 & 0 & 0 & & \end{pmatrix} \tag{3.16}$$

where $\lambda^c_{1\ldots8}$ are the matrices which generate the $SU(3)_c$ transformations, given in Lecture 2. As an illustration,

$$\lambda_3 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \tag{3.17}$$

$\mathbf{SU(2)_L \times U(1)}$

Similarly, $\lambda_{9\ldots11}$ generate $SU(2)_L$ rotations, i.e

$$\lambda_{9\ldots11} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \left(\ \tau_i \ \right) \end{pmatrix} \tag{3.18}$$

Again, as an example,

$$\lambda_{10} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -i \\ & & & i & 0 \end{pmatrix} \tag{3.19}$$

and λ_{11} was given before, see eq. (3.3).

The hypercharge generator is $Y/2 = \sqrt{\frac{5}{3}}\lambda_{11}/2$ and

$$\lambda_{12} = \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{2}{3} & & & & \\ & -\frac{2}{3} & & & \\ & & -\frac{2}{3} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \tag{3.20}$$

(X, Y) **generators**

Finally, the rest of the 24 matrices $\lambda_{13\ldots24}$ act in the off-diagonal (3,2) space (and will be shown to correspond to the X and Y gauge bosons); and are obtained by inserting the Pauli matrices appropriately; e.g.

$$\begin{array}{cc}
\lambda_{13} = \left(\begin{array}{ccc|cc} & & & 1 & 0 \\ & O & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & O \end{array}\right), & \lambda_{14} = \left(\begin{array}{ccc|cc} & & & -i & 0 \\ & O & & 0 & 0 \\ & & & 0 & 0 \\ i & 0 & 0 & & \\ 0 & 0 & 0 & & O \end{array}\right) \\
\vdots & \vdots \\
\lambda_{23} = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & O & & 0 & 0 \\ & & & 0 & 1 \\ 0 & 0 & 0 & & \\ 0 & 0 & 1 & & O \end{array}\right), & \lambda_{24} = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & O & & 0 & 0 \\ & & & 0 & -i \\ 0 & 0 & 0 & & \\ 0 & 0 & i & & O \end{array}\right)
\end{array} \tag{3.21}$$

You can see that we have four diagonal λ matrices: λ_3, λ_8, λ_{11} and λ_{12}, which is a reflection of the fact that the rank of $SU(5)$ is 4. This is yet another way of seeing that $SU(5)$ is big enough to unify the rank four group G_{st}.

Now, from the form of a covariant derivative for a fundamental representation

$$D_\mu = \partial_\mu - ig\sum_{a=1}^{24} \frac{\lambda_a}{2} A_\mu^a \equiv \partial_\mu - ig\mathcal{A}_\mu \tag{3.22}$$

the matrix of the gauge bosons becomes

$$
\mathcal{A}_\mu \equiv \frac{\lambda_a}{2} A^a_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} & & & \bar{X}_r & \bar{Y}_r \\ & gluons & & \bar{X}_g & \bar{Y}_g \\ & & & \bar{X}_b & \bar{Y}_b \\ X^r & X^g & X^b & \frac{1}{\sqrt{2}} W_3 & W^+ \\ Y^r & Y^g & Y^b & W^- & -\frac{1}{\sqrt{2}} W_3 \end{pmatrix}
+ \sqrt{\frac{3}{5}} \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix} B_\mu \tag{3.23}
$$

where the X and Y bosons clearly carry color. From the transformation properties

$$
\mathcal{A} \longrightarrow U \, \mathcal{A} \, U^\dagger \tag{3.24}
$$

under *global* $SU(5)$ rotations, which is equivalent to the fact that $1 + 24 = 5 \otimes \bar{5}$, one can deduce that the charges of X and Y are $4/3$ and $1/3$, respectively ($\bar{X}$ and $\bar{Y}$ are their antiparticles). You will find it useful to notice that

$$
(Q_{em}\mathcal{A})_{ij} = (Q_i - Q_j)\mathcal{A}_{ij} \tag{3.25}
$$

in reproducing correctly the charges of the gauge bosons. Finally, it is noteworthy that the $X(\bar{X})$ and $Y(\bar{Y})$ states correspond to $\lambda_{13} \pm i\lambda_{14}$, ..., $\lambda_{23} \pm i\lambda_{24}$.

$SU(5)$: Interactions

We are now fully armed to compute the interactions of fermions with gauge bosons. As we know, the Lagrangian of any gauge theory can be written as

$$
\mathcal{L} = \mathcal{L}_{gb} + \mathcal{L}_f + \mathcal{L}_\Phi + \mathcal{L}_Y \tag{3.26}
$$

where

$$
\mathcal{L}_{gb} = -\frac{1}{4} F^a_{\mu\nu} \cdot F^{\mu\nu,a} \tag{3.27}
$$

where

$$
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^C_\nu \tag{3.28}
$$

where f^{abc} are the structure constants

$$f_{abc} = \frac{1}{4i} Tr \left\{ \lambda_a, [\lambda_b, \lambda_c] \right\} \tag{3.29}$$

The Higgs and the Yukawa parts of the Lagrangian, $\mathcal{L}_\Phi$ and $\mathcal{L}_Y$ will be presented later. Now,

$$\mathcal{L}_f = i\bar{\psi}\gamma^\mu D_\mu \psi - itr\bar{\chi}\gamma^\mu D_\mu \chi \tag{3.30}$$

where from (3.10) and (3.22)

$$D_\mu \chi = \partial_\mu \chi - ig(\mathcal{A}_\mu \chi + \chi \mathcal{A}_\mu^T) \tag{3.31}$$

It is easy to see now why we needed a factor of $1/\sqrt{2}$ in the definition of χ, for it renders a correct normalization of the physical fields in χ.

From (3.30), (3.22) and (3.31), after some straightforward computational tedium one can derive all the gauge boson-fermion interactions.

There are of course the old QCD and $SU(2)_L \times U(1)$ interactions with $g_s = g_W = g$, and $sin^2\theta_W = 3/8$; these are the parameters of the (large) energy scale M_X where all the interactions are unified. Next, we have the X and Y boson interactions which we can readily write down

$$\begin{aligned} \mathcal{L}(X,Y) &= \frac{g}{\sqrt{2}} \bar{X}_\mu^\alpha \left[\bar{d}_{\alpha R}\gamma^\mu e_R^+ + \bar{d}_{\alpha L}\gamma^\mu e_L^+ + \epsilon_{\alpha\beta\gamma}\bar{u}_L^{c\gamma}\gamma_\mu u_{\beta L} \right] \\ &+ \frac{g}{\sqrt{2}} \bar{Y}_\mu^\alpha \left[-\bar{d}_{\alpha R}\gamma^\mu \nu_R^C + \bar{u}_{\alpha L}\gamma^\mu e_L^+ + \epsilon_{\alpha\beta\gamma}\bar{u}_L^{c\gamma}\gamma_\mu d_{\beta L} \right] + \quad h.c. \end{aligned} \tag{3.32}$$

The above interactions are simply the $SU(3)_c \times SU(2)_L \times U(1)$ gauge invariant terms of the gauge bosons X and Y with charges $+4/3$ and $1/3$ respectively.

As expected, due to the nontrivial color and flavor characteristics of the quarks, the X and Y couple to the quark-quark and quark-lepton states. Let us write down the baryon and lepton numbers associated with the vertices in (3.32)

$B:$	$2/3$	$-1/3$	$2/3$
$L:$	0	-1	0
$B-L:$	$2/3$	$2/3$	$2/3$

$B:$	$-1/3$	$-1/3$
$L:$	-1	-1
$B-L:$	$2/3$	$2/3$

where we used $B_q = -B_{\bar{q}} = 1/3$, $B_e = 0; L_q = 0$, $L_e = -L_{e^C} = 1$. It is clear that B and L are violated, although for some magic reason $B - L$ is conserved (more about it later).

The outcome of $\Delta B = \Delta L \neq 0$ is dramatic for it is easily seen to lead to the decay of the proton. From (3.32) we get

By analogy with the usual weak decay $n \to p + e + \bar{\nu}$, $\mu \to e + \bar{\nu}_e + \nu_\mu$ we know that the decay rate of the proton can be estimated as

$$\Gamma_p \simeq \frac{g^4}{M_X^4} m_p^5 \qquad (3.33)$$

From $(\tau_p)_{exp} > 10^{32} yr$ we get $M_X > 10^{15} GeV$; later we will show that we can actually compute M_X. In any case, the $SU(5)$ grand unification requires the existence of a new scale of physics associated with the mass of X and Y bosons some *fifteen orders of magnitude bigger than the electro-weak scale.* Furthermore, it predicts a "desert", i.e. no new physics whatsoever between M_W and M_X.

Before we turn to the detailed discussion of the predictions of M_X and $sin^2\theta_W$, let us first make sure that we can actually construct a realistic $SU(5)$ theory. Namely, we must break the symmetry spontaneously in order to ensure renormalizability and we must achieve it at the scale $M_X > 10^{15} M_W$. Can we do it at all, and if yes could we do it naturally?

Chapter 4

Symmetry Breaking of $SU(5)$

We have seen that the $SU(2)_L \times U(1)$ Higgs doublet ϕ has a natural embedding in a $\phi \equiv 5_H$ of $SU(5)$. Thus a nonvanishing vev $< \Phi >= \begin{pmatrix} 0 \\ < \phi > \end{pmatrix} \neq 0$ will lead to the breaking of $SU(2)_L \times U(1)$, but the question is: how to break $SU(5)$ down to $G_{st} = SU(3)_c \times SU(2)_L \times U(1)$?

Let us analize the possible Higgs representations by the degree of minimality, i.e. we are looking for the smallest representation Σ such that

$$G \ = SU(5) \quad \underset{< \Sigma >}{\longrightarrow} \quad G_{st}$$

a) $\Sigma = 5_H$

In this case $< \Sigma > \neq 0$ breaks either $SU(3)_c$ or $SU(2)_L$, an unacceptable feature

b)$\Sigma = 10_H$

Then the charge assignment in Σ is the same as of χ; thus there is no Higgs scalar which is both neutral and invariant under $SU(3)_c \times SU(2)_L$ (the analog of e_L^+ is an $SU(2)_L$ singlet, but it carries charge).

c)$\Sigma = 15_H$

Again, a simple analysis of the quantum numbers shows that it cannot work, much as in case **b)**.

d)$\Sigma = 24_H$: the case of an adjoint Higgs.

Now, Σ can be written as $\Sigma_{24} = (\lambda_a/2)\Sigma_a$, and obviously the direction

$$< \Sigma_{24} >= \frac{\lambda_{12}}{2}\Sigma_{12} \propto Y \qquad (4.1)$$

commutes with $SU(3)_c \times SU(2)_L \times U(1)$, i.e. (4.1) leaves G_{st} unbroken. Can it be achieved ?

Recalling the transformation property of Σ_{24} under $SU(5)$,

$$\Sigma \longrightarrow U\, \Sigma\, U^{\dagger}\ , \quad \text{where} \quad U = e^{-i\lambda_a/2\,\theta_a} \tag{4.2}$$

we can write down the most general renormalizable potential for Σ. Assume, only for the sake of simplicity, the discrete symmetry $\Sigma \to -\Sigma$ (we skip the index 24 from now onwards). Then we write,

$$V(\Sigma) = -\frac{\mu^2}{2}\, Tr\Sigma^2 + \frac{1}{4} a\, (Tr\Sigma^2)^2 + \frac{1}{2} b\, Tr\Sigma^4 \tag{4.3}$$

Now, since Σ is a Hermitean matrix it can be diagonalized by an $SU(5)$ rotation, thus we can always choose, without any loose of generality: $< \Sigma >=$ diagonal.

Assume now that it is in the same direction as the hypercharge: $\Sigma \propto Y$ or

$$< \Sigma >= v_X \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix} \tag{4.4}$$

From (4.3) you get then

$$\mu^2 = \frac{1}{2}(15a + 7b)\, v_X^2 \tag{4.5}$$

which, for $\mu^2 > 0$, implies

$$(15a + 7b) > 0 \tag{4.6}$$

In order to check that (4.6) is a local minimum, we must show that all the second derivatives are positive. Since Σ has exactly the same form as the gauge boson matrix in (3.23), we can write

$$\Sigma =< \Sigma > + \begin{pmatrix} \Sigma_8 + \sqrt{\frac{3}{5}}\left(-\frac{2}{3}\right) 1_c \Sigma_0 & \bar{\Sigma}_X & \bar{\Sigma}_Y \\ \Sigma_X & \sqrt{\frac{1}{2}}\Sigma_3 + \sqrt{\frac{3}{5}}\Sigma_0 & \Sigma^+ \\ \Sigma_Y & \Sigma^- & -\sqrt{\frac{1}{2}}\Sigma_3 + \sqrt{\frac{3}{5}}\Sigma_0 \end{pmatrix} \tag{4.7}$$

where Σ_8 are the analogs of gluons, Σ_X and Σ_Y the analogs of X and Y, Σ_3, Σ^+, Σ^- and Σ_0 the analogs of W^3, W^+, W^- and B, respectively. Now, computing second derivatives is equivalent to computing the effective mass terms for the Σ fields.

For example, let us compute the mass of the Σ_3 field. We know that $SU(2)_L$ is unbroken and thus Σ_3 cannot mix with any other field. In this case we can keep only

$$\Sigma_{(\text{the } \Sigma_3 \text{ part})} = \begin{pmatrix} v_X & & & & \\ & v_X & & O & \\ & & v_X & & \\ & O & & \frac{1}{2}\Sigma_3 - \frac{3}{2}v_X & \\ & & & & -\frac{1}{2}\Sigma_3 - \frac{3}{2}v_X \end{pmatrix} \tag{4.8}$$

and then

$$\begin{aligned} Tr\,\Sigma^2 &\longrightarrow \frac{1}{2}\Sigma_3^2 + \frac{15}{2}v_X^2 \\ \left(Tr\,\Sigma^2\right)^2 &\longrightarrow \frac{15}{2}v_X^2\Sigma_3^2 \\ Tr\,\Sigma^4 &\longrightarrow 2\cdot 6\cdot\frac{1}{4}\cdot\frac{9}{4}v_X^2\Sigma_3^2 \end{aligned} \tag{4.9}$$

where we keep only terms quadratic in the field. From (4.3) we get

$$\begin{aligned} V(\Sigma) \ &\longrightarrow \Sigma_3^2\left(-\tfrac{1}{4}\mu^2 + \tfrac{15}{8}a\,v_X^2 + \tfrac{27}{8}b\,v_X^2\right) \\ &\qquad = \tfrac{5}{2}b\,v_X^2 \end{aligned} \tag{4.10}$$

where we have used eq. (4.5).

Similarly, one can derive all the other particle masses in Σ and the result is

$$\begin{aligned} m^2(\Sigma_8) &= \frac{5}{4}b\,v_X^2 \\ m^2(\Sigma_3) &= m^2(\Sigma_\pm) \;=\; 5b\,v_X^2 \\ m^2(\Sigma_0) &= \frac{15a+7b}{2}\,v_X^2 \\ m^2(\Sigma_X) &= m^2(\Sigma_Y) \;=\; 0 \end{aligned} \tag{4.11}$$

Thus for

$$15a + 7b > 0 \quad , \quad b > 0 \tag{4.12}$$

the extremum (4.4) is a local minimum of the theory. Notice that Σ_X and Σ_Y are would-be Goldstone bosons of the theory; they get "eaten" by the X and Y gauge fields, i.e. they become their longitudinal components.

Finally, one can show that for (4.12), the vev (4.4) of Σ is actually a global minimum. In fact, other extrema can be shown to be at best saddle points (prove it).

Thus $SU(5)$ can be successfully broken down to the G_{st} group of the standard model, since as we said Y commutes with both the $SU(3)_c$ and $SU(2)_L \times U(1)_Y$ generators. This will be even more evident from the study of the gauge bosons mass matrix. Since Σ is in the adjoint representation

$$D_\mu \Sigma = \partial_\mu \Sigma - ig[\mathcal{A}_\mu, \Sigma] \tag{4.13}$$

and from $\frac{1}{2}(D_\mu \Sigma)^\dagger (D^\mu \Sigma)$ one obtains

$$\frac{1}{2}(D_\mu < \Sigma >)^\dagger (D^\mu < \Sigma >) = \frac{25}{8} g^2 v_X^2 \left[\bar{X}_\mu^a X_a^\mu + \bar{Y}_\mu^a Y_a^\mu \right] \tag{4.14}$$

where a as usual is the color index, $a = r,\ g,\ b$. As expected, the gluons and the electro-weak gauge bosons remain massless, but X and Y get equal masses

$$m_X^2 = m_Y^2 \equiv M_X^2 = \frac{25}{8} g^2 v_X^2 \tag{4.15}$$

as a consequence of both $SU(3)_c$ and $SU(2)_L$ remaining unbroken, X and Y must have the same masses, and equal for different colors. The original $SU(5)$ symmetry is broken down to

$$SU(5) \quad \underset{< \Sigma > \propto Y}{\longrightarrow} \quad SU(3)_c \times SU(2)_L \times U(1)_Y$$

The rest of the breaking will be completed by a 5-dimensional Higgs multiplet Φ_5 which contains the Standard Model doublet. Let us study this in some detail including the full $SU(5)$ invariant potential.

We can write

$$\begin{aligned} V(\Sigma, \Phi) &= -\frac{\mu_\Sigma^2}{2}\, Tr\Sigma^2 + \frac{1}{4} a(\, Tr\Sigma^2)^2 + \frac{1}{2} b\, Tr\Sigma^4 \\ &- \frac{\mu_\Phi^2}{2} \Phi^\dagger \Phi + \frac{\lambda}{4} (\Phi^\dagger \Phi)^2 \\ &+ \alpha \Phi^\dagger \Phi\, Tr\Sigma^2 - \beta \Phi^\dagger \Sigma^2 \Phi \end{aligned} \tag{4.16}$$

with $a > 0$, $\lambda > 0$, $15a + 7b > 0$ and $\beta > 0$ (see below). Thus $< \Sigma >$ is in the direction given by (4.4). Since both $SU(3)_c$ and $SU(2)_L$ are unbroken at this point, we can always rotate $< \Phi >$ into the form

$$< \Phi >= \begin{pmatrix} v_c \\ 0 \\ 0 \\ 0 \\ v_W \end{pmatrix} \tag{4.17}$$

it is only the β term that is sensitive to the direction of $< \Phi >$ and it gives $-\beta v_X^2(v_c^2 + 9/4 v_W^2)$, which for $\beta > 0$ forms the solution $v_W \neq 0$, $v_c = 0$ (in order to minimize the energy). Thus

$$SU(3)_c \times SU(2)_L \times U(1)_Y \qquad \longrightarrow \qquad SU(3)_c \times U(1)_{em}$$

$$< \Phi >= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_W \end{pmatrix}$$

It is an easy exercise to compute the mass of the colored triplet scalar h_a in $< \Phi >$

$$m_h^2 = \frac{5}{2}\beta v_X^2 \tag{4.18}$$

which again justifies the choice $\beta > 0$. Furthermore, you can show that $\phi^{\pm}$ and $\mathrm{Im}\phi^0$ are eaten by $W^{\pm}$ and Z respectively, and $\eta = \mathrm{Re}\phi^0$ is the physical Higgs with $m_\eta^2 = 2\lambda v_W^2$.

At this point we could almost feel euphoric. We have managed to construct what appears to be a fully realistic $SU(5)$ theory which furthermore is broken down in stages to $SU(3)_c \times U(1)_{em}$. However, not all is perfect. If you look at the value for v_W from (4.16)

$$\lambda v_W^2 = \mu_\Phi^2 + (\frac{9}{2}\beta - 15\alpha)v_X^2 \tag{4.19}$$

and recall that $M_W^2 = (g^2/4)v_W^2$, $M_X^2 = (25g^2/8)v_X^2$, you get

$$M_W^2 = \frac{g^2}{4\lambda}\left[\mu_\Phi^2 + \frac{8M_X^2}{25g^2}(-15\alpha + \frac{9}{2}\beta)\right] \tag{4.20}$$

But $M_X > 10^{15} GeV$, which implies an extraordinary fine-tuning in the above equation of at least 26 orders of magnitude. The number on the right hand side of (4.20) is naturally of order $M_X^2 > 10^{30} GeV^2$; instead it ends up being $\simeq (100GeV)^2$. This is known as *the hierarchy problem*.

In the next lecture we will see that the colored triplet h_a mediates proton decay and thus it must be very heavy: $m_h > 10^{11} GeV$, implying that β cannot be taken arbitrarily small. On the other hand, it's partner η weighs $< 1TeV$, and this aspect of the hierarchy problem is known as the *doublet-triplet splitting problem*.

Before we close this section, let us say a few words more on the hierarchy problem. The problem is that the mass term for the Higgs scalars cannot be made small (or zero) by any symmetry, unlike the case of fermions. There the limit $m_f = 0$ corresponds to the chiral symmetry $f \to \gamma_5 f$, and thus the higher order corrections must also vanish if $m_f = 0$ at the tree level. In other words, the higher order corrections are necessarily proportional to $m_f(tree)$, and so only logarithmically divergent. In the case of scalars the divergence is quadratic and thus in the context of grand unified theories (GUTs) such as $SU(5)$ the natural value for M_W is of order M_X.

Imagine then that you live in a world where there is a symmetry between each boson and every fermion. (this symmetry is called *Supersymmetry*). Then by definition the masses of scalars would be protected from the large scale M_X, just as the fermionic ones are.

Could our world be supersymmetric? The answer is no, since according to such theories there must be a charged scalar, called *selectron*, whose mass is $m_{\tilde{e}} = m_e = 0.5MeV$. Such a particle does not exist.

Can you think of a way out of this impasse, which could save the idea of supersymmetry ?

More on the Hierarchy Problem

Let us try to pay some more attention to this notorious issue in GUTs. We have seen that in eq. (4.20) we must fine-tune the parameters of the right-hand side to 26 decimal places: either $\mu_\phi^2 \sim v_X^2$ and then they must pretty much cancel, or $\mu_\phi^2 \sim v_W^2$ and then $-5\alpha + 9/2\beta$ must be of order 10^{-26}. In any case, it is a disgusting feature of the theory.

Well, you may say, let us do it anyway, since we are already used to the fact that the electron mass $m_e = .5MeV$ in the standard model is fine-tuned to about 5 decimal places compared to M_W. The point, of course, is the profound difference between the fermion and scalar masses: setting $m_f = 0$ increases the symmetry of the theory, whereas there is no symmetry which can forbid the mass term for the scalars.

Now, the great virtue of gauge theories is that spontaneous symmetry breaking does not induce any new divergences, in other words we need only include symmetric counterterms. Thus if you set $m_f = 0$ which amounts to the symmetry $f \to \gamma_5 f$, it cannot be infinitely renormalized. Let us illustrate this on a simple Lagrangian

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_Y \tag{4.21}$$

where

$$\begin{aligned} \mathcal{L}_\phi &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m_s^2}{2}\phi^2 + \frac{\lambda}{4}\phi^2 + \frac{M}{3}\phi^3 \\ \mathcal{L}_\psi &= \bar\psi\gamma_\mu\partial^\mu\psi + m_\psi\bar\psi\psi \\ \mathcal{L}_Y &= h\bar\psi\psi\,\phi \end{aligned} \tag{4.22}$$

and ϕ is the real scalar field.

Imagine now that you set $m_s = 0$. Of course, you get no symmetry increase ($\phi \to i\phi$ is not allowed, since ϕ is real), but still let us take it. You will find immediately a quadratic divergent contribution to ϕ^2 from a diagram

For a zero external momentum, we can estimate the above diagram as

$$\begin{aligned}
&\propto \quad \phi^2 \lambda \int d^4 k \frac{1}{k^2} \propto \phi^2 \lambda \int^{\Lambda^2} k^2 \, \frac{dk^2}{k^2} \\
&\propto \quad \phi^2 \lambda \Lambda^2
\end{aligned} \tag{4.23}$$

which for $\lambda \to \infty$ diverges quadratically as we said. Thus, it makes no sense to say that $m_s = 0$ for we need a counterterm to the above λ^2 term.

Let us now look at the same question for the fermion mass. At the one-loop level we have a diagram

whose divergence is easily seen to vanish

$$\bar{\psi} \psi \, h^2 \int d^4 k \frac{\not{k}}{k^2} \frac{1}{k^2} = 0 \tag{4.24}$$

since $\int f(k^2) \not{k} dk^2 = 0$ in the symmetric integration. In the above we have set $m_\psi = m_s = 0$.

However, there is a divergent two-loop diagram

which can be estimated as

$$\begin{aligned}\bar{\psi}\psi\, h^3\, M \int d^8k \frac{\not{k}\ \not{k}}{(k^2)^2}\frac{1}{(k^2)^3} \quad &\propto \quad \bar{\psi}\psi\, h^3\, M \int^{\Lambda^2} dk^2 \\ &= \quad \bar{\psi}\psi\, h^3\, M \ln \Lambda^2 \end{aligned} \tag{4.25}$$

The divergence is softer than in the scalar case being only logarithmic, but anyway there is no sense in setting $m_\psi = 0$ since we get an infinite correction. As before, the reason is the lack of a protective symmetry which could keep $m_\psi = 0$ in perturbation theory.

It is easy to find such a protective symmetry. Set both $m_\psi = 0$ and $M = 0$ and we get an extra symmetry D:

$$\begin{aligned} D \quad : \quad & \psi \to \gamma_5 \psi \ \ (\bar{\psi}\psi \to -\bar{\psi}\psi) \\ & \phi \to -\phi \end{aligned}$$

Now, obviously (4.25) vanishes and thus $m_\psi = 0$ is left intact.

The message we are trying to bring across here is that in order to keep some quantity small in perturbation theory, a protective symmetry is required. This is the source of the hierarchy problem: by what symmetry can we keep a mass of ϕ (or v_W) small in perturbation theory ? How to prevent it from diverging quadratically ?

In order to appreciate the point of the trouble that the quadratic divergences carry, let us take a simple example of two scalar fields, one light (ϕ_L) and one heavy (ϕ_H). Thus the Lagrangian would have the form

$$\mathcal{L}_(\phi_L, \phi_H) \quad = \quad \frac{1}{2}(\partial_\mu \phi_L)^2 + \frac{1}{2}(\partial_\mu \phi_H)^2 - V(\phi_L, \phi_L)$$

$$V(\phi_L, \phi_H) = \frac{1}{2}m_L^2\phi_L^2 + \frac{1}{2}m_L^2\phi_H^2 + \frac{1}{4}\lambda_L\phi_L^4 + \frac{1}{4}\lambda_H\phi_H^4 + \frac{1}{2}\lambda\phi_L^2\phi_H^2 \tag{4.26}$$

where we have assumed discrete symmetries $\phi_L \to -\phi_L$, $\phi_H \to -\phi_H$ (separately) for simplicity and $m_H \gg m_L$. Then, among others, we will find the following diagram for the mass of ϕ_L at the one-loop level

whose quadratic divergence will be given by

$$\begin{aligned} &\propto \phi_L^2\lambda \int d^4k \frac{1}{k^2 - m_H^2} \propto \phi_L^2\lambda \int^{\Lambda^2} k^2 \,\frac{dk^2}{k^2 - m_H^2} \\ &= \phi_L^2\lambda \int_{m_H^2}^{\Lambda^2} (k^2 + m_H^2)\frac{dk^2}{k^2} \propto \phi_L^2\lambda(\Lambda^2 - m_H^2) \end{aligned} \tag{4.27}$$

You see that the quadratic divergence brings along a sensitivity to a large scale m_H. In other words, the mass for the ϕ_L is naturally of order m_H and in order to keep it small, $\sim m_L$, we must fine-tune the parameters: $\lambda\, m_H^2 \simeq m_L^2$.

Of course, the same would be true for any heavy particle in the loop, independently of whether it is a boson or a fermion. If $m_H = M_X$, $m_L = M_W$, we have the infamous hierarchy problem of GUTs.

Now, imagine the existence of a fermion ψ coupled to ϕ_L as in (4.22). We then have an additional quadratically divergent contribution to m_L^2 for the following diagram which can be estimated as (the "-" sign is the consequence of Fermi statistics)

$$
\begin{aligned}
& - \quad \phi_L^2\, h^2 \int d^4k \frac{(\not k + m_f)(\not k + m_f)}{(k^2 - m_f^2)^2} \\
& \propto \quad -\phi_L^2\, h^2 \int_0^{\Lambda^2} dk^2 \frac{k^4}{(k^2 - m_f^2)^2} = -\phi_L^2\, h^2 \int_{m_f^2}^{\Lambda^2} dk^2 \frac{(k^2 - m_f^2)^2}{k^4} \\
& = \quad -\phi_L^2\, h^2 (\Lambda^2 - m_f^2)
\end{aligned}
\tag{4.28}
$$

Notice that for $h^2 = \lambda$ we have a cancellation of a quadratic divergence for the bosonic contribution of ϕ_H in (4.27).

What we obtain then is

$$
\phi_L^2\, h^2 (m_f^2 - m_H^2) \tag{4.29}
$$

which would vanish for $m_f = m_H$. This limit of bosonic and fermionic masses and couplings equal is the manifestation of *supersymmetry*, the symmetry between bosons and fermions. As we said this symmetry must be broken, for the mass of the selectron $\tilde{e}$, the supersymmetric partner of the electron, is different from the electron mass: $m_{\tilde{e}} \gg m_e$ (or better to say: we are yet to discover the selectron).

The question si at which scale is this symmetry broken, i.e. how large is the mass difference between m_f and m_H ?

Well, let $\phi_L = \eta$, the Higgs field of the standard model. But then $m(\phi_L) \simeq M_W$, and thus in order to keep this scale naturally small compared to $M_X = m_H$, we must demand

$$
m_f^2 - m_H^2 \simeq M_W^2 \tag{4.30}
$$

or

$$
\Lambda_{SS} \simeq M_W \tag{4.31}
$$

where Λ_{SS} is the scale of breaking of supersymmetry.

This is a remarkable result for it would imply a whole new world of the supersymmetric partners of the known particles e, ν, u, d, W, etc. to be found in the near future. An optimist like Glashow would say that we have already discovered a half of the predicted particles !

Chapter 5

Yukawa Couplings and Fermion masses in $SU(5)$

In order for $SU(5)$ to be a realistic theory, it must allow for the fermionic masses, mixing angles and phases. Since the fermions are in the 5 and 10 -dimensional representations ψ and χ, obviously no direct mass term can be gauge invariant, just like in the standard model. This is an important feature for otherwise there would be no explanation of why the fermions do not pick up a large mass of order M_X. This is to say that the fermions are chiral, i.e. the left and right fermions belong to different representations of the gauge group.

In the Standard Model the left-handed fermions are doublets and the right-handed fermions are singlets, and so their chiral property is more than manifest. In $SU(5)$ again, since the V-A structure of a family of fermions is left-intact, the same must obviously be true.

Using the decomposition properties (please check it)

$$\begin{aligned} \bar{5} \otimes 10 &= 5 \oplus \bar{45} \\ 10 \otimes 10 &= \bar{5} \oplus 45 \oplus 50 \\ \bar{5} \otimes \bar{5} &= \bar{10} \oplus \bar{15} \end{aligned} \tag{5.1}$$

the above statements are easily confirmed. In the minimal $SU(5)$ theory the fermion masses may originate only through the coupling to the 5-dimensional Higgs representation Φ_5 (since we assume no 10, 45 or 50 of Higgs). Obviously Σ_{24} decouples from the fermions as it should, for it would give them masses of order M_X.

From (5.1) we can then write the Yukawa couplings of fermions with the light Higgs Φ:

$$\mathcal{L}_Y = f_d \, \bar{\psi}_R \, \chi \, \Phi^\dagger + f_u \frac{1}{2} \chi^T \, C \, \chi \, \Phi + h.c. \tag{5.2}$$

where C is the Dirac conjugation matrix needed for the sake of Lorentz invariance. The symbolic notation of (5.2) should read in the $SU(5)$ notation as

$$\begin{aligned} \bar{\psi}_R \, \chi \, \Phi^\dagger &= \bar{\psi}_{R\,i} \, \chi^{ij} \, \Phi^\dagger_j \\ \chi^T \, C \, \chi \, \Phi &= \epsilon_{ijklm} \, (\chi^T)^{ij} \, C \, \chi^{kl} \, \Phi^m \end{aligned} \tag{5.3}$$

When Φ gets its vacuum expectation value $< \Phi >^T = (\ 0\ 0\ 0\ 0\ v_W\ )$, we get from (5.2) and (5.3) for the fermionic masses

$$\begin{aligned} \mathcal{L}_m &= f_d v_W (\bar{d}_R d_L + \bar{e}^+_R e^+_L) - f_u v_W (u^c)^T_L \, C \, u_L + h.c. \\ &= -[f_d v_W (\bar{d} d + \bar{e} e) - f_u v_W \bar{u} u] \end{aligned} \tag{5.4}$$

In other words we get just as in the Standard Model $m_f = h_f v_W$, except that we predict the electron and down quark masses being equal. Can you explain why this happens ? This prediction appears very bad (we know that $m_d \simeq 10 MeV$, $m_e \simeq 0.5 MeV$), but we must recall that it is valid only at the large scale M_X where the whole $SU(5)$ symmetry becomes operative.

More precisely, one has to "run" down to low energies by including the radiative corrections. Since the quarks have color their effective mass gets increased and a precise calculation shows

$$(\frac{m_d}{m_e})_{ren} \simeq 3 \tag{5.5}$$

Notice that this works very well for the third generation: $m_b \simeq 5GeV$, $m_\tau \simeq 1.8GeV$; not so well for the second one: $m_s \simeq 150MeV$, $m_\mu \simeq 100MeV$; and badly for the lightest one. It is hard to decide whether this prediction is a success or a failure of the $SU(5)$ unification. Some people argue that one should not worry about the lighter generations, since their masses may be sensitive to small perturbations.

Alternatively, you can include other Higgs representations that can contribute to the fermionic masses. For example, you can add 45_H and then argue that it's effect is important for the lighter fermions only. Before discussing this in some detail, let us understand as to why we ended up with a prediction $m_d = m_e$. Notice that d_α and e make an $SU(4)$ subgroup of $SU(5)$. This is often called the Pati-Salam $SU(4)_c$ symmetry, where the electron is just a fourth color (pink ?) of quarks. But then $< \Phi > \neq 0$ does not break this symmetry, for we chose only $< \phi^0 > \neq 0$, which is outside $SU(4)_c$; thus the equality $m_d = m_e$. Now, since in $SU(5)$, much as in the Standard Model, there is no ν_R (or ν_L^c), no similar situation occurs for the $\nu - u$ system.

It is interesting to take this $SU(4)_c$ more seriously as Pati and Salam did, for then you end up predicting the right-handed neutrino. But more about it later.

Let us now present the Higgs scalar interactions with fermions. Only Φ couples to fermions and in Φ only h_α ($\alpha = r,\ g,\ b$) and $\eta \equiv \mathrm{Re}\,\phi^0$ are the physical fields. It is easy to see that η is precisely the Standard Model Higgs particles whose couplings are the usual ones. The situation with h_α is rather interesting, since it carries color and has fractional charge $Q_h = -1/3$ (just like the Y gauge boson). From (5.2) and (5.3) it is easy to compute it's couplings to fermions

$$\mathcal{L}_h = f_d \bar{\psi}_{R\,i} \chi^{i\,\alpha} h_\alpha^+ + f_u \epsilon_{ijkl\alpha} (\chi^T)^{ij} C \chi^{kl} h^\alpha \tag{5.6}$$

which gives

$$\begin{aligned} \mathcal{L}_h \;\; = \;\; & \left\{ f_d \left(\epsilon^{\alpha\beta\gamma} \bar{u}^c_{L\,\beta}\, d_R^\gamma + \bar{u}_L^\alpha\, e_R^+ + \bar{d}_L^\alpha\, \nu_R^c \right) \right. \\ & \left. + f_u \left(\epsilon^{\alpha\beta\gamma} \bar{u}^c_{R\,\beta}\, d_L^\gamma + \bar{u}_R^\alpha\, e_L^+ \right) \right\} h^\alpha \end{aligned} \tag{5.7}$$

Notice that the structure of the above couplings (not the strength, though), is dictated by the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge invariance only. This follows simply from the fact that $(\psi^c)_L \equiv C\bar{\psi}_R^T$ and $(\psi^c)_R \equiv C\bar{\psi}_L^T$ for any fermion ψ_L and ψ_R. Thus for example we can write

$$\begin{aligned} \bar{u^c_L}\, d_R \;\; &= \;\; u_R^T\, C\, d_R \\ \bar{u^c_R}\, d_L \;\; &= \;\; u_L^T\, C\, d_L \end{aligned} \tag{5.8}$$

In the above notation (the color being suppressed for simplicity) it is easy to see that they are $SU(2)_L$ singlets ($T_{3W} u_R = T_{3W} d_R = 0$, $T_{3W} u_L = \frac{1}{2} u_L$, $T_{3W} d_L = -\frac{1}{2} d_L$) and they carry charge $-1/3$. It is also clear that they are Lorentz invariant, since the asymmetric matrix $C^T = -C$ plays the same role in the Lorentz group as $i\sigma_2$ for $SU(2)$. In $SU(2)$, we can form an invariant out of two spinors ϕ and χ as $\chi^T i\sigma_2 \phi$, since under $SU(2)$:

$$\chi \longrightarrow U\,\chi \;\; ; \quad \phi \longrightarrow U\,\phi \;\; ; \quad U = e^{-i\,\theta_\alpha\,\tau_\alpha/2} \tag{5.9}$$

Thus

$$\chi^T i\sigma_2 \phi \longrightarrow \chi^T U^T i\sigma_2 U \phi \;=\; \chi^T i\sigma_2 U^\dagger U \phi \;=\; \chi^T i\sigma_2 \phi \tag{5.10}$$

Similarly, for two Lorentz spinors ψ_1 and ψ_2 with

$$\psi_i \longrightarrow L\,\psi_i\ , \quad \text{where} \quad L = e^{-i\,(\theta_{ij}/2)\,\sigma_{ij}} \quad , \quad \sigma_{ij} \equiv \frac{i}{4}[\gamma_i, \gamma_j] \tag{5.11}$$

You can show

$$\psi_1^T\,C\,\psi_2 \longrightarrow \psi_1^T L^T\,C\,L\psi_2 \;=\; \psi_1^T\,C\,L^{-1}L\,\psi_2 \;=\; \psi_1^T\,C\,\psi_2 \tag{5.12}$$

(you should use the fact : $C\gamma_\mu^T C^T = -\gamma_u$, $C^T = -C$).

It is clear that the interactions of H break B and L, just like those of X and Y. Notice, though, that $B-L$ is again conserved. In a complete analogy with the situation encountered before for the X and Y bosons, we have the possible exchanges of h^α

which leads to the proton decay. Of course, the amplitude is proportional to the small Yukawa coupling and the corresponding limit on m_h is somewhat less strict. From $(\tau_p)_{\text{exp}} \geq 10^{32}$ yr, one can obtain the following lower limit on m_h

$$m_h \geq 10^{12} GeV \tag{5.13}$$

This is the infamous doublet-triplet splitting phenomenon: $m_h \gg m_\eta$. Why ?

5.0.1 Generations and their mixings

We know that in the standard model the neutral current interactions are flavor diagonal and that the charged current processes lead to flavor, or generation mixing and CP violation. How is this feature incorporated in the $SU(5)$ theory and what about new superweak interactions of the X and Y bosons ? The analysis is straightforward and it proceeds along the same lines as in the $SU(2)_L \times U(1)_Y$ theory.

If we assume N_G generations ($N_g \geq 3$), then the Yukawa couplings in (5.2) and (5.4) become matrices in the generation space, just as in the Standard Model. Thus we will have

$$\begin{aligned} (M_d)^{ab} &= (M_{e^+})^{ab} = (f_d)^{ab} v_W \\ (M_u)^{ab} &= (f_u)^{ab} v_W \end{aligned} \tag{5.14}$$

where the notation d, e^+, u denotes down quarks, positrons and up quarks respectively. Using the fact that $C^T = -C$, we get furthermore that the up quark mass matrix is symmetric: $M_u^T = -M_u$. We then diagonalize these matrices by bi-unitary transformations just as in the Standard Model

$$\mathcal{U}_{Lf}^\dagger M_f \mathcal{U}_{Rf} = D_f \tag{5.15}$$

where D_f is diagonal, with its elements being real, positive numbers. Furthermore, from the fact that M_u is symmetric you can show that

$$\mathcal{U}_{Ru} = \mathcal{U}_{Lu}^* K^* \tag{5.16}$$

where

$$K = \begin{pmatrix} e^{i\phi u} & & & \\ & e^{i\phi e} & & \\ & & e^{i\phi t} & \\ & & & ... \end{pmatrix} \tag{5.17}$$

is the matrix of phases needed to ensure that the elements of D_u are real and positive.

The above statements are equivalent to the redefinition of our original fermionic fields in the Lagrangian

$$f^a_{L,R} \to (\mathcal{U}^\dagger_{L,R})^{ab} f^b_{L,R} \tag{5.18}$$

where we will have $\mathcal{U}^d_{L,R} = \mathcal{U}^{e^+}_{L,R}$ due to (5.14). Since on the other hand the neutrinos are massless (no ν_R-just as in $SU(2)_L \times U(1)$), we can rotate them any which way we wish and so we chose $\nu^c_R \to \mathcal{U}^d_R \nu^c_R$.

Thus we can write for the 5-dimensional representation

$$\psi_R \to \mathcal{U}^d_R \psi_R \tag{5.19}$$

which means that $\mathcal{U}^d_R$ disappears since the Lagrangian (3.30) cannot depend on it. Suppressing the color index, we can write

$$\chi = \begin{bmatrix} u^c_L & -u_L & -d_L \\ & & \\ & & e^+_L \end{bmatrix} \tag{5.20}$$

so that under (5.18), using the fact that $u^c_L \equiv C\bar{u}^T_R = C\gamma_) u^*_R$ we get

$$\begin{aligned} \chi &\to \begin{bmatrix} \mathcal{U}_{Lu} K u^c & -\mathcal{U}_{Lu} u & -\mathcal{U}_{Ld} d \\ & & \\ & & -\mathcal{U}_{Ld} e^+ \end{bmatrix}_L \\ &= \mathcal{U}_{Ld} \begin{bmatrix} \mathcal{U}_{CKM} K u^c & -\mathcal{U}_{CKM} u & -d \\ & & \\ & & -e^+ \end{bmatrix}_L \end{aligned} \tag{5.21}$$

where $\mathcal{U}_{CKM} = \mathcal{U}^\dagger_{Ld}\mathcal{U}_{Lu}$. Again $\mathcal{U}_{Ld}$ is just an overall phase and so it will disappear from (3.30). We are left with the Cabibbo-Kobayashi-Maskawa unitary matrix and the phase matrix K only. Thus the leptonic interactions are flavor conserving just as in the Standard Model (since neutrinos are massless), and the weak quark interactions involve $\mathcal{U}_{CKM}$ only, again as in $SU(2)_L \times U(1)$. Finally and importantly, the X and Y boson interactions involve *no* new flavor mixings besides $\mathcal{U}_{CKM}$, however there will be new phases hidden in K. This means that the proton decay proceeds with full strength, i.e you cannot “rotate it away” by invoking the second or third generation in (3.32). More precisely the decay mode $p \to \pi^0 + e^+$ is not suppressed, while the energetically forbidden mode $p \to \pi^0 + \tau^+$ ($m_\tau > m_p$) becomes suppressed by the small mixings. The phases in K play an important role in the physics of the early universe, but that is beyond the present scope of our discussion.

Before closing this section, let us illustrate in some more detail the situation with proton decay. To see what happens precisely, we must rewrite the X and Y boson interactions, formula (3.32) in the basis of physical fields, i.e. the eigenstates of (5.19) and (5.21). Suppressing the color index, we get readily

$$\begin{aligned}\mathcal{L}(X,Y) &= \frac{g}{\sqrt{2}}\bar{X}_\mu\left[\bar{d}_R\,\gamma^\mu\,e^+_R + \bar{d}_L\,\gamma^\mu\,e^+_L + \bar{u}^c_L\,\gamma^\mu\,K^*\,u_L\right] \\ &+ \frac{g}{\sqrt{2}}\bar{Y}_\mu\left[-\bar{d}_R\,\gamma^\mu\,\nu^c_R + \bar{u}_L\,\gamma^\mu\,\mathcal{U}^\dagger_{CKM}\,e^+_L + \bar{u}^c_L\,\gamma^\mu\,\mathcal{U}^\dagger_{CKM}\,d_L\right] + h.c.\end{aligned} \tag{5.22}$$

Clearly, there are no mixing angles in the X interaction, only the phases K^* which play no role at this point. On the other hand, we see clearly the CKM rotation, much as it appears in the ordinary interactions. This provides us with a clear prediction for the relative branching ratios of the electron and the muon final states. From $\mathcal{U}_{11} \propto \cos\theta_c$, $\mathcal{U}_{12} \propto \sin\theta_c$ we expect

$$\frac{\Gamma(p \to \pi^0\mu^+)}{\Gamma(p \to \pi^0 e^+)} \propto \sin^2\theta_c \tag{5.23}$$

By observing the proton decay and measuring the different branching ratios we could obviously put $SU(5)$ to various tests. Of course, what remains is to see if we can predict the actual strength of these processes, or in other words the X and Y gauge boson masses.

Chapter 6

Low energy predictions from $SU(5)$ (without and with supersymmetry)

6.1 A. Minimal $SU(5)$

In section 3 we have given general equations for the running of the gauge coupling constants, and then applied them specifically to the $SU(3)_c$, $SU(2)_L$ and $U(1)_{em}$ gauge couplings. In order to study the question of the unification of these couplings, we choose to work with $U(1)_Y$ instead of $U(1)_{em}$, since we shall be interested in the regions of energies above M_W, where the whole $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry becomes effectively operative.

Thus, we have

$$\frac{1}{\alpha_G(M_W)} = \frac{1}{\alpha_U} - \frac{1}{2\pi} b_G \ln \frac{M_X}{M_W} \tag{6.1}$$

for the gauge group G; M_X is the energy where we imagine the unification to take place, and α_U is the value of the unified coupling at M_X. We have for the $SU(3)_C$ from (2.8)

$$b_3 = \frac{33}{3} - \frac{4}{3} n_g \tag{6.2}$$

where N_g is the number of generations; and for the $SU(2)_L$ we get

$$b_2 = \frac{22}{3} - \frac{4}{3} n_g - \frac{1}{6} n_H \tag{6.3}$$

where n_H is the number of Higgs doublets ($n_H = 1$ in the minimal standard model).

Now, for $U(1)_Y$ we find easily that for one generation of fermions

$$T_F(1\ gen) = Tr\left(\frac{Y}{2}\right)^2 (1\, gen.) = \frac{5}{3} \tag{6.4}$$

To obtain (6.4), use simply

$$\begin{aligned} \frac{Y}{2}\begin{pmatrix} u \\ d \end{pmatrix}_L &= \frac{1}{6}\begin{pmatrix} u \\ d \end{pmatrix}_L \\ \frac{Y}{2}\begin{pmatrix} \nu \\ e \end{pmatrix}_L &= -\frac{1}{2}\begin{pmatrix} \nu \\ e \end{pmatrix}_L \end{aligned} \tag{6.5}$$

$$\frac{Y}{2}u_R = \frac{2}{3}u_R \ ; \ \frac{Y}{2}d_R = -\frac{1}{3}d_R \ ; \ \frac{Y}{2}e_R = -e_R \tag{6.6}$$

Similarly, using $\frac{Y}{2}\Phi = \frac{1}{2}\Phi$ for a Higgs doublet Φ, we can write down

$$b_Y = -\frac{5}{3}\cdot\frac{4}{3}n_g - \frac{1}{6}n_H \tag{6.7}$$

Now, notice the difference of $5/3$ for the fermionic contributions of b_3 and b_2 on one side, and L_Y on the other side. Since the fermions in one generation belong to the full representations of $SU(5)$ **5** and **10**, their contributions to b_G must be equal for all the couplings equally normalized, i.e. for all the couplings which have the value α_U at M_X. For us α_U is the value of the unified coupling of the $SU(5)$ theory and thus we are told that the $U(1)_Y$ coupling is not properly normalized.

We have seen this before, in eq. (3.7) which tells us that the $U(1)_Y$ coupling g' is given by

$$g' = g_1\sqrt{\frac{3}{5}} \tag{6.8}$$

where g_1 is correctly normalized. To make it more manifest, let us take the **5**-dimensional fermionic representation as given in (3.9). You can easily see that

$$Tr\left(\frac{Y}{2}\right)^2 (5_F) = \left[\left(\frac{1}{3}\right)^2\cdot 3 + \left(\frac{1}{2}\right)^2\cdot 2\right]\cdot\frac{1}{2} = \frac{5}{12} \tag{6.9}$$

where the last factor $\frac{1}{2}$is due to $Tr(1+\gamma_5)/2 = 1/2$.

On the other hand, take say T_{3W} which gives

$$Tr(T_{3W})^2(5_F) = \left(\frac{1}{2}\right)^2 \cdot 2 \cdot \frac{1}{2} = \frac{1}{4} \tag{6.10}$$

and you recover the same factor $\frac{5}{3}$ as before.
Now, using (6.8) which says that $1/\alpha_Y = (5/3)(1/\alpha_1)$ we get

$$b_1 = \frac{3}{5} b_Y = -\frac{4}{3} n_g - \frac{1}{10} n_H \tag{6.11}$$

Notice that there is no gauge boson contribution in b_Y and b_1, since the gluons and $SU(2)_L$ gauge bosons carry no hypercharge Y.

We are now fully armed to check the evolution of these couplings above M_W. We can adopt two different strategies:

1. assume that they unify at the scale M_X and then derive the low energy predictions and check them against experiment, or
2. just follow the couplings above M_W and see if they meet at a single point.

We shall do both. Let us start with (1) to see how grand unification leads to low energy predictions.

(1) From (6.1), (6.2) and (6.3) you get

$$\frac{1}{\alpha_2(M_W)} - \frac{1}{\alpha_3(M_W)} = \frac{b_3 - b_2}{2\pi} \ln \frac{M_X}{M_W} \tag{6.12}$$

and from (6.1), (6.3) and (6.11) we get

$$\frac{1}{\alpha_1(M_W)} - \frac{1}{\alpha_2(M_W)} = \frac{b_2 - b_1}{2\pi} \ln \frac{M_X}{M_W} \tag{6.13}$$

In the above we have used $\alpha_1(M_X) = \alpha_2(M_X) = \alpha_3(M_X) = \alpha_U$. From $\alpha_{em} = \sin^2\theta_W \alpha_2 = \cos^2\theta_W \alpha_Y$ and $\alpha_Y = 3/5\alpha_1$ we get

$$\frac{1}{\alpha_1(M_W)} - \frac{1}{\alpha_2(M_W)} = \frac{3}{5}\frac{\cos^2\theta_W(M_W)}{\alpha_{em}(M_W)} - \frac{\sin^2\theta_W(M_W)}{\alpha_{em}(M_W)} \tag{6.14}$$

which combined with (6.13) gives in turn

$$\sin^2\theta_W(M_W) = \frac{3}{8} - \frac{5}{8}\alpha_{em}(M_W)\frac{b_2 - b_1}{2\pi} \ln \frac{M_X}{M_W} \tag{6.15}$$

Equations (6.12) and (6.15) form the basis for low energy predictions. Substituting the values for b_i we get

$$\begin{aligned} \frac{1}{\alpha_2(M_W)} - \frac{1}{\alpha_3(M_W)} &= \frac{22 + n_H}{12\pi} \ln \frac{M_X}{M_W} \\ \sin^2\theta_W(M_W) &= \frac{3}{8} - \frac{110 - n_H}{48\pi} \alpha_{em}(M_W) \ln \frac{M_X}{M_W} \end{aligned} \tag{6.16}$$

Notice the prediction $\sin^2\theta_W = \frac{3}{8}$ at M_X which we discussed before. Now, for $n_H = 1$ we get the minimal theory and by taking as inputs

$$\begin{aligned} \alpha_3(M_W) &\simeq .12 \\ \alpha_2(M_W) &\simeq \frac{1}{30} \end{aligned} \tag{6.17}$$

we find

$$M_X \simeq 10^{16} GeV \tag{6.18}$$

but

$$\sin^2\theta_W(M_W) \simeq 0.2 \tag{6.19}$$

To see better the above result, take for simplicity $n_H = 0$ (there is almost *no* dependence on n_H in the above equations) and by eliminating M_X from equation (6.16) we get

$$\sin^2\theta_W(M_W) = \frac{1}{6} - \frac{5}{9}\frac{\alpha_{em}(M_W)}{\alpha_3(M_W)} \tag{6.20}$$

and $\alpha_{em}(M_W) \simeq 1/128$. The measured value of

$$(\sin^2\theta_W(M_W))_{\exp} \simeq 0.23 \tag{6.21}$$

contradicts the predictions (6.19) (or (6.20). Equivalently, you may use (6.20) and take $\sin^2\theta_W(M_W) \simeq 0.23$, then you get $\alpha_3(M_W)$ too small.

The minimal $SU(5)$ theory thus fails to meet the experiment.

(2) It is probably more illustrative to just follow the couplings $\alpha_i(E)$ above M_W and see if they unify at a single point. The graph below is the result of a very careful such analysis and it shows manifestly that there can be *no* unification of gauge couplings at a single point.

This was the source of our failure above; simply speaking there is no single meeting M_X. The $U(1)$ coupling α_1 meets α_2 too soon.

6.2 B. Supersymmetry

We have already argued that the minimal $SU(5)$ may be a sick theory, since it has no way of keeping the gauge hierarchy $M_W \ll M_X$ small in perturbation theory. The solution seems to be supersymmetry, i.e. symmetry between bosons and fermions which guarantees the cancellation of quadratic divergences for the Higgs mass and thus can make M_W insensitive to M_X. That is, we do not know why M_W/M_X is small, but it is not a problem, since it will stay small in perturbation theory as long as the scale of supersymmetry breaking is small $\Lambda_{SS} \simeq M_W$.

This is an exciting possibility since it predicts a whole new world of supersymmetric partners of ordinary particles to be discovered at the new colliders. We cannot do justice here to this important topic, but we would like to show how the existence of these superpartners renders the unification in agreement with the experiment.

Imagine that for every particle of the standard model there is a supersymmetric partner of the opposite statistics, that is imagine the supersymmetry to imply

FERMIONS (quarks, leptons) $s = 1/2$	$\Longleftrightarrow$	SFERMIONS (squarks, sleptons) $s = 0$
GAUGE BOSONS ($W^{\pm}$, Z, γ, gluons) $s = 1$	$\Longleftrightarrow$	GAUGINOS (Wino, Zino, photino, gluinos) $s = 1/2$
HIGGS SCALAR $s = 0$	$\Longleftrightarrow$	HIGGSINO $s = 1/2$

It is easy to see that the formulas of section 3 for the running of the gauge couplings will be affected by the presence of the new particles. Write

$$b_G = \frac{11}{3}T_{GB} - \frac{4}{3}T_F - \frac{1}{3}T_H \tag{6.22}$$

where we assume some *chiral* fermions in order to apply the supersymmetric constraints. It is important to know that both gauginos and Higgsino are chiral fermions at fixed helicity, a fact we must take for granted here.

But then we will get

$$b_G^{SS} = \left(\frac{11}{3} - \frac{2}{3}\right) T_{GB} - \left(\frac{2}{3} + \frac{1}{3}\right) T_F - \left(\frac{1}{3} + \frac{2}{3}\right) T_H \tag{6.23}$$

or

$$b_G = 3T_{GB} - T_F - T_H \tag{6.24}$$

where the added contributions in (6.23) are due to the superpartners. Thus gauginos add $-2/3$, sfermions $-1/3$ and Higgsinos $-2/3$, using (6.22).

From (6.23) we get for the individual gauge couplings

$$\begin{aligned} b_3^{SS} &= 9 - 2n_g \\ b_2^{SS} &= 6 - 2n_g - \frac{1}{2} n_H \\ b_1^{SS} &= -2n_g - \frac{3}{10} n_H \end{aligned} \tag{6.25}$$

where n_H is again the number of Higgs doublets.

(1) In exactly the same way as before, assuming the unification of couplings at M_X, we find

$$\begin{aligned} \frac{1}{\alpha_2(M_W)} - \frac{1}{\alpha_3(M_W)} &= \frac{6 + n_H}{4\pi} \ln \frac{M_X}{M_W} \\ \sin^2 \theta_W(M_W) &= \frac{3}{8} - \frac{30 - n_H}{16\pi} \alpha_{em}(M_W) \ln \frac{M_X}{M_W} \end{aligned} \tag{6.26}$$

In the minimal supersymmetric model it turns out that $n_H = 2$, because of the cancellation of anomalies (more about it later) and we find

$$M_X \simeq 10^{16} GeV \tag{6.27}$$

and

$$\sin^2 \theta_W(M_W) = \frac{1}{5} + \frac{7}{15} \frac{\alpha_{em}(M_W)}{\alpha_3(M_W)} \simeq 0.23 \tag{6.28}$$

where the second equation follows from the elimination of M_X in (6.26) for $n_H = 2$. We see that the supersymmetric theory agrees perfectly well with the experiment and with the above value for M_X we predict the proton lifetime

$$\tau_p \simeq 10^{34-35} yr \tag{6.29}$$

which is somewhat above the experimental bound

$$(\tau_p)_{\text{exp}} \simeq 6 \cdot 10^{33} yr \tag{6.30}$$

Now, if we are to take supersymmetry seriously, all the way up to the scale M_X, we expect of course new gauginos $\tilde{X}$, $\tilde{Y}$, associated with the superheavy bosons X and Y of $SU(5)$; and also the heavy Higgsinos $\tilde{h}_\alpha$ from $\mathbf{5}$ of $SU(5)$. Their exchange can in principle also lead to the proton decay, and the precise analysis of τ_p in the supersymmetric $SU(5)$ theory becomes much richer. This is however beyond the scope of our course.

The search for supersymmetry has become one of the main efforts of the physics community and it almost parallels the importance of finding the Higgs boson. In a sense, the physics of the Higgs boson makes full sense in the context of supersymmetry.

(2) As before, in the case of ordinary $SU(5)$, we can just follow the couplings $\alpha_i(E)$ above M_W, using their measured values at M_W. The graph below shows clearly how now we can speak of a single unification point M_X.

Chapter 7

Topological Defects: Domain Walls

We have learned repeatedly that unifying known interactions, or better to say the quarks and leptons in a compact group implies *the quantization of charge*. By that we mean that the charges of quarks and leptons get related, which in turn tells us that the charges in nature are integer products of some basic charge. Recall that in the standard model we get

$$Q_u = Q_d + 1 \tag{7.1}$$
$$Q_\nu = Q_e + 1 \tag{7.2}$$

since the above particles make up $SU(2)_L$ doublets and we know that

$$Q_{em} = T_3 + \frac{Y}{2} \tag{7.3}$$

so that $\Delta Q_{em} = 2T_3 = 1$.
Unifying quarks and leptons in $SU(5)$, and recalling that

$$\psi_5 = \begin{pmatrix} d_r \\ d_g \\ d_b \\ e^+ \\ -\nu^c \end{pmatrix}_R \tag{7.4}$$

we get

$$3Q_d + Q_{e^+} + Q_{e^+} - 1 = 0 \tag{7.5}$$

or

$$3Q_d - 2Q_e - 1 = 0 \tag{7.6}$$

which as we promised relates the quark and lepton charges. This is what we call charge quantization.

On the other hand, it is well known that the existence of magnetic monopoles also implies quantization, as we will show later. We will see that these facts are intimately related to each other for we will show that unified theories predict the existence of magnetic monopoles, which is a prerequisite for the quantization of charge.

To see this, we wish to arm ourselves with some techniques of searching for classical solutions by studying some simple systems. We start with the domain walls.

7.1 A. Discrete symmetries and domain walls

Imagine a simple example of a single real scalar field ϕ with a discrete symmetry

D: $\phi \longrightarrow -\phi$

whose Lagrangian is then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{7.7}$$

The potential is chosen with $v^2 > 0$, $\lambda > 0$, which implies spontaneous symmetry breaking of **D**, since the minimum of the potential is at

$$< \phi >^2 = v^2 \quad \text{or} \quad < \phi > = \pm v \tag{7.8}$$

We have then the usual scenario in which we chose either of the vacua $< \phi > = +v$ *or* $< \phi > = -v$, and have our system live in one of them. This implies the spontaneous breakdown of the discrete symmetry **D**. To see this, choose $< \phi > = +v$; then we can write

$$\phi = v + \eta \tag{7.9}$$

and

$$\begin{aligned} V(\eta) &= \frac{\lambda}{4}\left[(v+\eta)^2 - v^2\right]^2 = \frac{\lambda}{4}\left[2v\eta + \eta^2\right]^2 \\ &= \frac{\lambda}{4}\left[\eta^4 + 4v^2\eta^2 + 4v\eta^3\right] \end{aligned} \tag{7.10}$$

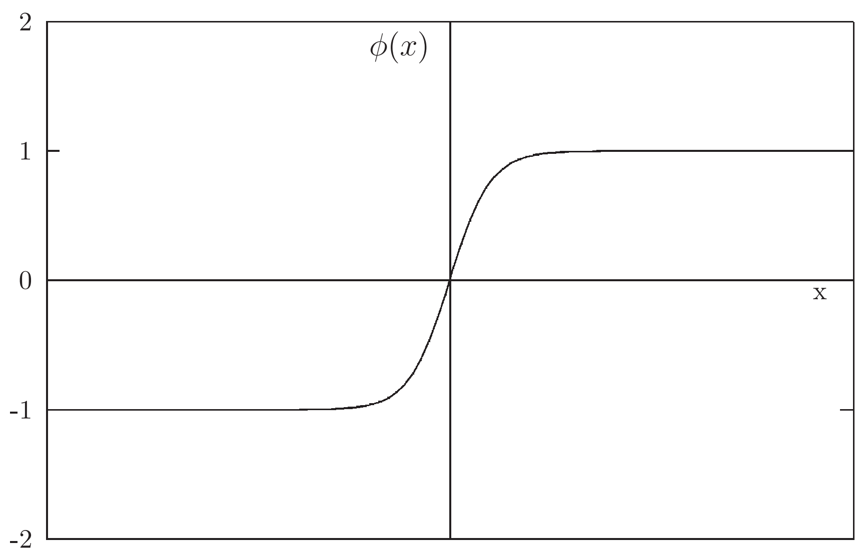
The presence of the term cubic in η clearly breaks the symmetry $\eta \rightarrow -\eta$. On the other hand, we could have chosen as well $<\phi>= -v$, and we can image another domain with that ground state. Since these ground states are *not* connected by any continuos symmetry, it should cost us energy to go from one to the other, or in other words, there should be a "wall" between these two different domains.

Thus we believe that there must exist a classical (static) solution which connects these two domains, i.e. the solution which interpolates between the two vacua: $<\phi>= +v$ and $<\phi>= -v$. Such a solution ϕ_{cl} must satisfy

$$\lim_{z \to +\infty} \phi_{cl}(z) = v \ \ ; \quad \lim_{z \to -\infty} \phi_{cl}(z) = -v \tag{7.11}$$

or vice versa.

But then, $\phi_{cl}(z)$ must vanish at some point, and due to the symmetry $\phi \longrightarrow -\phi$ it will happen at $z = 0$. Our solution should look like:



Of course, the shape is to be determined by the construction of the actual solution. However, before we do it, we can still say something about its properties. Suppose we wish to define the "width" of our solution, i.e. the distance in which the field carries energy. Namely, at $z = 0$ we know that $\phi = 0$ which is the local maximum of $V(\phi)$ and $V(0) = \frac{\lambda}{4}v^4 > 0$. Let us approximate our solution in the figure by a simple step function, so that

$$
\phi(z) = \begin{cases} v, & z \geq \delta \\ 0, & -\delta \leq z \leq \delta \\ 0, & \leq z \leq -\delta \end{cases} \tag{7.12}
$$

From (7.7), we get for the energy per unit area

$$
\frac{E}{S} = \int_{-\infty}^{+\infty} dz \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_z \phi)^2 + V(\phi) \right] \tag{7.13}
$$

And since we look for a static solution

$$
\frac{E}{S} = \int_{-\infty}^{+\infty} dz \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial z} \right)^2 + V(\phi) \right] \tag{7.14}
$$

From (7.12) we can write

$$
\frac{E}{S} = \frac{\lambda}{4} v^4 \cdot 2\delta + \frac{1}{2} \left(\frac{v^2}{\delta^2} \right) \cdot 2\delta \tag{7.15}
$$

or

$$
\frac{E}{S} \simeq \frac{\lambda v^4}{2} \delta + \frac{v^2}{\delta} \tag{7.16}
$$

The width of the wall is determined by minimizing the energy per unit area

$$
\frac{\partial}{\partial \delta} \left(\frac{E}{S} \right) = \frac{\lambda v^4}{2} - \frac{v^2}{\delta^2} = 0 \tag{7.17}
$$

and so

$$
\delta = \sqrt{\frac{2}{\lambda}} v^{-1} \tag{7.18}
$$

In what follows we shall construct an actual solution and we shall see that the above approximation of a thin wall works perfectly well. More precisely, we will find that the width of a domain wall is exceedingly small on the macroscopic scales.

7.2 B. The domain wall solution

Once again, we are searching for a static solution in the $x - y$ plane, i.e. we assume that $\phi(\vec{x}, t)$ depends only on z. Thus from the Euler-Lagrange equation

$$\Box\phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\phi = -\frac{\partial V}{\partial \phi} \tag{7.19}$$

we get

$$\frac{d^2\phi}{dz^2} = \frac{\partial V}{\partial \phi} \tag{7.20}$$

multiplying (7.20) by $d\phi/dz$ we get

$$\frac{d}{dz}\left[\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2\right] = \frac{dV}{dz} \tag{7.21}$$

or

$$\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 - V = C \tag{7.22}$$

where C is a constant.

On the other hand, from (7.14) the energy per unit area for our solution is

$$\frac{E}{S} = \int_{-\infty}^{+\infty} dz \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 + V(\phi)\right] \tag{7.23}$$

By demanding that E/S is finite (we are looking for a well defined finite energy solution), we must demand

$$V(\phi) \underset{z \to \pm\infty}{\longrightarrow} 0 \quad ; \quad \left(\tfrac{d\phi}{dz}\right)^2 \underset{z \to \pm\infty}{\longrightarrow} 0$$

This implies $C = 0$ in eq. (7.22) and thus

$$\frac{d\phi}{dz} = \pm\sqrt{2V} \tag{7.24}$$

We get a single first order equation which can be readily integrated. From (7.24) we can write for the energy per unit area

$$\frac{E}{S} = \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi \, \frac{dz}{d\phi} \, 2V = \pm \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi \sqrt{2V} \tag{7.25}$$

Formulas (7.24) and (7.25) are essential for what follows; for a given V they give the form and the energy of the solution.

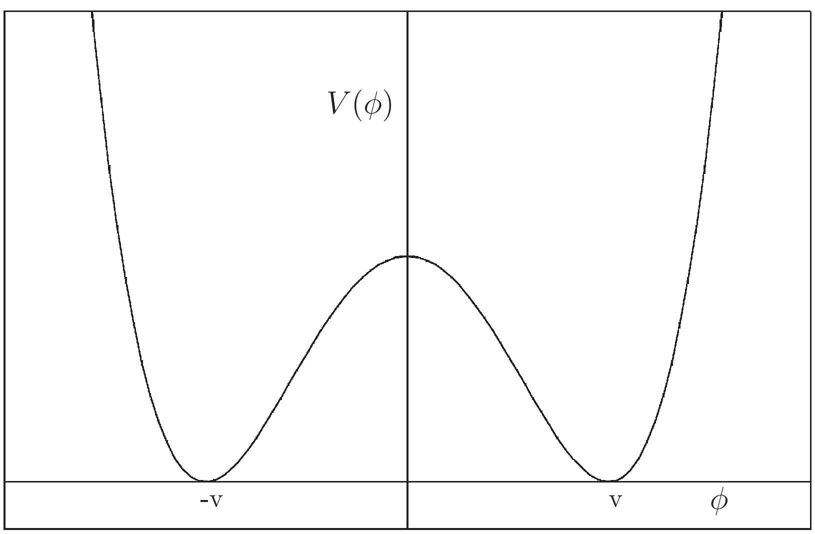
Notice that the above discussion offers a useful mechanical analogy with a particle moving in a potential $U = -V$. Namely, take

$$\begin{aligned} \phi &\leftrightarrow x \\ \frac{d\phi}{dz} &\leftrightarrow \frac{dx}{dt} \end{aligned} \tag{7.26}$$

Since

$$\frac{d^2x}{dt^2} = -\frac{dU}{dx} \tag{7.27}$$

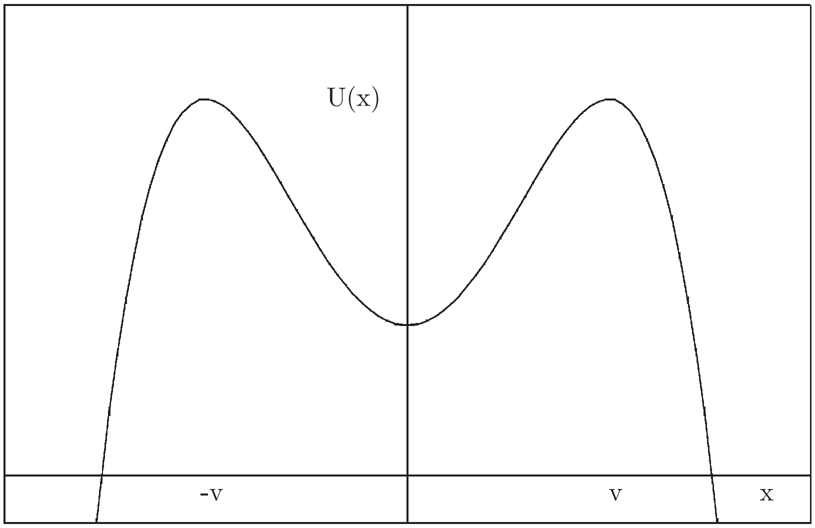
analogy with (7.20) gives $U = -V$. Thus for our potential $V = \lambda/4(\phi^2 - v^2)^2$ which can be depicted as



the inverted potential for the mechanical analogy is

The solution we are looking for with $\phi(-\infty) = -v$, $\phi(+\infty) = +v$, $\frac{d\phi}{dz}(\pm\infty) = 0$, corresponds to the motion of the particle above which starts with a zero velocity at $x(-\infty) = -v$ and obviously arrives at $x(+\infty) = +v$ again with a zero velocity. This encourages the existence of the solution we are looking for.

Inserting the form of V in (7.24) we get



$$\frac{d\phi}{dz} = \pm\sqrt{\frac{\lambda}{2}}(\phi^2 - v^2)^2 \tag{7.28}$$

If we chose $\phi(\pm\infty) = \pm v$, we get easily the domain wall solution

$$\begin{aligned} \phi_{cl} &= v \tanh \frac{z}{\delta} \\ \delta &= \sqrt{\frac{2}{\lambda}} v^{-1} \end{aligned} \tag{7.29}$$

notice that δ comes close to our approximate expression for a thin wall in the step function approximation. We call δ the thickness of the wall due to the exponential nature of the solution (7.29); for all practical purposes it is in the region $-\delta \leq z \leq \delta$ where the field ϕ_{cl} sits away from its boundary values $\pm v$. Notice furthermore that there exists a classical solution

$$\phi'_{cl} = -\phi_{cl} \tag{7.30}$$

which obviously satisfies the reversed boundary conditions $\phi'_{cl}(\pm\infty) = \mp v$. Furthermore, from (7.25) we can easily compute the energy per unit area of our solution

$$\frac{E}{S} = \frac{2\sqrt{2\lambda}}{3} v^3 \tag{7.31}$$

Clearly, since v is the only dimensional parameter in the theory, its value determines both the thickness and the energy of the solution. To get a feel for the scales, imagine that v is of order of the weak interaction physics, i.e. $v \simeq v_W \simeq 100GeV$. Taking $\lambda \simeq 1$ for simplicity, we find

$$\begin{aligned} \delta &\simeq 10^{-2}GeV^{-1} \simeq 10^{-16}cm \\ \frac{E}{S} &\simeq 10^{6}GeV^{3} \end{aligned} \tag{7.32}$$

The domain wall thickness n is given by the microscopic scales as expected and we can view the wall as situated at the origin for all practical macroscopic purposes.

7.2.1 The wall and the Universe

To get a further feel for the scales involved, let us imagine that there exists a large domain wall in the Universe, large in the sense of spreading throughout the visible universe. Such a wall is expected to result from a phase transition at high temperature in the early universe. Now, the size of the visible universe is its age

$$R_U \simeq 10^{10}yr \simeq 10^{18}sec \simeq 10^{28}cm \tag{7.33}$$

and the matter density o the universe is

$$\rho_U \simeq 10^{-7}GeV/cm^3 \tag{7.34}$$

we can then compute the ratio between the energies of the wall and the matter in the universe

$$\frac{E_{\text{wall}}}{E_U} \simeq \frac{10^6 GeV^3 R^3}{10^{-7}GeV/cm^3 R_U^3} \simeq \frac{10^{13}(GeV cm)^2}{10^2 8} \tag{7.35}$$

which for $GeV cm \simeq 10^{14}$ gives

$$\frac{E_{\text{wall}}}{E_U} \simeq 10^{13} \tag{7.36}$$

The energy of a wall corresponding to the weak interaction physics would be some thirteen orders of magnitude bigger than the observed one, and would dramatically affect the big-bang scenario. This is known as the domain wall problem and its solution is being actually searched for.

7.3 C. Topology and stability of domain walls

We have demonstrated the existence of the domain wall solution (also called "kink" in the literature), but have not proved its stability. In other words, we should show that that for $\phi(z) = \phi_{cl}(z) + \epsilon$, the energy of $\phi(z)$ is bigger or equal than the corresponding one for $\phi_{cl}(z)$ independently of what ϵ is. Instead of going through this tedious computation, we offer rather a topological argument for its stability.

If ϕ is only a function of z, we can view this as a 1+1 dimensional problem with coordinates t and z. It is readily seen that the current

$$j^{\mu} = \epsilon^{\mu\nu}\partial_{\nu}\phi \tag{7.37}$$

is conserved

$$\partial^{\mu}_{\mu} = \epsilon^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi = 0 \tag{7.38}$$

The above conservation law is not a product of a symmetry as in the Noether case; rather it is called a topological conservation law.

from (7.38), we know that the corresponding charge is conserved

$$\frac{dQ}{dt} = 0 \tag{7.39}$$

where

$$Q = \int_{-\infty}^{+\infty} dz j_o(z) = \int_{-\infty}^{+\infty} dz \frac{d\phi}{dz} = \phi(+\infty) - \phi(-\infty) \tag{7.40}$$

for the pure vacuum, characterized by $\phi(z) = v$, or $\phi(+\infty) = \phi(-\infty)$ the charge vanishes

$$Q_{\text{vac}} = 0 \tag{7.41}$$

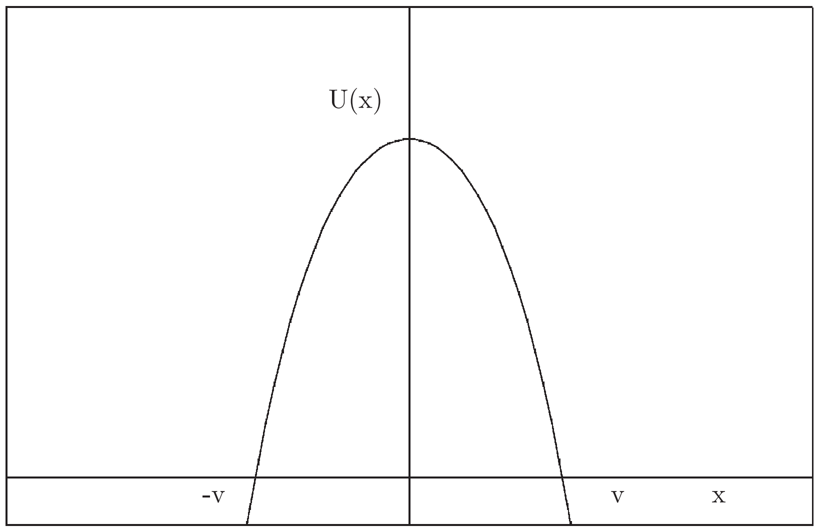
and for our domain wall solution with $\phi_{cl}(\pm\infty) = \pm v$

$$Q_{\text{wall}} = 2v \tag{7.42}$$

while for the inverted solution (anti wall) $\phi_{cl}(\pm\infty) = \mp v$, we get

$$Q_{\text{antiwall}} = -2v \tag{7.43}$$

This proves the stability of the domain wall solution. Namely, although the quantum vacuum carries less energy, the wall cannot "decay" into it, since $Q_{\text{wall}} = 2v$ is conserved. In other words, the boundary conditions we



have chosen prevent the wall from untwisting itself and taking the value $+V$ (or $-v$) everywhere, as would be preferable from the energy point of view.

Let us discuss at some length the origin and the criteria for the existence of our solution. Notice that it was crucial to choose nontrivial boundary conditions $\phi(-\infty) \neq \phi(+\infty)$ for its existence, but also it was necessary that the potential V allowed for such a nontrivial choice. To appreciate this, let us take a potential which does not break the symmetry **D**: $\phi \to -\phi$, i.e. the potential with a positive mass term

$$V'(\phi) = \frac{\lambda}{4}(\phi^2 + v^2)^2 \tag{7.44}$$

Now, $< \phi >= 0$ and the symmetry **D** remains unbroken. Since at infinity we must demand $V'(\phi(\infty)) \to 0$, this implies $\phi(\pm\infty) \to 0$, and thus there can be no nontrivial solution such as $\phi_{cl}(z)$ in (7.29).

It is again useful to use our mechanical analogy of a point particle moving in a potential $U = -V'$

Obviously now a particle that starts from the maximum of $-V'$ at $x = 0$ with a zero velocity will never turn back, it will simply continue falling down for ever. We must have a nontrivial potential $-V'$ with at least two different maxima.

Physically, we must have a spontaneously broken discrete symmetry, since a domain wall is simply a reflection of the fact that it costs us energy to go from one vacuum to another. Its stability is connected to the fact that the boundary conditions provide a nontrivial map $\phi_{cl}(\pm\infty) = \pm v$.

Let us denote by $\mathcal{M}_0$ the manifold of the zeroes of $V(\phi)$, i.e. the minima of the energy; and by $\mathcal{M}_\infty$ the manifold of the points at infinity. In our case

$$\begin{aligned}\mathcal{M}_\infty &= \{z \to -\infty\ ;\ \ z \to +\infty\} \\ \mathcal{M}_0 &= \{\phi_0 \because V(\phi_o) = 0 \Rightarrow \phi_0 = \pm v\}\end{aligned} \tag{7.45}$$

But $\mathcal{M}_0$ and $\mathcal{M}_\infty$ consist of two points. Thus the boundary conditions $\phi(\pm\infty)$ provide a well defined map from $\mathcal{M}_\infty$ to $\mathcal{M}_0$. A nontrivial solution demands at least two discrete points on $\mathcal{M}$.

For a further example, we could choose

$$V(\phi) = \frac{\lambda}{4}(\phi^4 - v^4)^2 \tag{7.46}$$

In which case we have a Z_4 discrete symmetry $\mathbf{D}$ $\phi \to \pm i\phi$, $\phi \to \pm\phi$. The manifold $\mathcal{M}_0$ contains 4 points

$$\mathcal{M}_0 = \{\phi_0 \because V(\phi_0) = 0 \Rightarrow \phi_0 = \pm iv,\, \pm v\} \tag{7.47}$$

and again we have a well-defined map $\mathcal{M}_\infty \overset{\phi(\pm\infty)}{\to} \mathcal{M}_0$. We expect stable domain walls connecting different vacua. Keep in mind though that the example (7.46) is not realistic, since it would imply a nonrenormalizable theory.

Let us take yet another example, now of a continuos $U(1)$ symmetry with a complex field ϕ and the potential

$$V(\phi) = \frac{\lambda}{4}(\phi^*\phi - v^2)^2 \tag{7.48}$$

Now the manifold $\mathcal{M}_0$ is a circle

$$\mathcal{M}_0 = \{\phi_0 \because V(\phi_0) = 0 \Rightarrow \phi_0 = ve^{i\alpha}\} \tag{7.49}$$

and the map $\mathcal{M}_\infty$ to $\mathcal{M}_0$ cannot guarantee the stability of the solution $\phi_{cl}(\infty)$. Namely, the points $-v$ and $+v$ can be simply connected by a continuous phase change in α from π to zero. The solution is not stable anymore and you can show that $\phi(z) = \phi_{cl}(z) + i\epsilon$ can lead to less energy than ϕ_{cl}.

Clearly, to get a stable solution we should map as circle on a circle. This will give a so called string solution, but more about it later.

7.4 D. The gravitational field of a domain wall

We shall close this section with an amusing example of what kind of gravitational field a static large domain wall would produce. We shall study it only in the Newtonian approximation of a weak field felt by slowly moving objects.

Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{7.50}$$

or

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{7.51}$$

where $R = R^{\mu}_{\mu}$, $T = T^{\mu}_{\mu}$; in the Newtonian limit reduce to $R_{00} = -\nabla^2 \mathcal{V}_{\mathrm{grav}}$, and thus

$$\nabla^2 \mathcal{V}_{\mathrm{grav}} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{7.52}$$

In the above $\mathcal{V}_{\mathrm{grav}}$ is the gravitational potential. The energy momentum tensor for a field ϕ is easily calculated from

$$T_{\mu\nu} = -\mathcal{L} g_{\mu\nu} + \partial_\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial^\nu \phi)} \tag{7.53}$$

and thus

$$T_{\mu\nu} = -\mathcal{L} g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi \tag{7.54}$$

for the Lagrangian given in (7.7). Since only $d\phi/dz \neq 0$, we can compute easily (for $\phi_{cl}(z)$) the nonvanishing components of $T_{\mu\nu}$

$$\begin{aligned}
T_{00} &= -\mathcal{L} = \frac{1}{2} \left(\frac{d\phi_{cl}}{dz} \right)^2 + V(\phi_{cl}) = \rho(\phi_{cl}) \\
T_{11} &= T_{22} = \mathcal{L} = -\rho(\phi_{cl}) \\
T_{33} &= \mathcal{L} + \left(\frac{d\phi_{cl}}{dz} \right)^2 = \frac{1}{2} \left(\frac{d\phi_{cl}}{dz} \right)^2 - V(\phi_{cl}) = 0
\end{aligned}$$

where $\rho(\phi_{cl})$ denotes the energy density of the classical solution in (7.25): $\rho(\phi_{cl}) = 2V(\phi_{cl})$. Notice that is is a perfectly well defined positive quantity. Since $T^1_1 = -T_{11}$, $T^2_2 = -T_{22}$, we get

$$T^{\mu}_{\mu} = 3T_{00} = 3\rho(\phi_{cl}) \tag{7.55}$$

This in turn leads to a negative sign in (7.52)

$$\nabla^2 \mathcal{V}_{\text{grav}} = -4\pi G\rho(\phi_{cl}) < 0 \tag{7.56}$$

The *gravitational field of a domain wall is repulsive* in spite of the positive mass, or energy density - *we have a remarkable phenomenon of antigravity.*

The source of this extraordinary fact lies in the nonvanishing (and negative) quantities T_{11} and T_{22}. In Newtonian gravity (*not* in a Newtonian limit of Einstein's gravity) one writes

$$\nabla^2 \mathcal{V}_{\text{newt}} = 4\pi G\rho(\phi_{cl}) > 0 \tag{7.57}$$

which is manifestly positive and implies an attractive gravitational force.

By analogy with a perfect fluid, the T^i_i are called $-p_i$, the pressure in the i direction, and thus $T_{11} = T_{22} < 0$, implies $T^1_1 = T^2_2 > 0$ or $p_1 = p_2 < 0$. The domain wall system behaves as if it carries negative pressure. This is the virtue of relativistic field theory. For example, if you imagine a nonvanishing energy in the vacuum, then obviously

$$T_{\mu\nu}(vacuum) = g_{\mu\nu} V(vacuum) \tag{7.58}$$

and thus $T^{\nu}_{\mu} = 4V = 4T^0_0 = 4T_{00}$ and we have a repulsive force

$$\nabla^2 \mathcal{V}_{\text{grav}} = -8\pi G T_{00} = -8\pi G V(vacuum) < 0 \tag{7.59}$$

This is the principle of the so-called inflationary cosmology which makes the universe grow by having the vacuum energy dominating over the matter and radiation one. But this is a different story we must leave aside.

Chapter 8

Realistic examples of Discrete Symmetries

The example we have studied based on the discrete symmetry $\phi \to -\phi$ is the simplest one and the prototype for more complicated and realistic particle physics systems. Before we turn to other examples, let us ask ourselves if the symmetry **D** exists in the Standard Model. Naively, one would be tempted to say yes, since $\phi \to -\phi$ (ϕ being the $SU(2)_L \times U(1)_Y$ doublet) is obviously a symmetry of the $SU(2)_L \times U(1)_Y$ symmetric theory. However it is a continuos gauge symmetry, since the gauge transformation

$$\phi \longrightarrow e^{i\tau_3\theta}\phi \tag{8.1}$$

reduces to **D** for $\theta = \pi$.

The most natural candidates for realistic discrete symmetries are clearly time reversal T and parity P. We discuss them in what follows.

8.1 A. T (or CP) symmetry

In relativistic field theory, the time reversal symmetry T is equivalent to CP; thus we study CP as a potential candidate for a spontaneously broken discrete symmetry.

Under CP a fermion field transforms as

$$\psi_{L,R} \overset{CP}{\longleftrightarrow} C\,\overline{\psi}^T_{L,R} \tag{8.2}$$

and thus

$$\overline{\psi}_L \psi R \overset{CP}{\longleftrightarrow} \overline{\psi}_R \psi L \tag{8.3}$$

From the general form of Yukawa couplings

$$\mathcal{L}_y = h\phi\bar{\psi}_L\psi_R + h^*\phi^\dagger\bar{\psi}_R\psi_L \tag{8.4}$$

we demand

$$\phi \overset{CP}{\longleftrightarrow} \phi^\dagger \tag{8.5}$$

and furthermore CP invariance would require $h = h^*$.

In the Standard Model ϕ is the Higgs doublet and the Yukawa couplings take the form given in (1.22). Since by a $SU(2)$ transformation we can always arrange

$$< \phi >= \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{8.6}$$

where v is a positive, real number, the spontaneous breaking of $SU(2)_L \times U(1)_Y$ preserves CP. Thus in the Standard Model or better in its minimal version) it is *not* possible to break CP spontaneously and one instead opts for complex Yukawa couplings which imply CP breaking already at the Lagrangian level.

Some more than two decades ago, T.D. Lee has suggested a simple extension of the Standard Model which can break CP spontaneously. He assumes simply the existence of two Higgs doublets, ϕ_1 and ϕ_2, and by a $SU(2)_L \times U(1)_Y$ puts their vevs in the form

$$< \phi_1 >= \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_1 \quad ; \quad < \phi_2 >= \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} v_2 e^{i\delta} \tag{8.7}$$

where v_1, v_2, θ and δ are real, positive numbers. Notice that we cannot demand $\theta = 0$ on the basis of gauge invariance. The values of θ and δ must be obtained by minimizing the Higgs potential.

The most general d=4 potential for two Higgs doublets ϕ_1 and ϕ_2 can be written as

$$\begin{aligned} V &= -\frac{\mu_i^2}{2}\phi_i^\dagger\phi_i + \frac{\lambda_i}{4}(\phi_i^\dagger\phi_i)^2 \\ &+ \frac{\lambda_3}{2}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \frac{\lambda_3'}{2}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\ &+ (\frac{\lambda_4}{2}\phi_1^\dagger\phi_1 + \frac{\lambda_5}{2}\phi_2^\dagger\phi_2)(\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1) \\ &+ \frac{\lambda_6}{2}[(\phi_1^\dagger\phi_2)^2 + (\phi_2^\dagger\phi_1)^2] \end{aligned} \tag{8.8}$$

Notice that the only term which depends on θ, but not on the phase δ, is λ'_3, and it becomes for the vevs in (8.7)

$$\frac{\lambda'_3}{2} v_i^2 \, v_2^2 \tag{8.9}$$

Obviously the sign of λ'_3 determines the value of θ at the minimum. Thus for

1. $\lambda'_3 > 0$, in order to minimize the positive value of (8.9) $\cos\theta$ is forced to vanish: $\theta_0 = \pi/2$.
2. $\lambda'_3 < 0$, the λ'_3 term is negative and the potential prefers to maximize the negative contribution from (8.9), thus $\theta_0 = 0$

Since $\theta_0 = \pi/2$ would imply the breaking of electromagnetic charge invariance, we choose 2), i.e. $\lambda'_3 < 0$. It is easy then to write down the potential (8.8) as a function of the phase δ

$$V = A - B\cos\delta + C\sin\delta \tag{8.10}$$

where A depends on $\mu_i, \lambda_i, \lambda_3$ and λ'_3, and

$$\begin{aligned} B &= -(\lambda_4 v_1^2 + \lambda_5 v_2^2) v_1 v_2 \\ C &= \lambda_6 v_1^2 v_2^2 \end{aligned} \tag{8.11}$$

We wish to minimize V as a function of δ

$$\frac{\partial V}{\partial \delta} = (B - 4C\cos\delta)\sin\delta = 0 \tag{8.12}$$

and thus either

$$\delta_0 = 0 \tag{8.13}$$

or

$$\cos\delta_1 = \frac{B}{4C} \tag{8.14}$$

are the extrema. It is easy to see that for

$$B > 0 \;\; ; \quad C > 0 \;\; ; \quad \frac{B}{4C} > 1 \tag{8.15}$$

δ_1 is a minimum and δ_0 is a maximum of the potential (8.10). In the range of parameters given by (8.15), which amounts to

$$\begin{aligned}\lambda_4 v_1^2 + \lambda_5 v_2^2 &< 0 \\ \lambda_6 &> 0\end{aligned} \tag{8.16}$$

CP symmetry is spontaneously broken.

Notice that, just like in our toy model with a **D** symmetry, there are two solutions δ_1 and $-\delta_1$ which satisfy (8.14) -this is due to the symmetry $\delta \to -\delta$ of the potential (8.10).

Thus the field ϕ of our toy model can be viewed as the phase field $\delta(x)$ and we can imagine a classical solution, a domain wall with

$$\begin{aligned}\delta(z \to -\infty) &= -\delta_1 \\ \delta(z \to +\infty) &= +\delta_1\end{aligned} \tag{8.17}$$

where again we take a wall in the $x-y$ plane. The precise form of the solution(8.17) is very hard to obtain and for the sake of simplicity we give an exact solution for the case $B=0$ or

$$\mathcal{L}(\delta) = \frac{1}{2} v^2 \partial_\mu \delta \partial^\mu \delta + m^4 [1 - \cos 2\delta] \tag{8.18}$$

where v and m are the scales given for dimensional reasons (in the $SU(2)_L \times U(1)$ we expect $v \sim m \simeq M_W$).

The minimum of (8.18) lies at $\delta = 0, \pi$ and the domain wall interpolates between these two degenerate vacua (the discrete symmetry is $\delta \to \delta + \pi$). From ()

$$\left(\frac{d\delta}{dz}\right)^2 = 2v \tag{8.19}$$

we get

$$\frac{d\delta}{\sin\delta} = \frac{m^2}{v} dz \tag{8.20}$$

and it is easy to show that for the boundary conditions

$$\begin{aligned}\delta(-\infty) &= 0 \\ \delta(+\infty) &= \pi\end{aligned} \tag{8.21}$$

we get

$$\delta(z) = 2 \tan^{-1} exp(\frac{m^2}{v} z) \tag{8.22}$$

Everything else we have said about domain walls applies to this example.

8.2 B. Parity

Equally fundamental is the symmetry between left and right, the parity. In the standard model P is broken explicitly and clearly, in order to break P spontaneously we must enlarge the gauge group. The minimal model is based on the gauge group

$$G_{LR} = SU(2)_L \times SU(2)_R \times U(1)_{Y'}$$

with the quarks and leptons completely symmetric under $L \leftrightarrow R$

$$\begin{aligned} Q_L &= \begin{pmatrix} u \\ d \end{pmatrix}_L \quad \overset{P}{\longleftrightarrow} \quad Q_R = \begin{pmatrix} u \\ d \end{pmatrix}_R \\ \ell_L &= \begin{pmatrix} \nu \\ e \end{pmatrix}_L \quad \overset{P}{\longleftrightarrow} \quad \ell_R = \begin{pmatrix} \nu \\ e \end{pmatrix}_R \end{aligned} \tag{8.23}$$

Notice that the requirement of left-right symmetry leads to the existence of the right-handed neutrino and now the neutrino mass becomes a dynamical issue, related to the pattern of symmetry breaking. In the Standard Model, where ν_R is absent, $m_\nu = 0$; here instead we shall need to explain why neutrinos are so much lighter than the corresponding charged leptons.

In this theory, the formula for the electromagnetic charge becomes

$$Q_{em} = I_{3L} + I_{3R} + \frac{Y'}{2} \tag{8.24}$$

with Y' being left-right symmetric.

It checks readily from $Y' = 2(Q_{em} - I_{3L} - I_{3R})$ that

$$\begin{aligned} Y' \begin{pmatrix} u \\ d \end{pmatrix}_L &= \frac{1}{3} \begin{pmatrix} u \\ d \end{pmatrix}_L \quad ; \quad Y' \begin{pmatrix} \nu \\ e \end{pmatrix}_L = - \begin{pmatrix} \nu \\ e \end{pmatrix}_L \\ Y' \begin{pmatrix} u \\ d \end{pmatrix}_R &= \frac{1}{3} \begin{pmatrix} u \\ d \end{pmatrix}_R \quad ; \quad Y' \begin{pmatrix} \nu \\ e \end{pmatrix}_R = - \begin{pmatrix} \nu \\ e \end{pmatrix}_R \end{aligned} \tag{8.25}$$

which tells us that Y' has a physical interpretation

$$Y' = B - L \tag{8.26}$$

This is in sharp contrast with the Standard Model, where the hypercharge Y was completely devoid of any physical meaning.

Our primary task is to break L-R symmetry, i.e. to account for the fact that $M_{W_R} \gg M_{W_L}$, W_R and W_L denoting right-handed and left-handed gauge bosons respectively. In order to do so we need a set of left-handed and right-handed Higgs scalars whose quantum numbers we will specify later. Imagine for the moment two scalars φ_L and φ_R with

$$\varphi_L \stackrel{P}{\longleftrightarrow} \varphi_R \tag{8.27}$$

Assume no terms linear in the fields (since φ_L and φ_R should carry quantum numbers under $SU(2)_L$ and $SU(2)_R$) we can write down the left-right symmetric potential

$$V = -\frac{\mu^2}{2}(\varphi_L^2 + \varphi_R^2) + \frac{\lambda}{4}(\varphi_L^4 + \varphi_R^4) + \frac{\lambda'}{2}\varphi_L^2\varphi_R^2 \tag{8.28}$$

where $\lambda > 0$ in order for V to be bounded from below, and we choose $\mu^2 > 0$ in order to achieve symmetry breaking in the usual manner. We rewrite the potential as

$$V = -\frac{\mu^2}{2}(\varphi_L^2 + \varphi_R^2) + \frac{\lambda}{4}(\varphi_L^2 + \varphi_R^2)^2 + \frac{\lambda' - \lambda}{2}\varphi_L^2\varphi_R^2 \tag{8.29}$$

which tells us that the pattern of symmetry breaking depends crucially on the sign of $\lambda' - \lambda$, since the first two terms do not depend on the direction of symmetry breaking (of course $\mu^2 > 0$ guarantees that $< \varphi_L >=< \varphi_R >= 0$ is a maximum and not a minimum of the potential).

Thus for

1. $\lambda' - \lambda > 0$, in order to minimize V we have either $< \varphi_L >= 0$, $< \varphi_R >\neq 0$, or vice versa. Due to the symmetry of V both solutions are equally probable.

2. $\lambda' - \lambda < 0$, we need $< \varphi_L >\neq 0 \neq< \varphi_R >$ and L-R symmetry implies $< \varphi_L >=< \varphi_R >$.

Obviously we choose 1), which implies that P is broken in nature. The phenomenology tells us that $< \varphi_L >= 0$, $< \varphi_R >\neq 0$; but the other, symmetric solution $< \varphi_L >\neq 0$, $< \varphi_R >= 0$ exists as well and it tells us that there must be domain walls between the two dependent minima. To make this more clear, let us introduce fields

$$\begin{aligned} \varphi = \varphi_L + \varphi_R &\stackrel{P}{\longrightarrow} \varphi \\ \varphi' = \varphi_L - \varphi_R &\stackrel{P}{\longrightarrow} -\varphi' \end{aligned} \tag{8.30}$$

and the solution 1) implies that $< \varphi' >\neq 0$. Then, this is precisely the situation that we have encountered before with the symmetry **D**, and the field φ' plays the role of our domain wall solution in section A of Lecture 8.

8.2.1 Left-Right and the phenomenology of weak interactions

What fields should we choose for the role of φ_L and φ_R ? It is not an easy question and only after long thought one arrives to the conclusion that the ideal candidates should be triplets, i.e.

$$\Delta_L(\bar{3}_L\,,\,1_R\,,\,2)\quad;\quad\Delta_R(\bar{1}_L\,,\,3_R\,,\,2) \tag{8.31}$$

where the quantum numbers denote $SU(2)_L$, $SU(2)_R$ and $B-L$ transformation properties. Simply speaking, Δ_L and Δ_R are $SU(2)_L$ and $SU(2)_R$ triplets, respectively, with $B-L$ numbers equal to two.

Writing $\Delta_{L,R}=\Delta^i_{L,R}\tau_i/2$ (τ_i being the Pauli matrices) as is usual for the adjoint representations, we find Yukawa couplings

$$\mathcal{L}_\Delta = h_\Delta(\ell^T_L\,C\,i\tau_2\,\Delta_L\,\ell_L + L\rightarrow R) + h.c. \tag{8.32}$$

To check the invariance of (8.32) under the Lorentz group and the gauge symmetry $SU(2)_L\times SU(2)_R\times U(1)_{B-L}$, recall

- that $\psi^T_L C\psi_L$ is a Lorentz invariant quantity for a chiral Weyl spinor ψ_L (and similarly for ψ_R).
- under the gauge symmetry $SU(2)_L$

$$\begin{aligned} \ell_L \longrightarrow \mathcal{U}_L\ell_L \quad , \quad \Delta_L &\longrightarrow \mathcal{U}_L\Delta_L\mathcal{U}^\dagger_L \\ \mathcal{U}^T_L(i\tau_2) &= (i\tau_2)\mathcal{U}^\dagger_L \end{aligned} \tag{8.33}$$

  and similarly for $SU(2)_R$

- the B-L number of the $\Delta_{L,R}$ fields is two.

This proves the invariance of (8.32) under all the relevant symmetries. Now, from their definition, the fields $\Delta_{L,R}$ have the following decomposition under the charge eigenstates

$$\Delta_{L,R} = \left[\begin{array}{cc} \Delta^+ & \Delta^{++} \\ \Delta^0 & -\Delta^+ \end{array}\right]_{L,R} \tag{8.34}$$

where we use the fact that $Tr\Delta_{L,R}=0$ and the charge is computed from $Q=I_{3L}+I_{3R}+(B-L)/2$.

Furthermore, Δ's being in the adjoint representation have

$$I_{3L}\Delta_L = \left[\frac{\tau_3}{2}\,,\,\Delta_L\right]$$
$$I_{3R}\Delta_R = \left[\frac{\tau_3}{2}\,,\,\Delta_R\right] \tag{8.35}$$

Notice the interesting consequence of doubly charged physical Higgs scalars in this theory.

From the general analysis of the spontaneous L-R symmetry breaking, we know that for a range of parameters of the potential the minimum of the theory can be chosen as

$$< \Delta_L >= 0 \quad , \quad < \Delta_R >= \left[\begin{array}{cc} 0 & 0 \\ v_R & 0 \end{array}\right] \tag{8.36}$$

From (8.32), we the obtain the mass for the right-handed neutrino ν_R

$$\mathcal{L}_m = h_\Delta \, v_R \, (\nu_R^T \, C \, \nu_R + \nu_R^\dagger \, C^\dagger \, \nu_R^*) \tag{8.37}$$

Thus the right-handed neutrino gets a large mass $m_N = h_\Delta v_R$, which corresponds to the scale of breaking of parity.

At the same time, $< \Delta_R > \neq 0$ breaks the $SU(2)_R$ gauge symmetry and furthermore $(B-L) < \Delta_R >= 2 < \Delta_R >$, thus using (8.36)

$$\frac{Y}{2} < \Delta_R >= (I_{3R} + \frac{B-L}{2}) < \Delta_R >= [\frac{\tau_3}{2}, < \Delta_R >] + < \Delta_R > \tag{8.38}$$

In other words, the original gauge symmetry is broken down to the Standard Model one

$$SU(2)_L \times SU(2)_R \times U(1)_{B-L} \xrightarrow{<\Delta_R>} SU(2)_L \times U(1)_Y \tag{8.39}$$

This can be checked by computing the gauge boson mass matrix. Use

$$D_\mu \Delta_R = (\partial_\mu - ig\vec{I}_R \cdot \vec{A}_\mu^R - ig_{B-L}A_\mu^{B-L})\Delta_R \tag{8.40}$$

where

$$\vec{I}_R \Delta_R = [\frac{\vec{\tau}}{2}\,,\,\delta_R] \tag{8.41}$$

and g_{B-L} and A_μ^{B-L} are the gauge coupling and the gauge potential of the $U(1)_{B-L}$ symmetry; and

$$\frac{1}{2}Tr(D_\mu < \Delta_R >)^\dagger (D^\mu < \Delta_R >) = Tr\frac{1}{2}\left|(g\vec{I}_R \cdot \vec{A}^R_\mu + g_{B-L}A^{B-L}_\mu)\Delta_R\right|^2 \tag{8.42}$$

The above computation is straightforward; by defining

$$W^\pm_R = \frac{A^1_R \mp iA^2_R}{\sqrt{2}} \tag{8.43}$$

we get

$$M^2_{W_R} = g^2\, v^2_R \tag{8.44}$$
$$M^2_{Z_R} = (g^2 + 4g^2_{B-L})\, v^2_R \tag{8.45}$$

where

$$Z_R = \frac{2g_{B-L}A^3_R + gA_{B-L}}{\sqrt{g^2 + 4g^2_{B-L}}} \tag{8.46}$$

is the massive neutral gauge field.

Thus the scale of parity breaking is related to the mass of the right-handed charged gauge bosons $W^\pm_R$. The predominant V-A nature of the weak interactions puts a lower limit on M_{W_R}

$$M_{W_R} > 1TeV \tag{8.47}$$

while the theory at this point cannot predict it. All we can say is that at $E \gg 1TeV$, one may be able to see a L-R symmetric world one day. This gives $m_N \simeq M_{W_R} > 1TeV$ and the right-handed neutrino becomes very heavy.

We shall not go any further into the phenomenological details of this theory, but we wish to close this section with some important comments on neutrino mass.

To complete the theory, one needs to include new Higgs fields (which contain the Weinberg-Salam Higgs doublet) that can give masses to quarks and leptons. This is done easily and it proceeds along the same lines as in the Standard Model. In the process we get the Dirac neutrino mass between ν_L and ν_R and in turn we end up with the see-saw mechanism for the neutrino mass described in the solution to problem #1. The important point here is that the mass of ν_R is determined by the scale of parity breaking and the smallness of the nuetrino mass is a reflection of the predominant V-A structure of the weak interaction.

In the see-saw mechanism, we have (assuming $m_D \simeq m_e$)

$$m_\nu \simeq \frac{m_e}{m_{\nu_R}} \simeq \frac{m_e}{M_{W_R}}$$

From $M_{W_R} \geq 1\,TeV$, and $m_e = 0.5\,MeV$ we get

$$m_\nu \leq 1\,eV$$

which explains clearly the observed smallness of m_ν. Obviously the measurement of the neutrino mass is an importnt probe of the idea of parity restoration at high energies $E > M_{W_R}$

Chapter 9

Strings

It is intuitively clear that the form of the solution, i.e. its symmetry, corresponds to the type of symmetry which is spontaneously broken. We have just learned that the spontaneous breaking of discrete symmetries leads to the existence of domain walls, topological defects with a discrete symmetry. It should not come out as a surprise that the spontaneous breakdown of a $U(1)$ symmetry allows for cylindrically symmetric solutions, the so called strings. Recall that the existence of domain walls was tied to the choice of boundary conditions; by asking that the solutions carry finite energy we demanded that the scalar field ϕ belongs to a set of zeros of the potential, i.e. we demanded $\phi(\pm\infty \in \mathcal{M}_0$, where $\mathcal{M}_0$ is the vacuum manifold. This provides, as we said, a map from $\mathcal{M}_\infty$ into $\mathcal{M}_0$, and the nontriviality of the map leads to the nontriviality of the solution.

Now, we wish to construct cylindrically symmetric solutions, and so we will ask (in cylindrical coordinates ρ, ϕ, z) that for $\rho \to \infty$, the scalar field again belongs to an approppiate $\mathcal{M}_0$. But now $\mathcal{M}_\infty$ is a circle and thus in order to have a nontrivial map $\mathcal{M}_\infty \to \mathcal{M}_0$, we need $\mathcal{M}_0$ to be a circle. This suggests obviously to take the group $G = U(1)$, and ϕ a complex field, charged under $U(1)$.

9.1 G = U(1)

The Lagrangian in this case is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - V(\phi) \tag{9.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$
D_\mu\phi \;=\; (\partial_\mu - igA_\mu)\phi \tag{9.2}
$$

and the potential we choose to have the usual Mexican hat form

$$
V = \frac{\lambda}{4}(\phi^*\phi - v^2)^2 \tag{9.3}
$$

Clearly, the vacuum manifold is a circle, since the minimum of V is at $V = 0$ for ϕ_0 which satisfies

$$
\begin{aligned}
|\phi_0|^2 &= v^2 \\
\phi_0 &= ve^{i\alpha}
\end{aligned} \tag{9.4}
$$

Thus $\mathcal{M}_0 = S^1$. Next, as we said, in cylindrical coordinates ρ, θ, z we look for a static, cylindrically symmetric solution, which defines $\mathcal{M}_\infty = \rho = R$, $R \to \infty$. Thus $\mathcal{M}_\infty = S^1$, too.

We look for a finite energy solution, or better to say for a solution with finite energy per unit length.

From (9.1), it is readily found that

$$
\frac{E}{L} = \int \rho\, d\rho \left[\frac{1}{2}(D_i\phi)^*(D^i\phi) + \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + V(\phi)\right] \tag{9.5}
$$

where

$$
\begin{aligned}
E_i &= F_{0i} \\
B_i &= \frac{1}{2}\epsilon_{ijk}F^{jk}
\end{aligned} \tag{9.6}
$$

Since each term in (9.5) is non-negative, we ask that for $R \to \infty$

$$
\begin{aligned}
V(\phi) &\longrightarrow 0 \\
D_i\phi &\longrightarrow 0 \\
E_i\phi &\longrightarrow 0\,, \quad B_i\phi \longrightarrow 0
\end{aligned} \tag{9.7}
$$

Thus

$$
\phi(R \to \infty) \in \mathcal{M}_0 \tag{9.8}
$$

which provides a map $\mathcal{M}_\infty \to \mathcal{M}_0$. It can be shown that all such maps are characterized by integers, i.e. we can write

$$\phi(R \to \infty) \longrightarrow v\, e^{i\theta} \tag{9.9}$$

The crucial demand is the single-valuedness of ϕ. We shall be mostly interested in the "minimal" solution with $n = 1$.

Notice that the form (9.9) appears to be a "pure gauge", since under $U(1)$ ϕ transforms as $\phi \to e^{-i\alpha}\phi$ and you could argue that by choosing $\alpha = n\theta$ we can cast ϕ in the form $\phi(R \to \infty) = v$. However, such a transformation is not well defined at the origin; alternatively you can say that (9.9) cannot hold to be true everywhere. More precisely, if we look for a solution in the form

$$\phi = v\, f(\rho)\, e^{in\theta} \tag{9.10}$$

then $f(\rho) \to 1$ when $R \to \infty$, and $f(\rho)$ must vanish at the origin. At that point ϕ is in the local maximum of the potential (9.3), and there will be some energy stored in the Higgs field. This is reminiscent of the situation encountered with domain walls before, when the nontrivial boundary conditions forced ϕ to vanish at $z = 0$.

From $D_i\phi \to 0$ at $\mathcal{M}_\infty$ we get

$$A_\mu \longrightarrow \frac{n}{g}\partial_\mu\theta \tag{9.11}$$

Again, at first glance A_μ appears to be a pure gauge, but this appearance is deceiving for the same reason that $\phi(\infty)$ is not a pure gauge.

Now, since the magnetic flux is given by

$$\int \vec{B} \cdot d\vec{S} = \oint_{R\to\infty} A_\mu dx^\mu = \frac{n}{g}\oint_{R\to\infty} \partial_\mu\theta dx^\mu = \frac{n}{g}\Delta\theta \tag{9.12}$$

we get

$$Flux = \frac{2\pi n}{g} \tag{9.13}$$

This means that there is (for $n \neq 0$) a nonvanishing magnetic flux "inside" our solution, and furthermore that the flux is quantized. It can also be shown that this flux is conserved in time. For the minimal solution, $n = 1$, we get that the flux $= 2\pi/g$. The symmetry of the problem dictates $\vec{B} = B_z$, and for $n = 1$ we get

$$2\pi \int_0^\infty B\,\rho\, d\rho = \frac{2\pi}{g} \tag{9.14}$$

Let us summarize the situation. By analogy with the domain wall solution, we expect an exponential die-off of $\phi(\rho)$ and $B(\rho)$, i.e. we expect to find the thickness to be $\delta \simeq, 1/v$ ($1/v$ is the only length scale in the theory), which characterizes the core of the string. In other words, for

$$\begin{aligned} r \leq \delta \quad &: \quad \phi = 0\,,\, B = \text{const} \\ r > \delta \quad &: \quad \phi \to v e^{i\theta}\,,\, B \to 0 \end{aligned} \tag{9.15}$$

If we take $B = const$, (9.14) gives

$$B \int_0^\delta \rho\, d\rho = \frac{1}{g} \tag{9.16}$$

or

$$B = \frac{2}{g\delta^2} \tag{9.17}$$

Then from (9.5), in the approximation (9.15)

$$\frac{E}{L} = 2\pi \left[\int_0^\delta \frac{1}{2} \frac{4}{(g\delta^2)^2}\, \rho\, d\rho + \int_0^\delta \frac{\lambda}{4} v^4\, \rho\, d\rho \right] \tag{9.18}$$

which gives

$$\frac{E}{L} = 2\pi \left[\frac{1}{g^2\delta^2} + \frac{\lambda}{8} v^4 \delta^2 \right] \tag{9.19}$$

You can see why there is a finite core: the energy in the magnetic field prefers $\delta \to \infty$ (due to (9.17)), whereas the energy in the Higgs field likes $\delta \to 0$. Minimizing the energy with respect to δ

$$\frac{\partial E/l}{\partial \delta} = 2\pi \left[-\frac{1}{g^2\delta^3} + \frac{\lambda}{4} v^4 \delta \right] = 0 \tag{9.20}$$

gives the thickness of the string

$$\delta = \left(\frac{8}{g^2\lambda} \right)^{1/4} v^{-1} \tag{9.21}$$

and the energy per unit length becomes

$$\frac{E}{L} = 4\pi 8^{1/4} \sqrt{\frac{\lambda}{g^2}}\, v^2 \tag{9.22}$$

where we should not trust the numerical factor

9.1.1 The stability of the string solution

We have remarked before that it can be shown that the flux of the string is conserved in time. Thus, for the minimal string $n = 1$ this implies the stability, since the state with lower energy than the string, i.e. the vacuum, has $n = 0$ and n cannot be changed in time. The interested reader will be referred to the original work by Nielsen and Olesen. We will, on the other hand, discuss in detail the analogous issue of the stability of monopoles.

9.1.2 Strings and Cosmology

Just like domain walls, strings are expected to be produced in the phase transitions of the early hot universe. We cannot do justice to this interesting subject here, but suffice it to mention that at very high temperatures one expects symmetries to be restored, i.e. for some $T > T_c$, we expect $< \phi(T) >= 0$. If so, there will be a phase transition when the Universe learns that $< \phi >\neq 0$ (as it cools down). It is clear that the whole Universe cannot manage to choose one and the same value of $< \phi >$; its variation is expected to lead to the production of strings. It is hoped that these objects may actually provide the seeds for the formation of clusters of galaxies.

9.2 Realistic models of strings

We have found that the domain walls do not exist in the Standard Model, but that some of its simple extensions which break P or CP spontaneously do contain them. What is the analogue situation with strings ? What about strings in the Standard Model ?

Well, the message of the last section is that we need a spontaneously broken $U(1)$ symmetry in order to have a string, since then $\mathcal{M}_0 = S^1$ and the map $\phi(R \to \infty) \in \mathcal{M}_0$ becomes a well-defined map $\mathcal{M}_\infty \to \mathcal{M}_0$ characterized by an integer n. Now, in the standard Model $SU(2)_L \times U(1)_Y$ is broken down to $U(1)_{em}$ through the vev of the doublet Φ, which means that it is actually $SU(2)$ which is broken; $U(1)$ is only redefined. But $\mathcal{M}_0(\Phi) = S^3$ (see the section on monopoles), and there is no nontrivial map $S^1 \to S^3$ (if you wish a circle on the sphere S^3 can be always shrunk to a point).

This, of course, may sound too formal or too complicated. After all, you can always look for an embedded string solution so that for $R \to \infty$ the doublet Φ has the form

$$\Phi \longrightarrow \begin{pmatrix} 0 \\ ve^{in\theta} \end{pmatrix} \tag{9.23}$$

This looks very much like a string, one could say. The trouble is that now we can make a full $SU(2)_L \times U(1)$ gauge transformation on Φ and it is not clear that we cannot rotate the phase θ away (in the $U(1)$ case such a transformation did not make sense at $\rho \to 0$). We shall not enter into the subtleties of this interesting question here. Suffice it to say that one can construct the same solution with (9.23) in the $U(1)$ case, but that such solution is not stable; the instability is found in the other direction of the $SU(2)$ space. Thus in the Standard Model there are no stable string solutions.

It can be shown that the same is true of a minimal $SU(5)$ theory. At the first stage of symmetry breaking we break $SU(5)$ down to $U(1)_{em} \times SU(2)_L \times SU(3)_c$ and no strings get formed (see however the discussion on monopoles below). To get strings one must resort to more complicated grand unified theories, such as $SO(10)$, that we did not discuss in this course. We turn rather to the more promising and for us more important issue of monopoles.

Chapter 10

Monopoles

The machinery for constructing classical solutions we have learned so far serves (to us) mainly to discuss the important topological defects which carry monopole charge. Such objects are expected to exist in GUTs based on simple groups, since these theories imply the quantization of charge. The connection between the quantization of charge and the existence of magnetic monopoles has been realized long ago by Dirac, who has studied the consistence of the quantum theory of monopoles.

We offer here a simplified, semi-classical argument in favor of this connection. Imagine that there exists *one* monopole in the universe with a monopole charge g_m. Its magnetic field is given by

$$\vec{B} = \frac{g_m}{4\pi}\frac{\hat{r}}{r^2} \tag{10.1}$$

where $\hat{r}$ is the unit vector in the $\vec{r}$ direction ($\hat{r}\,r = \vec{r}$). Notice that in the case of the magnetic monopole we cannot write $\vec{B} = \vec{\nabla} \times \vec{A}$, since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$, but now $\vec{\nabla} \cdot \vec{B} = g_m \delta^3(0)$. Thus we cannot ask in general $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will be able to say more about this when we construct a monopole solution below.

Furthermore, imagine a particle of arbitrary electric charge q and mass m moving in the field of a monopole; it will feel the force

$$\vec{F} = q\frac{d\vec{r}}{dt} \times \vec{B} \tag{10.2}$$

Let us now calculate the rate of change of angular momentum of such a particle. From $\vec{L} = m\vec{r} \times \frac{d\vec{r}}{dt}$

$$\frac{d\vec{L}}{dt} = m\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \vec{F} \tag{10.3}$$

and thus

$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= \frac{g_m q}{4\pi r^2}\vec{r}\times\frac{d\vec{r}}{dt}\times\hat{r} \\
&= \frac{g_m q}{4\pi r^2}\left[r\frac{d\vec{r}}{dt}-\hat{r}\left(\frac{d\vec{r}}{dt}\cdot\vec{r}\right)\right] \\
&= \frac{g_m q}{4\pi r^2}\left[r\frac{d\vec{r}}{dt}-\vec{r}\frac{d\vec{r}}{dt}\right]
\end{aligned} \tag{10.4}
$$

Now, from

$$
\frac{d}{dt}(\hat{r}) = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{r(\frac{d\vec{r}}{dt}) - \vec{r}\frac{dr}{dt}}{r^2} \tag{10.5}
$$

we find that the angular momentum is not conserved in time, its rate of change is

$$
\frac{d\vec{L}}{dt} = \frac{g_m q}{4\pi}\frac{d}{dt}(\hat{r}) \tag{10.6}
$$

We can define the "total" angular momentum $\vec{J}$ which is conserved in time

$$
\vec{J} = \vec{L} - \frac{g_m q}{4\pi}\hat{r} \tag{10.7}
$$

with $d\vec{J}/dt = 0$

Up to now our discussion was purely classical, but eventually we need to quantize the theory. Obviously, the consistency with the quantization of angular momentum requires

$$
\frac{g_m q}{4\pi} = n \tag{10.8}
$$

where n is an integer: $n = 0, 1, 2, ...$ But this is a remarkable finding: even a single monopole in the Universe would demand the quantization of charge, since $q = \frac{4\pi}{g_m}n$. Of course, the basic unit is not predictable, but the observed phenomena of charges being an integer multiple of the charge of the down quark would be reproduced. It is not surprising that the quest for those objects became a dream of both theorists and experimentalists.

I wish to stress here that the work of Dirac dealt with the full quantum mechanics of monopoles, not just our simple-minded argument. By demanding the single-valuedness of the wave function of the electron Dirac provided a quantum mechanical version of (10.8). On the other hand, the motivation to study these objects is inspired by the possibility of making Maxwell's theory symmetric between electric and magnetic charges.

Now, we have repeatedly stressed the fact that unified theories based on the simple group G imply the quantization of charge, since now

$$Q\mathrm{em} = \sum_i c_i T_i \tag{10.9}$$

where T_i are the generators of G with $TrT_i = 0$. But then $TrQ_{em} = 0$, and so Q_{em} can only come in integer units. Should we then expect the existence of magnetic monopoles in these theories? The answer is yes, as we now demonstrate.

10.1 Monopole solution

A prototype for a simple unified theory is the group $SO(3)$, the minimal and simplest such theory. The only neutral generator is T_3 and thus $Q_{em} = T_3$, and the eigenvalues of T_3 are quantized as we know. Ignoring for the moment the question of the theory being realistic or not, imagine simply an $SO(3)$ local gauge theory with a triplet of Higgs scalars. Its Lagrangian is then

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a} + \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) + V(\phi) \tag{10.10}$$

where

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu \\ D_\mu\phi^a &= \partial_\mu\phi^a + \epsilon_{abc}A^b_\mu\phi^c \\ V(\phi) &= \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2 \qquad a = 1,\,2,\,3 \end{aligned} \tag{10.11}$$

The theory consists of the triplet of gauge bosons in the adjoint representation and the similar triplet of Higgs scalars. In our notation, the $SO(3)$ generators are given by

$$(T_a)^{ij} = -i\epsilon^{aij} \tag{10.12}$$

Of course, one of them, say T_3, can be diagonalized ($SO(3)$ has rank 1), and for a triplet the eigenvalues are 1, 0, -1. In this representation

$$Q_{em} = (T_3)_{\mathrm{diag.}} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \tag{10.13}$$

which is nothing but the reflection of the act that the charge is quantized.

It is easy to find the vacuum manifold $\mathcal{M}_0$ from (10.11): $V \geq 0$ dictates at the minimum

$$\phi^a_o \phi^a_o = v^2 \tag{10.14}$$

or

$$\mathcal{M}_0 = \left\{ (\phi^1_0)^2 + (\phi^2_0)^2 + (\phi^3_0)^2 = v^2 \right\} = S^2 \tag{10.15}$$

which shows that $\mathcal{M}_0$ is a three-dimensional sphere S^2.

By now the procedure for constructing a static classical solution is straightforward.: in order to have a nontrivial map $\mathcal{M}_\infty \longrightarrow \mathcal{M}_0$, we choose the boundary conditions such as to have $\mathcal{M}_\infty = S^2$, too. This implies spherical symmetry, and in spherical coordinates r, θ, ϕ we define a sphere at "infinity" $r = R \to \infty$, or

$$\mathcal{M}_\infty = \{\text{fixed}\, r = R\ ,\quad R \to \infty\} = S^2 \tag{10.16}$$

Next, as before with domain walls and strings we ask that the energy of the solution be finite. From (10.10) and (10.11) the energy of a static configuration is

$$E = \int d^3x \left[\frac{1}{2}\vec{B}^a\vec{B}^a + \frac{1}{2}\vec{E}^a\vec{E}^a + \frac{1}{2}D_i\phi^a D_i\phi^a + V(\phi) \right] \tag{10.17}$$

where we expect $\vec{E}^a = 0$, but more about it later. Each of the terms in (10.17) is non-negative and thus at $\mathcal{M}_\infty$ we must have

$$\begin{aligned} \vec{B}^a \longrightarrow 0 \quad &, \qquad \vec{E}^a \longrightarrow 0 \\ D_i\phi^a \longrightarrow 0 \quad &, \qquad V(\phi) \longrightarrow 0 \end{aligned} \tag{10.18}$$

the last condition implying

$$\phi^a(\mathcal{M}_\infty) \in \mathcal{M}_0 \tag{10.19}$$

which is a map from S^2 onto S^2 and can be characterized by an integer. More precisely, the most general such map has the form

$$\begin{aligned} \phi^1_0 &= \sin\theta \cos n\varphi \\ \phi^2_0 &= \sin\theta \sin n\varphi \\ \phi^3_0 &= \cos\theta \end{aligned} \tag{10.20}$$

which covers a sphere $\mathcal{M}_0$ times. We are interested in the minimal case $n = 1$, which can also be written

$$\phi_0^a = v\frac{x_a}{r} \tag{10.21}$$

Thus, from now on we demand $\phi^a \to \phi_0^a$ in (10.21).

Now, before we go on studying the properties of the solution which satisfies (10.21), we should first construct the charge and the field tensor which correspond to it. Recall that for the vacuum, the state of energy zero, the field should point in one and the same direction. If we choose

$$< \phi_0 >^a = v\delta_{a3} \tag{10.22}$$

then the charge Q_{em} is given by T_3, since

$$\begin{aligned} (Q_{em} < \phi_0 >)^a &= (Q_{em})_{ab} < \phi_0 >^b = (T_3)_{ab} < \phi_0 >^b \\ &= -i\epsilon^{iab} v\delta^{b3} \end{aligned} \tag{10.23}$$

This means that Q_{em} annihilates the vacuum as it should.

It is equally easy to see that $T_1 < \phi_0 > \neq 0 \neq T_2 < \phi_0 >$, and thus the $SO(3)$ gauge symmetry is broken down to $U(1)$, which we wish to identify with $U(1)_{em}$. The same can be seen from (10.10) and (10.11), which give

$$\begin{aligned} \frac{1}{2}(D_\mu < \phi >)^{a\,*}(D_\mu < \phi >)^a &= \frac{1}{2}g^2v^2\epsilon_{ab3}\epsilon_{ac3}A_\mu^b A^{\mu\, c} \\ &= \frac{1}{2}g^2v^2(A_\mu^1 A^{\mu\, 1} + A_\mu^2 A^{\mu\, 2}) \end{aligned} \tag{10.24}$$

Thus two gauge bosons get the mass, and the third A_μ^3 remain massless. We identify A_μ^3 with the photon, and we can write $W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp A_\mu^2)$ for the massive gauge bosons of the fixed charge, with $m_W^2 = g^2v^2$.

For the case (10.21), we obviously need an analogous expression for Q_{em}. A little thought suggests

$$Q_{em} = T^a\frac{x_a}{r} \tag{10.25}$$

since now

$$\begin{aligned} (Q_{em}\phi)^a &= \frac{x_c}{r}(T^c)_{ab}\phi^b \\ &= -i\epsilon^{abc}v\frac{x_c x_b}{r^2} = 0 \end{aligned} \tag{10.26}$$

where we have inserted (10.21) for ϕ^b.

Similarly, we can show that the photon is now given by

$$A_\mu = \frac{x_a}{r} A_\mu^a \tag{10.27}$$

since the gauge boson mass matrix from (10.21) becomes

$$\begin{aligned}&\frac{1}{2} g^2 v^2 \epsilon_{abc}\epsilon_{ade}\frac{x_c x_e}{r^2} A_\mu^b A^{\mu\, d}\\ =& \frac{1}{2} g^2 v^2 [\delta_{bd} - \frac{x_b x_d}{r^2}] A_\mu^b A^{\mu\, d}\end{aligned} \tag{10.28}$$

or

$$M_{ab}^2(A) = g^2 v^2 \frac{r^2 \delta_{ab} - x_a x_b}{r^2} \tag{10.29}$$

But then $M_{ab}^2(A) x_b / r = 0$, which shows that A_μ in (10.27) remains massless and thus can be identified with the photon.

Finally, we need the $U(1)_{em}$ electromagnetic tensor, the analog of $F_{\mu\nu}(\text{vac}) = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$. You may say that we could take $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in (10.27), but we must be more careful about it, since as we said before this is only true when there are no magnetic monopoles. We need a form of $F_{\mu\nu}$ which satisfies the following two conditions

1. it is gauge invariant, i.e. it does not depend on any direction in the group space $SO(3)$; and
2. it reduces to $F_{\mu\nu} = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i$ for the vacuum configuration $< \phi_a >= v\delta_{a\, i}$ (more precisely, for any configuration $< \phi_a >= v\delta_{a\, 3}$, we should get $F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$).

Some thought tells us that it will take the form

$$F_{\mu\nu} = F_{\mu\nu}^a \frac{\phi^a}{|\phi|} - \frac{1}{g}\epsilon^{abc}\frac{(D_\mu \phi)^a (D_\nu \phi)^b \phi^c}{|\phi|^3} \tag{10.30}$$

where $|\phi|$ is the magnitude of $\vec{\phi}$: $|\phi|^2 = \phi^a \phi^a$.

The form in (10.30) is dictated by the conditions 1) and 2) above, and the requirement to be linear in the derivatives of A_μ^a and at most quadratic in A_μ^a. For the vacuum configuration $< \phi^a >= v\delta^{a3}$,

$$
\begin{aligned}
F_{\mu\nu}(<\phi>) &= F^3_{\mu\nu} - \frac{1}{g}\epsilon^{3bc}g^2 A^b_\mu A^c_\mu \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + g(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu) - g(A^1_\mu A^2_\nu - A^2_\mu A^1_\nu) \\
&= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu
\end{aligned}
\tag{10.31}
$$

Thus 2) is satisfied, and so is 1) as it is obviously gauge invariant.
Now, for our choice (10.1), i.e. $D_i\phi^a \to 0$ and $\phi^a \to \nu x_a/r$, we get

$$
F_{\mu\nu} \longrightarrow F^a_{\mu\nu}\frac{x_a}{r}
\tag{10.32}
$$

a form that we could have guessed by analogy with (10.27).
Now, from (10.27), i.e. from $D_i\phi^a = 0$, we get

$$
\partial_i\phi^a + g\epsilon^{abc}A^b_\mu\phi^c = 0
\tag{10.33}
$$

which using (10.21) gives

$$
A^a_i \stackrel{r\to\infty}{\longrightarrow} \epsilon^{aij}\frac{x_j}{gr^2}
\tag{10.34}
$$

It is easy to compute $F_{\mu\nu}$ in (10.32) for the above form of (10.34)

$$
\begin{aligned}
F_{ij} &\stackrel{R\to\infty}{\longrightarrow} (\partial_i A^a_j - \partial_j A^a_i + g\epsilon^{abc}A^b_i A^c_j)\frac{x^a}{r} \\
&= 2\epsilon^{aji}\frac{x_a}{gr^3} + \epsilon^{abc}\epsilon^{bik}\epsilon^{cjl}\frac{x_k x_l x_a}{gr^5} \\
&= 2\epsilon^{aji}\frac{x_a}{gr^3} - (\delta_{ai}\delta_{ck} - \delta_{ak}\delta_{ci})\epsilon^{cjl}\frac{x_k x_l x_a}{gr^5} \\
&= 2\epsilon^{aji}\frac{x_a}{gr^3} - \epsilon^{kjl}\frac{x_i x_k x_l}{gr^5} + \epsilon^{ijl}\frac{x_l(x_a x_a)}{gr^5} \\
&= -\epsilon^{ija}\frac{x_a}{gr^3}
\end{aligned}
\tag{10.35}
$$

From $F_{ij} = \epsilon ijkB_k$, we find that (10.35) corresponds to the field of a magnetic monopole

$$
B_k \stackrel{r\to\infty}{\longrightarrow} -\frac{x_a}{gr^3}
\tag{10.36}
$$

with a magnetic charge

$$
g_m = \frac{4\pi}{g}
\tag{10.37}
$$

This is the promised solution of a magnetic monopole and you explicitly have $F_{\mu\nu} \neq \partial_\mu A_\nu - \partial_\nu A_\mu$, since from (10.27) for A_i^a satisfying (10.34) we one gets $A_\mu \to 0$. Of course, by now the above fact should not come as a surprise, since the fact that there is a magnetic monopole means precisely $\vec{B} \neq \vec{\nabla} \times \vec{A}$.

Now, we should study the stability of our solution whose asymptotic form is given by (10.36), and we do it by showing that the magnetic charge is conserved in time.

It is easy to see that the following current

$$k_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\partial^\nu\hat{\phi}^a\,\partial^\rho\hat{\phi}^b\,\partial^\sigma\hat{\phi}^c \tag{10.38}$$

is conserved ($\hat{\phi}^a$ is the unit vector: $\hat{\phi}^a \equiv \phi^a/|\phi|$)

$$\partial^\mu k_\mu = 0 \tag{10.39}$$

due to the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$ and the symmetry of partial derivatives. Thus the corresponding charge

$$Q = \int d^3x k_0 \tag{10.40}$$

is conserved in time

$$\frac{dQ}{dt} = 0 \tag{10.41}$$

A simple computation gives

$$\begin{aligned} Q &= \frac{1}{8\pi}\int d^3x\epsilon_{ijk}\epsilon_{abc}\partial_i\hat{\phi}^a\,\partial_j\hat{\phi}^b\,\partial_k\hat{\phi}^c \\ &= \frac{1}{8\pi}\int d^3x\epsilon_{ijk}\epsilon_{abc}\partial_i(\hat{\phi}^a\,\partial_j\hat{\phi}^b\,\partial_k\hat{\phi}^c) \\ &= \frac{1}{8\pi}\int dS_i(\epsilon_{ijk}\epsilon_{abc}\hat{\phi}^a\,\partial_j\hat{\phi}^b\,\partial_k\hat{\phi}^c) \end{aligned} \tag{10.42}$$

where the last integral is just a surface one. We can turn it into an integral over θ, ϕ variables called α_i ($\alpha_1 = \theta$, $\alpha_2 = \phi$), using

$$dS_i = d\theta d\phi \sum_{mn}\sum_{pq}\frac{\partial x^m}{\partial\alpha_p}\frac{\partial x^n}{\partial\alpha_q} \tag{10.43}$$

In turn, this gives the conserved charge Q

$$Q = \frac{1}{8\pi} \int d\theta d\phi \left[2\hat{\phi}^1 \left(\frac{\partial\hat{\phi}^2}{\partial\theta} \frac{\partial\hat{\phi}^3}{\partial\phi} - \frac{\partial\hat{\phi}^3}{\partial\theta} \frac{\partial\hat{\phi}^2}{\partial\phi} \right) + \text{cyclic} \right] \tag{10.44}$$

which for the map (10.20) gives

$$\begin{aligned} Q &= \frac{n}{4\pi} \int_o^\pi d\theta \int_0^{2\pi} d\phi \left[\sin^3\theta (\cos^2 n\phi + \sin^2 n\phi) + \cos^2\theta \sin\theta (\cos^2 n\phi + sin^2 n\phi) \right] \\ &= \frac{n}{4\pi} \int_o^\pi d\theta \sin\theta \int_0^{2\pi} d\phi = n \end{aligned} \tag{10.45}$$

Thus the charge Q is an integer which counts how many times $\phi^a(R \to \infty)$ maps around the sphere $\mathcal{M}_0 = S^2$.

The physical meaning of the charge Q becomes clear if one proves (and we will not) the following expression

$$\epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = \frac{4\pi}{e} k_\mu \tag{10.46}$$

For $B_i = \frac{1}{2} \epsilon_{ijk} F^{jk}$, this means

$$\vec{\nabla} \cdot \vec{B} = \frac{4\pi}{e} k_0 \tag{10.47}$$

and thus using Stoke's theorem

$$g_m = \oint_{R\to\infty} \vec{B} d\vec{S} = \int \vec{\nabla} \cdot \vec{B} d^3x = \frac{4\pi}{e} Q \tag{10.48}$$

Thus the conserved charge Q is proportional to the magnetic charge and (10.45) together with (10.46) implies

1. that $dg_m/dt = 0$, i.e. the monopole charge is conserved.

2. that g_m is quantized

$$g_m = \frac{4\pi}{e} n \tag{10.49}$$

The importance of 1) should be clear, since it implies immediately the stability of a monopole solution with $n = 1$, it simply has no state to decay into. Of course, a solution with $n = 2$ can in principle decay into two $n = 1$ states and thus higher n solutions may not be stable.

The formula (10.49) may be somewhat of a surprise. It agrees with (10.8) which we offered by semiclassical arguments, asking for the quantization of the total angular momentum for the electric charge-magnetic monopole system. On the other hand, (10.49) was derived by purely classical arguments. It may be a bit more clear, if we notice that our condition (10.21) is invariant under the total angular momentum

$$\vec{J} = \vec{L}_{(A)} + \vec{T} \tag{10.50}$$

where $\vec{L}_{(A)}$ is the total orbital angular momentum in the presence of the electromagnetic field and $\vec{T}$ is an internal $SO(3)$ isospin. Now

$$\vec{L}_{(A)} = \vec{r} \times [\vec{p} + g\vec{A}^a T^a] \tag{10.51}$$

using the form for $\vec{A}$ in (10.34), we can write

$$L_{(A)\,i} = L_i + g\epsilon_{ijk} x_j A_k^a T^a = L_i + g\epsilon_{ijk}\epsilon_{akl}\frac{x_j x_l T_a}{gr^2} \tag{10.52}$$

This gives

$$L_{(A)\,i} = L_i - T_i + \frac{x_i x_a T_a}{r^2} \tag{10.53}$$

and thus we get from (10.50)

$$\vec{J} = \vec{L} + \hat{r}\frac{x_a T_a}{r} = \vec{L} + \hat{r} Q_{em} \tag{10.54}$$

Comparison with (10.7) gives

$$Q_{em} = -\frac{q g_m}{4\pi} \tag{10.55}$$

and since the eigenvalues of Q_{em} are integers, we again get the quantization condition (10.49).

In summary, it has been known for a long time that the existence of magnetic monopoles would imply the quantization of charge. On the other hand, a theory based on a simple group such as $SO(3)$ has already charge quantization built in, and it predicts the existence of monopoles as extended objects whose magnetic charge comes out properly quantized. I say extended objects, since the size of the monopole is nonzero. We can determine it from the minimization of the energy or the mass of the monopole, as we did in the case of domain walls and strings.

We know that at the origin $\phi^a \to 0$ and thus the potential is at its maximum: similarly the magnetic field vanishes. Define again the interior of the string $r \leq \delta$ as the region where this happens, i.e.

$$r \leq \delta; \quad \phi^a = 0, \quad \vec{B} = 0 \quad \vec{A} = 0$$

In this approximation (to be justified by the smallness of the thickness of the monopole δ) the energy of the monopole is given by

$$\begin{aligned} E_M &= \frac{\lambda}{4} v^4 \frac{4}{3} \pi \delta^3 + \frac{1}{2} 4\pi \int_\delta^\infty \vec{B}^2 r^2 dr \\ &= \frac{\lambda}{3} v^4 \pi \delta^3 + \frac{2\pi}{g^2 \delta} \end{aligned} \tag{10.56}$$

Just as in the case of the string, the energy stored in the Higgs field prefers $\delta \to 0$, while the magnetic field energy grows with small δ. The minimum of the energy is obtained by δ satisfying

$$0 = \frac{\partial E_m}{\partial \delta} = \lambda v^4 \pi \delta^2 - \frac{2\pi}{g^2 \delta^2} \tag{10.57}$$

Thus, we get

$$\delta \simeq (\lambda g^2)^{-1/4} \frac{1}{v} \tag{10.58}$$

as expected on purely dimensional grounds.

Now, what scale should we associate with v, the scale of the symmetry breaking in our $O(3)$ model ? More precisely, can v be M_W, the scale of the weak interactions, i.e. can there be monopoles in the standard model?

The answer, unfortunately, is no. Namely, in the standard electro-weak model the symmetry breaking is $SU(2)_L \times U(1)_Y \to U(1)_{em}$, through the vev of the Higgs doublet Φ, which is equivalent to a full braking of $SU(2)$, rather than $SU(2) \to U(1)$ which we needed in our example above. More precisely, the vacuum manifold in the case of a doublet is the three-sphere

$$\mathcal{M}_0 = \left\{ \Phi_0^\dagger \Phi_0 = v^2 \right\} = S^2 \tag{10.59}$$

since we can write

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \quad , \quad \phi_i \in R \tag{10.60}$$

and $\Phi^\dagger \Phi = \sum_{i=1}^4 \phi_i^2$

Thus

$$\mathcal{M}_0 = \left\{ (\phi_{01})^2 + (\phi_{02})^2 + (\phi_{03})^2 + (\phi_{04})^2 = v^2 \right\} \tag{10.61}$$

which is precisely the equation for a three-dimensional sphere embedded in a four-dimensional Euclidean space.

Since $\mathcal{M}_\infty = S^2$, the map $\mathcal{M}_\infty \to \mathcal{M}_0$ by our boundary conditions $\phi_i \to \phi_{0i}$ when $R \to \infty$ becomes a map $S^2 \to S^3$. It can be shown that no such nontrivial map exists; if you wish in this case the result is equivalent to the case $n = 0$.

Another way of understanding this result follows from the fact that the monopole solution exists when the electric charge is quantized, or when a simple group is broken down to a subgroup containing a $U(1)$ factor. In the case of the Standard Model, it is $U(1)_Y$ which is the culprit responsible for the absence of the quantization of charge or the absence of monopoles.

10.1.1 Grand unified Monopoles

Well, we know of a realistic example of a theory that provides quantization of charge; these are the grand unified theories whose minimal prototype is $SU(5)$, the theory which was object of our focus. In the first step of symmetry breaking, the $SU(5)$ symmetry is broken down to the standard model symmetry

$$SU(5) \xrightarrow{M_X} SU(3)_c \times SU(2)_L \times U(1)_Y \tag{10.62}$$

at a a large scale $M_X \simeq 10^{16} GeV$. This scale is simply the vacuum expectation value of the adjoint representation Σ

$$\Sigma = v_X \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{pmatrix} \tag{10.63}$$

with

$$M_X \simeq g v_X \tag{10.64}$$

As we have discussed in section 5, the above vev preserves obviously $SU(3)_c$ and $SU(2)_L \times U(1)$.

The neutral generators of $SU(2)_L \times U(1)$ are given in (3.3) and (3.4), which corresponds to the electro-magnetic charge operator

$$Q_{em} = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix} \tag{10.65}$$

Now, we wish to find a spherically symmetric solution, the monopole, as we did for the $SO(3)$ case. To start with, we should find an analogue of formula (10.25) for the $SU(5)$ theory. There is clearly more than one way of doing it; we choose

$$Q_{em}(r) = \begin{bmatrix} -\frac{1}{3} & & & \\ & -\frac{1}{3} & & \\ & & \frac{1}{3}1 - \frac{2}{3}\hat{r}\cdot\vec{\tau} & \\ & & & 0 \end{bmatrix} \tag{10.66}$$

where $\vec{\tau}$ are the usual $2\times$ 2 Pauli matrices and (10.66) will be valid for $r \to \infty$, i.e. very far from the monopole.

Notice that for the direction $\hat{r} = \hat{z}$, we get

$$\begin{aligned} Q_{em}(r) &= \begin{bmatrix} -\frac{1}{3} & & & \\ & -\frac{1}{3} & & \\ & & \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \\ & & & 0 \end{bmatrix} \\ &= \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix} \end{aligned} \tag{10.67}$$

which agrees with (10.65) for the vacuum as it should.

The next task is to generalize the formula (10.32) which gave us the electromagnetic field for the monopole configuration. To achieve this, notice that(10.32) can be rewritten in a somewhat different form

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{2}Tr(F^a_{\mu\nu}T^aQ_{em}) = \\ &= Tr\,\frac{1}{2}F^a_{\mu\nu}T^aT^b\frac{x_b}{r} = F^a_{\mu\nu}\frac{x_a}{r} \end{aligned} \tag{10.68}$$

We can use the same trick for the $SU(5)$ case, i.e. use the same formula above, but with $T_a = \frac{\lambda_a}{2}$ being the $SU(5)$ generators. We will not bother you with the details of the $SU(5)$ solution here, but it should be clear from the above that such a solution can be constructed in exactly the same manner as in the $SO(3)$ case. Besides having monopole charge, it can be shown that it carries color too, and of course, we believe that the color degrees of freedom will be confined.

The crucial point in all this is that in the realistic $SU(5)$ theory we expect the existence of a superheavy magnetic monople with a mass of the order of

$$M_m \simeq 10^{16} - 10^{17} GeV \tag{10.69}$$

The hunt for these particles is one of the central tasks of today's physics, and their discovery would be a confirmation of the ideas of unification and charge quantization.