Space-Time Approach to Quantum Electrodynamics

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In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a restatement of conventional electrodynamics, however, the matrix elements diverge for complex processes. (2) Electrodynamics is modified by altering the interaction of electrons at short distances. All matrix elements are now finite, with the exception of those relating to problems of vacuum polarization. The latter are evaluated in a manner suggested by Pauli and Bethe, which gives finite results for these matrices also. The only effects sensitive to the modification are changes in mass and charge of the electrons. Such changes could not be directly observed. Phenomena directly observable, are insensitive to the details of the modification used (except at extreme energies). For such phenomena, a limit can be taken as the range of the modification goes to zero. The results then agree with those of Schwinger. A complete, unambiguous, and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.

The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation of this to the more conventional Hamiltonian point of view is discussed. It would be very difficult to make the modification which is proposed if one insisted on having the equations in Hamiltonian form.

The methods apply as well to charges obeying the Klein-Gordon equation, and to the various meson theories of nuclear forces. Illustrative examples are given. Although a modification like that used in electrodynamics can make all matrices finite for all of the meson theories, for some of the theories it is no longer true that all directly observable phenomena are insensitive to the details of the modification used.

The actual evaluation of integrals appearing in the matrix elements may be facilitated, in the simpler cases, by methods described in the appendix.

THIS paper should be considered as a direct continuation of a preceding one[1] (I) in which the motion of electrons, neglecting interaction, was analyzed, by dealing directly with the *solution* of the Hamiltonian differential equations. Here the same technique is applied to include interactions and in that way to express in simple terms the solution of problems in quantum electrodynamics.

For most practical calculations in quantum electrodynamics the solution is ordinarily expressed in terms of a matrix element. The matrix is worked out as an expansion in powers of $e^2/\hbar c$, the successive terms corresponding to the inclusion of an increasing number of virtual quanta. It appears that a considerable simplification can be achieved in writing down these matrix elements for complex processes. Furthermore, each term in the expansion can be written down and understood directly from a physical point of view, similar to the space-time view in I. It is the purpose of this paper to describe how this may be done. We shall also discuss methods of handling the divergent integrals which appear in these matrix elements.

The simplification in the formulae results mainly from the fact that previous methods unnecessarily separated into individual terms processes that were closely related physically. For example, in the exchange of a quantum between two electrons there were two terms depending on which electron emitted and which absorbed the quantum. Yet, in the virtual states considered, timing relations are not significant. Olny the order of operators in the matrix must be maintained. We have seen (I), that in addition, processes in which virtual pairs are produced can be combined with others in which only positive energy electrons are involved. Further, the effects of longitudinal and transverse waves can be combined together. The separations previously made were on an unrelativistic basis (reflected in the circumstance that apparently momentum but not energy is conserved in intermediate states). When the terms are combined and simplified, the relativistic invariance of the result is self-evident.

We begin by discussing the solution in space and time of the Schrödinger equation for particles interacting instantaneously. The results are immediately generalizable to delayed interactions of relativistic electrons and we represent in that way the laws of quantum electrodynamics. We can then see how the matrix element for any process can be written down directly. In particular, the self-energy expression is written down.

So far, nothing has been done other than a restatement of conventional electrodynamics in other terms. Therefore, the self-energy diverges. A modification[2] in interaction between charges is next made, and it is shown that the self-energy is made convergent and corresponds to a correction to the electron mass. After the mass correction is made, other real processes are finite and insensitive to the "width" of the cut-off in the interaction.[3]

Unfortunately, the modification proposed is not completely satisfactory theoretically (it leads to some difficulties of conservation of energy). It does, however, seem consistent and satisfactory to define the matrix

[1] R. P. Feynman, Phys. Rev. **76**, 749 (1949), hereafter called I.

[2] For a discussion of this modification in classical physics see R. P. Feynman, Phys. Rev. **74** 939 (1948), hereafter referred to as A.

[3] A brief summary of the methods and results will be found in R. P. Feynman, Phys. Rev. **74**, 1430 (1948), hereafter referred to as B.

element for all real processes as the limit of that computed here as the cut-off width goes to zero. A similar technique suggested by Pauli and by Bethe can be applied to problems of vacuum polarization (resulting in a renormalization of charge) but again a strict physical basis for the rules of convergence is not known.

After mass and charge renormalization, the limit of zero cut-off width can be taken for all real processes. The results are then equivalent to those of Schwinger[4] who does not make explicit use of the convergence factors. The method of Schwinger is to identify the terms corresponding to corrections in mass and charge and, previous to their evaluation, to remove them from the expressions for real processes. This has the advantage of showing that the results can be strictly independent of particular cut-off methods. On the other hand, many of the properties of the integrals are analyzed using formal properties of invariant propagation functions. But one of the properties is that the integrals are infinite and it is not clear to what extent this invalidates the demonstrations. A practical advantage of the present method is that ambiguities can be more easily resolved; simply by direct calculation of the otherwise divergent integrals. Nevertheless, it is not at all clear that the convergence factors do not upset the physical consistency of the theory. Although in the limit the two methods agree, neither method appears to be thoroughly satisfactory theoretically. Nevertheless, it does appear that we now have available a complete and definite method for the calculation of physical processes to any order in quantum electrodynamics.

Since we can write down the solution to any physical problem, we have a complete theory which could stand by itself. It will be theoretically incomplete, however, in two respects. First, although each term of increasing order in $e^2/\hbar c$ can be written down it would be desirable to see some way of expressing things in finite form to all orders in $e^2/\hbar c$ at once. Second, although it will be physically evident that the results obtained are equivalent to those obtained by conventional electrodynamics the mathematical proof of this is not included. Both of these limitations will be removed in a subsequent paper (see also Dyson[4]).

Briefly the genesis of this theory was this. The conventional electrodynamics was expressed in the Lagrangian form of quantum mechanics described in the Reviews of Modern Physics.[5] The motion of the field oscillators could be integrated out (as described in Section 13 of that paper), the result being an expression of the delayed interaction of the particles. Next the modification of the delta-function interaction could be made directly from the analogy to the classical case.[2] This was still not complete because the Lagrangian method had been worked out in detail only for particles obeying the non-relativistic Schrödinger equation. It was then modified in accordance with the requirements of the Dirac equation and the phenomenon of pair creation. This was made easier by the reinterpretation of the theory of holes (I). Finally for practical calculations the expressions were developed in a power series in $e^2/\hbar c$. It was apparent that each term in the series had a simple physical interpretation. Since the result was easier to understand than the derivation, it was thought best to publish the results first in this paper. Considerable time has been spent to make these first two papers as complete and as physically plausible as possible without relying on the Lagrangian method, because it is not generally familiar. It is realized that such a description cannot carry the conviction of truth which would accompany the derivation. On the other hand, in the interest of keeping simple things simple the derivation will appear in a separate paper.

The possible application of these methods to the various meson theories is discussed briefly. The formulas corresponding to a charge particle of zero spin moving in accordance with the Klein Gordon equation are also given. In an Appendix a method is given for calculating the integrals appearing in the matrix elements for the simpler processes.

The point of view which is taken here of the interaction of charges differs from the more usual point of view of field theory. Furthermore, the familiar Hamiltonian form of quantum mechanics must be compared to the over-all space-time view used here. The first section is, therefore, devoted to a discussion of the relations of these viewpoints.

1. COMPARISON WITH THE HAMILTONIAN METHOD

Electrodynamics can be looked upon in two equivalent and complementary ways. One is as the description of the behavior of a field (Maxwell's equations). The other is as a description of a direct interaction at a distance (albeit delayed in time) between charges (the solutions of Lienard and Wiechert). From the latter point of view light is considered as an interaction of the charges in the source with those in the absorber. This is an impractical point of view because many kinds of sources produce the same kind of effects. The field point of view separates these aspects into two simpler problems, production of light, and absorption of light. On the other hand, the field point of view is less practical when dealing with close collisions of particles (or their action on themselves). For here the source and absorber are not readily distinguishable, there is an intimate exchange of quanta. The fields are so closely determined by the motions of the particles that it is just as well not to separate the question into two problems but to consider the process as a direct interaction. Roughly, the field point of view is most practical for problems involv-

[4] J. Schwinger, Phys. Rev. **74**, 1439 (1948), Phys. Rev. **75**, 651 (1949). A proof of this equivalence is given by F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[5] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948). The application to electrodynamics is described in detail by H. J. Groenewold, Koninklijke Nederlandsche Akademia van Weteschappen. Proceedings Vol. LII, 3 (226) 1949.

ing real quanta, while the interaction view is best for the discussion of the virtual quanta involved. We shall emphasize the interaction viewpoint in this paper, first because it is less familiar and therefore requires more discussion, and second because the important aspect in the problems with which we shall deal is the effect of virtual quanta.

The Hamiltonian method is not well adapted to represent the direct action at a distance between charges because that action is delayed. The Hamiltonian method represents the future as developing out of the present. If the values of a complete set of quantities are known now, their values can be computed at the next instant in time. If particles interact through a delayed interaction, however, one cannot predict the future by simply knowing the present motion of the particles. One would also have to know what the motions of the particles were in the past in view of the interaction this may have on the future motions. This is done in the Hamiltonian electrodynamics, of course, by requiring that one specify besides the present motion of the particles, the values of a host of new variables (the coordinates of the field oscillators) to keep track of that aspect of the past motions of the particles which determines their future behavior. The use of the Hamiltonian forces one to choose the field viewpoint rather than the interaction viewpoint.

In many problems, for example, the close collisions of particles, we are not interested in the precise temporal sequence of events. It is not of interest to be able to say how the situation would look at each instant of time during a collision and how it progresses from instant to instant. Such ideas are only useful for events taking a long time and for which we can readily obtain information during the intervening period. For collisions it is much easier to treat the process as a whole.[6] The Møller interaction matrix for the the collision of two electrons is not essentially more complicated than the nonrelativistic Rutherford formula, yet the mathematical machinery used to obtain the former from quantum electrodynamics is vastly more complicated than Schrödinger's equation with the e^2/r_{12} interaction needed to obtain the latter. The difference is only that in the latter the action is instantaneous so that the Hamiltonian method requires no extra variables, while in the former relativistic case it is delayed and the Hamiltonian method is very cumbersome.

We shall be discussing the solutions of equations rather than the time differential equations from which they come. We shall discover that the solutions, because of the over-all space-time view that they permit, are as easy to understand when interactions are delayed as when they are instantaneous.

As a further point, relativistic invariance will be self-evident. The Hamiltonian form of the equations develops the future from the instantaneous present. But for different observers in relative motion the instantaneous present is different, and corresponds to a different 3-dimensional cut of space-time. Thus the temporal analyses of different observers is different and their Hamiltonian equations are developing the process in different ways. These differences are irrelevant, however, for the solution is the same in any space time frame. By forsaking the Hamiltonian method, the wedding of relativity and quantum mechanics can be accomplished most naturally.

We illustrate these points in the next section by studying the solution of Schrödinger's equation for nonrelativistic particles interacting by an instantaneous Coulomb potential (Eq. 2). When the solution is modified to include the effects of delay in the interaction and the relativistic properties of the electrons we obtain an expression of the laws of quantum electrodynamics (Eq. 4).

2. THE INTERACTION BETWEEN CHARGES

We study by the same methods as in I, the interaction of two particles using the same notation as I. We start by considering the non-relativistic case described by the Schrödinger equation (I, Eq. 1). The wave function at a given time is a function $\psi(\mathbf{x}_a, \mathbf{x}_b, t)$ of the coordinates $\mathbf{x}_a$ and $\mathbf{x}_b$ of each particle. Thus call $K(\mathbf{x}_a, \mathbf{x}_b, t; \mathbf{x}_a', \mathbf{x}_b', t')$ the amplitude that particle a at $\mathbf{x}_a'$ at time t' will get to $\mathbf{x}_a$ at t while particle b at $\mathbf{x}_b'$ at t' gets to $\mathbf{x}_b$ at t. If the particles are free and do not interact this is

$$K(\mathbf{x}_a, \mathbf{x}_b, t; \mathbf{x}_a', \mathbf{x}_b', t') = K_{0a}(\mathbf{x}_a, t; \mathbf{x}_a', t') K_{0b}(\mathbf{x}_b, t; \mathbf{x}_b', t')$$

where K_{0a} is the K_0 function for particle a considered as free. In *this* case we can obviously define a quantity like K, but for which the time t need not be the same for particles a and b (likewise for t'); e.g.,

$$K_0(3, 4; 1, 2) = K_{0a}(3, 1) K_{0b}(4, 2) \tag{1}$$

can be thought of as the amplitude that particle a goes from $\mathbf{x}_1$ at t_1 to $\mathbf{x}_3$ at t_3 and that particle b goes from $\mathbf{x}_2$ at t_2 to $\mathbf{x}_4$ at t_4.

When the particles do interact, one can only define the quantity $K(3, 4; 1, 2)$ precisely if the interaction vanishes between t_1 and t_2 and also between t_3 and t_4. In a real physical system such is not the case. There is such an enormous advantage, however, to the concept that we shall continue to use it, imagining that we can neglect the effect of interactions between t_1 and t_2 and between t_3 and t_4. For practical problems this means choosing such long time intervals $t_3 - t_1$ and $t_4 - t_2$ that the extra interactions near the end points have small relative effects. As an example, in a scattering problem it may well be that the particles are so well separated initially and finally that the interaction at these times is negligible. Again energy values can be defined by the average rate of change of phase over such long time intervals that errors initially and finally can be neglected. Inasmuch as any physical problem can be defined in terms of scattering processes we do not lose much in

[6] This is the viewpoint of the theory of the S matrix of Heisenberg.

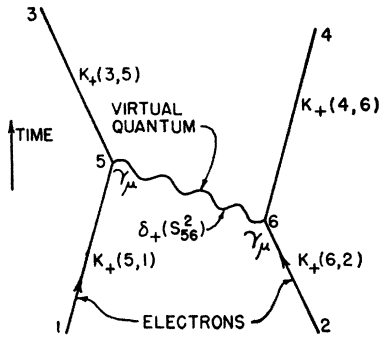

FIG. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.

a general theoretical sense by this approximation. If it is not made it is not easy to study interacting particles relativistically, for there is nothing significant in choosing $t_1=t_3$ if $\mathbf{x}_1\neq\mathbf{x}_3$, as absolute simultaneity of events at a distance cannot be defined invariantly. It is essentially to avoid this approximation that the complicated structure of the older quantum electrodynamics has been built up. We wish to describe electrodynamics as a delayed interaction between particles. If we can make the approximation of assuming a meaning to $K(3,4;1,2)$ the results of this interaction can be expressed very simply.

To see how this may be done, imagine first that the interaction is simply that given by a Coulomb potential e^2/r where r is the distance between the particles. If this be turned on only for a very short time Δt_0 at time t_0, the first order correction to $K(3,4;1,2)$ can be worked out exactly as was Eq. (9) of I by an obvious generalization to two particles:

$$K^{(1)}(3,4;1,2)=-ie^2\int\int K_{0a}(3,5)K_{0b}(4,6)r_{56}^{-1}\times K_{0a}(5,1)K_{0b}(6,2)d^3\mathbf{x}_5d^3\mathbf{x}_6\Delta t_0,$$

where $t_5=t_6=t_0$. If now the potential were on at all times (so that strictly K is not defined unless $t_4=t_3$ and $t_1=t_2$), the first-order effect is obtained by integrating on t_0, which we can write as an integral over both t_5 and t_6 if we include a delta-function $\delta(t_5-t_6)$ to insure contribution only when $t_5=t_6$. Hence, the first-order effect of interaction is (calling $t_5-t_6=t_{56}$):

$$K^{(1)}(3,4;1,2)=-ie^2\int\int K_{0a}(3,5)K_{0b}(4,6)r_{56}^{-1}\times\delta(t_{56})K_{0a}(5,1)K_{0b}(6,2)d\tau_5d\tau_6, \quad (2)$$

where $d\tau=d^3\mathbf{x}dt$.

We know, however, in classical electrodynamics, that the Coulomb potential does not act instantaneously, but is delayed by a time r_{56}, taking the speed of light as unity. This suggests simply replacing $r_{56}^{-1}\delta(t_{56})$ in (2) by something like $r_{56}^{-1}\delta(t_{56}-r_{56})$ to represent the delay in the effect of b on a.

This turns out to be not quite right,[7] for when this interaction is represented by photons they must be of only positive energy, while the Fourier transform of $\delta(t_{56}-r_{56})$ contains frequencies of both signs. It should instead be replaced by $\delta_+(t_{56}-r_{56})$ where

$$\delta_+(x)=\int_0^\infty e^{-i\omega x}d\omega/\pi=\lim_{\epsilon\to0}\frac{(\pi i)^{-1}}{x-i\epsilon}=\delta(x)+(\pi ix)^{-1}. \quad (3)$$

This is to be averaged with $r_{56}^{-1}\delta_+(-t_{56}-r_{56})$ which arises when $t_5<t_6$ and corresponds to a emitting the quantum which b receives. Since

$$(2r)^{-1}(\delta_+(t-r)+\delta_+(-t-r))=\delta_+(t^2-r^2),$$

this means $r_{56}^{-1}\delta(t_{56})$ is replaced by $\delta_+(s_{56}^2)$ where $s_{56}^2=t_{56}^2-r_{56}^2$ is the square of the relativistically invariant interval between points 5 and 6. Since in classical electrodynamics there is also an interaction through the vector potential, the complete interaction (see A, Eq. (1)) should be $(1-(\mathbf{v}_5\cdot\mathbf{v}_6)\delta_+(s_{56}^2)$, or in the relativistic case,

$$(1-\boldsymbol{\alpha}_a\cdot\boldsymbol{\alpha}_b)\delta_+(s_{56}^2)=\beta_a\beta_b\gamma_{a\mu}\gamma_{b\mu}\delta_+(s_{56}^2).$$

Hence we have for electrons obeying the Dirac equation,

$$K^{(1)}(3,4;1,2)=-ie^2\int\int K_{+a}(3,5)K_{+b}(4,6)\gamma_{a\mu}\gamma_{b\mu}\times\delta_+(s_{56}^2)K_{+a}(5,1)K_{+b}(6,2)d\tau_5d\tau_6, \quad (4)$$

where $\gamma_{a\mu}$ and $\gamma_{b\mu}$ are the Dirac matrices applying to the spinor corresponding to particles a and b, respectively (the factor $\beta_a\beta_b$ being absorbed in the definition, I Eq. (17), of K_+).

This is our fundamental equation for electrodynamics. It describes the effect of exchange of one quantum (therefore first order in e^2) between two electrons. It will serve as a prototype enabling us to write down the corresponding quantities involving the exchange of two or more quanta between two electrons or the interaction of an electron with itself. It is a consequence of conventional electrodynamics. Relativistic invariance is clear. Since one sums over μ it contains the effects of both longitudinal and transverse waves in a relativistically symmetrical way.

We shall now interpret Eq. (4) in a manner which will permit us to write down the higher order terms. It can be understood (see Fig. 1) as saying that the amplitude for "a" to go from 1 to 3 and "b" to go from 2 to 4 is altered to first order because they can exchange a quantum. Thus, "a" can go to 5 (amplitude $K_+(5,1)$)

[7] It, and a like term for the effect of a on b, leads to a theory which, in the classical limit, exhibits interaction through half-advanced and half-retarded potentials. Classically, this is equivalent to purely retarded effects within a closed box from which no light escapes (e.g., see A, or J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945)). Analogous theorems exist in quantum mechanics but it would lead us too far astray to discuss them now.

emit a quantum (longitudinal, transverse, or scalar $\gamma_{a\mu}$) and then proceed to 3 ($K_+(3, 5)$). Meantime "b" goes to 6 ($K_+(6, 2)$), absorbs the quantum ($\gamma_{b\mu}$) and proceeds to 4 ($K_+(4, 6)$). The quantum meanwhile proceeds from 5 to 6, which it does with amplitude $\delta_+(s_{56}^2)$. We must sum over all the possible quantum polarizations μ and positions and times of emission 5, and of absorption 6. Actually if $t_5 > t_6$ it would be better to say that "a" absorbs and "b" emits but no attention need be paid to these matters, as all such alternatives are automatically contained in (4).

The correct terms of higher order in e^2 or involving larger numbers of electrons (interacting with themselves or in pairs) can be written down by the same kind of reasoning. They will be illustrated by examples as we proceed. In a succeeding paper they will all be deduced from conventional quantum electrodynamics.

Calculation, from (4), of the transition element between positive energy free electron states gives the Möller scattering of two electrons, when account is taken of the Pauli principle.

The exclusion principle for interacting charges is handled in exactly the same way as for non-interacting charges (I). For example, for two charges it requires only that one calculate $K(3, 4; 1, 2) - K(4, 3; 1, 2)$ to get the net amplitude for arrival of charges at 3 and 4. It is disregarded in intermediate states. The interference effects for scattering of electrons by positrons discussed by Bhabha will be seen to result directly in this formulation. The formulas are interpreted to apply to positrons in the manner discussed in I.

As our primary concern will be for processes in which the quanta are virtual we shall not include here the detailed analysis of processes involving real quanta in initial or final state, and shall content ourselves by only stating the rules applying to them.[8] The result of the analysis is, as expected, that they can be included by the same line of reasoning as is used in discussing the virtual processes, provided the quantities are normalized in the usual manner to represent single quanta. For example, the amplitude that an electron in going from 1 to 2 absorbs a quantum whose vector potential, suitably normalized, is $c_\mu \exp(-ik\cdot x) = C_\mu(x)$ is just the expression (I, Eq. (13)) for scattering in a potential with $\boldsymbol{A}$ (3) replaced by $\boldsymbol{C}$ (3). Each quantum interacts only once (either in emission or in absorption), terms like (I, Eq. (14)) occur only when there is more than one quantum involved. The Bose statistics of the quanta can, in all cases, be disregarded in intermediate states. The only effect of the statistics is to change the weight of initial or final states. If there are among quanta, in the initial state, some n which are identical then the weight of the state is $(1/n!)$ of what it would be if these quanta were considered as different (similarly for the final state).

3. THE SELF-ENERGY PROBLEM

Having a term representing the mutual interaction of a pair of charges, we must include similar terms to represent the interaction of a charge with itself. For under some circumstances what appears to be two distinct electrons may, according to I, be viewed also as a single electron (namely in case one electron was created in a pair with a positron destined to annihilate the other electron). Thus to the interaction between such electrons must correspond the possibility of the action of an electron on itself.[9]

This interaction is the heart of the self energy problem. Consider to first order in e^2 the action of an electron on itself in an otherwise force free region. The amplitude $K(2, 1)$ for a single particle to get from 1 to 2 differs from $K_+(2, 1)$ to first order in e^2 by a term

$$K^{(1)}(2, 1) = -ie^2 \int\int K_+(2, 4)\gamma_\mu K_+(4, 3)\gamma_\mu \times K_+(3, 1) d\tau_3 d\tau_4 \delta_+(s_{43}^2). \quad (6)$$

It arises because the electron instead of going from 1 directly to 2, may go (Fig. 2) first to 3, ($K_+(3, 1)$), emit a quantum (γ_μ), proceed to 4, ($K_+(4, 3)$), absorb it (γ_μ), and finally arrive at 2 ($K_+(2, 4)$). The quantum must go from 3 to 4 ($\delta_+(s_{43}^2)$).

This is related to the self-energy of a free electron in the following manner. Suppose initially, time t_1, we have an electron in state $f(1)$ which we imagine to be a positive energy solution of Dirac's equation for a free particle. After a long time $t_2 - t_1$ the perturbation will alter

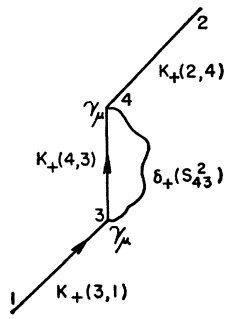

FIG. 2. Interaction of an electron with itself, Eq. (6).

[8] Although in the expressions stemming from (4) the quanta are virtual, this is not actually a theoretical limitation. One way to deduce the correct rules for real quanta from (4) is to note that in a closed system all quanta can be considered as virtual (i.e., they have a known source and are eventually absorbed) so that in such a system the present description is complete and equivalent to the conventional one. In particular, the relation of the Einstein A and B coefficients can be deduced. A more practical direct deduction of the expressions for real quanta will be given in the subsequent paper. It might be noted that (4) can be rewritten as describing the action on a, $K^{(1)}(3, 1) = i\int K_+(3, 5) \times \boldsymbol{A}(5) K_+(5, 1) d\tau_5$ of the potential $A_\mu(5) = e^2 \int K_+(4, 6)\delta_+(s_{56}^2)\gamma_\mu \times K_+(6, 2) d\tau_6$ arising from Maxwell's equations $-\Box^2 A_\mu = 4\pi j_\mu$ from a "current" $j_\mu(6) = e^2 K_+(4, 6)\gamma_\mu K_+(6, 2)$ produced by particle b in going from 2 to 4. This is virtue of the fact that δ_+ satisfies

$$-\Box_2^2 \delta_+(s_{21}^2) = 4\pi\delta(2, 1). \quad (5)$$

[9] These considerations make it appear unlikely that the contention of J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945), that electrons do not act on themselves, will be a successful concept in quantum electrodynamics.

the wave function, which can then be looked upon as a superposition of free particle solutions (actually it only contains f). The amplitude that $g(2)$ is contained is calculated as in (I, Eq. (21)). The diagonal element ($g=f$) is therefore

$$\int\int \bar{f}(2)\beta K^{(1)}(2,\,1)\beta f(1)d^3\mathbf{x}_1 d^3\mathbf{x}_2. \tag{7}$$

The time interval $T=t_2-t_1$ (and the spatial volume V over which one integrates) must be taken very large, for the expressions are only approximate (analogous to the situation for two interacting charges).[10] This is because, for example, we are dealing incorrectly with quanta emitted just before t_2 which would normally be reabsorbed at times after t_2.

If $K^{(1)}(2,\,1)$ from (6) is actually substituted into (7) the surface integrals can be performed as was done in obtaining I, Eq. (22) resulting in

$$-ie^2\int\int \bar{f}(4)\gamma_\mu K_+(4,\,3)\gamma_\mu f(3)\delta_+(s_{43}{}^2)d\tau_3 d\tau_4. \tag{8}$$

Putting for $f(1)$ the plane wave $u\exp(-ip\cdot x_1)$ where p_μ is the energy (p_4) and momentum of the electron ($p^2=m^2$), and u is a constant 4-index symbol, (8) becomes

$$-ie^2\int\int (\bar{u}\gamma_\mu K_+(4,\,3)\gamma_\mu u) \times\exp(ip\cdot(x_4-x_3))\delta_+(s_{43}{}^2)d\tau_3 d\tau_4,$$

the integrals extending over the volume V and time interval T. Since $K_+(4,\,3)$ depends only on the difference of the coordinates of 4 and 3, $x_{43\mu}$, the integral on 4 gives a result (except near the surfaces of the region) independent of 3. When integrated on 3, therefore, the result is of order VT. The effect is proportional to V, for the wave functions have been normalized to unit volume. If normalized to volume V, the result would simply be proportional to T. This is expected, for if the effect were equivalent to a change in energy ΔE, the amplitude for arrival in f at t_2 is altered by a factor $\exp(-i\Delta E(t_2-t_1))$, or to first order by the difference $-i(\Delta E)T$. Hence, we have

$$\Delta E=e^2\int (\bar{u}\gamma_\mu K_+(4,\,3)\gamma_\mu u)\exp(ip\cdot x_{43})\delta_+(s_{43}{}^2)d\tau_4, \tag{9}$$

integrated over all space-time $d\tau_4$. This expression will be simplified presently. In interpreting (9) we have tacitly assumed that the wave functions are normalized so that $(u^*u)=(\bar{u}\gamma_4 u)=1$. The equation may therefore be made independent of the normalization by writing the left side as $(\Delta E)(\bar{u}\gamma_4 u)$, or since $(\bar{u}\gamma_4 u)=(E/m)(\bar{u}u)$ and $m\Delta m=E\Delta E$, as $\Delta m(\bar{u}u)$ where Δm is an equivalent change in mass of the electron. In this form invariance is obvious.

One can likewise obtain an expression for the energy shift for an electron in a hydrogen atom. Simply replace K_+ in (8), by $K_+{}^{(V)}$, the exact kernel for an electron in the potential, $V=\beta e^2/r$, of the atom, and f by a wave function (of space and time) for an atomic state. In general the ΔE which results is not real. The imaginary part is negative and in $\exp(-i\Delta ET)$ produces an exponentially decreasing amplitude with time. This is because we are asking for the amplitude that an atom initially with no photon in the field, will still appear after time T with no photon. If the atom is in a state which can radiate, this amplitude must decay with time. The imaginary part of ΔE when calculated does indeed give the correct rate of radiation from atomic states. It is zero for the ground state and for a free electron.

In the non-relativistic region the expression for ΔE can be worked out as has been done by Bethe.[11] In the relativistic region (points 4 and 3 as close together as a Compton wave-length) the $K_+{}^{(V)}$ which should appear in (8) can be replaced to first order in V by K_+ plus $K_+{}^{(1)}(2,\,1)$ given in I, Eq. (13). The problem is then very similar to the radiationless scattering problem discussed below.

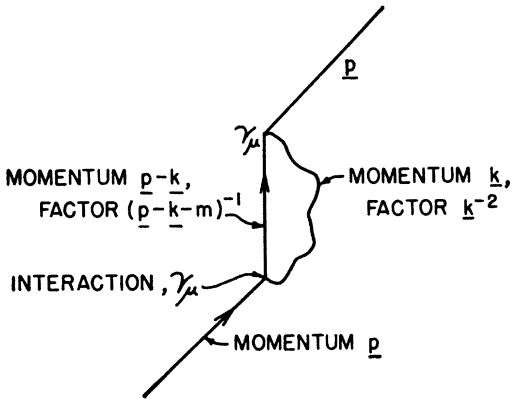

FIG. 3. Interaction of an electron with itself. Momentum space, Eq. (11).

4. EXPRESSION IN MOMENTUM AND ENERGY SPACE

The evaluation of (9), as well as all the other more complicated expressions arising in these problems, is very much simplified by working in the momentum and energy variables, rather than space and time. For this we shall need the Fourier Transform of $\delta_+(s_{21}{}^2)$ which is

$$-\delta_+(s_{21}{}^2)=\pi^{-1}\int \exp(-ik\cdot x_{21})k^{-2}d^4k, \tag{10}$$

which can be obtained from (3) and (5) or from I, Eq. (32) noting that $I_+(2,\,1)$ for $m^2=0$ is $\delta_+(s_{21}{}^2)$ from

[10] This is discussed in reference 5 in which it is pointed out that the concept of a wave function loses accuracy if there are delayed self-actions.

[11] H. A. Bethe, Phys. Rev. **72**, 339 (1947).

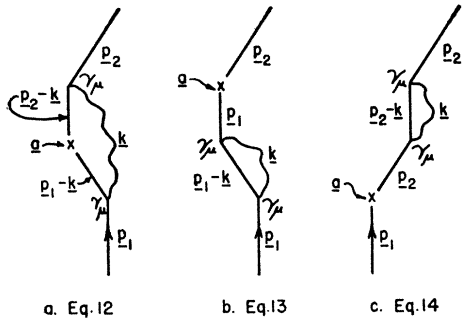

FIG. 4. Radiative correction to scattering, momentum space.

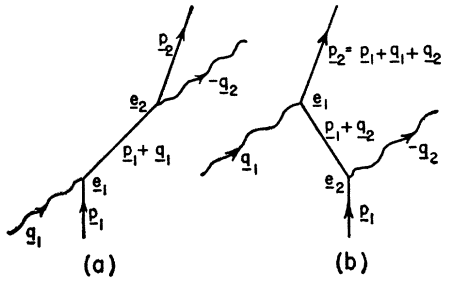

FIG. 5. Compton scattering, Eq. (15).

I, Eq. (34). The k^{-2} means $(k\cdot k)^{-1}$ or more precisely the limit as $\delta\to 0$ of $(k\cdot k+i\delta)^{-1}$. Further d^4k means $(2\pi)^{-2}dk_1dk_2dk_3dk_4$. If we imagine that quanta are particles of zero mass, then we can make the general rule that all poles are to be resolved by considering the masses of the particles and quanta to have infinitesimal negative imaginary parts.

Using these results we see that the self-energy (9) is the matrix element between $\bar{u}$ and u of the matrix

$$(e^2/\pi i)\int \gamma_\mu(p-k-m)^{-1}\gamma_\mu k^{-2}d^4k, \qquad (11)$$

where we have used the expression (I, Eq. (31)) for the Fourier transform of K_+. This form for the self-energy is easier to work with than is (9).

The equation can be understood by imagining (Fig. 3) that the electron of momentum p emits (γ_μ) a quantum of momentum k, and makes its way now with momentum $p-k$ to the next event (factor $(p-k-m)^{-1}$) which is to absorb the quantum (another γ_μ). The amplitude of propagation of quanta is k^{-2}. (There is a factor $e^2/\pi i$ for each virtual quantum). One integrates over all quanta. The reason an electron of momentum p propagates as $1/(p-m)$ is that this operator is the reciprocal of the Dirac equation operator, and we are simply solving this equation. Likewise light goes as $1/k^2$, for this is the reciprocal D'Alembertian operator of the wave equation of light. The first γ_μ represents the current which generates the vector potential, while the second is the velocity operator by which this potential is multiplied in the Dirac equation when an external field acts on an electron.

Using the same line of reasoning, other problems may be set up directly in momentum space. For example, consider the scattering in a potential $A=A_\mu\gamma_\mu$ varying in space and time as $a\exp(-iq\cdot x)$. An electron initially in state of momentum $p_1=p_{1\mu}\gamma_\mu$ will be deflected to state p_2 where $p_2=p_1+q$. The zero-order answer is simply the matrix element of a between states 1 and 2. We next ask for the first order (in e^2) radiative correction due to virtual radiation of one quantum. There are several ways this can happen. First for the case illustrated in Fig. 4(a), find the matrix:

$$(e^2/\pi i)\int \gamma_\mu(p_2-k-m)^{-1}a(p_1-k-m)^{-1}\gamma_\mu k^{-2}d^4k. \qquad (12)$$

For in this case, first[12] a quantum of momentum k is emitted (γ_μ), the electron then having momentum p_1-k and hence propagating with factor $(p_1-k-m)^{-1}$. Next it is scattered by the potential (matrix a) receiving additional momentum q, propagating on then (factor $(p_2-k-m)^{-1}$) with the new momentum until the quantum is reabsorbed (γ_μ). The quantum propagates from emission to absorption (k^{-2}) and we integrate over all quanta (d^4k), and sum on polarization μ. When this is integrated on k_4, the result can be shown to be exactly equal to the expressions (16) and (17) given in B for the same process, the various terms coming from residues of the poles of the integrand (12).

Or again if the quantum is both emitted and reabsorbed before the scattering takes place one finds (Fig. 4(b))

$$(e^2/\pi i)\int a(p_1-m)^{-1}\gamma_\mu(p_1-k-m)^{-1}\gamma_\mu k^{-2}d^4k, \qquad (13)$$

or if both emission and absorption occur after the scattering, (Fig. 4(c))

$$(e^2/\pi i)\int \gamma_\mu(p_2-k-m)^{-1}\gamma_\mu(p_2-m)^{-1}ak^{-2}d^4k. \qquad (14)$$

These terms are discussed in detail below.

We have now achieved our simplification of the form of writing matrix elements arising from virtual processes. Processes in which a number of real quanta is given initially and finally offer no problem (assuming correct normalization). For example, consider the Compton effect (Fig. 5(a)) in which an electron in state p_1 absorbs a quantum of momentum q_1, polarization vector $e_{1\mu}$ so that its interaction is $e_{1\mu}\gamma_\mu=e_1$, and emits a second quantum of momentum $-q_2$, polarization e_2 to arrive in final state of momentum p_2. The matrix for

[12] First, next, etc., here refer not to the order in true time but to the succession of events along the trajectory of the electron. That is, more precisely, to the order of appearance of the matrices in the expressions.

this process is $e_2(p_1+q_1-m)^{-1}e_1$. The total matrix for the Compton effect is, then,

$$e_2(p_1+q_1-m)^{-1}e_1+e_1(p_1+q_2-m)^{-1}e_2, \quad (15)$$

the second term arising because the emission of e_2 may also precede the absorption of e_1 (Fig. 5(b)). One takes matrix elements of this between initial and final electron states ($p_1+q_1=p_2-q_2$), to obtain the Klein Nishina formula. Pair annihilation with emission of two quanta, etc., are given by the same matrix, positron states being those with negative time component of p. Whether quanta are absorbed or emitted depends on whether the time component of q is positive or negative.

5. THE CONVERGENCE OF PROCESSES WITH VIRTUAL QUANTA

These expressions are, as has been indicated, no more than a re-expression of conventional quantum electrodynamics. As a consequence, many of them are meaningless. For example, the self-energy expression (9) or (11) gives an infinite result when evaluated. The infinity arises, apparently, from the coincidence of the δ-function singularities in $K_+(4, 3)$ and $\delta_+(s_{43}^2)$. Only at this point is it necessary to make a real departure from conventional electrodynamics, a departure other than simply rewriting expressions in a simpler form.

We desire to make a modification of quantum electrodynamics analogous to the modification of classical electrodynamics described in a previous article, A. There the $\delta(s_{12}^2)$ appearing in the action of interaction was replaced by $f(s_{12}^2)$ where $f(x)$ is a function of small width and great height.

The obvious corresponding modification in the quantum theory is to replace the $\delta_+(s^2)$ appearing the quantum mechanical interaction by a new function $f_+(s^2)$. We can postulate that if the Fourier transform of the classical $f(s_{12}^2)$ is the integral over all k of $F(k^2)\exp(-ik\cdot x_{12})d^4k$, then the Fourier transform of $f_+(s^2)$ is the same integral taken over only positive frequencies k_4 for $t_2>t_1$ and over only negative ones for $t_2<t_1$ in analogy to the relation of $\delta_+(s^2)$ to $\delta(s^2)$. The function $f(s^2)=f(x\cdot x)$ can be written* as

$$f(x\cdot x)=(2\pi)^{-2}\int_{k_4=0}^{\infty}\int \sin(k_4|x_4|)\times\cos(\mathbf{K}\cdot\mathbf{x})dk_4d^3\mathbf{K}g(k\cdot k),$$

where $g(k\cdot k)$ is k_4^{-1} times the density of oscillators and may be expressed for positive k_4 as (A, Eq. (16))

$$g(k^2)=\int_0^{\infty}(\delta(k^2)-\delta(k^2-\lambda^2))G(\lambda)d\lambda,$$

where $\int_0^{\infty}G(\lambda)d\lambda=1$ and G involves values of λ large compared to m. This simply means that the amplitude for propagation of quanta of momentum k is

$$-F_+(k^2)=\pi^{-1}\int_0^{\infty}(k^{-2}-(k^2-\lambda^2)^{-1})G(\lambda)d\lambda,$$

rather than k^{-2}. That is, writing $F_+(k^2)=-\pi^{-1}k^{-2}C(k^2)$,

$$-f_+(s_{12}^2)=\pi^{-1}\int \exp(-ik\cdot x_{12})k^{-2}C(k^2)d^4k. \quad (16)$$

Every integral over an intermediate quantum which previously involved a factor d^4k/k^2 is now supplied with a convergence factor $C(k^2)$ where

$$C(k^2)=\int_0^{\infty}-\lambda^2(k^2-\lambda^2)^{-1}G(\lambda)d\lambda. \quad (17)$$

The poles are defined by replacing k^2 by $k^2+i\delta$ in the limit $\delta\to 0$. That is λ^2 may be assumed to have an infinitesimal negative imaginary part.

The function $f_+(s_{12}^2)$ may still have a discontinuity in value on the light cone. This is of no influence for the Dirac electron. For a particle satisfying the Klein Gordon equation, however, the interaction involves gradients of the potential which reinstates the δ function if f has discontinuities. The condition that f is to have no discontinuity in value on the light cone implies $k^2C(k^2)$ approaches zero as k^2 approaches infinity. In terms of $G(\lambda)$ the condition is

$$\int_0^{\infty}\lambda^2G(\lambda)d\lambda=0. \quad (18)$$

This condition will also be used in discussing the convergence of vacuum polarization integrals.

The expression for the self-energy matrix is now

$$(e^2/\pi i)\int \gamma_\mu(p-k-m)^{-1}\gamma_\mu k^{-2}d^4kC(k^2), \quad (19)$$

which, since $C(k^2)$ falls off at least as rapidly as $1/k^2$, converges. For practical purposes we shall suppose hereafter that $C(k^2)$ is simply $-\lambda^2/(k^2-\lambda^2)$ implying that some average (with weight $G(\lambda)d\lambda$) over values of λ may be taken afterwards. Since in all processes the quantum momentum will be contained in at least one extra factor of the form $(p-k-m)^{-1}$ representing propagation of an electron while that quantum is in the field, we can expect all such integrals with their convergence factors to converge and that the result of all such processes will now be finite and definite (excepting the processes with closed loops, discussed below, in which the diverging integrals are over the momenta of the electrons rather than the quanta).

The integral of (19) with $C(k^2)=-\lambda^2(k^2-\lambda^2)^{-1}$ noting that $p^2=m^2$, $\lambda\gg m$ and dropping terms of order m/λ, is (see Appendix A)

$$(e^2/2\pi)[4m(\ln(\lambda/m)+\tfrac{1}{2})-p(\ln(\lambda/m)+5/4)]. \quad (20)$$

* This relation is given incorrectly in A, equation just preceding 16.

When applied to a state of an electron of momentum $\boldsymbol{p}$ satisfying $\boldsymbol{p}u=mu$, it gives for the change in mass (as in B, Eq. (9))

$$\Delta m=m(e^2/2\pi)(3\ln(\lambda/m)+\tfrac{3}{4}). \tag{21}$$

6. RADIATIVE CORRECTIONS TO SCATTERING

We can now complete the discussion of the radiative corrections to scattering. In the integrals we include the convergence factor $C(\boldsymbol{k}^2)$, so that they converge for large $\boldsymbol{k}$. Integral (12) is also not convergent because of the well-known infra-red catastrophy. For this reason we calculate (as discussed in B) the value of the integral assuming the photons to have a small mass $\lambda_{\min}\ll m\ll\lambda$. The integral (12) becomes

$$(e^2/\pi i)\int\gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}\times\gamma_\mu(\boldsymbol{k}^2-\lambda_{\min}{}^2)^{-1}d^4kC(\boldsymbol{k}^2-\lambda_{\min}{}^2),$$

which when integrated (see Appendix B) gives $(e^2/2\pi)$ times

$$\left[2\left(\ln\frac{m}{\lambda_{\min}}-1\right)\left(1-\frac{2\theta}{\tan2\theta}\right)+\theta\tan\theta+\frac{4}{\tan2\theta}\int_0^\theta\alpha\tan\alpha d\alpha\right]\boldsymbol{a}+\frac{1}{4m}(\boldsymbol{qa}-\boldsymbol{aq})\frac{2\theta}{\sin2\theta}+r\boldsymbol{a}, \tag{22}$$

where $(\boldsymbol{q}^2)^{\frac{1}{2}}=2m\sin\theta$ and we have assumed the matrix to operate between states of momentum $\boldsymbol{p}_1$ and $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$ and have neglected terms of order $\lambda_{\min}/m$, m/λ, and $\boldsymbol{q}^2/\lambda^2$. Here the only dependence on the convergence factor is in the term $r\boldsymbol{a}$, where

$$r=\ln(\lambda/m)+9/4-2\ln(m/\lambda_{\min}). \tag{23}$$

As we shall see in a moment, the other terms (13), (14) give contributions which just cancel the $r\boldsymbol{a}$ term. The remaining terms give for small $\boldsymbol{q}$,

$$(e^2/4\pi)\left(\frac{1}{2m}(\boldsymbol{qa}-\boldsymbol{aq})+\frac{4\boldsymbol{q}^2}{3m^2}\boldsymbol{a}\left(\ln\frac{m}{\lambda_{\min}}-\frac{3}{8}\right)\right), \tag{24}$$

which shows the change in magnetic moment and the Lamb shift as interpreted in more detail in B.[13]

We must now study the remaining terms (13) and (14). The integral on $\boldsymbol{k}$ in (13) can be performed (after multiplication by $C(\boldsymbol{k}^2)$) since it involves nothing but the integral (19) for the self-energy and the result is allowed to operate on the initial state u_1, (so that $\boldsymbol{p}_1u_1=mu_1$). Hence the factor following $\boldsymbol{a}(\boldsymbol{p}_1-m)^{-1}$ will be just Δm. But, if one now tries to expand $1/(\boldsymbol{p}_1-m)=(\boldsymbol{p}_1+m)/(\boldsymbol{p}_1{}^2-m^2)$ one obtains an infinite result, since $\boldsymbol{p}_1{}^2=m^2$. This is, however, just what is expected physically. For the quantum can be emitted and absorbed at any time previous to the scattering. Such a process has the effect of a change in mass of the electron in the state 1. It therefore changes the energy by ΔE and the amplitude to first order in ΔE by $-i\Delta E\cdot t$ where t is the time it is acting, which is infinite. That is, the major effect of this term would be canceled by the effect of change of mass Δm.

The situation can be analyzed in the following manner. We suppose that the electron approaching the scattering potential $\boldsymbol{a}$ has not been free for an infinite time, but at some time far past suffered a scattering by a potential $\boldsymbol{b}$. If we limit our discussion to the effects of Δm and of the virtual radiation of one quantum between two such scatterings each of the effects will be finite, though large, and their difference is determinate. The propagation from $\boldsymbol{b}$ to $\boldsymbol{a}$ is represented by a matrix

$$\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}, \tag{25}$$

in which one is to integrate possibly over $\boldsymbol{p}'$ (depending on details of the situation). (If the time is long between $\boldsymbol{b}$ and $\boldsymbol{a}$, the energy is very nearly determined so that $\boldsymbol{p}'{}^2$ is very nearly m^2.)

We shall compare the effect on the matrix (25) of the virtual quanta and of the change of mass Δm. The effect of a virtual quantum is

$$(e^2/\pi i)\int\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\gamma_\mu(\boldsymbol{p}'-\boldsymbol{k}-m)^{-1}\times\gamma_\mu(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}\boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2), \tag{26}$$

while that of a change of mass can be written

$$\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\Delta m(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}, \tag{27}$$

and we are interested in the difference (26)–(27). A simple and direct method of making this comparison is just to evaluate the integral on $\boldsymbol{k}$ in (26) and subtract from the result the expression (27) where Δm is given in (21). The remainder can be expressed as a multiple $-r(\boldsymbol{p}'{}^2)$ of the unperturbed amplitude (25);

$$-r(\boldsymbol{p}'{}^2)\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}. \tag{28}$$

This has the same result (to this order) as replacing the potentials $\boldsymbol{a}$ and $\boldsymbol{b}$ in (25) by $(1-\frac{1}{2}r(\boldsymbol{p}'{}^2))\boldsymbol{a}$ and

[13] That the result given in B in Eq. (19) was in error was repeatedly pointed out to the author, in private communication, by V. F. Weisskopf and J. B. French, as their calculation, completed simultaneously with the author's early in 1948, gave a different result. French has finally shown that although the expression for the radiationless scattering B, Eq. (18) or (24) above is correct, it was incorrectly joined onto Bethe's non-relativistic result. He shows that the relation $\ln2k_{\max}-1=\ln\lambda_{\min}$ used by the author should have been $\ln2k_{\max}-5/6=\ln\lambda_{\min}$. This results in adding a term $-(1/6)$ to the logarithm in B, Eq. (19) so that the result now agrees with that of J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949) and N. H. Kroll and W. E. Lamb, Phys. Rev. **75**, 388 (1949). The author feels unhappily responsible for the very considerable delay in the publication of French's result occasioned by this error. This footnote is appropriately numbered.

$(1-\frac{1}{2}r(\not{p}'^2))b$. In the limit, then, as $\not{p}'^2\to m^2$ the net effect on the scattering is $-\frac{1}{2}ra$ where r, the limit of $r(\not{p}'^2)$ as $\not{p}'^2\to m^2$ (assuming the integrals have an infrared cut-off), turns out to be just equal to that given in (23). An equal term $-\frac{1}{2}ra$ arises from virtual transitions after the scattering (14) so that the entire ra term in (22) is canceled.

The reason that r is just the value of (12) when $q^2=0$ can also be seen without a direct calculation as follows: Let us call $\not{p}$ the vector of length m in the direction of $\not{p}'$ so that if $\not{p}'^2=m(1+\epsilon)^2$ we have $\not{p}'=(1+\epsilon)\not{p}$ and we take ϵ as very small, being of order T^{-1} where T is the time between the scatterings b and a. Since $(\not{p}'-m)^{-1}=(\not{p}'+m)/(\not{p}'^2-m^2)\approx(\not{p}+m)/2m^2\epsilon$, the quantity (25) is of order ϵ^{-1} or T. We shall compute corrections to it only to its own order (ϵ^{-1}) in the limit $\epsilon\to 0$. The term (27) can be written approximately[14] as

$$(e^2/\pi i)\int a(\not{p}'-m)^{-1}\gamma_\mu(\not{p}-\not{k}-m)^{-1}\times\gamma_\mu(\not{p}'-m)^{-1}b\not{k}^{-2}d^4kC(k^2),$$

using the expression (19) for Δm. The net of the two effects is therefore approximately[15]

$$-(e^2/\pi i)\int a(\not{p}'-m)^{-1}\gamma_\mu(\not{p}-\not{k}-m)^{-1}\epsilon\not{p}(\not{p}-\not{k}-m)^{-1}\times\gamma_\mu(\not{p}'-m)^{-1}b\not{k}^{-2}d^4kC(k^2),$$

a term now of order $1/\epsilon$ (since $(\not{p}'-m)^{-1}\approx(\not{p}+m)\times(2m^2\epsilon)^{-1}$) and therefore the one desired in the limit. Comparison to (28) gives for r the expression

$$(\not{p}_1+m/2m)\int\gamma_\mu(\not{p}_1-\not{k}-m)^{-1}(\not{p}_1m^{-1})(\not{p}_1-\not{k}-m)^{-1}\times\gamma_\mu\not{k}^{-2}d^4kC(k^2). \quad (29)$$

The integral can be immediately evaluated, since it is the same as the integral (12), but with $q=0$, for a replaced by $\not{p}_1/m$. The result is therefore $r\cdot(\not{p}_1/m)$ which when acting on the state u_1 is just r, as $\not{p}_1u_1=mu_1$. For the same reason the term $(\not{p}_1+m)/2m$ in (29) is effectively 1 and we are left with $-r$ of (23).[16]

In more complex problems starting with a free electron the same type of term arises from the effects of a virtual emission and absorption both previous to the other processes. They, therefore, simply lead to the same factor r so that the expression (23) may be used directly and these renormalization integrals need not be computed afresh for each problem.

In this problem of the radiative corrections to scattering the net result is insensitive to the cut-off. This means, of course, that by a simple rearrangement of terms previous to the integration we could have avoided the use of the convergence factors completely (see for example Lewis[17]). The problem was solved in the manner here in order to illustrate how the use of such convergence factors, even when they are actually unnecessary, may facilitate analysis somewhat by removing the effort and ambiguities that may be involved in trying to rearrange the otherwise divergent terms.

The replacement of δ_+ by f_+ given in (16), (17) is not determined by the analogy with the classical problem. In the classical limit only the real part of δ_+ (i.e., just δ) is easy to interpret. But by what should the imaginary part, $1/(\pi is^2)$, of δ_+ be replaced? The choice we have made here (in defining, as we have, the location of the poles of (17)) is arbitrary and almost certainly incorrect. If the radiation resistance is calculated for an atom, as the imaginary part of (8), the result depends slightly on the function f_+. On the other hand the light radiated at very large distances from a source is independent of f_+. The total energy absorbed by distant absorbers will not check with the energy loss of the source. We are in a situation analogous to that in the classical theory if the entire f function is made to contain only retarded contributions (see A, Appendix). One desires instead the analogue of $\langle F\rangle_{\text{ret}}$ of A. This problem is being studied.

One can say therefore, that this attempt to find a consistent modification of quantum electrodynamics is incomplete (see also the question of closed loops, below). For it could turn out that any correct form of f_+ which will guarantee energy conservation may at the same time not be able to make the self-energy integral finite. The desire to make the methods of simplifying the calculation of quantum electrodynamic processes more widely available has prompted this publication before an analysis of the correct form for f_+ is complete. One might try to take the position that, since the energy discrepancies discussed vanish in the limit $\lambda\to\infty$, the correct physics might be considered to be that obtained by letting $\lambda\to\infty$ after mass renormalization. I have no proof of the mathematical consistency of this procedure, but the presumption is very strong that it is satisfactory. (It is also strong that a satisfactory form for f_+ can be found.)

7. THE PROBLEM OF VACUUM POLARIZATION

In the analysis of the radiative corrections to scattering one type of term was not considered. The potential

[14] The expression is not exact because the substitution of Δm by the integral in (19) is valid only if $\not{p}$ operates on a state such that $\not{p}$ can be replaced by m. The error, however, is of order $a(\not{p}'-m)^{-1}(\not{p}-m)(\not{p}'-m)^{-1}b$ which is $a((1+\epsilon)\not{p}+m)(\not{p}-m)\times((1+\epsilon)\not{p}+m)\not{p}(2\epsilon+\epsilon^2)^{-2}m^{-4}$. But since $\not{p}^2=m^2$, we have $\not{p}(\not{p}-m)=-m(\not{p}-m)=(\not{p}-m)\not{p}$ so the net result is approximately $a(\not{p}-m)b/4m^2$ and is not of order $1/\epsilon$ but smaller, so that its effect drops out in the limit.

[15] We have used, to first order, the general expansion (valid for any operators A, B)

$$(A+B)^{-1}=A^{-1}-A^{-1}BA^{-1}+A^{-1}BA^{-1}BA^{-1}-\cdots$$

with $A=\not{p}-\not{k}-m$ and $B=\not{p}'-\not{p}=\epsilon\not{p}$ to expand the difference of $(\not{p}'-\not{k}-m)^{-1}$ and $(\not{p}-\not{k}-m)^{-1}$.

[16] The renormalization terms appearing B, Eqs. (14), (15) when translated directly into the present notation do not give twice (29) but give this expression with the central $\not{p}_1m^{-1}$ factor replaced by $m\gamma_4/E_1$ where $E_1=p_{1\mu}$ for $\mu=4$. When integrated it therefore gives $ra((\not{p}_1+m)/2m)(m\gamma_4/E_1)$ or $ra-ra(m\gamma_4/E_1)(\not{p}_1-m)/2m$. (Since $\not{p}_1\gamma_4+\gamma_4\not{p}_1=2E_1$) which gives just ra, since $\not{p}_1u_1=mu_1$.

[17] H. W. Lewis, Phys. Rev. 73, 173 (1948).

which we can assume to vary as $a_\mu \exp(-iq\cdot x)$ creates a pair of electrons (see Fig. 6), momenta $\boldsymbol{p}_a$, $-\boldsymbol{p}_b$. This pair then reannihilates, emitting a quantum $\boldsymbol{q}=\boldsymbol{p}_b-\boldsymbol{p}_a$, which quantum scatters the original electron from state 1 to state 2. The matrix element for this process (and the others which can be obtained by rearranging the order in time of the various events) is

$$-(e^2/\pi i)(\bar{u}_2\gamma_\mu u_1)\int Sp[(\boldsymbol{p}_a+\boldsymbol{q}-m)^{-1}\times\gamma_\nu(\boldsymbol{p}_a-m)^{-1}\gamma_\mu]d^4p_a\boldsymbol{q}^{-2}C(\boldsymbol{q}^2)a_\nu. \quad (30)$$

This is because the potential produces the pair with amplitude proportional to $a_\nu\gamma_\nu$, the electrons of momenta $\boldsymbol{p}_a$ and $-(\boldsymbol{p}_a+\boldsymbol{q})$ proceed from there to annihilate, producing a quantum (factor γ_μ) which propagates (factor $\boldsymbol{q}^{-2}C(\boldsymbol{q}^2)$) over to the other electron, by which it is absorbed (matrix element of γ_μ between states 1 and 2 of the original electron $(\bar{u}_2\gamma_\mu u_1)$). All momenta $\boldsymbol{p}_a$ and spin states of the virtual electron are admitted, which means the spur and the integral on d^4p_a are calculated.

One can imagine that the closed loop path of the positron-electron produces a current

$$4\pi j_\mu = J_{\mu\nu}a_\nu, \quad (31)$$

which is the source of the quanta which act on the second electron. The quantity

$$J_{\mu\nu}=-(e^2/\pi i)\int Sp[(\boldsymbol{p}+\boldsymbol{q}-m)^{-1}\times\gamma_\nu(\boldsymbol{p}-m)^{-1}\gamma_\mu]d^4p, \quad (32)$$

is then characteristic for this problem of polarization of the vacuum.

One sees at once that $J_{\mu\nu}$ diverges badly. The modification of δ to f alters the amplitude with which the current j_μ will affect the scattered electron, but it can do nothing to prevent the divergence of the integral (32) and of its effects.

One way to avoid such difficulties is apparent. From one point of view we are considering all routes by which a given electron can get from one region of space-time to another, i.e., from the source of electrons to the apparatus which measures them. From this point of view the closed loop path leading to (32) is unnatural. It might be assumed that the only paths of meaning are those which start from the source and work their way in a continuous path (possibly containing many time reversals) to the detector. Closed loops would be excluded. We have already found that this may be done for electrons moving in a fixed potential.

Such a suggestion must meet several questions, however. The closed loops are a consequence of the usual hole theory in electrodynamics. Among other things, they are required to keep probability conserved. The probability that no pair is produced by a potential is not unity and its deviation from unity arises from the imaginary part of $J_{\mu\nu}$. Again, with closed loops excluded, a pair of electrons once created cannot annihilate one another again, the scattering of light by light would be zero, etc. Although we are not experimentally sure of these phenomena, this does seem to indicate that the closed loops are necessary. To be sure, it is always possible that these matters of probability conservation, etc., will work themselves out as simply in the case of interacting particles as for those in a fixed potential. Lacking such a demonstration the presumption is that the difficulties of vacuum polarization are not so easily circumvented.[18]

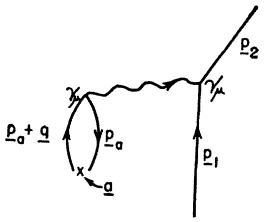

FIG. 6. Vacuum polarization effect on scattering, Eq. (30).

An alternative procedure discussed in B is to assume that the function $K_+(2,1)$ used above is incorrect and is to be replaced by a modified function K_+' having no singularity on the light cone. The effect of this is to provide a convergence factor $C(\boldsymbol{p}^2-m^2)$ for *every* integral over electron momenta.[19] This will multiply the integrand of (32) by $C(\boldsymbol{p}^2-m^2)C((\boldsymbol{p}+\boldsymbol{q})^2-m^2)$, since the integral was originally $\delta(\boldsymbol{p}_a-\boldsymbol{p}_b+\boldsymbol{q})d^4p_ad^4p_b$ and both $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ get convergence factors. The integral now converges but the result is unsatisfactory.[20]

One expects the current (31) to be conserved, that is $q_\mu j_\mu=0$ or $q_\mu J_{\mu\nu}=0$. Also one expects no current if a_ν is a gradient, or $a_\nu=q_\nu$ times a constant. This leads to the condition $J_{\mu\nu}q_\nu=0$ which is equivalent to $q_\mu J_{\mu\nu}=0$ since $J_{\mu\nu}$ is symmetrical. But when the expression (32) is integrated with such convergence factors it does not satisfy this condition. By altering the kernel from K to another, K', which does not satisfy the Dirac equation we have lost the gauge invariance, its consequent current conservation and the general consistency of the theory.

One can see this best by calculating $J_{\mu\nu}q_\nu$ directly from (32). The expression within the spur becomes $(\boldsymbol{p}+\boldsymbol{q}-m)^{-1}\boldsymbol{q}(\boldsymbol{p}-m)^{-1}\gamma_\mu$ which can be written as the difference of two terms: $(\boldsymbol{p}-m)^{-1}\gamma_\mu-(\boldsymbol{p}+\boldsymbol{q}-m)^{-1}\gamma_\mu$. Each of these terms would give the same result if the integration d^4p were without a convergence factor, for

[18] It would be very interesting to calculate the Lamb shift accurately enough to be sure that the 20 megacycles expected from vacuum polarization are actually present.

[19] This technique also makes self-energy and radiationless scattering integrals finite even without the modification of δ_+ to f_+ for the radiation (and the consequent convergence factor $C(\boldsymbol{k}^2)$ for the quanta). See B.

[20] Added to the terms given below (33) there is a term $\frac{1}{4}(\lambda^3-2\mu^2+\frac{1}{3}\boldsymbol{q}^2)\delta_{\mu\nu}$ for $C(\boldsymbol{k}^2)=-\lambda^2(\boldsymbol{k}^2-\lambda^2)^{-1}$, which is not gauge invariant. (In addition the charge renormalization has $-7/6$ added to the logarithm.)

the first can be converted into the second by a shift of the origin of $\boldsymbol{p}$, namely $\boldsymbol{p}'=\boldsymbol{p}+\boldsymbol{q}$. This does not result in cancelation in (32) however, for the convergence factor is altered by the substitution.

A method of making (32) convergent without spoiling the gauge invariance has been found by Bethe and by Pauli. The convergence factor for light can be looked upon as the result of superposition of the effects of quanta of various masses (some contributing negatively). Likewise if we take the factor $C(\boldsymbol{p}^2-m^2)=-\lambda^2(\boldsymbol{p}^2-m^2-\lambda^2)^{-1}$ so that $(\boldsymbol{p}^2-m^2)^{-1}C(\boldsymbol{p}^2-m^2)=(\boldsymbol{p}^2-m^2)^{-1}-(\boldsymbol{p}^2-m^2-\lambda^2)^{-1}$ we are taking the difference of the result for electrons of mass m and mass $(\lambda^2+m^2)^{\frac{1}{2}}$. But we have taken this difference for *each* propagation between interactions with photons. They suggest instead that once created with a certain mass the electron should continue to propagate with this mass through all the potential interactions until it closes its loop. That is if the quantity (32), integrated over some finite range of $\boldsymbol{p}$, is called $J_{\mu\nu}(m^2)$ and the corresponding quantity over the same range of $\boldsymbol{p}$, but with m replaced by $(m^2+\lambda^2)^{\frac{1}{2}}$ is $J_{\mu\nu}(m^2+\lambda^2)$ we should calculate

$$J_{\mu\nu}{}^P=\int_0^\infty [J_{\mu\nu}(m^2)-J_{\mu\nu}(m^2+\lambda^2)]G(\lambda)d\lambda, \quad (32')$$

the function $G(\lambda)$ satisfying $\int_0^\infty G(\lambda)d\lambda=1$ and $\int_0^\infty G(\lambda)\lambda^2 d\lambda=0$. Then in the expression for $J_{\mu\nu}{}^P$ the range of $\boldsymbol{p}$ integration can be extended to infinity as the integral now converges. The result of the integration using this method is the integral on $d\lambda$ over $G(\lambda)$ of (see Appendix C)

$$J_{\mu\nu}{}^P=-\frac{e^2}{\pi}(q_\mu q_\nu-\delta_{\mu\nu}\boldsymbol{q}^2)\left(-\frac{1}{3}\ln\frac{\lambda^2}{m^2}-\left[\frac{4m^2+2\boldsymbol{q}^2}{3\boldsymbol{q}^2}\left(1-\frac{\theta}{\tan\theta}\right)-\frac{1}{9}\right]\right), \quad (33)$$

with $\boldsymbol{q}^2=4m^2\sin^2\theta$.

The gauge invariance is clear, since $q_\mu(q_\mu q_\nu-\boldsymbol{q}^2\delta_{\mu\nu})=0$. Operating (as it always will) on a potential of zero divergence the $(q_\mu q_\nu-\delta_{\mu\nu}\boldsymbol{q}^2)a_\nu$ is simply $-q^2a_\mu$, the D'Alembertian of the potential, that is, the current producing the potential. The term $-\frac{1}{3}(\ln(\lambda^2/m^2))(q_\mu q_\nu-\boldsymbol{q}^2\delta_{\mu\nu})$ therefore gives a current proportional to the current producing the potential. This would have the same effect as a change in charge, so that we would have a difference $\Delta(e^2)$ between e^2 and the experimentally observed charge, $e^2+\Delta(e^2)$, analogous to the difference between m and the observed mass. This charge depends logarithmically on the cut-off, $\Delta(e^2)/e^2=-(2e^2/3\pi)\ln(\lambda/m)$. After this renormalization of charge is made, no effects will be sensitive to the cut-off.

After this is done the final term remaining in (33), contains the usual effects[21] of polarization of the vacuum. It is zero for a free light quantum ($\boldsymbol{q}^2=0$). For small $\boldsymbol{q}^2$ it behaves as $(2/15)\boldsymbol{q}^2$ (adding $-\frac{1}{5}$ to the logarithm in the Lamb effect). For $\boldsymbol{q}^2>(2m)^2$ it is complex, the imaginary part representing the loss in amplitude required by the fact that the probability that no quanta are produced by a potential able to produce pairs ($(\boldsymbol{q}^2)^{\frac{1}{2}}>2m$) decreases with time. (To make the necessary analytic continuation, imagine m to have a small negative imaginary part, so that $(1-\boldsymbol{q}^2/4m^2)^{\frac{1}{2}}$ becomes $-i(\boldsymbol{q}^2/4m^2-1)^{\frac{1}{2}}$ as $\boldsymbol{q}^2$ goes from below to above $4m^2$. Then $\theta=\pi/2+iu$ where $\sinh u=+(\boldsymbol{q}^2/4m^2-1)^{\frac{1}{2}}$, and $-1/\tan\theta=i\tanh u=+i(\boldsymbol{q}^2-4m^2)^{\frac{1}{2}}(\boldsymbol{q}^2)^{-\frac{1}{2}}$.)

Closed loops containing a number of quanta or potential interactions larger than two produce no trouble. Any loop with an odd number of interactions gives zero (I, reference 9). Four or more potential interactions give integrals which are convergent even without a convergence factor as is well known. The situation is analogous to that for self-energy. Once the simple problem of a single closed loop is solved there are no further divergence difficulties for more complex processes.[22]

8. LONGITUDINAL WAVES

In the usual form of quantum electrodynamics the longitudinal and transverse waves are given separate treatment. Alternately the condition $(\partial A_\mu/\partial x_\mu)\Psi=0$ is carried along as a supplementary condition. In the present form no such special considerations are necessary for we are dealing with the solutions of the equation $-\Box^2 A_\mu=4\pi j_\mu$ with a current j_μ which is conserved $\partial j_\mu/\partial x_\mu=0$. That means at least $\Box^2(\partial A_\mu/\partial x_\mu)=0$ and in fact our solution also satisfies $\partial A_\mu/\partial x_\mu=0$.

To show that this is the case we consider the amplitude for emission (real or virtual) of a photon and show that the divergence of this amplitude vanishes. The amplitude for emission for photons polarized in the μ direction involves matrix elements of γ_μ. Therefore what we have to show is that the corresponding matrix elements of $q_\mu\gamma_\mu=\boldsymbol{q}$ vanish. For example, for a first order effect we would require the matrix element of $\boldsymbol{q}$ between two states $\boldsymbol{p}_1$ and $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$. But since $\boldsymbol{q}=\boldsymbol{p}_2-\boldsymbol{p}_1$ and $(\bar{u}_2\boldsymbol{p}_1u_1)=m(\bar{u}_2u_1)=(\bar{u}_2\boldsymbol{p}_2u_1)$ the matrix element vanishes, which proves the contention in this case. It also vanishes in more complex situations (essentially because of relation (34), below) (for example, try putting $\boldsymbol{e}_2=\boldsymbol{q}_2$ in the matrix (15) for the Compton Effect).

To prove this in general, suppose $\boldsymbol{a}_i$, $i=1$ to N are a set of plane wave disturbing potentials carrying momenta $\boldsymbol{q}_i$ (e.g., some may be emissions or absorptions of the same or different quanta) and consider a matrix for the transition from a state of momentum $\boldsymbol{p}_0$ to $\boldsymbol{p}_N$ such

[21] E. A. Uehling, Phys. Rev. **48**, 55 (1935), R. Serber, Phys. Rev. **48**, 49 (1935).

[22] There are loops completely without external interactions. For example, a pair is created virtually along with a photon. Next they annihilate, absorbing this photon. Such loops are disregarded on the grounds that they do not interact with anything and are thereby completely unobservable. Any indirect effects they may have via the exclusion principle have already been included.

as $\boldsymbol{a}_N\prod_{i=1}^{N-1}(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i$ where $\boldsymbol{p}_i=\boldsymbol{p}_{i-1}+\boldsymbol{q}_i$ (and in the product, terms with larger i are written to the left). The most general matrix element is simply a linear combination of these. Next consider the matrix between states $\boldsymbol{p}_0$ and $\boldsymbol{p}_N+\boldsymbol{q}$ in a situation in which not only are the $\boldsymbol{a}_i$ acting but also another potential $\boldsymbol{a}\exp(-iq\cdot x)$ where $\boldsymbol{a}=\boldsymbol{q}$. This may act previous to all $\boldsymbol{a}_i$, in which case it gives $\boldsymbol{a}_N\prod(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i(\boldsymbol{p}_0+\boldsymbol{q}-m)^{-1}\boldsymbol{q}$ which is equivalent to $+\boldsymbol{a}_N\prod(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i$ since $+(\boldsymbol{p}_0+\boldsymbol{q}-m)^{-1}\boldsymbol{q}$ is equivalent to $(\boldsymbol{p}_0+\boldsymbol{q}-m)^{-1}\times(\boldsymbol{p}_0+\boldsymbol{q}-m)$ as $\boldsymbol{p}_0$ is equivalent to m acting on the initial state. Likewise if it acts after all the potentials it gives $\boldsymbol{q}(\boldsymbol{p}_N-m)^{-1}\boldsymbol{a}_N\prod(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i$ which is equivalent to $-\boldsymbol{a}_N\prod(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i$ since $\boldsymbol{p}_N+\boldsymbol{q}-m$ gives zero on the final state. Or again it may act between the potential $\boldsymbol{a}_k$ and $\boldsymbol{a}_{k+1}$ for each k. This gives

$$\sum_{k=1}^{N-1}\boldsymbol{a}_N\prod_{i=k+1}^{N-1}(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i(\boldsymbol{p}_k+\boldsymbol{q}-m)^{-1}\times\boldsymbol{q}(\boldsymbol{p}_k-m)^{-1}\boldsymbol{a}_k\prod_{j=1}^{k-1}(\boldsymbol{p}_j-m)^{-1}\boldsymbol{a}_j.$$

However,

$$(\boldsymbol{p}_k+\boldsymbol{q}-m)^{-1}\boldsymbol{q}(\boldsymbol{p}_k-m)^{-1}=(\boldsymbol{p}_k-m)^{-1}-(\boldsymbol{p}_k+\boldsymbol{q}-m)^{-1},\quad(34)$$

so that the sum breaks into the difference of two sums, the first of which may be converted to the other by the replacement of k by $k-1$. There remain only the terms from the ends of the range of summation,

$$+\boldsymbol{a}_N\prod_{i=1}^{N-1}(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i-\boldsymbol{a}_N\prod_{i=1}^{N-1}(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i.$$

These cancel the two terms originally discussed so that the entire effect is zero. Hence any wave emitted will satisfy $\partial A_\mu/\partial x_\mu=0$. Likewise longitudinal waves (that is, waves for which $A_\mu=\partial\phi/\partial x_\mu$ or $\boldsymbol{a}=\boldsymbol{q}$) cannot be absorbed and will have no effect, for the matrix elements for emission and absorption are similar. (We have said little more than that a potential $A_\mu=\partial\varphi/\partial x_\mu$ has no effect on a Dirac electron since a transformation $\psi'=\exp(-i\phi)\psi$ removes it. It is also easy to see in coordinate representation using integrations by parts.)

This has a useful practical consequence in that in computing probabilities for transition for unpolarized light one can sum the squared matrix over all four directions rather than just the two special polarization vectors. Thus suppose the matrix element for some process for light polarized in direction e_μ is $e_\mu M_\mu$. If the light has wave vector q_μ we know from the argument above that $q_\mu M_\mu=0$. For unpolarized light progressing in the z direction we would ordinarily calculate $M_x{}^2+M_y{}^2$. But we can as well sum $M_x{}^2+M_y{}^2+M_z{}^2-M_t{}^2$ for $q_\mu M_\mu$ implies $M_t=M_z$ since $q_t=q_z$ for free quanta. This shows that unpolarized light is a relativistically invariant concept, and permits some simplification in computing cross sections for such light.

Incidentally, the virtual quanta interact through terms like $\gamma_\mu\cdots\gamma_\mu\boldsymbol{k}^{-2}d^4k$. Real processes correspond to poles in the formulae for virtual processes. The pole occurs when $\boldsymbol{k}^2=0$, but it looks at first as though in the sum on all four values of μ, of $\gamma_\mu\cdots\gamma_\mu$ we would have four kinds of polarization instead of two. Now it is clear that only two perpendicular to $\boldsymbol{k}$ are effective.

The usual elimination of longitudinal and scalar virtual photons (leading to an instantaneous Coulomb potential) can of course be performed here too (although it is not particularly useful). A typical term in a virtual transition is $\gamma_\mu\cdots\gamma_\mu\boldsymbol{k}^{-2}d^4k$ where the $\cdots$ represent some intervening matrices. Let us choose for the values of μ, the time t, the direction of vector part $\mathbf{K}$, of $\boldsymbol{k}$, and two perpendicular directions 1, 2. We shall not change the expression for these two 1, 2 for these are represented by transverse quanta. But we must find $(\gamma_t\cdots\gamma_t)-(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})$. Now $\boldsymbol{k}=k_4\gamma_t-K\gamma_{\mathbf{K}}$, where $K=(\mathbf{K}\cdot\mathbf{K})^{\frac{1}{2}}$, and we have shown above that $\boldsymbol{k}$ replacing the γ_μ gives zero.[23] Hence $K\gamma_{\mathbf{K}}$ is equivalent to $k_4\gamma_t$ and

$$(\gamma_t\cdots\gamma_t)-(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})=((K^2-k_4{}^2)/K^2)(\gamma_t\cdots\gamma_t),$$

so that on multiplying by $\boldsymbol{k}^{-2}d^4k=d^4k(k_4{}^2-K^2)^{-1}$ the net effect is $-(\gamma_t\cdots\gamma_t)d^4k/K^2$. The γ_t means just scalar waves, that is, potentials produced by charge density. The fact that $1/K^2$ does not contain k_4 means that k_4 can be integrated first, resulting in an instantaneous interaction, and the $d^3\mathbf{K}/K^2$ is just the momentum representation of the Coulomb potential, $1/r$.

9. KLEIN GORDON EQUATION

The methods may be readily extended to particles of spin zero satisfying the Klein Gordon equation,[24]

$$\Box^2\psi-m^2\psi=i\partial(A_\mu\psi)/\partial x_\mu+iA_\mu\partial\psi/\partial x_\mu-A_\mu A_\mu\psi.\quad(35)$$

[23] A little more care is required when both γ_μ's act on the same particle. Define $\boldsymbol{x}=k_4\gamma_t+K\gamma_{\mathbf{K}}$, and consider $(\boldsymbol{k}\cdots\boldsymbol{x})+(\boldsymbol{x}\cdots\boldsymbol{k})$. Exactly this term would arise if a system, acted on by potential $\boldsymbol{x}$ carrying momentum $-\boldsymbol{k}$, is disturbed by an added potential $\boldsymbol{k}$ of momentum $+\boldsymbol{k}$ (the reversed sign of the momenta in the intermediate factors in the second term $\boldsymbol{x}\cdots\boldsymbol{k}$ has no effect since we will later integrate over all $\boldsymbol{k}$). Hence as shown above the result is zero, but since $(\boldsymbol{k}\cdots\boldsymbol{x})+(\boldsymbol{x}\cdots\boldsymbol{k})=k_4{}^2(\gamma_t\cdots\gamma_t)-K^2(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})$ we can still conclude $(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})=k_4{}^2K^{-2}(\gamma_t\cdots\gamma_t)$.

[24] The equations discussed in this section were deduced from the formulation of the Klein Gordon equation given in reference 5, Section 14. The function ψ in this section has only one component and is not a spinor. An alternative formal method of making the equations valid for spin zero and also for spin 1 is (presumably) by use of the Kemmer-Duffin matrices β_μ, satisfying the commutation relation

$$\beta_\mu\beta_\nu\beta_\sigma+\beta_\sigma\beta_\nu\beta_\mu=\delta_{\mu\nu}\beta_\sigma+\delta_{\sigma\nu}\beta_\mu.$$

If we interpret $\boldsymbol{a}$ to mean $a_\mu\beta_\mu$, rather than $a_\mu\gamma_\mu$, for any a_μ, all of the equations in momentum space will remain formally identical to those for the spin 1/2; with the exception of those in which a denominator $(\boldsymbol{p}-m)^{-1}$ has been rationalized to $(\boldsymbol{p}+m)(\boldsymbol{p}^2-m^2)^{-1}$ since $\boldsymbol{p}^2$ is no longer equal to a number, $p\cdot p$. But $\boldsymbol{p}^3$ does equal $(p\cdot p)\boldsymbol{p}$ so that $(\boldsymbol{p}-m)^{-1}$ may now be interpreted as $(m\boldsymbol{p}+m^2+\boldsymbol{p}^2-p\cdot p)(p\cdot p-m^2)^{-1}m^{-1}$. This implies that equations in coordinate space will be valid of the function $K_+(2,1)$ is given as $K_+(2,1)=[(i\nabla_2+m)-m^{-1}(\nabla_2{}^2+\Box_2{}^2)]iI_+(2,1)$ with $\nabla_2=\beta_\mu\partial/\partial x_{2\mu}$. This is all in virtue of the fact that the many component wave function ψ (5 components for spin 0, 10 for spin 1) satisfies $(i\nabla-m)\psi=\boldsymbol{A}\psi$ which is formally identical to the Dirac Equation. See W. Pauli, Rev. Mod. Phys. **13**, 203 (1940).

The important kernel is now $I_+(2, 1)$ defined in (I, Eq. (32)). For a free particle, the wave function $\psi(2)$ satisfies $+\Box^2\psi - m^2\psi = 0$. At a point, 2, inside a space time region it is given by

$$\psi(2) = \int [\psi(1)\partial I_+(2, 1)/\partial x_{1\mu} - (\partial\psi/\partial x_{1\mu}) I_+(2, 1)] N_\mu(1) d^3V_1,$$

(as is readily shown by the usual method of demonstrating Green's theorem) the integral being over an entire 3-surface boundary of the region (with normal vector N_μ). Only the positive frequency components of ψ contribute from the surface preceding the time corresponding to 2, and only negative frequencies from the surface future to 2. These can be interpreted as electrons and positrons in direct analogy to the Dirac case.

The right-hand side of (35) can be considered as a source of new waves and a series of terms written down to represent matrix elements for processes of increasing order. There is only one new point here, the term in $A_\mu A_\mu$ by which two quanta can act at the same time. As an example, suppose three quanta or potentials, $a_\mu \exp(-iq_a\cdot x)$, $b_\mu \exp(-iq_b\cdot x)$, and $c_\mu \exp(-iq_c\cdot x)$ are to act in that order on a particle of original momentum $p_{0\mu}$ so that $\boldsymbol{p}_a = \boldsymbol{p}_0 + \boldsymbol{q}_a$ and $\boldsymbol{p}_b = \boldsymbol{p}_a + \boldsymbol{q}_b$; the final momentum being $\boldsymbol{p}_c = \boldsymbol{p}_b + \boldsymbol{q}_c$. The matrix element is the sum of three terms ($\boldsymbol{p}^2 = p_\mu p_\mu$) (illustrated in Fig. 7)

$$\begin{aligned}&(p_c\cdot c + p_b\cdot c)(\boldsymbol{p}_b^2 - m^2)^{-1}(p_b\cdot b + p_a\cdot b)\\ &\qquad\times(\boldsymbol{p}_a^2 - m^2)^{-1}(p_a\cdot a + p_0\cdot a)\\ &-(p_c\cdot c + p_b\cdot c)(\boldsymbol{p}_b^2 - m^2)^{-1}(b\cdot a)\\ &-(c\cdot b)(\boldsymbol{p}_a^2 - m^2)^{-1}(p_a\cdot a + p_0\cdot a).\end{aligned} \tag{36}$$

The first comes when each potential acts through the perturbation $i\partial(A_\mu\psi)/\partial x_\mu + iA_\mu\partial\psi/\partial x_\mu$. These gradient operators in momentum space mean respectively the momentum after and before the potential A_μ operates. The second term comes from b_μ and a_μ acting at the same instant and arises from the $A_\mu A_\mu$ term in (a). Together b_μ and a_μ carry momentum $q_{b\mu} + q_{a\mu}$ so that after $b\cdot a$ operates the momentum is $\boldsymbol{p}_0 + \boldsymbol{q}_a + \boldsymbol{q}_b$ or $\boldsymbol{p}_b$. The final term comes from c_μ and b_μ operating together in a similar manner. The term $A_\mu A_\mu$ thus permits a new type of process in which two quanta can be emitted (or absorbed, or one absorbed, one emitted) at the same time. There is no $a\cdot c$ term for the order a, b, c we have assumed. In an actual problem there would be other terms like (36) but with alterations in the order in which the quanta a, b, c act. In these terms $a\cdot c$ would appear.

As a further example the self-energy of a particle of momentum p_μ is

$$(e^2/2\pi im)\int [(2p-k)_\mu((\boldsymbol{p}-\boldsymbol{k})^2 - m^2)^{-1} \times (2p-k)_\mu - \delta_{\mu\mu}] d^4k\boldsymbol{k}^{-2}C(\boldsymbol{k}^2),$$

where the $\delta_{\mu\mu} = 4$ comes from the $A_\mu A_\mu$ term and represents the possibility of the simultaneous emission and absorption of the same virtual quantum. This integral without the $C(\boldsymbol{k}^2)$ diverges quadratically and would not converge if $C(\boldsymbol{k}^2) = -\lambda^2/(\boldsymbol{k}^2 - \lambda^2)$. Since the interaction occurs through the gradients of the potential, we must use a stronger convergence factor, for example $C(\boldsymbol{k}^2) = \lambda^4(\boldsymbol{k}^2 - \lambda^2)^{-2}$, or in general (17) with $\int_0^\infty \lambda^2 G(\lambda) d\lambda = 0$. In this case the self-energy converges but depends quadratically on the cut-off λ and is not necessarily small compared to m. The radiative corrections to scattering after mass renormalization are insensitive to the cut-off just as for the Dirac equation.

When there are several particles one can obtain Bose statistics by the rule that if two processes lead to the same state but with two electrons exchanged, their amplitudes are to be added (rather than subtracted as for Fermi statistics). In this case equivalence to the second quantization treatment of Pauli and Weisskopf should be demonstrable in a way very much like that given in I (appendix) for Dirac electrons. The Bose statistics mean that the sign of contribution of a closed loop to the vacuum polarization is the opposite of what it is for the Fermi case (see I). It is ($\boldsymbol{p}_b = \boldsymbol{p}_a + \boldsymbol{q}$)

$$J_{\mu\nu} = \frac{e^2}{2\pi im}\int [(p_{b\mu} + p_{a\mu})(p_{b\nu} + p_{a\nu})(\boldsymbol{p}_a^2 - m^2)^{-1} \times (\boldsymbol{p}_b^2 - m^2)^{-1} - \delta_{\mu\nu}(\boldsymbol{p}_a^2 - m^2)^{-1} - \delta_{\mu\nu}(\boldsymbol{p}_b^2 - m^2)^{-1}] d^4p_a$$

giving,

$$J_{\mu\nu}{}^P = \frac{e^2}{\pi}(q_\mu q_\nu - \delta_{\mu\nu}\boldsymbol{q}^2)\left[\frac{1}{6}\ln\frac{\lambda^2}{m^2} + \frac{1}{9} - \frac{4m^2 - \boldsymbol{q}^2}{3\boldsymbol{q}^2}\left(1 - \frac{\theta}{\tan\theta}\right)\right],$$

the notation as in (33). The imaginary part for $(\boldsymbol{q}^2)^{\frac{1}{2}} > 2m$ is again positive representing the loss in the probability of finding the final state to be a vacuum, associated with the possibilities of pair production. Fermi statistics would give a gain in probability (and also a charge renormalization of opposite sign to that expected).

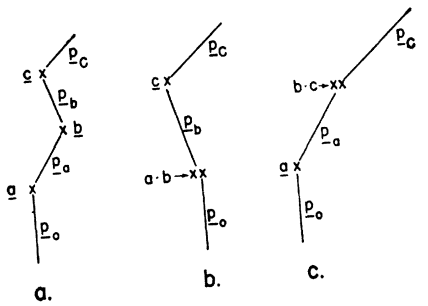

FIG. 7. Klein-Gordon particle in three potentials, Eq. (36). The coupling to the electromagnetic field is now, for example, $p_0\cdot a + p_a\cdot a$, and a new possibility arises, (b), of simultaneous interaction with two quanta $a\cdot b$. The propagation factor is now $(p\cdot p - m^2)^{-1}$ for a particle of momentum p_μ.

10. APPLICATION TO MESON THEORIES

The theories which have been developed to describe mesons and the interaction of nucleons can be easily expressed in the language used here. Calculations, to lowest order in the interactions can be made very easily for the various theories, but agreement with experimental results is not obtained. Most likely all of our present formulations are quantitatively unsatisfactory. We shall content ourselves therefore with a brief summary of the methods which can be used.

The nucleons are usually assumed to satisfy Dirac's equation so that the factor for propagation of a nucleon of momentum $\boldsymbol{p}$ is $(\boldsymbol{p}-M)^{-1}$ where M is the mass of the nucleon (which implies that nucleons can be created in pairs). The nucleon is then assumed to interact with mesons, the various theories differing in the form assumed for this interaction.

First, we consider the case of neutral mesons. The theory closest to electrodynamics is the theory of vector mesons with vector coupling. Here the factor for emission or absorption of a meson is $g\gamma_\mu$ when this meson is "polarized" in the μ direction. The factor g, the "mesonic charge," replaces the electric charge e. The amplitude for propagation of a meson of momentum $\boldsymbol{q}$ in intermediate states is $(\boldsymbol{q}^2-\mu^2)^{-1}$ (rather than $\boldsymbol{q}^{-2}$ as it is for light) where μ is the mass of the meson. The necessary integrals are made finite by convergence factors $C(\boldsymbol{q}^2-\mu^2)$ as in electrodynamics. For scalar mesons with scalar coupling the only change is that one replaces the γ_μ by 1 in emission and absorption. There is no longer a direction of polarization, μ, to sum upon. For pseudoscalar mesons, pseudoscalar coupling replace γ_μ by $\gamma_5=i\gamma_x\gamma_y\gamma_z\gamma_t$. For example, the self-energy matrix of a nucleon of momentum $\boldsymbol{p}$ in this theory is

$$(g^2/\pi i)\int \gamma_5(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\gamma_5 d^4k(\boldsymbol{k}^2-\mu^2)^{-1}C(\boldsymbol{k}^2-\mu^2).$$

Other types of meson theory result from the replacement of γ_μ by other expressions (for example by $\frac{1}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$ with a subsequent sum over all μ and ν for virtual mesons). Scalar mesons with vector coupling result from the replacement of γ_μ by $\mu^{-1}\boldsymbol{q}$ where $\boldsymbol{q}$ is the final momentum of the nucleon minus its initial momentum, that is, it is the momentum of the meson if absorbed, or the negative of the momentum of a meson emitted. As is well known, this theory with neutral mesons gives zero for all processes, as is proved by our discussion on longitudinal waves in electrodynamics. Pseudoscalar mesons with pseudo-vector coupling corresponds to γ_μ being replaced by $\mu^{-1}\gamma_5\boldsymbol{q}$ while vector mesons with tensor coupling correspond to using $(2\mu)^{-1}(\gamma_\mu\boldsymbol{q}-\boldsymbol{q}\gamma_\mu)$. These extra gradients involve the danger of producing higher divergencies for real processes. For example, $\gamma_5\boldsymbol{q}$ gives a logarithmically divergent interaction of neutron and electron.[25] Although these divergencies can be held by strong enough convergence factors, the results then are sensitive to the method used for convergence and the size of the cut-off values of λ. For low order processes $\mu^{-1}\gamma_5\boldsymbol{q}$ is equivalent to the pseudoscalar interaction $2M\mu^{-1}\gamma_5$ because if taken between free particle wave functions of the nucleon of momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$, we have

$$(\bar{u}_2\gamma_5\boldsymbol{q}u_1)=(\bar{u}_2\gamma_5(\boldsymbol{p}_2-\boldsymbol{p}_1)u_1)=-(\bar{u}_2\boldsymbol{p}_2\gamma_5u_1)-(\bar{u}_2\gamma_5\boldsymbol{p}_1u_1)=-2M(\bar{u}_2\gamma_5u_1)$$

since γ_5 anticommutes with $\boldsymbol{p}_2$ and $\boldsymbol{p}_2$ operating on the state 2 equivalent to M as is $\boldsymbol{p}_1$ on the state 1. This shows that the γ_5 interaction is unusually weak in the non-relativistic limit (for example the expected value of γ_5 for a free nucleon is zero), but since $\gamma_5^2=1$ is not small, pseudoscalar theory gives a more important interaction in second order than it does in first. Thus the pseudoscalar coupling constant should be chosen to fit nuclear forces including these important second order processes.[26] The equivalence of pseudoscalar and pseudovector coupling which holds for low order processes therefore does not hold when the pseudoscalar theory is giving its most important effects. These theories will therefore give quite different results in the majority of practical problems.

In calculating the corrections to scattering of a nucleon by a neutral vector meson field (γ_μ) due to the effects of virtual mesons, the situation is just as in electrodynamics, in that the result converges without need for a cut-off and depends only on gradients of the meson potential. With scalar (1) or pseudoscalar (γ_5) neutral mesons the result diverges logarithmically and so must be cut off. The part sensitive to the cut-off, however, is directly proportional to the meson potential. It may thereby be removed by a renormalization of mesonic charge g. After this renormalization the results depend only on gradients of the meson potential and are essentially independent of cut-off. This is in addition to the mesonic charge renormalization coming from the production of virtual nucleon pairs by a meson, analogous to the vacuum polarization in electrodynamics. But here there is a further difference from electrodynamics for scalar or pseudoscalar mesons in that the polarization also gives a term in the induced current proportional to the meson potential representing therefore an additional renormalization of the *mass of the meson* which usually depends quadratically on the cut-off.

Next consider charged mesons in the absence of an electromagnetic field. One can introduce isotopic spin operators in an obvious way. (Specifically replace the neutral γ_5, say, by $\tau_i\gamma_5$ and sum over $i=1, 2$ where $\tau_1=\tau_++\tau_-$, $\tau_2=i(\tau_+-\tau_-)$ and τ_+ changes neutron to proton (τ_+ on proton$=0$) and τ_- changes proton to neutron.) It is just as easy for practical problems simply to keep track of whether the particle is a proton or a neutron on a diagram drawn to help write down the

[25] M. Slotnick and W. Heitler, Phys. Rev. **75**, 1645 (1949).

[26] H. A. Bethe, Bull. Am. Phys. Soc. **24**, 3, Z3 (Washington, 1949).

matrix element. This excludes certain processes. For example in the scattering of a negative meson from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ by a neutron, the meson $\boldsymbol{q}_2$ must be emitted first (in order of operators, not time) for the neutron cannot absorb the negative meson $\boldsymbol{q}_1$ until it becomes a proton. That is, in comparison to the Klein Nishina formula (15), only the analogue of second term (see Fig. 5(b)) would appear in the scattering of negative mesons by neutrons, and only the first term (Fig. 5(a)) in the neutron scattering of positive mesons.

The source of mesons of a given charge is not conserved, for a neutron capable of emitting negative mesons may (on emitting one, say) become a proton no longer able to do so. The proof that a perturbation $\boldsymbol{q}$ gives zero, discussed for longitudinal electromagnetic waves, fails. This has the consequence that vector mesons, if represented by the interaction γ_μ would not satisfy the condition that the divergence of the potential is zero. The interaction is to be taken[27] as $\gamma_\mu - \mu^{-2}q_\mu\boldsymbol{q}$ in emission and as γ_μ in absorption if the real emission of mesons with a non-zero divergence of potential is to be avoided. (The correction term $\mu^{-2}q_\mu\boldsymbol{q}$ gives zero in the neutral case.) The asymmetry in emission and absorption is only apparent, as this is clearly the same thing as subtracting from the original $\gamma_\mu\cdots\gamma_\mu$, a term $\mu^{-2}\boldsymbol{q}\cdots\boldsymbol{q}$. That is, if the term $-\mu^{-2}q_\mu\boldsymbol{q}$ is omitted the resulting theory describes a combination of mesons of spin one and spin zero. The spin zero mesons, coupled by vector coupling $\boldsymbol{q}$, are removed by subtracting the term $\mu^{-2}\boldsymbol{q}\cdots\boldsymbol{q}$.

The two extra gradients $\boldsymbol{q}\cdots\boldsymbol{q}$ make the problem of diverging integrals still more serious (for example the interaction between two protons corresponding to the exchange of two charged vector mesons depends quadratically on the cut-off if calculated in a straightforward way). One is tempted in this formulation to choose simply $\gamma_\mu\cdots\gamma_\mu$ and accept the admixture of spin zero mesons. But it appears that this leads in the conventional formalism to negative energies for the spin zero component. This shows one of the advantages of the method of second quantization of meson fields over the present formulation. There such errors of sign are obvious while here we seem to be able to write seemingly innocent expressions which can give absurd results. Pseudovector mesons with pseudovector coupling correspond to using $\gamma_5(\gamma_\mu - \mu^{-2}q_\mu\boldsymbol{q})$ for absorption and $\gamma_5\gamma_\mu$ for emission for both charged and neutral mesons.

In the presence of an electromagnetic field, whenever the nucleon is a proton it interacts with the field in the way described for electrons. The meson interacts in the scalar or pseudoscalar case as a particle obeying the Klein-Gordon equation. It is important here to use the method of calculation of Bethe and Pauli, that is, a virtual meson is assumed to have the same "mass" during all its interactions with the electromagnetic field. The result for mass μ and for $(\mu^2+\lambda^2)^{\frac{1}{2}}$ are subtracted and the difference integrated over the function $G(\lambda)d\lambda$. A separate convergence factor is not provided for each meson propagation between electromagnetic interactions, otherwise gauge invariance is not insured. When the coupling involves a gradient, such as $\gamma_5\boldsymbol{q}$ where $\boldsymbol{q}$ is the final minus the initial momentum of the nucleon, the vector potential $\boldsymbol{A}$ must be subtracted from the momentum of the proton. That is, there is an additional coupling $\pm\gamma_5\boldsymbol{A}$ (plus when going from proton to neutron, minus for the reverse) representing the new possibility of a simultaneous emission (or absorption) of meson and photon.

Emission of positive or absorption of negative virtual mesons are represented in the same term, the sign of the charge being determined by temporal relations as for electrons and positrons.

Calculations are very easily carried out in this way to lowest order in g^2 for the various theories for nucleon interaction, scattering of mesons by nucleons, meson production by nuclear collisions and by gamma-rays, nuclear magnetic moments, neutron electron scattering, etc., However, no good agreement with experiment results, when these are available, is obtained. Probably all of the formulations are incorrect. An uncertainty arises since the calculations are only to first order in g^2, and are not valid if $g^2/\hbar c$ is large.

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APPENDIX

In this appendix a method will be illustrated by which the simpler integrals appearing in problems in electrodynamics can be directly evaluated. The integrals arising in more complex processes lead to rather complicated functions, but the study of the relations of one integral to another and their expression in terms of simpler integrals may be facilitated by the methods given here.

[27] The vector meson field potentials φ_μ satisfy

$$-\partial/\partial x_\nu(\partial\varphi_\mu/\partial x_\nu - \partial\varphi_\nu/\partial x_\mu) - \mu^2\varphi_\mu = -4\pi s_\mu,$$

where s_μ, the source for such mesons, is the matrix element of γ_μ between states of neutron and proton. By taking the divergence $\partial/\partial x_\mu$ of both sides, conclude that $\partial\varphi_\nu/\partial x_\nu = 4\pi\mu^{-2}\partial s_\nu/\partial x_\nu$ so that the original equation can be rewritten as

$$\Box^2\varphi_\mu - \mu^2\varphi_\mu = -4\pi(s_\mu + \mu^{-2}\partial/\partial x_\mu(\partial s_\nu/\partial x_\nu)).$$

The right hand side gives in momentum representation $\gamma_\mu - \mu^{-2}q_\mu q_\nu\gamma_\nu$ the left yields the $(\boldsymbol{q}^2-\mu^2)^{-1}$ and finally the interaction $s_\mu\varphi_\mu$ in the Lagrangian gives the γ_μ on absorption.

Proceeding in this way find generally that particles of spin one can be represented by a four-vector u_μ (which, for a free particle of momentum q satisfies $q\cdot u=0$). The propagation of virtual particles of momentum q from state ν to μ is represented by multiplication by the 4–4 matrix (or tensor) $P_{\mu\nu} = (\delta_{\mu\nu} - \mu^{-2}q_\mu q_\nu) \times (q^2-\mu^2)^{-1}$. The first-order interaction (from the Proca equation) with an electromagnetic potential $a\exp(-ik\cdot x)$ corresponds to multiplication by the matrix $E_{\mu\nu} = (q_2\cdot a + q_1\cdot a)\delta_{\mu\nu} - q_{2\nu}a_\mu - q_{1\mu}a_\nu$ where q_1 and $q_2 = q_1 + k$ are the momenta before and after the interaction. Finally, two potentials a, b may act simultaneously, with matrix $E'_{\mu\nu} = -(a\cdot b)\delta_{\mu\nu} + b_\mu a_\nu$.

As a typical problem consider the integral (12) appearing in the first order radiationless scattering problem:

$$\int \gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}\gamma_\mu \boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2), \tag{1a}$$

where we shall take $C(\boldsymbol{k}^2)$ to be typically $-\lambda^2(\boldsymbol{k}^2-\lambda^2)^{-1}$ and d^4k means $(2\pi)^{-2}dk_1dk_2dk_3dk_4$. We first rationalize the factors $(\boldsymbol{p}-\boldsymbol{k}-m)^{-1}=(\boldsymbol{p}-\boldsymbol{k}+m)((\boldsymbol{p}-\boldsymbol{k})^2-m^2)^{-1}$ obtaining,

$$\int \gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}+m)\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}+m)\gamma_\mu \boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2) \times((\boldsymbol{p}_1-\boldsymbol{k})^2-m^2)^{-1}((\boldsymbol{p}_2-\boldsymbol{k})^2-m^2)^{-1}. \tag{2a}$$

The matrix expression may be simplified. It appears to be best to do so *after* the integrations are performed. Since $\boldsymbol{AB}=2A\cdot B-\boldsymbol{BA}$ where $A\cdot B=A_\mu B_\mu$ is a number commuting with all matrices, find, if R is any expression, and $\boldsymbol{A}$ a vector, since $\gamma_\mu \boldsymbol{A}=-\boldsymbol{A}\gamma_\mu+2A_\mu$,

$$\gamma_\mu \boldsymbol{A}R\gamma_\mu=-\boldsymbol{A}\gamma_\mu R\gamma_\mu+2R\boldsymbol{A}. \tag{3a}$$

Expressions between two γ_μ's can be thereby reduced by induction. Particularly useful are

$$\begin{aligned}\gamma_\mu\gamma_\mu&=4\\ \gamma_\mu \boldsymbol{A}\gamma_\mu&=-2\boldsymbol{A}\\ \gamma_\mu \boldsymbol{AB}\gamma_\mu&=2(\boldsymbol{AB}+\boldsymbol{BA})=4A\cdot B\\ \gamma_\mu \boldsymbol{ABC}\gamma_\mu&=-2\boldsymbol{CBA}\end{aligned} \tag{4a}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are any three vector-matrices (i.e., linear combinations of the four γ's).

In order to calculate the integral in (2a) the integral may be written as the sum of three terms (since $\boldsymbol{k}=k_\sigma\gamma_\sigma$),

$$\gamma_\mu(\boldsymbol{p}_2+m)\boldsymbol{a}(\boldsymbol{p}_1+m)\gamma_\mu J_1-[\gamma_\mu\gamma_\sigma \boldsymbol{a}(\boldsymbol{p}_1+m)\gamma_\mu +\gamma_\mu(\boldsymbol{p}_2+m)\boldsymbol{a}\gamma_\sigma\gamma_\mu]J_2+\gamma_\mu\gamma_\sigma\boldsymbol{a}\gamma_\tau\gamma_\mu J_3, \tag{5a}$$

where

$$J_{(1;2;3)}=\int(1;k_\sigma;k_\sigma k_\tau)\boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2) \times((\boldsymbol{p}_2-\boldsymbol{k})^2-m^2)^{-1}((\boldsymbol{p}_1-\boldsymbol{k})^2-m^2)^{-1}. \tag{6a}$$

That is for J_1 the $(1;k_\sigma;k_\sigma k_\tau)$ is replaced by 1, for J_2 by k_σ, and for J_3 by $k_\sigma k_\tau$.

More complex processes of the first order involve more factors like $((\boldsymbol{p}_3-\boldsymbol{k})^2-m^2)^{-1}$ and a corresponding increase in the number of k's which may appear in the numerator, as $k_\sigma k_\tau k_\nu\cdots$. Higher order processes involving two or more virtual quanta involve similar integrals but with factors possibly involving $\boldsymbol{k}+\boldsymbol{k}'$ instead of just $\boldsymbol{k}$, and the integral extending on $\boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2)\boldsymbol{k}'^{-2}d^4k'C(\boldsymbol{k}'^2)$. They can be simplified by methods analogous to those used on the first order integrals.

The factors $(\boldsymbol{p}-\boldsymbol{k})^2-m^2$ may be written

$$(\boldsymbol{p}-\boldsymbol{k})^2-m^2=\boldsymbol{k}^2-2p\cdot k-\Delta, \tag{7a}$$

where $\Delta=m^2-\boldsymbol{p}^2$, $\Delta_1=m_1{}^2-\boldsymbol{p}_1{}^2$, etc., and we can consider dealing with cases of greater generality in that the different denominators need not have the same value of the mass m. In our specific problem (6a), $\boldsymbol{p}_1{}^2=m^2$ so that $\Delta_1=0$, but we desire to work with greater generality.

Now for the factor $C(\boldsymbol{k}^2)/\boldsymbol{k}^2$ we shall use $-\lambda^2(\boldsymbol{k}^2-\lambda^2)^{-1}\boldsymbol{k}^{-2}$. This can be written as

$$-\lambda^2/(\boldsymbol{k}^2-\lambda^2)\boldsymbol{k}^2=\boldsymbol{k}^{-2}C(\boldsymbol{k}^2)=-\int_0^{\lambda^2}dL(\boldsymbol{k}^2-L)^{-2}. \tag{8a}$$

Thus we can replace $\boldsymbol{k}^{-2}C(\boldsymbol{k}^2)$ by $(\boldsymbol{k}^2-L)^{-2}$ and at the end integrate the result with respect to L from zero to λ^2. We can for many practical purposes consider λ^2 very large relative to m^2 or $\boldsymbol{p}^2$. When the original integral converges even without the convergence factor, it will be obvious since the L integration will then be convergent to infinity. If an infra-red catastrophe exists in the integral one can simply assume quanta have a small mass $\lambda_{\min}$ and extend the integral on L from $\lambda^2{}_{\min}$ to λ^2, rather than from zero to λ^2.

We then have to do integrals of the form

$$\int(1;k_\sigma;k_\sigma k_\tau)d^4k(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2p_1\cdot k-\Delta_1)^{-1} \times(\boldsymbol{k}^2-2p_2\cdot k-\Delta_2)^{-1}, \tag{9a}$$

where by $(1;k_\sigma;\cdot k_\sigma k_\tau)$ we mean that in the place of this symbol either 1, or k_σ, or $k_\sigma k_\tau$ may stand in different cases. In more complicated problems there may be more factors $(\boldsymbol{k}^2-2p_i\cdot k-\Delta_i)^{-1}$ or other powers of these factors (the $(\boldsymbol{k}^2-L)^{-2}$ may be considered as a special case of such a factor with $\boldsymbol{p}_i=0$, $\Delta_i=L$) and further factors like $k_\sigma k_\tau k_\rho\cdots$ in the numerator. The poles in all the factors are made definite by the assumption that L, and the Δ's have infinitesimal negative imaginary parts.

We shall do the integrals of successive complexity by induction. We start with the simplest convergent one, and show

$$\int d^4k(\boldsymbol{k}^2-L)^{-3}=(8iL)^{-1}. \tag{10a}$$

For this integral is $\int(2\pi)^{-2}dk_4d^3\mathbf{K}(k_4{}^2-\mathbf{K}\cdot\mathbf{K}-L)^{-3}$ where the vector $\mathbf{K}$, of magnitude $K=(\mathbf{K}\cdot\mathbf{K})^{\frac{1}{2}}$ is k_1, k_2, k_3. The integral on k_4 shows third order poles at $k_4=+(K^2+L)^{\frac{1}{2}}$ and $k_4=-(K^2+L)^{\frac{1}{2}}$. Imagining, in accordance with our definitions, that L has a small negative imaginary part only the first is below the real axis. The contour can be closed by an infinite semi-circle below this axis, without change of the value of the integral since the contribution from the semi-circle vanishes in the limit. Thus the contour can be shrunk about the pole $k_4=+(K^2+L)^{\frac{1}{2}}$ and the resulting k_4 integral is $-2\pi i$ times the residue at this pole. Writing $k_4=(K^2+L)^{\frac{1}{2}}+\epsilon$ and expanding $(k_4{}^2-K^2-L)^{-3}=\epsilon^{-3}(\epsilon+2(K^2+L)^{\frac{1}{2}})^{-3}$ in powers of ϵ, the residue, being the coefficient of the term ϵ^{-1}, is seen to be $6(2(K^2+L)^{\frac{1}{2}})^{-5}$ so our integral is

$$-(3i/32\pi)\int_0^\infty 4\pi K^2dK(K^2+L)^{-5/2}=(3/8i)(1/3L)$$

establishing (10a).

We also have $\int k_\sigma d^4k(\boldsymbol{k}^2-L)^{-3}=0$ from the symmetry in the k space. We write these results as

$$(8i)\int(1;k_\sigma)d^4k(\boldsymbol{k}^2-L)^{-3}=(1;0)L^{-1}, \tag{11a}$$

where in the brackets $(1;k_\sigma)$ and $(1;0)$ corresponding entries are to be used.

Substituting $\boldsymbol{k}=\boldsymbol{k}'-\boldsymbol{p}$ in (11a), and calling $L-\boldsymbol{p}^2=\Delta$ shows that

$$(8i)\int(1;k_\sigma)d^4k(\boldsymbol{k}^2-2p\cdot k-\Delta)^{-3}=(1;p_\sigma)(\boldsymbol{p}^2+\Delta)^{-1}. \tag{12a}$$

By differentiating both sides of (12a) with respect to Δ, or with respect to p_τ there follows directly

$$(24i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(\boldsymbol{k}^2-2p\cdot k-\Delta)^{-4} =-(1;p_\sigma;p_\sigma p_\tau-\tfrac{1}{2}\delta_{\sigma\tau}(\boldsymbol{p}^2+\Delta))(\boldsymbol{p}^2+\Delta)^{-2}. \tag{13a}$$

Further differentiations give directly successive integrals including more k factors in the numerator and higher powers of $(\boldsymbol{k}^2-2p\cdot k-\Delta)$ in the denominator.

The integrals so far only contain one factor in the denominator. To obtain results for two factors we make use of the identity

$$a^{-1}b^{-1}=\int_0^1 dx(ax+b(1-x))^{-2}, \tag{14a}$$

(suggested by some work of Schwinger's involving Gaussian integrals). This represents the product of two reciprocals as a parametric integral over one and will therefore permit integrals with two factors to be expressed in terms of one. For other powers of a, b, we make use of all of the identities, such as

$$a^{-2}b^{-1}=\int_0^1 2xdx(ax+b(1-x))^{-3}, \tag{15a}$$

deducible from (14a) by successive differentiations with respect to a or b.

To perform an integral, such as

$$(8i)\int(1;k_\sigma)d^4k(\boldsymbol{k}^2-2p_1\cdot k-\Delta_1)^{-2}(\boldsymbol{k}^2-2p_2\cdot k-\Delta_2)^{-1}, \tag{16a}$$

write, using (15a),

$$(k^2-2p_1\cdot k-\Delta_1)^{-2}(k^2-2p_2\cdot k-\Delta_2)^{-1}=\int_0^1 2xdx(k^2-2p_x\cdot k-\Delta_x)^{-3},$$

where

$$p_x=xp_1+(1-x)p_2 \quad \text{and} \quad \Delta_x=x\Delta_1+(1-x)\Delta_2, \tag{17a}$$

(note that Δ_x is *not* equal to $m^2-p_x^2$) so that the expression (16a) is $(8i)\int_0^1 2xdx\int(1;k_\sigma)d^4k(k^2-2p_x\cdot k-\Delta_x)^{-3}$ which may now be evaluated by (12a) and is

$$(16a)=\int_0^1(1;p_{x\sigma})2xdx(p_x^2+\Delta_x)^{-1}, \tag{18a}$$

where p_x, Δ_x are given in (17a). The integral in (18a) is elementary, being the integral of ratio of polynomials, the denominator of second degree in x. The general expression although readily obtained is a rather complicated combination of roots and logarithms.

Other integrals can be obtained again by parametric differentiation. For example differentiation of (16a), (18a) with respect to Δ_2 or $p_{2\tau}$ gives

$$(8i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(k^2-2p_1\cdot k-\Delta_1)^{-2}(k^2-2p_2\cdot k-\Delta_2)^{-2}$$
$$=-\int_0^1(1;p_{x\sigma};p_{x\sigma}p_{x\tau}-\tfrac{1}{2}\delta_{\sigma\tau}(p_x^2+\Delta_x))\times 2x(1-x)dx(p_x^2+\Delta_x)^{-2}, \tag{19a}$$

again leading to elementary integrals.

As an example, consider the case that the second factor is just $(k^2-L)^{-2}$ and in the first put $p_1=p$, $\Delta_1=\Delta$. Then $p_x=xp$, $\Delta_x=x\Delta+(1-x)L$. There results

$$(8i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(k^2-L)^{-2}(k^2-2p\cdot k-\Delta)^{-2}$$
$$=-\int_0^1(1;xp_\sigma;x^2p_\sigma p_\tau-\tfrac{1}{2}\delta_{\sigma\tau}(x^2p^2+\Delta_x))\times 2x(1-x)dx(x^2p^2+\Delta_x)^{-2}. \tag{20a}$$

Integrals with three factors can be reduced to those involving two by using (14a) again. They, therefore, lead to integrals with two parameters (e.g., see application to radiative correction to scattering below).

The methods of calculation given in this paper are deceptively simple when applied to the lower order processes. For processes of increasingly higher orders the complexity and difficulty increases rapidly, and these methods soon become impractical in their present form.

A. Self-Energy

The self-energy integral (19) is

$$(e^2/\pi i)\int\gamma_\mu(p-k-m)^{-1}\gamma_\mu k^{-2}d^4kC(k^2), \tag{19}$$

so that it requires that we find (using the principle of (8a)) the integral on L from 0 to λ^2 of

$$\int\gamma_\mu(p-k+m)\gamma_\mu d^4k(k^2-L)^{-2}(k^2-2p\cdot k)^{-1},$$

since $(p-k)^2-m^2=k^2-2p\cdot k$, as $p^2=m^2$. This is of the form (16a) with $\Delta_1=L$, $p_1=0$, $\Delta_2=0$, $p_2=p$ so that (18a) gives, since $p_x=(1-x)p$, $\Delta_x=xL$,

$$(8i)\int(1;k_\sigma)d^4k(k^2-L)^{-2}(k^2-2p\cdot k)^{-1}=\int_0^1(1;(1-x)p_\sigma)2xdx((1-x)^2m^2+xL)^{-1},$$

or performing the integral on L, as in (8),

$$(8i)\int(1;k_\sigma)d^4kk^{-2}C(k^2)(k^2-2p\cdot k)^{-1}=\int_0^1(1;(1-x)p_\sigma)2dx\ln\frac{x\lambda^2+(1-x)^2m^2}{(1-x)^2m^2}.$$

Assuming now that $\lambda^2\gg m^2$ we neglect $(1-x)^2m^2$ relative to $x\lambda^2$ in the argument of the logarithm, which then becomes $(\lambda^2/m^2)(x/(1-x)^2)$. Then since $\int_0^1dx\ln(x(1-x)^{-2})=1$ and $\int_0^1(1-x)dx\ln(x(1-x)^{-2})=-(1/4)$ find

$$(8i)\int(1;k_\sigma)k^{-2}C(k^2)d^4k(k^2-2p\cdot k)^{-1}=\left(2\ln\frac{\lambda^2}{m^2}+2;\ p_\sigma\left(\ln\frac{\lambda^2}{m^2}-\frac{1}{2}\right)\right),$$

so that substitution into (19) (after the $(p-k-m)^{-1}$ in (19) is replaced by $(p-k+m)(k^2-2p\cdot k)^{-1}$) gives

$$(19)=(e^2/8\pi)\gamma_\mu[(p+m)(2\ln(\lambda^2/m^2)+2)-p(\ln(\lambda^2/m^2)-\tfrac{1}{2})]\gamma_\mu$$
$$=(e^2/8\pi)[8m(\ln(\lambda^2/m^2)+1)-p(2\ln(\lambda^2/m^2)+5)], \tag{20}$$

using (4a) to remove the γ_μ's. This agrees with Eq. (20) of the text, and gives the self-energy (21) when p is replaced by m.

B. Corrections to Scattering

The term (12) in the radiationless scattering, after rationalizing the matrix denominators and using $p_1^2=p_2^2=m^2$ requires the integrals (9a), as we have discussed. This is an integral with three denominators which we do in two stages. First the factors $(k^2-2p_1\cdot k)$ and $(k^2-2p_2\cdot k)$ are combined by a parameter y;

$$(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=\int_0^1 dy(k^2-2p_y\cdot k)^{-2},$$

from (14a) where

$$p_y=yp_1+(1-y)p_2. \tag{21a}$$

We therefore need the integrals

$$(8i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(k^2-L)^{-2}(k^2-2p_y\cdot k)^{-2}, \tag{22a}$$

which we will then integrate with respect to y from 0 to 1. Next we do the integrals (22a) immediately from (20a) with $p=p_y$, $\Delta=0$:

$$(22a)=-\int_0^1\int_0^1(1;xp_{y\sigma};x^2p_{y\sigma}p_{y\tau}-\tfrac{1}{2}\delta_{\sigma\tau}(x^2p_y^2+(1-x)L))2x(1-x)dx(x^2p_y^2+L(1-x))^{-2}dy.$$

We now turn to the integrals on L as required in (8a). The first term, (1), in $(1;k_\sigma;k_\sigma k_\tau)$ gives no trouble for large L, but if L is put equal to zero there results $x^{-2}p_y^{-2}$ which leads to a diverging integral on x as $x\to 0$. This infra-red catastrophe is analyzed by using $\lambda_{\min}^2$ for the lower limit of the L integral. For the last term the upper limit of L must be kept as λ^2. Assuming $\lambda_{\min}^2\ll p_y^2\ll\lambda^2$ the x integrals which remain are trivial, as in the self-energy case. One finds

$$-(8i)\int(k^2-\lambda_{\min}^2)^{-1}d^4kC(k^2-\lambda_{\min}^2)(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=\int_0^1 p_y^{-2}dy\ln(p_y^2/\lambda_{\min}^2) \tag{23a}$$

$$-(8i)\int k_\sigma k^{-2}d^4kC(k^2)(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=2\int_0^1 p_{y\sigma}p_y^{-2}dy, \tag{24a}$$

$$-(8i)\int k_\sigma k_\tau k^{-2}d^4kC(k^2)(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=\int_0^1 p_{y\sigma}p_{y\tau}p_y^{-2}dy-\tfrac{1}{2}\delta_{\sigma\tau}\int_0^1 dy\ln(\lambda^2p_y^{-2})+\tfrac{1}{4}\delta_{\sigma\tau}. \tag{25a}$$

The integrals on y give,

$$\int_0^1 p_y^{-2}dy\ln(p_y^2\lambda_{\min}^{-2})=4(m^2\sin2\theta)^{-1}\left[\theta\ln(m\lambda_{\min}^{-1})-\int_0^\theta\alpha\tan\alpha d\alpha\right], \tag{26a}$$

$$\int_0^1 p_{y\sigma}p_y^{-2}dy=\theta(m^2\sin2\theta)^{-1}(p_{1\sigma}+p_{2\sigma}), \tag{27a}$$

$$\int_0^1 p_{y\sigma}p_{y\tau}p_y^{-2}dy=\theta(2m^2\sin2\theta)^{-1}(p_{1\sigma}+p_{1\tau})(p_{2\sigma}+p_{2\tau})+q^{-2}q_\sigma q_\tau(1-\theta\operatorname{ctn}\theta), \tag{28a}$$

$$\int_0^1 dy\ln(\lambda^2p_y^{-2})=\ln(\lambda^2/m^2)+2(1-\theta\operatorname{ctn}\theta). \tag{29a}$$

These integrals on y were performed as follows. Since $\not p_2=\not p_1+\not q$ where $\not q$ is the momentum carried by the potential, it follows from $p_2^2=p_1^2=m^2$ that $2p_1\cdot q=-q^2$ so that since $\not p_y=\not p_1+\not q(1-y)$, $p_y^2=m^2-q^2y(1-y)$. The substitution $2y-1=\tan\alpha/\tan\theta$ where θ is defined by $4m^2\sin^2\theta=q^2$ is useful for it means $p_y^2=m^2\sec^2\alpha/\sec^2\theta$ and $p_y^{-2}dy=(m^2\sin 2\theta)^{-1}d\alpha$ where α goes from $-\theta$ to $+\theta$.

These results are substituted into the original scattering formula (2a), giving (22). It has been simplified by frequent use of the fact that $\not p_1$ operating on the initial state is m, and likewise $\not p_2$ when it appears at the left is replacable by m. (Thus, to simplify:

$$\gamma_\mu \not p_2 a \not p_1 \gamma_\mu = -2\not p_1 a \not p_2 \text{ by (4a)},$$
$$= -2(\not p_2-\not q)a(\not p_1+\not q)=-2(m-\not q)a(m+\not q).$$

A term like $\not q a \not q = -q^2 a+2(a\cdot q)\not q$ is equivalent to just $-q^2 a$ since $\not q=\not p_2-\not p_1=m-m$ has zero matrix element.) The renormalization term requires the corresponding integrals for the special case $q=0$.

C. Vacuum Polarization

The expressions (32) and (32′) for $J_{\mu\nu}$ in the vacuum polarization problem require the calculation of the integral

$$J_{\mu\nu}(m^2)=-\frac{e^2}{\pi i}\int Sp[\gamma_\mu(\not p-\tfrac{1}{2}\not q+m)\gamma_\nu(\not p+\tfrac{1}{2}\not q+m)]d^4p$$
$$\times((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}, \quad (32)$$

where we have replaced p by $p-\frac{1}{2}q$ to simplify the calculation somewhat. We shall indicate the method of calculation by studying the integral,

$$I(m^2)=\int p_\sigma p_\tau d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}.$$

The factors in the denominator, $p^2-p\cdot q-m^2+\frac{1}{4}q^2$ and $p^2+p\cdot q-m^2+\frac{1}{4}q^2$ are combined as usual by (8a) but for symmetry we substitute $x=\frac{1}{2}(1+\eta)$, $(1-x)=\frac{1}{2}(1-\eta)$ and integrate η from -1 to $+1$:

$$I(m^2)=\int_{-1}^{+1} p_\sigma p_\tau d^4p(p^2-\eta p\cdot q-m^2+\tfrac{1}{4}q^2)^{-2}d\eta/2. \quad (30a)$$

But the integral on p will not be found in our list for it is badly divergent. However, as discussed in Section 7, Eq. (32′) we do not wish $I(m^2)$ but rather $\int_0^\infty[I(m^2)-I(m^2+\lambda^2)]G(\lambda)d\lambda$. We can calculate the difference $I(m^2)-I(m^2+\lambda^2)$ by first calculating the derivative $I'(m^2+L)$ of I with respect to m^2 at m^2+L and later integrating L from zero to λ^2. By differentiating (30a), with respect to m^2 find,

$$I'(m^2+L)=\int_{-1}^{+1} p_\sigma p_\tau d^4p(p^2-\eta p\cdot q-m^2-L+\tfrac{1}{4}q^2)^{-3}d\eta.$$

This still diverges, but we can differentiate again to get

$$I''(m^2+L)=3\int_{-1}^{+1} p_\sigma p_\tau d^4p(p^2-\eta p\cdot q-m^2-L+\tfrac{1}{4}q^2)^{-4}d\eta$$
$$=-(8i)^{-1}\int_{-1}^{+1}(\tfrac{1}{4}\eta^2 q_\sigma q_\tau D^{-2}-\tfrac{1}{2}\delta_{\sigma\tau}D^{-1})d\eta \quad (31a)$$

(where $D=\frac{1}{4}(\eta^2-1)q^2+m^2+L$), which now converges and has been evaluated by (13a) with $p=\frac{1}{2}\eta q$ and $\Delta=m^2+L-\frac{1}{4}q^2$. Now to get I' we may integrate I'' with respect to L as an indefinite integral and *we may choose any convenient arbitrary constant.* This is because a constant C in I' will mean a term $-C\lambda^2$ in $I(m^2)-I(m^2+\lambda^2)$ which vanishes since we will integrate the results times $G(\lambda)d\lambda$ and $\int_0^\infty\lambda^2G(\lambda)d\lambda=0$. This means that the logarithm appearing on integrating L in (31a) presents no problem. We may take

$$I'(m^2+L)=(8i)^{-1}\int_{-1}^{+1}[\tfrac{1}{4}\eta^2 q_\sigma q_\tau D^{-1}+\tfrac{1}{2}\delta_{\sigma\tau}\ln D]d\eta+C\delta_{\sigma\tau},$$

a subsequent integral on L and finally on η presents no new problems. There results

$$-(8i)\int p_\sigma p_\tau d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}$$
$$=(q_\sigma q_\tau-\delta_{\sigma\tau}q^2)\left[\frac{1}{9}-\frac{4m^2-q^2}{3q^2}\left(1-\frac{\theta}{\tan\theta}\right)+\tfrac{1}{6}\ln\frac{\lambda^2}{m^2}\right]$$
$$+\delta_{\sigma\tau}[(\lambda^2+m^2)\ln(\lambda^2m^{-2}+1)-C'\lambda^2], \quad (32a)$$

where we assume $\lambda^2\gg m^2$ and have put some terms into the arbitrary constant C' which is independent of λ^2 (but in principle could depend on q^2) and which drops out in the integral on $G(\lambda)d\lambda$. We have set $q^2=4m^2\sin^2\theta$.

In a very similar way the integral with m^2 in the numerator can be worked out. It is, of course, necessary to differentiate this m^2 also when calculating I' and I''. There results

$$-(8i)\int m^2d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}$$
$$=4m^2(1-\theta\operatorname{ctn}\theta)-q^2/3+2(\lambda^2+m^2)\ln(\lambda^2m^{-2}+1)-C''\lambda^2), \quad (33a)$$

with another unimportant constant C''. The complete problem requires the further integral,

$$-(8i)\int(1;\,p_\sigma)d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}$$
$$=(1,0)(4(1-\theta\operatorname{ctn}\theta)+2\ln(\lambda^2m^{-2})). \quad (34a)$$

The value of the integral (34a) times m^2 differs from (33a), of course, because the results on the right are not actually the integrals on the left, but rather equal their actual value minus their value for $m^2=m^2+\lambda^2$.

Combining these quantities, as required by (32), dropping the constants C', C'' and evaluating the spur gives (33). The spurs are evaluated in the usual way, noting that the spur of any odd number of γ matrices vanishes and $Sp(AB)=Sp(BA)$ for arbitrary A, B. The $Sp(1)=4$ and we also have

$$\tfrac{1}{4}Sp[(\not p_1+m_1)(\not p_2-m_2)]=p_1\cdot p_2-m_1m_2, \quad (35a)$$

$$\tfrac{1}{4}Sp[(\not p_1+m_1)(\not p_2-m_2)(\not p_3+m_3)(\not p_4-m_4)]$$
$$=(p_1\cdot p_2-m_1m_2)(p_3\cdot p_4-m_3m_4)$$
$$-(p_1\cdot p_3-m_1m_3)(p_2\cdot p_4-m_2m_4)$$
$$+(p_1\cdot p_4-m_1m_4)(p_2\cdot p_3-m_2m_3), \quad (36a)$$

where $\not p_i$, m_i are arbitrary four-vectors and constants.

It is interesting that the terms of order $\lambda^2\ln\lambda^2$ go out, so that the charge renormalization depends only logarithmically on λ^2. This is not true for some of the meson theories. Electrodynamics is suspiciously unique in the mildness of its divergence.

D. More Complex Problems

Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-

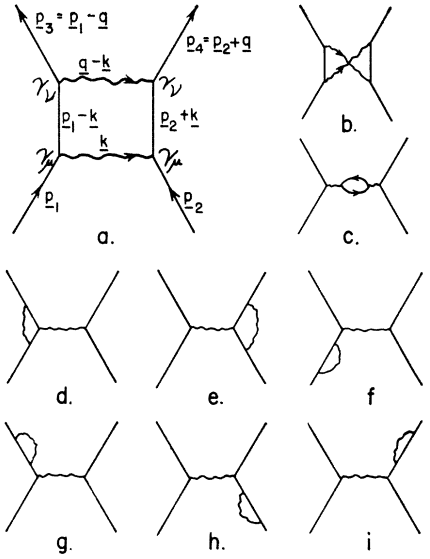

FIG. 8. The interaction between two electrons to order $(e^2/\hbar c)^2$. One adds the contribution of every figure involving two virtual quanta, Appendix D.

ing, to the Compton scattering, and the interaction of a neutron with an electromagnetic field.

For the Møller scattering, consider two electrons, one in state u_1 of momentum p_1 and the other in state u_2 of momentum p_2. Later they are found in states u_3, p_3 and u_4, p_4. This may happen (first order in $e^2/\hbar c$) because they exchange a quantum of momentum $q=p_1-p_3=p_4-p_2$ in the manner of Eq. (4) and Fig. 1. The matrix element for this process is proportional to (translating (4) to momentum space)

$$(\bar{u}_4\gamma_\mu u_2)(\bar{u}_3\gamma_\mu u_1)q^{-2}. \tag{37a}$$

We shall discuss corrections to (37a) to the next order in $e^2/\hbar c$. (There is also the possibility that it is the electron at 2 which finally arrives at 3, the electron at 1 going to 4 through the exchange of quantum of momentum p_3-p_2. The amplitude for this process, $(\bar{u}_4\gamma_\mu u_1)(\bar{u}_3\gamma_\mu u_2)(p_3-p_2)^{-2}$, must be subtracted from (37a) in accordance with the exclusion principle. A similar situation exists to each order so that we need consider in detail only the corrections to (37a), reserving to the last the subtraction of the same terms with 3, 4 exchanged.)

One reason that (37a) is modified is that two quanta may be exchanged, in the manner of Fig. 8a. The total matrix element for all exchanges of this type is

$$(e^2/\pi i)\int(\bar{u}_3\gamma_\nu(p_1-k-m)^{-1}\gamma_\mu u_1)(\bar{u}_4\gamma_\nu(p_2+k-m)^{-1}\gamma_\mu u_2)\cdot k^{-2}(q-k)^{-2}d^4k, \tag{38a}$$

as is clear from the figure and the general rule that electrons of momentum p contribute in amplitude $(p-m)^{-1}$ between interactions γ_μ, and that quanta of momentum k contribute k^{-2}. In integrating on d^4k and summing over μ and ν, we add all alternatives of the type of Fig. 8a. If the time of absorption, γ_μ, of the quantum k by electron 2 is later than the absorption, γ_ν, of $q-k$, this corresponds to the virtual state p_2+k being a positron (so that (38a) contains over thirty terms of the conventional method of analysis).

In integrating over all these alternatives we have considered all possible distortions of Fig. 8a which preserve the order of events along the trajectories. We have not included the possibilities corresponding to Fig. 8b, however. Their contribution is

$$(e^2/\pi i)\int(\bar{u}_3\gamma_\nu(p_1-k-m)^{-1}\gamma_\mu u_1)\times(\bar{u}_4\gamma_\mu(p_2+q-k-m)^{-1}\gamma_\nu u_2)k^{-2}(q-k)^{-2}d^4k, \tag{39a}$$

as is readily verified by labeling the diagram. The contributions of all possible ways that an event can occur are to be added. This means that one adds with equal weight the integrals corresponding to each topologically distinct figure.

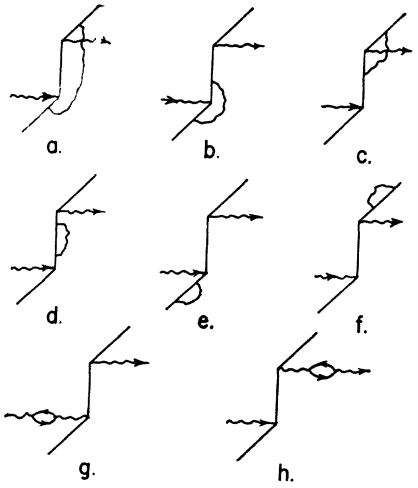

FIG. 9. Radiative correction to the Compton scattering term (a) of Fig. 5. Appendix D.

To this same order there are also the possibilities of Fig. 8d which give

$$(e^2/\pi i)\int(\bar{u}_3\gamma_\nu(p_3-k-m)^{-1}\gamma_\mu(p_1-k-m)^{-1}\gamma_\nu u_1)\times(\bar{u}_4\gamma_\mu u_2)k^{-2}q^{-2}d^4k.$$

This integral on k will be seen to be precisely the integral (12) for the radiative corrections to scattering, which we have worked out. The term may be combined with the renormalization terms resulting from the difference of the effects of mass change and the terms, Figs. 8f and 8g. Figures 8e, 8h, and 8i are similarly analyzed.

Finally the term Fig. 8c is clearly related to our vacuum polarization problem, and when integrated gives a term proportional to $(\bar{u}_4\gamma_\mu u_2)(\bar{u}_3\gamma_\nu u_1)J_{\mu\nu}q^{-4}$. If the charge is renormalized the term $\ln(\lambda/m)$ in $J_{\mu\nu}$ in (33) is omitted so there is no remaining dependence on the cut-off.

The only new integrals we require are the convergent integrals (38a) and (39a). They can be simplified by rationalizing the denominators and combining them by (14a). For example (38a) involves the factors $(k^2-2p_1\cdot k)^{-1}(k^2+2p_2\cdot k)^{-1}k^{-2}(q^2+k^2-2q\cdot k)^{-2}$. The first two may be combined by (14a) with a parameter x, and the second pair by an expression obtained by differentiation (15a) with respect to b and calling the parameter y. There results a factor $(k^2-2p_x\cdot k)^{-2}(k^2+yq^2-2yq\cdot k)^{-4}$ so that the integrals on d^4k now involve two factors and can be performed by the methods given earlier in the appendix. The subsequent integrals on the parameters x and y are complicated and have not been worked out in detail.

Working with charged mesons there is often a considerable reduction of the number of terms. For example, for the interaction between protons resulting from the exchange of two mesons only the term corresponding to Fig. 8b remains. Term 8a, for example, is impossible, for if the first proton emits a positive meson the second cannot absorb it directly for only neutrons can absorb positive mesons.

As a second example, consider the radiative correction to the Compton scattering. As seen from Eq. (15) and Fig. 5 this scattering is represented by two terms, so that we can consider the corrections to each one separately. Figure 9 shows the types of terms arising from corrections to the term of Fig. 5a. Calling k the momentum of the virtual quantum, Fig. 9a gives an integral

$$\int\gamma_\mu(p_2-k-m)^{-1}e_2(p_1+q_1-k-m)^{-1}e_1(p_1-k-m)^{-1}\gamma_\mu k^{-2}d^4k,$$

convergent without cut-off and reducible by the methods outlined in this appendix.

The other terms are relatively easy to evaluate. Terms b and c of Fig. 9 are closely related to radiative corrections (although somewhat more difficult to evaluate, for one of the states is not that of a free electron, $(p_1+q)^2\neq m^2$). Terms e, f are renormalization terms. From term d must be subtracted explicitly the effect of mass Δm, as analyzed in Eqs. (26) and (27) leading to (28) with $p'=p_1+q$, $a=e_2$, $b=e_1$. Terms g, h give zero since the vacuum polarization has zero effect on free light quanta, $q_1^2=0$, $q_2^2=0$. The total is insensitive to the cut-off λ.

The result shows an infra-red catastrophe, the largest part of the effect. When cut-off at $\lambda_{\min}$, the effect proportional to $\ln(m/\lambda_{\min})$ goes as

$$(e^2/\pi)\ln(m/\lambda_{\min})(1-2\theta\operatorname{ctn}2\theta), \tag{40a}$$

times the uncorrected amplitude, where $(p_2-p_1)^2=4m^2\sin^2\theta$. This is the same as for the radiative correction to scattering for a deflection p_2-p_1. This is physically clear since the long wave quanta are not effected by short-lived intermediate states. The infra-red effects arise[28] from a final adjustment of the field from the asymptotic coulomb field characteristic of the electron of

[28] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

momentum $\boldsymbol{p}_1$ before the collision to that characteristic of an electron moving in a new direction $\boldsymbol{p}_2$ after the collision.

The complete expression for the correction is a very complicated expression involving transcendental integrals.

As a final example we consider the interaction of a neutron with an electromagnetic field in virtue of the fact that the neutron may emit a virtual negative meson. We choose the example of pseudoscalar mesons with pseudovector coupling. The change in amplitude due to an electromagnetic field $\boldsymbol{A}=\boldsymbol{a}\exp(-i\boldsymbol{q}\cdot x)$ determines the scattering of a neutron by such a field. In the limit of small $\boldsymbol{q}$ it will vary as $\boldsymbol{qa}-\boldsymbol{aq}$ which represents the interaction of a particle possessing a magnetic moment. The first-order interaction between an electron and a neutron is given by the same calculation by considering the exchange of a quantum between the electron and the nucleon. In this case a_μ is q^{-2} times the matrix element of γ_μ between the initial and final states of the electron, the states differing in momentum by $\boldsymbol{q}$.

The interaction may occur because the neutron of momentum $\boldsymbol{p}_1$ emits a negative meson becoming a proton which proton interacts with the field and then reabsorbs the meson (Fig. 10a). The matrix for this process is ($\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$),

$$\int(\gamma_5\boldsymbol{k})(\boldsymbol{p}_2-\boldsymbol{k}-M)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}(\gamma_5\boldsymbol{k})(\boldsymbol{k}^2-\mu^2)^{-1}d^4k. \quad (41a)$$

Alternatively it may be the meson which interacts with the field. We assume that it does this in the manner of a scalar potential satisfying the Klein Gordon Eq. (35), (Fig. 10b)

$$-\int(\gamma_5\boldsymbol{k}_2)(\boldsymbol{p}_1-\boldsymbol{k}_1-M)^{-1}(\gamma_5\boldsymbol{k}_1)(\boldsymbol{k}_2^2-\mu^2)^{-1}\times(k_2\cdot a+k_1\cdot a)(\boldsymbol{k}_1^2-\mu^2)^{-1}d^4k_1, \quad (42a)$$

where we have put $\boldsymbol{k}_2=\boldsymbol{k}_1+\boldsymbol{q}$. The change in sign arises because the virtual meson is negative. Finally there are two terms arising from the $\gamma_5\boldsymbol{a}$ part of the pseudovector coupling (Figs. 10c, 10d)

$$\int(\gamma_5\boldsymbol{k})(\boldsymbol{p}_2-\boldsymbol{k}-M)^{-1}(\gamma_5\boldsymbol{a})(\boldsymbol{k}^2-\mu^2)^{-1}d^4k, \quad (43a)$$

and

$$\int(\gamma_5\boldsymbol{a})(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}(\gamma_5\boldsymbol{k})(\boldsymbol{k}^2-\mu^2)^{-1}d^4k. \quad (44a)$$

Using convergence factors in the manner discussed in the section on meson theories each integral can be evaluated and the results combined. Expanded in powers of $\boldsymbol{q}$ the first term gives the magnetic moment of the neutron and is insensitive to the cut-off, the next gives the scattering amplitude of slow electrons on neutrons, and depends logarithmically on the cut-off.

The expressions may be simplified and combined somewhat before integration. This makes the integrals a little easier and also shows the relation to the case of pseudoscalar coupling. For example in (41a) the final $\gamma_5\boldsymbol{k}$ can be written as $\gamma_5(\boldsymbol{k}-\boldsymbol{p}_1+M)$ since $\boldsymbol{p}_1=M$ when operating on the initial neutron state. This is

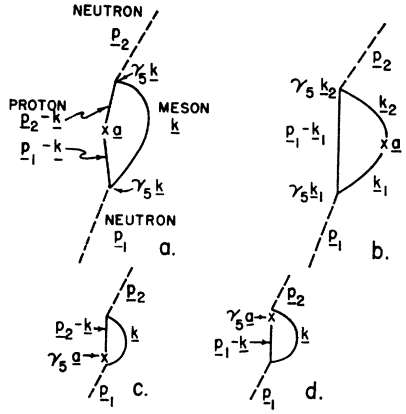

FIG. 10. According to the meson theory a neutron interacts with an electromagnetic potential $\boldsymbol{a}$ by first emitting a virtual charged meson. The figure illustrates the case for a pseudoscalar meson with pseudovector coupling. Appendix D.

$(\boldsymbol{p}_1-\boldsymbol{k}-M)\gamma_5+2M\gamma_5$ since γ_5 anticommutes with $\boldsymbol{p}_1$ and $\boldsymbol{k}$. The first term cancels the $(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}$ and gives a term which just cancels (43a). In a like manner the leading factor $\gamma_5\boldsymbol{k}$ in (41a) is written as $-2M\gamma_5-\gamma_5(\boldsymbol{p}_2-\boldsymbol{k}-M)$, the second term leading to a simpler term containing no $(\boldsymbol{p}_2-k-M)^{-1}$ factor and combining with a similar one from (44a). One simplifies the $\gamma_5\boldsymbol{k}_1$ and $\gamma_5\boldsymbol{k}_2$ in (42a) in an analogous way. There finally results terms like (41a), (42a) but with pseudoscalar coupling $2M\gamma_5$ instead of $\gamma_5\boldsymbol{k}$, no terms like (43a) or (44a) and a remainder, representing the difference in effects of pseudovector and pseudoscalar coupling. The pseudoscalar terms do not depend sensitively on the cut-off, but the difference term depends on it logarithmically. The difference term affects the electron-neutron interaction but not the magnetic moment of the neutron.

Interaction of a proton with an electromagnetic potential can be similarly analyzed. There is an effect of virtual mesons on the electromagnetic properties of the proton even in the case that the mesons are neutral. It is analogous to the radiative corrections to the scattering of electrons due to virtual photons. The sum of the magnetic moments of neutron and proton for charged mesons is the same as the proton moment calculated for the corresponding neutral mesons. In fact it is readily seen by comparing diagrams, that for arbitrary $\boldsymbol{q}$, the scattering matrix to *first order in the electromagnetic potential* for a proton according to neutral meson theory is equal, if the mesons were charged, to the sum of the matrix for a neutron and the matrix for a proton. This is true, for any type or mixtures of meson coupling, to all orders in the coupling (neglecting the mass difference of neutron and proton).