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General relativity from three-forms in seven dimensions

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A B S T R A C T

We consider a certain theory of 3-forms in 7 dimensions, and study its dimensional reduction to 4D, compactifying the 7-dimensional manifold on the 3-sphere of a fixed radius. We show that the resulting 4D theory is (Riemannian) General Relativity (GR) in Plebanski formulation, modulo corrections that are negligible for curvatures smaller than Planckian. Possibly the most interesting point of this construction is that the dimensionally reduced theory is GR with a non-zero cosmological constant, and the value of the cosmological constant is directly related to the size of S^3. Realistic values of Λ correspond to S^3 of Planck size.

It is almost universally agreed that General Relativity (GR) should be viewed as a low energy approximation to some other theory. The usual view is that this is some quantum gravity theory that gives the ultra-violet (UV) completion of perturbatively quantised GR.

It is also possible that GR arises as the low energy limit of some other classical theory, and it is only this distinct from GR classical theory that is to be UV completed quantum mechanically. A scenario of this sort is part of string theory, where 4D gravity arises by compactification from 11D supergravity. The latter is to be completed quantum mechanically by M-theory.

Scenarios embedding 4D GR into other classical theories are interesting for several reasons. First, what comes out from such an embedding is usually more than just GR. There are typically more degrees of freedom, and this is of interest both phenomenologically as well as for the question of the UV completion. Indeed, UV incomplete theories may arise as the low energy approximation of UV complete theories, the latter having more degrees of freedom than the former.

In the extensively studied Kaluza–Klein scenarios 4D gravity arises from a higher dimensional theory that is again gravity (possibly with extra fields). The purpose of this letter is to point out that (Riemannian signature) General Relativity in four dimensions may be obtained by compactification from a higher-dimensional theory of a very different nature. Thus, we show that a certain dynamical theory of 3-forms in 7 dimensions, when compactified on the 3-sphere (of a fixed size) gives a 4D gravity theory that at low energies is indistinguishable from GR.

Mathematically, the idea of our construction is rather simple and natural and can be explained already here. It is well-known that gravity can be usefully described using the language of frame fields instead of the metric. A frame e^I is a collection of one-forms that are declared orthonormal. This defines the metric via $ds^2 = \eta_{IJ} e^I e^J$, where η_{IJ} is the flat metric. When the frame field is used to describe geometry, it is also very natural to allow the spin connection w^{IJ} (later used to construct the curvature) to become an independent object. Its relation to derivatives of the frame is then fixed by its field equation.

The (spin) connection w^{IJ} can locally be viewed as a one-form with values in the Lie algebra of Lorentz group, or orthogonal group SO(D) of appropriate dimension D if one wants to describe metrics of Riemannian signature. On the other hand, the frame one-forms are not $\mathfrak{so}(D)$ Lie algebra valued (apart from the case of 3D gravity where the frame index can be identified with the Lie algebra index). However, one can construct two-forms that are Lie algebra valued, taking the wedge product of frame with itself $B^{IJ} := e^I \wedge e^J$. This gives rise to a formalism for gravity in which the basic fields are Lie algebra valued one-forms w^{IJ} and two-forms B^{IJ}. The two-forms must however be appropriately constrained in order to guarantee that they come from the frame field. This formalism is particularly simple in 4 dimensions, see below, but can also be used to describe gravity in any dimension, see [1].

The main idea behind the constructions of this letter is that it is natural to combine the one- and two-forms of such a for-

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mulation of gravity into a single object — a 3-form in a higher dimensional space. The space in question is (locally) the product of the spacetime and the SO(D) group manifolds, i.e. it is the total space of the principal SO(D) bundle over space(time). Let m^{IJ} be Lie algebra valued one-forms on the SO(D) group manifold, i.e. Maurer–Cartan forms. To exhibit the fields w^{IJ} and B^{IJ} as components of a 3-form in the total space of the bundle, we use the matrix notation in which m^{IJ}, w^{IJ} are one-forms and B^{IJ} are two-forms valued in the space of anti-symmetric $D \times D$ matrices. We then consider the following 3-form

$$C = \mathrm{Tr}\,(m \wedge m \wedge m + m \wedge m \wedge w + m \wedge B)\,. \tag{1}$$

Thus, the spin connection 1-forms w^{IJ} arise as two vertical one horizontal components of the 3-form C above, and the fields B^{IJ} arise as the one vertical two horizontal components. This recovers all fields of the 2-form formulation of gravity as components of a 3-form in the total space of the SO(D) group bundle over the spacetime. In [2,3] it was explained how 3D gravity can be usefully described in this fashion. The purpose of this letter is to outline the construction that works for 4D gravity.

While the idea described is general enough to work in any dimension, the 3-forms are most interesting objects in 6 and 7 dimensions, as we review below. On the other hand, the total space of the SO(4) group principal bundle over a 4-dimensional manifold is 10 dimensional. But instead of working with the full 4D rotation group one can take one of its chiral halves, i.e. either its self-dual or anti-self-dual part, both of which are 3-dimensional. There is a formalism for 4D gravity [4,5] that works with one- and two-forms with values in the Lie algebra of the chiral half of the rotation group. We will base our construction on this formalism. In this case the total space of the bundle is 7-dimensional, and we are in the setting where 3-forms are most natural and interesting.

The fundamental fact about a generic (or using the terminology of [6] stable) 3-form C on a 7-dimensional manifold $\mathcal{M}$ is that it defines a metric g_C. The metric is explicitly given by the following formula

$$g_C(\xi,\eta)\mathrm{vol}_C = -\frac{1}{6} i_\xi C \wedge i_\eta C \wedge C. \tag{2}$$

Here $g_C(\xi,\eta)$ is the result of the metric contraction of two vector fields ξ,η, vol_C is the volume form for g_C, and $i_{\xi,\eta}$ is the operation of insertion of a vector field into a form. The metric (2) has been known for more than a century, see e.g. [7] for a historical perspective. It is ultimately related to the geometry of spinors in 7 and 8 dimensions, see e.g. [8] for the discussion of the spinor aspect, and to octonions, see e.g. [9].

Generic 3-forms in 7 dimensions are related to the exceptional group G_2. This can be defined as the subgroup of GL(7) that stabilises a generic 3-form, see [10] and more recently [6]. The space of generic 3-forms (at a point) can then be identified with the coset GL(7)/G_2. It is easy to see that the spaces GL(7)/G_2 and $\Lambda^3\mathbb{R}^7$ have the same dimension 35. The fact that C defines g_C explains why G_2 is a subgroup of SO(7).

The volume form vol_C, playing an important role below, can also be described explicitly as a homogeneity degree 7/3 object built from C. Thus, let $\tilde{\epsilon}^{a_1\ldots a_7}$ be the densitiesed completely anti-symmetric tensor taking values ± 1 in any coordinate system. Here $a = 1,\ldots,7$. We can then construct the following degree 7 and weight 3 scalar:

$$\tilde{\epsilon}^{a_1\ldots a_7}\tilde{\epsilon}^{b_1\ldots b_7}\tilde{\epsilon}^{c_1\ldots c_7} C_{a_1 b_1 c_1}\ldots C_{a_7 b_7 c_7}. \tag{3}$$

The cube root of this expression is a multiple of vol_C. This is not the most useful in practise way of computing the volume form — it is usually much more effective to compute the volume vol_C from the determinant of g_C.

Let us now make C dynamical. Consider the following action principle

$$S[C] = \frac{1}{2}\int_{\mathcal{M}} C \wedge dC + 6\lambda\,\mathrm{vol}_C. \tag{4}$$

The first term here (i.e. the case $\lambda = 0$) describes a topological field theory considered in [11]. The Euler–Lagrange equations following from (4) are

$$dC = \lambda\, {}^*C. \tag{5}$$

Here ${}^*C$ is the Hodge dual of C computed using g_C. The numerical coefficient on the right-hand-side here is simplest verified by noticing that $\mathrm{vol}_C = -(1/7)C \wedge {}^*C$, and then using the homogeneity to compute the variation of vol_C with respect to C. We note that, because the two terms in (4) scale differently, by rescaling C we can always achieve $\lambda = 1$ at the expense of introducing a parameter in front of the action. We will do so from now on. Thus, there are no free parameters in the theory (4).

Real 3-forms C are of two possible types, see e.g. [7]. Forms of one type give g_C of signature (4, 3). Forms of the other type give the Riemannian signature metrics g_C. Such 3-forms satisfying (5) describe what [8] call nearly parallel G_2 structures. Note that (5) implies that ${}^*C$ is closed. However, this equation also says that $dC \neq 0$. Thus, the critical points of (4) are *not* the possibly more familiar in this context torsion-free G_2 structures satisfying $dC = 0$, $d^*C = 0$ and describing G_2 holonomy manifolds. A related observation is that the equation (5) implies that the metric g_C is Einstein with non-zero scalar curvature, see proposition 3.10 from [8]. In contrast, G_2-holonomy manifolds are Ricci flat, see e.g. [6].

Equations (5) have been studied in the literature [8]. The variational principle (4) is a subcase of a more general action (29) in [12], with 1- and 5-forms set to zero and the auxiliary metric integrated out. It is also similar to prepotentials appearing in the literature on G_2 compactifications of M-theory, see e.g. [13]. Nevertheless, it seems that the theory of 3-forms (4) has not been studied in the literature. Note that action (4) is different from the ones considered by Hitchin [6]. The simplest Hitchin action is the last term in (4), restricted to 3-forms in a fixed cohomology class. Our action is the sum of those in [11] and [6], with the only constraint on C being that it is generic (or stable), which is an open condition.

Additional information about the theory (4) is given in a recent paper [14]. In particular, this reference demonstrates that (4) is a 7D theory with 3 propagating degrees of freedom. Also, some solutions of the equations (5) are described, in particular those that can be obtained as S^3 bundles over S^4.

We now describe a relation to 4D General Relativity (GR). We claim that (4) dimensionally reduced on S^3 (of a fixed radius) is a 4D theory of gravity that is for all practical purposes indistinguishable from (Riemannian signature) GR. This means that while the reduced theory is, strictly speaking, not GR, it coincides with GR for Weyl curvatures smaller than Planckian, which is anyway the regime where we can trust GR as a classical theory. All this is to be explained in more details below.

To explain why the outlined embedding of 4D GR into a theory of 3-forms in 7D may be interesting, let us remind the reader the basics of Kaluza–Klein (KK) theory. Here one starts with GR in higher dimensions and dimensionally reduces to 4 dimensions. In the simplest and also historically the first setup one starts with GR (with zero cosmological constant) in 5 dimensions and dimensionally reduces on S^1. If one fixes the size of S^1, as was done in the first treatments, the dimensionally reduced 4D theory is GR coupled to Maxwell. Allowing the size of the circle to become dynamical gives rise to an additional massless scalar field in 4D, and to avoid conflict with observations this must be given a mass, or stabilised in some other way.

The Kaluza–Klein mechanism gives a geometrically compelling unification of gravity with electromagnetism. Also, as pointed out by Kaluza, it relates the quantum of electric charge to the size of the compact extra dimension. The Kaluza–Klein mechanism can be generalised to describe non-Abelian gauge fields. A comprehensive review on KK is e.g. [15].

Let us return to our story. We claimed that 4D GR arises as the dimensional reduction of the theory (4) of 3-forms in 7D. Unlike the KK case, no unification is achieved here, the reduced theory is pure gravity. Also unlike KK, gravity in 4D arises from a 7D theory of a very different sort — the theory (4) is a dynamical theory of 3-forms, not metrics. While it may be amusing that 4D GR admits a lift to a theory of such a different nature as (4), is this a useful perspective on 4D gravity?

Now comes what we believe is the most interesting physics point about our construction. As we will show, the dimensionally reduced theory is GR with non-zero cosmological constant, and the value of the cosmological constant is directly related to the size of the S^3. As the 5D Kaluza–Klein story makes the electric charge a dynamically determinable quantity, at least in principle, via some “spontaneous compactification” mechanism, in our setup the 4D cosmological constant becomes in principle determinable by the dynamics of the extra dimensions.

Thus, our 7D lift of 4D GR makes the 4D cosmological constant a dynamical object. Having said this, we must also say that in this letter we limit ourselves to just demonstrating the relation between the radius of S^3 and Λ. No attempt at studying the dynamics of the extra dimensions (and thus predicting Λ) will be made. Still, this should be kept in mind as one of the motivations for our construction, in addition to those described in the beginning of this text.

After these motivational remarks, we are ready to describe the dimensional reduction. We phrase the discussion that follows in terms of real objects. In this case the dimensionally reduced theory is the Riemannian signature GR. All objects can also be complexified, in this case one obtains complexified GR. The subtler issue of reality conditions relevant for the Lorentzian signature theory will be treated elsewhere.

To carry out the dimensional reduction, we need to assume that the 3-form C in 7D is “independent” of 3 of the 7 coordinates. The appropriate for our purposes way of doing this is to assume that we have the group SU(2) that acts on $\mathcal{M}$ freely. This gives $\mathcal{M}$ the structure of an SU(2) principal bundle over a 4-dimensional base M. Our considerations here are local, over a region in M. We parametrise fibre points as $g \in \mathrm{SU}(2)$, with the group action on the fibre being the right action of SU(2) on itself. Denote by $\mathbf{m} = g^{-1}dg$ the Maurer–Cartan one-forms on SU(2). These forms transform covariantly under the right action of SU(2) on itself. To establish further notations, let $\mathbf{A}$ be an SU(2) connection on the base M, i.e. a 2×2 anti-Hermitian matrix valued one-form on M, and let $A = g^{-1}\mathbf{A}g$ be its lift into the total space of the bundle. Then $W = \mathbf{m} + A$ is the connection one-form in the total space of the bundle. Simple standard computation shows that $F := dW + WW$ is a 2-form that is purely horizontal $F = g^{-1}\mathbf{F}g$, where $\mathbf{F} = d\mathbf{A} + \mathbf{AA}$ is the curvature of the connection one-form on the base. Here and in what follows, for brevity, we omit the wedge product symbol.

A general SU(2) invariant 3-form on X can be written as $C = \mathrm{Tr}(\phi\mathbf{m}^3 + A\mathbf{m}^2 + B\mathbf{m}) + c$. Here $\phi \in \Lambda^0(M), c \in \Lambda^3(M)$, while A, B are lifts to the total space of the bundle of Lie algebra valued 1- and 2-forms on the base M respectively. Note that none of the 35 components of C has been lost here, as a simple count of components in ϕ, A, B, c shows. The above parametrisation of C is however not the one most suited for computations. We note that the term quadratic in $\mathbf{m}$ can always be eliminated by shifting $\mathbf{m}$ by a Lie algebra valued 1-form. This also redefines B, c. This suggests we parametrise

$$C = -2\,\mathrm{Tr}\left(\frac{1}{3}\phi^3 W^3 + \phi W B\right) + c. \tag{6}$$

Here $W = \mathbf{m} + A$ is a connection in the total space of the bundle. Geometrically, an SU(2) invariant 3-form C in the total space of an SU(2) bundle over M defines a connection by declaring the horizontal vector fields to be those that are in the kernel of the 1-form $i_{\xi_V} i_{\eta_V} C$ for arbitrary vertical vector fields ξ_V, η_V. In (6) we simply chose to parametrise the 3-form C by this connection. The parametrisation (6) is most suited for practical computations. Numerical prefactors are for future convenience.

A simple computation then gives

$$dC = -2\,\mathrm{Tr}\Big(\phi^2 d\phi W^3 + (\phi^3 F + \phi B)W^2 + (d\phi B + \phi d_A B)W + \phi F B\Big) + dc. \tag{7}$$

Here $d_A B = g^{-1}(d\mathbf{B} + \mathbf{AB} - \mathbf{BA})g$ is the lift to the bundle of the covariant derivative of Lie algebra-valued 2-form $\mathbf{B}$ with respect to the connection $\mathbf{A}$. Another simple computation using some trace identities gives

$$\frac{1}{2}\int_{\mathcal{M}} C dC = \int_{\mathrm{SU}(2)} -\frac{2}{3}\mathrm{Tr}(\mathbf{m}^3) \times \int_M -2\,\mathrm{Tr}(\phi^4 \mathbf{BF} + (\phi^2/2)\mathbf{BB}) + \phi^3 dc. \tag{8}$$

We learn that the dimensional reduction of the first, topological term in (4), modulo the prefactor equal to the volume of SU(2), is the so-called BF theory with a Λ-term, coupled to the scalar and 3-form fields. We find this result interesting in its own right. The dimensional reduction of the topological theory is topological. Thus, if there is no second term in (4), varying with respect to c gives $\phi = const$, and we recover the usual Lagrangian of the topological BF theory with the Λ term.

Let us now understand the dimensional reduction of the second term in (4). This is a no-derivative term, so it only changes the “potential” for the ϕ, B, c fields. In this paper we will set $\phi = const$ and $c = 0$. The complete dimensional reduction is carried out in [14]. Setting the size of the extra dimensions to a constant corresponds to $\phi = const$. At the same time, it is clear from (8) that the 3-form field c is “conjugate” to ϕ and so setting this field to constant justifies setting c to zero.

To compute the volume form corresponding to (6) (with $c = 0$) we need to write this 3-form in SO(3) notations. This is achieved by decomposing all matrix-valued fields in terms of the SU(2) generators $\tau^i = -(\mathrm{i}/2)\sigma^i$, where σ^i are the usual Pauli matrices. So, we write $W = W^i\tau^i$ etc. This gives

$$C = \frac{\phi^3}{6}\epsilon^{ijk} W^i W^j W^k + \phi W^i B^i. \tag{9}$$

The metric g_C and thus the volume form vol_C are then easiest computed by putting this C into its canonical form. As the canonical form we take

$$C = \frac{1}{6}\epsilon^{ijk} e^i e^j e^k + e^i \Sigma^i. \tag{10}$$

Here Σ^i is the basis of anti-self-dual 2-forms

$$\Sigma^1 = e^{45} - e^{67}, \quad \Sigma^2 = e^{46} - e^{75}, \quad \Sigma^3 = e^{47} - e^{56}.$$

The notation here is $e^{ab} = e^a e^b$, with wedge product implied. It is then easy to check that for C in its canonical form (10), the metric defined by C via (2) is $ds^2_C = \sum_{a=1}^7 (e^a)^2$.

To compute the metric for (9) we need to rewrite it in the canonical form (10). This is done by choosing a convenient parametrisation of B^i fields. To establish this parametrisation, we note that the triple of 2-forms B^i defines a metric on the base in which these forms are anti-self-dual (ASD). This is the Urbantke metric [16]. In fact, a simple calculation with the formula (2) shows that the Urbantke formula

$$g_\Sigma(\xi, \eta)\mathrm{vol}_\Sigma = -\frac{1}{6}\epsilon^{ijk} i_\xi \Sigma^i \wedge i_\eta \Sigma^j \wedge \Sigma^k \tag{11}$$

arises as the metric on the base from (2), with C in its canonical form (10). This clearly points towards a 7-dimensional origin of the Urbantke formula. The 2-forms B^i can then always be parametrised as

$$B^i = \sqrt{X}^{ij} \Sigma^j, \tag{12}$$

where one uses an arbitrary branch of the square root of a symmetric matrix X^{ij}, and two-forms Σ^i are orthonormal $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$. Here Σ^i is an orthonormal basis of ASD 2-forms for the metric defined (via Urbantke formula) by B^i. The matrix X^{ij} is defined as that of the wedge products of B^i. We have $B^i \wedge B^j = -2X^{ij}\mathrm{vol}_\Sigma$, where vol_Σ is the volume form of the metric whose ASD 2-forms are Σ^i. Substituting the parametrisation (12) into (9) we see that the 3-form can be written in the following way

$$C = \rho\left(\frac{1}{6}\epsilon^{ijk} e^i e^j e^k + e^i \Sigma^i\right), \tag{13}$$

with

$$\rho = (\det(X))^{1/4}, \quad e^i = \frac{\phi}{\rho}\sqrt{X}^{ij} W^j. \tag{14}$$

This puts C into a form that is a multiple of the canonical. The metric g_C is then $\rho^{2/3}$ time the metric for which the above e^a is the frame. This gives

$$ds^2_C = \phi^2 W^i \frac{X^{ij}}{(\det(X))^{1/3}} W^j + (\det(X))^{1/6} ds^2_\Sigma. \tag{15}$$

The volume form for this metric is

$$\mathrm{vol}_C = \frac{\phi^3}{6}\epsilon^{ijk} W^i W^j W^k (\det(X))^{1/3}\mathrm{vol}_\Sigma. \tag{16}$$

We now put all pieces together and write the dimensionally reduced 4D Lagrangian, which is (4) on the ansatz (6) (with $c = 0$), divided by the volume of the fibre. We have

$$L_{4\mathrm{D}} = \phi^4 B^i F^i + \frac{\phi^2}{2} B^i B^i + 3\phi^3 (\det(X))^{1/3}\mathrm{vol}_\Sigma. \tag{17}$$

This is a Lagrangian of the type "BF theory plus a potential for the B field". From general considerations in [17] we know that this is a 4D gravity theory, in the sense that the only degrees of freedom it describes are, as in GR, the two polarisations of the graviton. We also know [18] that for sufficiently small Weyl curvatures (i.e. sufficiently low energy) any such theory of gravity reduces to GR.

We would now like to quantify the regime in which the above theory reduces to GR. To this end, let us rewrite the last term in (17) by introducing two auxiliary fields. We have

$$6(\det(X))^{1/3}\mathrm{vol}_\Sigma \hat{=} -H^{ij} B^i B^j - 2\mu(\det(H) - 1)\mathrm{vol}_\Sigma,$$

where the meaning of the hat symbol over the equality sign is "on-shell". Indeed, varying the right-hand-side with respect to H^{ij} we get $X^{ij} = \mu \det(H)(H^{-1})^{ij}$. The condition $\det(H) = 1$ imposed by the Lagrange multiplier μ then sets $\mu = (\det(X))^{1/3}$. Substituting the resulting solution $H^{ij} = (\det(X))^{1/3}(X^{-1})^{ij}$ into the first term we reproduce the left-hand-side.

Using the above way of writing the last term in (17) we can rewrite the effective 4D Lagrangian as follows

$$L_{4\mathrm{D}}/\phi^4 = B^i F^i - \frac{1}{2} M^{ij} B^i B^j \tag{18}$$
$$- 2\mu(\det(\mathbb{I} + \phi^2 M) - \phi^3)\mathrm{vol}_\Sigma.$$

Here we defined a new matrix M^{ij} so that $\phi H = \mathbb{I} + \phi^2 M$, and redefined the Lagrange multiplier μ. The key point now is that the constraint $\det(\mathbb{I} + \phi^2 M) = const$, when expanded in powers of M, approximates the constraint $\mathrm{Tr}(M) = const$, and this is known to give General Relativity in its Plebanski formulation [4,5].

For readers lost in the sequence of field redefinitions performed to achieve (18) we describe in words the origin of all the fields. The field ϕ is the scalar field parametrising the size of the extra dimensional S^3 in the 3-form (6). The 2-form field B^i is also explicitly present in (6) as $B = g^{-1} B^i \tau^i g$. The object $F^i := dA^i + (1/2)\epsilon^{ijk} A^j A^k$ is the curvature of the SO(3) connection A^i defined by C, with W from (6) given by $W = \mathbf{m} + g^{-1} A^i \tau^i g$. The auxiliary fields M^{ij} and μ where introduced in such a way that their elimination by their algebraic field equations produces the dimensional reduction of the "potential term", i.e. the last term in (4). The volume form vol_Σ is defined by B^i via formulas (12) and (11).

We now claim that (18) describes Riemannian signature GR, plus higher order corrections immaterial in the regime of not too high Weyl curvatures. To see this, let us remind the reader the Plebanski Lagrangian [4]

$$L'_{\mathrm{Pleb}} = M_p^2\left(B^i F^i - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right) B^i B^j\right). \tag{19}$$

Here $M_p^2 = 1/8\pi G$ is the Planck mass, G is the Newton's, and Λ is the cosmological constant. Here B^i is a dimensionless field that describes the metric (via Urbantke formula). We now absorb the Planck mass so as to make B^i (and thus the metric) dimensionful. The dimensionful metric measures distances in units of the Planck length. Thus, we redefine $B^i \to B^i/M_p^2$, $\Psi \to M_p^2 \Psi$, $\Lambda \to M_p^2 \Lambda$.

$$L''_{\mathrm{Pleb}} = B^i F^i - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right) B^i B^j. \tag{20}$$

The new Ψ, Λ are dimensionless. The object Ψ is the (anti-self-dual part of) the Weyl curvature, measured in Planck units. So, it satisfies $\Psi \ll 1$ in all situations in which GR has been tested, or can be trusted.

To exhibit close similarly to (18) we further rewrite the Plebanski Lagrangian by introducing an auxiliary field μ to impose the constraint that Ψ^{ij} is tracefree. Thus, we write

$$L_{\mathrm{Pleb}} = B^i F^i - \frac{1}{2} M^{ij} B^i B^j - 2\mu\,(\mathrm{Tr}(M) - \Lambda)\,\mathrm{vol}_\Sigma. \tag{21}$$

The only difference between this Lagrangian and (18), apart from the overall factor, is that the constraints imposed on the auxiliary matrix M^{ij} are different.

To quantify the difference, we parametrise the matrix M^{ij} via

$$M^{ij} = \Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}. \tag{22}$$

Here Ψ^{ij} is the tracefree part of M^{ij} and $\Lambda(\Psi)$ is the function to be found by imposing the constraint obtained by varying the

Lagrangian with respect to μ. Thus, in the case of GR in Plebanski formalism the constraint simply states that Λ is a constant — the cosmological constant. In the case of the theory (18) one obtains Λ as a non-trivial function of Ψ^{ij} instead. This is the parametrisation in which this class of 4D gravity theories was discovered in [18]. For (18) the constraint gives

$$\left(1+\frac{\Lambda\phi^2}{3}\right)^3 - \frac{3}{2}\left(1+\frac{\Lambda\phi^2}{3}\right)\phi^4 \mathrm{Tr}(\Psi^2) + \phi^6 \det(\Psi) = \phi^3. \tag{23}$$

We can use this as an equation to solve for Λ in terms of the tracefree part. Under assumption that $\Psi \ll 1$ and to order Ψ^2 the solution is

$$\frac{\Lambda(\Psi)}{3} = \frac{\phi-1}{\phi^2} + \frac{\phi}{2}\mathrm{Tr}(\Psi^2) + O(\Psi^3). \tag{24}$$

It is now clear that (18) is the theory of the same type as (20), the only difference being that Λ is not a constant, but a function $\Lambda(\Psi)$ given by (24). However, for $\Psi \ll 1$ the dependence of $\Lambda(\Psi)$ on Ψ in (24) can be neglected and $\Lambda(\Psi)$ becomes a constant. This shows that the theory (18) is indistinguishable from GR in its form (21) in all situations where GR has been tested and/or can be trusted. Note that the $O(\Psi^2)$ term in (24) is neglected as compared to Ψ^{ij} term in (22), not as compared to the constant, which can be small. Detailed study of effects of modification of GR such as (24) on e.g. the spherically symmetric solution can be found in [19].

The above argument can be rephrased as follows. The theory (4) or (18) is a classical theory with no scale in it. The metric that the field B^i determines measures distances that are dimensionless. As (24) shows, this theory behaves as GR with non-zero dimensionless cosmological constant $(\phi-1)/\phi^2$ for small $\Psi \ll 1$ values of the dimensionless Weyl curvature. To compare this with the usual GR in which the distances are dimensionful, one needs to introduce a scale into theory (18). Alternatively one can redefine all GR fields using some scale to make GR quantities dimensionless, as we have done in the passage from (19) to (21). This scale can be chosen to be the Planck scale, and then the theory (18) will be indistinguishable from GR for curvatures smaller than Planckian.

Thus, choosing the scale for identifying (18) with GR to be the Planck scale, the dimensionless Λ in (24) becomes the cosmological constant measured in Planck units, and is the extraordinary small number $\Lambda \sim 10^{-120}$ that embodies the cosmological constant problem. Our theory (18) gives small cosmological constant for values of radius of compactification ϕ close to unity. This must hold to extraordinary high accuracy

$$\phi - 1 = \frac{\Lambda}{3M_p^2}, \tag{25}$$

where we now reinstated the Planck mass so that this is the usual dimensionful Λ, and omitted higher order terms. In (24) one can also get small Λ for large ϕ, but this gives large GR modifications as in this case the $\mathrm{Tr}(\Psi^2)$ term in (24) can no longer be ignored.

There are several open questions that need to be addressed to convert the model studied here into a realistic theory. First and foremost, one must find a dynamical mechanism for driving the compactification radius ϕ to unity to produce a small cosmological constant. Similar issue is present in the usual Kaluza–Klein scenarios where one needs to provide a mechanism for spontaneous compactification.

We note, however, that the situation in theory (4) is somewhat better than in the usual KK setup. In the latter case, apart from the case of compactification on S^1, the pure gravity theory in $4+D$ dimensions usually *does not* have solutions of the form of the product of Minkowski spacetime and (compact) internal manifolds. For this reason one usually extends the pure gravity theory in $4+D$ dimensions with extra fields, e.g. by considering the Einstein–Yang–Mills system. The stress–energy tensor of these extra fields then allows for solutions of the required product form, see e.g. [20], Section 3. Probably the most famous compactification mechanism is that due to Freund and Rubin [21], where the 3-form field of the 11D supergravity is doing the job. In contrast, the theory (4) admits the solution that is the S^3 fibration over S^4, see [14] for an explicit description. Thus, at least there is a solution of (4) of the desired type without having to introduce extra fields. However, the cosmological constant for the S^3 fibration over S^4 solution is too large, see [14]. This is similar to the situation with the Freund–Rubin solution.

Thus, a compactification mechanism that would result in an appropriately small cosmological constant is a very serious open issue for our setup. It is possible that the only way forward is to add other fields. We then remark that there is a very natural extension of the theory (4) that adds forms of all odd degrees. This is the theory that appeared in [12], formula (29). It would be interesting to study 4D compactifications of this more general theory. We hope to analyse this in the future.

Another open problem of the present approach is that of coupling to matter. Again, a natural way to proceed is suggested by supergravity. One does not couple supergravity to extra fields, one simply studies what the modes already present become when viewed from the 4D perspective. In particular, when compactifying on a coset manifold all modes related to isometries of the internal space are known to be important. Indeed, recall that the gauge group that arises in the KK compactification is the group of isometries of the internal manifold, and its dimension may be larger than the dimension of the internal space itself. In this paper we have considered a compactification on a group manifold, but only retained half of the relevant isometries by considering the invariant dimensional reduction ansatz. It is clear that additional fields will arise by enlarging the ansatz by taking into account all the isometries. In this case, however, one must be careful about the issue of consistent truncation, see e.g. [22] for a clear description of all the issues arising. We leave a study of the dimensional reduction on S^3 viewed as a coset $S^3 = \mathrm{SO}(4)/\mathrm{SO}(3)$ to future research.

Third, there is a question of how to describe Lorentzian signature metrics using this formalism. To do this one must make the 3-form C complex-valued, and then impose some appropriate reality conditions. Similar issues exist in all Plebanski-related formulations. We postpone their resolution to future work.

Finally, to avoid confusion, we would like to say that our present use of G_2 structures (3-forms in 7D) is different from what one can find in the literature on Kaluza–Klein compactifications of supergravity. In our approach a 3-form is not an object that exist in addition to the metric — it is the only object that exist. The metric, and in particular the 4D metric, is defined by the 3-form via (2). Also, in the supergravity context a 7D manifold with a G_2 structure is used for compactifying the 11D supergravity down to 4D. In contrast, we compactify from 7D to 4D.

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