

11D Supergravity and Hidden Symmetries

Henning Samtleben

Abstract We review the structure of maximal $D = 11$ and $D = 10$ supergravities. Upon dimensional reduction, these theories give rise to the unique maximal supergravities in all lower spacetime dimensions $D < 10$. In D dimensions, maximal supergravity exhibits the exceptional global symmetry group E_{11-D} , part of which is realized as hidden symmetries and only manifest after proper dualization of the fields. We also briefly review the reformulation of $D = 11$ supergravity as an exceptional field theory which renders the appearance of hidden symmetries manifest.

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Supersymmetry, Supergravity, Higher dimensions, Compactification, Kaluza-Klein theory, Gauge symmetry, Hidden symmetries, Exceptional groups, Exceptional field theory

1 Introduction

The interest to study supergravity theories in higher dimensions $D > 4$, and their dimensional reduction is (at least) twofold. On the one hand supergravity theories appear as the low-energy effective action for string theories, which generically live in higher dimensions. On the other hand, from a purely four-dimensional point of view, the dimensional reduction of higher-dimensional supergravities can be considered as a powerful technique in order to construct extended $D = 4$ supergravity theories, i.e., supergravities with $\mathcal{N} > 1$ supercharges.

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Under dimensional reduction of a minimally supersymmetric theory (i.e., a theory with a single supercharge), the spinor supercharge in general breaks into a number of lower-dimensional spinors, according to the fundamental spinor representations of the corresponding Poincaré groups.¹ Minimal supersymmetry in higher dimensions thus induces extended supersymmetry in the reduced theory. For globally supersymmetric theories, this led to the direct construction of maximally supersymmetric $D = 4$, $\mathcal{N} = 4$ Yang-Mills theory upon dimensional reduction of the ten-dimensional minimal $\mathcal{N} = 1$ super Yang-Mills theory [2].

For supergravity, eleven is the highest dimension in which the minimal supersymmetric extension of the Poincaré algebra allows for a supermultiplet of fields of spin smaller or equal to 2 [3]. For fields with spin larger than 2, interacting theories with a finite number of fields in general do not exist. Eleven is thus the highest dimension in which supergravity can be constructed. The associated interacting theory is unique and has been found by Cremmer, Julia, and Scherk [4]. Upon dimensional reduction to $D = 4$ dimensions, this theory gives rise to maximally supersymmetric $\mathcal{N} = 8$ supergravity [5, 6]. This is arguably the most remarkable extension of $D = 4$ Einstein gravity due to its high degree of symmetry and the finiteness properties of its higher loop amplitudes [7–14].

The detour via eleven dimensions has in fact been an indispensable tool in the construction of this theory [5, 6], in particular in order to determine the complicated non-linear interactions between its 70 scalar fields, which had proven an extremely challenging task within the $D = 4$ perturbative approach [15]. The other key ingredient in the construction of $\mathcal{N} = 8$ supergravity has been the seminal observation that the theory admits an unexpectedly large global symmetry group, the exceptional group $E_{7(7)}$ [5, 6].² While part of these global symmetries can be understood from the gauge symmetries of the theory’s higher-dimensional ancestor, a considerable part of the exceptional symmetries has no direct higher-dimensional interpretation and is often referred to as *hidden symmetries*. Their presence is essential for the realization of the exceptional symmetry group which in turn allows to organize the couplings of the theory in a remarkably compact way. For example, the scalar sector of the theory is most concisely described as a non-linear sigma model on the coset space $E_{7(7)}/SU(8)$.

Subsequently, the presence of hidden symmetries and the appearance of exceptional global symmetry groups in maximal supergravity were recognized as part of a general pattern that has been dubbed the *silver rules of supergravity* [16–19]. Maximal supergravity in $D = 11 - d$ dimensions exhibits a global symmetry group $E_{d(d)}$, realizing the series of exceptional Lie groups in the Dynkin classification, with the Dynkin diagram of the associated algebra given in Figure 1. For small values of d , the exceptional series degenerates into the classical Lie groups

¹ To get the correct counting, the reality conditions of spinors in the different dimensions of space-time have to be taken into account, see e.g., [1].

² The subscript in parentheses in this notation specifies the particular real form of the group: for $E_{7(7)}$, the associated Lie algebra has 70 non-compact generators and 63 compact generators with the latter spanning the compact Lie algebra $\mathfrak{su}(8)$.

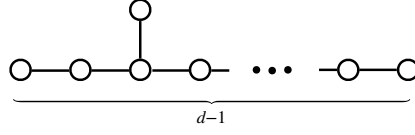


Fig. 1 The Dynkin diagram of the exceptional Lie algebra e_d .

$$E_{5(5)} \simeq \text{SO}(5,5), \quad E_{4(4)} \simeq \text{SL}(5), \quad E_{3(3)} \simeq \text{SL}(3) \times \text{SL}(2), \quad (1)$$

as can be found from properly extrapolating the general Dynkin diagram. Let us also note that discrete versions $E_{d(d)}(\mathbb{Z})$ of the exceptional symmetry groups of supergravity survive in the toroidal compactification of the full string theories [20].

In this chapter, we review the structure of maximal supergravity in eleven dimensions, its dimensional reduction, and the appearance of hidden symmetries. To this end, we first review in section 2 in some detail the field content and dynamics of the maximal supergravities in ten and eleven dimensions. In section 3, we discuss the toroidal compactification of these theories to lower-dimensional maximal supergravities. In particular, we determine the *geometric* symmetries of the lower-dimensional theories, i.e., the global symmetries that descend from particular diffeomorphism and gauge transformations in higher dimensions. Section 4 then reviews the appearance of *hidden* global symmetries in lower dimensions. The central example is $D = 4$, $\mathcal{N} = 8$ supergravity with its global symmetry group enhanced to the full exceptional group $E_{7(7)}$. Finally, in section 5, we review the formulation of 11D supergravity as an *exceptional field theory* [21], which highlights the role of the full exceptional group in the full 11D supergravity before dimensional reduction.

Throughout this chapter, our discussion of hidden symmetries will mostly be restricted to the bosonic sectors of the supergravity theories. Although the fermionic field content and couplings are at the very origin of all these theories, the symmetry enhancement and the appearance of the exceptional symmetry groups can be realized and studied entirely within their bosonic sectors. In particular, even in presence of fermions, the exceptional global symmetry algebras do not extend to larger superalgebras.

2 Maximal supergravity in $D = 11$ and $D = 10$ dimensions

2.1 Field content

The highest-dimensional supergravity theory lives in eleven spacetime dimensions and was constructed by Cremmer, Julia, and Scherk [4]. Its field content is given by the lowest massless representation of the supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = 2P_M (\Gamma^M \Gamma^0)_{\alpha\beta}, \quad (2)$$

where Q_α are the 32 independent real supercharges in eleven-dimensional Minkowski spacetime. P_M with $M = 0, 1, \dots, 10$, is the $D = 11$ momentum, and Γ^M are the $\mathfrak{so}(1, 10)$ gamma matrices. For massless states which fulfill $P_M P^M = 0$, the r.h.s. of (2) describes a projector of half-maximal rank in spinor space. It follows that only 16 out of the 32 supercharges act non-trivially on massless states and satisfy the Clifford algebra \mathcal{C}_{16} , which admits a $2^8 = 256$ -dimensional irreducible representation. The spacetime interpretation of these states is inferred from the embedding of the massless little group

$$\mathcal{C}_{16} \supset \mathfrak{so}(16) \supset \mathfrak{so}(9), \quad (3)$$

according to

$$256 \longrightarrow 128_s + 128_c \longrightarrow 44 + 84 + 128. \quad (4)$$

The 44 corresponds to the symmetric traceless product of two vectors: these are the degrees of freedom of a massless spin-2 field, the graviton G_{MN} . The $84 = \binom{9}{3}$ on the other hand counts the degrees of freedom a totally antisymmetric massless 3-form field C_{KMN} , i.e., a field with local tensor gauge symmetry

$$\delta C_{KMN} = 3 \partial_{[K} \Lambda_{MN]}. \quad (5)$$

The 128 finally corresponds to the degrees of freedom of a massless spin-3/2 field in eleven dimensions, the gravitino Ψ_M . The full 11D supergravity multiplet thus is given by

$$\{G_{MN}, \Psi_M, C_{KMN}\}. \quad (6)$$

The interacting theory has been constructed in [4] and is reviewed in section 2.2. The same reasoning shows why supergravity theories do not exist beyond eleven dimensions: repeating the above analysis for, say, a twelve-dimensional spacetime with 64 supercharges yields a minimal field content that includes fields with spin larger than two. No consistent interacting theory for such fields can be constructed (unless infinitely many fields are included).

A similar analysis yields the field content of $D = 10$ supergravity. In this case, the smallest massless representation descends from the Clifford algebra \mathcal{C}_8 and comprises $2^4 = 16$ states. In analogy to (4), they transform as vector and spinor under the little group $\text{SO}(8)$

$$16 \longrightarrow 8_v + 8_s, \quad (7)$$

counting degrees of freedom of a ten-dimensional massless vector and a matter fermion. This is the minimal $\mathcal{N} = 1$ vector multiplet in ten dimensions. The higher massless multiplets can be found by tensoring (7) with the fundamental representations of $\text{SO}(8)$. This results in the $\mathcal{N} = 1$ supergravity multiplet and two inequivalent gravitino multiplets:

$$\begin{aligned} \text{sugra} : 8_v \otimes (8_v + 8_s) &= 1 + 28 + 35_v + 8_c + 56_c, \\ \text{gravitino A} : 8_c \otimes (8_v + 8_s) &= 8_v + 56_v + 8_s + 56_s, \\ \text{gravitino B} : 8_s \otimes (8_v + 8_s) &= 1 + 28 + 35_s + 8_c + 56_c. \end{aligned} \quad (8)$$

Similar to (4), the physical field content of these multiplets may be inferred from the representations of the little group. The $\mathcal{N} = 1$ supergravity multiplet carries the metric, a scalar field (the dilaton) and an antisymmetric 2-form together with a gravitino and a matter fermion. The interacting theory exists, and upon coupling to the vector multiplet (7) in the adjoint representation of the gauge group, this describes the low-energy effective theory of the heterotic string [22].

The first gravitino multiplet in (8) carries a spacetime vector and a 3-form together with a gravitino and a matter fermion of chirality opposite to the fermions of the supergravity multiplet. Instead, the second gravitino multiplet carries another scalar (the axion) and a 2-form together with a self-dual 4-form (the 35_s in (8)), see (30) below. Its fermions are a gravitino and a matter fermion of the same chirality as the fermions of the supergravity multiplet. Coupling of a massless gravitino multiplet requires supersymmetry enhancement. The two resulting theories exist as maximally supersymmetric interacting theories and are denoted as the non-chiral $\mathcal{N} = (1, 1)$ type IIA and the chiral $\mathcal{N} = (2, 0)$ type IIB theory, respectively. They are the low-energy effective theories for the massless spectrum of IIA and IIB strings, respectively. The type IIA theory can be obtained by compactifying 11D supergravity on a circle S^1 as will be discussed in section 2.3. In contrast, the type IIB supergravity does not have a higher-dimensional origin and has been constructed in [23–25]. We will review it in section 2.4.

2.2 11D supergravity

The action of eleven-dimensional supergravity with field content given in (6) has been constructed in [4]. To quadratic order in the fermions, its Lagrangian extends the standard combination of Einstein-Hilbert and Rarita-Schwinger term

$$\mathcal{L}_0[E, \psi] = |E| R[\omega] - \frac{1}{2} |E| \bar{\psi}_K \Gamma^{KMN} D[\omega]_M \psi_N, \quad (9)$$

by a kinetic, a topological, and a fermionic interaction term for the antisymmetric 3-form field C_{KMN} , given by

$$\begin{aligned} \mathcal{L}_C[E, C, \psi] = & -\frac{1}{48} |E| F_{KLMN} F^{KLMN} + \frac{1}{1442} \varepsilon^{N_0 N_1 \dots N_{10}} F_{N_0 \dots N_3} F_{N_4 \dots N_7} C_{N_8 N_9 N_{10}} \\ & + \frac{1}{192} |E| \left(\bar{\psi}_P \Gamma^{KLMNPQ} \psi_Q + 12 \bar{\psi}^K \Gamma^{LM} \psi^N \right) F_{KLMN}. \end{aligned} \quad (10)$$

Here, $|E|$ denotes the determinant of the eleven-bein E_M^A , related to the metric as

$$G_{MN} = E_M^A E_{NA}, \quad (11)$$

with flat Lorentz indices A .³ The gauge invariant abelian field strength is defined as $F_{KLMN} = 4\partial_{[K}C_{LMN]}$. The totally antisymmetric (numerical) $\varepsilon^{N_0N_1\dots N_{10}}$ is the Levi-Civita density. The Chern-Simons term $F \wedge F \wedge C$ is invariant under tensor gauge transformations (5) up to a total derivative. The appearance of such topological terms is a generic feature in higher-dimensional supergravity theories. The full 11D supergravity Lagrangian is given by

$$\mathcal{L}_{11D} = \mathcal{L}_0 + \mathcal{L}_C + \mathcal{L}_{\psi^4}, \quad (12)$$

where the quartic fermion terms of \mathcal{L}_{ψ^4} can be formally absorbed into an appropriate modification of the spin connection ω and the 4-form field strength in the ψ^2 terms in (9), (10). The full Lagrangian (12) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta E_M^A &= \frac{1}{4} \bar{\varepsilon} \Gamma^A \psi_M, & \delta C_{KLM} &= \frac{3}{4} \bar{\varepsilon} \Gamma_{[KL} \psi_{M]}, \\ \delta \psi_M &= D[\omega]_M \varepsilon - \frac{1}{288} F^{KLPQ} \Gamma_{MKLPQ} \varepsilon + \frac{1}{36} F_{MNPQ} \Gamma^{NPQ} \varepsilon, \end{aligned} \quad (13)$$

given up to cubic terms in the fermions. In turn, supersymmetry uniquely fixes all the terms in (12) and in particular requires the presence of the topological Chern-Simons term in (10).

Let us still point out two interesting properties of the Lagrangian (12), which will be important for the appearance of hidden symmetries. First, the field equations descending from the Lagrangian (12), scale homogeneously under the following transformation

$$\delta E_M^A = \zeta E_M^A, \quad \delta C_{KMN} = 3\zeta C_{KMN}, \quad \delta \psi_M = \frac{\zeta}{2} \psi_M, \quad (14)$$

with constant ζ . In general spacetime dimension D , this so-called *trombone symmetry* is not a symmetry of the action but rescales the Lagrangian as

$$\delta \mathcal{L} = (D-2) \zeta \mathcal{L}. \quad (15)$$

It still plays an important role, e.g., among the spectrum-generating symmetries for the fundamental BPS solutions [26]. The trombone symmetry is present in all two-derivative supergravity theories, but is in general broken by higher-order corrections.

Second, the field equations for the 3-form C_{KLM} may be used as integrability relations which ensure the consistency of the definition of an antisymmetric 6-form potential $C_{N_1\dots N_6}$ by means of the first-order duality equation

$$F_{N_1N_2\dots N_7} = -\frac{1}{24} |E| \varepsilon_{N_1N_2\dots N_7M_1M_2M_3M_4} F^{M_1M_2M_3M_4} + \text{fermions}. \quad (16)$$

Here, the 7-form field strength is defined as

³ Throughout this chapter, we use spacetime signature $(- + + \dots +)$.

$$F_{N_1 N_2 \dots N_7} = 7 \partial_{[N_1} C_{N_2 N_3 \dots N_7]} + \frac{35}{2} C_{[N_1 N_2 N_3} F_{N_4 N_5 N_6 N_7]} , \quad (17)$$

with a non-trivial Bianchi identity

$$8 \partial_{[N_1} F_{N_2 N_3 \dots N_8]} = 35 F_{[N_1 N_2 N_3 N_4} F_{N_5 N_6 N_7 N_8]} . \quad (18)$$

Indeed, hitting (16) with another external derivative and using (18), this equation reduces to the field equations for the 3-form C_{KMN} , obtained by variation of (10). In particular, the contribution from the Chern-Simons term in (10) requires the second term in (17). As a consequence, the 6-form $C_{N_1 \dots N_6}$ transforms non-trivially under the gauge transformations (5). Specifically, its field strength (17) is invariant under the gauge transformations

$$\delta C_{N_1 N_2 \dots N_6} = 6 \partial_{[N_1} \Lambda_{N_2 N_3 \dots N_6]} - 30 C_{[N_1 N_2 N_3} \partial_{N_4} \Lambda_{N_5 N_6]} , \quad (19)$$

with a new gauge parameter $\Lambda_{N_1 \dots N_5}$, and Λ_{MN} from (5). This shows that the full algebra of 11D gauge symmetries is actually non-abelian

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{\Lambda_{12}} , \quad (20)$$

with the commutator of two 3-form gauge transformations (5), (19), resulting in a 6-form gauge transformation with parameter

$$\Lambda_{12, N_1 N_2 \dots N_5} = 15 \Lambda_{2, [N_1 N_2} \partial_{N_3} \Lambda_{1, N_4 N_5]} - 15 \Lambda_{1, [N_1 N_2} \partial_{N_3} \Lambda_{2, N_4 N_5]} . \quad (21)$$

The introduction of dual fields, defined by first-order duality equations such as (16) is a general feature of supergravity theories. In general spacetime dimension D , the field equations for a given p -form allow the introduction of its dual form of degree $(D - p - 2)$, in terms of which Bianchi identities and equations of motion become exchanged. As a consequence, supergravity theories may admit different Lagrangian formulations, which are on-shell equivalent only after relating their fields by means of duality equations such as (16).⁴ In particular, there is always a Lagrangian formulation of the theory in which all forms are dualized to the lowest possible degree. It is in this form that the largest global symmetry group becomes visible, as we shall see below.

2.3 IIA supergravity

Dimensional reduction of 11D supergravity on a circle S^1 yields maximal type IIA supergravity in ten dimensions [27, 28]. Explicitly, the reduction amounts to imposing independence of all fields on the eleventh coordinate

⁴ Specifically, in dimensional reductions from 11D supergravity, the lower-dimensional duality equations are precisely obtained by dimensional reduction of (16).

$$\partial_{10}G_{MN} = 0 = \partial_{10}C_{KMN} , \quad (22)$$

together with a standard Kaluza-Klein ansatz for the eleven-dimensional metric⁵

$$ds_{(11)}^2 = e^{-\phi/6} g_{\mu\nu} dx^\mu dx^\nu + e^{4\phi/3} (dy + A_\mu dx^\mu)^2 , \quad (23)$$

in terms of the ten-dimensional metric $g_{\mu\nu}$, (with indices $\mu = 0, \dots, 9$, labelling the coordinates of the ten-dimensional spacetime), a dilaton ϕ , and a Kaluza-Klein vector A_μ . Similarly, the components of the eleven-dimensional 3-form are parametrized as

$$C_{\mu\nu 10} = B_{\mu\nu} , \quad C_{\mu\nu\rho} = A_{\mu\nu\rho} + 3A_{[\mu} B_{\nu\rho]} , \quad (24)$$

in terms of a 2-form $B_{\mu\nu}$ and a 3-form $A_{\mu\nu\rho}$. Under the eleven-dimensional gauge transformations (5) these forms transform as

$$\delta B_{\mu\nu} = 2\partial_{[\mu} \Lambda_{\nu]} , \quad \delta A_{\mu\nu\rho} = 3\partial_{[\mu} \Lambda_{\nu\rho]} - 6A_{[\mu} \partial_\nu \Lambda_{\rho]} , \quad (25)$$

respectively, where we denote $\Lambda_\mu = \Lambda_{\mu 10}$. Accordingly, their gauge invariant field strengths are defined as

$$H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} , \quad F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]} + 6F_{[\mu\nu} B_{\rho\sigma]} , \quad (26)$$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ denotes the abelian field strength of the Kaluza-Klein vector from (23). Plugging the reduction ansatz (22), (23), (24), into the supergravity Lagrangian (12), yields the ten-dimensional Lagrangian of type IIA supergravity

$$\begin{aligned} \mathcal{L}_{\text{IIA}} = & |e| R_{(10)} - \frac{1}{4} |e| e^{3\phi/2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} |e| \partial_\mu \phi \partial^\mu \phi \\ & - \frac{1}{48} |e| e^{\phi/2} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} - \frac{1}{12} |e| e^{-\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & + \frac{1}{144^2} \varepsilon^{\mu_0\mu_1\dots\mu_9} F_{\mu_0\mu_1\mu_2\mu_3} (3F_{\mu_4\mu_5\mu_6\mu_7} B_{\mu_8\mu_9} - 8H_{\mu_4\mu_5\mu_6} A_{\mu_7\mu_8\mu_9}) \\ & + \text{fermions} . \end{aligned} \quad (27)$$

Here, $|e|$ and $R_{(10)}$ are the determinant of the ten-dimensional vielbein, and the ten-dimensional Ricci scalar, respectively. The topological term is invariant under gauge transformations (25) up to a total derivative. The field content of (27) matches the IIA supergravity multiplet of (8). In particular, reduction of the 11D gravitino gives rise to two ten-dimensional gravitini of opposite chirality. The IIA supergravity Lagrangian may be equivalently expressed in terms of different fundamental fields (e.g., using the original components $C_{\mu\nu\rho}$ rather than the $A_{\mu\nu\rho}$ of (24)). However, all formulations share the property that the field strength $F_{\mu\nu\rho\sigma}$ building the kinetic term for the 3-form, satisfies a non-trivial Bianchi identity

⁵ Compared to the general Kaluza-Klein parametrization (45) given below, this ansatz uses a rescaled dilaton $\phi \rightarrow \frac{2}{3}\phi$ in order to match some standard conventions of IIA supergravity.

$$5 \partial_{[\mu} F_{\nu\rho\sigma\tau]} = 10 F_{[\mu\nu} H_{\rho\sigma\tau]} . \quad (28)$$

Let us also note that the Lagrangian (27) admits a 1-parameter massive deformation upon deforming the field strengths

$$F_{\mu\nu} \rightarrow F_{\mu\nu} + m B_{\mu\nu} , \quad F_{\mu\nu\rho\sigma} \rightarrow F_{\mu\nu\rho\sigma} + 3m B_{[\mu\nu} B_{\rho\sigma]} , \quad (29)$$

thereby inducing a mass term for the 2-form $B_{\mu\nu}$ [29]. The resulting theory has an equivalent description in terms of a 9-form gauge potential which reflects the presence of D8-branes in IIA string theory [30, 31].

2.4 IIB supergravity

As shown in equation (8) above, the bosonic field content of the ten-dimensional type IIB supergravity comprises the metric, two scalar fields, two 2-form gauge potentials $C_{\mu\nu}^\alpha$, $\alpha = 1, 2$, and a selfdual 4-form potential $C_{\mu\nu\rho\sigma}$. Specifically, the latter satisfies a first-order selfduality equations

$$F_{\mu\nu\rho\sigma\tau} = \frac{1}{5!} |e| \varepsilon_{\mu\nu\rho\sigma\tau\mu_1\mu_2\mu_3\mu_4\mu_5} F^{\mu_1\mu_2\mu_3\mu_4\mu_5} , \quad (30)$$

in terms of the field strength

$$F_{\mu_1\dots\mu_5} = 5 \partial_{[\mu_1} C_{\mu_2\dots\mu_5]} - \frac{5}{4} \varepsilon_{\alpha\beta} C_{[\mu_1\mu_2}^\alpha F_{\mu_3\mu_4\mu_5]}^\beta . \quad (31)$$

Here, $\varepsilon_{\alpha\beta}$ is the antisymmetric tensor in two indices, and

$$F_{\mu\nu\rho}^\alpha = 3 \partial_{[\mu} C_{\nu\rho]}^\alpha , \quad (32)$$

is the abelian field strength for the doublet of 2-forms. IIB supergravity has been constructed in [23–25]. Selfduality equations such as (30) cannot be derived from a standard action principle. As a consequence, these equations are often imposed separately, while the remaining field equations of IIB supergravity are most conveniently derived from a so-called pseudo-action with Lagrangian given by

$$\begin{aligned} \mathcal{L}_{\text{IIB}} = & |e| R_{(10)} + \frac{1}{4} |e| \partial_\mu m_{\alpha\beta} \partial^\mu m^{\alpha\beta} - \frac{1}{12} |e| F_{\mu_1\mu_2\mu_3}^\alpha F^{\mu_1\mu_2\mu_3\beta} m_{\alpha\beta} \\ & - \frac{1}{30} |e| F_{\mu_1\mu_2\mu_3\mu_4\mu_5} F^{\mu_1\mu_2\mu_3\mu_4\mu_5} \\ & - \frac{1}{864} \varepsilon_{\alpha\beta} \varepsilon^{\mu_1\dots\mu_{10}} C_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_6\mu_7\mu_8}^\alpha F_{\mu_8\mu_9\mu_{10}}^\beta \\ & + \text{fermions} . \end{aligned} \quad (33)$$

Here the matrix $m_{\alpha\beta}$ denotes a symmetric 2×2 matrix of unit determinant which is parametrized by the two scalars, ϕ , C_0 , of the theory

$$m_{\alpha\beta} = \frac{1}{\Im\tau} \begin{pmatrix} |\tau|^2 & -\Re\tau \\ -\Re\tau & 1 \end{pmatrix}_{\alpha\beta}, \quad \tau = C_0 + ie^{-\phi}. \quad (34)$$

Its inverse is denoted as $m^{\alpha\beta} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}m_{\gamma\delta}$, such that the kinetic term in (33) describes the sigma model on the coset space $\text{SL}(2)/\text{SO}(2)$, c.f. section 4.1,

$$\frac{1}{4}\partial_\mu m_{\alpha\beta}\partial^\mu m^{\alpha\beta} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu C_0\partial^\mu C_0. \quad (35)$$

Type IIB supergravity is manifestly invariant under a global $\text{SL}(2)$ symmetry, acting on all indices α, β , inducing a non-linear action on the scalars ϕ , C_0 via (34). Furthermore, the Lagrangian (33) is invariant under gauge transformations

$$\begin{aligned} \delta C_{\mu\nu}{}^\alpha &= 2\partial_{[\mu}\Lambda_{\nu]}{}^\alpha, \\ \delta C_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}\Lambda_{\nu\rho\sigma]} + \frac{1}{2}\varepsilon_{\alpha\beta}\Lambda_{[\mu}{}^\alpha F_{\nu\rho\sigma]}{}^\beta, \end{aligned} \quad (36)$$

up to total derivatives from variation of the topological term. It is straightforward to verify that the integrability conditions of the selfduality equations (30) coincide with the second-order field equations obtained by variation of (33). Various alternative action principles for IIB supergravity have been put forward in the literature in order to also derive the selfduality equations (30) from a variational principle, typically at the expense of introducing additional fields and/or sacrificing manifest (9+1)-dimensional Lorentz invariance [32–38].

The $\text{SL}(2)$ conventions of (33), (34) can be translated into the $\text{SU}(1,1)/\text{U}(1)$ conventions of [24] by combining the real components of the doublet $F_{\mu\nu\rho}{}^\alpha$ into a complex field strength

$$F_{\mu\nu\rho} \equiv F_{\mu\nu\rho}{}^1 + iF_{\mu\nu\rho}{}^2, \quad (37)$$

and parametrizing the matrix $m_{\alpha\beta}$ in terms of a single complex scalar field B as

$$m_{\alpha\beta} \equiv (1 - BB^*)^{-1} \begin{pmatrix} (1-B)(1-B^*) & i(B-B^*) \\ i(B-B^*) & (1+B)(1+B^*) \end{pmatrix}_{\alpha\beta}. \quad (38)$$

In terms of the complex combinations

$$G_{\mu\nu\rho} \equiv (1 - BB^*)^{-1/2}(F_{\mu\nu\rho} - BF_{\mu\nu\rho}^*), \quad P_\mu \equiv (1 - BB^*)^{-1}\partial_\mu B, \quad (39)$$

the kinetic terms of (33) translate into those of [24] with

$$m_{\alpha\beta}F_{\mu\nu\rho}{}^\alpha F^{\mu\nu\rho\beta} = G_{\mu\nu\rho}^* G^{\mu\nu\rho}, \quad \frac{1}{4}\partial_\mu m_{\alpha\beta}\partial^\mu m^{\alpha\beta} = -2P_\mu^* P^\mu. \quad (40)$$

Let us finally note, that the Lagrangians (27), (33) of type IIA and type IIB supergravity coincide when truncated to the common field content $\{g_{\mu\nu}, \phi, B_{\mu\nu} = C_{\mu\nu}^1\}$, which is the bosonic part of the $\mathcal{N} = 1$ supergravity multiplet from (8) — or the NS-NS sector.

3 Toroidal reduction

Maximal supergravities can be obtained by Kaluza-Klein reduction from 11D and type IIB supergravity. In particular, the reduction of 11D supergravity on a d -dimensional torus T^d yields the maximal ungauged supergravity in $D = 11 - d$ dimensions. More precisely, for $D < 10$, this is the unique supergravity theory in D dimensions with 32 real supercharges and no fields charged under the abelian gauge group.

In this section, we first discuss the global geometric $GL(d)$ symmetry appearing after reduction to D dimensions as a remnant of the higher-dimensional diffeomorphisms acting on the torus T^d . We explicitly perform the dimensional reduction on a torus, first for pure gravity and next for the p -forms, which typically span the matter sector of higher-dimensional supergravities. In section 4, we then discuss the enhancement of the geometric symmetry group by the so-called hidden symmetries.

3.1 Geometric symmetries

The dimensional reduction of supergravity can be performed most conveniently by using the vielbein formalism, see e.g., [39]. We will first consider the reduction of pure gravity in an $(D + d)$ -dimensional spacetime on a d -dimensional torus T^d down to D dimensions. The coordinates of $(D + d)$ -dimensional spacetime are split according to

$$x^M \rightarrow (x^\mu, y^m), \quad \mu = 0, \dots, D-1, \quad m = 1, \dots, d, \quad (41)$$

and similarly we split the flat Lorentz indices as

$$A \rightarrow (a, \underline{a}), \quad a = 0, \dots, D-1, \quad \underline{a} = 1, \dots, d. \quad (42)$$

In toroidal dimensional reduction, all fields are taken to be independent of the coordinates y^m of the d -torus

$$\partial_m \Phi = 0. \quad (43)$$

One may think of a normal mode expansion of the fields and drop all modes other than the zero modes.

The local Lorentz invariance in $(D + d)$ dimensions can be used to bring the vielbein into a triangular form

$$E_M^A = \begin{pmatrix} E_\mu^a & E_\mu^a \\ 0 & E_m^a \end{pmatrix}, \quad (44)$$

which breaks the Lorentz group $\text{SO}(1, D+d-1)$ down to $\text{SO}(1, D-1) \times \text{SO}(d)$. It turns out to be convenient to further parametrize (44) as

$$E_M^A = \begin{pmatrix} e^{\gamma\phi} e_\mu^a & e^{\phi/d} V_m^a A_\mu^m \\ 0 & e^{\phi/d} V_m^a \end{pmatrix}, \quad (45)$$

with a matrix $V_m^a \in \text{SL}(d)$ of unit determinant, such that $e^\phi = \det E_m^a$. The constant

$$\gamma = -\frac{1}{D-2}, \quad (46)$$

is chosen such that plugging (45) into the $(D+d)$ -dimensional Einstein-Hilbert Lagrangian, one finds

$$\mathcal{L}_{\text{EH}}^{(D+d)} = |E| R_{(D+d)} \longrightarrow |e| R_{(D)} + \dots, \quad (47)$$

where $R_{(D)}$ is the Ricci scalar computed from the D -dimensional vielbein e_μ^a . I.e., the reduced theory is directly obtained in the Einstein frame (without a dilaton pre-factor). The ellipsis in (47) represents the matter couplings in the D -dimensional theory, i.e., the couplings of vector fields A_μ^m and scalar fields ϕ , V_m^a , from (45). Before working out the explicit form of these terms, it is instructive to analyze the symmetries of the lower-dimensional theory (47).

The D -dimensional theory inherits a number of symmetries from its higher-dimensional ancestor. Namely, with the vielbein (45) transforming under infinitesimal diffeomorphisms as

$$\delta_\xi E_M^A = \xi^N \partial_N E_M^A + E_N^A \partial_M \xi^N, \quad (48)$$

it is straightforward to see that such diffeomorphisms survive in the truncated theory (43) if the diffeomorphism parameter ξ^M itself satisfies (43). In particular, diffeomorphisms of the type $\xi^M = \{\xi^\mu(x), 0\}$ generate the D -dimensional diffeomorphisms on the fields e_μ^a , A_μ^n , ϕ , and V_m^a . On the other hand, under diffeomorphisms of the type $\xi^M = \{0, \xi^m(x)\}$, the fields A_μ^m transform as

$$\delta A_\mu^m = \partial_\mu \xi^m(x), \quad (49)$$

whereas the graviton and the scalar fields are left inert. This shows that the resulting theory is an abelian $U(1)^d$ gauge theory with gauge fields A_μ^m , while none of the matter is charged under the gauge group. Accordingly, the vector fields will couple with a Maxwell-type term in the reduced theory.

A different type of symmetry can be inferred from internal diffeomorphisms linear in the compactified coordinates y^m , i.e., of the form

$$\xi^m(y) = \Lambda_n^m y^n. \quad (50)$$

Despite this dependence on the internal coordinates, the action (48) of such a diffeomorphism remains compatible with the truncation (43). Explicitly, this induces a global symmetry $\text{SL}(d)$ acting on the D -dimensional matter fields parametrizing (45) as

$$\delta V_m^a = \Lambda^n_m V_n^a, \quad \delta A_\mu^m = -\Lambda^n_m A_\mu^n, \quad (51)$$

where we have taken the matrix Λ^n_m to be traceless $\Lambda^m_m \equiv 0$. The trace part of such transformation is slightly more subtle. With the parametrization (45), an internal diffeomorphism $\xi^m(y) \propto y^m$ also induces a non-trivial action on the D -dimensional vielbein. It has to be accompanied by the action of the trombone rescaling symmetry of the $(D+d)$ -dimensional theory, c.f. (14), in order to yield a proper off-shell symmetry of the D -dimensional theory. The combination of these transformations induce the action

$$\delta\phi = -\lambda, \quad \delta A_\mu^m = \lambda \beta A_\mu^m, \quad \beta = \frac{D+d-2}{d(D-2)}, \quad (52)$$

with constant λ , on the D -dimensional fields. Together, the transformations (51) and (52) generate a global $\text{GL}(d)$ symmetry of the D -dimensional theory. We will refer to this group as the geometric symmetries of the theory, as they have their origin in the diffeomorphisms on the internal torus. Let us stress, that the enhancement from $\text{SL}(d)$ to $\text{GL}(d)$ requires the higher-dimensional scaling symmetry (14) and is no longer realized in the presence of higher curvature corrections.

Finally, local Lorentz invariance is also a symmetry of the higher-dimensional theory. As mentioned above, the upper triangular form (44) of the vielbein breaks the original $\text{SO}(1, D+d-1)$ down to $\text{SO}(1, D-1) \times \text{SO}(d)$, of which the first factor acts as D -dimensional Lorentz transformation on e_μ^a and the second factor acts as an additional local symmetry on V_m^a

$$\delta V_m^a = V_m^b \Lambda_b^a(x), \quad \Lambda(x) \in \mathfrak{so}(N). \quad (53)$$

This shows that not all components of the matrix V_m^a correspond to physical scalars as we make explicit in the following.

3.2 Reduction of pure gravity

For pure gravity, the fields of the D -dimensional theory are the various components of the higher-dimensional vielbein (45). The global and local symmetries of the D -dimensional theory, identified in the previous subsection, almost uniquely fix the form of the resulting two-derivative action. Explicitly, plugging the ansatz (43), (45), into the $(D+d)$ -dimensional Einstein-Hilbert Lagrangian, yields the completion of (47)

$$\begin{aligned}
\mathcal{L}_{\text{EH}}^{(D+d)} = |E| R_{(D+d)} \longrightarrow & |e| R_{(D)} - |e| \text{Tr} [P_\mu P^\mu] - \beta |e| \partial_\mu \phi \partial^\mu \phi \\
& - \frac{1}{4} |e| e^{2\beta \phi} M_{mn} F_{\mu\nu}{}^m F^{\mu\nu n} \\
& + \text{total derivatives} ,
\end{aligned} \tag{54}$$

with β from (52). The vector fields appear with abelian field strengths given by $F_{\mu\nu}{}^m = 2 \partial_{[\mu} A_{\nu]}{}^m$ and a Maxwell term with the scalar dependent metric

$$M_{mn} = V_m{}^a V_n{}^b \delta_{ab} . \tag{55}$$

The scalar fields $V_m{}^a$ describe a coset space sigma model

$$\text{G/K} = \text{GL}(d)/\text{SO}(d) , \tag{56}$$

with the $\mathfrak{sl}(d)$ currents defined by

$$\delta^{ac} (V^{-1})_{\underline{c}}{}^m \partial_\mu V_m{}^b \equiv Q_\mu{}^{[ab]} + P_\mu{}^{(ab)} , \tag{57}$$

and decomposed into their antisymmetric and symmetric parts. The target space $\text{SL}(d)/\text{SO}(d)$ is given by

$$-\text{Tr} [P_\mu P^\mu] = \frac{1}{4} \partial_\mu M_{mn} \partial^\mu (M^{-1})^{mn} , \tag{58}$$

with the matrix M_{mn} from (55). Indeed, this kinetic term is invariant under the local symmetry transformations (53), showing that the matrix $V_m{}^a$ carries

$$\dim (\text{SL}(d)/\text{SO}(d)) = \frac{1}{2} (d-1)(d+2) , \tag{59}$$

physical scalar fields.

Up to its relative coefficients, the Lagrangian (54) is the unique two-derivative Lagrangian for this field content, which is compatible with the global symmetries (51), (52), as well as with the gauge symmetries (49), (53), whose presence was deduced from the higher-dimensional diffeomorphism and Lorentz symmetries.

3.3 Reduction of p -forms

The bosonic sector of higher-dimensional supergravities typically combines Einstein gravity with p -form matter couplings, such as the 3-form couplings (10) of 11D supergravity. Upon dimensional reduction on a torus T^d , this matter sector gives rise to additional fields, couplings, and symmetries in the lower-dimensional theory. Before performing the explicit toroidal reduction of the Lagrangian (10), it is instructive to first study the behavior of the additional fields w.r.t. the global $\text{GL}(d)$ symmetry (51), (52).

Although we will mostly be interested in the reduction of a 3-form from 11 dimensions, the analysis is straightforward in general spacetime dimensions. Let us consider a p -form $C_{M_1 \dots M_p}$ in $(D+d)$ spacetime dimensions, such that its various components give rise to D -dimensional $p, p-1, \dots, (p-d)$ -forms. The precise reduction ansatz corresponds to a split in the flat basis (42), i.e., the D -dimensional k -forms are built as

$$A_{\mu_1 \dots \mu_k m_{k+1} \dots m_p} = P_{\mu_1}^{M_1} \dots P_{\mu_k}^{M_k} C_{M_1 \dots M_k m_{k+1} \dots m_p}, \quad (60)$$

with $P_{\mu}^M \equiv \{\delta_{\mu}^v, -A_{\mu}^m\}$. This ansatz (60) is such that the lower-dimensional fields remain invariant under the Kaluza-Klein gauge transformations (49). The transformation behavior of these fields under the global $SL(d)$ symmetry from (51) can be computed from the action of diffeomorphisms (50) and follows from their index structure in the internal indices m_1, m_2, \dots , e.g.,

$$\delta A_{\mu mn} = \Lambda^k_m A_{\mu kn} + \Lambda^k_n A_{\mu mk}, \quad \text{etc..} \quad (61)$$

Their charge under the $GL(1)$ symmetry from (52) is slightly more tedious to determine, since it involves the $(D+d)$ -dimensional trombone symmetry as discussed above. One obtains (see e.g., [40])

$$\delta_{\lambda} A_{\mu_1 \dots \mu_k m_{k+1} \dots m_p} = -\lambda \left(p\gamma + (p-k)\beta \right) A_{\mu_1 \dots \mu_k m_{k+1} \dots m_p}, \quad (62)$$

with γ and β from (46), (52). The higher-dimensional tensor gauge symmetries

$$\delta C_{M_1 \dots M_p} = p \partial_{[M_1} \Lambda_{M_2 \dots M_p]}, \quad (63)$$

give rise to the lower-dimensional gauge symmetries of the k -forms. Due to the reduction ansatz (60), these symmetries in general mix forms of different degree with a non-linear action.

For a sufficiently large torus, i.e., for $d \geq p$ the reduction (60) adds $\binom{d}{p}$ scalar fields $A_{m_1 \dots m_p}$ to the scalar sector of the D -dimensional theory. In analogy to (50), the higher-dimensional tensor gauge transformations linear in the compactified coordinates

$$\Lambda_{m_1 \dots m_{p-1}}(y) = \xi_{m_1 \dots m_{p-1}} y^{m_p}, \quad (64)$$

induce additional global shift symmetries

$$\delta_{\xi} A_{m_1 \dots m_p} = \xi_{m_1 \dots m_p}, \quad (65)$$

on these scalar fields. These symmetries enhance the global $\mathfrak{gl}(d)$ from (51), (52), to a non-semisimple algebra of the type

$$\mathfrak{g}_{\text{nss}} = \mathfrak{g}_0 \oplus \mathfrak{n}_+, \quad (66)$$

where \mathfrak{g}_0 combines the geometric $\mathfrak{gl}(d)$ with other potential global symmetries of the higher-dimensional theory, while the nilpotent \mathfrak{n}_+ combines all the shifts of type

(65). For example, for reductions from 11D supergravity, $\mathfrak{g}_0 = \mathfrak{gl}(d)$, whereas for reductions from IIB supergravity $\mathfrak{g}_0 = \mathfrak{gl}(d) \oplus \mathfrak{sl}(2)$. The algebra (66) is graded w.r.t. the $\mathfrak{gl}(1) \subset \mathfrak{gl}(d)$ of (52), under which \mathfrak{g}_0 are the zero modes. Moreover, it follows from (62) that all generators of \mathfrak{n}_+ have positive charge under $\mathfrak{gl}(1)$. We emphasize once more, that all the global symmetries (66) of the lower-dimensional theory have a direct geometrical origin by the higher-dimensional local diffeomorphism and tensor gauge symmetries.

Let us also note that part of the shift symmetries (65) may arise from the dual higher-dimensional p -forms. For example, we have noted in (16) the existence of the dual 6-form in 11D supergravity. Upon toroidal reduction on a sufficiently large torus T^d with $d \geq 6$, the associated gauge symmetries induce shift symmetries (65) on the scalar fields descending from the 6-form. This indicates that the full symmetry algebra (66) in general is only visible after taking into account all the fields together with their duals.

Furthermore, the symmetry induced by the gauge transformations (64) does not only act on scalar fields via the shift (65) but may in general also have a non-trivial action on some of the p -forms. Consider the gauge transformations (64)

$$\Lambda_{m_1 m_2}(y) = \xi_{m_1 m_2 m_3} y^{m_3} , \quad (67)$$

in 11D supergravity. While they induce the shifts (65) on the scalar fields $A_{m_1 m_2 m_3}$, they also have a nontrivial action on the p -forms descending from the dual 6-form according to its gauge transformation (19). For example, the D -dimensional 3-forms $C_{\mu\nu\rho m_1 m_2 m_3}$, transform as

$$\delta C_{\mu\nu\rho m_1 m_2 m_3} = -\frac{3}{2} C_{\mu\nu\rho} \xi_{m_1 m_2 m_3} , \quad (68)$$

and similar for the lower-rank forms.

As an illustration for the reduction of p -forms, let us perform the explicit reduction of the 3-form Lagrangian (10) of 11D supergravity. The reduction ansatz (60) identifies the D -dimensional scalars, 1-forms, 2-forms, and 3-form as

$$\begin{aligned} A_{mnk} &= C_{mnk} , \\ A_{\mu mn} &= C_{\mu mn} - A_{\mu}{}^k C_{kmn} , \\ A_{\mu\nu m} &= C_{\mu\nu m} - 2A_{[\mu}{}^n C_{\nu]mn} + A_{\mu}{}^n A_{\nu}{}^k C_{mnk} , \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho} - 3A_{[\mu}{}^m C_{\nu\rho]m} + 3A_{[\mu}{}^m A_{\nu}{}^n C_{\rho]mn} - A_{\mu}{}^m A_{\nu}{}^n A_{\rho}{}^k C_{mnk} , \end{aligned} \quad (69)$$

in terms of the components of C_{KLM} . The y -independent eleven-dimensional gauge transformations (63) translate into

$$\begin{aligned} \delta A_{\mu mn} &= \partial_{\mu} \Lambda_{mn} , \\ \delta A_{\mu\nu m} &= 2\partial_{[\mu} \Lambda_{\nu]m} - F_{\mu\nu}{}^n \Lambda_{mn} , \\ \delta A_{\mu\nu\rho} &= 3\partial_{[\mu} \Lambda_{\nu\rho]} - 3F_{[\mu\nu}{}^m \Lambda_{\rho]m} , \end{aligned} \quad (70)$$

where the lower-dimensional gauge parameters have been embedded into Λ_{MN} in analogy to (60). Likewise, the gauge invariant field strengths in D dimensions are defined via

$$F_{\mu_1 \dots \mu_k m_{k+1} \dots m_4} = 4 P_{\mu_1}^{M_1} \dots P_{\mu_k}^{M_k} \partial_{[M_1} C_{\dots M_k m_{k+1} \dots m_4]} , \quad (71)$$

and take the explicit form

$$\begin{aligned} F_{\mu nkl} &= \partial_{\mu} A_{nkl} , \\ F_{\mu\nu mn} &= 2\partial_{[\mu} A_{\nu]mn} + F_{\mu\nu}{}^k A_{kmn} , \\ F_{\mu\nu\rho m} &= 3\partial_{[\mu} A_{\nu\rho]m} + 3F_{[\mu\nu}{}^n A_{\rho]mn} , \\ F_{\mu\nu\rho\sigma} &= 4\partial_{[\mu} A_{\nu\rho\sigma]} + 6F_{[\mu\nu}{}^m A_{\rho\sigma]m} . \end{aligned} \quad (72)$$

They satisfy non-standard non-linear Bianchi identities

$$\begin{aligned} 3\partial_{[\mu} F_{\nu\rho]mn} &= 3F_{[\mu\nu}{}^k F_{\rho]kmn} , \\ 4\partial_{[\mu} F_{\nu\rho\sigma]m} &= 6F_{[\mu\nu}{}^n F_{\rho\sigma]mn} , \\ 5\partial_{[\mu} F_{\nu\rho\sigma\lambda]} &= 10F_{[\mu\nu}{}^m F_{\rho\sigma\lambda]m} . \end{aligned} \quad (73)$$

Putting everything together, the kinetic term for the 3-form (10) reduces to

$$\begin{aligned} \mathcal{L}_{\text{kin}} &= -\frac{1}{48} |E| F^{KLMN} F_{KLMN} \\ &= -\frac{1}{48} |e| \left(e^{-6\gamma\phi} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} + 4e^{-(6\gamma+2\beta)\phi} M^{mn} F^{\mu\nu\rho}{}_m F_{\mu\nu\rho n} \right. \\ &\quad \left. + 6e^{-(6\gamma+4\beta)\phi} M^{mn} M^{kl} F^{\mu\nu}{}_{mk} F_{\mu\nu nl} \right. \\ &\quad \left. + 4e^{-(6\gamma+6\beta)\phi} M^{mn} M^{kl} M^{pq} \partial^{\mu} A_{mnp} \partial_{\mu} A_{nlq} \right) , \end{aligned} \quad (74)$$

with γ and β from (46) and (52), and the scalar dependent matrix M^{mn} denoting the inverse of (55). The reduced Lagrangian provides the kinetic terms for the lower-dimensional forms. It is straightforward to check that the dilaton powers precisely ensure invariance of the action under the GL(1) scaling symmetry (52), (62). Also the invariance under constant shifts (65) is manifest.

Finally, reduction of the Chern-Simons term in (10) gives rise to a lengthy topological term in D dimensions. It is most compactly described by writing the original Chern-Simons term as the boundary contribution of some twelve-dimensional integral of

$$d\mathcal{L}_{\text{top}} = F^{(4)} \wedge F^{(4)} \wedge F^{(4)} , \quad (75)$$

and to reduce the r.h.s. of this equation in terms of the different components (72).

The straightforward toroidal reduction of 11D supergravity thus gives a lower-dimensional theory with a global symmetry group of the type (66) with semisimple part $\mathfrak{g}_0 = \mathfrak{gl}(d)$. Before we discuss the further enhancement of the global symmetry

group by hidden symmetries in the next section, let us briefly spell out the case $d = 2$, i.e., the reduction to $D = 9$ maximal supergravity.

3.4 Maximal $D = 9$ supergravity

We first consider the case $d = 2$, i.e., the reduction of 11D supergravity on a two-torus T^2 . From (46) and (52), we find the values $\beta = \frac{9}{14}$, $\gamma = -\frac{1}{7}$. From the general structure given in the previous sections, we read off the resulting Lagrangian in $D = 9$ dimensions as the sum of (54), (74), and a nine-dimensional CS term as

$$\begin{aligned} \mathcal{L}_{\text{EH}}^{(9)} = & |e| R_{(9)} - |e| \text{Tr} [P_\mu P^\mu] - \frac{7}{2} |e| \partial_\mu \phi \partial^\mu \phi \\ & - \frac{1}{4} |e| e^{3\phi} M_{mn} F_{\mu\nu}{}^m F^{\mu\nu n} - \frac{1}{8} |e| e^{-4\phi} F^{\mu\nu} F_{\mu\nu} \\ & - \frac{1}{12} |e| e^{-\phi} M^{mn} F^{\mu\nu\rho}{}_m F_{\mu\nu\rho n} - \frac{1}{48} |e| e^{2\phi} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \\ & + \mathcal{L}_{\text{top}} , \end{aligned} \quad (76)$$

where for convenience we have rescaled the dilaton as $\phi \rightarrow \frac{7}{3} \phi$ and furthermore set

$$A_{\mu mn} = A_\mu \varepsilon_{mn} , \quad F_{\mu\nu mn} = F_{\mu\nu} \varepsilon_{mn} . \quad (77)$$

As discussed above, the global symmetry group of this theory is given by

$$\text{GL}(2) = \text{SL}(2) \times \text{GL}(1) , \quad (78)$$

and there is no further symmetry enhancement, as shift symmetries of the type (65) are absent.

The remarkable property of the theory (76) is the fact that the very same theory is obtained by reducing ten-dimensional IIB supergravity on S^1 . This is consistent with the fact that only a single maximal supermultiplet exists in $D = 9$ dimensions. For the IIB reduction, the origin of the global symmetry (78) is the geometric $\text{GL}(1)$ of the circle, together with the $\text{SL}(2)$ symmetry of the IIB theory. The fields from (76) have different higher-dimensional origin according to the scheme discussed in sections 3.2, and 3.3 above, specifically

$$\begin{aligned} \text{11D : metric: } & \{g_{\mu\nu}, A_\mu{}^m, \phi, M_{mn}\} , \\ & \text{3-form: } \{A_\mu, A_{\mu\nu m}, A_{\mu\nu\rho}\} , \\ \text{IIB : metric: } & \{g_{\mu\nu}, A_\mu, \phi\} , \quad \text{scalars: } \{M_{mn}\} , \\ & \text{2-form: } \{A_\mu{}^m, A_{\mu\nu m}\} , \quad \text{4-form: } \{A_{\mu\nu\rho}\} . \end{aligned} \quad (79)$$

The presence of Chern-Simons terms in (12) and (33) is indispensable for the equivalence of the two theories after toroidal reduction. It gives rise to non-trivial

Bianchi identities for dual fields (18) akin to those appearing after dimensional reduction (73), thus allowing for the identification of fields of different higher-dimensional origin (79). A detailed discussion of $D = 9$ supergravity and its higher-dimensional embeddings is given in [41].

4 Hidden symmetries

We have seen in the previous section that toroidal reduction of higher-dimensional supergravity theories induces lower-dimensional supergravity theories with manifest global symmetries descending from higher-dimensional diffeomorphism and tensor gauge symmetries, spanning an algebra of the type (66). It is one of the most remarkable facts about these theories that on top of the geometric symmetries (66), the lower-dimensional supergravities possess further so-called hidden global symmetries which only become apparent after toroidal reduction and proper redefinition of the fields [6, 16–18].

As part of the general pattern, dubbed the ‘silver rules of supergravity’ [19], the full algebra of global symmetries of the lower-dimensional supergravity is given by the extension of (66) into a semisimple algebra

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_+ . \quad (80)$$

The hidden symmetries combine into a nilpotent algebra \mathfrak{n}_- , which completes (66) into a semisimple algebra. W.r.t. $\mathfrak{gl}(d)$, the generators of \mathfrak{n}_- transform in the representation dual to the generators of \mathfrak{n}_+ . In particular, they carry negative charge under $\mathfrak{gl}(1) \subset \mathfrak{gl}(d)$. The fact that the dimension and structure of the algebra of hidden symmetries precisely fits the expansion (80) depends of course strongly on the field content and the couplings of the higher-dimensional supergravity theory. For generic couplings, no such symmetry enhancement would occur. Already a different pre-factor in front of the Chern-Simons term (10) of 11D supergravity would prevent the symmetry enhancement in all lower-dimensional theories. This is where supersymmetry comes to play its role, although here we only focus on the bosonic sectors of theory. For example, the coefficients in (10) allowing the symmetry enhancement (80) are precisely the coefficients that were fixed by supersymmetry of the 11D action.

The global symmetry algebra (80) acts on all fields of the theory. While on the p -forms its action is necessarily linear as imposed by compatibility with gauge symmetry, the action of the hidden symmetries on the scalar fields is in general non-linear. It is most elegantly described by the isometries of the coset space

$$G/K , \quad (81)$$

with $G = \text{Lie } \mathfrak{g}$, and K its maximal compact subgroup. For maximal supergravity, the resulting global symmetry groups G build the $E_{d(d)}$ series of non-compact exceptional Lie groups in the Dynkin classification, with Dynkin diagram given in

D	G/K
9	$GL(2)/SO(2)$
8	$(SL(2) \times SL(3))/(SO(2) \times SO(3))$
7	$SL(5)/SO(5)$
6	$SO(5,5)/(SO(5) \times SO(5))$
5	$E_{6(6)}/USp(8)$
4	$E_{7(7)}/SU(8)$
3	$E_{8(8)}/SO(16)$
2	$E_{9(9)}/K(E_9)$

Table 1 Global symmetry groups G and their compact subgroups K in maximal supergravity in the various spacetime dimensions. For $D = 2$, the group $E_{9(9)}$ is the (centrally extended) affine extension of the group $E_{8(8)}$, $K(E_9)$ denotes its maximal compact subgroup.

Figure 1 above.⁶ For small values of d , the series degenerates into the classical Lie groups

$$E_{5(5)} \simeq SO(5,5), \quad E_{4(4)} \simeq SL(5), \quad E_{3(3)} \simeq SL(3) \times SL(2), \quad (82)$$

as can be extrapolated from the general Dynkin diagram of Figure 1. The full set of coset spaces is listed in Table 1.

Before going through the various cases, we first briefly review the structure of such coset spaces and their isometries.

4.1 Coset spaces

After toroidal reduction, the scalar fields of maximal supergravity theories are most conveniently described by a coset space sigma model. We have already encountered this structure in section 3.2 in the reduction of pure gravity on a torus T^d with the scalars parametrizing the target space $GL(d)/SL(d)$ according to (58). Including the higher-dimensional p -forms, this space gets enhanced by the additional scalar fields into a larger coset space

$$GL(d)/SL(d) \hookrightarrow G/K. \quad (83)$$

Here, G is the Lie group associated to the algebra \mathfrak{g} in (80), and K denotes its maximal compact subgroup. The coset space is described by a representative (or vielbein) $\mathcal{V} \in G$ with local gauge invariance

$$\delta \mathcal{V} = \mathcal{V} k(x), \quad k(x) \in \mathfrak{k}, \quad (84)$$

⁶ The subscripts in parentheses in $E_{d(d)}$ specify the particular real form of the exceptional groups. Specifically, it denotes the difference between non-compact and compact generators of the associated algebra. For maximal supergravity the global symmetry groups always appear in their split form, i.e., the maximally non-compact form of the group.

with $K = \text{Lie } \mathfrak{k}$. In analogy to (57), (58), the Lagrangian for the coset space sigma model is built by decomposing the left invariant scalar currents as

$$\mathcal{V}^{-1} \partial_\mu \mathcal{V} = Q_\mu + P_\mu, \quad (85)$$

with $Q_\mu \in \mathfrak{k}$ and $P_\mu \in \mathfrak{p}$, according to the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \quad (86)$$

The Lagrangian

$$\mathcal{L}_{\text{coset}} = -\frac{1}{2} |e| \text{Tr}(P_\mu P^\mu), \quad (87)$$

then is invariant under the gauge transformations (84), under which

$$\delta P_\mu = [P_\mu, k(x)]. \quad (88)$$

The gauge symmetry (84) can be fixed by imposing a sufficient number of conditions on the vielbein \mathcal{V} , such that it is uniquely parametrized by

$$n = \dim G - \dim K, \quad (89)$$

scalar fields, corresponding to a choice of coordinates on the target space (83). The Lagrangian (87) remains invariant under the global symmetry

$$\delta_g \mathcal{V} = g \mathcal{V} - \mathcal{V} k_g, \quad g \in \mathfrak{g}, \quad k_g \in \mathfrak{k}, \quad (90)$$

combining left multiplication on \mathcal{V} with a compensating gauge transformation (84) in order to restore the fixed gauge. The action (90) describes the infinitesimal action of the isometry group G on the n coordinates of the target space (83). In particular, it encodes the action of the global symmetry group G on the fermionic sector of the theory. Before gauge fixing the local symmetry (84), the fermions of the theory appear as singlets under G but transform under local K transformations (84). Derivatives are covariantized with the composite connection Q_μ from (85). After gauge fixing, the fermions inherit a non-trivial action of the global symmetry group G by means of the compensating \mathfrak{k} -transformation $k_g \in \mathfrak{k}$ of (90).

In the context of toroidal reduction of supergravity, a natural gauge fixing for the vielbein \mathcal{V} is the triangular gauge, in which this matrix is put to the form

$$\mathcal{V} = \exp(\phi^a N_a) V_{G_0}, \quad V_{G_0} \in G_0 = \text{Lie } \mathfrak{g}_0, \quad (91)$$

where the right factor V_{G_0} lives in the Lie group associated with the algebra of zero charge generators \mathfrak{g}_0 in (80) and the N_a denote the generators of the nilpotent algebra \mathfrak{n}_+ . The matrix V_{G_0} is built from the internal part of the higher-dimensional vielbein V_m^a introduced in (45) (together with the other scalars of the higher-dimensional theory), while the scalars ϕ^a describe the scalars descending from the higher-dimensional p -forms.

As a result, the action (90) of \mathfrak{g}_0 induces a linear action on the scalar fields ϕ^a , in accordance with the representation of the associated generators N_a . The action (90) of \mathfrak{n}_+ does not require a compensating gauge transformation, $k_{\mathfrak{n}_+} = 0$, and induces shift symmetries on the scalars ϕ^a , generating the transformations (65). In contrast, the action (90) of the hidden symmetries \mathfrak{n}_- induces a compensating gauge transformation and thereby a non-linear action on the scalar fields.

As an illustration, let us evaluate the transformation (90) for the coset space $\text{SL}(2)/\text{SO}(2)$ which appears in the matter sector of various supergravity theories. With the $\mathfrak{sl}(2)$ algebra generators given by

$$\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (92)$$

the decomposition (80) of the algebra corresponds to

$$\mathfrak{sl}(2) = \mathfrak{n}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_+ = \langle \mathbf{f} \rangle \oplus \langle \mathbf{h} \rangle \oplus \langle \mathbf{e} \rangle. \quad (93)$$

Accordingly, the matrix \mathcal{V} in triangular gauge (91) can be parametrized as

$$\mathcal{V} = \exp(C\mathbf{e}) \exp\left(\frac{1}{2}\phi\mathbf{h}\right) = \begin{pmatrix} e^{\phi/2} & e^{-\phi/2}C \\ 0 & e^{-\phi/2} \end{pmatrix}. \quad (94)$$

The action (90) then induces the transformation

$$\delta_{\mathbf{h}}\phi = 2, \quad \delta_{\mathbf{h}}C = 2C, \quad \delta_{\mathbf{e}}C = 1, \quad \delta_{\mathbf{f}}\phi = -2C, \quad \delta_{\mathbf{f}}C = e^{2\phi} - C^2, \quad (95)$$

on the scalars ϕ, C . This shows how the algebra $\mathfrak{g}_0 = \langle \mathbf{h} \rangle$ acts as a scaling symmetry on the fields, whereas $\mathfrak{n}_+ = \langle \mathbf{e} \rangle$ acts as a shift symmetry on C . The hidden symmetries in this example are generated by $\mathfrak{n}_- = \langle \mathbf{f} \rangle$, inducing a non-linear action on the scalar fields. Let us also note, that the Lagrangian (87) for this example is given by

$$\mathcal{L}_{\text{coset}} = -\frac{1}{4}|e|\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{4}|e|e^{-2\phi}\partial_{\mu}C\partial^{\mu}C, \quad (96)$$

which is invariant under the transformations (95).

This is the coset space that already appears in the ten-dimensional IIB supergravity (35) (with $C_0 = C$, $\phi \rightarrow -\phi$). In lower-dimensional supergravities, this coset space shows up, for example, in the reduction of $D = 5$ minimal supergravity on a circle S^1 . The bosonic field content of the $D = 5$ theory comprises the metric and a single vector field. Its S^1 reduction follows the scheme described in sections 3.2, 3.3, with $\gamma = -\frac{1}{2}, \beta = \frac{3}{2}$. In $D = 4$ dimensions, it gives rise to gravity coupled to two vectors and two scalar fields. According to the symmetry enhancement (80), the two scalars build an $\text{SL}(2)/\text{SO}(2)$ sigma model, given by (96). The geometric symmetries (66) in this example contain the

$$\mathfrak{gl}(1) = \mathfrak{g}_0 = \langle \mathbf{h} \rangle, \quad (97)$$

whose action (52), (62) on the scalar fields is reproduced by $-\frac{1}{2}\delta_{\mathbf{h}}$ in (95). The same formulas show that the two vector fields arising in this reduction carry charges $(+\frac{3}{2})$ and $(+\frac{1}{2})$, respectively, under $\mathfrak{gl}(1)$. Together with their dual vectors (defined via the duality equation $\tilde{F}_{\mu\nu} = \frac{1}{2}|e|\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ in $D = 4$ dimensions), they fill the spin- $\frac{3}{2}$ representation of $\text{SL}(2)$. This illustrates the fact the realization of the full symmetry group in general involves the original and the dual fields of the theory.

In a similar way, the coset space $\text{SL}(2)/\text{SO}(2)$ appears as one of the factors of the scalar target space in $D = 8$ maximal supergravity, c.f. Table 1, as we shall discuss in more detail in section 4.2 below.

Yet another example featuring the same global symmetry group $\text{SL}(2)$ comes from the S^1 reduction of four-dimensional Einstein gravity. In this case, the geometric scaling symmetry is still (97), but the shift symmetry δ_e acts on the scalar that is obtained by dualizing the three-dimensional Kaluza-Klein vector A_μ . The non-linear action of the hidden symmetry $\delta_{\mathbf{f}}$ in this example was originally discovered in [42], and this $\text{SL}(2)$ group goes under the name of the *Ehlers group*. It is instructive to spell out some details of this first example of hidden symmetries. The S^1 reduction of $D = 4$ gravity is described by an ansatz (45) with $\gamma = -1$, $\beta = 2$. According to (54), it results in a three-dimensional Lagrangian⁷

$$\mathcal{L}^{(3)} = |e|R_{(3)} - \frac{1}{2}|e|\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}|e|e^{2\phi}F_{\mu\nu}F^{\mu\nu}, \quad (98)$$

which exhibits the $\text{GL}(1)$ symmetry (52), scaling the scalar and the vector field. In three dimensions, vector fields are dual to scalar fields. This is most conveniently implemented in the Lagrangian (98) by treating $F_{\mu\nu}$ as a fundamental field (instead of the gauge potential A_μ) and implementing its Bianchi identity by means of a Lagrange multiplier C as

$$\mathcal{L}_{\text{parent}}^{(3)} = |e|R_{(3)} - \frac{1}{2}|e|\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}|e|e^{2\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\varepsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}C. \quad (99)$$

The resulting field equations for $F_{\mu\nu}$ are algebraic and can be used to eliminate this field, leading to the dual Lagrangian

$$\mathcal{L}_{\text{dual}}^{(3)} = |e|R_{(3)} - \frac{1}{2}|e|\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|e|e^{-2\phi}\partial_\mu C\partial^\mu C, \quad (100)$$

in terms of two scalar fields. This is precisely the coset space sigma model (96). After dualizing the three-dimensional vector A_μ into a scalar field, the symmetry of the Lagrangian is thus enhanced to $\text{SL}(2)$. Uplifting its action (95) back to $D = 4$ dimensions then induces a ‘hidden’ symmetry of general relativity acting on solutions with a $\text{U}(1)$ isometry.

⁷ W.r.t. the conventions of (45), (54), we have rescaled $\phi \rightarrow \phi/2$.

4.2 $D = 8, 7, 6, 5$ maximal supergravities

We have seen in section 3.4 that the reduction of 11D supergravity on a two-torus T^2 leads to maximal $D = 9$ supergravity whose global symmetries (78) do not include any hidden symmetries but are limited to the geometric symmetries \mathfrak{g}_0 in (66). Hidden symmetries appear upon reduction on larger tori.

Let us start with $D = 8$ maximal supergravity, obtained by reduction of 11D supergravity on T^3 [43]. The necessity of a symmetry enhancement in this theory can already be deduced from simple group theory considerations. As discussed above, toroidal reduction of 11D supergravity on a three-torus T^3 yields a theory with manifest geometric symmetries (66)

$$\mathfrak{gl}(3) \oplus 1_+ , \quad (101)$$

where 1_+ denotes the one-dimensional algebra \mathfrak{n}_+ generated by shifts (65) on the scalar descending from the 11D 3-form. On the other hand, it follows from the discussion of section 3.4 that the same $D = 8$ supergravity is obtained by reducing IIB supergravity on a two-torus T^2 . In this case, the higher-dimensional origin implies a global symmetry group (66)

$$\left(\mathfrak{gl}(2) \oplus \mathfrak{sl}(2) \right) \oplus 2_+ , \quad (102)$$

where $\mathfrak{sl}(2)$ is the symmetry of the IIB theory and 2_+ denotes the two-dimensional algebra \mathfrak{n}_+ generated by shifts (65) on the scalars descending from the doublet of IIB 2-forms. Thus, the full global symmetry algebra of $D = 8$ supergravity must unite both, (101) and (102). This is realized by the group

$$E_{3(3)} \simeq \mathrm{SL}(3) \times \mathrm{SL}(2) , \quad (103)$$

whose algebra admits the decompositions

$$\begin{aligned} \mathfrak{e}_{3(3)} &= \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \xrightarrow{11\mathrm{D}} 1_- \oplus \mathfrak{gl}(3) \oplus 1_+ , \\ \mathfrak{e}_{3(3)} &= \mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \xrightarrow{\mathrm{IIB}} 2_- \oplus \left(\mathfrak{gl}(2) \oplus \mathfrak{sl}(2) \right) \oplus 2_+ , \end{aligned} \quad (104)$$

embedding both, (101) and (102), in accordance with (80). With the labelling (92) of $\mathfrak{sl}(2)$ generators, the geometric symmetries (101) from 11D supergravity are identified as

$$\langle \mathbf{h} \rangle = \mathfrak{gl}(1) \subset \mathfrak{gl}(3) , \quad \langle \mathbf{e} \rangle = 1_+ . \quad (105)$$

The scalars of $D = 8$ maximal supergravity descend from the internal block of the 11D metric together with the single scalar C descending from the 11D 3-form. The symmetry enhancement (103) corresponds to the enhancement of the coset space (56) to

$$\mathrm{GL}(3)/\mathrm{SO}(3) \hookrightarrow (\mathrm{SL}(3)/\mathrm{SO}(3)) \times (\mathrm{SL}(2)/\mathrm{SO}(2)) . \quad (106)$$

In particular, the determinant of the internal metric combines with the scalar C into an $\text{SL}(2)/\text{SO}(2)$ coset space sigma model, similar to the example of minimal $D = 5$ supergravity discussed after (96) above. As another non-trivial consequence of the symmetry enhancement, the vector fields of $D = 8$ supergravity

$$\{A_\mu{}^m, A_{\mu mn}\}, \quad m, n = 1, 2, 3, \quad (107)$$

descending from the 11D metric and 3-form, respectively, combine into an $\text{SL}(2)$ doublet, i.e., span the $(3, 2)$ representation of $\text{SL}(3) \times \text{SL}(2)$. Indeed, one may check with (52), (62), and the identification (105) that these fields have opposite charges $\pm \frac{1}{2}$ under $\mathfrak{h} \in \mathfrak{sl}(2)$. Moreover, under \mathfrak{e} , acting as shift symmetry (65) on the scalar C , the vectors $A_{\mu mn}$ transform into $A_\mu{}^m$, as follows from the higher-dimensional embedding (69). The $D = 8$, 2-forms $A_{\mu\nu m}$ have zero charge under $\mathfrak{h} \in \mathfrak{sl}(2)$ and remain $\text{SL}(2)$ singlets. For the $D = 8$, 3-form, the symmetry enhancement can only be made visible upon including the dual fields. To this end, consider the 3-form $A_{\mu\nu\rho}$ descending from the 11D three form, together with its dual

$$\tilde{A}_{\mu\nu\rho} = \frac{1}{6} A_{\mu\nu\rho kmn} \epsilon^{kmn}, \quad (108)$$

descending from the 11D 6-form.⁸ Equation (62) shows that they have opposite charges $\pm \frac{1}{2}$ under \mathfrak{h} while (68) shows how they are mapped into each other under the action of \mathfrak{e} . This shows that the 3-form $A_{\mu\nu\rho}$ together with its dual, forms a doublet under the $\text{SL}(2)$. Once more, this illustrates that the realization of the full enhanced symmetry group in general involves the original and the dual fields of the theory.

For the lower-dimensional maximal supergravities, the symmetry enhancement proceeds in an analogous way. For $D = 7$ maximal supergravity [44], the geometric symmetries enhance to an $E_{4(4)} = \text{SL}(5)$ global symmetry group, with the decompositions (80) corresponding to⁹

$$\begin{aligned} \mathfrak{e}_{4(4)} &= \mathfrak{sl}(5) \xrightarrow{\text{11D}} 4'_- \oplus \mathfrak{gl}(4) \oplus 4_+, \\ \mathfrak{e}_{4(4)} &= \mathfrak{sl}(5) \xrightarrow{\text{IIB}} (3', 2)_- \oplus (\mathfrak{gl}(3) \oplus \mathfrak{sl}(2)) \oplus (3, 2)_+, \end{aligned} \quad (109)$$

respectively, depending on the higher-dimensional origin.

For $D = 6$ maximal supergravity [45], the geometric symmetries enhance to an $E_{5(5)} = \text{SO}(5, 5)$ global symmetry group, with the decompositions (80) corresponding to

$$\begin{aligned} \mathfrak{e}_{5(5)} &= \mathfrak{so}(5, 5) \xrightarrow{\text{11D}} 10'_- \oplus \mathfrak{gl}(5) \oplus 10_+, \\ \mathfrak{e}_{5(5)} &= \mathfrak{so}(5, 5) \xrightarrow{\text{IIB}} (1, 1)_{-2} \oplus (6, 2)_{-1} \oplus (\mathfrak{gl}(4) \oplus \mathfrak{sl}(2)) \oplus (6, 2)_{+1} \oplus (1, 1)_{+2}, \end{aligned} \quad (110)$$

⁸ Equivalently, in $D = 8$ dimensions the two 3-forms are related by a first-order duality equation of the form $\tilde{F}_{\mu_1 \dots \mu_4} = \frac{1}{24} |e| \epsilon_{\mu_1 \dots \mu_8} F^{\mu_5 \dots \mu_8}$, obtained by dimensional reduction of (16).

⁹ For the $\text{SL}(d)$ groups, we use the notation R' to denote the dual representation to R .

respectively, depending on the higher-dimensional origin. The shift symmetries $(6, 2)_{+1}$ and $(1, 1)_{+2}$ in the IIB decomposition are realized on the scalars descending from the IIB 2-forms and 4-form, respectively.

For $D = 5$ maximal supergravity [46], the geometric symmetries enhance to an $E_{6(6)}$ global symmetry group, with the decompositions (80) corresponding to

$$\begin{aligned} \epsilon_{6(6)} &\xrightarrow{\text{11D}} 1_{-2} \oplus 20_{-1} \oplus \mathfrak{gl}(6) \oplus 20_{+1} \oplus 1_{+2}, \\ \epsilon_{6(6)} &\xrightarrow{\text{IIB}} (5', 1)_{-2} \oplus (10, 2)_{-1} \oplus \left(\mathfrak{gl}(5) \oplus \mathfrak{sl}(2) \right) \oplus (10', 2)_{+1} \oplus (5, 1)_{+2}, \end{aligned} \quad (111)$$

respectively, depending on the higher-dimensional origin. The shift symmetry 1_{+2} is realized on the scalar descending from the 11D 6-form. Recall that this form is not present in the original 11D Lagrangian (12). I.e., after reduction of (12) to $D = 5$ dimensions, the full scalar coset space sigma model $E_{6(6)}/\text{USp}(8)$ can only be realized after dualizing the 3-form $A_{\mu\nu\rho}$, descending from the 11D 3-form, into a scalar field.¹⁰

We refer to [47] for a systematic discussion of the maximal supergravities in various dimensions, together with their symmetries, and their eleven-dimensional origin.

4.3 $D = 4$ maximal supergravity

Let us discuss in a little more detail the case of $D = 4$ maximal supergravity. Historically, this was the first example of exceptional symmetry groups appearing in supergravity theories and playing an essential role in their explicit construction. The field content of this theory is the massless $\mathcal{N} = 8$ supergravity multiplet

$$\left\{ g_{\mu\nu}, \psi_{\mu}^i, A_{\mu}^{\Lambda}, \chi^{ijk}, \phi^{ijkl} \right\}, \quad i = 1, \dots, 8, \quad \Lambda = 1, \dots, 28, \quad (112)$$

which comprises the graviton, 8 gravitinos, 28 vector fields, 56 spin-1/2 fermions, and 70 scalar fields. The complete theory was obtained in [6] by dimensional reduction of the 11D supergravity on a seven-torus T^7 and realizing the exceptional symmetry group $E_{7(7)}$. As discussed in section 3.2, the reduction of pure gravity from eleven dimensions down to $D = 4$ dimensions yields a gravitational theory with seven abelian vector fields A_{μ}^n , $n = 1, \dots, 7$, and $1 + 27$ scalar fields, parametrizing the coset space $\text{GL}(7)/\text{SO}(7)$. The dimensional reduction of the antisymmetric 3-form to $D = 4$ dimensions as described in section 3.3 gives rise to one 3-form field, seven 2-form fields, $\binom{7}{2} = 21$ vectors and additional $\binom{7}{3} = 35$ scalar fields. A priori, the field content thus looks quite different from the $\mathcal{N} = 8$ multiplet (112). Including the (normalized) $\text{GL}(1)$ charges from (52), (62), we find the four-dimensional bosonic field content

¹⁰ The corresponding $D = 5$ duality equation follows from dimensional reduction of (16).

$$\begin{aligned}
g_{\mu\nu} &: 1_0, \\
\phi &: 1_0 + 27_0 + 35_{-2} \\
A_\mu &: 7'_{+3} + 21_{+1}, \\
A_{\mu\nu} &: 7_{+4}, \\
A_{\mu\nu\rho} &: 1_{+7},
\end{aligned} \tag{113}$$

with the fields (other than the $GL(7)/SO(7)$ scalars) falling into linear $GL(7)$ representations. Reduction of the 11D Lagrangian (12) yields the sum of (54) and (74), together with the reduction of the 11D CS term. The symmetries of the resulting action span the algebra

$$\mathfrak{gl}(7) \oplus 35'_{+2}, \tag{114}$$

where the $35'_{+2}$ generators induce the shift symmetries (65) on the scalars descending from the 11D 3-form. In order to make contact with the $\mathcal{N} = 8$ supergravity multiplet, and realize the symmetry enhancement (80), we need to first dualize the 2-forms $A_{\mu\nu m}$ into scalar fields (put equivalently, we trade them for the corresponding scalars $A_{m_1 \dots m_6}$ descending from the dual 11D 6-form). With the associated shift symmetries (65), the algebra of manifest global symmetries enhances to (66)

$$\mathfrak{gl}(7) \oplus 35'_{+2} \oplus 7_{+4}. \tag{115}$$

The non-trivial algebra structure of the charged generators directly descends from the 11D gauge algebra (21).

The dynamics of the resulting set of 70 scalar fields is described by a sigma model on the coset space

$$G/K = E_{7(7)}/SU(8) \supset GL(7)/SO(7), \tag{116}$$

according to (87). The symmetry enhancement from (115) to $\mathfrak{e}_{7(7)}$ is realized along the lines discussed in the previous sections: decomposing $E_{7(7)}$ according to its $\mathfrak{gl}(1)$ grading, the algebra splits into the form (80)

$$\mathfrak{e}_{7(7)} \xrightarrow{11D} 7'_{-4} \oplus 35_{-2} \oplus \mathfrak{gl}(7) \oplus 35'_{+2} \oplus 7_{+4}, \tag{117}$$

with the generators of non-negative charge spanning (115). The negative grading generators correspond to the hidden symmetries that are present after reduction to four dimensions but have no manifest origin in 11D supergravity. An explicit parametrization of the coset space (116) in terms of the 11D fields is given in the triangular gauge (91) as

$$\mathcal{V} \equiv \exp \left[\varepsilon^{klmnpqr} A_{klmnpq} t_{(+4)r} \right] \exp \left[A_{kmn} t_{(+2)}^{kmn} \right] V_{GL(7)}. \tag{118}$$

Here, $V_{GL(7)} \in GL(7)$ is the internal block of the 11D metric (up to some power of its determinant), whereas the $t_{(+n)}$ refer to the $E_{7(7)}$ generators of positive grad-

ing in (117). For convenience, all generators are evaluated in the fundamental 56 representation. Under the global symmetry $e_{7(7)}$, the vielbein transforms as (90), inducing a non-linear action on the scalar fields. The bosonic sector of the theory can be formulated in terms of the symmetric, positive definite matrix

$$\mathcal{M}_{MN} = \mathcal{V}_M^{\Lambda} \mathcal{V}_N^{\Lambda}, \quad (119)$$

on which $e_{7(7)}$ acts by conjugation. In particular, the coset space sigma model (87) can be written as

$$\mathcal{L} = \frac{1}{8} |e| \partial_{\mu} (\mathcal{M}^{-1})^{MN} \partial^{\mu} \mathcal{M}_{MN}. \quad (120)$$

In order to realize the global $E_{7(7)}$ symmetry on the vector fields, we have to combine the 28 vector fields of (113) with their magnetic duals into the irreducible 56 of $E_{7(7)}$

$$56 \rightarrow 7_{-3} + 21'_{-1} + 21_{+1} + 7'_{+3}. \quad (121)$$

Once more, this illustrates that the realization of the full enhanced symmetry group involves the original and the dual fields of the theory. The dynamics of the vector fields is described by the $E_{7(7)}$ covariant twisted self-duality equation

$$F_{\mu\nu}{}^M = -\frac{1}{2} |e| \varepsilon_{\mu\nu\rho\sigma} \Omega^{MN} \mathcal{M}_{NK} F^{\rho\sigma K}, \quad (122)$$

for the abelian field strengths $F_{\mu\nu}{}^M = 2 \partial_{[\mu} A_{\nu]}{}^M$, and \mathcal{M}_{NK} from (119). Here, Ω^{MN} is the constant antisymmetric $E_{7(7)}$ -invariant symplectic tensor which establishes the embedding $E_{7(7)} \subset \text{Sp}(28, 28)$. In particular, the relation

$$\Omega^{KL} \mathcal{M}_{LM} \Omega^{MN} \mathcal{M}_{NP} = -\delta^K{}_P, \quad (123)$$

is necessary for consistency of (122). The full bosonic sector of maximal $D = 4$ supergravity can be compactly described by a pseudo-Lagrangian

$$\mathcal{L}_{D=4} = |e| \left(R + \frac{1}{48} \partial_{\mu} (\mathcal{M}^{-1})^{MN} \partial^{\mu} \mathcal{M}_{MN} - \frac{1}{8} \mathcal{M}_{MN} F^{\mu\nu M} F_{\mu\nu}{}^N \right), \quad (124)$$

combined with the twisted self-duality equation (122). Both, the Lagrangian (124) and equation (122) are manifestly $E_{7(7)}$ covariant. However, (124) yields only a pseudo-Lagrangian of the theory in that the twisted self-duality equation (122) does not follow from the variational principle but has to be imposed separately. Only its derivative coincides with the second-order equation for the vector fields that is obtained by variation of (124).

A standard action principle of the theory can only be spelled out after sacrificing the manifest $E_{7(7)}$ invariance and splitting the 56 vector fields into 28+28 as

$$A_{\mu}{}^M = \{A_{\mu}{}^{\Lambda}, A_{\mu\Lambda}\}. \quad (125)$$

An action can then be constructed in terms of half of the fields A_μ^Λ considered as independent propagating (electric) fields, while the $A_{\mu\Lambda}$ are defined via (122) as their on-shell (magnetic) duals [48]. This is achieved by replacing the Maxwell term of (124) by the Lagrangian

$$\mathcal{L}_{\text{vector}} = \frac{1}{4} |e| \mathcal{I}_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\mu\nu\Sigma} + \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Sigma, \quad (126)$$

in terms of the 28 abelian field strengths $F_{\mu\nu}^\Lambda$, with the symmetric kinetic matrices $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ related to the matrix \mathcal{M} from (119) as

$$\mathcal{M} \equiv - \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \quad (127)$$

in the split (125). In particular, $\mathcal{I}_{\Lambda\Sigma}$ is negative definite, such that the kinetic term in (126) comes with the correct sign. This yields a true action principle for the bosonic sector of $D=4$ maximal supergravity, whose off-shell symmetry however is reduced to a subgroup of $E_{7(7)}$ whereas the full $E_{7(7)}$ global symmetry is realized only on-shell. A similar pattern is observed in all even-dimensional maximal supergravities in $D < 10$.

Different choices for the electric/magnetic split (125) correspond to different electric frames and can be related by symplectic rotation. These give rise to different off-shell formulations of the theory. In particular, the off-shell symmetry group depends on the particular choice of the electric frame. Choosing the 28 electric vectors to be

$$7'_{+3} + 21'_{-1} : \quad A_\mu^\Lambda = \{A_\mu^m, A_\mu^{mn}\}, \quad (128)$$

among the 56 vectors (121), the symmetry of the Lagrangian (126) is given by the electric subgroup $SL(8) \subset E_{7(7)}$, with the vectors (128) spanning its irreducible 28-dimensional representation.

Finally, the 3-form in (113) is non-propagating and can be consistently set to zero in the dimensional reduction. Strictly speaking, however, its field equations only imply that its field strength is constant and may be set to an arbitrary value. Keeping this integration constant produces a one-parameter deformation of the maximally supersymmetric theory [49]. It breaks the global $E_{7(7)}$ symmetry and closer inspection shows that this integration constant is only one component of an irreducible 912-dimensional representation $E_{7(7)}$ [50]. Switching on other parameters within this representation leads to different maximally supersymmetric theories which generically have non-abelian gauge groups and matter charged under the gauge group; these deformations are the so-called gauged supergravities and may correspond to more complicated compactifications in the presence of background fluxes and/or geometric fluxes, see [51] for a review. In particular, these theories include the compactification of eleven-dimensional supergravity on the seven-sphere S^7 , which gives rise to a four-dimensional theory with compact non-abelian gauge group $SO(8)$ [52].

Let us finally note that maximal $D = 4$ supergravity can of course also be obtained by reduction of the IIB theory on a six-torus T^6 . In this case, the decomposition (80) of the algebra is given by

$$\begin{aligned} \mathfrak{e}_{7(7)} \xrightarrow{\text{IIB}} & (1,2)_{-3} \oplus (15',1)_{-2} \oplus (15,2)_{-1} \oplus \left(\mathfrak{gl}(6) \oplus \mathfrak{sl}(2) \right) \\ & \oplus (15',2)_{+1} \oplus (15,1)_{+2} \oplus (1,2)_{+3}, \end{aligned} \quad (129)$$

and the shift symmetries $(15',2)_{+1}$, $(15,1)_{+2}$, and $(1,2)_{+3}$ are realized on the scalars descending from the IIB 2-forms, 4-form and dual 6-forms, respectively.

4.4 Lower-dimensional supergravities

The symmetry enhancement and appearance of exceptional symmetry groups continues and becomes even more intricate with maximal supergravities in lower dimensions. For $D = 3$ maximal supergravity [53], the geometric symmetries enhance to an $E_{8(8)}$ global symmetry group, with the decomposition (80) corresponding to

$$\mathfrak{e}_{8(8)} \xrightarrow{\text{11D}} 8_{-3} \oplus 28'_{-2} \oplus 56_{-2} \oplus \mathfrak{gl}(8) \oplus 56'_{+1} \oplus 28_{+2} \oplus 8'_{+3}. \quad (130)$$

The shift symmetries $56'_{+1}$, and 28_{+2} are realized on the scalars descending from the 11D 3-form and dual 6-form, respectively. A new feature arising in the reduction to $D = 3$ dimensions is the fact that the realization of the full symmetry algebra requires the dualization of the 8 Kaluza-Klein vector fields $A_\mu{}^m$ (45) into scalar fields ϕ_m . We have already encountered this in the example of the $SL(2)$ Ehlers symmetry (100) which is embedded as a subgroup into the $E_{8(8)}$ of (130). The dual scalar fields encountered in $D > 3$ supergravities have all been identified among the components of the 11D dual 6-form. In contrast, the higher-dimensional interpretation of the ϕ_m is more subtle, as they should be identified with components of the 11D ‘dual graviton’ [54, 55], whose proper definition beyond the linearized theory and before dimensional reduction remains ambiguous. The shift symmetries $8'_{+3}$ in (130) act on these dual scalars ϕ_m , and the full bosonic sector of the theory is given by a gravity coupled sigma model on the coset space $E_{8(8)}/SO(16)$. For completeness, let us also note the decomposition (80) of $E_{8(8)}$ w.r.t. IIB supergravity

$$\begin{aligned} \mathfrak{e}_{8(8)} \xrightarrow{\text{IIB}} & (7,1)_{-4} \oplus (7',2)_{-3} \oplus (35',1)_{-2} \oplus (21,2)_{-1} \oplus \left(\mathfrak{gl}(7) \oplus \mathfrak{sl}(2) \right) \\ & \oplus (21',2)_{+1} \oplus (35,1)_{+2} \oplus (7,2)_{+3} \oplus (7',1)_{+4}. \end{aligned} \quad (131)$$

In the reduction to $D = 2$, the structures become even richer. Extrapolating the exceptional series of Lie algebras with Dynkin diagram given by Figure 1 defines the infinite-dimensional algebra $\mathfrak{e}_{9(9)}$ as the (centrally extended) loop algebra $\widehat{\mathfrak{e}_{8(8)}}$. This algebra naturally acts on infinite-dimensional representations which are built by the infinite towers of dual scalar fields that are defined on-shell by repeatedly

dualizing the 128 scalar fields that appear in the reduction of 11D supergravity. These physical scalars together with the infinite tower of dual potentials can be cast into the coset space $E_{9(9)}/K(E_9)$ with a well-defined action of the $e_{9(9)}$ symmetry algebra. We refer to [17, 56–61] for details.

The large global symmetry groups appearing in low dimensions have been of practical use in order to generate solutions of the higher-dimensional theories. While we have seen that part of the global symmetries (66) after toroidal reduction descends from the action of particular higher-dimensional diffeomorphisms and gauge transformations, the hidden symmetries combined in the algebra n_- in (80) do not have a direct higher-dimensional interpretation. A solution of the higher-dimensional field equations with a sufficient number of commuting Killing vector fields induces a solution of the lower-dimensional theory on which the action of the full symmetry group associated with (80) can be explicitly computed. Lifting the result back to higher dimensions then produces a genuinely new solution to the higher-dimensional field equations. It is in this context of solution generating methods in Einstein gravity that hidden symmetries have first been discovered in $D = 3$ reductions [42, 62–64], as well as in the infinite-dimensional case in $D = 2$ reductions [65–69], see [70] for a review. The larger the group of hidden symmetries, the larger is the orbit of newly generated solutions. In supergravity, the hidden symmetries from $D = 3$ reductions have been exploited as solution generating techniques for the construction of black hole and black ring solutions in higher dimensions, see e.g., [71–73].

Let us finally mention that the extrapolation to yet higher $d > 9$ leads to the over-extended and very-extended Kac-Moody algebras e_{10} and e_{11} , respectively, which each have been conjectured in different context to play a fundamental role in the full 11D supergravity [55, 74].

5 Exceptional field theory

We have seen in the preceding sections that the global symmetry groups of lower-dimensional maximal supergravity theories are only partially explained by the diffeomorphism and gauge symmetries of 11D supergravity. In particular, after dimensional reduction of (12), it is only after dualization of some of the lower-dimensional fields that the full global symmetry group $E_{d(d)}$ becomes manifest. In this final section, we briefly review the reformulation of 11D supergravity as an exceptional field theory (ExFT) [21]. As an illustration, we focus on the example of $E_{7(7)}$ ExFT [75]. In this formulation, dimensional reduction of the higher-dimensional theory directly leads to the Lagrangian (124) and equations of motion (122), in which the global exceptional symmetry $E_{7(7)}$ is manifest.

Starting from 11D supergravity, we may perform a decomposition of fields (45) and (60), as appropriate for dimensional reduction on T^7 , and rewrite the theory in terms of the various components, without however imposing (43), i.e., keeping the full eleven-dimensional coordinate dependence of all fields. This is akin to a Kaluza-

Klein compactification of the 11D theory in which all the Kaluza-Klein towers of massive fields are kept. From the four-dimensional point of view, this corresponds to a theory with infinitely many fields, compactly encoded in the dependence of all fields on the internal coordinates y^m . As a standard structure of Kaluza-Klein theory, the resulting theory comes with an infinite-dimensional non-abelian gauge structure, corresponding to the diffeomorphisms and gauge symmetries on the internal space. After dualization of fields, following the steps of section 4.3,¹¹ this leads to a formulation of the bosonic field content in terms of the $E_{7(7)}$ fields

$$\{g_{\mu\nu}, \mathcal{M}_{MN}, \mathcal{A}_\mu^M\}, \quad \mu, \nu = 0, \dots, 3, \quad M = 1, \dots, 56, \quad (132)$$

in alignment with the field content of $D = 4$ maximal supergravity, except for all fields keeping their dependence on the internal coordinates y^m . In particular, \mathcal{M}_{MN} still is a matrix of type (119), representing the coset space $E_{7(7)}/SU(8)$. Its vielbein (119) is parametrized as (118) in terms of the 11D fields. On top of these fields, the formulation requires 2-forms of the type

$$\{\mathcal{B}_{\mu\nu\alpha}, \mathcal{B}_{\mu\nu M}\}, \quad \alpha = 1, \dots, 133, \quad (133)$$

also depending on all coordinates. Here, α is an index in the adjoint representation of $E_{7(7)}$, while the 56 2-forms $\mathcal{B}_{\mu\nu M}$ satisfy algebraic constraints

$$(t_\alpha)_K^M \Omega^{NK} \mathcal{B}_{\mu\nu M} \mathcal{B}_{\rho\sigma N} = 0 = \Omega^{MN} \mathcal{B}_{\mu\nu M} \mathcal{B}_{\rho\sigma N}, \quad (134)$$

where $(t_\alpha)_K^M$ denote the generators of the algebra $\mathfrak{e}_{7(7)}$.

The 11D field equations can be written in terms of the objects (132), (133). Remarkably, the resulting equations can equivalently be derived from first principles based on the infinite-dimensional gauge structure of the theory. To this end, the internal coordinates are embedded into an extended spacetime with coordinates transforming in the fundamental 56 of $E_{7(7)}$. The original physical coordinates are recovered as the solution of an $E_{7(7)}$ -covariant section constraint. On the extended spacetime, the original diffeomorphisms and gauge symmetries are unified into generalized diffeomorphisms [77–86], which provide the organizing structure for the construction of the theory.

Specifically, the action of generalized diffeomorphisms on the scalar matrix \mathcal{M}_{MN} is of the form [81, 82]

$$\delta_\Lambda \mathcal{M}_{MN} = \mathcal{L}_\Lambda \mathcal{M}_{MN} = \Lambda^K \partial_K \mathcal{M}_{MN} + 24 \partial_L \Lambda^K \mathbb{P}^K_L{}^P{}_{(M} \mathcal{M}_{N)P}. \quad (135)$$

¹¹ In the reduction to $D = 4$ discussed in section 4.3, dualization always refers to abelian vectors and p -forms. Here, the non-abelian structure related to the dependence on internal coordinates does not pose an obstruction to the dualization but can be compensated by Stückelberg-type couplings to higher degree forms, as is common in gauged supergravity [51, 76]. In particular, this is the underlying reason for extending the bosonic field content to (133).

Here, $\mathbb{P}^K{}_L{}^P{}_M$ is the projector on the adjoint representation of $E_{7(7)}$, which takes the explicit form¹²

$$\begin{aligned}\mathbb{P}^K{}_M{}^L{}_N &= (t_\alpha)_M{}^K (t^\alpha)_N{}^L \\ &= \frac{1}{24} \delta_M{}^K \delta_N{}^L + \frac{1}{12} \delta_M{}^L \delta_N{}^K + (t_\alpha)_{MN} (t^\alpha)^{KL} - \frac{1}{24} \Omega_{MN} \Omega^{KL} .\end{aligned}\quad (136)$$

The 56 components of the gauge parameter Λ^M in (135) combine the gauge symmetries associated with the vector fields (121). I.e., 7 + 21 of the gauge symmetries descend from the 11D diffeomorphisms and tensor gauge transformations, respectively, while the other half is associated with the dual vector fields.

The coordinate dependence of all fields and gauge parameters is constrained by the so-called section constraint. The latter imposes an embedding of the physical coordinates $\partial_m \leftrightarrow \partial_M$ imposing that every couple of fields (Φ_1, Φ_2) satisfies

$$(t_\alpha)_K{}^M \Omega^{NK} \partial_M \Phi_1 \partial_N \Phi_2 = 0 = \Omega^{MN} \partial_M \Phi_1 \partial_N \Phi_2 , \quad (137)$$

similar to (134). This is an $E_{7(7)}$ -covariant way of stating that only 7 out of the formally 56 derivatives ∂_M appearing in (135) actually have a non-trivial action, as compatible with the eleven-dimensional nature of the original theory. It is straightforward to check that the algebra of generalized diffeomorphisms (135) only closes under the assumption of (137).

Not unexpectedly, there are in fact two inequivalent maximal solutions to (137) that restrict the dependence of all fields to 7 and 6 coordinates, respectively. They correspond to 11D and IIB supergravity, respectively, which are thus both embedded into the same exceptional field theory. Specifically, the solutions to the section constraint (137) are based on the decompositions (117) and (129), respectively, of $\mathfrak{e}_{7(7)}$ and correspond to the embedding of internal coordinates $\partial_m \leftrightarrow \partial_M$ realized according to¹³

$$\begin{aligned}\text{11D} : \mathfrak{e}_{7(7)} &\longrightarrow \mathfrak{gl}(7) \\ &56 \longrightarrow \boxed{7_{-3}} + 21'_{-1} + 21_{+1} + 7'_{+3} , \\ \text{IIB} : \mathfrak{e}_{7(7)} &\longrightarrow \mathfrak{gl}(6) \oplus \mathfrak{sl}(2) \\ &56 \longrightarrow \boxed{(6, 1)_{-4}} + (6', 2)_{-2} + (20, 1)_0 + (6, 2)_{+2} + (6', 1)_{+4} .\end{aligned}\quad (138)$$

Invariance under local gauge transformations (135) in ExFT is implemented by covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu - \mathcal{L}_{\mathcal{A}_\mu} , \quad (139)$$

¹² For the raising and lowering of symplectic indices, we use North-West South-East conventions, i.e., $Z^M = \Omega^{MN} Z_N$ and $Z_M = Z^N \Omega_{NM}$.

¹³ The fact that these embeddings provide solutions to (137) can immediately be inferred from the $\mathfrak{gl}(1)$ gradings.

with the vector fields \mathcal{A}_μ^M from (132) transforming as

$$\begin{aligned}\delta_\Lambda \mathcal{A}_\mu^M &= \partial_\mu \Lambda^M - \mathcal{A}_\mu^K \partial_K \Lambda^M + 12 \partial_K \mathcal{A}_\mu^L \mathbb{P}^K{}_{L^M} \Lambda_\mu^N - \frac{1}{2} \Lambda^M \partial_K \mathcal{A}_\mu^K \\ &= \mathcal{D}_\mu \Lambda^M.\end{aligned}\quad (140)$$

In turn, the gauge covariant field strength is given by

$$\begin{aligned}\mathcal{F}_{\mu\nu}^M &= 2 \partial_{[\mu} \mathcal{A}_{\nu]}^M - 2 \mathcal{A}_{[\mu}^K \partial_K \mathcal{A}_{\nu]}^M - 12 (t^\alpha)^{MK} (t^\alpha)_{NL} \mathcal{A}_{[\mu}^N \partial_K \mathcal{A}_{\nu]}^L \\ &\quad - \frac{1}{2} \Omega^{MK} \mathcal{A}_{[\mu}^N \partial_K \mathcal{A}_{\nu]}^N - 12 (t^\alpha)^{MN} \partial_N \mathcal{B}_{\mu\nu\alpha} - \frac{1}{2} \Omega^{MN} \mathcal{B}_{\mu\nu N}.\end{aligned}\quad (141)$$

While the non-abelian part of the field strength immediately follows from the algebra of generalized diffeomorphisms (135), the Stückelberg-type couplings to the 2-forms of (133) are required by gauge covariance since the algebra is a Leibniz rather than a Lie algebra.

The dynamics of $E_{7(7)}$ ExFT is described by a twisted self-duality equation for these non-abelian field strengths which directly generalizes the corresponding equation of the $D = 4$ theory (122)

$$\mathcal{F}_{\mu\nu}^M = -\frac{1}{2} |e| \varepsilon_{\mu\nu\rho\sigma} \Omega^{MN} \mathcal{M}_{NK} \mathcal{F}^{\rho\sigma K}.\quad (142)$$

Similar to the $D = 4$ theory, the remaining field equations of $E_{7(7)}$ ExFT are obtained from a pseudo-action whose Lagrangian is directly modeled after (124) as

$$\begin{aligned}\mathcal{L}_{\text{ExFT7}} &= |e| \left(\mathcal{R} + \frac{1}{48} g^{\mu\nu} \mathcal{D}_\mu (\mathcal{M}^{-1})^{MN} \mathcal{D}_\nu \mathcal{M}_{MN} - \frac{1}{8} \mathcal{M}_{MN} \mathcal{F}^{\mu\nu M} \mathcal{F}_{\mu\nu}^N \right) \\ &\quad + \mathcal{L}_{\text{top}} - |e| V(g, \mathcal{M}),\end{aligned}\quad (143)$$

upon introducing an internal coordinate dependence for all fields and rendering all terms invariant under the action of generalized diffeomorphisms (135), (140).

Here, the Einstein-Hilbert term is constructed from the modified Ricci scalar \mathcal{R} , constructed from the external metric $g_{\mu\nu}$ however using covariant derivatives

$$\mathcal{D}_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \mathcal{A}_\mu^K \partial_K g_{\nu\rho} - \partial_K \mathcal{A}_\mu^K g_{\nu\rho}.\quad (144)$$

Similarly, the scalar kinetic term in (143) is a gauged sigma model with covariant derivatives defined by (139), (135). The Yang-Mills term is built from the field strengths (141), while the non-abelian topological term is most compactly defined as the boundary contribution of a five-dimensional integral over

$$d\mathcal{L}_{\text{top}} \propto \Omega_{MN} \mathcal{F}^M \wedge \mathcal{D} \mathcal{F}^N,\quad (145)$$

with the covariant derivative $\mathcal{D} \mathcal{F}^N$ defined as in (140). Finally, the potential term $V(g, \mathcal{M})$ in (143) is given by

$$\begin{aligned}
V(g, \mathcal{M}) = & -\frac{1}{48} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} + \frac{1}{2} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK} \\
& -\frac{1}{4} |e|^{-1} \partial_M |e| \partial_N \mathcal{M}^{MN} - \frac{1}{16} \mathcal{M}^{MN} |e|^{-2} \partial_M |e| \partial_N |e| \\
& -\frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\nu} \partial_N g_{\mu\nu} .
\end{aligned} \tag{146}$$

Although not manifest, this term can be shown to be invariant under generalized diffeomorphisms (135) up to total derivatives. It may be thought of as the analogue of a Ricci scalar on the extended internal spacetime [81].

By construction, all terms in (143) are separately invariant under generalized diffeomorphisms (135), (140). Moreover, the relative coefficients among these terms are uniquely fixed by further demanding invariance under properly defined external diffeomorphisms. The latter are generated by vectors ξ^μ , which however also depend on the internal coordinates. Invariance under generalized internal and external diffeomorphisms thus uniquely fixes the dynamics of $E_{7(7)}$ ExFT.

After picking a solution (138) of the section constraint, the field equations obtained from (142) and (143) precisely reproduce the field equations of 11D and IIB supergravity, respectively. Moreover, also massive IIA supergravity, c.f. (29), can be reproduced upon further deformation of the gauge structures [87, 88]. All these theories are thus united within a common framework. Moreover, after dimensional reduction $\partial_M \rightarrow 0$ this formulation directly yields the $D = 4$ theory (122), (124), with the global symmetry group $E_{7(7)}$ manifest. This makes ExFT a natural framework for the study of duality and solution generating transformations, see e.g., [89–91]. Moreover, with the higher-dimensional fields already re-arranged such as to fit the fields of the lower-dimensional theory, $E_{7(7)}$ ExFT (143) is precisely tailored for the study of reductions to $D = 4$ dimensions. It has been a particularly powerful tool for the construction of consistent truncations [92, 93] and the computation of Kaluza-Klein spectra around four-dimensional backgrounds [94].

Exceptional field theories have been constructed for all finite-dimensional duality groups $E_{d(d)}$ (i.e., for $d \leq 8$) [21, 36, 75, 95–99]. Just as (143), the respective actions are modeled after the structure of the $(11 - d)$ -dimensional maximal supergravities, lifting all fields to an extended spacetime (subject to the section constraint), with the non-abelian gauge structure induced by the infinite-dimensional algebraic structure of generalized diffeomorphisms. For $d > 8$, the exceptional field theory based on the infinite-dimensional affine algebra $\mathfrak{e}_{9(9)}$ has been constructed in [100, 101]. Extrapolating the structures all the way to the very-extended Kac-Moody algebra \mathfrak{e}_{11} , a master formulation has been given in [102]. This also allows to make contact with the E_{11} conjectures of [55, 103, 104] and the E_{10} conjecture of [74].

Let us finally note that although we have restricted here to a discussion of the bosonic sector, the ExFT construction can be extended to the fermionic sector in a unique way such that supersymmetry can be realized [83, 105–108]. It is however interesting to note that, as stated above, in this framework, the bosonic sector is already uniquely determined by imposing purely bosonic symmetries, the generalized internal and external diffeomorphisms. This of course is directly related to the observation discussed after (80) that the appearance of hidden symmetries relies

on the exact bosonic couplings determined by supersymmetry. Why supersymmetry precisely selects the couplings that give rise to the symmetry enhancement and the exceptional groups still remains somewhat of a mystery and continues to challenge our understanding of the fundamental symmetries of supergravity.

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