Twist-four gravitational form factor $\bar{C}_{q,g}$ at NNLO QCD from trace anomaly constraints

Kazuhiro Tanaka (Juntendo U)

KT, JHEP 03 ('23) 013, arXiv:2212.09417

(Belinfante-improved) energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi \ + \ F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^{2} \ + \ (\text{ghost}) \ + \ (\text{gauge fix})$$

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)Mg^{\mu\nu}\Big]u(p)$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

$$t = \Delta^2$$

$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{\langle N(p)\,S|J^i|N(p)\,S\rangle}{\langle N(p)\,S|N(p)\,S\rangle} = \frac{1}{2}S^i$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$B_q(0) + B_g(0) = 0$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

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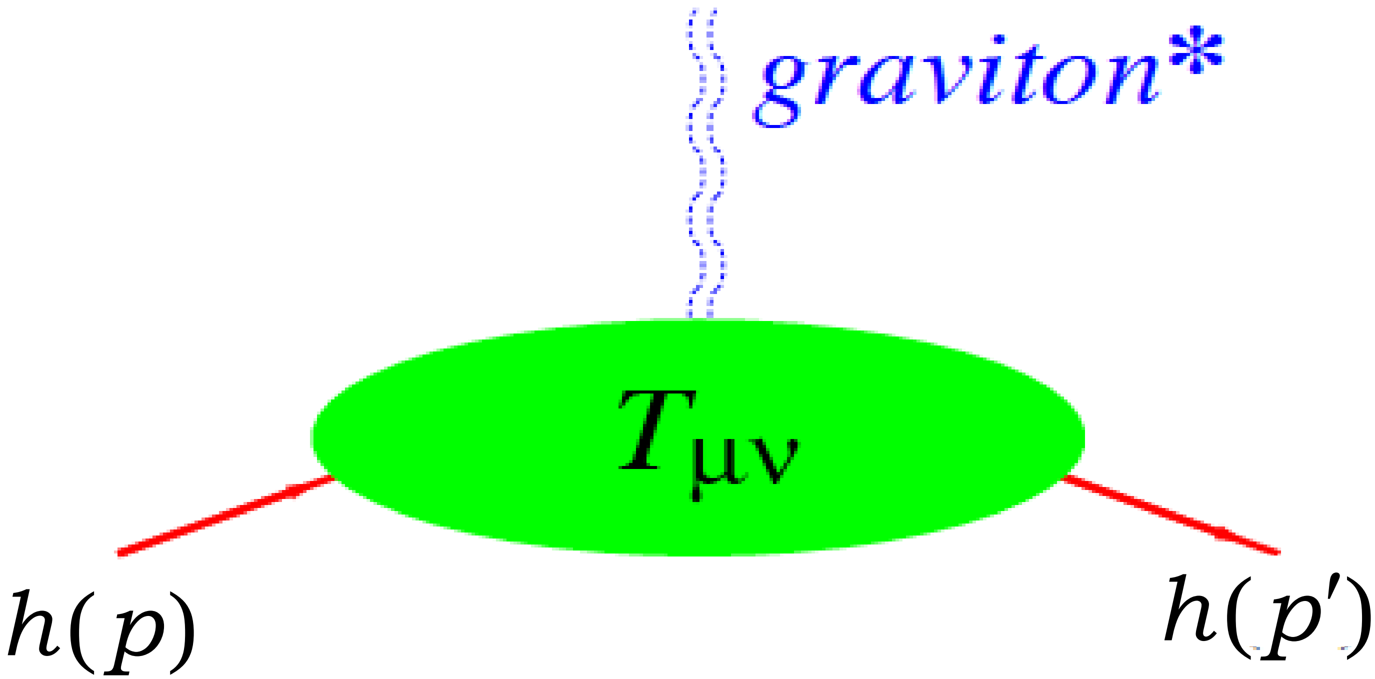

$$T^{\mu\nu} = \frac{1}{2}\overline{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi \; + \; F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^{2}$$

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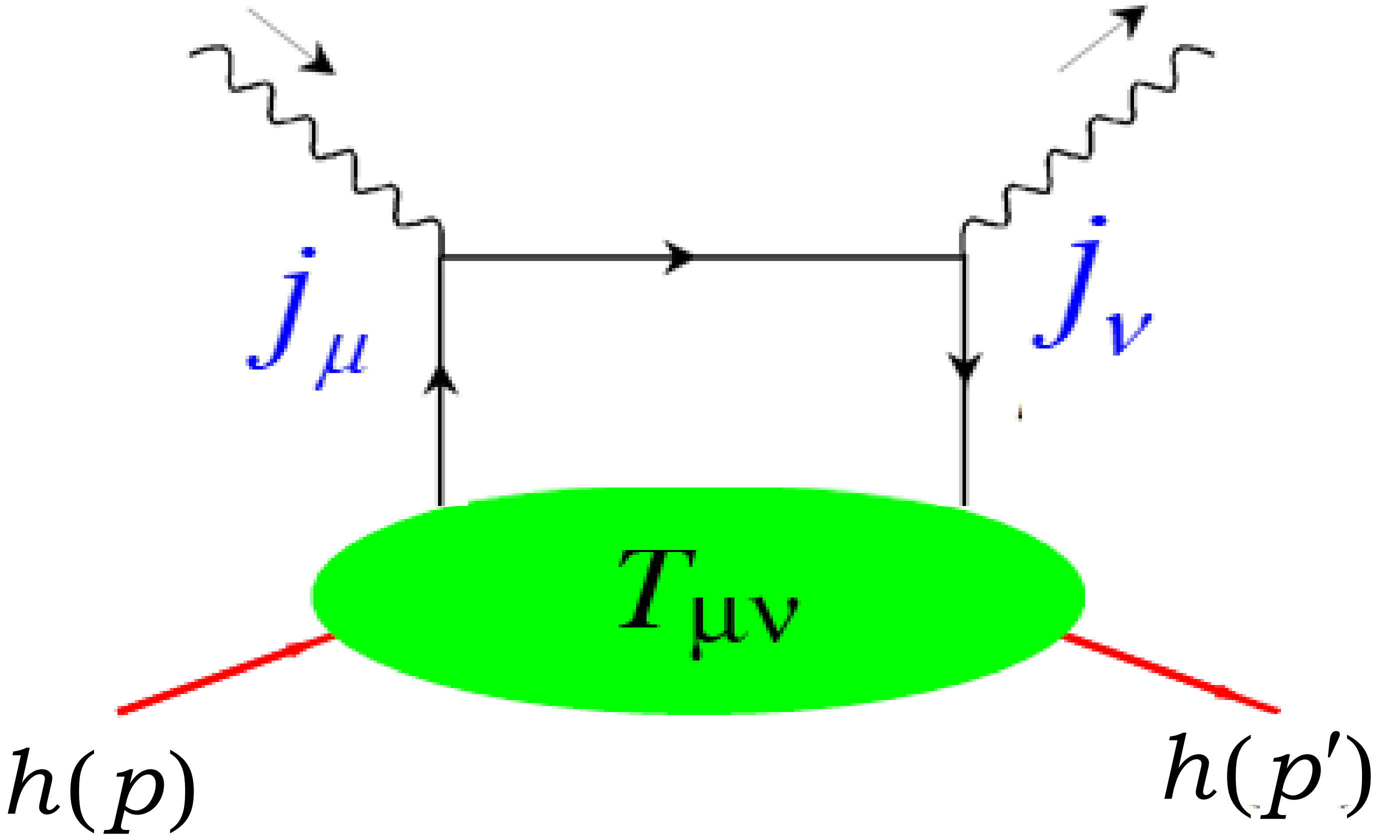

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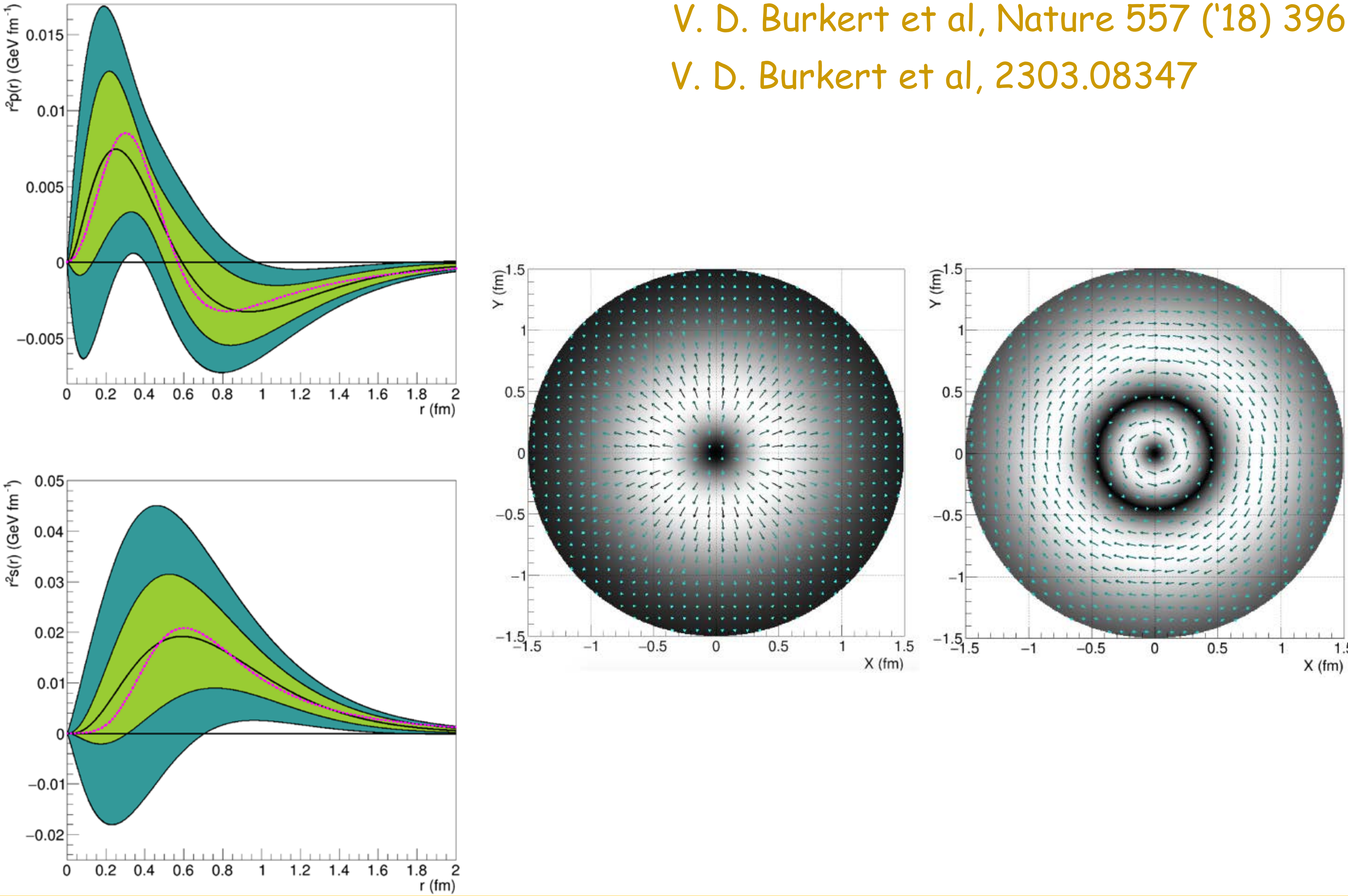
V. D. Burkert et al, Nature 557 ('18) 396

V. D. Burkert et al, 2303.08347



$$\langle N(p')|T^{ik}(0)|N(p)\rangle \sim \left(\Delta^i\Delta^k - \delta^{ik}\boldsymbol{\Delta}^2\right)D(t), \quad T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}\,p(r)$$

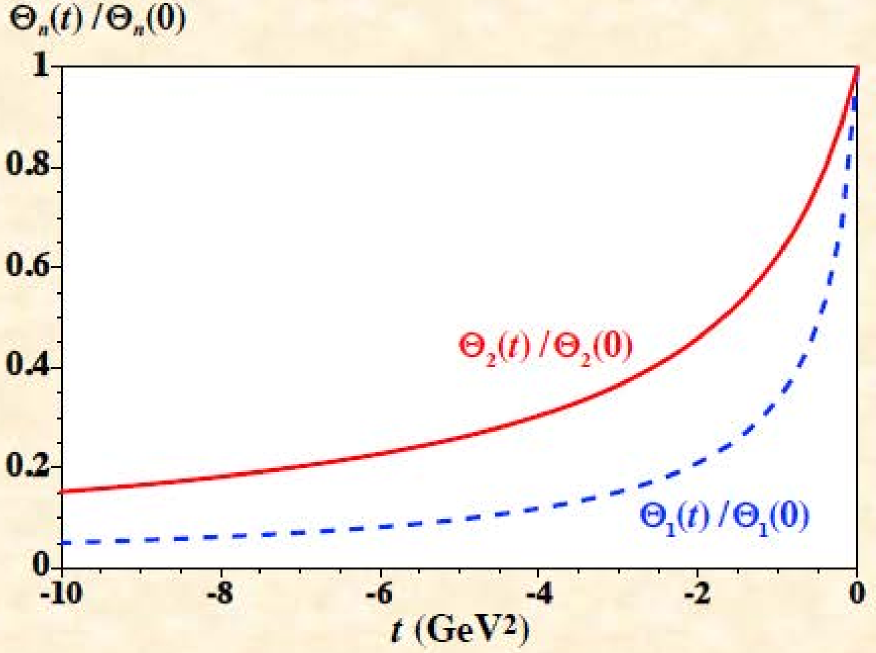
Spacelike gravitational form factors and radii for pion

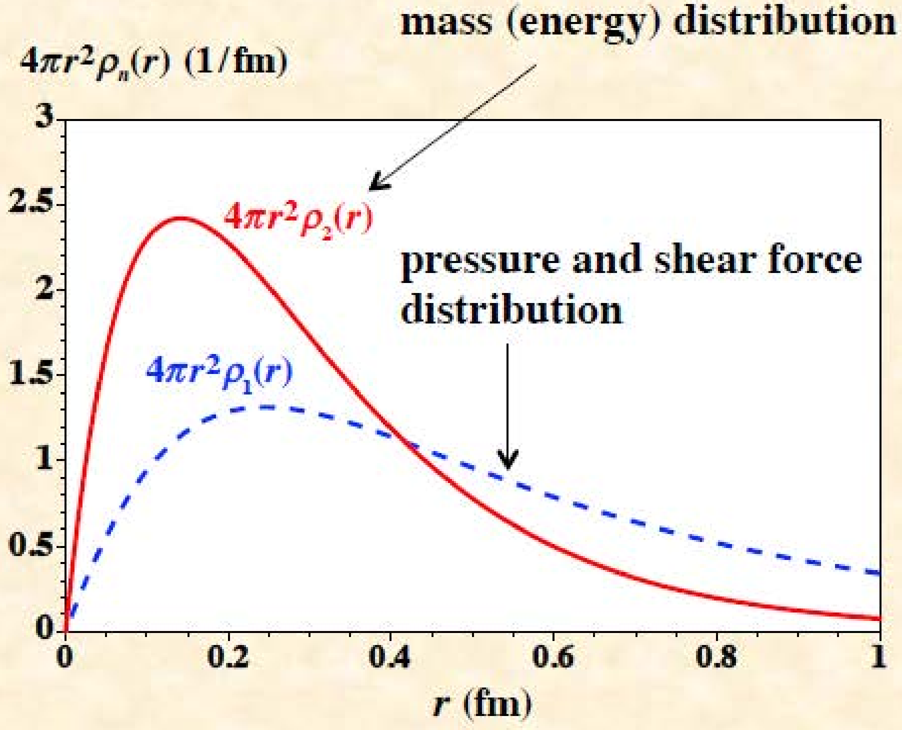
$$F(s)=\Theta_1(s),\ \Theta_1(s),\quad F(t)=\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}\,F(s)}{\pi(s-t-i\varepsilon)},\quad \rho(r)=\frac{1}{(2\pi)^3}\int d^3q e^{-i\vec{q}\cdot\vec{r}}F(q)=\frac{1}{4\pi^2}\frac{1}{r}\int_{4m_\pi^2}^{\infty} ds\, e^{-\sqrt{s}r}\,\mathrm{Im}\,F(s)$$

This is the first report on gravitational radii of hadrons from actual experimental measurements.

$\sqrt{\langle r^2\rangle_{mass}}=0.32\sim0.39$ fm, $\sqrt{\langle r^2\rangle_{mech}}=0.82\sim0.88$ fm ← First finding on gravitational radius from actual experimental measurements

$\Leftrightarrow\ \sqrt{\langle r^2\rangle_{charge}}=0.672\pm0.008$ fm





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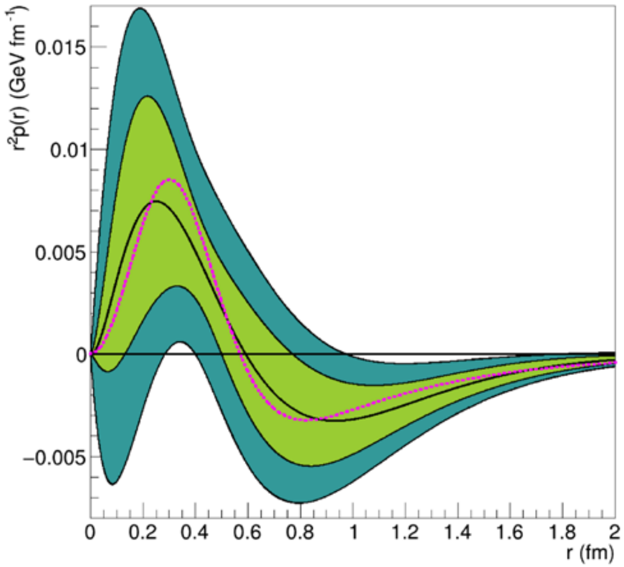

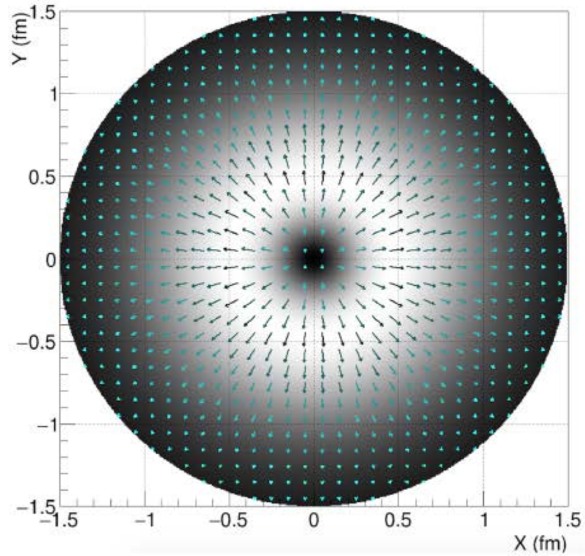

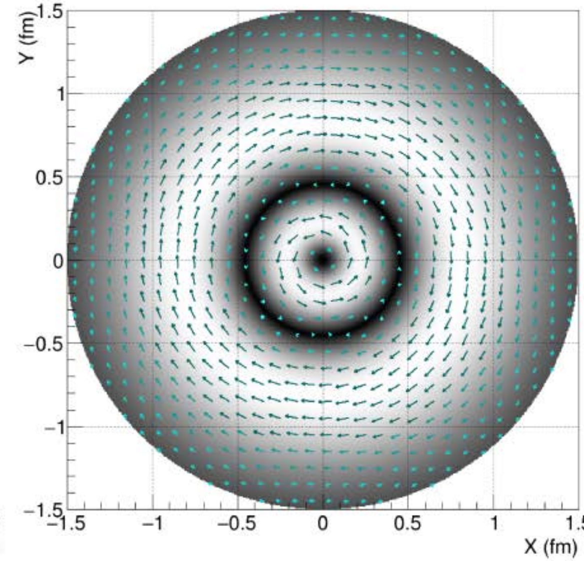

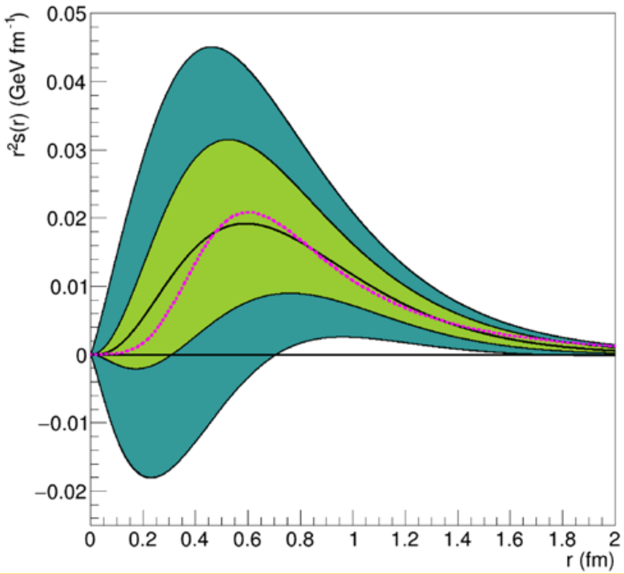

$$\langle N(p')\,|\,T^{ik}(0)\,|\,N(p)\rangle \sim \left(\Delta^i\Delta^k - \delta^{ik}\boldsymbol{\Delta}^2\right)D(t),\quad T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}\,p(r)$$

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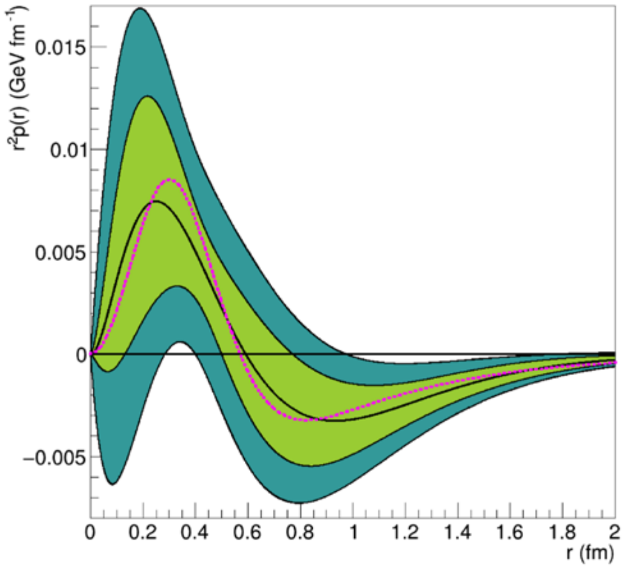
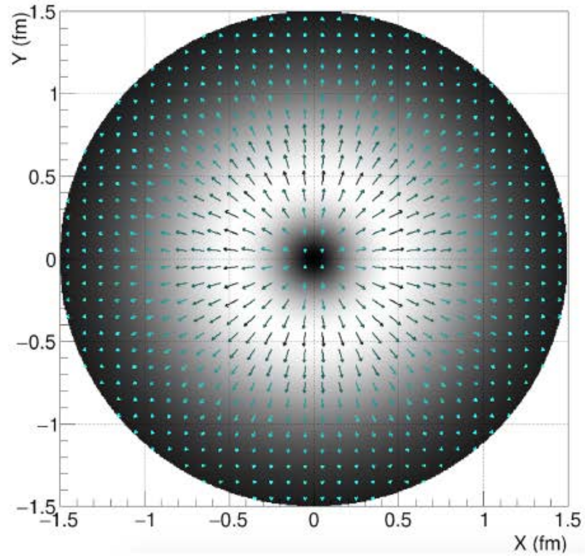
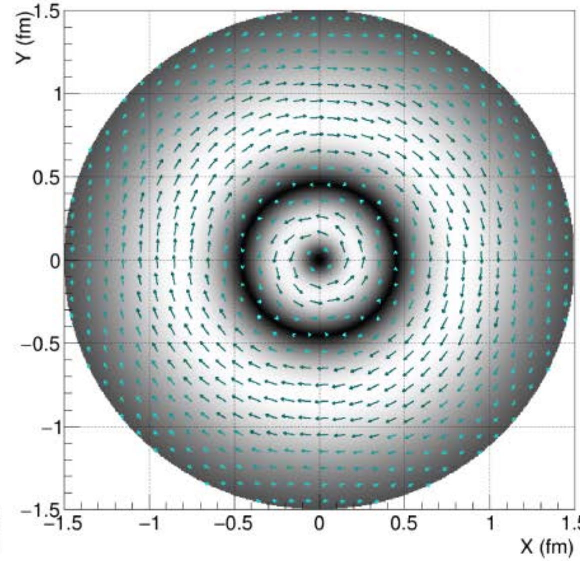
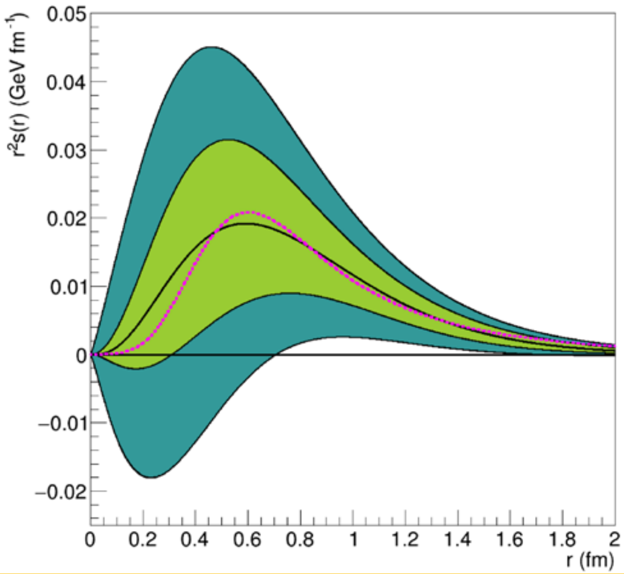








$$\langle N(p')|T_q^{ik}(0)|N(p)\rangle \sim \left(\Delta^i\Delta^k - \delta^{ik}\boldsymbol{\Delta}^2\right)D_q(t) - 4M^2\delta^{ik}\bar{C}_q(t)$$

① mass decomposition

Ji, PRD52 271 ('95)

Lorce, EPJC78, 120 ('18)

Metz, Pasquini, Rodini, PRD102, 114042 ('20)

Ji, Liu, Schafer, NPB971, 115537 ('21)

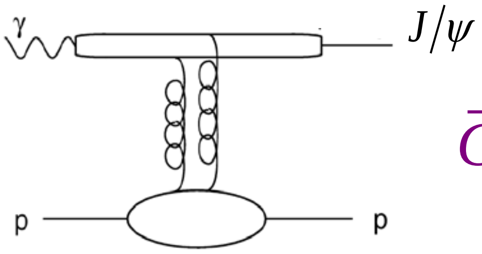
Liu, PRD104, 076010 ('21)

② nucleon's transverse spin sum rule

Hatta, KT,Yoshida, JHEP 02 ('13) 003

$$J_{q,g} = \frac{1}{2}(A_{q,g} + B_{q,g}) + \frac{p^3}{2(p^0 + M)}\bar{C}_{q,g}$$

③ $\gamma p \to J/\psi\ p$ near threshold JLab, EIC



$\bar{C}_g \ \left(= -\bar{C}_q\right)$

Y. Hatta, D. Yang, PRD98, 074003

Y. Hatta, A. Rajan, D. Yang, PRD100, 014032

Studies for $\bar{C}_{q,g}$ themselves

QCD EOMs $(i\not{D}-m)\psi=0, \quad D_\nu F^{\mu\nu}=g\bar{\psi}\gamma^\mu\psi$

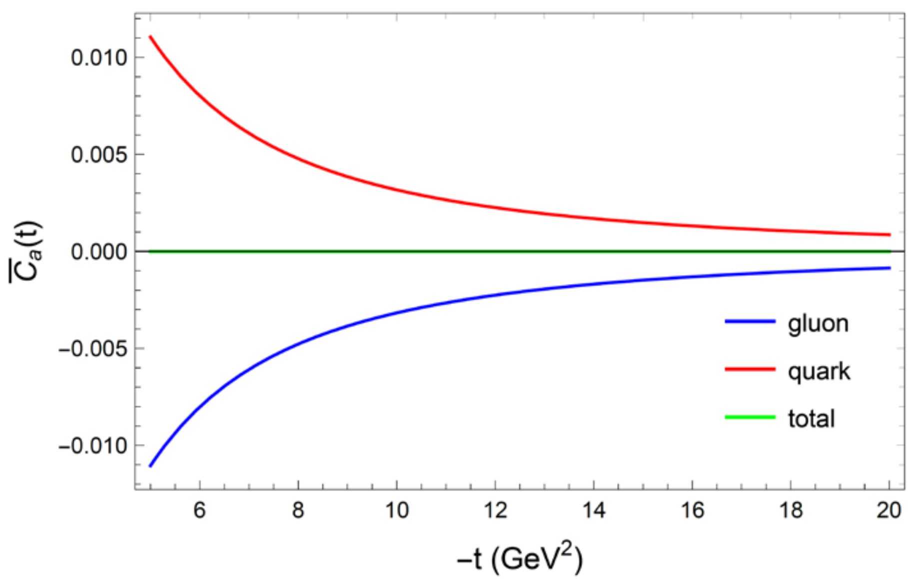
$$\partial_\nu T_q^{\mu\nu}=-\bar{\psi}gF^{\mu\nu}\gamma_\nu\psi, \quad \partial_\nu T_g^{\mu\nu}=-F_a^{\mu\nu}D_{ab}^{\rho}F_{\rho\nu}^{b}$$

KT, PRD98, 034009 ('18)

$$\Delta^\mu\bar{u}(p')u(p)M\bar{C}_q(t)=\langle N(p')|\bar{\psi}igF^{\mu\nu}\gamma_\nu\psi|N(p)\rangle$$

$$\Delta^\mu\bar{u}(p',S')u(p,S)M\bar{C}_g(t)=\langle N(p')|F_a^{\mu\nu}iD_{ab}^{\rho}F_{\rho\nu}^{b}|N(p)\rangle$$

pQCD for large t



Tong, Ma, Yuan, PLB823, 136751 ('21)

Tong, Ma, Yuan, JHEP10, 046 ('22)

at $t = 0$:

$\bar{C}_q(0, \mu \sim 0.4 \text{ GeV}) = 0.25$ **Bag model** [Ji, Melnitchouk, Song, PRD56, 5511 ('97)]

$\bar{C}_q(0, \mu = 2 \text{ GeV}) \approx -0.11$ **Phenomenological** [Lorce, EPJC78, 120 ('18)]

$\bar{C}_q(0, \mu \sim 0.63 \text{ GeV}) = 0.014$ **Instanton** [Polyakov, Son, JHEP09, 156 ('18)]

$\bar{C}_q(0, \mu = 1 \text{ GeV}) = -0.021 \pm 0.008$ **LCSR** [Azizi,Ozdem, EPJC80, 104 ('20)]

$\bar{C}_q(0, \mu \to \infty) \simeq -0.15$ **Trace anomaly** [Hatta, Rajan, KT, JHEP12, 008 ('18)]

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$g_{\mu\nu}T^{\mu\nu}_{q,g}$ **1&2-loop** Hatta, Rajan, KT, JHEP 12 ('18) 008

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$g_{\mu\nu}T^{\mu\nu}_{q,g}$

1&2-loop Hatta, Rajan, KT, JHEP 12 ('18) 008

3-loop (& all orders) KT, JHEP 01 ('19) 120

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$g_{\mu\nu}T_{q,g}^{\mu\nu}$	**1&2-loop**	Hatta, Rajan, KT, JHEP 12 ('18) 008
	3-loop (& all orders)	KT, JHEP 01 ('19) 120
	4-loop	Ahmed, Chen, Czakon, JHEP 01 ('23) 077

trace anomaly separately for q, g

Hatta, Rajan, KT, JHEP 12 ('18) 008

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{\psi}\psi + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi + \frac{1}{3}n_f F^2\right)$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{\psi}\psi - \frac{11}{6}C_A F^2\right)$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$C_A = N_c\ , \quad C_F = \frac{N_c^2-1}{2N_c}$$

trace anomaly separately for q, g $\overline{\text{MS}}$ scheme Hatta, Rajan, KT, JHEP

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{\psi}\psi + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right]$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{\psi}\psi - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right]$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad C_A = N_c\,, \quad C_F = \frac{N_c^2-1}{2N_c}$$

trace anomaly separately for q, g $\overline{\text{MS}}$ scheme   KT, JHEP01 ('19)120

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{\psi}\psi + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{\psi}\psi + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{64\zeta(3)}{9} - \frac{8305}{729}\right)C_F^2 - \frac{2}{243}(864\zeta(3)+1079)C_A C_F\right) - \frac{8}{729}(972\zeta(3)+143)C_A C_F^2\right.\right.$$
$$\left.+\left(\frac{32\zeta(3)}{9} + \frac{6611}{729}\right)C_A^2 C_F - \frac{76}{243}C_F n_f^2 + \frac{8}{729}(648\zeta(3)-125)C_F^3\right\}m\bar{\psi}\psi$$
$$\left.+\left\{n_f\left(\left(\frac{52\zeta(3)}{9} - \frac{401}{324}\right)C_A C_F + \left(\frac{134}{27} - 4\zeta(3)\right)C_A^2 + \left(\frac{2407}{1458} - \frac{16\zeta(3)}{9}\right)C_F^2\right) + n_f^2\left(-\frac{697C_A}{729} - \frac{169C_F}{1458}\right)\right\}F^2\right]$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{\psi}\psi - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{\psi}\psi + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right]$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^3\left[\left\{n_f\left(\left(\frac{368\zeta(3)}{9} - \frac{25229}{729}\right)C_F^2 - \frac{2}{243}(4968\zeta(3)+1423)C_A C_F\right) + \left(\frac{32\zeta(3)}{3} - \frac{91753}{1458}\right)C_A C_F^2\right.\right.$$
$$\left.+\left(\frac{294929}{1458} - \frac{32\zeta(3)}{9}\right)C_A^2 C_F - \frac{554}{243}C_F n_f^2 + \left(\frac{95041}{729} - \frac{64\zeta(3)}{9}\right)C_F^3\right\}m\bar{\psi}\psi$$
$$\left.+\left\{n_f\left(\left(\frac{1123}{162} - \frac{52\zeta(3)}{9}\right)C_A C_F + \left(4\zeta(3) + \frac{293}{36}\right)C_A^2 + \frac{16}{729}(81\zeta(3)-98)C_F^2\right) + n_f^2\left(\frac{655C_A}{2916} - \frac{361C_F}{729}\right) - \frac{2857C_A^3}{108}\right\}F^2\right]$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad C_A = N_c\,, \qquad C_F = \frac{N_c^2-1}{2N_c}$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)Mg^{\mu\nu}\right]u(p)$$

$$g_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|g_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_\rho{}^\nu + \frac{g^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p)\left[A_{q,g}(0)\gamma^{(\mu}p^{\nu)} + \bar{C}_{q,g}(0)Mg^{\mu\nu}\right]u(p)$$

$$g_{\mu\nu}\langle N(p)|T^{\mu\nu}_{q,g}|N(p)\rangle = 2M^2\left(A_{q,g}(0) + 4\bar{C}_{q,g}(0)\right)$$

$$\bar{C}_q(0)\ \left(= -\bar{C}_g(0)\right) = -\frac{1}{4}A_q(0) + \frac{1}{8M^2}\langle N(p)|g_{\mu\nu}T^{\mu\nu}_q|N(p)\rangle$$

$$T^{\mu\nu} = \boxed{\frac{1}{2}\bar{\psi}\gamma^{(\mu}i\overleftrightarrow{D}^{\nu)}\psi} + \boxed{F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2} \equiv T^{\mu\nu}_q + T^{\mu\nu}_g$$

$$g_{\mu\nu}T^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + \left(1+\gamma_m(g)\right)m\bar{\psi}\psi \qquad \left(\beta(g) \equiv \mu\frac{dg}{d\mu},\ \gamma_m(g) = -\frac{\mu}{m}\frac{dm}{d\mu}\right)$$

$$\int_0^1 dx\,x\left[q(x,\mu) + \bar{q}(x,\mu)\right] = A_q(t=0,\mu)$$

$$A_q(0,\mu) = \frac{n_f}{4C_F + n_f} + \frac{4C_F A_q(0,\mu_0) + n_f\left(A_q(0,\mu_0) - 1\right)}{4C_F + n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F + 2n_f}{3\beta_0}} + \cdots$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)|\,g_{\mu\nu}T_q^{\mu\nu}\,|N(p)\rangle$$

$$=-\frac{1}{4}A_q(0)+\frac{1}{2M^2}\langle N(p)|\,m\bar{\psi}\psi\,|N(p)\rangle$$

$$+\frac{1}{8M^2}\langle N(p)|\frac{\alpha_s}{4\pi}\frac{4}{3}C_F m\bar{\psi}\psi\,|N(p)\rangle+\frac{1}{8M^2}\langle N(p)|\frac{\alpha_s}{4\pi}\frac{1}{3}n_f F^2\,|N(p)\rangle+\cdots$$

$$g_{\mu\nu}T^{\mu\nu}=\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi$$

$$\int_0^1 dx\,x\left[q(x,\mu)+\bar{q}(x,\mu)\right]=A_q(t=0,\mu)$$

$$A_q(0,\mu)=\frac{n_f}{4C_F+n_f}+\frac{4C_F A_q(0,\mu_0)+n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}+\cdots$$

$$\bar{C}_q(0)\ \left(=-\bar{C}_g(0)\right)=-\frac{1}{4}A_q(0)+\frac{1}{8M^2}\langle N(p)\,|\,g_{\mu\nu}T_q^{\mu\nu}\,|\,N(p)\rangle$$

$$=-\frac{1}{4}A_q(0)+\frac{1}{2M^2}\langle N(p)\,|\,m\bar{\psi}\psi\,|\,N(p)\rangle$$

$$+\frac{1}{8M^2}\langle N(p)\,|\,\frac{\alpha_s}{4\pi}\frac{4}{3}C_F m\bar{\psi}\psi\,|\,N(p)\rangle+\frac{1}{8M^2}\langle N(p)\,|\,\frac{\alpha_s}{4\pi}\frac{1}{3}n_f F^2\,|\,N(p)\rangle+\cdots$$

$$g_{\mu\nu}T^{\mu\nu}=\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{q}q$$

$$2M^2=\langle N(p)\,|\left(\frac{\beta(g)}{2g}F^2+\left(1+\gamma_m(g)\right)m\bar{\psi}\psi\right)|\,N(p)\rangle$$

$$=-\langle N(p)\,|\,\frac{\beta_0}{2}\frac{\alpha_s}{4\pi}F^2\,|\,N(p)\rangle+\left(1+\gamma_{m0}\frac{\alpha_s}{4\pi}\right)\langle N(p)\,|\,m\bar{\psi}\psi\,|\,N(p)\rangle+\cdots$$

$$\int_0^1 dx\,x\left[q(x,\mu)+\bar{q}(x,\mu)\right]=A_q(t=0,\mu)$$

$$A_q(0,\mu)=\frac{n_f}{4C_F+n_f}+\frac{4C_F A_q(0,\mu_0)+n_f\left(A_q(0,\mu_0)-1\right)}{4C_F+n_f}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}+\cdots$$

$$\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$-\frac{4C_F A_q(\mu_0)+n_f\left(A_q(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}$$

$$+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)+\frac{\beta_1 n_f}{6\beta_0^2}\right]+\cdots$$

$$\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}$$

$$-\frac{4C_F A_q(\mu_0)+n_f\left(A_q(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}$$

$$+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)+\frac{\beta_1 n_f}{6\beta_0^2}\right]\cdot$$

$$+\left[\frac{n_f\,(34C_A+157C_F)}{108\beta_0}+\frac{C_F}{3}-\frac{\beta_1 n_f}{6\beta_0^2}\right]\frac{\langle N(p)|\, m\bar{\psi}\psi \,|N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\mathrm{NLO}}(\mu)$$

$$\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle N(p)|\, m\bar{\psi}\psi\, |N(p)\rangle}{2M^2}$$

$$-\frac{4C_F A_q(\mu_0)+n_f\left(A_q(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}$$

$$+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)+\frac{\beta_1 n_f}{6\beta_0^2}\right]\cdot$$

$$+\left[\frac{n_f\,(34C_A+157C_F)}{108\beta_0}+\frac{C_F}{3}-\frac{\beta_1 n_f}{6\beta_0^2}\right]\frac{\langle N(p)|\, m\bar{\psi}\psi\, |N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\rm NLO}(\mu)$$

$$+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\Bigg(\frac{n_f^2}{\beta_0}\left[\frac{697C_A}{1458}+\frac{169C_F}{2916}\right]+n_f\left[\frac{17\beta_1 C_A}{54\beta_0^2}+\frac{\beta_2}{6\beta_0^2}+\frac{49\beta_1 C_F}{108\beta_0^2}\right.$$

$$\left.+\frac{1}{\beta_0}\left\{\left(\frac{401}{648}-\frac{26\zeta(3)}{9}\right)C_AC_F+\left(2\zeta(3)-\frac{67}{27}\right)C_A^2+\left(\frac{8\zeta(3)}{9}-\frac{2407}{2916}\right)C_F^2\right\}-\frac{\beta_1^2}{6\beta_0^3}\right]$$

$$+\left[-\frac{n_f^2}{\beta_0}\left(\frac{697C_A}{1458}+\frac{1789C_F}{2916}\right)+n_f\left(-\frac{17\beta_1 C_A}{54\beta_0^2}-\frac{\beta_2}{6\beta_0^2}-\frac{157\beta_1 C_F}{108\beta_0^2}+\frac{\beta_1^2}{6\beta_0^3}-\frac{17C_F}{27}\right)\right.$$

$$+\frac{n_f}{\beta_0}\left\{\left(\frac{26\zeta(3)}{9}+\frac{4315}{648}\right)C_AC_F+\left(\frac{67}{27}-2\zeta(3)\right)C_A^2+\left(\frac{11803}{2916}-\frac{8\zeta(3)}{9}\right)C_F^2\right\}$$

$$\left.+\frac{61C_AC_F}{108}-\frac{C_F^2}{27}\right]\frac{\langle N(p)|\, m\bar{\psi}\psi\, |N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\rm NNLO}(\mu)\ ,$$

$$\bar{C}_q(0,\mu) = -\frac{1}{4}\left(\frac{n_f}{4C_F+n_f}+\frac{2n_f}{3\beta_0}\right)+\frac{1}{4}\left(\frac{2n_f}{3\beta_0}+1\right)\frac{\langle N(p)|\, m\bar{\psi}\psi\, |N(p)\rangle}{2M^2}$$

$$-\frac{4C_F A_q(\mu_0)+n_f\left(A_q(\mu_0)-1\right)}{4(4C_F+n_f)}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{8C_F+2n_f}{3\beta_0}}$$

$$+\frac{\alpha_s(\mu)}{4\pi}\left[\frac{n_f}{4\beta_0}\left(-\frac{34C_A}{27}-\frac{49C_F}{27}\right)+\frac{\beta_1 n_f}{6\beta_0^2}\right]-$$

$$+\left[\frac{n_f\,(34C_A+157C_F)}{108\beta_0}+\frac{C_F}{3}-\frac{\beta_1 n_f}{6\beta_0^2}\right]\frac{\langle N(p)|\, m\bar{\psi}\psi\, |N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\mathrm{NLO}}(\mu)$$

$$+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\Bigg(\frac{n_f^2}{\beta_0}\left[\frac{697C_A}{1458}+\frac{169C_F}{2916}\right]+n_f\Bigg[\frac{17\beta_1 C_A}{54\beta_0^2}+\frac{\beta_2}{6\beta_0^2}+\frac{49\beta_1 C_F}{108\beta_0^2}$$

$$+\frac{1}{\beta_0}\left\{\left(\frac{401}{648}-\frac{26\zeta(3)}{9}\right)C_AC_F+\left(2\zeta(3)-\frac{67}{27}\right)C_A^2+\left(\frac{8\zeta(3)}{9}-\frac{2407}{2916}\right)C_F^2\right\}-\frac{\beta_1^2}{6\beta_0^3}\Bigg]$$

$$+\Bigg[-\frac{n_f^2}{\beta_0}\left(\frac{697C_A}{1458}+\frac{1789C_F}{2916}\right)+n_f\Bigg(-\frac{17\beta_1 C_A}{54\beta_0^2}-\frac{\beta_2}{6\beta_0^2}-\frac{157\beta_1 C_F}{108\beta_0^2}+\frac{\beta_1^2}{6\beta_0^3}-\frac{17C_F}{27}\Bigg)$$

$$+\frac{n_f}{\beta_0}\left\{\left(\frac{26\zeta(3)}{9}+\frac{4315}{648}\right)C_AC_F+\left(\frac{67}{27}-2\zeta(3)\right)C_A^2+\left(\frac{11803}{2916}-\frac{8\zeta(3)}{9}\right)C_F^2\right\}$$

$$+\frac{61C_AC_F}{108}-\frac{C_F^2}{27}\Bigg]\frac{\langle N(p)|\, m\bar{\psi}\psi\, |N(p)\rangle}{2M^2}\Bigg)-\frac{1}{4}A_q^{\mathrm{NNLO}}(\mu)\ ,$$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.145556 + 0.305556\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$+\left(0.09-0.25A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$+\alpha_s(\mu)\Big[0.00553609+0.0803962\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$+\left(0.0127684-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}-\left(0.0279651-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}\Big]$$

$$+\left(\alpha_s(\mu)\right)^2\Big[0.00174426+0.0312256\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$-\left(0.0059729-0.0165914A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$-\left(0.00396745-0.00503187A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}$$

$$+\left(0.0237481-0.0216233A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{112}{81}}\Big]$$

$$
\begin{aligned}
\bar{C}_q(0,\mu)\Big|_{n_f=3} &= -0.145556 + 0.305556\frac{\langle N(p)\,|\,m\bar{\psi}\psi\,|\,N(p)\rangle}{2M^2} \\
&+\left(0.09-0.25A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} \\
&+\alpha_s(\mu)\Big[0.00553609+0.0803962\frac{\langle N(p)\,|\,m\bar{\psi}\psi\,|\,N(p)\rangle}{2M^2} \\
&+\left(0.0127684-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} - \left(0.0279651-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}\Big] \\
&+\left(\alpha_s(\mu)\right)^2\Big[0.00174426+0.0312256\frac{\langle N(p)\,|\,m\bar{\psi}\psi\,|\,N(p)\rangle}{2M^2} \\
&-\left(0.0059729-0.0165914A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} \\
&-\left(0.00396745-0.00503187A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}} \\
&+\left(0.0237481-0.0216233A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{112}{81}}\Big]
\end{aligned}
$$

$$A_q(\mu_0) = \int_0^1 dx\, x\left[q(x,\mu_0)+\bar{q}(x,\mu_0)\right]$$

$$
\begin{aligned}
&\langle N(p)\,|\,m\bar{\psi}\psi\,|\,N(p)\rangle \\
&= \langle N(p)\,|\,m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s\,|\,N(p)\rangle
\end{aligned}
$$

$$A_q(\mu_0) = \int_0^1 dx x\left[q(x,\mu_0) + \overline{q}(x,\mu_0)\right]$$

global QCD analysis at NNLO

$$A_q\left(\mu_0 = 1.3\,\text{GeV}\right) = 0.613$$

CT18

(MMHT2014,NNPDF)

$$\langle N(p)|m\overline{\psi}\psi|N(p)\rangle = \langle N(p)|m_u\overline{u}u + m_d\overline{d}d + m_s\overline{s}s|N(p)\rangle = 2M\left(\sigma_{\pi N} + \sigma_s\right)$$

$$\sigma_{\pi N} = \frac{1}{2M}\langle N(p)|\frac{m_u + m_d}{2}\left(\overline{u}u + \overline{d}d\right)|N(p)\rangle = 59.1 \pm 3.5\,\text{MeV}$$

Hoferichter, Elvira, Kubis, Meißner, PRL115, 092301

$$\sigma_s = \frac{1}{2M}\langle N(p)|m_s\overline{s}s|N(p)\rangle = 45.6 \pm 6.2\,\text{MeV}$$

Alexandrou, et al., PRD102, 054517

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.145556 + 0.305556\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$+\left(0.09-0.25A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$+\alpha_s(\mu)\Big[0.00553609+0.0803962\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$+\left(0.0127684-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}-\left(0.0279651-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}\Big]$$

$$+\left(\alpha_s(\mu)\right)^2\Big[0.00174426+0.0312256\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

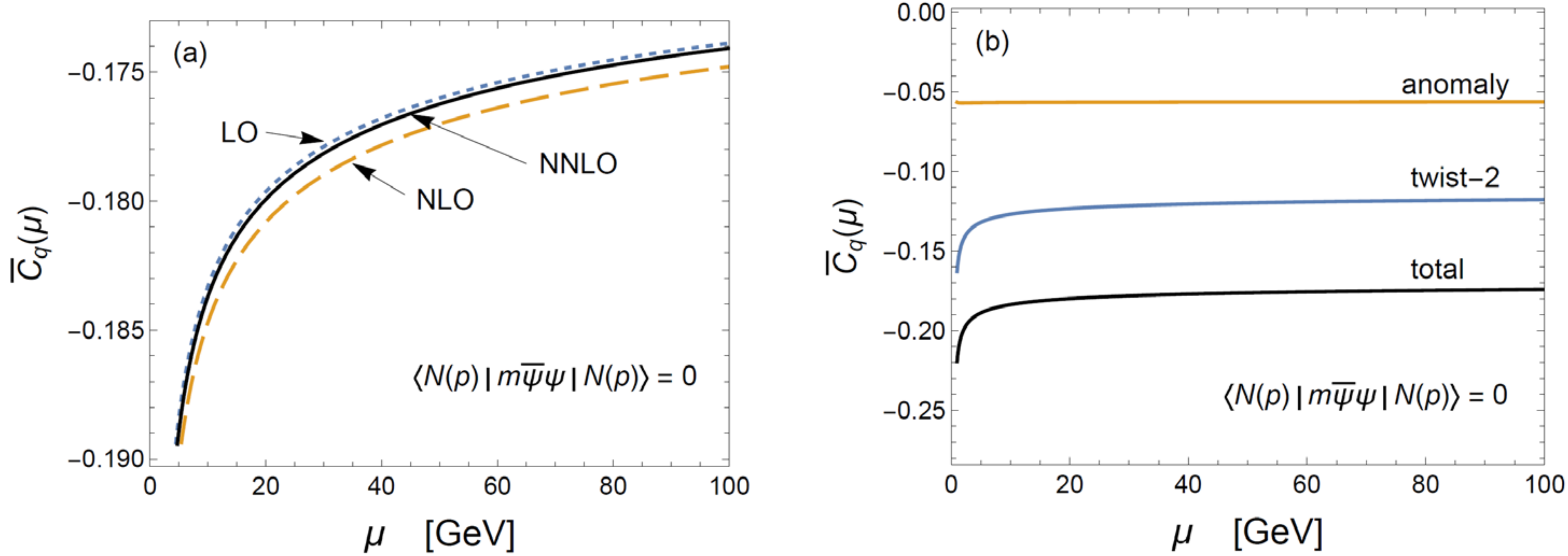
$$-\left(0.0059729-0.0165914A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$A_q(\mu_0)=\int_0^1 dx\,x\left[q(x,\mu_0)+\bar{q}(x,\mu_0)\right]$$

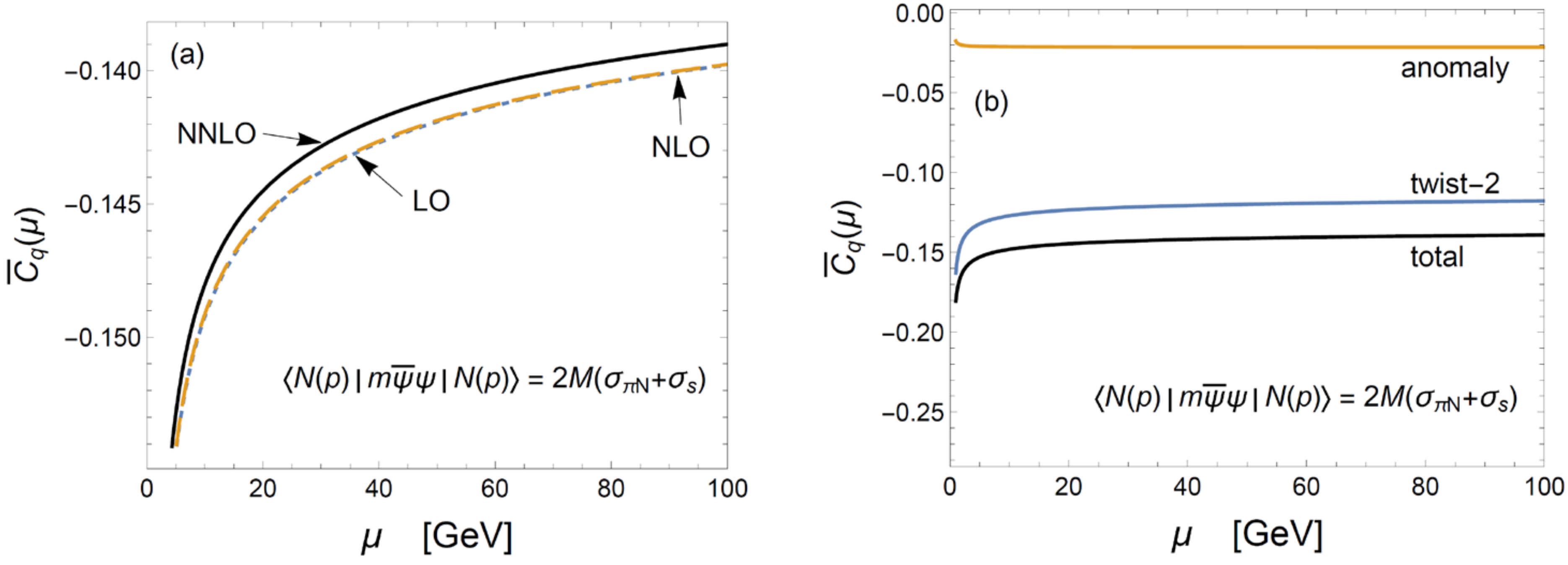
$$-\left(0.00396745-0.00503187A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}$$

$$\langle N(p)|m\bar{\psi}\psi|N(p)\rangle = \langle N(p)|m_u\bar{u}u+m_d\bar{d}d+m_s\bar{s}s|N(p)\rangle$$

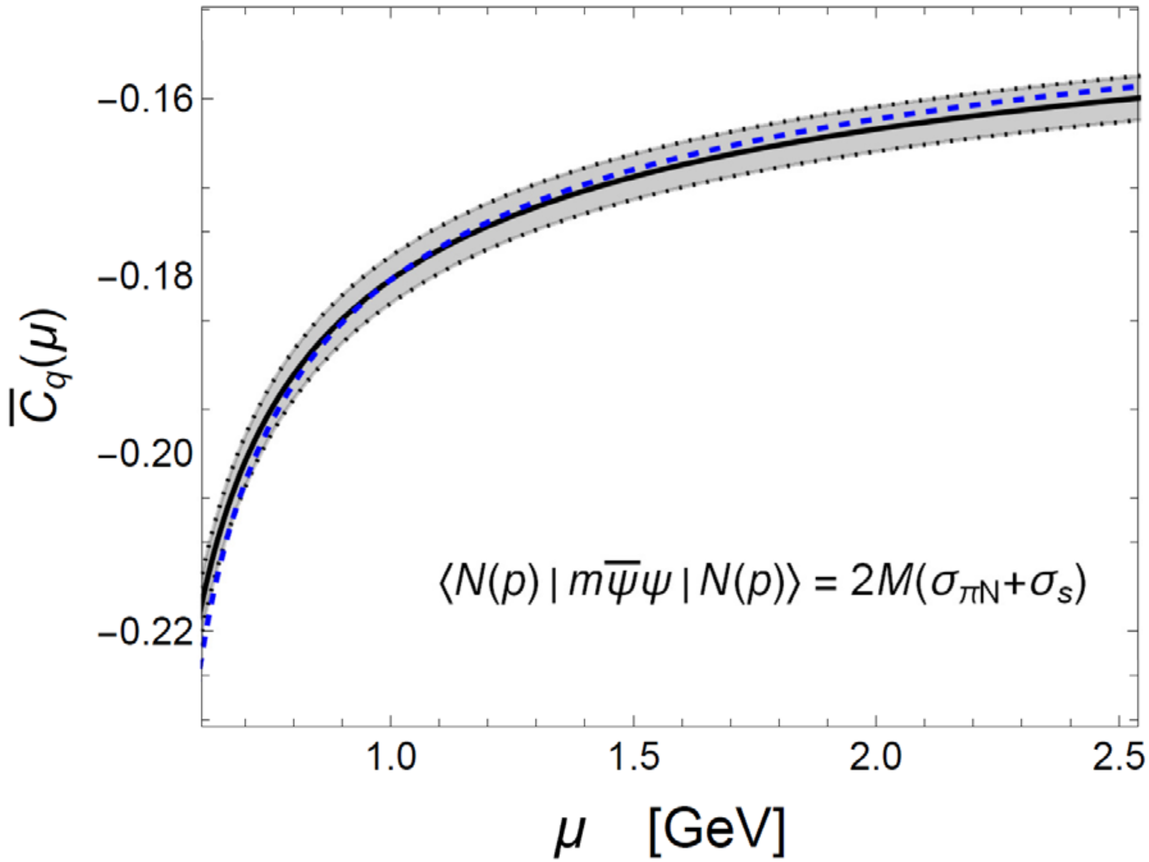
$$+\left(0.0237481-0.0216233A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{112}{81}}\Big]$$



(a)
LO
NNLO
NLO
$\overline{C}_q(\mu)$
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 0$
μ [GeV]
(b)
anomaly
twist−2
total
$\overline{C}_q(\mu)$
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 0$
μ [GeV]



(a)
NNLO
LO
NLO
$\overline{C}_q(\mu)$
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 2M(\sigma_{\pi N}+\sigma_s)$
μ [GeV]
(b)
anomaly
twist−2
total
$\overline{C}_q(\mu)$
$\langle N(p)\,|\,m\overline{\psi}\psi\,|\,N(p)\rangle = 2M(\sigma_{\pi N}+\sigma_s)$
μ [GeV]



$$\bar{C}_q(\mu = 0.7\ \mathrm{GeV})\big|_{n_f=3} = -0.201 \pm 0.003$$

$$\bar{C}_q(\mu = 1\ \mathrm{GeV})\big|_{n_f=3} = -0.180 \pm 0.003$$

$$\bar{C}_q(\mu = 2\ \mathrm{GeV})\big|_{n_f=3} = -0.163 \pm 0.003$$

$\overline{\mathrm{MS}}$ scheme

$$\bar{C}_q(\mu)\big|_{n_f=3} \simeq -0.108 - 0.114\,[\alpha_s(\mu)]^{\frac{50}{81}}$$

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\gamma^{(\mu} i\overleftrightarrow{D}^{\nu)}\psi + F^{\mu\rho}F_{\rho}{}^{\nu} + \frac{g^{\mu\nu}}{4}F^2 \equiv T_q^{\mu\nu} + T_g^{\mu\nu}$$

$$\langle N(p')|T_{q,g}^{\mu\nu}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)Mg^{\mu\nu}\Big]u(p)$$

$$P = \frac{p+p'}{2}$$

$$\Delta = p' - p$$

$$t = \Delta^2$$

$$A_q(0) + A_g(0) = 1$$

$$\langle N(p)|T^{\mu\nu}|N(p)\rangle = 2p^\mu p^\nu$$

$$\frac{\langle N(p)\,S\,|J^i|\,N(p)\,S\rangle}{\langle N(p)\,S\,|\,N(p)\,S\rangle} = \frac{1}{2}S^i$$

$$\frac{1}{2}\Big(A_q(0) + B_q(0) + A_g(0) + B_g(0)\Big) = \frac{1}{2}$$

$$B_q(0) + B_g(0) = 0$$

$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{+jk}$$

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

$$\bar{C}_q(t) + \bar{C}_g(t) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$\bar{C}_q(0,\mu)\Big|_{n_f=3} = -0.145556 + 0.305556\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$+\left(0.09-0.25A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$+\alpha_s(\mu)\Big[0.00553609+0.0803962\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$+\left(0.0127684-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}} -\left(0.0279651-0.0354678A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}\Big]$$

$$+\left(\alpha_s(\mu)\right)^2\Big[0.00174426+0.0312256\frac{\langle N(p)|m\bar{\psi}\psi|N(p)\rangle}{2M^2}$$

$$-\left(0.0059729-0.0165914A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{50}{81}}$$

$$A_q(\mu_0)=\int_0^1 dx\, x\left[q(x,\mu_0)+\bar{q}(x,\mu_0)\right]$$

$$-\left(0.00396745-0.00503187A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{31}{81}}$$

$$\langle N(p)|m\bar{\psi}\psi|N(p)\rangle = \langle N(p)|m_u\bar{u}u+m_d\bar{d}d+m_s\bar{s}s|N(p)\rangle$$

$$+\left(0.0237481-0.0216233A_q(\mu_0)\right)\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{112}{81}}\Big]$$

$$A_q(\mu_0) = \int_0^1 dx x\left[q(x,\mu_0) + \overline{q}(x,\mu_0)\right]$$ global QCD analysis at NNLO

$$A_q\left(\mu_0 = 1.3\,\mathrm{GeV}\right) = 0.613$$

CT18

(MMHT2014,NNPDF)

$$\langle N(p)|m\overline{\psi}\psi|N(p)\rangle = \langle N(p)|m_u\overline{u}u + m_d\overline{d}d + m_s\overline{s}s|N(p)\rangle = 2M\left(\sigma_{\pi N} + \sigma_s\right)$$

$$\sigma_{\pi N} = \frac{1}{2M}\langle N(p)|\frac{m_u + m_d}{2}\left(\overline{u}u + \overline{d}d\right)|N(p)\rangle = 59.1 \pm 3.5\,\mathrm{MeV}$$

Hoferichter, Elvira, Kubis, Meißner, PRL115, 092301

$$\sigma_s = \frac{1}{2M}\langle N(p)|m_s\overline{s}s|N(p)\rangle = 45.6 \pm 6.2\,\mathrm{MeV}$$

Alexandrou, et al., PRD102, 054517

$$A_q^{\pi}(\mu_0) = \int_0^1 dx\, x\left[q^{\pi}(x,\mu_0) + \bar{q}^{\pi}(x,\mu_0)\right]$$

global QCD analysis at NLO

$$A_q^{\pi}\left(\mu_0 = 1.3\,\mathrm{GeV}\right) = \begin{cases} 0.70 \pm 0.02 \\ 0.81 \pm 0.16 \\ 0.61 \pm 0.08 \end{cases}$$

JAM ('18)

xFitter ('20)

JAM ('21)

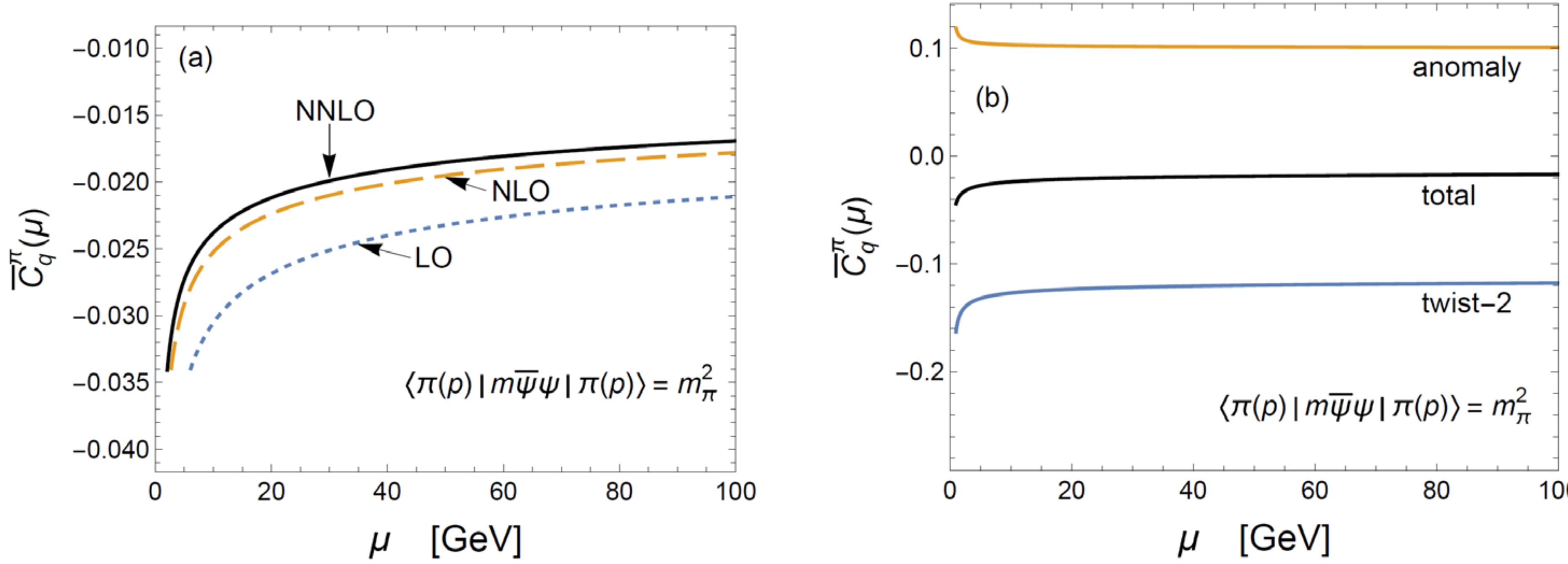
$$\langle \pi(p) | m\bar{\psi}\psi | \pi(p) \rangle$$

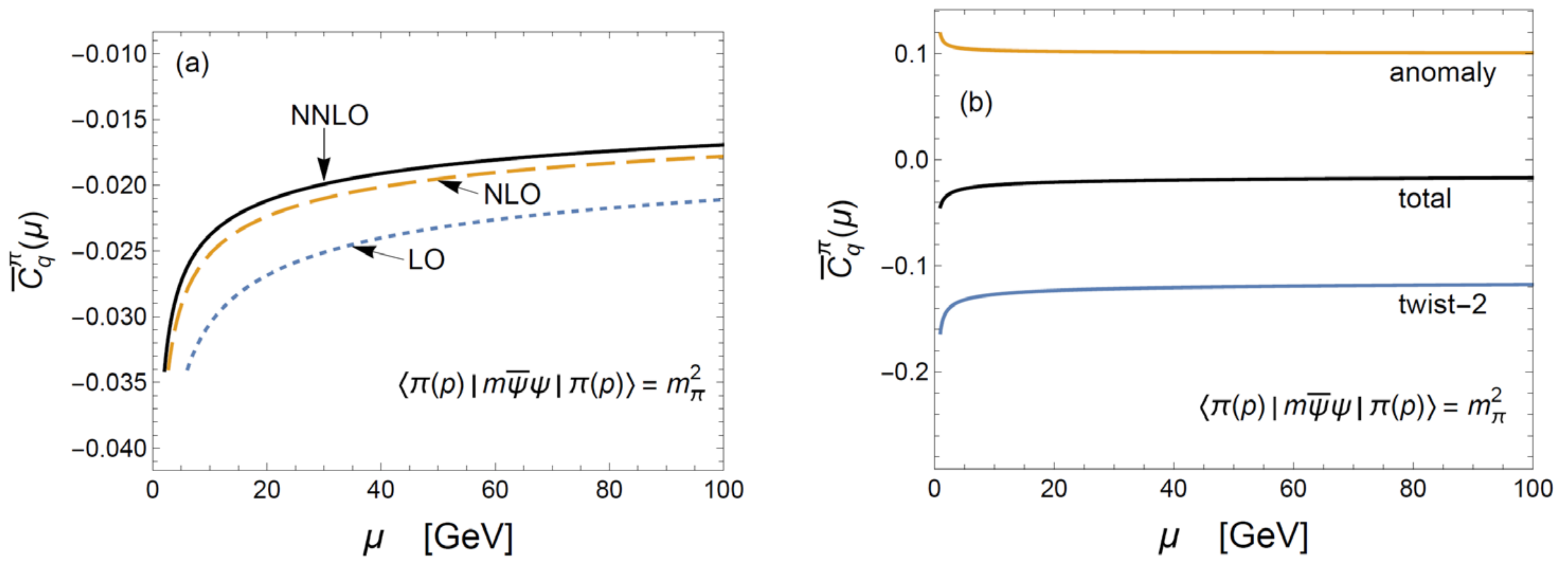
$$= m_{\pi}^2 + O(6\%)$$

χPT

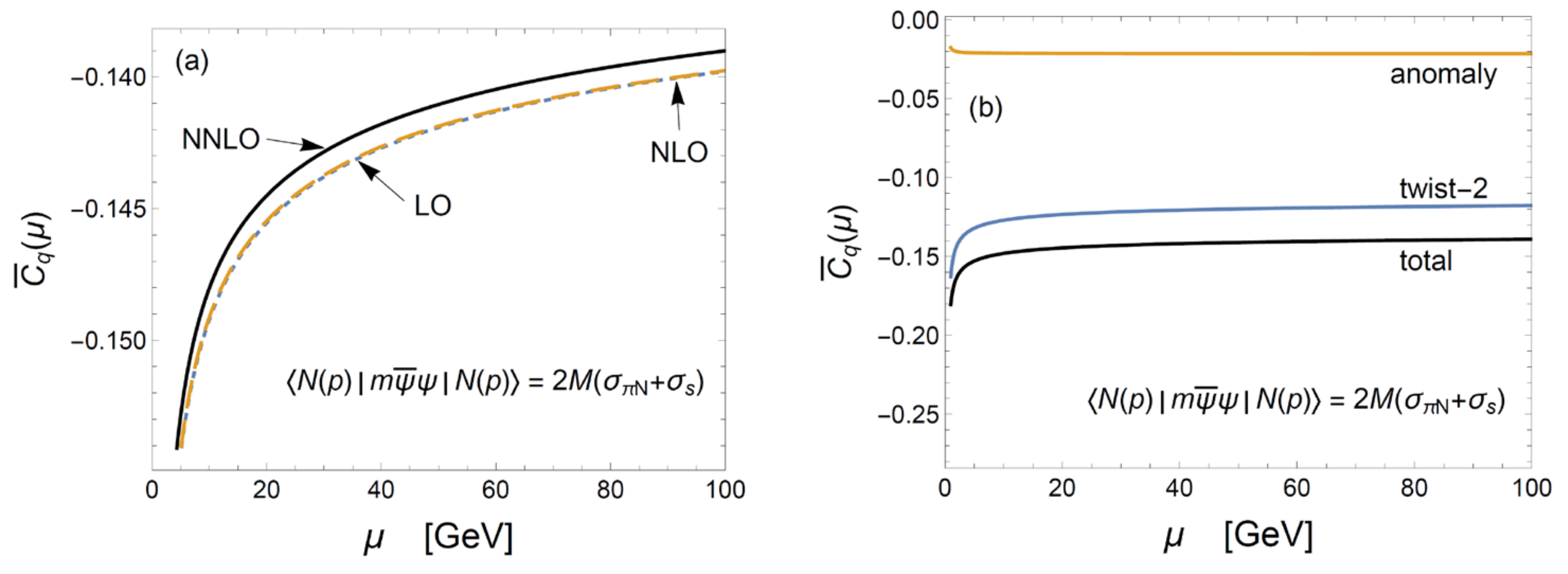
Gasser, Leutwyler, Annals Phys. 158, 142

Colangelo, Gasser, Leutwyler, PRL86, 5008

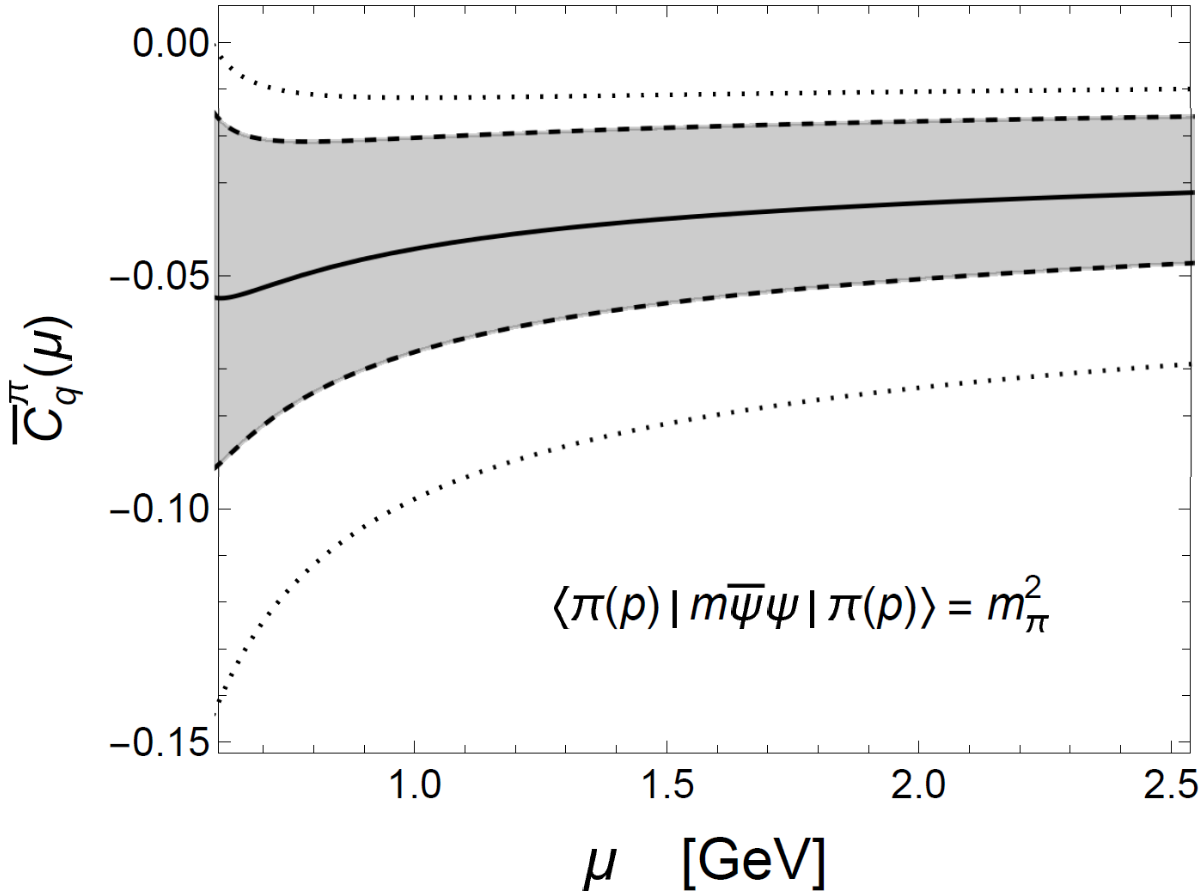

(a)
NNLO
NLO
LO
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
μ [GeV]
(b)
anomaly
total
twist-2
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
μ [GeV]



(a)
NNLO
NLO
LO
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
−0.010
−0.015
−0.020
−0.025
−0.030
−0.035
−0.040
0
20
40
60
80
100
μ [GeV]
(b)
anomaly
total
twist−2
$\langle \pi(p) | m\overline{\psi}\psi | \pi(p) \rangle = m_\pi^2$
$\overline{C}_q^\pi(\mu)$
0.1
0.0
−0.1
−0.2
0
20
40
60
80
100
μ [GeV]



(a)
NNLO
LO
NLO
$\langle N(p) | m\overline{\psi}\psi | N(p) \rangle = 2M(\sigma_{\pi N}+\sigma_s)$
$\overline{C}_q(\mu)$
−0.140
−0.145
−0.150
0
20
40
60
80
100
μ [GeV]
(b)
anomaly
twist−2
total
$\langle N(p) | m\overline{\psi}\psi | N(p) \rangle = 2M(\sigma_{\pi N}+\sigma_s)$
$\overline{C}_q(\mu)$
0.00
−0.05
−0.10
−0.15
−0.20
−0.25
0
20
40
60
80
100
μ [GeV]



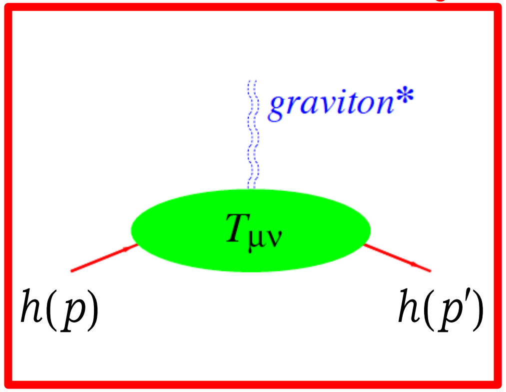
$$\bar{C}_q^\pi(\mu = 0.7\ \mathrm{GeV})\big|_{n_f=3} = -0.05 \pm 0.03$$

$$\bar{C}_q^\pi(\mu = 1\ \mathrm{GeV})\big|_{n_f=3} = -0.04 \pm 0.02$$

$$\bar{C}_q^\pi(\mu = 2\ \mathrm{GeV})\big|_{n_f=3} = -0.03 \pm 0.02$$

$\overline{\mathrm{MS}}$ scheme

Summary



mass & energy distribution

angular momentum distribution

force & pressure distribution

trace anomaly

$$\langle N(p')|T^{\mu\nu}_{q,g}|N(p)\rangle = \bar{u}(p')\Big[A_{q,g}(t)\gamma^{(\mu}P^{\nu)} + B_{q,g}(t)\frac{P^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha}{2M} + D_{q,g}(t)\frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M} + \bar{C}_{q,g}(t)Mg^{\mu\nu}\Big]u(p)$$

$\bar{C}_q$, $\bar{C}_g$ — related to trace anomaly for q/g part of energy-momentum tensor

$$\bar{C}_q(0,\mu) = -\bar{C}_g(0,\mu) = \mathrm{LO} + \mathrm{NLO} + \mathrm{NNLO}$$

$$A_q(\mu_0) = \int_0^1 dx\, x[q(x,\mu_0) + \bar{q}(x,\mu_0)]$$

$$\langle N(p)|m\bar{\psi}\psi|N(p)\rangle$$

- NNLO term is ~1% level
- The approach to the asymptotic value is quite slow
- $\langle N(p)|m\bar{\psi}\psi|N(p)\rangle$ gives impotant contribution
- quite different behaviors between nucleon and pion

nucleon

$\bar{C}_q(0, \mu \sim 0.4 \text{ GeV}) = 0.25$ **Bag model** [Ji, Melnitchouk, Song, PRD56, 5511 ('97)]

$\bar{C}_q(0, \mu = 2 \text{ GeV}) \approx -0.11$ **Phenomenological** [Lorce, EPJC78, 120 ('18)]

$\bar{C}_q(0, \mu \sim 0.63 \text{ GeV}) = 0.014$ **Instanton** [Polyakov, Son, JHEP09, 156 ('18)]

$\bar{C}_q(0, \mu = 1 \text{ GeV}) = -0.021 \pm 0.008$ **LCSR** [Azizi,Ozdem, EPJC80, 104 ('20)]

$\bar{C}_q(0, \mu \to \infty) \simeq -0.15$ **Trace anomaly** [Hatta, Rajan, KT, JHEP12, 008 ('18)]

$\bar{C}_q(0, \mu = 1 \text{ GeV}) = -0.180 \pm 0.003$
$\bar{C}_q(0, \mu = 2 \text{ GeV}) = -0.163 \pm 0.003$
NNLO QCD [this work]

pion

$\bar{C}_q^{\pi}(0, \mu = 1 \text{ GeV}) = -0.04 \pm 0.02$ **NNLO QCD with NLO input** [this work] backup

Explicit quark/gluon separation of QCD trace anomaly is available at 3-loop in the MSbar scheme

$$g_{\mu\nu}T_q^{\mu\nu} = m\bar{q}q + \frac{\alpha_s}{4\pi}\left(\frac{4}{3}C_F m\bar{q}q + \frac{1}{3}n_f F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\left(C_F\left(\frac{61C_A}{27} - \frac{68n_f}{27}\right) - \frac{4C_F^2}{27}\right)m\bar{q}q + \left(\frac{17C_A n_f}{27} + \frac{49C_F n_f}{54}\right)F^2\right] + \left(\frac{\alpha_s}{4\pi}\right)^3[\cdots]$$

$$g_{\mu\nu}T_g^{\mu\nu} = \frac{\alpha_s}{4\pi}\left(\frac{14}{3}C_F m\bar{q}q - \frac{11}{6}C_A F^2\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\left(C_F\left(\frac{812C_A}{27} - \frac{22n_f}{27}\right) + \frac{85C_F^2}{27}\right)m\bar{q}q + \left(\frac{28C_A n_f}{27} - \frac{17C_A^2}{3} + \frac{5C_F n_f}{54}\right)F^2\right] + \left(\frac{\alpha_s}{4\pi}\right)^3[\cdots]$$

$$g_{\mu\nu}T_q^{\mu\nu} + g_{\mu\nu}T_g^{\mu\nu} = \frac{\beta(g)}{2g}F^2 + (1+\gamma_m(g))m\bar{q}q$$

1. hadron mass formula $2M_h^2 = g_{\mu\nu}\langle h|T_q^{\mu\nu}|h\rangle + g_{\mu\nu}\langle h|T_g^{\mu\nu}|h\rangle$

nucleon	-1	:	5
pion	1	:	1

2. gravitational form factor $\bar{C}_{q/g}(t)$

$$\bar{C}_q(0,\mu) = -\bar{C}_g(0,\mu) = \mathrm{LO} + \mathrm{NLO} + \mathrm{NNLO}$$

NNLO ~% level corr.

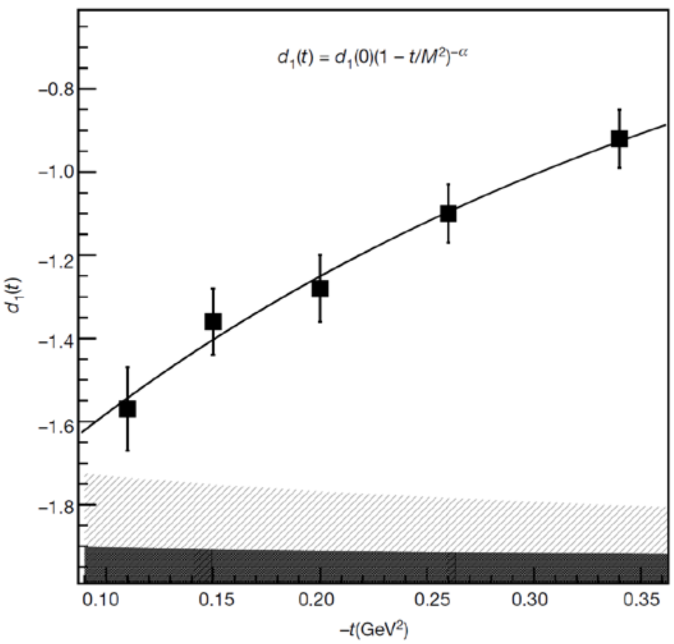
``asymptotic'' is quite slow

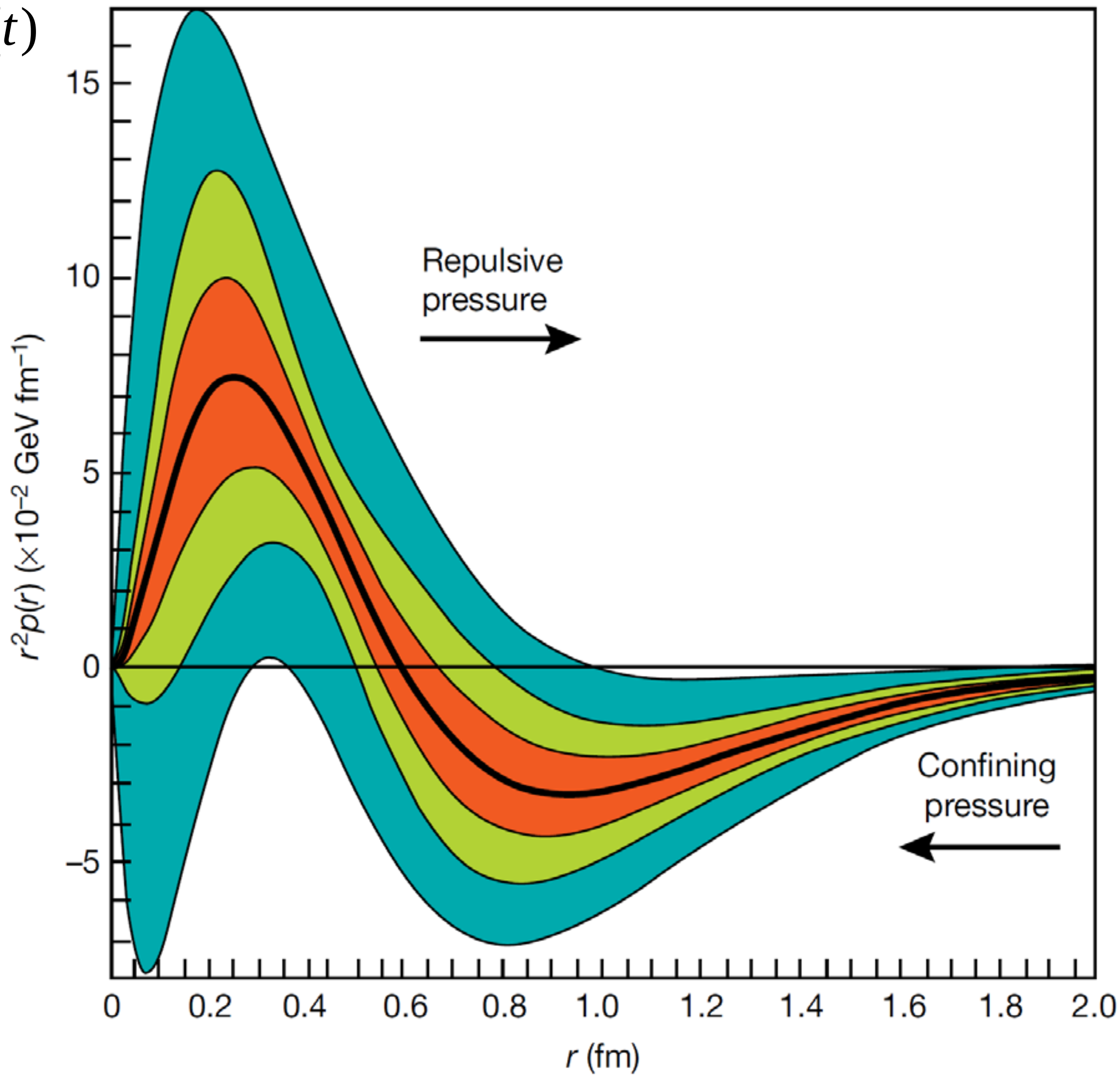
pion (Nambu-Goldstone boson) ⟷ nucleon

V. D. Burkert et al, Nature 557 ('18) 396

$$\int_{-1}^{1} dx\,x H^q(x,\eta,t) = A_q(t) + 4\eta^2 D_q(t)$$

$$5\sum_q e_q^2 D_q(t) \sim d_1(t)$$





$$\langle N(p')|T^{ik}(0)|N(p)\rangle \sim \left(\Delta^i\Delta^k - \delta^{ik}\boldsymbol{\Delta}^2\right)D(t),\quad T^{ij}(r) = \left(\frac{r^i r^j}{r^2} - \frac{1}{3}\delta^{ij}\right)s(r) + \delta^{ij}\,p(r)$$