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Domain Wall from Gauged $d = 4, \mathcal{N} = 8$ Supergravity: Part I

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abstract

By studying already known extrema of non-semi-simple Inonu-Wigner contraction $CSO(p,q)^+$ and non-compact $SO(p,q)^+ (p+q=8)$ gauged $\mathcal{N}=8$ supergravity in 4-dimensions developed by Hull sometime ago, one expects there exists nontrivial flow in the 3-dimensional boundary field theory. We find that these gaugings provide first-order domain-wall solutions from direct extremization of energy-density.

We also consider the most general $CSO(p,q,r)^+$ with $p+q+r=8$ gauging of $\mathcal{N}=8$ supergravity by two successive $SL(8,\mathbf{R})$ transformations of the de Wit-Nicolai theory, that is, compact $SO(8)$ gauged supergravity. The theory found earlier has local $SU(8) \times CSO(p,q,r)^+$ gauge symmetry as well as local $\mathcal{N}=8$ supersymmetry. The gauge group $CSO(p,q,r)^+$ is spontaneously reduced to its maximal compact subgroup $SO(p)^+ \times SO(q)^+ \times U(1)^{+r(r-1)/2}$. The T-tensor we obtain describes a two-parameter family of gauged $\mathcal{N}=8$ supergravity from which one can construct A_1 and A_2 tensors. The effective nontrivial scalar potential can be written as the difference of positive definite terms. We examine the scalar potential for critical points at which the expectation value of the scalar field is $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant. It turns out that there is no new extra critical point. However, we do have flow equations and domain-wall solutions for the scalar fields are the gradient flow equations of the superpotential that is one of the eigenvalues of A_1 tensor.

1 Introduction

One of the interesting issues in recent research is the domain wall(DW)/quantum field theory(QFT) correspondence initiated by [1] between supergravity, in the near horizon region of the corresponding supergravity brane solution, compactified on domain wall spacetimes that are locally isometric to Anti-de Sitter(AdS) space but different from it globally and quantum(nonconformal) field theories describing the internal dynamics of branes and living on the boundary of such spacetimes. DW/QFT correspondence was motivated by the fact that the AdS metric in horospherical coordinates is a special case of the domain wall metric [2, 3]. R-symmetry of the supersymmetric QFT on the boundary of domain worldvolume should match with the gauge group of the corresponding gauged supergravity. Compact gaugings are not the only ones for extended supergravities but there exists a rich structure of non-compact and non-semi-simple gaugings (Note that the unitarity property is preserved since in all extrema of scalar potential, non-compact gauge symmetry is reduced to some residual compact subgroup). Such a theory plays a fundamental role in the description of the DW/QFT correspondence as the maximally compact gauged supergravity has played in the AdS/conformal field theory(CFT) duality [4, 5, 6] that is a correspondence between certain compact gauged supergravities and certain conformal field theories. It would be interesting to identify the appropriate non-compact and non-semi-simple gauged supergravities corresponding to each choice of brane configuration.

One of the questions we addressed was whether the maximal supergravity theories with non-compact gauge groups can be obtained from higher dimensional theory. $\mathcal{N}=8$ gauged supergravity theories have been constructed in 4-dimensions with gauge groups $SO(p, 8-p)$ where $p=0,1,2,3,$ and 4 or with non-semi-simple contractions of these gauge groups [7, 8, 9, 10, 11, 12, 13, 14]. In 7-dimensions, $\mathcal{N}=4$ gauged supergravity theories have been constructed with gauge group $SO(p, 5-p)$ with $p=0,1,2$ [15]. In five-dimensions there exist gauged $\mathcal{N}=8$ supergravity theories with gauge groups $SO(p, 6-p)$ with $p=0,1,2,3$ or $SU(3,1)$ [16]. Although odd-dimensional gauged supergravity theories do not appear to allow gaugings of non-semi-simple contractions, some researchers have attempted to resolve the difficulties in five-dimensions [17, 18]. It has been shown that the $SO(p,q)$ gaugings and their non-semi-simple contractions can be obtained from the appropriate higher dimensional supergravity theories. The spheres used to compactify the $SO(p)$ gaugings are replaced by hyperboloids for the non-compact $SO(p,q)$ gaugings and generalized cylinders for the non-semi-simple contractions [14].

Since embedding or consistent truncation of gauged supergravity is known to characterize $\mathbf{S}^7$ compactification of eleven-dimensional supergravity[1], we also are interested in the domain-

[1] By generalizing compactification vacuum ansatz to the nonlinear level, solutions of eleven-dimensional supergravity were obtained directly from the scalar and pseudo-scalar expectation values at various critical points of the $\mathcal{N}=8$ supergravity potential [19]. They reproduced all known Kaluza-Klein solutions of the eleven-dimensional supergravity: round $\mathbf{S}^7$ [20], $SO(7)^-$-invariant, *parallelized* $\mathbf{S}^7$ [21], $SO(7)^+$-invariant vacuum

wall solution in four-dimensional gauged supergravity. In [28], a renormalization group flow from $\mathcal{N} = 8$, $SO(8)$ invariant UV fixed point to $\mathcal{N} = 2$, $SU(3) \times U(1)$ invariant IR fixed point was found by studying the de Wit-Nicolai potential which is invariant in the $SU(3) \times U(1)$ group. For this interpretation it was crucial to know the form of superpotential that was encoded in the structure of the T-tensor of the theory. Very recently, the lift to M-theory of the solution described in [28] was constructed [27](See also [29]). Moreover, it was natural and illuminating to ask whether one can construct the most general superpotential for critical points in four-dimensional $\mathcal{N} = 8$ gauged supergravity: 1) $SU(3)$-invariant sectors, 2) $SO(5)$-invariant sectors and 3) $SO(3) \times SO(3)$-invariant sector [30]. In order to find and study BPS domain-wall solutions by minimization of the energy-functional, one has to reorganize it into complete squares. Then one should expect that the scalar potential takes the form of squares of physical quantities. One important feature of the de Wit-Nicolai $d = 4, \mathcal{N} = 8$ supergravity is that the scalar potential can be written as the difference of two positive square terms. Together with kinetic terms this implies that one may construct the energy-functional in terms of complete squares.

The other gaugings of $\mathcal{N} = 8$ supergravity could be obtained in the same way the $SO(8)$ gauging. One could proceed in the same way as the de Wit-Nicolai theory by changing the supersymmetry transformations and adding them to the Lagrangian. Contrary to $\mathcal{N} = 4$ supergravity in four or seven dimensions, as a result of the complicated nonlinear tensorial structure, it is necessary to prove that the modified A_1 and A_2 tensors satisfy a number of rather involved and lengthy quantities as in [31], to demonstrate the supersymmetry of the theory. However, in [7, 8, 9, 10, 11, 12, 13, 14], an indirect and simple method which uses some results known in the de Wit-Nicolai theory was found to generate different gaugings from $SO(8)$ compact-gauged supergravity theory in such a way that one obtains the full nonlinear structure automatically and is guaranteed gauge invariance and supersymmetry. The first step was to construct a real, self-dual anti-symmetric $SO(p)^+ \times SO(q)^+$-invariant four-form tensor using both the generator of $SL(8, \mathbf{R})$ and $SO(8)$ Γ matrices. Next, it was necessary to describe the projectors that project the $SO(8)$ Lie algebra onto its subalgebras in terms of a four-form tensor in order to provide a convenient way to deal explicitly with the $SL(8, \mathbf{R})$ transformation. Then exhaustive manipulations of the invariance of four-form tensor were crucial for the existence of those gaugings and finiteness of coupling constant-dependent, covariant derivative terms as we take the infinity limit of some real parameter. Then we possess an explicit form for the

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[22], $SU(4)^-$-invariant vacuum [23], and a supersymmetric one with G_2 invariance. Among them, round $\mathbf{S}^7$- and G_2-invariant vacua are stable, while $SO(7)^{\pm}$-invariant ones are known to be unstable [24]. In [25, 26] three dimensional conformal field theories were classified by using AdS/CFT correspondence. In particular, researchers [27] have studied the $SU(3) \times U(1)$ critical point, from the point of higher dimensional analysis, which does not belong to the classification [19] but is a supersymmetric critical point of four-dimensional gauged supergravity.

T-tensor in terms of the standard parametrization of the scalar coset space.

In this paper, in section 2, we analyze known vacua of four-dimensional $\mathcal{N} = 8$ non-compact and non-semi-simple gauged supergravity developed by Hull [7, 8, 9, 10, 11, 12, 13, 14]. We claim no originality for most of the results presented in sections 2.1-2.3, although our derivations are hopefully illuminating. What we will do is to find out

- a superpotential from given T-tensor for known gauged supergravity theory and
- BPS domain-wall solutions from an energy-functional written in terms of complete squares.

In section 3, we will consider most general gaugings $CSO(p, q, r)^+$ initiated by Hull where $p + q + r = 8$ by using two successive $SL(8, \mathbf{R})$ transformations on the compact gauged supergravity. What we will do is to find

- a T-tensor in $SU(8)$-basis,
- a superpotential and a scalar potential from our findings of T-tensor and
- domain-wall solutions.

In section 4, we present our main results. In appendices, we present some details which are necessary for the calculations in sections 2 and 3.

2 Domain Wall from $SO(p)^+ \times SO(q)^+$ Sectors of $\mathcal{N} = 8$ Supergravity

Let us consider an ungauged supergravity theory with $\mathcal{N}$ local Majorana supersymmetries, $4 \leq \mathcal{N} \leq 8$ given by Cremmer-Julia theory [32] who constructed it by dimensionally reducing 11-dimensional supergravity. Recall that since a Majorana spinor in four-dimensions has four real components, the total number of supercharges for the maximal $\mathcal{N} = 8$ theory becomes 32. Note that there is no scalar field in the graviton multiplet for $\mathcal{N} < 4$. If the maximum spin is to be two, the number $\mathcal{N}$ can not be larger than 8. The scalar fields lie in a coset space G/H where G is some non-compact group and H its maximal compact subgroup. Group H is a local symmetry of the whole action while group G is a global symmetry of the equations of motion only(not the action) because it acts on the spin-1 fields through duality transformations. However, there exists a non-compact subgroup L of G, which is a global(rigid) symmetry of the action. One can gauge subgroup K of the global symmetry group L of the action where the dimension of K cannot exceed the number of vector fields in the model. To gauge the theory, one adds minimal Yang-Mills couplings for K both to the Lagrangian $\mathcal{L}_0$, which is the Lagrangian of the ungauged theory, and to the supersymmetry transformation rules of the ungauged theory with the vector fields of the theory acting as gauge connections. One should add coupling constant dependent terms to both the action and supersymmetry transformation laws in such a way that local supersymmetry is restored and gauge invariance is maintained.

Then one obtains a theory with Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_g$ where $\mathcal{L}_g$ consists of minimal gauge couplings with coupling constant g, fermionic bilinear terms proportional to g, and a scalar potential proportional to g^2. The minimal couplings and scalar potential break the symmetry G of the equations of motion and the symmetry L of the action down to K while leaving the local symmetry H unchanged. Then the gauge theory has both $H \times K$ local gauge symmetry and $\mathcal{N}$-extended local supersymmetry.

The ungauged $\mathcal{N} = 8$ supergravity(in this paper, we restrict ourselves to $\mathcal{N} = 8$ theory) has a symmetry $G \times H = E_{7\text{global}} \times SU(8)_{\text{local}}$ of the equations of motion where the 28 vectors correspond to a global Abelian symmetry between particles. Motivated by the fact that realistic theories of fundamental interactions are based on local, non-Abelian symmetries, de Wit and Nicolai [33, 31] gauged the subgroup $K = SO(8)$ of (the $L = SL(8, \mathbf{R})$ subgroup of) E_7 that is a global symmetry of $\mathcal{L}_0$, and obtained a theory with a local $K \times H = SO(8) \times SU(8)$ symmetry. The gauge group $K \subset L$ is a local symmetry: $\mathcal{L} \to \mathcal{L}$ under K while the remainder $L \backslash K$ of the non-compact group L is a global symmetry of $\mathcal{L}_0$ but not of $\mathcal{L}_g$:$\mathcal{L} \to \mathcal{L}' = \mathcal{L}_0 + \mathcal{L}'_g$ under $L \backslash K$. In other words, acting with $L \backslash K$ changes the gauge covariantizations, fermion bilinear terms, and scalar potential in $\mathcal{L}_g$ while keeping the $\mathcal{L}_0$ unaffected. This is an invertible field-redefinition for finite value of t which appears in (1) that leads to an equivalent theory, invariant under the local supersymmetries and local gauge symmetry.

The contraction procedure involves a sequence of change of basis transformations depending on the parameter [34]. Although the transformation becomes singular in the zero limit of a parameter, the Lie bracket exists and is well defined in this singular limit. The original and contracted algebras are not isomorphic. Note that non-singular changes of bases can never lead to new algebras because under such a transformation the new structure constant tensor possesses exactly as much information as the original. Let us consider a sequence of non-singular elements $E(\xi)$ of L with ξ real parameter and $E(1) = 1$, and an identity transformation, whose limit point $E(0)$ is singular and not in L. As long as $E(\xi)$ remains nonsingular($\xi \neq 0$), the structure constants have the usual tensor properties. Acting on the Lagrangian with $E(\xi)$ yields a sequence of Lagrangian:$\mathcal{L} \to \mathcal{L}'(\xi) = \mathcal{L}_0 + {\mathcal{L}_g}'(\xi)$. If one also rescales the coupling constant g to an ξ-dependent one through $g \to g'(\xi)$ for some choices of the sequence $E(\xi)$ in L, the limit of ${\mathcal{L}_g}'(\xi)$ as $\xi \to 0 (\equiv {\mathcal{L}_g}'(0))$ exists and is well defined(the new structure constants characterize a Lie algebra) so that $\mathcal{L}'(0) = \mathcal{L}_0 + {\mathcal{L}_g}'(0)$ gives the Lagrangian for a gauge-invariant supersymmetric theory. The gauge group corresponding to $\mathcal{L}'(0)$ is not $K = SO(8)$ itself but an Inonu-Wigner [35] contraction of K denoted by $CSO(p, q)^+$ with $p + q = 8$ [10, 11]. A new(different from the de Wit-Nicolai compact gauged supergravity theory) gauging, inequivalent to the original one, is obtained by a singular, noninvertible field redefinition. One can also continue the Lagrangian $\mathcal{L}'(\xi)$ to negative values of ξ. In this case, $\mathcal{L}'(-1)$ is the Lagrangian for another gauging and the gauge group is non-compact $SO(p, q)^+$ with $p + q = 8$

[10, 11].

In section 2.1, starting with the action of $L = SL(8, \mathbf{R})$ element on the de Wit-Nicolai theory, we present a superpotential which is an eigenvalue of A_1 tensor, that is, a partially contracted T-tensor. In section 2.2, with explicit ξ-dependence on the T-tensor, one recovers more general scalar potetential which will be reduced to the one in section 2.1 when we put $\xi = 0$ and obtains a more general superpotential. In section 2.3, we present other cases, $CSO(p,q)^+$ and $SO(p,q)^+$ gaugings where $p = 6, 5, 4, 3, 2, 1$ and $q = 8 - p$ and review their critical points in a scalar potential. In section 2.4, we construct a domain-wall solution from an energy-functional. Finally in section 2.5, as an aside, we will concentrate on the construction of a scalar potential for the vacuum expectation value given in terms of real, anti-self-dual(not self-dual), totally anti-symmetric tensor. The parametrization for this singlet-space is invariant under the $SO(p)^- \times SO(q)^-$ where $p + q = 8$.

2.1 Superpotential in $CSO(7,1)^+ = ISO(7)^+$ Gauging [7]

Following the procedure we have introduced, the action of the non-compact part of $SL(8, \mathbf{R})$, $L \backslash K$, on the theory can be used to other gauged $\mathcal{N} = 8$ supergravity. Let us consider the acting with the $L = SL(8, \mathbf{R}) \subset E_{7(+7)}$ element

$$E(t) = \exp \begin{pmatrix} 0 & tX^{+IJKL} \\ tX^{+}_{IJKL} & 0 \end{pmatrix}, \tag{1}$$

on the de Wit-Nicolai theory where t is a real parameter proportional to $-\ln \xi$ where ξ was introduced before and X^{+IJKL} is some real and self-dual totally antisymmetric tensor that satisfies

$$X^{+IJKL} = \overline{X}^{+}_{IJKL} = \frac{\eta}{24} \epsilon^{IJKLMNPQ} X^{+MNPQ}.$$

Since $E(t)$ is in the real $SL(8, \mathbf{R})$ subgroup of $E_{7(+7)}$, the ungauged Cremmer-Julia action $\mathcal{L}_0$ remains unchanged but the g-dependent part $\mathcal{L}_g$ is modified nontrivially(changes the minimal couplings and rotates the A_1 and A_2 tensors into one another). This gives one-parameter family of Lagrangian related to the de Wit-Nicolai theory($t = 0$ where $E(0) = 1$, identity transformation, or equivalently $\xi = 1$ and $E(\xi = 1) = 1$) by the $SL(8, \mathbf{R})$ field-redefinition given by $E(t)$. For all finite values of t, this yields a theory which is equivalent to the de Wit-Nicolai theory by field-redefinition. However, a different gauging might be found in the limit $t \to \infty$(equivalent to $\xi \to 0$) if it exists. For many choices of the four-form X^{+IJKL}, the limit does not exist. The simplest and special choice for which this limit exists [7] is[2]

$$X^{+IJKL} = Y^{IJKL} + \frac{\eta}{24} \epsilon^{IJKLMNPQ} Y^{MNPQ}, \tag{2}$$

[2] We emphasize that the way we have chosen X^{+IJKL} here is different from the one in [30] in the sense that in [30] the $SU(2)$ matrix of $SU(8)$ appears in the last 2×2 block diagonal. However, in this paper, we take it as the first 2×2 block diagonal matrix. The nonzero-component of X^{+IJKL} is either $1/2$ or $-1/2$ as in [7].

where

$$Y^{IJKL} = \frac{1}{2}\left(\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,2\,5\,6} + \delta^{IJKL}_{1\,2\,7\,8} + \delta^{IJKL}_{1\,3\,7\,5} + \delta^{IJKL}_{1\,3\,6\,8} + \delta^{IJKL}_{1\,4\,5\,8} + \delta^{IJKL}_{1\,4\,6\,7}\right).$$

Here $\eta = +1$ for $SO(7)^+$-invariant X^{+IJKL} and δ^{IJKL}_{MNPQ} has 1 when I, J, K and L form an even permutation of M, N, P, Q and -1 when they form an odd permutation of M, N, P, Q and vanishes. We will come to the case in which $\eta = -1$ later in section 2.5 which holds for $SO(7)^-$-invariant X^{-IJKL}. The four-form tensor X^{+IJKL} is closely related to the torsion parallelizing seven-sphere $\mathbf{S}^7$ [24, 21, 36, 37, 38] and invariant under the $SO(7)^+$-subgroup of $SO(8)$. Turning on the vacuum expectation value proportional to X^{+IJKL} in the de Wit-Nicolai theory gives rise to spontaneous symmetry breaking of $SO(8)$ into $SO(7)^+$. Regarded as 28×28 matrices, X^{+IJKL} has 21 eigenvalues of -1 and 7 eigenvalues of $+3$. Introducing the projector P_+ into the 21-dimensional eigenspace(P_+ projects the generators of $SO(8)$ onto those of $SO(7)^+$ while P_- projects the generators of $SO(8)$ into the remainder $SO(8)\backslash SO(7)^+$), [3] they are given in terms of X^{+IJKL}

$$P_+^{IJKL} = \frac{3}{4}\left(\delta^{IJ}_{KL} - \frac{1}{3}X^{+IJKL}\right),$$

and[4]

$$P_-^{IJKL} = \delta^{IJ}_{KL} - P_+^{IJKL} = \frac{1}{4}\left(\delta^{KL}_{IJ} + X^{+IJKL}\right).$$

Therefore, one has

$$X^{+IJKL} = -P_{+IJKL} + 3P_{-IJKL}. \tag{3}$$

One can easily check that the projectors have the following properties[5] which will be used throughout this paper

$$P_\pm^2 = P_\pm, \qquad P_\pm P_\mp = 0.$$

Here the product $P_\pm^2$ is that of 28×28 matrices, $(P_\pm^2)^{IJKL} = P_\pm^{IJMN} P_\pm^{MNKL}$. The 28 $SO(8)$ generators Λ^{IJ} are projected onto a 21-dimensional subspace by P_+, $\Lambda_+^{IJ} = P_+^{IJKL}\Lambda^{KL}$ and this subspace is the Lie algebra for the $SO(7)^+$-subgroup of $SO(8)$; in other words, the subgroup stabilizes a right-handed side $SO(8)$ spinor(See the appendix B). Similarly, the remaining 7 generators are generated by $\Lambda_-^{IJ} = P_-^{IJKL}\Lambda^{KL}$.

[3] Note that although the subscript minus sign in P_- is nothing to do with the anti-self dual part $SO(7)^-$ of $SO(8)$, we will follow the same notation as in the previous literature [7]. In section 2.5, we take those projectors as P_1 and P_2.

[4] δ^{IJ}_{KL} is defined as $\delta^{IJ}_{KL} = \frac{1}{2!2!}\left(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K - \delta^J_K\delta^I_L + \delta^J_L\delta^I_K\right) = \frac{1}{2}\left(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K\right)$.

[5] In terms of X^{+IJKL}, we have the following relation, $\left(\delta^{IJ}_{KL} - \frac{1}{3}X^{+IJKL}\right)\left(\delta^{KL}_{IJ} + X^{+IJKL}\right) = 0$.

The change of the minimal couplings under supersymmetry gives a net change of the action under an infinitesimal local supersymmetry that can be parametrized by a T-tensor [7]. An expression for the T-tensor, $T_i'^{\,jkl}$ can be obtained by realizing that a variation of A_μ^{IJ} leads to a variation of the $SU(8)$-connection $\mathcal{B}_{\mu i}{}^{j}$

$$\begin{aligned} T_i'^{\,jkl} &= \left(\overline{u}^{kl}{}_{IJ} + \overline{v}^{klIJ}\right)\Big[M_{IJKL}\left(u_{im}{}^{KM}\overline{u}^{jm}{}_{LM} - v_{imKM}\overline{v}^{jmLM}\right) \\ &\quad + N_{IJ}{}^{KLMN}\left(v_{imKL}\overline{u}^{jm}{}_{MN} - u_{im}{}^{KL}\overline{v}^{jmMN}\right)\Big] \end{aligned} \tag{4}$$

where M_{IJKL} and $N_{IJ}{}^{KLMN}$ are defined in terms of projectors

$$\begin{aligned} M_{IJKL} &= P_{+IJKL} + \frac{1}{2}P_{-IJKL}, \\ N_{IJ}{}^{KLMN} &= \frac{1}{2}P_{-}^{IJ[K}{}_{[P}\delta^{L]}{}_{Q]}\left(P_{-}^{PQMN} - P_{+}^{PQMN}\right). \end{aligned}$$

The supersymmetry of the theory is restored by adding $\mathcal{L}'_g$ to the ungauged action $\mathcal{L}_0$ and to the supersymmetry transformation rules where A_1, A_2 tensors, that appear in $\mathcal{L}'_g$, have a functional dependence on the scalar field but with T' tensor. That is, for example,

$$A_1'^{\,ij} = -\frac{4}{21}T_m'^{\,ijm}, \quad A_{2l}'^{\,ijk} = -\frac{4}{3}T_l'^{\,[ijk]}. \tag{5}$$

The parametrization for the $SO(7)^+$-singlet space[6] that is an invariant subspace under a particular $SO(7)^+$ subgroup of $SO(8)$ becomes

$$\phi_{IJKL} = 4\sqrt{2}sX^{+}_{IJKL}$$

where s is a real scalar field.

Therefore, 56-beins $\mathcal{V}(x)$ can be written as a 56×56 matrix whose elements are the function of scalar s by exponentiating the vacuum expectation value ϕ_{IJKL}. On the other hand, 28-beins $u_{ij}{}^{KL}$ and v_{ijKL} are elements of this $\mathcal{V}(x)$. One can explicitly construct 28-beins $u_{ij}{}^{KL}$ and v_{ijKL} in terms of scalar s and they are given in the appendix E (60). Now the complete expression for A_1' and A_2' tensors are given in terms of s using (4) and (5). It turns out from (5) that A_1' tensor has a single real eigenvalue, z_1 with degeneracies 8 and has the following form

$$A_1'^{\,ij} = \mathrm{diag}\left(z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1\right), \qquad z_1 = \frac{7}{8}e^s. \tag{6}$$

Similarly, A_2' tensor can be obtained from the triple product of $u_{ij}{}^{KL}$ and v_{ijKL} fields, that is, from (5). They can be written as

$$A_{2i}'^{\,jkl} = \frac{1}{4}e^s X^{+ijkl}. \tag{7}$$

[6]The 35-dimensional fourth rank self-dual antisymmetric tensor representation of $SO(8)$ splits into the $SO(7)^+$ representation $\mathbf{35} \to \mathbf{27} + \mathbf{7} + \mathbf{1}$ where the singlet $\mathbf{1}$ is nothing but $SO(7)^+$-invariant tensor X^{+IJKL}.

Finally, the scalar potential together with new A_1' and A_2' tensors can be written, by combining all the components of A_1', A_2' tensors, as [7, 9]

$$V_{7,1,\xi=0} = -g^2 \left(\frac{3}{4} |A_1'^{\ ij}|^2 - \frac{1}{24} |A_{2i}'^{\ jkl}|^2 \right) = -\frac{35}{8} g^2 e^{2s} \tag{8}$$

which implies that there is no $SO(7)^+$-invariant critical point of potential by differentiating this scalar potential with repect to a field s. The eigenvalue z_1 provides a superpotential which will be analyzed in detail in section 2.3-2.4. The scalar potential can be written as

$$V_{7,1,\xi=0} = g^2 \left[\frac{2}{7} (\partial_s z_1)^2 - 6 z_1^2 \right] = g^2 \left[4 (\partial_{\widetilde{s}} z_1)^2 - 6 z_1^2 \right]$$

where $\widetilde{s} = \sqrt{14} s$. The theory [7] constitutes a gauging of the 28-dimensional, non-compact $ISO(7)^+$ symmetry of the Cremmer-Julia action $\mathcal{L}_0$. The theory has $\mathcal{N} = 8$ local supersymmetry and $H \times K_{\xi=0,p=7,q=1} = SU(8) \times ISO(7)^+$ local gauge symmetry where $ISO(7)^+$ is the isometry group of Euclidean 7 space, $\mathbf{R}^7$. In the symmetric gauge the diagonal $SO(7)^+$ subgroup is manifest. The non-compact gauged $\mathcal{N} = 8$ supergravity theories can be obtained by compactification of 11-dimensional supergravity on hyperboloids of a constant negative curvature. The contracted version corresponds to a limit in which the hyperboloid degenerates to an infinite cylinder [14]. Thus, the $ISO(7)^+$ theory corresponds to a compactification on the cylinder $\mathbf{S}^6 \times \mathbf{R}^1$ that can be replaced by $\mathbf{S}^6 \times \mathbf{S}^1$ because the near-horizon limit of the D2-brane is different from that of M2-brane [1]. As near-horizon limits of the k torus T^k reduction of the M2-brane, one expects that the corresponding theory is $CSO(8-k,k)^+$ gauged $\mathcal{N} = 8$, $d = 4$ supergravity.

2.2 Superpotential in Non-compact $SO(7,1)^+$ Gauging [8]

A suitable one-parameter family (that cannot be more than 28-dimensional) where ξ is a real parameter of 28-dimensional subgroups of $L = SL(8, \mathbf{R})$ each parametrized by some real antisymmetric generator, is generated with nonzero ξ. The commutation relations of the generators are given in the previous subsection and the only difference is that there exists another nonzero commutator. Then the 21 Λ_+ generate $SO(7)^+$ group in which the seven linearly independent Λ_- transforms as a **7** representation. When $\xi > 0$, the algebra is that of $SO(8)$ and normalization is obtained when $\xi = 1$. When $\xi < 0$, one obtains $SO(7,1)^+$, the normalization being obtained when $\xi = -1$. When $\xi = 0$, it gives $ISO(7)^+$ as done previously. Then the one parameter family of gauged $\mathcal{N} = 8$ supergravities can be described by inserting ξ-dependent terms where the T-tensor is given by [8]

$$\begin{aligned} T_i'^{\ jkl}(\xi) &= T_i^{\ jkl} - (1-\xi) \left(\overline{u}^{kl}{}_{IJ} + \overline{v}^{klIJ} \right) \\ &\times \left[\frac{1}{2} P_-^{IJKL} \left(u_{im}{}^{KM} \overline{u}^{jm}{}_{LM} - v_{imKM} \overline{v}^{jmLM} \right) \right. \end{aligned}$$

$$+N_{IJ}{}^{KLMN}\left(v_{imKL}\overline{u}^{jm}{}_{MN}-u_{im}{}^{KL}\overline{v}^{jmMN}\right)\Big]. \quad (9)$$

When $\xi = 1$, one gets the de Wit-Nicolai model with $SU(8)\times SO(8)$ gauge symmetry. When $\xi = 0$, one has $SU(8)\times ISO(7)^+$ gauge symmetry. Moreover, when $\xi = -1$, a different theory with $SU(8)\times SO(7,1)^+$ gauge symmetry was obtained. All the $\xi < 0$ theories are equivalent to $\xi = -1$ model related to the $SL(8, \mathbf{R})$ transformation and all the $\xi > 0$ theories are equivalent to $\xi = 1$ de Wit-Nicolai theory related to the $SL(8, \mathbf{R})$ transformation. Moreover, the $\xi = 0$ theory was obtained by limiting either $\xi > 0$ or $\xi < 0$ models, under which $SO(8)$ or $SO(7,1)^+$ is transformed from an Inonu-Wigner contraction to $ISO(7)^+$. As we have done before, we can describe 28-beins in terms of s. It turns out that the A_1' tensor has a single eigenvalue z_1 with a multiplicity of 8 which will provide a superpotential of a scalar potential and has the following form generalizing (6)

$$A_1'^{\ ij} = \mathrm{diag}\left(z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1\right), \qquad z_1 = \frac{1}{8}\left(7e^{s} + \xi e^{-7s}\right). \quad (10)$$

Additionally, we can construct an A_2' tensor generalizing (7) which is the combination of the triple product of 28 beins, as given in

$$A_{2i}'^{\ jkl} = \frac{1}{4}\left(e^{s} - \xi e^{-7s}\right) X^{+ijkl}. \quad (11)$$

Therefore, the scalar potential generalizing (8) in the $SO(7)^+$-invariant direction by summing all the components of A_1' and A_2' tensors and counting the degeneracies correctly is given by [8, 9]

$$V_{7,1,\xi} = -g^2\left(\frac{3}{4}|A_1'^{\ ij}|^2 - \frac{1}{24}|A_{2i}'^{\ jkl}|^2\right) = \frac{1}{8}g^2\left(-35e^{2s} - 14\xi e^{-6s} + \xi^2 e^{-14s}\right).$$

This can be written as a superpotential:$V_{7,1,\xi} = g^2\left[\frac{2}{7}(\partial_s z_1)^2 - 6z_1^2\right] = g^2\left[4(\partial_{\widetilde{s}} z_1)^2 - 6z_1^2\right]$ where $\widetilde{s} = \sqrt{14}s$. It is easily determined that there are no $SO(7)^+$-invariant critical points. The theory [8] has $\mathcal{N} = 8$ local supersymmetry and $H \times K_{\xi=-1,p=7,q=1} = SU(8)\times SO(7,1)^+$ local gauge symmetry. $SO(7,1)^+$ gauge symmetry is broken down to its compact subgroup.

2.3 Superpotential in Other $CSO(p,q)^+$ and $SO(p,q)^+$ Gaugings [10, 11]

Starting from the $SO(8)$ gauging, the $ISO(7)^+$ and $SO(7,1)^+$ gaugings were obtained by exploiting the transformations generated by the $SO(7)^+$-invariant fourth rank antisymmetric tensor. Now if one uses the $SO(p)^+\times SO(8-p)^+$-invariant fourth rank tensor to generate transformations, one expects an $SO(p, 8-p)^+$ gauging and a gauging of a certain contraction of $SO(p, 8-p)^+$ about its compact subgroup $SO(p)^+$ [10, 11]. Let us consider the $SO(p)^+\times SO(q)^+$

invariant generator of $SL(8,\mathbf{R})$,

$$X_{ab} = \begin{pmatrix} \alpha \mathbf{1}_{p\times p} & 0 \\ 0 & \beta \mathbf{1}_{q\times q} \end{pmatrix}$$

with

$$\alpha p + \beta q = 0, \qquad p + q = 8$$

where $\mathbf{1}_{p\times p}$ is the $p \times p$ identity matrix. The embedding of this $SL(8,\mathbf{R})$ in E_7 is such that X_{ab} corresponds to the 56×56 E_7 generator which is a non-compact $SO(p)^+ \times SO(q)^+$ invariant element of the $SL(8,\mathbf{R})$ subalgebra of E_7

$$\begin{pmatrix} 0 & X^{+IJKL} \\ X^+_{IJKL} & 0 \end{pmatrix},$$

where the real, self-dual totally anti-symmetric $SO(p)^+ \times SO(q)^+$ invariant four-form tensor X^+_{IJKL} can be written in terms of a symmetric, trace-free, 8×8 matrix with $SO(8)$ right-handed spinor indices, X_{ab} using $SO(8)$ Γ matrices(See appendix B)

$$X^+_{IJKL} = -\frac{1}{8} \left(\Gamma_{IJKL}\right)^{ab} X_{ab} \tag{12}$$

where $\Gamma_{IJKL} = \Gamma_{[I}\Gamma_J\Gamma_K\Gamma_{L]}$ and an arbitrary $SO(8)$ generator L_{IJ} acts in the right-handed spinor representation by $(L_{IJ}\Gamma_{IJ})^{ab}$. When $p = 7$ and $q = 1$, this expression of (12) through Γ matrix coincides exactly with the one in (2). We also present (12) explicitly in Appendix A for various p and q.

Regarded as a 28×28 matrix, X^{+IJKL} has eigenvalues α, β and $\gamma = (\alpha + \beta)/2$ with degeneracies d_α, d_β and d_γ respectively. Let it be recalled that $SO(7)^+$-invariant four-form tensor has eigenvalues of -1 and $+3$. The eigenvalues and eigenspaces of the $SO(p)^+ \times SO(q)^+$ invariant tensor are summarized in Table 1, including the case of $(p, q) = (7, 1)$. By introducing projectors as done in previous cases, P_α, P_β and P_γ onto corresponding eigenspaces, we have a 28×28 matrix equation that generalizes (3) to arbitrary p and q

$$X^{+IJKL} = \alpha P_\alpha^{IJKL} + \beta P_\beta^{IJKL} + \gamma P_\gamma^{IJKL}.$$

Projector $P_\alpha(P_\beta)$ projects the $SO(8)$ Lie algebra onto its $SO(p)^+(SO(q)^+)$ subalgebra while P_γ does onto the remainder $SO(8)/(SO(p)^+ \times SO(q)^+)$.

p	q	α	β	$\gamma = (\alpha+\beta)/2$	$d_\alpha = p(p-1)/2$	$d_\beta = q(q-1)/2$	$d_\gamma = pq$	$\lvert X^+\rvert^2$
7	1	-1	7	3	21	0	7	84
6	2	-1	3	1	15	1	12	36
5	3	-1	5/3	1/3	10	3	15	20
4	4	-1	1	0	6	6	16	12
3	5	-1	3/5	$-1/5$	3	10	15	36/5
2	6	-1	1/3	$-1/3$	1	15	12	4
1	7	-1	1/7	$-3/7$	0	21	7	12/7

Table 1. *Eigenvalues and eigenspaces of the $SO(p)^+ \times SO(q)^+$ invariant tensor, X^+ where $|X^+|^2 = d_\alpha|\alpha|^2 + d_\beta|\beta|^2 + d_\gamma|\gamma|^2$. We have taken this table from [10]. In [39], they displayed the signature of the Killing-Cartan form by writing the numbers n_+, n_- and n_0 of its positive, negative and zero eigenvalues. Here we identify $d_\alpha + d_\beta$ with n_+ and d_γ with n_-.*

Then the ξ-dependent T-tensor [10, 11] has a much more complicated expression that generalizes (9)

$$\begin{aligned} T_i^{\prime\ jkl}(\xi) &= T_i^{\ jkl} - (1-\xi)\left(\overline{u}^{kl}{}_{IJ} + \overline{v}^{klIJ}\right) \\ &\times \left[\left(P_\beta^{IJKL} + \frac{1}{2}P_\gamma^{IJKL}\right)\left(u_{im}{}^{KM}\overline{u}^{jm}{}_{LM} - v_{imKM}\overline{v}^{jmLM}\right)\right. \\ &\left.+P_\gamma^{IJRS} Z_{RS}^{KLMN}\left(-v_{imKL}\overline{u}^{jm}{}_{MN} + u_{im}{}^{KL}\overline{v}^{jmMN}\right)\right] \end{aligned} \tag{13}$$

where we introduce the new quantity Z_{IJKL}^{MN}

$$Z_{IJKL}^{MN} = \frac{1}{2}\left[(P_\alpha - P_\beta)_{IJMP} P_\gamma^{NPKL} - P_\gamma^{IJMP}(P_\alpha - P_\beta)_{NPKL}\right]. \tag{14}$$

The 28-beins $u_{ij}{}^{KL}$ and v_{ijKL} are given in Appendix E and the projectors $P_\sigma^{IJKL}(\sigma = \alpha, \beta, \gamma)$ are given in Appendix F. This T' tensor [10] defines new A_1' and A_2' tensors. These models will have $\mathcal{N} = 8$ local supersymmetry and local $SU(8) \times K_{\xi,p,q}$ invariance. The gauge groups are

$$SO(7,1)^+, \quad SO(6,2)^+, \quad SO(5,3)^+ \quad \text{and} \quad SO(4,4)^+,$$

when $\xi = -1(t = i\pi/(\alpha - \beta))$. When $\xi = 0(t = \infty)$ there exist the inhomogeneous groups

$$\begin{aligned} &CSO(7,1)^+ = ISO(7)^+, \quad CSO(6,2)^+, \quad CSO(5,3)^+, \quad CSO(4,4)^+, \\ &CSO(3,5)^+, \quad CSO(2,6)^+ \quad \text{and} \quad CSO(1,7)^+. \end{aligned}$$

Any other choice of $\xi > 0(\xi < 0)$ gives a model equivalent to the $SO(8)(SO(p,q)^+)$ gauging by field-redefinition. The gauge symmetry $K_{\xi,p,q}$ is broken down to its maximal compact subgroup or some subgroup thereof. There are three inequivalent distinct gaugings. From the expression (13) one gets a single eigenvalue z_1 with degeneracies 8 which has the following form

$$A_1^{\prime\ ij} = \text{diag}\left(z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1\right), \qquad z_1 = \frac{1}{8}\left(pe^s + q\xi e^{-\frac{p}{q}s}\right) \tag{15}$$

which include all the cases p, q and ξ and generalize (10). Similarly, one can construct A_2' generalizing (11)

$$A_{2i}^{\prime\ jkl} = \frac{q}{4}\left(e^s - \xi e^{-\frac{p}{q}s}\right) X^{+ijkl}. \tag{16}$$

Finally the $K_{\xi,p,q}$-invariant scalar potential as a function of p, q, ξ and s by counting the degeneracies correctly can be written as[7]

$$\begin{aligned} V &= -g^2\left(\frac{3}{4}|A_1'^{\,ij}|^2 - \frac{1}{24}|A_{2i}'^{\,jkl}|^2\right) \\ &= -g^2\left(\frac{3}{4}\times 8\times\left(\frac{1}{8}\left(pe^s+q\xi e^{-\frac{p}{q}s}\right)\right)^2 - \frac{1}{24}\times\left(\frac{q}{4}\left(e^s-\xi e^{-\frac{p}{q}s}\right)|X^{+ijkl}|\right)^2\right) \end{aligned}$$

with

$$|X^+|^2 = \frac{1}{2}p(p-1)|\alpha|^2 + \frac{1}{2}q(q-1)|\beta|^2 + pq|\gamma|^2.$$

The potentials $V_{p,q,\xi}$ for the $K_{\xi,p,q}$ gauging are given by [11]

$$\begin{aligned} V_{7,1,\xi} &= \frac{1}{8}g^2\left(-35e^{2s} - 14\xi e^{-6s} + \xi^2 e^{-14s}\right), \\ V_{6,2,\xi} &= -3g^2\left(e^{2s} + \xi e^{-2s}\right), \\ V_{5,3,\xi} &= -\frac{3}{8}g^2\left(5e^{2s} + 10\xi e^{-2s/3} + \xi^2 e^{-10s/3}\right), \\ V_{4,4,\xi} &= -g^2\left(e^{2s} + 4\xi + \xi^2 e^{-2s}\right), \\ V_{3,5,\xi} &= -\frac{3}{8}g^2\left(e^{2s} + 10\xi e^{2s/5} + 5\xi^2 e^{-6s/5}\right), \\ V_{2,6,\xi} &= -3g^2\xi\left(e^{2s/3} + \xi e^{-2s/3}\right), \\ V_{1,7,\xi} &= \frac{1}{8}g^2\left(e^{2s} - 14\xi e^{6s/7} - 35\xi^2 e^{-2s/7}\right). \end{aligned} \tag{17}$$

Of course, the potential $V_{7,1,\xi}$ is identical to the one in previous sections 2.1 and 2.2 and is obtained by putting $p = 7$ and $q = 1$ into the general expression of a scalar potential. Note that for $\xi = -1$, the potentials for the $SO(p,q)^+$ gauging and the $SO(q,p)^+$ gauging coincide with each other due to the fact that the potential $V_{p,q,\xi}$ can be obtained from $V_{q,p,\xi}$ by rescaling $s \to -ps/q$. However, this is not true for $\xi = 0$ because $V_{p,q,\xi=0} \neq V_{q,p,\xi=0}$.

From the above effective non-trivial scalar potential one expects that the superpotential W maybe encoded in either the A_1' or A_2' tensors. It turns out that the eigenvalue of the A_1' tensor z_1 provides a superpotential and one can check that the scalar potential can be written in terms of a superpotential as follows, observed newly in this paper

$$\begin{aligned} W_{p,q}(\xi;s) &= z_1 = \frac{1}{8}\left(pe^s + q\xi e^{-\frac{p}{q}s}\right) = \frac{1}{8}\left(pe^{\sqrt{\frac{q}{2p}}\widetilde{s}} + q\xi e^{-\sqrt{\frac{p}{2q}}\widetilde{s}}\right), \\ V_{p,q}(\xi;s) &= g^2\left[\frac{2q}{p}\left(\partial_s W_{p,q}(\xi;\widetilde{s})\right)^2 - 6W_{p,q}(\xi;\widetilde{s})^2\right] \\ &= g^2\left[4\left(\partial_{\widetilde{s}} W_{p,q}(\xi;\widetilde{s})\right)^2 - 6W_{p,q}(\xi;\widetilde{s})^2\right] \end{aligned} \tag{18}$$

[7] It is known [12] that for finite real t, the T-tensor can be obtained from the old one, de Wit-Nicolai T-tensor by replacing $\mathcal{V}$ with $\mathcal{V}E(t)^{-1}$ and scaling by a factor of $e^{\alpha t}$:$T_i'^{\;jkl}(\mathcal{V}) = e^{\alpha t}T_i^{\;jkl}(\mathcal{V}E(t)^{-1})$. This can be used to give a simple calculation of the potential in the $SO(p)^+ \times SO(q)^+$ invariant direction in the space of scalar field.

where $\widetilde{s} = \sqrt{\frac{2p}{q}} s$. The scalar potential has critical points at 1) critical points of superpotential and at 2) points for which superpotential satisfies some differential equation. By differentiating W with respect to field s, one finds that there are no critical points of superpotential corresponding to supersymmetric critical points except the trivial critical point which has $\mathcal{N} = 8$ supersymmetry and whose cosmological constant $\Lambda = -6g^2$ for which $W = 1$. The other critical points of scalar potential yield nonsupersymmetric vacua that may or may not be stable. The superpotential has the following values at the various critical points.

Gauge symmetry	$\mathcal{N}$	p	$q = 8 - p$	ξ	s	W	V
$SO(8)$	8	any	any	1	0	1	$-6g^2$
$SO(7)^+ \times SO(1)^+$	0	7	1	1	$-\frac{1}{8}\ln 5$	$\frac{3}{2} \times 5^{-1/8}$	$-2 \times 5^{3/4} g^2$
$SO(5)^+ \times SO(3)^+$	0	5	3	-1	$-\frac{3}{8}\ln 3$	$-\frac{1}{2} \times 3^{-3/8}$	$2 \times 3^{1/4} g^2$
$SO(4)^+ \times SO(4)^+$	0	4	4	-1	0	0	$2g^2$
$SO(3)^+ \times SO(5)^+$	0	3	5	-1	$\frac{5}{8}\ln 3$	$\frac{1}{2} \times 3^{-3/8}$	$2 \times 3^{1/4} g^2$
$SO(2)^+ \times U(1)^{+15}$	0	2	6	0	any	$e^s/4$	0
$SO(1)^+ \times SO(7)^+$	0	1	7	1	$\frac{7}{8}\ln 5$	$\frac{3}{2} \times 5^{-1/8}$	$-2 \times 5^{3/4} g^2$

Table 2. *Summary of various critical points [11] in the context of superpotential observed in this paper first : Gauge symmetry, supersymmetry, vacuum expectation value of field, superpotential and cosmological constants. For $SO(3)^+ \times SO(5)^+$ case, one can check it by the change of variable of $SO(5)^+ \times SO(3)^+$ case, $s \to -3s/5$ that corresponding potential of $SO(3)^+ \times SO(5)^+$ is obtained while by change of variable, $s \to -s/7$, the potential of $SO(1)^+ \times SO(7)^+$ can be found from $SO(7)^+ \times SO(1)^+$ case. Although the corresponding superpotential of these two cases may be different from the original ones, the scalar potentials are the same.*

• $SO(8)$ case: $\mathcal{N} = 8$

By differentiating the scalar potential with respect to real scalar field s, there exists a solution of $s = 0$ when $\xi = 1$ for all possible values of p and q. This is nothing but de Wit-Nicolai's $SO(8)$-invariant critical point and vacuum, which is fully supersymmetric(because in this case, $\partial_s W|_{s=0} = 0$ implying that $V = -6g^2 W^2$. In other words, $|W| = \sqrt{-V/6g^2}$. All the eight eigenvalues of the A_1' tensor give rise to the number of supersymmetries) and hence are stable. All the scalar potential $V_{p,q,\xi}$ becomes $-6g^2$ when $s = 0$ for $\xi = 1$.

• $SO(7)^+ \times SO(1)^+$ case: $\mathcal{N} = 0$

This is exactly the $SO(7)^+$-invariant critical point of the $SO(8)$ theory. As in Table 2, it has no supersymmetry and is unstable.

• $SO(5)^+ \times SO(3)^+$ case: $\mathcal{N} = 0$

In this case, the value of the scalar potential gives a positive cosmological constant where the eigenvalue of the A_1' tensor is $-\frac{1}{2} \times 3^{-3/8}$ and the A_2' tensor has a value of $2 \times 3^{5/8} X^{+ijkl}$. It is known to be unstable.

• $SO(4)^+ \times SO(4)^+$ case: $\mathcal{N}=0$

At this critical point, the value of the scalar potential gives a positive cosmological constant where the A_1' tensor vanishes and the A_2' tensor has a value of $4X^{+ijkl}$. It is known to be unstable. The positivity of the cosmological constant from the analysis of 11-dimensional field equations for $SO(5,3)^+$ and $SO(4,4)^+$ theories was confirmed in [14].

• $SO(2)^+ \times U(1)^{+15}$ case: $\mathcal{N}=0$

When $\xi = 0$, the potential vanishes implying that for any value of s, there exists a zero cosmological constant critical point. In addition, the potential is also flat in the $SO(2)^+ \times SO(6)^+$-invariant direction. Nonetheless, global $SO(6)^+$ symmetry remains unbroken by the vacuum. In this case, the eigenvalue of the A_1' tensor is equal to $e^s/4$ and the A_2' tensor is $3e^s X^{+ijkl}$.

2.4 Domain Wall in $CSO(p,q)^+$ and $SO(p,q)^+$ Gaugings [10, 11]

Let us begin with the resulting Lagrangian of the scalar-gravity sector by explicitly determining the scalar kinetic terms appearing in the action in terms of s. The scalar kinetic term is $-\frac{1}{96}\left|A_\mu{}^{ijkl}\right|^2$ where the generalized g-dependent $A_\mu{}^{ijkl}$ can be obtained

$$\begin{aligned} A_\mu{}^{ijkl} &= -2\sqrt{2}\left(\overline{u}^{ij}{}_{IJ}\partial_\mu \overline{v}^{klIJ} - \overline{v}^{ijIJ}\partial_\mu \overline{u}^{kl}{}_{IJ}\right) \\ &\quad +4\sqrt{2}(1-\xi) g A_{\mu IJ}\left[\left(P_\beta^{IJKL} + \frac{1}{2}P_\gamma^{IJKL}\right)\left(-\overline{u}^{ij}{}_{KM}\overline{v}^{klLM} + \overline{v}^{ijKM}\overline{u}^{kl}{}_{LM}\right)\right. \\ &\quad \left. +P_\gamma^{IJRS} Z_{RS}^{KLMN}\left(\overline{u}^{ij}{}_{KL}\overline{u}^{kl}{}_{MN} - \overline{v}^{ijKL}\overline{v}^{klMN}\right)\right]. \end{aligned} \tag{19}$$

By taking the product of $A_\mu{}^{IJKL}$ and its complex conjugation and taking into account the multiplicity with vanishing $A_{\mu IJ}$, we arrive at the following expression for $(p, 8-p)$ where $p = 7,6,5,4,3,2,1$

$$-\frac{1}{96}\left|A_\mu{}^{IJKL}\right|^2 = -\left(7,3,5/3,1,3/5,1/3,1/7\right)\partial^\mu s \partial_\mu s.$$

Let us define a new variable $\widetilde{s}$, in order to have usual canonical kinetic terms, normalized by 1/2, as

$$\widetilde{s} = \sqrt{\frac{2p}{q}}\, s.$$

Therefore, the resulting Lagrangian of scalar-gravity sector takes the form:

$$\int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial^\mu \widetilde{s}\partial_\mu \widetilde{s} - V_{p,q}(\xi;\widetilde{s})\right), \tag{20}$$

together with (17) where s replaced by $\widetilde{s}$. Having established the holographic duals of both supergravity critical points, and examined small perturbations around the corresponding fixed point field theories, one can proceed the supergravity description. The supergravity scalar whose vacuum expectation value leads to the new critical point tells us what relevant operators in the dual field theory would drive a flow to the fixed point in the IR. To construct the kink corresponding to the supergravity description of the nonconformal (in special case: RG) flow from one scale to two other connecting critical points in $d = 3$ field theories, the form of a 3d Poincare invariant metric breaking the full conformal group invariance takes the form [40]:

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}dr^2, \quad \eta_{\mu\nu} = (-,+,+), \tag{21}$$

characteristic of space-time with a domain wall where r is the coordinate transverse to the wall(can be interpreted as an energy scale) and $A(r)$ is the scale factor in the four-dimensional metric.

Our interest in domain wall space-times arises from their relevance to dual field theories. The distance from horizon $U = \infty$ corresponds to long distance in the bulk(UV in the dual field theory) and $U = 0$(near horizon corresponds to short distances in the bulk(IR in the dual field theory). We are looking for "interpolating" solutions. We will show how supergravity can provide a description of the entire flow from the maximal supersymmetric UV theory to the IR fixed point. With the above ansatz (21) the equations of motion for the scalars and the metric from (20) read

$$\begin{aligned}
&\partial_r^2 A - \partial_r A \partial_r B + \frac{3}{2}(\partial_r A)^2 + \frac{1}{4}(\partial_r \widetilde{s})^2 + \frac{1}{2}e^{2B}V_{p,q,\xi} = 0, \\
&\partial_r^2 \widetilde{s} + 3\partial_r A \partial_r \widetilde{s} - \partial_r B \partial_r \widetilde{s} - e^{2B}\partial_{\widetilde{s}} V_{p,q,\xi} = 0.
\end{aligned} \tag{22}$$

By substituting the domain-wall ansatz (21) into the Lagrangian (20), the energy-density $E[A, \widetilde{s}]$ [41], with the integration by parts on the term of $\partial_r^2 A$, per unit area transverse to r-direction is given by

$$E[A, \widetilde{s}] \;\; = \;\; -\int_{-\infty}^{\infty} dr e^{3A+B}\left[-3e^{-2B}\left(2(\partial_r A)^2 + \partial_r^2 A - \partial_r A \partial_r B\right) - \frac{1}{2}e^{-2B}\left(\partial_r \widetilde{s}\right)^2 - V_{p,q,\xi}(\widetilde{s})\right].$$

We are looking for a nontrivial configuration along r-direction in order to find out the first-order differential equations satisfying the domain-wall, let us rewrite and reorganize the energy-density by complete squares plus others due to usual squaring-procedure as follows:

$$\begin{aligned}
&E[A, \widetilde{s}] = \\
&\frac{1}{2}\int_{-\infty}^{\infty} dr e^{3A+B}\Big[-6\left(e^{-B}\partial_r A + \sqrt{2}gW_{p,q}(\xi;\widetilde{s})\right)^2 + \left(e^{-B}\partial_r \widetilde{s} - 2\sqrt{2}g\partial_{\widetilde{s}} W_{p,q}(\xi;\widetilde{s})\right)^2 \\
&12\sqrt{2}ge^{-B}W_{p,q}(\xi;\widetilde{s})\partial_r A + 4\sqrt{2}ge^{-B}\partial_r W_{p,q}(\xi;\widetilde{s})\Big]
\end{aligned}$$

where superpotential $W_{p,q}(\xi;\widetilde{s})$ is given by (18). Then one can easily check the last two terms in the above can be combined as $4\sqrt{2}g\partial_r(e^{3A}W_{p,q}(\xi;\widetilde{s}))$. Therefore, one arrives at

$$\frac{1}{2}\int_{-\infty}^{\infty} dr e^{3A+B}\left[-6\left(e^{-B}\partial_r A + \sqrt{2}gW_{p,q}(\xi;\widetilde{s})\right)^2 + \left(e^{-B}\partial_r \widetilde{s} - 2\sqrt{2}g\partial_{\widetilde{s}} W_{p,q}(\xi;\widetilde{s})\right)^2\right]$$

$$+2\sqrt{2}g\left(e^{3A}W_{p,q}(\xi;\widetilde{s})\right)|_{-\infty}^{\infty}.$$

Finally, we find BPS bound, inequality of the energy-density

$$E[A,\widetilde{s}] \geq 2\sqrt{2}g\left(e^{3A(\infty)}W_{p,q}(\xi;\widetilde{s})(\infty) - e^{3A(-\infty)}W_{p,q}(\xi;\widetilde{s})(-\infty)\right). \tag{23}$$

Then $E[A,\widetilde{s}]$ is extremized by the following so-called BPS domain-wall solutions. The first order differential equations for the scalar field are the gradient flow equations of a superpotential defined on a restricted slice of the scalar manifold and simply related to the potential of gauged supergravity on this slice. The equations describing the flow are then

$$\begin{aligned}\partial_r\widetilde{s} &= \pm 2\sqrt{2}e^{B}g\partial_{\widetilde{s}}W_{p,q}(\xi;\widetilde{s}),\\ \partial_r A &= \mp\sqrt{2}e^{B}gW_{p,q}(\xi;\widetilde{s}).\end{aligned} \tag{24}$$

There exists a supersymmetry [41, 42] of the background with a nonvanishing metric and a single scalar field, for each spinor satisfying the Killing spinor condition. The background satisfying (24) preserve half the supersymmetry. It is straightforward to verify that any solutions $\widetilde{s}(r), A(r)$ of (24) satisfy the gravitational and scalar equations of motion given by the second order differential equations (22). Embedding or consistent truncation means that the flow is entirely determined by the equations of motion of supergravity in four-dimensions and any solution of the truncated theory can be lifted to a solution of untruncated theory [43]. Using (24), the monotonicity [44] of $\partial_r A$ which is related to the local potential energy of the kink leads to $\partial_r^2 A \leq 0$ when B is constant. Note that the value of superpotential at either end of a kink may be thought of as determining the topological sector. The analytic solutions of (24) for $(p,q) = (4,4)$ when B is a constant become

$$\widetilde{s}(r) = \sqrt{2}\log\left[\sqrt{\xi}\frac{(e^{\sqrt{2\xi}g(c-r)}-1)}{(e^{\sqrt{2\xi}g(c-r)}+1)}\right], \qquad A(r) = \left(1+\sqrt{2\xi}g\right)c + \log\left[2\sinh\sqrt{2\xi}g(r-c)\right]$$

where c is some constant. For other values of (p,q), the analytic solutions exist only for $\xi = 0$.

2.5 $SO(7)^-$ Invariant Sector from $SO(8)$ Gauging

The four-form tensor[8] X^{-IJKL} is invariant under the $SO(7)^-$ subgroup of $SO(8)$. Turning on the vacuum expectation value proportional to X^{-IJKL} in the de Wit-Nicolai theory gives rise to spontaneous symmetry breaking $SO(8)$ into $SO(7)^-$. Regarded as a 28×28 matrix, X^{-IJKL} has 21 eigenvalues of 1 and 7 eigenvalues of -3. Introducing the projector P_1 onto the

[8] The $SL(8,\mathbf{R})$ does act on the vector potential and is generated by the $SO(8)$ and self-dual part. The remainder of E_7 including the anti-self-dual part does not act on the vector potentials but does act on the field strengths. Therefore, contrary to the self-dual case we have discussed in previous sections, the anti-self-dual case does not act on the vector potential. We thank C.M. Hull for pointing out this to us.

21-dimensional eigenspace(P_1 projects the generators of $SO(8)$ onto those of $SO(7)^-$ while P_2 projects the generators of $SO(8)$ onto the remainder $SO(8)\backslash SO(7)^-$), they are given in terms of X^{-IJKL}

$$
\begin{aligned}
P_1^{IJKL} &= \frac{3}{4}\left(\delta_{KL}^{IJ} + \frac{1}{3}X^{-IJKL}\right), \\
P_2^{IJKL} &= \delta_{KL}^{IJ} - P_1^{IJKL} = \frac{1}{4}\left(\delta_{IJ}^{KL} - X^{-IJKL}\right).
\end{aligned}
$$

One can easily check that they satisfy

$$
P_1^2 = P_1, \qquad P_2^2 = P_2, \qquad P_1P_2 = P_2P_1 = 0.
$$

The 28 $SO(8)$ generators Λ^{IJ} are projected onto a 21-dimensional subspace by P_1, $\Lambda_1^{IJ} = P_1^{IJKL}\Lambda^{KL}$ and this subspace is the Lie algebra for the $SO(7)^-$ subgroup of $SO(8)$; in other words, the subgroup stabilizes a left-handed $SO(8)$ spinor(See the appendix B). The remaining 7 generators are $\Lambda_2^{IJ} = P_2^{IJKL}\Lambda^{KL}$. The usual commutation relations for $SO(8)$ are given in terms of Λ_1^{IJ} and Λ_2^{IJ}.

Viewed as a 28×28 matrix, X^{-IJKL} has eigenvalues α, β and $\gamma = (\alpha+\beta)/2$ with degeneracies d_α, d_β and d_γ respectively(For the explicit construction of X^{-IJKL} see the Appendix A). The eigenvalues and eigenspaces of the $SO(p)^- \times SO(q)^-$ invariant tensor are summarized similarly. By introducing projectors as done in previous cases, P_α, P_β and P_γ onto corresponding eigenspaces, we have a 28×28 matrix equation to arbitrary p and q. The parametrization for the $SO(p)^- \times SO(q)^-$-singlet space that is invariant subspace under a particular $SO(p)^- \times SO(q)^-$ subgroup of $SO(8)$ becomes

$$
\phi_{IJKL} = 4\sqrt{2}isX^-_{IJKL}
$$

where s is a real scalar field. Note the presence of imaginary number i. As in the previous consideration, the A'_1 tensor we obtained is a single complex eigenvalue with degeneracies 8

$$
\begin{aligned}
A_1'^{\,ij} &= \mathrm{diag}\left(z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1\right), \\
z_1 &= \frac{1}{16}(1+i)\left(pe^{s} + qe^{-\frac{p}{q}s}\right) + \frac{1}{16}(1-i)\left(pe^{-s} + qe^{\frac{p}{q}s}\right).
\end{aligned} \tag{25}
$$

For the A'_2 tensor we get

$$
A_{2i}'^{\;jkl} = \frac{p}{8}\left[(1+i)\left(e^{-\frac{p}{q}s} - e^{s}\right) + (1-i)\left(e^{\frac{p}{q}s} - e^{-s}\right)\right]X^{-ijkl}.
$$

Therefore, we are now ready to calculate the full expression of a scalar potential and it turns out

$$
V_{7,1} = \frac{1}{16}g^2e^{-14s}\left(1+e^{4s}\right)^5\left(1-5e^{4s}+e^{8s}\right),
$$

$$
\begin{aligned}
V_{6,2} &= -3g^2 e^{-2s}\left(1+e^{4s}\right), \\
V_{5,3} &= -\frac{3}{16} g^2 e^{-10s/3}\left(1+e^{4s/3}\right)^5, \\
V_{4,4} &= -g^2\left(4+e^{-2s}+e^{2s}\right), \\
V_{3,5} &= -\frac{3}{16} g^2 e^{-2s}\left(1+e^{4s/5}\right)^5, \\
V_{2,6} &= -3g^2 e^{-2s/3}\left(1+e^{4s/3}\right), \\
V_{1,7} &= \frac{1}{16} g^2 e^{-2s}\left(1+e^{4s/7}\right)^5\left(1-5e^{4s/7}+e^{8s/7}\right).
\end{aligned}
$$

Note that the potential $V_{p,q}$ can be obtained from $V_{q,p}$ by rescaling $s \to ps/q$. The eigenvalue of the A_1' tensor z_1 provides a superpotential and one can check that the scalar potential can be written in terms of superpotential:

$$
\begin{aligned}
W_{p,q}(s) &= |z_1|, \\
V_{p,q}(s) &= g^2\left[\frac{2q}{p}\left(\partial_s W_{p,q}(s)\right)^2 - 6W_{p,q}(s)^2\right] = g^2\left[4\left(\partial_{\tilde{s}} W_{p,q}(s)\right)^2 - 6W_{p,q}(s)^2\right]
\end{aligned}
$$

where $\tilde{s} = \sqrt{\frac{2p}{q}}s$ and z_1 is given by (25). The kinetic terms are equivalent to the previous cases. In this case, there are no such first order differential equations for either a flow between $SO(8)$ fixed point and $SO(7)^- \times SO(1)^-$ fixed point or a flow between $SO(8)$ and $SO(1)^- \times SO(7)^-$, contrary to the previous $SO(p)^+ \times SO(q)^+$ embedding case. The superpotential has the following values at the two critical points.

Gauge symmetry	$\mathcal{N}$	p	$q = 8 - p$	s	W	V
$SO(8)$	8	any	any	1	0	$-6g^2$
$SO(7)^- \times SO(1)^-$	0	7	1	$\frac{1}{2}\ln\frac{1}{2}(\pm 1+\sqrt{5})$	$\frac{3\times 5^{3/4}}{8}$	$-\frac{25\sqrt{5}}{8}g^2$
$SO(1)^- \times SO(7)^-$	0	1	7	$\frac{7}{2}\ln\frac{1}{2}(\pm 1+\sqrt{5})$	$\frac{3\times 5^{3/4}}{8}$	$-\frac{25\sqrt{5}}{8}g^2$

Table 3. *Summary of critical points in the context of superpotential : symmetry group, supersymmetry, vacuum expectation values of field, superpotential, and cosmological constants. For either case, it is exactly $SO(7)^-$-invariant critical point of the $SO(8)$ theory. It has no supersymmetry and is unstable.*

3 Domain Wall from $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ Sectors of $\mathcal{N} = 8$ Supergravity

Let us consider a sequence of non-singular elements $E(\xi)$ of $L = SL(8, \mathbf{R})$ with ξ real parameter and $E(1) = 1$, identity transformation, whose limit point $E(0)$ is singular and not in L. As long as $E(\xi)$ remains nonsingular($\xi \neq 0$), the structure constants have the usual tensor properties.

Acting on the Lagrangian with $E(\xi)$ yields a sequence of Lagrangian:$\mathcal{L} \to \mathcal{L}'(\xi) = \mathcal{L}_0 + \mathcal{L}_g{}'(\xi)$. If one also rescales the coupling constant g by a ξ-dependent one through $g \to g'(\xi)$ for some choices of the sequence $E(\xi)$ in L, a limit of $\mathcal{L}_g{}'(\xi)$ exists and is well defined. One can continue the Lagrangian $\mathcal{L}'(\xi)$ to negative values of ξ. In this case, $\mathcal{L}'(-1)$ is the Lagrangian for different gauging and the gauge group is non-compact $SO(p, q+r)^+$ with $p+q+r=8$ which will be discussed in section 3.2. One continues to consider a sequence of non-singular elements $F(\zeta)$ of L with ζ real parameter and $F(1)=1$, identity transformation, whose limit point $F(0)$ is singular and not in L. As long as $F(\zeta)$ remains nonsingular($\zeta \neq 0$), the structure constants have the usual tensor properties. Acting on the Lagrangian $\mathcal{L}'$ with $F(\zeta)$ yields a sequence of Lagrangian:$\mathcal{L}' \to \mathcal{L}''(\zeta) = \mathcal{L}_0 + \mathcal{L}_g{}''(\zeta, \xi)$. If one also rescales the coupling constant g' by a ζ-dependent one through $g' \to g''(\zeta)$ for some choices of the sequence $F(\zeta)$ in L, the limit of $\mathcal{L}_g{}''(\zeta)$ exists as $\zeta \to 0$ so that $\mathcal{L}''(\zeta=0) = \mathcal{L}_0 + \mathcal{L}_g{}''(\zeta=0)$ gives the Lagrangian. The gauge group corresponding to $\mathcal{L}''(\zeta=0, \xi=-1)$ is an Inonu-Wigner contraction of $K_{\xi,\zeta,p,q,r}$ denoted by $CSO(p,q,r)^+$ with $p+q+r=8$ [11].

In section 3.1, we start with the most general gaugings which generalize previous considerations by introducing two parameters, ξ and ζ. The gauging denoted by $CSO(p,q,r)^+$ preserves a metric with p positive eigenvalues, q negative eigenvalues and r zero eigenvalues. In section 3.2, by analyzing two successive $SL(8,\mathbf{R})$ transformations(repeating twice) in the context of $SO(p,q+r)^+$ and $SO(p+q,r)^+$ gaugings, we discover a T' tensor which depends on these two parameters, ξ and ζ. As done in previous sections, the A_1 and A_2 tensors can be easily determined by realizing that 56-beins are product of each 56-bein for each parametrization of the singlet-space. In section 3.3 it turns out that one has a scalar potential which can be written as a superpotential in very simple form and in section 3.4, we find the domain-wall solutions. In section 3.5, by starting with $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant generator of $SL(8,\mathbf{R})$ directly, one can construct the projectors corresponding to this invariant four-form tensor, which we will compare with the approach given in section 3.2-3.4.

3.1 Non-semi-simple and Non-compact Gaugings [11]

It is possible to gauge the 28-dimensional subgroup $K_{\xi,\zeta,p,q,r}$ of $L = SL(8,\mathbf{R})$ whose algebra

$$\begin{aligned}
[\Lambda_{ab}, \Lambda_{cd}]_{\xi,\zeta} &= \Lambda_{ad}\eta_{bc} - \Lambda_{ac}\eta_{bd} - \Lambda_{bd}\eta_{ac} + \Lambda_{bc}\eta_{ad}, \\
\eta_{ab} &= \begin{pmatrix} \mathbf{1}_{p\times p} & 0 & 0 \\ 0 & \xi\mathbf{1}_{q\times q} & 0 \\ 0 & 0 & \xi\zeta\mathbf{1}_{r\times r} \end{pmatrix}, \qquad p+q+r=8
\end{aligned}$$

where $a, b = 1, \cdots, 8$ and $\Lambda_{ab} = -\Lambda_{ba}$.

• When $(\xi,\zeta) = (1,1)$, this leads to the algebra of $SO(8)$ and the de Wit-Nicolai gauging is recovered. When $(\xi,\zeta) = (1,0)$ it will give $CSO(p+q,r)^+$ algebra which was discussed in the previous section and the maximal compact subgroup is $SO(p+q)^+ \times U(1)^{+r(r-1)/2}$. Moreover, when $(\xi,\zeta) = (1,-1)$, one gets $SO(p+q,r)^+$ algebra which was already considered and the maximal compact subgroup is $SO(p+q)^+ \times SO(r)^+$.

• When $(\xi,\zeta) = (-1,1)$, it will give non-compact $SO(p,q+r)^+$ gauging whose maximal compact subgroup is $SO(p)^+ \times SO(q+r)^+$. When $(\xi,\zeta) = (-1,0)$, it gives a certain non-semi-simple algebra of the Inonu-Wigner contraction of $SO(8)$ about its $SO(p,q)^+$ subgroup, denoted by $CSO(p,q,r)^+$ [11]. The maximal compact subgroup is $SO(p)^+ \times SO(q)^+ \times U(1)^{+r(r-1)/2}$. Note that $CSO(p,q,1)^+ = ISO(p,q)^+$, the inhomogeneous group. For $(\xi,\zeta) = (-1,-1)$, one gets $SO(p+r,q)^+$ algebra.

• When $\xi = 0$, it gives Inonu-Wigner contraction $CSO(p,q+r)^+$ which was already considered.

The $CSO(p,q,r)^+$ gauging initiated by Hull which preserves a metric with p positive eigenvalues, q negative eigenvalues and r zero eigenvalues can be obtained by group contractions of $SO(8)$ as follows. One decomposes each $SO(8)$ generator Λ into the part $\Lambda_{(\alpha)}$ in the $SO(p)^+$ sub-algebra, the part $\Lambda_{(\beta)}$ in the $SO(q)^+$ sub-algebra, the part $\Lambda_{(\gamma)}$ in the $SO(r)^+$ sub-algebra, and the remainders $\Lambda_{(\delta)}, \Lambda_{(\lambda)}$, and $\Lambda_{(\rho)}$ where $\Lambda = \Lambda_{(\alpha)} + \Lambda_{(\beta)} + \Lambda_{(\gamma)} + \Lambda_{(\delta)} + \Lambda_{(\lambda)} + \Lambda_{(\rho)}$. See also the discussion around in (32). One performs the rescaling as

$$\Lambda \to \Lambda_{(\alpha)} + \xi \left(\Lambda_{(\beta)} + \zeta \Lambda_{(\gamma)} + \sqrt{\zeta} \Lambda_{(\rho)} \right) + \sqrt{\xi} \left(\Lambda_{(\delta)} + \sqrt{\zeta} \Lambda_{(\lambda)} \right).$$

The rescaled algebra can be represented as

$$\begin{aligned}
&[\Lambda_{(\alpha)}, \Lambda_{(\alpha)}] \approx \Lambda_{(\alpha)}, \qquad [\Lambda_{(\alpha)}, \Lambda_{(\delta)}] \approx \Lambda_{(\delta)}, \qquad [\Lambda_{(\alpha)}, \Lambda_{(\lambda)}] \approx \Lambda_{(\lambda)}, \\
&[\Lambda_{(\beta)}, \Lambda_{(\beta)}] \approx \xi \Lambda_{(\beta)}, \qquad [\Lambda_{(\beta)}, \Lambda_{(\delta)}] \approx \xi \Lambda_{(\delta)}, \qquad [\Lambda_{(\beta)}, \Lambda_{(\rho)}] \approx \xi \Lambda_{(\rho)}, \\
&[\Lambda_{(\gamma)}, \Lambda_{(\gamma)}] \approx \xi\zeta \Lambda_{(\gamma)}, \qquad [\Lambda_{(\gamma)}, \Lambda_{(\rho)}] \approx \xi\zeta \Lambda_{(\rho)}, \qquad [\Lambda_{(\gamma)}, \Lambda_{(\lambda)}] \approx \xi\zeta \Lambda_{(\lambda)}, \\
&[\Lambda_{(\delta)}, \Lambda_{(\delta)}] \approx \xi \Lambda_{(\alpha)} + \Lambda_{(\beta)}, \qquad [\Lambda_{(\delta)}, \Lambda_{(\rho)]} \approx \xi \Lambda_{(\lambda)}, \qquad [\Lambda_{(\delta)}, \Lambda_{(\lambda)}] \approx \Lambda_{(\rho)}, \\
&[\Lambda_{(\rho)}, \Lambda_{(\rho)}] \approx \xi\zeta \Lambda_{(\beta)} + \xi \Lambda_{(\gamma)}, \qquad [\Lambda_{(\rho)}, \Lambda_{(\lambda)}] \approx \xi\zeta \Lambda_{(\delta)}, \qquad [\Lambda_{(\lambda)}, \Lambda_{(\lambda)}] \approx \xi\zeta \Lambda_{(\alpha)} + \Lambda_{(\gamma)},
\end{aligned}$$

with other commutators vanishing. By taking the contraction, $\zeta \to 0$, the $SO(r)^+$ subgroup generated by $\Lambda_{(\gamma)}$ collapses to an abelian group $U(1)^{+r(r-1)/2}$ and the maximal compact subgroup of $CSO(p,q,r)^+$ is $SO(p)^+ \times SO(q)^+ \times U(1)^{+r(r-1)/2}$. The generators $\Lambda_{(\gamma)}$ commute all the generators except appearing on the right hand sides of $[\Lambda_{(\lambda)}, \Lambda_{(\lambda)}]$ and $[\Lambda_{(\rho)}, \Lambda_{(\rho)}]$. The methods described in previous section will be used to obtain a $CSO(p,q,r)^+$ gaugings.

3.2 T-tensor in $CSO(p,q,r)^+$ Gaugings [11]

The $CSO(p,q,r)^+$ gaugings can be obtained by acting on the $SO(p,q+r)^+$ gauging first. Some idea in this direction was already given in the paper of [14]. Let us consider the $SO(p)^+ \times$

$SO(q+r)^+$ invariant generator of $SL(8,\mathbf{R})$ we have discussed in previous section,

$$X_{ab} = \begin{pmatrix} \alpha \mathbf{1}_{p\times p} & 0 \\ 0 & \beta \mathbf{1}_{(q+r)\times(q+r)} \end{pmatrix} \tag{26}$$

with

$$\alpha p + \beta(q+r) = 0, \qquad p+q+r = 8. \tag{27}$$

Regarded as a 28×28 matrix, real, self-dual totally anti-symmetric $SO(p)^+ \times SO(q+r)^+$-invariant four-form tensor X_t^{+IJKL} has eigenvalues α, β and $\gamma = (\alpha+\beta)/2$ with degeneracies d_α, d_β and d_γ respectively. The eigenvalues and eigenspaces of the $SO(p)^+ \times SO(q+r)^+$ invariant tensor are summarized in Table 1. By introducing projectors, $P_{\alpha,t}, P_{\beta,t}$ and $P_{\gamma,t}$ onto corresponding eigenspaces, we have a 28×28 matrix equation. Projector $P_{\alpha,t}(P_{\beta,t})$ projects the $SO(8)$ Lie algebra onto its $SO(p)^+(SO(q+r)^+)$ subalgebra while $P_{\gamma,t}$ does onto the remainder $SO(8)/(SO(p)^+ \times SO(q+r)^+)$. Note that q over there is replaced by $q+r$ here. The projectors can be constructed from X_t^{+IJKL}. The combination gA_μ^{IJ} in the minimal couplings will be finite as $t \to \infty$ if g is rescaled to

$$g(t) = ge^{\alpha t}$$

for constant α which we have chosen as -1 so that

$$\begin{aligned} g(t)A_\mu^{IJ}(t) &= g\left(A_{\mu(\alpha)}^{IJ} + e^{(\alpha-\beta)t}A_{\mu(\beta)}^{IJ} + e^{(\alpha-\gamma)t}A_{\mu(\gamma)}^{IJ}\right) \\ &= g\left(A_{\mu(\alpha)}^{IJ} + \xi A_{\mu(\beta)}^{IJ} + \sqrt{\xi}A_{\mu(\gamma)}^{IJ}\right), \end{aligned} \tag{28}$$

where $\xi = e^{(\alpha-\beta)t}$ as before. One finds that on taking the limit $t \to \infty (\xi \to 0)$ one obtains a gauging with gauge group contraction of $SO(8)$ about its $SO(p)^+$ subgroup. If, instead, one analytically continues to $t = i\pi/(\alpha-\beta)$, one obtains a gauging of $SO(p, q+r)^+$.

Let us consider the additional, second $SL(8,\mathbf{R})$ transformation using the $SO(p+q)^+ \times SO(r)^+$ invariant generator of $SL(8,\mathbf{R})$,

$$X_{ab} = \begin{pmatrix} \alpha' \mathbf{1}_{(p+q)\times(p+q)} & 0 \\ 0 & \beta' \mathbf{1}_{r\times r} \end{pmatrix} \tag{29}$$

with

$$\alpha'(p+q) + \beta' r = 0, \qquad p+q+r = 8. \tag{30}$$

Regarded as a 28×28 matrix, real, self-dual totally anti-symmetric $SO(p+q)^+ \times SO(r)^+$-invariant four-form tensor X_s^{+IJKL} has eigenvalues α', β' and $\gamma' = (\alpha'+\beta')/2$ with degeneracies $d_{\alpha'}, d_{\beta'}$ and $d_{\gamma'}$ respectively. The eigenvalues and eigenspaces of the $SO(p+q)^+ \times SO(r)^+$

invariant tensor are summarized in Table 1. By introducing projectors, $P_{\alpha',s}, P_{\beta',s}$ and $P_{\gamma',s}$ onto corresponding eigenspaces, we have a 28×28 matrix equation. Projector $P_{\alpha',s}(P_{\beta',s})$ projects the $SO(8)$ Lie algebra onto its $SO(p+q)^+(SO(r)^+)$ subalgebra while $P_{\gamma',s}$ does onto the remainder $SO(8)/(SO(p+q)^+ \times SO(r)^+)$. Note that p over there is replaced by $p+q$ here. The projectors can be constructed from X_s^{+IJKL} similarly. The combination gA_μ^{IJ} in the minimal couplings will be finite as $s \to \infty$ if g is rescaled to

$$g(s) = ge^{\alpha' s}$$

for some constant α'(taken as -1) so that by acting $[\exp(-sX_s^+)]^{IJKL}$ on the right hand side of (28)

$$\begin{aligned} g(s,t)A_\mu^{IJ}(s,t) &= g\left(P_{\alpha',s}^{IJKL} + e^{(\alpha'-\beta')s}P_{\beta',s}^{IJKL} + e^{(\alpha'-\gamma')s}P_{\gamma',s}^{IJKL}\right) \\ &\quad \times \left(A_{\mu(\alpha)}^{KL} + e^{(\alpha-\beta)t}A_{\mu(\beta)}^{KL} + e^{(\alpha-\gamma)t}A_{\mu(\gamma)}^{KL}\right) \\ &= g\left(A_{\mu(\alpha'\alpha)}^{IJ} + \xi A_{\mu(\alpha'\beta)}^{IJ} + \sqrt{\xi}A_{\mu(\alpha'\gamma)}^{IJ} + \zeta\xi A_{\mu(\beta'\beta)}^{IJ} + \sqrt{\zeta}\xi A_{\mu(\gamma'\beta)}^{IJ} + \sqrt{\zeta\xi}A_{\mu(\gamma'\gamma)}^{IJ}\right) \end{aligned}$$

where $\zeta = e^{(\alpha'-\beta')s}$ as before. Here we used the fact that

$$P_{\beta',s}P_{\alpha,t} = P_{\beta',s}P_{\gamma,t} = P_{\gamma',s}P_{\alpha,t} = 0 \tag{31}$$

which can be shown by the explicit expression of projectors given in Appendix F and we denote the simplified notations for $A_{\mu(\sigma'\sigma)}^{IJ}$, where $\sigma' = \alpha', \beta', \gamma', \sigma = \alpha, \beta, \gamma$ as follows:

$$\begin{aligned} A_{\mu(\alpha'\alpha)}^{IJ} &\equiv (P_{\alpha',s}P_{\alpha,t})^{IJMN}A_\mu^{MN}, \qquad & A_{\mu(\alpha'\beta)}^{IJ} &\equiv (P_{\alpha',s}P_{\beta,t})^{IJMN}A_\mu^{MN}, \\ A_{\mu(\alpha'\gamma)}^{IJ} &\equiv (P_{\alpha',s}P_{\gamma,t})^{IJMN}A_\mu^{MN}, & A_{\mu(\beta'\beta)}^{IJ} &\equiv (P_{\beta',s}P_{\beta,t})^{IJMN}A_\mu^{MN}, \\ A_{\mu(\gamma'\beta)}^{IJ} &\equiv (P_{\gamma',s}P_{\beta,t})^{IJMN}A_\mu^{MN}, & A_{\mu(\gamma'\gamma)}^{IJ} &\equiv (P_{\gamma',s}P_{\gamma,t})^{IJMN}A_\mu^{MN}. \end{aligned}$$

Now we can think of the product of these projectors, $P_{\sigma',s}^{IJKL}P_{\sigma,t}^{KLMN}$, as a single projector. Therefore, let us define them, to satisfy the usual property of projectors, as

$$\begin{aligned} P_{\alpha',s}P_{\alpha,t} &\equiv P_\alpha, \qquad & P_{\alpha',s}P_{\beta,t} &\equiv P_\beta, \qquad & P_{\alpha',s}P_{\gamma,t} &\equiv P_\delta, \\ P_{\beta',s}P_{\beta,t} &\equiv P_\gamma, & P_{\gamma',s}P_{\beta,t} &\equiv P_\rho, & P_{\gamma',s}P_{\gamma,t} &\equiv P_\lambda. \end{aligned} \tag{32}$$

We will see that $\delta = (\alpha+\beta)/2, \lambda = (\alpha+\gamma)/2$ and $\rho = (\beta+\gamma)/2$ and α and β are related to α's in (27) and (30). Projector $P_\alpha(P_\beta)[P_\gamma]$ projects the $SO(8)$ Lie algebra onto its $SO(p)^+(SO(q)^+)[SO(r)^+]$ subalgebra while $P_\delta(P_\lambda)[P_\rho]$ projects onto the remainder

$$\frac{SO(8)}{SO(p)^+ \times SO(q)^+}\left(\frac{SO(8)}{SO(p)^+ \times SO(r)^+}\right)\left[\frac{SO(8)}{SO(q)^+ \times SO(r)^+}\right]$$

which will be discussed in next section 3.5. One obtains these projectors explicitly from the relation (32) where the projectors in $SO(p)^+ \times SO(q)^+$-invariant sector are given in the appendix F. In terms of these new projectors, one can write the combination $g(s,t)A^{IJ}_\mu(s,t)$ as

$$A^{IJ}_{\mu(\alpha)} + \xi\left(A^{IJ}_{\mu(\beta)} + \zeta A^{IJ}_{\mu(\gamma)} + \sqrt{\zeta}A^{IJ}_{\mu(\rho)}\right) + \sqrt{\xi}\left(A^{IJ}_{\mu(\delta)} + \sqrt{\zeta}A^{IJ}_{\mu(\lambda)}\right). \tag{33}$$

By expanding $g(t,s)D(A_\mu,\xi,\zeta)$ with respect to both t and s, there exist many terms that seem to diverge as $t \to \infty$ or $s \to \infty$. However, by exploiting some identities of the generators given in appendix D, it implies that those divergent terms vanish identically and therefore a limit of $t \to \infty$ or $s \to \infty$ exists.

By simplifying the expressions appearing in $g(t,s)D(A_\mu,\xi,\zeta)$, one gets, for example, the first 28×28 block diagonal terms given by

$$\begin{aligned}
&A_{(\alpha)\mu} + P_\alpha A_{(\delta)\mu}P_\delta + P_\delta A_{(\delta)\mu}P_\beta + P_\lambda A_{(\delta)\mu}P_\rho + P_\lambda A_{(\lambda)\mu}P_\gamma + +P_\alpha A_{(\lambda)\mu}P_\lambda + P_\delta A_{(\lambda)\mu}P_\rho \\
&+\xi\left(A_{(\beta)\mu} + P_\delta A_{(\delta)\mu}P_\alpha + P_\beta A_{(\delta)\mu}P_\delta + P_\rho A_{(\delta)\mu}P_\lambda + P_\beta A_{(\rho)\mu}P_\rho + P_\delta A_{(\rho)\mu}P_\lambda\right. \\
&\left.+P_\rho A_{(\rho)\mu}P_\gamma\right) + \xi\zeta\left(A_{(\gamma)\mu} + P_\rho A_{(\rho)\mu}P_\beta + P_\lambda A_{(\rho)\mu}P_\delta + P_\gamma A_{(\rho)\mu}P_\rho + P_\lambda A_{(\lambda)\mu}P_\alpha\right. \\
&\left.+P_\rho A_{(\lambda)\mu}P_\delta + P_\gamma A_{(\lambda)\mu}P_\lambda\right)
\end{aligned} \tag{34}$$

where we used the properties between projectors and vector fields:

$$P_\alpha A_{(\rho)\mu}P_\alpha = P_\gamma A_{(\delta)\mu}P_\gamma = P_\beta A_{(\lambda)\mu}P_\beta = 0.$$

One can prove that (34) becomes the one we have considered for $SO(p,q+r)^+$ gauging when $\zeta = 1$ by combining ξ-dependent terms with $\xi\zeta$-dependent terms[9] and removing the projectors $P_{\sigma',s}(\sigma' = \alpha',\beta',\gamma')$ with (32) under the extensive manipulation of properties of projectors. On the other hand, when $\xi = 1$, it becomes the one in $SO(p+q,r)^+$ gauging by combining the ξ,ζ-independent terms with ξ-depedent terms and removing the projectors $P_{\sigma,t}(\sigma = \alpha,\beta,\gamma)$. In this case, we can write it similarly[10].

Collecting all other terms by simplifying other three 28×28 blocks we get

$$g(t,s)D(A_\mu,\xi,\zeta) = gD(A_\mu)$$

[9] When $\zeta = 1$, (34) becomes,

$$\underline{A}_{(\alpha)\mu} + P_{\alpha,t}\underline{A}_{(\gamma)\mu}P_{\gamma,t} + P_{\gamma,t}\underline{A}_{(\gamma)\mu}P_{\beta,t} + \xi\left(\underline{A}_{(\beta)\mu} + P_{\beta,t}\underline{A}_{(\gamma)\mu}P_{\gamma,t} + P_{\gamma,t}\underline{A}_{(\gamma)\mu}P_{\alpha,t}\right).$$

[10] When $\xi = 1$, (34) becomes

$$\underline{A}_{(\alpha')\mu} + P_{\alpha',s}\underline{A}_{(\gamma')\mu}P_{\gamma',s} + P_{\gamma',s}\underline{A}_{(\gamma')\mu}P_{\beta',s} + \zeta\left(\underline{A}_{(\beta')\mu} + P_{\beta',s}\underline{A}_{(\gamma')\mu}P_{\gamma',s} + P_{\gamma',s}\underline{A}_{(\gamma')\mu}P_{\alpha',s}\right).$$

$$-(1-\xi)g\left(\begin{array}{cc} \underline{A}_{(\beta)\mu}+\frac{1}{2}\left(\underline{A}_{(\delta)\mu}+\underline{A}_{(\rho)\mu}\right), & Z^{MN}_{(\rho)IJKL}A^{MN}_{(\rho)\mu}-Z^{MN}_{(\delta)IJKL}A^{MN}_{(\delta)\mu} \\ Z^{MN}_{(\rho)IJKL}A^{MN}_{(\rho)\mu}-Z^{MN}_{(\delta)IJKL}A^{MN}_{(\delta)\mu}, & \underline{A}_{(\beta)\mu}+\frac{1}{2}\left(\underline{A}_{(\delta)\mu}+\underline{A}_{(\rho)\mu}\right) \end{array}\right)$$
$$-(1-\xi\zeta)g\left(\begin{array}{cc} \underline{A}_{(\gamma)\mu}+\frac{1}{2}\left(\underline{A}_{(\lambda)\mu}+\underline{A}_{(\rho)\mu}\right), & Z^{MN}_{(\lambda)IJKL}A^{MN}_{(\lambda)\mu}-Z^{MN}_{(\rho)IJKL}A^{MN}_{(\rho)\mu} \\ Z^{MN}_{(\lambda)IJKL}A^{MN}_{(\lambda)\mu}-Z^{MN}_{(\rho)IJKL}A^{MN}_{(\rho)\mu}, & \underline{A}_{(\gamma)\mu}+\frac{1}{2}\left(\underline{A}_{(\lambda)\mu}+\underline{A}_{(\rho)\mu}\right) \end{array}\right)$$

where $Z^{MN}_{(\sigma)IJKL}$ are quadratic forms of projectors

$$\begin{aligned} Z^{MN}_{(\delta)IJKL} &= \frac{1}{2}\Big[\left(P_\alpha-P_\beta\right)_{IJMP}P_\delta^{NPKL}-P_\delta^{IJMP}\left(P_\alpha-P_\beta\right)_{NPKL} \\ &\quad -\left(P_{\rho IJMP}P_\lambda^{NPKL}-P_{\lambda IJMP}P_\rho^{NPKL}\right)\Big], \end{aligned} \tag{35}$$

and $Z^{MN}_{(\lambda)IJKL}$ can be written by performing the change of the above indices in (35) as $\alpha\to\gamma,\beta\to\alpha,\delta\to\lambda,\rho\to\delta,\lambda\to\rho$ and $Z^{MN}_{(\rho)IJKL}$ which can be expressed by changing the indices in (35) as $\alpha\to\beta,\beta\to\gamma,\delta\to\rho,\rho\to\lambda,\lambda\to\delta$. Then our $SU(8)$ T' tensor encoding the structure of the scalar sector of the $\mathcal{N}=8$ supergravity can be read off and one arrives at the final complicated expression:

$$\begin{aligned} T'^{\ jkl}_{i}(\xi,\zeta) &= T_i^{\ jkl}-(1-\xi)\left(\overline{u}^{kl}{}_{IJ}+\overline{v}^{klIJ}\right) \\ &\times\Big[\left(P_\beta^{IJKL}+\frac{1}{2}\left(P_\delta^{IJKL}+P_\rho^{IJKL}\right)\right)\left(u_{im}{}^{KM}\overline{u}^{jm}{}_{LM}-v_{imKM}\overline{v}^{jmLM}\right) \\ &+\left(P_\delta^{IJRS}Z^{KLMN}_{(\delta)RS}-P_\rho^{IJRS}Z^{KLMN}_{(\rho)RS}\right)\left(-v_{imKL}\overline{u}^{jm}{}_{MN}+u_{im}{}^{KL}\overline{v}^{jmMN}\right)\Big] \\ &-(1-\xi\zeta)\left(\overline{u}^{kl}{}_{IJ}+\overline{v}^{klIJ}\right) \\ &\times\Big[\left(P_\gamma^{IJKL}+\frac{1}{2}\left(P_\lambda^{IJKL}+P_\rho^{IJKL}\right)\right)\left(u_{im}{}^{KM}\overline{u}^{jm}{}_{LM}-v_{imKM}\overline{v}^{jmLM}\right) \\ &+\left(P_\rho^{IJRS}Z^{KLMN}_{(\rho)RS}-P_\lambda^{IJRS}Z^{KLMN}_{(\lambda)RS}\right)\left(-v_{imKL}\overline{u}^{jm}{}_{MN}+u_{im}{}^{KL}\overline{v}^{jmMN}\right)\Big]. \end{aligned} \tag{36}$$

Let us examine the structure of T'-tensor[11]. When $\xi=1$, it consists of ζ-independent part plus ζ-dependent part. One can prove that by plugging $P_\sigma(\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho)$ into the product of $P_{\sigma',s}(\sigma'=\alpha',\beta',\gamma')$ and $P_{\sigma,t}(\sigma=\alpha,\beta,\gamma)$. According to (32), the expressions of projectors proportional to $1-\zeta$ are identical to those in (13) for $SO(p+q)^+\times SO(r)^+$-invariant sector. On the other hand, when $\zeta=1$, the above (36) will consist of ξ-independent part plus ξ-dependent part. By substituting $P_\sigma(\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho)$ back into $P_{\sigma',s}(\sigma'=\alpha',\beta',\gamma')$ and $P_{\sigma,t}(\sigma=\alpha,\beta,\gamma)$. According to (32), the expressions of projectors proportional to $1-\xi$ are the same as those in (13) for $SO(p)^+\times SO(q+r)^+$-invariant sector. Therefore, the expressions of projectors proportional to $1-\xi$ in (36) are the difference between the one in $(p,q+r)$ and the one in $(p+q,r)$. One can easily see that the expressions of projectors proportional to $1-\xi\zeta$ in (36)

[11] It is more convenient to use the $SL(8,\mathbf{R})$ basis in order to compare operators in the dual CFT. This is nothing but a triality rotation of the $SU(8)$ basis we have considered in which the two representations of $SO(8)$ are converted to each other using gamma matrices. In an $SL(8,\mathbf{R})$ basis, the expression of T-tensor was already given as Eq. (21) in [11].

are the one in $(p+q,r)$. This implies that one can use the projectors in (36) for those in $SO(p)^+ \times SO(q)^+$ invariant sector. Alternatively, one can exploit those projectors from (48) directly.

When $(\xi,\zeta) = (1,1)$, this leads to the algebra of $SO(8)$ and one obtains de Wit-Nicolai gauging with $SU(8) \times SO(8)$ gauge symmetry. When $(\xi,\zeta) = (1,0)$ one has $CSO(p+q,r)^+$ algebra with $SU(8) \times CSO(p+q,r)^+$ gauge symmetry. Moreover, when $(\xi,\zeta) = (1,-1)$, one gets $SO(p+q,r)^+$ algebra with $SU(8) \times SO(p+q,r)^+$ gauge symmetry. When $(\xi,\zeta) = (-1,1)$, it will give non-compact $SO(p,q+r)^+$ gauging with $SU(8) \times SO(p,q+r)^+$ gauge symmetry. When $(\xi,\zeta) = (-1,0)$, it yields a nontrivial non-semi-simple algebra of the Inonu-Wigner contraction of $SO(8)$ about its $SO(p,q)^+$ subgroup, denoted by $CSO(p,q,r)^+$ with $SU(8) \times CSO(p,q,r)^+$ gauge symmetry. For $(\xi,\zeta) = (-1,-1)$, one gets $SO(p+r,q)^+$ algebra with gauge symmetry $SU(8) \times SO(p+r,q)^+$. Finally, when $\xi = 0$, it gives Inonu-Wigner contraction $CSO(p,q+r)^+$ with gauge symmetry $SU(8) \times CSO(p,q+r)^+$. The gauge group will be spontaneously broken to its maximal compact subgroup.

3.3 Superpotential and Scalar Potential in $CSO(p,q,r)^+$ Gaugings [11]

The parametrization for the $SO(p)^+ \times SO(q)^+ \times SO(r)^+$-singlet space that is invariant subspace under a particular $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ subgroup of $SO(8)$ becomes

$$\phi_{IJKL} = 4\sqrt{2}\left(m X^+_{IJKL,s} + n X^+_{IJKL,t}\right)$$

where m, n are two real scalar fields. The two scalar fields parametrize an $SO(p)^+ \times SO(q)^+ \times SO(r)^+$-invariant subspace of the full scalar manifold $E_{7(7)}/SU(8)$. The 56-beins $\mathcal{V}$ can be written as a 56×56 matrix by exponentiating the vacuum expectation value ϕ_{IJKL}. One can construct 28-beins $u_{ij}{}^{KL}$ and v_{ijKL} in terms of scalars m, n explicitly, which can be given in terms of the products of $u_{ij,t}^{\;KL}$, $v_{ijKL,t}$, $u_{ij,s}^{\;KL}$ and $v_{ijKL,s}$, as given in the Appendix E[12]. Now the full expression for A_1' and A_2' tensors are given in terms of m, n using (5) and (36) with T' tensor.

$$\mathcal{V}(x) = \exp\begin{pmatrix} 0 & -\frac{1}{2\sqrt{2}}\,\phi_{IJPQ} \\ -\frac{1}{2\sqrt{2}}\,\overline{\phi}^{MNKL} & 0 \end{pmatrix}$$

[12]One can express $u_{IJ}{}^{KL}$ and v_{IJKL} in terms of sum of product of 4×4 matrices as follows:

$$\begin{aligned} u_{IJ}{}^{KL} &= \text{diag}\,(u_{1,t}u_{1,s} + v_{1,t}\overline{v}_{1,s}, u_{2,t}u_{2,s} + v_{2,t}\overline{v}_{2,s}, u_{3,t}u_{3,s} + v_{3,t}\overline{v}_{3,s}, \\ &\quad u_{4,t}u_{4,s} + v_{4,t}\overline{v}_{4,s}, u_{5,t}u_{5,s} + v_{5,t}\overline{v}_{5,s}, u_{6,t}u_{6,s} + v_{6,t}\overline{v}_{6,s}, u_{7,t}u_{7,s} + v_{7,t}\overline{v}_{7,s})\,, \\ v_{IJKL} &= \text{diag}\,(u_{1,t}v_{1,s} + v_{1,t}\overline{u}_{1,s}, u_{2,t}v_{2,s} + v_{2,t}\overline{u}_{2,s}, u_{3,t}v_{3,s} + v_{3,t}\overline{u}_{3,s}, \\ &\quad u_{4,t}v_{4,s} + v_{4,t}\overline{u}_{4,s}, u_{5,t}v_{5,s} + v_{5,t}\overline{u}_{5,s}, u_{6,t}v_{6,s} + v_{6,t}\overline{u}_{6,s}, u_{7,t}v_{7,s} + v_{7,t}\overline{u}_{7,s}) \end{aligned}$$

where each $u_{i,t}$ and $u_{i,s}$ corresponds to seven 4×4 block diagonal matrices for $u_{IJ,t}^{\;KL}$ and $u_{IJ,s}^{\;KL}$ respectively as in Appendix E and $v_{i,t}$ and $v_{i,s}$ for $v_{IJKL,t}$ and $v_{IJKL,s}$ respectively. Their complex conjugations hold similarly.

$$
\begin{aligned}
&= \exp\begin{pmatrix} 0 & -\frac{1}{2\sqrt{2}}\,\phi_{IJPQ,t} \\ -\frac{1}{2\sqrt{2}}\,\overline{\phi}^{MNKL,t} & 0 \end{pmatrix} \times \exp\begin{pmatrix} 0 & -\frac{1}{2\sqrt{2}}\,\phi_{IJPQ,s} \\ -\frac{1}{2\sqrt{2}}\,\overline{\phi}^{MNKL,s} & 0 \end{pmatrix} \\
&= \begin{pmatrix} u_{ij,t}{}^{IJ} & v_{ijKL,t} \\ \overline{v}_t^{klIJ} & \overline{u}^{kl}{}_{KL,t} \end{pmatrix} \times \begin{pmatrix} u_{ij,s}{}^{IJ} & v_{ijKL,s} \\ \overline{v}_s^{klIJ} & \overline{u}^{kl}{}_{KL,s} \end{pmatrix} = \begin{pmatrix} u_{ij}{}^{IJ} & v_{ijKL} \\ \overline{v}^{klIJ} & \overline{u}^{kl}{}_{KL} \end{pmatrix} \qquad (37)
\end{aligned}
$$

where $\phi_{IJKL,s} = 4\sqrt{2}mX^{+}_{IJKL,s}$ and $\phi_{IJKL,t} = 4\sqrt{2}nX^{+}_{IJKL,t}$, which commute each other. It turns out from (36) that the A_1' tensor has a single real eigenvalue, z_1, with degeneracies 8 that has the following form

$$
\begin{aligned}
A_1'^{ij} &= \mathrm{diag}(z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_1), \\
z_1 &= \frac{1}{8}\left(pe^{m+n} + qe^{m-\frac{p}{(q+r)}n}\xi + re^{-\frac{p+q}{r}m-\frac{p}{(q+r)}n}\xi\zeta\right). \qquad (38)
\end{aligned}
$$

It is now straightforward to verify that this yields (15) for $(p+q,r)$ gauging when $\xi = 1$ and $n = 0$. However, when $\zeta = 1$ and $m = 0$ it becomes (15) for $(p, q+r)$ gauging. In particular the superpotential, W, for the flow is found as one of the eigenvalues of the this symmetric tensor. Additionally, we can construct the A_2' tensor from (36) which are the combinations of the triple product of 28-beins as given in

$$
\begin{aligned}
A_{2i}'{}^{jkl} &= \frac{1}{4}(q+r)e^{m}\left(e^{n} - \xi e^{-\frac{p}{q+r}n}\right)X_t^{+ijkl} + \frac{1}{4}r\xi e^{-\frac{p}{q+r}n}\left(e^{m} - \zeta e^{-\frac{p+q}{r}m}\right)X_s^{+ijkl} \\
&= e^{m}(A_{2,t})_i{}^{jkl} + \xi e^{-\frac{p}{q+r}n}(A_{2,s})_i{}^{jkl} \qquad (39)
\end{aligned}
$$

where $(A_{2,t})_i{}^{jkl}$ is the same as the one (16) for $SO(p)^+ \times SO(q+r)^+$ sector and $(A_{2,s})_i{}^{jkl}$ for $SO(p+q)^+ \times SO(r)^+$. Moreover X_t^{+ijkl} is $\sum_{\sigma=\alpha,\beta,\gamma}\sigma P_{\sigma,t}$ while X_s^{+ijkl} is $\sum_{\sigma'=\alpha',\beta',\gamma'}\sigma' P_{\sigma',s}$[13]. Finally the $K_{\xi,\zeta,p,q,r}$-invariant scalar potential as a function of p, q, r, ξ, ζ and m, n by combining all the components can be written as

$$
V_{p,q,r}(\xi,\zeta;m,n) = -g^2\left(\frac{3}{4}|A_1'{}^{ij}|^2 - \frac{1}{24}|A_{2i}'{}^{jkl}|^2\right)
$$

[13] One can prove A_1' and A_2' can be obtained by analytic continuation. The T' tensor we obtained is $T_i'{}^{jkl}(E(-n) \times F(-m), \xi, \zeta)$. By considering only the $SL(8,\mathbf{R})$ transformation by ξ, this can be reduced to $e^{\alpha t}T_i'{}^{jkl}(E(t+n)^{-1} \times F(-m), 0, \zeta)$. Moreover, this becomes $e^{\alpha(t-n)}T_i'{}^{jkl}(E(t)^{-1} \times F(-m), e^{(\alpha-\beta)n}, \zeta)$. Now we arrive at the following intermediate expression: $e^{-\alpha n}T_i'{}^{jkl}(\mathbf{1} \times F(-m), \xi e^{(\alpha-\beta)n}, \zeta)$. Next we apply $SL(8,\mathbf{R})$ transformation by ζ. Then by doing a similar procedure we arrive at the final expression:

$$
T_i'{}^{jkl}(E(-n) \times F(-m), \xi, \zeta) = e^{-\alpha' m}e^{-\alpha n}T_i'{}^{jkl}(\mathbf{1}, \xi e^{(\alpha-\beta)n}, \zeta e^{(\alpha'-\beta')m}).
$$

At the origin, $\phi_{IJKL} = 0, \mathcal{V} = \mathbf{1}$, the T' tensor is from (36)

$$
T_i'{}^{jkl}(\mathbf{1}, \xi, \zeta) = \frac{3}{2}[1 - (1-\xi)a_1^{-1} - \xi(1-\zeta)a_2^{-1}]\delta_{ij}^{kl} - \frac{3}{2}(1-\xi)a_1^{-1}X_t^{ijkl} - \frac{3}{2}\xi(1-\zeta)a_2^{-1}X_s^{ijkl}. \qquad (40)
$$

Finally we possess all the information of $T_i'{}^{jkl}(E(-n) \times F(-m), \xi, \zeta)$ because by transforming $\xi \to \xi e^{(\alpha-\beta)n}, \zeta \to \zeta e^{(\alpha'-\beta')m}$ as in (40) we get $T_i'{}^{jkl}(\mathbf{1}, \xi e^{(\alpha-\beta)n}, \zeta e^{(\alpha'-\beta')m})$. From this, one can obtain A_1' tensor which is $\frac{4}{21}T_i'{}^{jkl}(E(-n) \times F(-m), \xi, \zeta)$. It turns out that it coincides with the one in (38). We used the numerical values: $\alpha' = -1 = \alpha, \beta = \frac{p}{q+r}, \beta' = \frac{p+q}{r}$ and $a_1 = \frac{p}{q+r} + 1$, $a_2 = \frac{p+q}{r} + 1$. Additionally, we have checked that $A_{2,i}'{}^{jkl} = -\frac{4}{3}T_i'{}^{[jkl]}(E(-n) \times F(-m), \xi, \zeta)$ is identical to the one in (39).

$$= \frac{1}{1536} e^{-\frac{2(p+q)m}{r} - \frac{2pn}{q+r}} \left(V_1 + V_2\xi + V_3\xi\zeta + V_4\xi^2 + V_5\xi^2\zeta + V_6\xi^2\zeta^2 \right)$$

where we introduce intermediate functions V_i's as the coefficients of the ξ and ζ

$$\begin{aligned}
V_1 &= e^{\frac{16m}{r}+\frac{16n}{q+r}} p \Big(p^2(q+r) + (-2+q+r)(q+r)^2 \\
&\quad + 2p(-72+q^2-r+r^2+q(-1+2r)) \Big) , \\
V_2 &= -2e^{\frac{16m}{r}+\frac{8n}{q+r}} pq \left(144+p^2+q^2+2q(-1+r)-2r+r^2+2p(-1+q+r) \right) , \\
V_3 &= -2e^{\frac{8m}{r}+\frac{8n}{q+r}} pr \left(144+p^2+q^2+2q(-1+r)-2r+r^2+2p(-1+q+r) \right) , \\
V_4 &= e^{\frac{16m}{r}} q \Big(p^3+q^2r+(-2+r)r^2+p^2(-2+2q+3r)+2q(-72-r+r^2) \\
&\quad + p(q^2+r(-4+3r)+q(-2+4r)) \Big) , \\
V_5 &= -2e^{\frac{8m}{r}} qr \left(144+p^2+q^2+2q(-1+r)-2r+r^2+2p(-1+q+r) \right) , \\
V_6 &= r \Big(p^3+q^3+2q^2(-1+r)-144r+q(-2+r)r+p^2(-2+3q+2r) \\
&\quad + p(3q^2+4q(-1+r)+(-2+r)r) \Big) .
\end{aligned}$$

By looking at the form of scalar potential, it is easy to see that $V_{r,q,p}(\xi=-1,\zeta=-1;m,n)$ can be obtained from $V_{p,q,r}(\xi=-1,\zeta=-1;-\frac{r}{p+q}n,-\frac{q+r}{p}m)$. Under the change of real fields, they are equivalent to each other. Moreover, the potential $V_{r,q,p}(\xi=-1,\zeta=1;m,n)$ can be obtained from $V_{p,q,r}(\xi=1,\zeta=-1;-\frac{r}{p+q}n,-\frac{q+r}{p}m)$. On this basis, the kinetic terms are not the usual ones but there exists a cross term, $\partial^\mu m\partial_\mu n$ which makes it difficult to find first-order differential equations for domain-wall solutions. Now we have to change the basis for which one has usual kinetic terms. We calculated all the quantities for 21 possible cases of $CSO(p,q,r)^+$ gaugings and summarized them in Appendix G:kinetic terms in terms of old fields[14], change of variables, superpotential, and scalar potential as new fields. From the results in Appendix G, one can describe a superpotential and scalar potential in terms of new real scalar fields $\widetilde{m}$ and

[14] One can generalize the kinetic terms (19) of $SO(p)^+ \times SO(q)^+$-invariant sector to write down

$$\begin{aligned}
&A_\mu{}^{ijkl} = -2\sqrt{2}\left(\overline{u}^{ij}{}_{IJ}\partial_\mu \overline{v}^{klIJ} - \overline{v}^{ijIJ}\partial_\mu \overline{u}^{kl}{}_{IJ} \right) \\
&+4\sqrt{2}(1-\xi)gA_{\mu IJ}\Big[\left(P_\beta^{IJKL} + \frac{1}{2}\left(P_\delta^{IJKL}+P_\rho^{IJKL}\right) \right) \left(-\overline{u}^{ij}{}_{KM}\overline{v}^{klLM} + \overline{v}^{ijKM}\overline{u}^{kl}{}_{LM} \right) \\
&+ \left(P_\delta^{IJRS} Z^{KLMN}_{(\delta)RS} - P_\rho^{IJRS} Z^{KLMN}_{(\rho)RS} \right) \left(\overline{u}^{ij}{}_{KL}\overline{u}^{kl}{}_{MN} - \overline{v}^{ijKL}\overline{v}^{klMN} \right) \Big] \\
&+4\sqrt{2}(1-\xi\zeta)gA_{\mu IJ}\Big[\left(P_\gamma^{IJKL} + \frac{1}{2}\left(P_\lambda^{IJKL}+P_\rho^{IJKL}\right) \right) \left(-\overline{u}^{ij}{}_{KM}\overline{v}^{klLM} + \overline{v}^{ijKM}\overline{u}^{kl}{}_{LM} \right) \\
&\left(P_\rho^{IJRS} Z^{KLMN}_{(\rho)RS} - P_\lambda^{IJRS} Z^{KLMN}_{(\lambda)RS} \right) \left(\overline{u}^{ij}{}_{KL}\overline{u}^{kl}{}_{MN} - \overline{v}^{ijKL}\overline{v}^{klMN} \right) \Big] .
\end{aligned}$$

Of course, in this case we put A_μ^{IJ} to zero because we are interested in the scalar plus gravity parts of the Lagrangian.

$\widetilde{n}$ that are related to the old fields m and n as follows:

$$m = -\frac{r\sqrt{pq(p+q)}}{4q(p+q)}\widetilde{m} - \frac{\sqrt{2r(p+q)}}{2(p+q)}\widetilde{n}, \qquad n = \frac{(q+r)\sqrt{pq(p+q)}}{4pq}\widetilde{m}.$$

Then in terms of new fields the superpotential can be written as

$$W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}) = \frac{1}{8}\left(pe^{2\sqrt{\frac{q}{p(p+q)}}\widetilde{m}+\sqrt{\frac{r}{2(p+q)}}\widetilde{n}} + qe^{-2\sqrt{\frac{p}{q(p+q)}}\widetilde{m}-\sqrt{\frac{r}{2(p+q)}}\widetilde{n}}\xi + re^{\sqrt{\frac{p+q}{2r}}\widetilde{n}}\xi\zeta\right), \quad (41)$$

and the supergravity potential is then given by

$$\begin{aligned} V_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= g^2\Big[4\left(\partial_{\widetilde{m}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2 + 4\left(\partial_{\widetilde{n}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2 \\ &\quad -6W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})^2\Big]. \end{aligned} \quad (42)$$

Note that the coefficients, 4 and 4, in the first and second terms in the above are a simple generalization of (18) for two scalar fields.

The superpotential has the following values at the various critical points in Table 4.

• The first row corresponds to the maximal supersymmetric case of de Wit-Nicolai's $SO(8)$-invariant trivial critical point.

• The second row corresponds to the $SO(7)^+$-invariant critical point of the $SO(8)$ theory that is equivalent to the second one in Table 2. We find $V_{r,q,p}(\xi = 1, \zeta = 1; m, n) = V_{p,q,r}(\xi = 1, \zeta = 1; -\frac{r}{p+q}n, -\frac{q+r}{p}m)$. This means that by a change of variables, the solutions of $(p,q,r) = (4,1,3), (5,1,2), (6,1,1)$ are obtained from $(p,q,r) = (3,1,4), (2,1,5), (1,1,6)$, respectively.

• The third row is the gauging of $CSO(p+q,r)^+ = CSO(2,6)^+$ that corresponds to the sixth in Table 2.

• The fourth row implies $SO(p+q,r)^+ = SO(5,3)^+$ that corresponds to negative superpotential, being equivalent to the third one in Table 2, or $SO(p+q,r)^+ = SO(3,5)^+$ that corresponds to positive superpotential, being equal to the fifth in Table 2. In each case, the potentials are the same, although the superpotentials are different at the critical points.

• The fifth row implies $SO(p+q,r)^+ = SO(4,4)^+$, being equivalent to the fourth one in Table 2.

• For the sixth row one has $SO(p,q+r)^+ = SO(3,5)^+$ gauging with positive superpotential and $SO(p,q+r)^+ = SO(5,3)^+$ with negative superpotential. According to the symmetry betwen the potential, one can see that all the critical points in this row can be obtained from those in fourth row: $V_{r,q,p}(\xi = -1, \zeta = 1; m, n) = V_{p,q,r}(\xi = 1, \zeta = -1; -\frac{r}{p+q}n, -\frac{q+r}{p}m)$. This implies that within a given class with the same scalar potential, these solutions are coming from the same critical point in the fourth row but are viewed along different directions in scalar space.

• For the seventh row, we have $SO(p,q+r)^+ = SO(4,4)^+$ gauging. Additionally, in this case, all the critical points are similarly obtained from those in the fifth row. These solutions are coming from the same critical point in the fifth row but are viewed along different directions in scalar space.

• For the eighth row, one has either $SO(p+r,q)^+ = SO(5,3)^+$ gaugings with negative superpotential or $SO(p+r,q)^+ = SO(3,5)^+$ with positive superpotential.

• In the ninth row $SO(p+r,q)^+ = SO(4,4)^+$ gauging.

• Finally the last row corresponds to $CSO(p,q+r)^+ = CSO(2,6)^+$ gauging.

For the eighth and ninth rows we have the following symmetry in the potential: $V_{r,q,p}(\xi = -1, \zeta = -1; m, n) = V_{p,q,r}(\xi = -1, \zeta = -1; -\frac{r}{p+q}n, -\frac{q+r}{p}m)$. In other words, the solutions corresponding to $(p,q,r) = (2,5,1), (3,3,2), (4,3,1)$ can be obtained from $(p,q,r) = (1,5,2), (2,3,3), (1,3,4)$ respectively in the eighth row. Similarly, in the ninth row the solution of $(p,q,r) = (1,4,3)$ is related to $(p,q,r) = (3,4,1)$.

3.4 Domain Wall in $CSO(p,q,r)^+$ Gaugings [11]

The resulting Lagrangian of scalar-gravity sector takes

$$\int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial^\mu\widetilde{m}\partial_\mu\widetilde{m} - \frac{1}{2}\partial^\mu\widetilde{n}\partial_\mu\widetilde{n} - V_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right). \tag{43}$$

With the ansatz (21) the equations of motion for the scalars and metric read

$$\begin{aligned}
&\partial_r^2 A - \partial_r A\partial_r B + \frac{3}{2}(\partial_r A)^2 + \frac{1}{4}(\partial_r\widetilde{m})^2 + \frac{1}{4}(\partial_r\widetilde{n})^2 + \frac{1}{2}e^{2B}V_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}) = 0,\\
&\partial_r^2\widetilde{m} + 3\partial_r A\partial_r\widetilde{m} - \partial_r B\partial_r\widetilde{m} - e^{2B}\partial_{\widetilde{m}}V_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}) = 0,\\
&\partial_r^2\widetilde{n} + 3\partial_r A\partial_r\widetilde{n} - \partial_r B\partial_r\widetilde{n} - e^{2B}\partial_{\widetilde{n}}V_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}) = 0.
\end{aligned} \tag{44}$$

By plugging the domain-wall ansatz (21) into the Lagrangian (43), the energy-density per unit area transverse to r-direction with the integration by parts on the term of $\partial_r^2 A$ can be expressed similarly and after rewriting and recombining the energy-density by summation of complete squares plus other terms, one gets

$$\begin{aligned}
E[A,\widetilde{m},\widetilde{n}] = \frac{1}{2}\int_{-\infty}^{\infty} dr e^{3A+B}&\left[-6\left(e^{-B}\partial_r A + \sqrt{2}gW_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2\right.\\
&+\left(e^{-B}\partial_r\widetilde{m} - 2\sqrt{2}g\partial_{\widetilde{m}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2 + \left(e^{-B}\partial_r\widetilde{n} - 2\sqrt{2}g\partial_{\widetilde{n}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2\\
&\left.+12\sqrt{2}ge^{-B}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\partial_r A + 4\sqrt{2}ge^{-B}\partial_r W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right].
\end{aligned}$$

By recognizing that the last two terms can be written as $4\sqrt{2}g\partial_r(e^{3A}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}))$ we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty} dr e^{3A+B}\left[-6\left(e^{-B}\partial_r A + \sqrt{2}gW_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2\right.\\
&\left.+\left(e^{-B}\partial_r\widetilde{m} - 2\sqrt{2}g\partial_{\widetilde{m}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2 + \left(e^{-B}\partial_r\widetilde{n} - 2\sqrt{2}g\partial_{\widetilde{n}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)^2\right]\\
&+2\sqrt{2}g\left(e^{3A}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})\right)|_{-\infty}^{\infty}.
\end{aligned}$$

Therefore, one finds the energy-density bound

$$E[A,\widetilde{m},\widetilde{n}] \geq 2\sqrt{2}g\left(e^{3A(\infty)}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})(\infty) - e^{3A(-\infty)}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n})(-\infty)\right).$$

This $E[A,\widetilde{m},\widetilde{n}]$ is extremized by domain-wall solutions and the first-order differential equations for the scalar fields one finds are the gradient flow equations of the superpotential (41):

$$\begin{aligned}
\partial_r\widetilde{m} &= \pm 2\sqrt{2}e^B g\partial_{\widetilde{m}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}),\\
\partial_r\widetilde{n} &= \pm 2\sqrt{2}e^B g\partial_{\widetilde{n}}W_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}),\\
\partial_r A &= \mp\sqrt{2}e^B gW_{p,q,r}(\xi,\zeta;\widetilde{m},\widetilde{n}).
\end{aligned} \tag{45}$$

$\mathcal{N}$	p	q	r	ξ	ζ	m	n	W	V
8	any	any	any	1	1	0	0	1	$-6g^2$
	1	1	6			$\frac{3}{4}\ln 5$	$-\frac{7}{8}\ln 5$		
	2	1	5			$\frac{5}{8}\ln 5$	$-\frac{3}{4}\ln 5$		
0	3	1	4	1	1	$\frac{1}{2}\ln 5$	$-\frac{5}{8}\ln 5$	$\frac{3}{2}\times 5^{-1/8}$	$-2\times 5^{3/4}g^2$
	4	1	3			$\frac{3}{8}\ln 5$	$-\frac{1}{2}\ln 5$		
	5	1	2			$\frac{1}{4}\ln 5$	$-\frac{3}{8}\ln 5$		
	6	1	1			$\frac{1}{8}\ln 5$	$-\frac{1}{4}\ln 5$		
0	1	1	6	1	0	any	0	$e^m/4$	0
	1	2	5			$\frac{5}{8}\ln 3$		$\frac{1}{2}\times 3^{-3/8}$	
	1	4	3			$-\frac{3}{8}\ln 3$		$-\frac{1}{2}\times 3^{-3/8}$	
0	2	1	5	1	-1	$\frac{5}{8}\ln 3$	0	$\frac{1}{2}\times 3^{-3/8}$	$2\times 3^{1/4}g^2$
	2	3	3			$-\frac{3}{8}\ln 3$		$-\frac{1}{2}\times 3^{-3/8}$	
	3	2	3			$-\frac{3}{8}\ln 3$		$-\frac{1}{2}\times 3^{-3/8}$	
	4	1	3			$-\frac{3}{8}\ln 3$		$-\frac{1}{2}\times 3^{-3/8}$	
	1	3	4						
0	2	2	4	1	-1	0	0	0	$2g^2$
	3	1	4						
	3	1	4				$\frac{5}{8}\ln 3$	$\frac{1}{2}\times 3^{-3/8}$	
	3	2	3				$\frac{5}{8}\ln 3$	$\frac{1}{2}\times 3^{-3/8}$	
0	3	3	2	-1	1	0	$\frac{5}{8}\ln 3$	$\frac{1}{2}\times 3^{-3/8}$	$2\times 3^{1/4}g^2$
	3	4	1				$\frac{5}{8}\ln 3$	$\frac{1}{2}\times 3^{-3/8}$	
	5	1	2				$-\frac{3}{8}\ln 3$	$-\frac{1}{2}\times 3^{-3/8}$	
	5	2	1				$-\frac{3}{8}\ln 3$	$-\frac{1}{2}\times 3^{-3/8}$	
	4	1	3						
0	4	2	2	-1	1	0	0	0	$2g^2$
	4	3	1						
	1	3	4			$\frac{1}{2}\ln 3$	$-\frac{7}{8}\ln 3$	$-\frac{1}{2}\times 3^{-3/8}$	
	1	5	2			$-\frac{1}{4}\ln 3$	$\frac{7}{8}\ln 3$	$\frac{1}{2}\times 3^{-3/8}$	
0	2	3	3	-1	-1	$\frac{3}{8}\ln 3$	$-\frac{3}{4}\ln 3$	$-\frac{1}{2}\times 3^{-3/8}$	$2\times 3^{1/4}g^2$
	2	5	1			$-\frac{1}{8}\ln 3$	$\frac{3}{4}\ln 3$	$\frac{1}{2}\times 3^{-3/8}$	
	3	3	2			$\frac{1}{4}\ln 3$	$-\frac{5}{8}\ln 3$	$-\frac{1}{2}\times 3^{-3/8}$	
	4	3	1			$\frac{1}{8}\ln 3$	$-\frac{1}{2}\ln 3$	$-\frac{1}{2}\times 3^{-3/8}$	
	1	4	3						
0	3	4	1	-1	-1	0	0	0	$2g^2$
	2	4	2						
	2	1	5						
	2	2	4						
0	2	3	3	0	$1, 0, -1$	any	any	$e^{m+n}/4$	0
	2	4	2						
	2	5	1						

Table 4. *Summary of various critical points in the context of superpotential : supersymmetry, vacuum expectation values of fields, superpotential and cosmological constants. There is no*

$SO(p)^+ \times SO(q)^+ \times U(1)^{+r(r-1)/2}$ *critical point of potential for* $\xi = -1$ *and* $\zeta = 0$. *The nontrivial* $CSO(p,q,r)^+$ *gauging in this section does not provide any new extra critical points.*

It is easy to check whether solutions $\widetilde{m}(r), \tilde{n}(r)$ and $A(r)$ of (45) satisfy the gravitational and scalar equations of motion in the second order differential equations (44). The analytic solutions of (45) for $\xi = 0$ when B is a constant become

$$\begin{aligned}
\widetilde{m}(x) &= -\frac{2q\left(\sqrt{\frac{2r}{p+q}}c_1 + 2\log\left[\frac{g(8q+pr)x+c_2}{4\sqrt{2}(p+q)}\right]\right)}{\sqrt{\frac{q}{p(p+q)}}\,(8q+pr)}, \\
\tilde{n}(x) &= \frac{8q\sqrt{\frac{r}{p+q}}c_1 - \sqrt{2}pr\log\left[\frac{g(8q+pr)x+c_2}{4\sqrt{2}(p+q)}\right]}{\sqrt{\frac{r}{(p+q)}}\,(8q+pr)} \\
A(x) &= c_1 + \frac{p(p+q)\log\left[\frac{g(8q+pr)x+c_2}{4\sqrt{2}(p+q)}\right]}{8q+pr}
\end{aligned}$$

where we change variable r into x in order not to confuse it with the integer r and c_1 and c_2 are constant. One expects to have nontrivial analytic solutions for nonzero ξ for particular (p,q,r) and ζ, as in section 2. In particular, for $CSO(1,1,6)^+$ gauging where we fix $\xi = -1$ and $\zeta = 0$, we have

$$\begin{aligned}
\widetilde{m}(x) &= \frac{2}{7}\left(-\sqrt{3}c_1 + \sqrt{2}\log\left(\tan\left[\frac{-7gx+c_2}{4\sqrt{2}}\right]\right)\right), \\
\tilde{n}(x) &= \frac{2}{14}\left(4c_1 + \sqrt{6}\log\left(\tan\left[\frac{-7gx+c_2}{4\sqrt{2}}\right]\right)\right), \\
A(x) &= \frac{1}{7}\left(7c_1\tan\left[\frac{-7gx+c_2}{4\sqrt{2}}\right] + 1\right)\log\left(\frac{1}{2}\sin\left[\frac{-7gx+c_2}{2\sqrt{2}}\right]\right).
\end{aligned}$$

Similarly, one also has an analytic solution of $CSO(3,3,2)^+$ gauging where $\xi = -1$ and $\zeta = 0$.

3.5 $CSO(p,q,r)^+$ Gaugings from $SO(8)$ Gaugings

Thus far, the values of p, q and r are greater than or equal to 1. If we allow those values to have zero, then one can classify them as follows: 1) $CSO(p,0,0)^+ = SO(p)^+$, 2) $CSO(p,q,0)^+ = SO(p,q)^+$, 3) $CSO(p,0,r)^+ = CSO(p,r)^+$, and 4) $CSO(p,q,r)^+$. In this section, we take a different route from previous the case. Some idea in this direction was already given in the paper of [14]. Let us consider the $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant generator of $SL(8,\mathbf{R})$,

$$X_{ab} = \begin{pmatrix} \alpha\mathbf{1}_{p\times p} & 0 & 0 \\ 0 & \beta\mathbf{1}_{q\times q} & 0 \\ 0 & 0 & \gamma\mathbf{1}_{r\times r} \end{pmatrix} \tag{46}$$

with

$$\alpha p + \beta q + \gamma r = 0, \qquad p+q+r = 8$$

where $\mathbf{1}_{p\times p}$ is $p \times p$ identity matrix. The embedding of this $SL(8,\mathbf{R})$ in E_7 is such that X_{ab} corresponds to the 56×56 E_7 generator which is a non-compact $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant element of the $SL(8,\mathbf{R})$ subalgebra of E_7

$$\begin{pmatrix} 0 & X^{+IJKL} \\ X^+_{IJKL} & 0 \end{pmatrix},$$

where the real, self-dual totally anti-symmetric $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant four-form tensor X^+_{IJKL} can be written in terms of a symmetric, trace-free, 8×8 matrix with $SO(8)$ right-handed spinor indices, X_{ab} using $SO(8)$ Γ matrices(See Appendix B)

$$X^+_{IJKL} = -\frac{1}{8}\left(\Gamma_{IJKL}\right)^{ab} X_{ab} \tag{47}$$

where $\Gamma_{IJKL} = \Gamma_{[I}\Gamma_J\Gamma_K\Gamma_{L]}$ and an arbitrary $SO(8)$ generator L_{IJ} acts in the right-handed spinor representation by $(L_{IJ}\Gamma_{IJ})^{ab}$. One can show that X^{+IJKL} (47) can be decomposed into X_t^{+IJKL} and X_s^{+IJKL}:

$$X^{+IJKL} = X_t^{+IJKL} + X_s^{+IJKL}$$

where the real, self-dual totally anti-symmetric $SO(p)^+ \times SO(q+r)^+$ invariant four-form tensor X_t^{+IJKL} was expressed in the previous subsection as Γ matrices with (26) and $SO(p+q)^+ \times SO(r)^+$ invariant four-form tensor X_s^{+IJKL} with (29). Moreover, α and β in (46) consist of α_t that was defined as (26) and (27)(we replace α by α_t) and α_s as (29) and (30). We also replace α' with α_s. Therefore we have

$$\alpha = \alpha_t + \alpha_s, \qquad \beta = \alpha_s - \frac{p}{q+r}\alpha_t.$$

Regarded as a 28×28 matrix, X^{+IJKL} has eigenvalues $\alpha, \beta, \gamma,\ \delta = (\alpha+\beta)/2, \lambda = (\alpha+\gamma)/2, \rho = (\beta+\gamma)/2$ with degeneracies $d_\alpha, d_\beta, d_\gamma, d_\delta, d_\lambda$ and d_ρ respectively. The eigenvalues and eigenspaces of the $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant tensor are summarized in Table 5. By introducing projectors, $P_\alpha, P_\beta, P_\gamma, P_\delta, P_\lambda$ and P_ρ onto corresponding eigenspaces, we have a 28×28 matrix equation

$$X^{+IJKL} = \sum_{\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho} \sigma P_\sigma^{IJKL}.$$

Projector $P_\alpha(P_\beta)[P_\gamma]$ projects the $SO(8)$ Lie algebra onto its $SO(p)^+(SO(q)^+)[SO(r)^+]$ subalgebra while $P_\delta(P_\lambda)[P_\rho]$ projects onto the remainder $\frac{SO(8)}{SO(p)^+\times SO(q)^+}(\frac{SO(8)}{SO(p)^+\times SO(r)^+})[\frac{SO(8)}{SO(q)^+\times SO(r)^+}]$. The projectors can be constructed from X^{+IJKL},

$$P_\sigma \;=\; \prod_{\sigma'\neq\sigma} \frac{1}{(\sigma'-\sigma)}\left(\sigma'\delta_{IJ}^{MN} - X^{+IJMN}\right), \qquad \text{for} \qquad \sigma = \alpha,\beta,\gamma,\delta,\lambda,\rho \tag{48}$$

and it is easily checked that they satisfy

$$P_\sigma^2 = P_\sigma, \qquad P_\sigma P_{\sigma'} = 0 (\sigma \neq \sigma') \qquad \text{where} \qquad \sigma, \sigma' = \alpha, \beta, \gamma, \delta, \lambda, \rho. \tag{49}$$

Then using the relation obtained by the properties of projectors above

$$\Big[\exp(-sX_s^+)\Big]^{IJKL} \Big[\exp(-tX_t^+)\Big]^{KLMN} = \sum_{\sigma'=\alpha',\beta',\gamma'} e^{-\sigma' s} P_{\sigma'}^{IJKL} \sum_{\sigma=\alpha,\beta,\gamma} e^{-\sigma t} P_\sigma^{KLMN}$$

one gets

$$\begin{aligned} g(s,t) A_\mu^{IJ}(s,t) &\equiv g e^{\alpha t} e^{\alpha' s} \Big[\exp(-sX_s^+)\Big]^{IJKL} \Big[\exp(-tX_t^+)\Big]^{KLMN} A_\mu^{MN} \\ &= g e^{\alpha t} e^{\alpha' s} \sum_{\sigma'=\alpha',\beta',\gamma'} e^{-\sigma' s} P_{\sigma',s}^{IJKL} \sum_{\sigma=\alpha,\beta,\gamma} e^{-\sigma t} P_{\sigma,t}^{KLMN} A_\mu^{MN} \end{aligned}$$

which will be the same as (33) together with $A_{\mu(\sigma)}^{IJ} \equiv P_\sigma^{IJKL} A_\mu^{KL}$ for $\sigma = \alpha, \beta, \gamma, \delta, \lambda, \rho$. In this section, the main difference with the previous section is that we started with projectors directly constructed from $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant four-form tensor. Of course, these projectors are very complicated expressions because they are fifth power of X^{+IJKL} or δ_{IJ}^{KL} given in (48). In the previous section, according to (32), we identified the product of projectors in $SO(p, q+r)^+$ and $SO(p+q, r)^+$ with a single projector (48) in this section.

4 Conclusion

In summary,

• the main results in section 2 is described by (24). There are BPS domain-wall solutions interpolating between a maximally supersymmetric $SO(8)$ critical point and various nonsupersymmetric ones.

• The analytic solution is available for only $p = q = 4$ with general ξ. For $\xi = 0$, we also have solutions for general (p, q). That is, for $SO(4,4)^+$ and $CSO(p, 8-p)^+$ gaugings where $p = 1, \cdots, 7$ there exist analytic solutions. Among these gaugings, only $SO(4,4)^+$ and $CSO(2,6)^+$ cases contain critical points according to Table 2. Note that the presence of domain-wall solutions do not have any critical points.

• In section 3, the crucial part is to obtain a T-tensor as found in (36). Although it is rather complicated and involved, all the components of a T-tensor can be obtained from the information on both the projectors and 28-beins established by $SO(p)^+ \times SO(q)^+ \times SO(r)^+$-singlet space. The only nontrivial gauging comparable to that in Section 2 corresponds to both $\xi = -1$ and $\zeta = 0$ that gives rise to $CSO(p, q, r)^+$ gauging. The other values of ξ and ζ gave us the previous gaugings discussed in Section 2.

• Finally, we arrived at (41) and (42) that is a general expression, our new findings, for two scalar fields as the one (18) for one scalar field.

• Moreover, similar domain-wall solutions are described by (45). Although the scalar potential for this case looks different from the case of $SO(p)^+ \times SO(q)^+$, the structure of the critical points are reduced to those in $SO(p)^+ \times SO(q)^+$-invariant sector. We emphasize that although $CSO(p,q,r)^+$ gaugings(in this case $\xi = -1$ and $\zeta = 0$) do not have any critical points analyzed in section 3.3, they *do* have domain wall solutions and even possess analytic solutions for particular (p,q,r) combinations.

p	q	r	α	β	γ	δ	λ	ρ	d_α	d_β	d_γ	d_δ	d_λ	d_ρ	$\|X^+\|^2$
1	1	6	−2	6/7	10/21	−10/7	−16/21	−4/21	0	0	15	1	6	6	64/7
1	2	5	−2	−6/7	26/35	−10/7	−22/35	−2/35	0	1	10	2	5	10	432/35
2	1	5	−2	−2/3	14/15	−4/3	−8/15	2/15	1	0	10	2	10	5	96/5
1	3	4	−2	−6/7	8/7	−10/7	−3/7	1/7	0	3	6	3	4	12	120/7
2	2	4	−2	−2/3	2	−4/3	0	2/3	1	1	6	4	8	8	32
3	1	4	−2	−2/5	8/5	−6/5	−1/5	3/5	3	0	6	3	12	4	168/5
1	4	3	−2	−6/7	38/21	−10/7	−2/21	10/21	0	6	3	4	3	12	176/7
2	3	3	−2	−2/3	2	−4/3	0	2/3	1	3	3	6	6	9	32
3	2	3	−2	−2/5	34/15	−6/5	2/15	14/15	3	1	3	6	9	6	208/5
4	1	3	−2	0	8/3	−1	1/3	4/3	6	0	3	4	12	3	56
1	5	2	−2	−6/7	22/7	−10/7	4/7	8/7	0	10	1	5	2	10	288/7
2	4	2	−2	−2/3	10/3	−4/3	2/3	4/3	1	6	1	8	4	8	48
3	3	2	−2	−2/5	18/5	−6/5	4/5	8/5	3	3	1	9	6	6	288/5
4	2	2	−2	0	4	−1	1	2	6	1	1	8	8	4	72
5	1	2	−2	2/3	14/3	−2/3	4/3	8/3	10	0	1	5	10	2	96
1	6	1	−2	−6/7	50/7	−10/7	18/7	22/7	0	15	0	6	1	6	624/7
2	5	1	−2	−2/3	22/3	−4/3	8/3	10/3	1	10	0	10	2	5	96
3	4	1	−2	−2/5	38/5	−6/5	14/5	18/5	3	6	0	12	3	4	528/5
4	3	1	−2	0	8	−1	3	4	6	3	0	12	4	3	120
5	2	1	−2	2/3	26/3	−2/3	10/3	14/3	10	1	0	10	5	2	144
6	1	1	−2	2	10	0	4	6	15	0	0	6	6	1	192

Table 5. *Eigenvalues and eigenspaces of the $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ invariant tensor, X^+ where $|X^+|^2 = \sum_{\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho} d_\sigma |\sigma|^2$. The degeneracies are given in $d_\alpha = p(p-1)/2, d_\beta = q(q-1)/2, d_\gamma = r(r-1)/2, d_\delta = pq, d_\lambda = pr$ and $d_\rho = qr$. In [39], they displayed the signature of the Killing-Cartan form by writing the numbers n_+, n_- and n_0 of its positive, negative and zero eigenvalues. Here we identify $d_\alpha + d_\beta$ with n_+, d_δ with n_- and $d_\gamma + d_\lambda + d_\rho$ with n_0.*

Recently, 11-dimensional embedding [27, 45] of supersymmetric vacua of compact-gauged supergravity was found. For solutions with varying scalars(due to the r-dependence of vacuum expectation values), the ansatz for the field strength was more complicated. In this direction it was crucial to know about the 11-dimensional analog of superpotential, so-called geometric superpotential, in order to achieve the M-theory lift of the RG flow. Provided that the r-dependence of the vevs is controlled by the RG flow equations, an exact solution to the 11-

dimensional Einstein-Maxwell equations was obtained. As mentioned in the introduction, the 11-dimensional origin of $SO(p,q)^+$ and $CSO(p,q)^+$ gaugings was found in [14] for constant scalars. In this paper, we describe explict r-dependence on the vevs by domain-wall solutions. It is natural to ask whether 11-dimensional embedding of various vacua we have considered of non-compact and non-semi-simple gauged supergravity can be obtained. In a recent paper [46], the metric on the 7-dimensional internal space and domain wall in 11-dimensions was found. However, they did not provide an ansatz for an 11-dimensional three-form gauge field. It would be interesting to study the geometric superpotential, 11-dimensional analog of superpotential we have obtained. We expect that the nontrivial r-dependence of vevs makes Einstein-Maxwell equations consistent not only at the critical points but also along the supersymmetric RG flow connecting two critical points.

In [38], all critical points of the scalar potential of the $\mathcal{N}=8$ supergravity with $SO(8)$ gauge symmetry that break the local $SO(8)$ down to a solution with symmetry that is at least some specified subgroup of $SO(8)$ were found. One considers only those scalars which are singlets of that subgroup and searches critical points of the potential restricted to be a function only of the singlets. Schurr's lemma tells us that any critical point of restricted potential will be a critical point of the original complete scalar potential. Then the problem of finding critical points of the potential is reduced to the simpler one of finding critical points of the restricted potential which is a singlet sector. In this paper, we applied similar techniques to the non-compact and non-semi-simple gauged supergravities and the subgroup is to be $SO(p)^+ \times SO(q)^+$ for the $SO(p,q)^+$ gaugings and $CSO(p,q)^+$ gaugings while that will be $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ for the $CSO(p,q,r)^+$ gaugings.

In [38], the specified subgroup H was taken to be $SU(3)$ for $SO(8)$ gauged supergravity. One can think of the H subgroup as a compact subgroup of $SO(p,q)^+$ gauged model because this is necessary to the validity of Schurr's lemma. When 56-beins commute the $SL(8,\mathbf{R})$ transformation $E(t)$, it is rather easy to calculate the scalar potential. However, it may happen that for the noncommutativity of 56-beins $\mathcal{V}$ and $E(t)$, it will be rather complicated to find the scalar potential because of the presence of additional Baker-Hausdorff terms appearing in the calculations of exponentials of matrices. According to [13], it was found that no G_2-invariant critical points exist for $SO(7,1)^+$ gauging, no $SU(3)$-invariant critical points for $SO(6,2)^+$ gauging and a $SO(5)$-invariant critical point with positive cosmological constant, and no supersymmetry for $SO(5,3)^+$ gauging. It would be interesting to investigate whether there exist any critical points of the potential restricted to the H-singlet sector for the most general $CSO(p,q,r)^+$ gaugings we have considered in this paper. Here group H is a compact subgroup of this model.

5 Appendix A: Four-form (Anti)Self-dual Tensors in 28×28 Matrices

Let us consider the $SO(p)^- \times SO(q)^-$ invariant generator of $SL(8,\mathbf{R})$,

$$X_{\dot a\dot b} \quad = \quad \begin{pmatrix} \alpha\mathbf{1}_{p\times p} & 0 \\ 0 & \beta\mathbf{1}_{q\times q} \end{pmatrix} \quad \text{with} \quad \alpha p + \beta q = 0, \qquad p+q = 8,$$

where $\mathbf{1}_{p\times p}$ is a $p\times p$ identity matrix. The embedding of this $SL(8,\mathbf{R})$ in E_7 is such that $X_{\dot a\dot b}$ corresponds to the 56×56 E_7 generator with X^{-IJKL}

$$\begin{pmatrix} 0 & X^{-IJKL} \\ X^-_{IJKL} & 0 \end{pmatrix},$$

where the real, anti-self-dual totally anti-symmetric tensor X^{-IJKL} is given by the following form through the $\widetilde{\Gamma}$ matrix

$$X^-_{IJKL} = -\frac{1}{8}\left(\widetilde{\Gamma}_{IJKL}\right)^{\dot a\dot b} X_{\dot a\dot b} \tag{50}$$

where $\widetilde{\Gamma}_{IJKL} = \widetilde{\Gamma}_{[I}\widetilde{\Gamma}_J\widetilde{\Gamma}_K\widetilde{\Gamma}_{L]}$ as in section 2.5 and an arbitrary $SO(8)$ generator L_{IJ} acts in the left-handed spinor representation(See Appendix B for this representation) by $(L_{IJ}\widetilde{\Gamma}_{IJ})^{\dot a\dot b}$. When $p=7$ and $q=1$, one can see that this expression of (50) through $\widetilde{\Gamma}$ matrix coincides exactly with the one in section 2.6 or $X^{-IJKL}_{7,1}$ presented below explicitly.

We have seen real (anti) self-dual tensors in the $SU(8)$-basis through Γ matrices in (12) and (50). Now one can express them as the following forms which will be a useful and illuminating description, viewed as a 28×28 matrix representation, after doing the Γ matrix algebra

$$X^{\pm IJKL}_{p,q} = Y^{IJKL}_{p,q} + \frac{\eta}{24}\epsilon^{IJKLMNPQ}Y^{MNPQ}_{p,q},$$

where self-duality $+$ corresponds to $\eta = 1$ and anti-self-duality $-$ corresponds to $\eta = -1$ and $Y^{IJKL}_{p,q}$ tensors are given for each p and q in

$$\begin{aligned}
Y^{IJKL}_{7,1} &= \frac{1}{2}\left(\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,2\,5\,6} + \delta^{IJKL}_{1\,2\,7\,8} + \delta^{IJKL}_{1\,3\,7\,5} + \delta^{IJKL}_{1\,3\,6\,8} + \delta^{IJKL}_{1\,4\,5\,8} + \delta^{IJKL}_{1\,4\,6\,7}\right),\\
Y^{IJKL}_{6,2} &= \frac{1}{2}\left(\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,2\,5\,6} + \delta^{IJKL}_{1\,2\,7\,8}\right),\\
Y^{IJKL}_{5,3} &= \frac{1}{6}\left(3\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,2\,5\,6} + \delta^{IJKL}_{1\,2\,7\,8} + \delta^{IJKL}_{1\,5\,3\,7} + \delta^{IJKL}_{1\,3\,6\,8} + \delta^{IJKL}_{1\,5\,4\,8} + \delta^{IJKL}_{1\,6\,4\,7}\right),\\
Y^{IJKL}_{4,4} &= \frac{1}{2}\delta^{IJKL}_{1\,2\,3\,4},\\
Y^{IJKL}_{3,5} &= \frac{1}{10}\left(3\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,5\,2\,6} + \delta^{IJKL}_{1\,2\,7\,8} + \delta^{IJKL}_{1\,3\,5\,7} + \delta^{IJKL}_{1\,3\,6\,8} + \delta^{IJKL}_{1\,4\,5\,8} + \delta^{IJKL}_{1\,6\,4\,7}\right),\\
Y^{IJKL}_{2,6} &= \frac{1}{6}\left(\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,5\,2\,6} + \delta^{IJKL}_{1\,2\,7\,8}\right),\\
Y^{IJKL}_{1,7} &= \frac{1}{14}\left(\delta^{IJKL}_{1\,2\,3\,4} + \delta^{IJKL}_{1\,5\,2\,6} + \delta^{IJKL}_{1\,2\,7\,8} + \delta^{IJKL}_{1\,3\,5\,7} + \delta^{IJKL}_{1\,3\,6\,8} + \delta^{IJKL}_{1\,5\,4\,8} + \delta^{IJKL}_{1\,4\,6\,7}\right).
\end{aligned}$$

Actually the case of $X_{5,3}^{\pm IJKL}$ can be identified with $SO(5)^{\pm}$-singlets among six scalars [47] when restricted to equal real parameters(ϕ_{IJKL} depends on only three real parameters because of $SO(3)^{\pm}$ rotation).

6 Appendix B: $SO(8)$ Γ Matrices and Its Representations

The 28 $SO(8)$ generators are denoted by Λ_{MN} where $M, N = 1, 2, \cdots, 8$ and they can be decomposed into $\Lambda_{MN} = (\Lambda_{mn}, \Lambda_{m1})$. Here $\Lambda_{mn} = -\Lambda_{nm}$ where $m, n = 2, 3, \cdots, 8$ are the 21 generators of $SO(7)$. Then the 8×8 $SO(7)$ gamma matrices satisfy $\{\Gamma_m, \Gamma_n\} = -2\delta_{mn}$ and the generators act on the 8-dimensional spinor representation of $SO(7)$ by $\frac{1}{4}\Lambda^{mn}\Gamma_{mn}$. Then the 16×16 $SO(8)$ gamma matrices have the following form, $\gamma_{MN} = \text{diag}((\Gamma_{MN})^{ab}, (\widetilde{\Gamma}_{MN})^{\dot{a}\dot{b}})$ where

$$\Gamma_{MN} = \widetilde{\Gamma}_{MN} = \Gamma_{mn}, \qquad M, N = 2, 3, \cdots, 8, \qquad \Gamma_{M1} = -\widetilde{\Gamma}_{M1} = \Gamma_m$$

and a, b are right-handed spinor indices and $\dot{a}, \dot{b}$ are left-handed spinors. The $SO(8)$ has three different eight-dimensional representations: the vector representation $\mathbf{8}_v$ generated by Λ_{MN}, the right-handed spinor representation $\mathbf{8}_s$ generated by $\frac{1}{4}\Lambda^{mn}\Gamma_{mn}$, and the left-handed spinor representation $\mathbf{8}_c$ generated by $\frac{1}{4}\Lambda^{mn}\widetilde{\Gamma}_{mn}$. This induces three inequivalent $SO(7)$ subgroups of $SO(8)$. That is, the stability group of the vector, $SO(7)$ is generated by Λ_{MN}, $M, N = 2, 3, \cdots, 8$, the stabilizer of a right -handed spinor, $SO(7)^+$ is generated by $\Lambda^{MN}\Gamma_{MN}$, and the stabilizer of a left-handed spinor, $SO(7)^-$ is generated by $\Lambda^{MN}\widetilde{\Gamma}_{MN}$. The $SO(7)^+$-singlet under the branching rule of 35-dimensional fourth rank self-dual antisymmetric tensor representation of $SO(8)$ into $SO(7)^+$ corresponds to the $SO(7)^+$-invariant tensor X^{+IJKL} given in Section 2.3. Moreover, we present explicit realizations of Γ matrices we are using here as follows [48, 32]:

$$\Gamma^2 = \begin{pmatrix} \alpha^3 & 0 \\ 0 & -\alpha^3 \end{pmatrix}, \Gamma^3 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix}, \Gamma^4 = \begin{pmatrix} \alpha^1 & 0 \\ 0 & -\alpha^1 \end{pmatrix},$$
$$\Gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \Gamma^6 = \begin{pmatrix} 0 & -\beta^3 \\ -\beta^3 & 0 \end{pmatrix}, \Gamma^7 = \begin{pmatrix} 0 & \beta^2 \\ \beta^2 & 0 \end{pmatrix}, \Gamma^8 = \begin{pmatrix} 0 & \beta^1 \\ \beta^1 & 0 \end{pmatrix},$$

where α^i's and β^i's are given in terms of usual 2×2 Pauli matrices σ^i's

$$\alpha^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \alpha^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \alpha^3 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix},$$
$$\beta^1 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \beta^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \beta^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}.$$

7 Appendix C: Some Identities between Invariant Generators and Projectors in $SO(p)^+ \times SO(q)^+$ Sectors

For any $SO(p)^+ \times SO(q)^+$ generator $\Lambda^{IJ}_{(\alpha)}$, the invariance of X^{+IJKL} under the $SO(p)^+$ implies

$$E(t)^{-1} \begin{pmatrix} \underline{\Lambda}_{(\alpha)} & 0 \\ 0 & \underline{\Lambda}_{(\alpha)} \end{pmatrix} E(t) = \begin{pmatrix} \underline{\Lambda}_{(\alpha)} & 0 \\ 0 & \underline{\Lambda}_{(\alpha)} \end{pmatrix},$$

which is equivalent to

$$[P_\alpha, \underline{\Lambda}_{(\alpha)}] = [P_\beta, \underline{\Lambda}_{(\alpha)}] = [P_\gamma, \underline{\Lambda}_{(\alpha)}] = 0.$$

Similarly, the invariance of X^{+IJKL} under the $SO(q)^+$ implies

$$E(t)^{-1} \begin{pmatrix} \underline{\Lambda}_{(\beta)} & 0 \\ 0 & \underline{\Lambda}_{(\beta)} \end{pmatrix} E(t) = \begin{pmatrix} \underline{\Lambda}_{(\beta)} & 0 \\ 0 & \underline{\Lambda}_{(\beta)} \end{pmatrix},$$

which will lead to vanishing of commutators between $P_{\alpha,\beta,\gamma}$ and $\underline{\Lambda}_{(\beta)}$

$$[P_\alpha, \underline{\Lambda}_{(\beta)}] = [P_\beta, \underline{\Lambda}_{(\beta)}] = [P_\gamma, \underline{\Lambda}_{(\beta)}] = 0.$$

One gets the following identities

$$\begin{aligned} &P_\alpha \underline{\Lambda}_{(\alpha)} P_\gamma = P_\beta \underline{\Lambda}_{(\alpha)} P_\gamma = P_\alpha \underline{\Lambda}_{(\beta)} P_\gamma = P_\beta \underline{\Lambda}_{(\beta)} P_\gamma = 0, \\ &P_\gamma \underline{\Lambda}_{(\alpha)} P_\alpha = P_\gamma \underline{\Lambda}_{(\alpha)} P_\beta = P_\gamma \underline{\Lambda}_{(\beta)} P_\alpha = P_\gamma \underline{\Lambda}_{(\beta)} P_\beta = 0, \\ &P_\beta \underline{\Lambda}_{(\alpha)} P_\alpha = P_\beta \underline{\Lambda}_{(\beta)} P_\alpha = P_\alpha \underline{\Lambda}_{(\alpha)} P_\beta = P_\alpha \underline{\Lambda}_{(\beta)} P_\beta = 0. \end{aligned} \tag{51}$$

Moreover, one gets for the $SO(8)/(SO(p)^+ \times SO(q)^+)$ generator $\Lambda_{(\gamma)}$

$$P_\alpha \underline{\Lambda}_{(\gamma)} P_\alpha = P_\beta \underline{\Lambda}_{(\gamma)} P_\alpha = P_\alpha \underline{\Lambda}_{(\gamma)} P_\beta = P_\beta \underline{\Lambda}_{(\gamma)} P_\beta = P_\gamma \underline{\Lambda}_{(\gamma)} P_\gamma = 0. \tag{52}$$

With $1 = P_\alpha + P_\beta + P_\gamma$, the combinations of (52) will give us

$$\begin{aligned} P_\alpha \underline{\Lambda}_{(\gamma)} P_\gamma &= P_\alpha \underline{\Lambda}_{(\gamma)}, \qquad P_\gamma \underline{\Lambda}_{(\gamma)} P_\alpha = \underline{\Lambda}_{(\gamma)} P_\alpha, \\ P_\beta \underline{\Lambda}_{(\gamma)} P_\gamma &= P_\beta \underline{\Lambda}_{(\gamma)}, \qquad P_\gamma \underline{\Lambda}_{(\gamma)} P_\beta = \underline{\Lambda}_{(\gamma)} P_\beta. \end{aligned} \tag{53}$$

By combining the first(second) and third(fourth) relations of (53) respectively and using (52) it is easily checked that

$$(P_\alpha + P_\beta) \underline{\Lambda}_{(\gamma)} = \underline{\Lambda}_{(\gamma)} P_\gamma, \qquad \underline{\Lambda}_{(\gamma)} (P_\alpha + P_\beta) = P_\gamma \underline{\Lambda}_{(\gamma)}.$$

8 Appendix D: Some Identities between Invariant Generators and Projectors in $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ Sectors

For any $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ generator $\Lambda^{IJ}_{(\alpha)}$, the invariance of X^{+IJKL} under the $SO(p)^+$ implies

$$E(t)^{-1} \left(\begin{array}{cc} \underline{\Lambda}_{(\alpha)} & 0 \\ 0 & \underline{\Lambda}_{(\alpha)} \end{array} \right) E(t) = \left(\begin{array}{cc} \underline{\Lambda}_{(\alpha)} & 0 \\ 0 & \underline{\Lambda}_{(\alpha)} \end{array} \right),$$

which is equivalent to

$$[P_\sigma, \underline{\Lambda}_{(\alpha)}] = 0 \qquad \text{for} \qquad \sigma = \alpha, \beta, \gamma, \delta, \lambda, \rho.$$

Similarly, the invariance of X^{+IJKL} under the $SO(q)^+$ implies

$$E(t)^{-1} \left(\begin{array}{cc} \underline{\Lambda}_{(\beta)} & 0 \\ 0 & \underline{\Lambda}_{(\beta)} \end{array} \right) E(t) = \left(\begin{array}{cc} \underline{\Lambda}_{(\beta)} & 0 \\ 0 & \underline{\Lambda}_{(\beta)} \end{array} \right),$$

which will lead to

$$[P_\sigma, \underline{\Lambda}_{(\beta)}] = 0 \qquad \text{for} \qquad \sigma = \alpha, \beta, \gamma, \delta, \lambda, \rho.$$

Similarly, the invariance of X^{+IJKL} under the $SO(r)^+$ implies

$$E(t)^{-1} \left(\begin{array}{cc} \underline{\Lambda}_{(\gamma)} & 0 \\ 0 & \underline{\Lambda}_{(\gamma)} \end{array} \right) E(t) = \left(\begin{array}{cc} \underline{\Lambda}_{(\gamma)} & 0 \\ 0 & \underline{\Lambda}_{(\gamma)} \end{array} \right),$$

which will lead to

$$[P_\sigma, \underline{\Lambda}_{(\gamma)}] = 0 \qquad \text{for} \qquad \sigma = \alpha, \beta, \gamma, \delta, \lambda, \rho.$$

Using the relations (49), one gets the following identities

$$P_\sigma \underline{\Lambda}_{(\alpha)} P_{\sigma'} = P_\sigma \underline{\Lambda}_{(\beta)} P_{\sigma'} = P_\sigma \underline{\Lambda}_{(\gamma)} P_{\sigma'} = 0, \qquad \text{for} \qquad \sigma, \sigma' = \alpha, \beta, \gamma, \delta, \lambda, \rho \qquad \sigma \neq \sigma'.$$

Moreover, one gets for the $SO(8)/(SO(p)^+ \times SO(q)^+)$ generator $\Lambda^{IJ}_{(\delta)}$

$$\begin{aligned}
P_\alpha \underline{\Lambda}_{(\delta)} P_\sigma &= 0, \qquad \sigma = \alpha, \beta, \gamma, \lambda, \rho \\
P_\beta \underline{\Lambda}_{(\delta)} P_\sigma &= 0, \qquad \sigma = \alpha, \beta, \gamma, \lambda, \rho \\
P_\gamma \underline{\Lambda}_{(\delta)} P_\sigma &= 0, \qquad \sigma = \alpha, \beta, \delta, \lambda, \rho \\
P_\delta \underline{\Lambda}_{(\delta)} P_\sigma &= 0, \qquad \sigma = \gamma, \delta, \lambda, \rho \\
P_\lambda \underline{\Lambda}_{(\delta)} P_\sigma &= 0, \qquad \sigma = \alpha, \beta, \gamma, \delta, \lambda \\
P_\rho \underline{\Lambda}_{(\delta)} P_\sigma &= 0, \qquad \sigma = \alpha, \beta, \gamma, \delta, \rho.
\end{aligned} \qquad (54)$$

With $1=\sum_{\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho} P_\sigma$, the combinations of (54) will give us

$$\begin{aligned}
&P_\alpha \underline{\Delta}_{(\delta)} P_\delta = P_\alpha \underline{\Delta}_{(\delta)}, \qquad P_\beta \underline{\Delta}_{(\delta)} P_\delta = P_\beta \underline{\Delta}_{(\delta)}, \qquad P_\gamma \underline{\Delta}_{(\delta)} P_\gamma = P_\gamma \underline{\Delta}_{(\delta)},\\
&P_\delta \underline{\Delta}_{(\delta)} P_\alpha = \underline{\Delta}_{(\delta)} P_\alpha, \qquad P_\delta \underline{\Delta}_{(\delta)} P_\beta = \underline{\Delta}_{(\delta)} P_\beta, \qquad P_\lambda \underline{\Delta}_{(\delta)} P_\rho = P_\lambda \underline{\Delta}_{(\delta)},\\
&P_\rho \underline{\Delta}_{(\delta)} P_\lambda = P_\rho \underline{\Delta}_{(\delta)}.
\end{aligned} \tag{55}$$

Moreover, one gets for the $SO(8)/(SO(p)^+ \times SO(r)^+)$ generator $\Lambda^{IJ}_{(\lambda)}$

$$\begin{aligned}
P_\alpha \underline{\Delta}_{(\lambda)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\delta,\rho\\
P_\beta \underline{\Delta}_{(\lambda)} P_\sigma &= 0, \qquad \sigma=\alpha,\gamma,\delta,\lambda,\rho\\
P_\gamma \underline{\Delta}_{(\lambda)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\delta,\rho\\
P_\delta \underline{\Delta}_{(\lambda)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\delta,\lambda\\
P_\lambda \underline{\Delta}_{(\lambda)} P_\sigma &= 0, \qquad \sigma=\beta,\delta,\lambda,\rho\\
P_\rho \underline{\Delta}_{(\lambda)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\lambda,\rho.
\end{aligned} \tag{56}$$

With $1=\sum_{\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho} P_\sigma$, the combinations of (56) will give us

$$\begin{aligned}
&P_\alpha \underline{\Delta}_{(\lambda)} P_\lambda = P_\alpha \underline{\Delta}_{(\lambda)}, \qquad P_\beta \underline{\Delta}_{(\lambda)} P_\beta = P_\beta \underline{\Delta}_{(\lambda)}, \qquad P_\gamma \underline{\Delta}_{(\lambda)} P_\lambda = P_\gamma \underline{\Delta}_{(\lambda)},\\
&P_\delta \underline{\Delta}_{(\lambda)} P_\rho = P_\delta \underline{\Delta}_{(\lambda)}, \qquad P_\lambda \underline{\Delta}_{(\lambda)} P_\alpha = \underline{\Delta}_{(\lambda)} P_\alpha, \qquad P_\lambda \underline{\Delta}_{(\lambda)} P_\gamma = \underline{\Delta}_{(\lambda)} P_\gamma,\\
&P_\rho \underline{\Delta}_{(\lambda)} P_\delta = P_\rho \underline{\Delta}_{(\lambda)}.
\end{aligned} \tag{57}$$

Moreover, one gets for the $SO(8)/(SO(q)^+ \times SO(r)^+)$ generator $\Lambda^{IJ}_{(\rho)}$

$$\begin{aligned}
P_\alpha \underline{\Delta}_{(\rho)} P_\sigma &= 0, \qquad \sigma=\beta,\gamma,\delta,\lambda,\rho\\
P_\beta \underline{\Delta}_{(\rho)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\delta,\lambda\\
P_\gamma \underline{\Delta}_{(\rho)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\delta,\lambda\\
P_\delta \underline{\Delta}_{(\rho)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\delta,\rho\\
P_\lambda \underline{\Delta}_{(\rho)} P_\sigma &= 0, \qquad \sigma=\alpha,\beta,\gamma,\lambda,\rho\\
P_\rho \underline{\Delta}_{(\rho)} P_\sigma &= 0, \qquad \sigma=\alpha,\delta,\lambda,\rho.
\end{aligned} \tag{58}$$

With $1=\sum_{\sigma=\alpha,\beta,\gamma,\delta,\lambda,\rho} P_\sigma$, the combinations of (58) will give us

$$\begin{aligned}
&P_\alpha \underline{\Delta}_{(\rho)} P_\alpha = P_\alpha \underline{\Delta}_{(\rho)}, \qquad P_\beta \underline{\Delta}_{(\rho)} P_\alpha = P_\beta \underline{\Delta}_{(\rho)}, \qquad P_\gamma \underline{\Delta}_{(\rho)} P_\rho = P_\gamma \underline{\Delta}_{(\rho)},\\
&P_\delta \underline{\Delta}_{(\rho)} P_\lambda = P_\delta \underline{\Delta}_{(\rho)}, \qquad P_\lambda \underline{\Delta}_{(\rho)} P_\delta = P_\lambda \underline{\Delta}_{(\rho)}, \qquad P_\rho \underline{\Delta}_{(\rho)} P_\gamma = \underline{\Delta}_{(\rho)} P_\gamma,\\
&P_\rho \underline{\Delta}_{(\rho)} P_\beta = \underline{\Delta}_{(\rho)} P_\beta.
\end{aligned} \tag{59}$$

Using (55), (57) and (59) it is easily checked that

$$(P_\alpha + P_\beta)\,\underline{\Delta}_{(\delta)} = \underline{\Delta}_{(\delta)} P_\delta, \qquad \underline{\Delta}_{(\delta)}\,(P_\alpha + P_\beta) = P_\delta \underline{\Delta}_{(\delta)}, \qquad (P_\alpha + P_\gamma)\,\underline{\Delta}_{(\lambda)} = \underline{\Delta}_{(\lambda)} P_\lambda,$$

$$\underline{\Delta}_{(\lambda)}\left(P_\alpha+P_\gamma\right)=P_\lambda\underline{\Delta}_{(\lambda)},\qquad \left(P_\beta+P_\gamma\right)\underline{\Delta}_{(\rho)}=\underline{\Delta}_{(\rho)}P_\rho,\qquad \underline{\Delta}_{(\rho)}\left(P_\beta+P_\gamma\right)=P_\rho\underline{\Delta}_{(\rho)},$$
$$P_\gamma\underline{\Delta}_{(\delta)}=\underline{\Delta}_{(\delta)}P_\gamma,\qquad P_\lambda\underline{\Delta}_{(\delta)}=\underline{\Delta}_{(\delta)}P_\rho,\qquad P_\rho\underline{\Delta}_{(\delta)}=\underline{\Delta}_{(\delta)}P_\lambda,$$
$$P_\beta\underline{\Delta}_{(\lambda)}=\underline{\Delta}_{(\lambda)}P_\beta,\qquad P_\delta\underline{\Delta}_{(\lambda)}=\underline{\Delta}_{(\lambda)}P_\rho,\qquad P_\rho\underline{\Delta}_{(\lambda)}=\underline{\Delta}_{(\lambda)}P_\delta,$$
$$P_\alpha\underline{\Delta}_{(\rho)}=\underline{\Delta}_{(\rho)}P_\alpha,\qquad P_\delta\underline{\Delta}_{(\rho)}=\underline{\Delta}_{(\rho)}P_\lambda,\qquad P_\lambda\underline{\Delta}_{(\rho)}=\underline{\Delta}_{(\rho)}P_\delta.$$

9 Appendix E: 28-beins $u^{IJ}{}_{KL}$ and v^{IJKL} for Each Invariant Sector

The 28-beins $u_{IJ}{}^{KL}$ and v_{IJKL} fields can be obtained by exponentiating the vacuum expectation values ϕ_{IJKL}. The nonzero components of those have the following seven 4×4 block diagonal matrices respectively

$$\begin{aligned} u_{IJ}{}^{KL} &= \mathrm{diag}\left(u_1,u_2,u_3,u_4,u_5,u_6,u_7\right), \\ v_{IJKL} &= \mathrm{diag}\left(v_1,v_2,v_3,v_4,v_5,v_6,v_7\right). \end{aligned}$$

Each hermitian submatrix is a 4×4 matrix and we denote antisymmetric indices explicitly for convenience. For simplicity, we make an empty space corresponding to lower triangle elements. We also denote $\varepsilon_+=1$(self-dual), $\varepsilon_-=i$(anti-self-dual) and $\eta=1$ corresponding to self-dual case or -1 anti-self dual case. We write down here each hermitian matrices.

- $SO(7)^\pm\times SO(1)^\pm$ Invariant Sectors:

$$u_1=\begin{pmatrix} & [12] & [34] & [56] & [78] \\ [12] & A & \eta B & \eta B & \eta B \\ [34] & & A & B & B \\ [56] & & & A & B \\ [78] & & & & A \end{pmatrix}, u_2=\begin{pmatrix} & [13] & [24] & [57] & [68] \\ [13] & A & -\eta B & -\eta B & \eta B \\ [24] & & A & B & -B \\ [57] & & & A & -B \\ [68] & & & & A \end{pmatrix},$$
$$u_3=\begin{pmatrix} & [14] & [23] & [58] & [67] \\ [14] & A & \eta B & \eta B & \eta B \\ [23] & & A & B & B \\ [58] & & & A & B \\ [67] & & & & A \end{pmatrix}, u_4=\begin{pmatrix} & [15] & [26] & [37] & [48] \\ [15] & A & -\eta B & \eta B & -\eta B \\ [26] & & A & -B & B \\ [37] & & & A & -B \\ [48] & & & & A \end{pmatrix},$$
$$u_5=\begin{pmatrix} & [16] & [25] & [38] & [47] \\ [16] & A & \eta B & -\eta B & -\eta B \\ [25] & & A & -B & -B \\ [38] & & & A & B \\ [47] & & & & A \end{pmatrix}, u_6=\begin{pmatrix} & [17] & [28] & [35] & [46] \\ [17] & A & -\eta B & -\eta B & \eta B \\ [28] & & A & B & -B \\ [35] & & & A & -B \\ [46] & & & & A \end{pmatrix},$$

$$u_7 = \left(\begin{array}{ccccc} & [18] & [27] & [36] & [45] \\ [18] & A & \eta B & \eta B & \eta B \\ [27] & & A & B & B \\ [36] & & & A & B \\ [45] & & & & A \end{array}\right), v_1 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [12] & [34] & [56] & [78] \\ [12] & F & \eta G & \eta G & \eta G \\ [34] & & F & G & G \\ [56] & & & F & G \\ [78] & & & & F \end{array}\right),$$

$$v_2 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [13] & [24] & [57] & [68] \\ [13] & F & -\eta G & -\eta G & \eta G \\ [24] & & F & G & -G \\ [57] & & & F & -G \\ [68] & & & & F \end{array}\right), v_3 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [14] & [23] & [58] & [67] \\ [14] & F & \eta G & \eta G & \eta G \\ [23] & & F & G & G \\ [58] & & & F & G \\ [67] & & & & F \end{array}\right),$$

$$v_4 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [15] & [26] & [37] & [48] \\ [15] & F & -\eta G & \eta G & -\eta G \\ [26] & & F & -G & G \\ [37] & & & F & -G \\ [48] & & & & F \end{array}\right), v_5 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [16] & [25] & [38] & [47] \\ [16] & F & \eta G & -\eta G & -\eta G \\ [25] & & F & -G & -G \\ [38] & & & F & G \\ [47] & & & G & F \end{array}\right),$$

$$v_6 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [17] & [28] & [35] & [46] \\ [17] & F & -\eta G & -\eta G & \eta G \\ [28] & & F & G & -G \\ [35] & & & F & -G \\ [46] & & & & F \end{array}\right), v_7 = -\varepsilon_\pm \left(\begin{array}{ccccc} & [18] & [27] & [36] & [45] \\ [18] & F & \eta G & \eta G & \eta G \\ [27] & & F & G & G \\ [36] & & & F & G \\ [45] & & & & F \end{array}\right), \tag{60}$$

where

$$\begin{aligned} A &= \cosh^3 s, \qquad B = \cosh s \sinh^2 s, \\ F &= \sinh^3 s, \qquad G = \sinh s \cosh^2 s. \end{aligned}$$

From now on, we do not include the index pairs into the 4×4 matrices u_i and v_i, for simplicity. For example, when we write $u_2 = u_3$ below, this implies that although the indices they possess are different, the corresponding matrix elements are identical.

- $SO(6)^\pm \times SO(2)^\pm$ Invariant Sectors:

$$u_1 = \begin{pmatrix} A & \eta B & \eta B & \eta B \\ & A & B & B \\ & & A & B \\ & & & A \end{pmatrix}, u_2 = C\mathbf{1}_{4\times 4} = u_3 = u_4 = u_5 = u_6 = u_7,$$

$$v_1 = -\varepsilon_\pm \begin{pmatrix} F & \eta G & -\eta G & \eta G \\ & F & G & G \\ & & F & -G \\ & & & F \end{pmatrix},$$

$$v_2 = \varepsilon_\pm \begin{pmatrix} 0 & \eta H & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & H \\ & & & 0 \end{pmatrix} = -v_3 = v_4 = -v_5 = v_6 = -v_7$$

where

$$A = \cosh^3 s, \qquad B = \cosh s \sinh^2 s, \qquad C = \cosh s,$$
$$F = \sinh^3 s, \qquad G = \sinh s \cosh^2 s, \qquad H = \sinh s.$$

- $SO(5)^\pm \times SO(3)^\pm$ Invariant Sectors:

$$u_1 = \begin{pmatrix} A & \eta B & \eta C & \eta C \\ & A & C & C \\ & & A & B \\ & & & A \end{pmatrix}, u_2 = \begin{pmatrix} A & -\eta B & -\eta C & \eta C \\ & A & C & -C \\ & & A & -B \\ & & & A \end{pmatrix},$$

$$u_3 = \begin{pmatrix} A & \eta B & -\eta C & -\eta C \\ & A & -C & -C \\ & & A & B \\ & & & A \end{pmatrix}, u_4 = \begin{pmatrix} D & \eta E & -\eta E & -\eta E \\ & D & -E & -E \\ & & D & E \\ & & & D \end{pmatrix},$$

$$u_5 = \begin{pmatrix} D & -\eta E & \eta E & -\eta E \\ & D & -E & E \\ & & D & -E \\ & & & D \end{pmatrix}, u_6 = \begin{pmatrix} D & \eta E & \eta E & \eta E \\ & D & E & E \\ & & D & E \\ & & & D \end{pmatrix},$$

$$u_7 = \begin{pmatrix} D & -\eta E & -\eta E & \eta E \\ & D & E & -E \\ & & D & -E \\ & & & D \end{pmatrix}, v_1 = -\varepsilon_\pm \begin{pmatrix} F & \eta G & -\eta H & -\eta H \\ & F & -H & -H \\ & & F & G \\ & & & F \end{pmatrix},$$

$$v_2 = -\varepsilon_\pm \begin{pmatrix} F & -\eta G & \eta H & -\eta H \\ & F & -H & H \\ & & F & -G \\ & & & F \end{pmatrix}, v_3 = -\varepsilon_\pm \begin{pmatrix} F & \eta G & \eta H & \eta H \\ & F & H & H \\ & & F & G \\ & & & F \end{pmatrix},$$

$$v_4 = \varepsilon_\pm \begin{pmatrix} I & \eta J & -\eta J & -\eta J \\ & I & -J & -J \\ & & I & J \\ & & & I \end{pmatrix}, v_5 = \varepsilon_\pm \begin{pmatrix} I & -\eta J & \eta J & -\eta J \\ & I & -J & J \\ & & I & -J \\ & & & I \end{pmatrix},$$

$$v_6 = \varepsilon_\pm \begin{pmatrix} I & \eta J & \eta J & \eta J \\ & I & J & J \\ & & I & J \\ & & & I \end{pmatrix}, v_7 = \varepsilon_\pm \begin{pmatrix} I & -\eta J & -\eta J & \eta J \\ & I & J & -J \\ & & I & -J \\ & & & I \end{pmatrix},$$

where

$$A = \left(-1 + 2\cosh\left(\frac{2s}{3}\right)\right)\cosh^3\left(\frac{s}{3}\right), \qquad B = \cosh(s)\sinh^2\left(\frac{s}{3}\right),$$

$$
\begin{aligned}
C &= \left(2\cosh\left(\frac{s}{3}\right)+\cosh(s)\right)\sinh^2\left(\frac{s}{3}\right), \qquad D=\cosh^3\left(\frac{s}{3}\right),\\
E &= \cosh\left(\frac{s}{3}\right)\sinh^2\left(\frac{s}{3}\right), \qquad F=\left(1+2\cosh\left(\frac{2s}{3}\right)\right)\sinh^3\left(\frac{s}{3}\right),\\
G &= \cosh^2\left(\frac{s}{3}\right)\sinh(s), \qquad H=\frac{1}{4}\left(\sinh\left(\frac{s}{3}\right)-\sinh\left(\frac{5s}{3}\right)\right),\\
I &= \sinh^3\left(\frac{s}{3}\right), \qquad J=\frac{1}{4}\left(\sinh\left(\frac{s}{3}\right)+\sinh(s)\right).
\end{aligned}
$$

- $SO(4)^{\pm}\times SO(4)^{\pm}$ Invariant Sectors:

$$
u_1 = A\mathbf{1}_{4\times4} = u_2 = u_3, \qquad u_4=u_5=u_6=u_7=\mathbf{1}_{4\times4},
$$

$$
v_1 = -\varepsilon_\pm \begin{pmatrix} 0 & \eta B & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & B \\ & & & 0 \end{pmatrix} = -v_2 = v_3, \qquad v_4=v_5=v_6=v_7=0,
$$

where

$$
A=\cosh s, \qquad B=\sinh s.
$$

- $SO(3)^{\pm}\times SO(5)^{\pm}$ Invariant Sectors:

$$
u_1 = \begin{pmatrix} A & -\eta B & \eta C & -\eta C \\ & A & -C & C \\ & & A & -B \\ & & & A \end{pmatrix}, u_2 = \begin{pmatrix} A & \eta B & -\eta C & -\eta C \\ & A & -C & -C \\ & & A & B \\ & & & A \end{pmatrix},
$$

$$
u_3 = \begin{pmatrix} A & -\eta B & -\eta C & \eta C \\ & A & C & -C \\ & & A & -B \\ & & & A \end{pmatrix}, u_4 = \begin{pmatrix} D & \eta E & -\eta E & -\eta E \\ & D & -E & -E \\ & & D & E \\ & & & D \end{pmatrix},
$$

$$
u_5 = \begin{pmatrix} D & -\eta E & -\eta E & \eta E \\ & D & E & -E \\ & & D & -E \\ & & & D \end{pmatrix}, u_6 = \begin{pmatrix} D & -\eta E & \eta E & -\eta E \\ & D & -E & E \\ & & D & -E \\ & & & D \end{pmatrix},
$$

$$
u_7 = \begin{pmatrix} D & \eta E & \eta E & \eta E \\ & D & E & E \\ & & D & E \\ & & & D \end{pmatrix}, v_1 = \varepsilon_\pm \begin{pmatrix} F & -\eta G & \eta H & -\eta H \\ & F & -H & H \\ & & F & -G \\ & & & F \end{pmatrix},
$$

$$
v_2 = \varepsilon_\pm \begin{pmatrix} F & \eta G & -\eta H & -\eta H \\ & F & -H & -H \\ & & F & G \\ & & & F \end{pmatrix}, v_3 = \varepsilon_\pm \begin{pmatrix} F & -\eta G & -\eta H & \eta H \\ & F & H & -H \\ & & F & -G \\ & & & F \end{pmatrix},
$$

$$v_4=-\varepsilon_\pm\begin{pmatrix} I & \eta J & -\eta J & -\eta J\\ & I & -J & -J\\ & & I & J\\ & & & I\end{pmatrix},\ v_5=-\varepsilon_\pm\begin{pmatrix} I & -\eta J & -\eta J & \eta J\\ & I & J & -J\\ & & I & -J\\ & & & I\end{pmatrix},$$

$$v_6=-\varepsilon_\pm\begin{pmatrix} I & -\eta J & \eta J & -\eta J\\ & I & -J & J\\ & & I & -J\\ & & & I\end{pmatrix},\ v_7=-\varepsilon_\pm\begin{pmatrix} I & \eta J & \eta J & \eta J\\ & I & J & J\\ & & I & J\\ & & & I\end{pmatrix},$$

where

$$\begin{aligned}
A &= \left(-1+2\cosh\left(\frac{2s}{5}\right)\right)\cosh^3\left(\frac{s}{5}\right), \qquad B=\cosh\left(\frac{3s}{5}\right)\sinh^2\left(\frac{s}{5}\right),\\
C &= \frac{1}{4}\left(\cosh(s)-\cosh\left(\frac{s}{5}\right)\right), \qquad D=\cosh^3\left(\frac{s}{5}\right),\\
E &= \cosh\left(\frac{s}{5}\right)\sinh^2\left(\frac{s}{5}\right), \qquad F=\left(1+2\cosh\left(\frac{2s}{5}\right)\right)\sinh^3\left(\frac{s}{5}\right),\\
G &= \cosh^2\left(\frac{s}{5}\right)\sinh\left(\frac{3s}{5}\right), \qquad H=\frac{1}{4}\left(\sinh(s)-\sinh\left(\frac{s}{5}\right)\right),\\
I &= \sinh^3\left(\frac{s}{5}\right), \qquad J=\cosh^2\left(\frac{s}{5}\right)\sinh\left(\frac{s}{5}\right).
\end{aligned}$$

All these functions of s can be obtained from those in $SO(5)^\pm\times SO(3)^\pm$ by replacing s with $3s/5$ and using the properties of hyperbolic functions. For example, each C that seems to look different is the same by a simple change of variable.

- $SO(2)^\pm\times SO(6)^\pm$ Invariant Sectors:

$$u_1=\begin{pmatrix} A & -\eta B & \eta B & -\eta B\\ & A & -B & B\\ & & A & -B\\ & & & A\end{pmatrix},\ u_2=C\mathbf{1}_{4\times4}=u_3=u_4=u_5=u_6=u_7,$$

$$v_1=\varepsilon_\pm\begin{pmatrix} F & -\eta G & \eta G & -\eta G\\ & F & -G & G\\ & & F & -G\\ & & & F\end{pmatrix},$$

$$v_2=\varepsilon_\pm\begin{pmatrix} 0 & \eta H & 0 & 0\\ & 0 & 0 & 0\\ & & 0 & H\\ & & & 0\end{pmatrix}=-v_3=-v_4=v_5=v_6=-v_7,$$

where

$$\begin{aligned}
A &= \cosh^3\left(\frac{s}{3}\right), \qquad B=\cosh\left(\frac{s}{3}\right)\sinh^2\left(\frac{s}{3}\right), \qquad C=\cosh\left(\frac{s}{3}\right),\\
F &= \sinh^3\left(\frac{s}{3}\right), \qquad G=\sinh\left(\frac{s}{3}\right)\cosh^2\left(\frac{s}{3}\right), \qquad H=\sinh\left(\frac{s}{3}\right).
\end{aligned}$$

All these functions of s can be obtained from those in $SO(6)^{\pm} \times SO(2)^{\pm}$ by replacing s with $s/3$.

- $SO(1)^{\pm} \times SO(7)^{\pm}$ Invariant Sectors:

$$u_1 = \begin{pmatrix} A & -\eta B & \eta B & -\eta B \\ & A & -B & B \\ & & A & -B \\ & & & A \end{pmatrix} = u_3 = u_4, u_2 = \begin{pmatrix} A & \eta B & -\eta B & -\eta B \\ & A & -B & -B \\ & & A & B \\ & & & A \end{pmatrix} = u_6,$$

$$u_5 = \begin{pmatrix} A & \eta B & \eta B & \eta B \\ & A & B & B \\ & & A & B \\ & & & A \end{pmatrix}, u_7 = \begin{pmatrix} A & -\eta B & -\eta B & \eta B \\ & A & B & -B \\ & & A & -B \\ & & & A \end{pmatrix},$$

$$v_1 = \varepsilon_{\pm} \begin{pmatrix} F & -\eta G & \eta G & -\eta G \\ & F & -G & G \\ & & F & -G \\ & & & F \end{pmatrix} = v_3 = v_4, v_2 = \varepsilon_{\pm} \begin{pmatrix} F & \eta G & -\eta G & -\eta G \\ & F & -G & -G \\ & & F & G \\ & & & F \end{pmatrix} = v_6,$$

$$v_5 = \varepsilon_{\pm} \begin{pmatrix} F & \eta G & \eta G & \eta G \\ & F & G & G \\ & & F & G \\ & & & F \end{pmatrix}, v_7 = \varepsilon_{\pm} \begin{pmatrix} F & -\eta G & -\eta G & \eta G \\ & F & G & -G \\ & & F & -G \\ & & & F \end{pmatrix},$$

where

$$\begin{aligned} A &= \cosh^3\left(\frac{s}{7}\right), \qquad B = \cosh\left(\frac{s}{7}\right)\sinh^2\left(\frac{s}{7}\right), \\ F &= \sinh^3\left(\frac{s}{7}\right), \qquad G = \sinh\left(\frac{s}{7}\right)\cosh^2\left(\frac{s}{7}\right). \end{aligned}$$

All these functions of s can be obtained from those in $SO(7)^{\pm} \times SO(1)^{\pm}$ by replacing s with $s/7$.

10 Appendix F: Projectors of $SO(p)^+ \times SO(q)^+$ Sectors in 28×28 Matrices

The projectors $P^{IJKL}_{\sigma,p,q}(\sigma = \alpha, \beta, \gamma)$ of $SO(p)^+ \times SO(q)^+$-invariant sectors can be obtained explicitly. We list $P^{IJKL}_{\alpha,p,q}$ and $P^{IJKL}_{\beta,p,q}$ only because $P^{IJKL}_{\gamma,p,q}$ can be obtained from those:$P^{IJKL}_{\gamma,p,q} = 1 - P^{IJKL}_{\alpha,p,q} - P^{IJKL}_{\beta,p,q}$.

$$\begin{aligned} P^{IJKL}_{\alpha,7,1} &= \mathrm{diag}\left(F_1, F_2, F_1, F_3, F_4, F_2, F_1\right), \\ P^{IJKL}_{\beta,7,1} &= 0, \\ P^{IJKL}_{\alpha,6,2} &= \mathrm{diag}\left(F_1, F_9, F_{10}, F_9, F_{10}, F_9, F_{10}\right), \end{aligned}$$

$$
\begin{aligned}
P^{IJKL}_{\beta,6,2} &= \mathrm{diag}\,(F_5, 0, 0, 0, 0, 0, 0)\,, \\
P^{IJKL}_{\alpha,5,3} &= \mathrm{diag}\,(F_{10}, F_9, F_{10}, F_8, F_7, F_5, F_6)\,, \\
P^{IJKL}_{\beta,5,3} &= \mathrm{diag}\,(F_5, F_6, F_8, 0, 0, 0, 0)\,, \\
P^{IJKL}_{\alpha,4,4} &= \mathrm{diag}\,(F_{10}, F_9, F_{10}, 0, 0, 0, 0)\,, \\
P^{IJKL}_{\beta,4,4} &= \mathrm{diag}\,(F_9, F_{10}, F_9, 0, 0, 0, 0)\,, \\
P^{IJKL}_{\alpha,3,5} &= \mathrm{diag}\,(F_7, F_8, F_6, 0, 0, 0, 0)\,, \\
P^{IJKL}_{\beta,3,5} &= \mathrm{diag}\,(F_9, F_{10}, F_9, F_8, F_6, F_7, F_5)\,, \\
P^{IJKL}_{\alpha,2,6} &= \mathrm{diag}\,(F_7, 0, 0, 0, 0, 0, 0)\,, \\
P^{IJKL}_{\beta,2,6} &= \mathrm{diag}\,(F_3, F_{10}, F_9, F_9, F_{10}, F_{10}, F_9)\,, \\
P^{IJKL}_{\alpha,1,7} &= 0, \\
P^{IJKL}_{\beta,1,7} &= \mathrm{diag}\,(F_3, F_4, F_3, F_3, F_1, F_4, F_2)\,,
\end{aligned}
$$

where the 4×4 block diagonal matrices F_i's are

$$
F_1 = \frac{1}{8}\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}, \qquad
F_2 = \frac{1}{8}\begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{pmatrix},
$$

$$
F_3 = \frac{1}{8}\begin{pmatrix} 3 & 1 & -1 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{pmatrix}, \qquad
F_4 = \frac{1}{8}\begin{pmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{pmatrix},
$$

$$
F_5 = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad
F_6 = \frac{1}{8}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},
$$

$$
F_7 = \frac{1}{8}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \qquad
F_8 = \frac{1}{8}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix},
$$

$$
F_9 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad
F_{10} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.
$$

11 Appendix G: Kinetic Terms, Superpotential and Potential in $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ Sectors

We list here 1) the kinetic terms in terms of original variables, m and n, 2) new variables, $\widetilde{m}$ and $\tilde{n}$ in order to have usual canonical expression of kinetic terms, 3) superpotential in terms of new fields, and 4) scalar potential in $SO(p)^+ \times SO(q)^+ \times SO(r)^+$ sectors. In all cases, the scalar potential can be expressed in terms of superpotential as (42). In this revised version, we list only six cases due to space limitations and refer to the original version in the hep-th archive for remaining cases.

- $SO(1,7)^+ \to SO(2,6)^+ \to CSO(1,1,6)^+$:

$$\begin{aligned}
K_{1,1,6}(m,n) &= -\frac{1}{3}\partial^\mu m\partial_\mu m - \frac{2}{7}\partial^\mu m\partial_\mu n - \frac{1}{7}\partial^\mu n\partial_\mu n, \\
m &= -\frac{3\sqrt{2}}{4}\widetilde{m} - \frac{\sqrt{6}}{2}\widetilde{n}, \\
n &= \frac{7\sqrt{2}}{4}\widetilde{m}, \\
W_{1,1,6}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{2\sqrt{2}\widetilde{m}+\sqrt{6}\widetilde{n}}{2}}\left(e^{2\sqrt{2}\widetilde{m}} + \xi + 6e^{\sqrt{2}\widetilde{m}+\frac{2\sqrt{6}}{3}\widetilde{n}}\xi\zeta\right), \\
V_{1,1,6}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-2\sqrt{2}\widetilde{m}-\sqrt{6}\widetilde{n}}\left(e^{4\sqrt{2}\widetilde{m}} - 2e^{2\sqrt{2}\widetilde{n}}\xi - 12e^{3\sqrt{2}\widetilde{m}+\frac{2\sqrt{6}}{3}\widetilde{n}}\xi\zeta + \xi^2\right. \\
&\quad \left. -12e^{\sqrt{2}\widetilde{m}+\frac{2\sqrt{6}}{3}\widetilde{n}}\xi^2\zeta - 24e^{2\sqrt{2}\widetilde{m}+\frac{4\sqrt{6}}{3}\widetilde{n}}\zeta^2\xi^2\right).
\end{aligned}$$

There exists a $SO(7)^+$-invariant critical point of $SO(8)$ theory for $\xi = 1$ and $\zeta = 1$ and a $SO(2)^+ \times SO(6)^+$-invariant critical point for $\xi = 1$ and $\zeta = 0$.

- $SO(1,7)^+ \to SO(3,5)^+ \to CSO(1,2,5)^+$:

$$\begin{aligned}
K_{1,2,5}(m,n) &= -\frac{3}{5}\partial^\mu m\partial_\mu m - \frac{2}{7}\partial^\mu m\partial_\mu n - \frac{1}{7}\partial^\mu n\partial_\mu n, \\
m &= -\frac{5\sqrt{6}}{24}\widetilde{m} - \frac{5\sqrt{6}}{6}\widetilde{n}, \\
n &= \frac{7\sqrt{6}}{8}\widetilde{m}, \\
W_{1,2,5}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{2\sqrt{6}\widetilde{m}+\sqrt{30}\widetilde{n}}{6}}\left(e^{\sqrt{6}\widetilde{m}} + 2\xi + 5e^{\frac{\sqrt{6}}{3}\widetilde{m}+\frac{4\sqrt{30}}{15}\widetilde{n}}\xi\zeta\right), \\
V_{1,2,5}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{\sqrt{6}}{3}\widetilde{m}-\frac{\sqrt{30}}{3}\widetilde{n}}\left(e^{\frac{5\sqrt{6}}{3}\widetilde{m}} - 4e^{\frac{2\sqrt{6}}{3}\widetilde{n}}\xi - 10e^{\sqrt{6}\widetilde{m}+\frac{4\sqrt{30}}{15}\widetilde{n}}\xi\zeta - 20e^{\frac{4\sqrt{30}}{15}\widetilde{n}}\xi^2\zeta\right. \\
&\quad \left. -15e^{\frac{\sqrt{6}}{3}\widetilde{m}+\frac{8\sqrt{30}}{15}\widetilde{n}}\xi^2\zeta^2\right).
\end{aligned}$$

There exists an $SO(3)^+ \times SO(5)^+$-invariant critical point for $\xi = 1$ and $\zeta = -1$.

- $SO(1,7)^+ \to SO(4,4)^+ \to CSO(1,3,4)^+$:

$$\begin{aligned}
K_{1,3,4}(m,n) &= -\partial^\mu m\partial_\mu m - \frac{2}{7}\partial^\mu m\partial_\mu n - \frac{1}{7}\partial^\mu n\partial_\mu n, \\
m &= -\frac{\sqrt{3}}{6}\widetilde{m} - \frac{\sqrt{2}}{2}\widetilde{n}, \\
n &= \frac{7\sqrt{3}}{6}\widetilde{m},
\end{aligned}$$

$$
\begin{aligned}
W_{1,3,4}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{2\sqrt{3}\widetilde{m}+3\sqrt{2}\widetilde{n}}{6}}\left(e^{\frac{4\sqrt{3}}{3}\widetilde{m}}+3\xi+4e^{\frac{\sqrt{3}}{3}\widetilde{m}+\sqrt{2}\widetilde{n}}\xi\zeta\right),\\
V_{1,3,4}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}\left(e^{2\sqrt{3}\widetilde{m}-\sqrt{2}\widetilde{n}}-6e^{\frac{2\sqrt{3}}{3}\widetilde{m}-\sqrt{2}\widetilde{n}}\xi-8e^{\sqrt{3}\widetilde{m}}\xi\zeta-e^{-\frac{2\sqrt{3}}{3}\widetilde{m}-\sqrt{2}\widetilde{n}}\xi^2\right.\\
&\left.-24e^{-\frac{\sqrt{3}}{3}\widetilde{m}}\xi^2\zeta-8e^{\sqrt{2}\widetilde{n}}\xi^2\zeta^2\right).
\end{aligned}
$$

There exists an $SO(4)^+\times SO(4)^+$-invariant critical point for $\xi=1$ and $\zeta=-1$, and an $SO(5)^+\times SO(3)^+$-invariant critical point for $\xi=-1$ and $\zeta=-1$.

- $SO(1,7)^+\to SO(5,3)^+\to CSO(1,4,3)^+$:

$$
\begin{aligned}
K_{1,4,3}(m,n) &= -\frac{5}{3}\partial^\mu m\partial_\mu m-\frac{2}{7}\partial^\mu m\partial_\mu n-\frac{1}{7}\partial^\mu n\partial_\mu n,\\
m &= -\frac{3\sqrt{5}}{40}\widetilde{m}-\frac{\sqrt{30}}{10}\widetilde{n},\\
n &= \frac{7\sqrt{5}}{8}\widetilde{m},\\
W_{1,4,3}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{2\sqrt{5}\widetilde{m}+\sqrt{30}\widetilde{n}}{10}}\left(e^{\sqrt{5}\widetilde{m}}+4\xi+3e^{\frac{\sqrt{5}}{5}\widetilde{m}+\frac{4\sqrt{30}}{15}\widetilde{n}}\xi\zeta\right),\\
V_{1,4,3}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{2\sqrt{5}\widetilde{m}+\sqrt{30}\widetilde{n}}{5}}\left(e^{2\sqrt{5}\widetilde{m}}-8e^{\sqrt{5}\widetilde{m}}\xi-6e^{\frac{6\sqrt{5}}{5}\widetilde{m}+\frac{4\sqrt{30}}{15}\widetilde{n}}\xi\zeta-8\xi^2\right.\\
&\left.-24e^{-\frac{\sqrt{5}}{5}\widetilde{m}+\frac{4\sqrt{30}}{15}\widetilde{n}}\xi^2\zeta-3e^{\frac{2\sqrt{5}}{5}\widetilde{m}+\frac{8\sqrt{30}}{15}\widetilde{n}}\xi^2\zeta^2\right).
\end{aligned}
$$

There exists an $SO(5)^+\times SO(3)^+$-invariant critical point for $\xi=1$ and $\zeta=-1$, and a $SO(4)^+\times SO(4)^+$-invariant critical point for $\xi=-1$ and $\zeta=-1$.

- $SO(1,7)^+\to SO(6,2)^+\to CSO(1,5,2)^+$:

$$
\begin{aligned}
K_{1,5,2}(m,n) &= -3\partial^\mu m\partial_\mu m-\frac{2}{7}\partial^\mu m\partial_\mu n-\frac{1}{7}\partial^\mu n\partial_\mu n,\\
m &= -\frac{\sqrt{30}}{60}\widetilde{m}-\frac{\sqrt{6}}{6}\widetilde{n},\\
n &= \frac{7\sqrt{30}}{20}\widetilde{m},\\
W_{1,5,2}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{\sqrt{30}}{15}\widetilde{m}-\frac{\sqrt{6}}{6}\widetilde{n}}\left(e^{\frac{2\sqrt{30}}{5}\widetilde{m}}+5\xi+2e^{\frac{\sqrt{30}}{15}\widetilde{m}+\frac{2\sqrt{6}}{3}\widetilde{n}}\xi\zeta\right),\\
V_{1,5,2}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= -\frac{1}{8}e^{-\frac{2\sqrt{30}}{15}\widetilde{m}-\frac{\sqrt{6}}{3}\widetilde{n}}\left(-e^{4\frac{\sqrt{30}}{5}\widetilde{m}}+10e^{\frac{2\sqrt{30}}{5}\widetilde{m}}\xi+4e^{\frac{7\sqrt{30}}{15}\widetilde{m}+\frac{2\sqrt{6}}{3}\widetilde{n}}\xi\zeta+15\xi^2\right.\\
&\left.+20e^{\frac{\sqrt{30}}{15}\widetilde{m}+\frac{2\sqrt{6}}{3}\widetilde{n}}\xi^2\zeta\right).
\end{aligned}
$$

There exists an $SO(3)^+\times SO(5)^+$-invariant critical point for $\xi=-1$ and $\zeta=-1$.

- $SO(1,7)^+\to SO(7,1)^+\to CSO(1,6,1)^+$:

$$
K_{1,6,1}(m,n) = -7\partial^\mu m\partial_\mu m-\frac{2}{7}\partial^\mu m\partial_\mu n-\frac{1}{7}\partial^\mu n\partial_\mu n,
$$

$$
\begin{aligned}
m &= -\frac{\sqrt{42}}{168}\widetilde{m} - \frac{\sqrt{14}}{14}\widetilde{n}, \\
n &= \frac{7\sqrt{42}}{24}\widetilde{m}, \\
W_{1,6,1}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{\sqrt{42}}{21}\widetilde{m}-\frac{\sqrt{14}}{14}\widetilde{n}}\left(e^{\frac{\sqrt{42}}{3}\widetilde{m}} + 6\xi + e^{\frac{\sqrt{42}}{21}(\widetilde{m}+4\sqrt{3}\widetilde{n})}\xi\zeta\right), \\
V_{1,6,1}(\xi,\zeta;\widetilde{m},\widetilde{n}) &= \frac{1}{8}e^{-\frac{2\sqrt{42}}{21}\widetilde{m}-\frac{\sqrt{14}}{7}\widetilde{n}}\left(e^{2\frac{\sqrt{42}}{3}\widetilde{m}} - 12e^{\frac{\sqrt{42}}{3}\widetilde{m}}\xi - 2e^{\frac{4\sqrt{42}}{21}(2\widetilde{m}+\sqrt{3}\widetilde{n})}\xi\zeta - 24\xi^2\right. \\
&\left. -12e^{\frac{\sqrt{42}}{21}(\widetilde{m}+4\sqrt{3}\widetilde{n})}\xi^2\zeta + e^{\frac{2\sqrt{42}}{21}(\widetilde{m}+4\sqrt{3}\widetilde{n})}\xi^2\zeta^2\right).
\end{aligned}
$$

There are no critical points, in this case.

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