

130 Final Exam Cheat Sheet

Heating Problem

$$\frac{dT}{dt} = -k(T - T_o)$$

T_o = outside temperature

Mixing Problem

$$\frac{dA}{dt} = c_1 r_1 - \frac{A}{V} r_2$$

$$V = V_0 + (r_1 - r_2)t$$

c_1 , solution mixture in
 r_1 , in rate
 r_2 , out rate

Inner Product Spaces

- $\langle v, v \rangle \geq 0$ Furthermore, $\langle v, v \rangle = 0 \leftrightarrow v = 0$
 - $\langle v, u \rangle = \langle u, v \rangle$
 - $\langle ku, v \rangle = k \langle u, v \rangle$
 - $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $$\|v\| = \sqrt{\langle v, v \rangle}$$
- $$\cos^{-1}\left(\frac{\langle v, u \rangle}{\|v\| \|u\|}\right)$$

Gram-Schmidt

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$\vdots$$

$$v_n = x_n - \sum_{k=1}^{n-1} \frac{\langle x_n, v_k \rangle}{\|v_k\|^2} v_k$$

Variation of Parameters

$$F(x) = y'' + y'$$

$y_h = b_1 y_1(x) + b_2 y_2(x)$, y_1, y_2 are L.I.

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$u_1 = \int^t -\frac{y_2 F(t) dt}{w[y_1, y_2](t)}$$

$$u_2 = \int^t \frac{y_1 F(t) dt}{w[y_1, y_2](t)}$$

$$y = y_h + y_p$$

ODEs

<i>1st Order Linear</i>	Use integrating factor, $I = e^{\int P(x) dx}$
<i>Separable:</i>	$\int P(y) dy / dx = \int Q(x)$
<i>Homogeneous:</i>	$dy/dx = f(x, y) = f(xt, yt)$ sub $y = xV$ solve, then sub $V = y/x$
<i>Exact:</i>	If $M(x, y) + N(x, y) dy/dx = 0$ and $M_y = N_x$ i.e. $\langle M, N \rangle = \nabla F$ then $\int_x M + \int_y N = F$
<i>Order Reduction</i>	Let $v = dy/dx$ then check other types If purely a function of y , $\frac{dv}{dx} = v \frac{dv}{dy}$
<i>Variation of Parameters:</i>	When $y'' + a_1 y' + a_2 y = F(x)$ F contains $\ln x$, $\sec x$, $\tan x$, \div
<i>Bernoulli</i>	$y' + P(x)y = Q(x)y^n$ $\div y^n$ $y^{-n} y' + P(x)y^{1-n} = Q(x)$ Let $U(x) = y^{1-n}(x)$ $\frac{dU}{dx} = (1-n)y^{-n} \frac{dy}{dx}$ $\frac{1}{1-n} \frac{dU}{dx} + P(x)U(x) = Q(x)$ solve as a 1st order
<i>Cauchy-Euler</i>	$x^n y^n + a_1 x^{n-1} y^{n-1} + \dots + a_{n-1} y^{n-2} + a_n y = 0$ guess $y = x^r$
<i>3 Cases:</i>	
1) Distinct real roots	$y = ax^{r_1} + bx^{r_2}$
2) Repeated real roots	$y = Ax^r + y_2$ Guess $y_2 = x^r u(x)$ Solve for $u(x)$ and choose one ($A = 1, C = 0$)
3) Distinct complex roots	$y = B_1 x^a \cos(b \ln x) + B_2 x^a \sin(b \ln x)$

Series Solution

$$y'' + p(x)y' + q(x)y = 0$$

Useful when $p(x), q(x)$ not constant

Guess $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Systems

$$\vec{x}' = A\vec{x}$$

A is diagonalizable

$$\vec{x}(t) = a_1 e^{\lambda_1 t} \vec{v}_1 + \dots + a_n e^{\lambda_n t} \vec{v}_n$$

A is not diagonalizable

$$\vec{x}(t) = a_1 e^{\lambda_1 t} \vec{v}_1 + a_2 e^{\lambda t} (\vec{w} + t\vec{v})$$

where $(A - \lambda I)\vec{w} = \vec{v}$
 \vec{v} is an Eigenvector w/ value λ
 i.e. \vec{w} is a generalized Eigenvector

$$\vec{x}' = A\vec{x} + \vec{B}$$

Solve y_h
 $\vec{x}_1 = e^{\lambda_1 t} \vec{v}_1, \vec{x}_2 = e^{\lambda_2 t} \vec{v}_2$
 $\vec{X} = [\vec{x}_1, \vec{x}_2]$
 $\vec{X} \vec{u}' = \vec{B}$
 $y_p = \vec{X} \vec{u}$
 $y = y_h + y_p$

Matrix Exponentiation

$$A^n = S D^n S^{-1}$$

D is the diagonalization of A

Laplace Transforms

$$L[f](s) = \int_0^{\infty} e^{-sx} f(x) dx$$

$f(t) = t^n, n \geq 0$	$F(s) = \frac{n!}{s^{n+1}}, s > 0$
$f(t) = e^{at}, a \text{ constant}$	$F(s) = \frac{1}{s-a}, s > a$
$f(t) = \sin bt, b \text{ constant}$	$F(s) = \frac{b}{s^2 + b^2}, s > 0$
$f(t) = \cos bt, b \text{ constant}$	$F(s) = \frac{s}{s^2 + b^2}, s > 0$
$f(t) = t^{-1/2}$	$F(s) = \frac{\pi}{s^{1/2}}, s > 0$
$f(t) = \delta(t-a)$	$F(s) = e^{-as}$
f'	$L[f'] = sL[f] - f(0)$
f''	$L[f''] = s^2 L[f] - sf(0) - f'(0)$
$L[e^{at} f(t)]$	$L[f](s-a)$
$L[u_a(t) f(t-a)]$	$L[f] e^{-as}$

Gaussian Integral

$$\int_{-\infty}^{+\infty} e^{-1/2(\vec{x}^T A \vec{x})} = \frac{\sqrt{2\pi}^n}{\sqrt{\det A}}$$

Complex Numbers

Systems of equations

If $\vec{w}_1 = u(\vec{t}) + iv(\vec{t})$ is a solution, $\vec{x}_1 = u(\vec{t}), \vec{x}_2 = v(\vec{t})$ are solutions
i.e. $\vec{x}_h = c_1\vec{x}_1 + c_2\vec{x}_2$

Euler's Identity

$e^{ix} = \cos x + i \sin x$

Vector Spaces

- $v_1, v_2 \in V$
1. $v_1 + v_2 \in V$

2. $k \in \mathbb{F}, kv_1 \in V$

3. $v_1 + v_2 = v_2 + v_1$

4. $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

5. $\forall v \in V, 0 \in V \mid 0 + v_1 = v_1 + 0 = v_1$

6. $\forall v \in V, \exists -v \in V \mid v + (-v) = (-v) + v = 0$

7. $\forall v \in V, 1 \in \mathbb{F} \mid 1 * v = v$

8. $\forall v \in V, k, l \in \mathbb{F}, (kl)v = k(lv)$

9. $\forall k \in \mathbb{F}, k(v_1 + v_2) = kv_1 + kv_2$

10. $\forall v \in V, k, l \in \mathbb{F}, (k + l)v = kv + lv$