Assignment 5:

Waiting:

1. midnight - 1 am (60 mins)

original functions

PDF:

$$f_x(x) = \begin{cases} \frac{1}{60} & x \in [0,60] \\ 0 & x \notin [0,60] \end{cases}$$

$$f_y(y) = \begin{cases} \frac{1}{60} & x \in [0,60] \\ 0 & x \notin [0,60] \end{cases}$$

CDF:

$$F_x(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{60} & x \in [0,60] \\ 1 & x > 60 \end{cases}$$

$$F_y(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{60} & y \in [0,60] \\ 1 & y > 60 \end{cases}$$

Joint

CDF:

by $F_x(x), F_y(y)$ values we can built the joint cdf:

$$F_{xy}(x,y) = P(X<x, Y<y) =$$

$$= F_x(x) \cdot F_y(y) = \begin{cases} 0 & x<60, y<60 \\ \frac{xy}{3600} & x\in[0,60] \wedge y\in[0,60] \\ \frac{x}{60} & x\in[0,60] \wedge y>60 \\ \frac{y}{60} & y\in[0,60] \wedge x>60 \\ 1 & x>60 \wedge y>60 \end{cases}$$

PDF:

$$f_{xy}(x,y) = \frac{\partial^2 F_{xy}(x,y)}{\partial x \, \partial y} = \begin{cases} \frac{1}{3600} & x\in[0,60] \wedge y\in[0,60] \\ 0 & \text{otherwise} \end{cases}$$

2. We need to calculate:

$$P(|x-y|>10)=P(x+10<y)+P(y+10<x) =$$

$$\underset{\text{Symmetry}}{=} 2P(x+10<y)$$

$$P(x+10<y)=\int_{10}^{60}\left[\int_{0}^{y-10}\frac{1}{3600}dx\right]dy=\frac{1}{3600}\int_{10}^{60}\left[\int_{0}^{y-10}dx\right]dy =$$

$$=\frac{1}{3600}\int_{10}^{60}\left[y-10\right]dy=\frac{1}{3600}\left[\frac{y^2}{2}-10y\right]_{10}^{60}=$$

$$=\frac{1800-600-50+100}{3600}=\frac{1250}{3600}$$

The limitation of $x<y$ wasn't clear to me in the question, if we assume $x<y$ then the answer is: $P(x+10<y)=\frac{1250}{3600}=\frac{125}{360}$

Otherwise if $x<y$ or $y<x$ can both occur the answer is:

$$2P(x+10<y)=\frac{2500}{3600}=\frac{25}{36}$$

Scanned with CamScanner

3. reminder: $f_{xy}(x,y) = \begin{cases} \frac{1}{3600} & (x,y) \in [0,60] \times [0,60] \\ 0 & \text{otherwise} \end{cases}$

marginal PDFs:

X: $f_{x_m}(x) = \int_0^{60} f_{xy}(x,y)\,dy = \begin{cases} \frac{1}{60} & x \in [0,60] \\ 0 & \text{otherwise} \end{cases}$

Y: $f_{y_m}(y) = \int_0^{60} f_{xy}(x,y)\,dx = \begin{cases} \frac{1}{60} & y \in [0,60] \\ 0 & \text{otherwise} \end{cases}$

Check they are PDFs by integration:

X:

$$\int_0^{60} f_{x_m}(x)\,dx = \frac{x}{60}\Big|_0^{60} = \frac{60}{60} = 1 \quad \checkmark$$

y:

$$\int_0^{60} f_{y_m}(y)\,dy = \frac{y}{60}\Big|_0^{60} = \frac{60}{60} = 1 \quad \checkmark$$

4. $X \sim \text{Uniform}(0,60)$

$$\Rightarrow E\{X\} = \frac{1}{2}(a+b) = 30$$

$$Var(X) = \frac{1}{12}(b-a)^2 = 300$$

$Y \sim \text{Uniform}(0,60)$

$$\Rightarrow E\{Y\} = \frac{1}{2}(a+b) = 30$$

$$Var(Y) = \frac{1}{12}(b-a)^2 = 300$$

5. $$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{3600}}{\frac{1}{60}} = \begin{cases} \frac{1}{60} & (x \in [0,60]) \\ 0 & \text{otherwise} \end{cases}$$

It is indeed the PDF since X, Y are independent and we see that

$$f_{X|Y}(x|y) = f_X(x)$$

Diagnostics:

$$P(ill) = 0.01 \quad \rightarrow P(\neg ill) = 0.99$$

$$P(x \mid ill) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma}\right)^2} \underset{\substack{\mu_1 = 25 \\ \sigma^2 = 1}}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-25)^2}$$

$$P(x \mid \neg ill) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma}\right)^2} \underset{\substack{\mu_2 = 20 \\ \sigma^2 = 1}}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-20)^2}$$

$$P(x) = P(x \mid ill) \cdot P(ill) + P(x \mid \neg ill) \cdot P(\neg ill) =$$

$$= 0.01 \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x-25)^2} + 0.99 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(x-20)^2}$$

We need to find $P(ill \mid x)$:

$$P(ill \mid x) = \frac{P(x \mid ill) \cdot P(ill)}{P(x)} = \frac{0.01\, e^{-\frac{1}{2}(x-25)^2}}{0.01 e^{-\frac{1}{2}(x-25)^2} + 0.99\, e^{-\frac{1}{2}(x-20)^2}}$$

$$P(ill \mid x=20) = \frac{0.01\, e^{-\frac{25}{2}}}{0.01 e^{-\frac{25}{2}} + 0.99} = 3.76 \cdot 10^{-8}$$

$$P(ill \mid x=23) = \frac{0.01 \cdot e^{-2}}{0.01 \cdot e^{-2} + 0.99 \cdot e^{-\frac{9}{2}}} = 0.1095$$

$$P(ill \mid x=25) = \frac{0.01}{0.01 + 0.99 \cdot e^{-\frac{25}{2}}} = 0.9996$$

other questions are
in jupyter notebook!