A few problems to practice discrete probability

Ants. Three ants are sitting at the three corners of an equilateral triangle. Each ant randomly picks a direction and starts to move along the edge of the triangle. What is the probability that none of the ants collide? 0.25

Passports. Russian passport number can consist of any 10 digits. What is the probability that a randomly chosen Russian passport number starts with 4508 and contains 53? 5x10^-6

Coins. I have 9 fair coins and one phony coin with heads on both sides. I picked one coin out of these 10 at random, and flipped it 3 times. It landed with heads up all the 3 times.

- What is the probability that this coin is the phony one? 8/17 = 0.4706
- What is the probability that this coin will show heads after the next flip? 25/34 = 0.7353

Family planning. In a country where everyone wants a boy, each family continues having babies till they have a boy. After some time, what is the proportion of boys to girls in the country? (Assuming probability of having a boy or a girl is the same). 1/ 1 (fifty - fifty)

Hitchhiker. The probability of at least one car passing a certain road intersection in a 20-minute window is 0.9. What is the probability of at least one car passing the intersection in a 5-minute window, assuming a constant probability throughout? 0.4377

Sociability. In a certain social network, probability that a randomly chosen user has k subscribers is inversely proportional to $k(k+1)$. Each user has at least one subscriber.

- Express PMF and CDF for the distribution of the number of subscribers. PMF=1/k(k+1) CDF=k/(k+1)
- Let us call *celebrities* the users that have more subscribers than 99% of all users. How many subscribers do you need in order to become a celebrity? At least 100 subscribers
- Try to find the expected number of subscribers per person in this network. 1 subscriber

King of the hill. Consider a game of "ladder climbing". There are 5 levels in the game, level 1 is the lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, a fair coin is tossed. If it turns up heads, the player moves up one rank. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if a tail turns up, and stays at the top if head turns up. How much time (on average) does the player spend on each level? 1 - 50% 2 - 25% 3 - 12.5% 4 - 6.25% 5 - 6.25%

Snail. A cube hangs on a chain attached to one corner (see the picture). A pet snail has got out of the hole in the bottom corner and is wandering from one corner to another along the edges (without treading on faces), one edge per minute. If the snail picks each next edge randomly, then how long (on average) will it wander before reaching the chain and escaping? 7 Minutes (on average)

Credit scoring. A new client applies to a bank for a consumer loan. She has a recent application in another bank, and she has repaid a mortgage loan two years ago. We want to estimate the probability that she is a "bad" client (she won't repay the loan). We believe that recent applications and repaid loans are independent conditionally on the client being bad/good. We know that:

- average share of bad clients is 5%;
- 80% of bad clients have recent applications in other banks, but only 30% of good clients have such applications;
- 10% of good clients have a mortgage repaid, but only 3% of bad clients have it.

So what is the probability that she is bad? 4/99=0.0404

Ants

$P(\text{right}) = P(\text{left}) = 0.5$

no collision = all right | all left

$\Rightarrow P(\text{no collision}) = 2 \cdot 0.5^3 = 0.25$

Passports:

Starts with "4508" $\rightarrow \left(\frac{1}{10}\right)^4$

10^{10} - total passport options

10^4 - passports with 4508 prefix + "53" in them

+ "53" has 5 optional positions

$\longrightarrow P = \frac{5 \cdot 10^4}{10^{10}} = 5 \cdot 10^{-6}$

Coins

	Picked	3H
fake	0.1	0.1
real	0.9	0.1125

1. $P(\text{fake} \cap 3H) = 0.1$

$P(\text{real} \cap 3H) = 0.9 \cdot 0.5^3 = 0.1125$

$\Rightarrow P(\text{fake} | 3H) = \frac{0.1}{0.2125} = \frac{8}{17}$

2. $P(\text{real} | 3H) = 1 - P(\text{fake} | 3H) = \frac{9}{17}$

$\Rightarrow P(4H | 3H) = \frac{8}{17} + 0.5 \cdot \frac{9}{17} = \frac{25}{34}$

Family planning

Proportion would be 1/1 with enough time and families.
P(boy) = P(girl) = 0.5 and every birth maintains this ratio (girl births just adds more births opportunities).

Hitchhiker:

X - Car passed in a 5 mins window
Y - Car passed in a 20 mins window

$$P(Y) = 0.9 = 1 - P(\neg Y) = 1 - P(\neg X)^4 =$$

$$= 1 - (1 - P(X))^4$$

$$\Rightarrow \quad (1 - P(X))^4 = 0.1 \Rightarrow P(X) = 0.4377$$

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Sociability:

1.

$$PMF_x(k) = \frac{1}{k(k+1)}$$

$$CDF_x(k) = \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

telescoping series

2. Celebrities:

$CDF = 0.99 \Rightarrow k = 99$

$\Rightarrow$ every celebrity has at least 100 subscribers.

3. AVG:

$CDF = 0.5 \Rightarrow k = 1$

$\Rightarrow$ the expected num of subscribers per person is 1.

King of the hill:

Transition matrix:

$P_0 = (1, 0, 0, 0, 0)$

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

$P_5 = P_4 = P_0 \cdot M^4 = (0.5, 0.25, 0.125, 0.0625, 0.0625)$

$\rightarrow$ Stationary $\Rightarrow$ 1- 50%

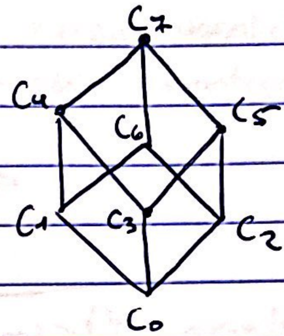
2- 25%

3- 12.5%

4- 6.25%

5- 6.25%

Snail



transition matrix:

$$M = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_0 = (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$$

$$P_6 = P_0 \cdot M^6 = (0.2'', 0, 0, 0, 0.13'', 0.13'', 0.13'', 0.395'')$$
$$P_7 = P_0 \cdot M^7 = (0, 0.157'', 0.157'', 0.157'', 0, 0, 0, 0.5294'')$$

$\Rightarrow$ after 7 mins probability of getting to C_7 (exit) is greater than 0.5

$\Rightarrow$ The average time would be 7 minutes.

Credit scoring

ra - recent application
rm - repaid mortgage

$$P(bad \mid ra \cap rm) = \frac{P(bad \cap ra \cap rm)}{P(ra \cap rm)} =$$

$$= \frac{P(bad) \cdot P(ra \cap rm \mid bad)}{P(bad) \cdot P(ra \cap rm \mid bad) + P(good) \cdot P(ra \cap rm \mid good)}$$ *1

$$P(bad) \cdot P(ra \cap rm \mid bad) = 0.05 \cdot P(ra \mid bad) \cdot P(rm \mid bad) = 0.05 \cdot 0.8 \cdot 0.03 = \frac{3}{2500}$$

In a similar manner:

$$P(good) \cdot P(ra \cap rm \mid good) = 0.95 \cdot 0.3 \cdot 0.1 = 0.0285$$

$$\Rightarrow P(bad \mid ra \cap rm) = \frac{\frac{3}{2500}}{\left(\frac{3}{2500} + 0.0285\right)} = \frac{4}{99} = 0.0404''$$