LECTURE X: THE TIETZE EXTENSION THEOREM

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Abstract.

The Tietze Extension Theorem explores the extension of the domain of a real-valued, continuous, bounded function $f : F \to \mathbb{R}$ to encompasses the entire metric space (X, d) in which F lives. This extension $\tilde{f} : X \to \mathbb{R}$ preserves that it is not only bounded and continuous but also that it has the same bounds as f.

This lecture aims to cover the proof of the Tietze Extension Theorem using Urysohn's Lemma and also to discuss the importance of this theorem through applications. One such application will give a characterization of compact metric spaces.

1. Introduction

Suppose we are given a real-valued, continuous, bounded function $f : F \to \mathbb{R}$, where $F \subseteq X$ is closed and (X, d) is a metric space. The Tietze Extension Theorem guarantees that this function can be extended to $\tilde{f} : X \to \mathbb{R}$ while retaining both its continuity and bounds. This is closely related to Urysohn's Lemma, which we proved in Lecture 10.

In fact, it turns out in Normal Spaces, a subset of a generalized Topological Space, Urysohn's Lemma turns out to be equivalent to the Tietze Extension Theorem, although this paper will not discuss Topological spaces.

2. Background and notation

Definition 2.1. Let (X, d) be a metric space. Let $x \in X$, and $\emptyset \neq A \subseteq X$. Then, define

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}$$

Definition 2.2. Let $f : X \to Y$ be a function from sets X to Y. Denote by the *image of* f the set

$$f(X) := \{y \in Y \mid \exists\ x \in X, f(x) = y\}$$

Definition 2.3. Let $f : X \to Y$ be a function from sets X to Y. Let $A \subseteq Y$. Denote by the *pre-image of* f the set

$$f^{-1}(A) := \{x \in X \mid f(x) \in A\}$$

Lemma 2.4 (Urysohn's Lemma)**.** *In a metric space (X, d), if $A, B \subseteq X$ are closed with $A \cap B = \emptyset$, then there is a continuous function $f : X \to \mathbb{R}$ such that*

(i) *For all $x \in X$, $f(x) \in [0, 1]$*
(ii) $f^{-1}(\{0\}) = A$

(iii) $f^{-1}(\{1\}) = B$

Remark 2.5. In our proof of Urysohn's Lemma, we found a function conveniently fit the required conditions, hence why the Professor called the proof of Urysohn's Lemma "Shamefully Easy", as the proof reduced to checking the three criteria. The function we defined was $g : X \to \mathbb{R}$ by

$$g(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad \forall\, x \in X$$

This will inspire us to define a similar function (Lemma 3.2) that will be used in the Tietze Extension Theorem.

3. Proof of the main result of the lecture

First, I will state the Tietze Extension Theorem before proving some lemmas we will need.

Theorem 3.1 (Tietze Extension Theorem)**.** *Let (X, d) be a metric space. Let $F \subseteq X$ be closed. Let $f : F \to \mathbb{R}$ be continuous and bounded: that is, that there is $M > 0 \in \mathbb{R}$ so that*

$$f(X) \subseteq [-M, M] \subseteq \mathbb{R}$$

Then, there is continuous $\tilde{f} : X \to \mathbb{R}$ so that

(1) *For all $x \in F$,*
$$\tilde{f}(x) = f(x)$$
(2) *For all $x \in X$,*
$$\tilde{f}(X) \subseteq [-M, M]$$
(3) *For all $x \in X \setminus F$,*
$$\tilde{f}(X) \subseteq (-M, M)$$

Lemma 3.2. *Let (X, d) be a metric space. Let $F \subseteq X$ be closed, and let $f : F \to \mathbb{R}$ be a function which satisfies the same conditions as in Theorem 3.1: that it is continuous and there is $M > 0 \in \mathbb{R}$ so that*

$$f(F) \subseteq [-M, M]$$

Then, there is $g : X \to \mathbb{R}$ so that

(1)
$$g(F) \subseteq [-M/3, M/3]$$
(2)
$$g(X \setminus F) \subseteq (-M/3, M/3)$$
(3) *For all $x \in F$,*
$$|f(x) - g(x)| \leq \frac{2}{3}M$$

Proof. The essence of this proof is in the conditions given in Urysohn's Lemma, so we will find a way to apply Urysohn's Lemma.

Define

$$A := \left\{ x \in F : f(x) \leq -\frac{M}{3} \right\}$$

and

$$B := \left\{ x \in F : f(x) \geq \frac{M}{3} \right\}$$

If A and B are both non-empty, then define

$$g(x) = \frac{M}{3} \underbrace{\frac{d(x,A) - d(x,B)}{d(x,A) + d(x,B)}}_{=:\varphi}$$

Checking this function with the one defined in Remark 2.5, we see that it is a scalar multiple and sum of continuous functions, so it suffices to check the three criteria.

(1) It is easy to see that $\varphi \in [-1, 1]$ because the numerator has smaller magnitude than denominator, so we obtain that

$$g(F) \subseteq g(X) \subseteq [-M/3, M/3]$$

(2) Observe that

$$\varphi = \pm 1 \iff d(x,A) - d(x,B) = \pm(d(x,A) + d(x,B)) \iff x \in A \vee x \in B$$

hence if $x \in X \setminus F$, then $x \notin A$ and $x \notin B$, so

$$g(x) \in (-M/3, M/3)$$

(3) Let $x \in X$. WLOG, if $x \in A$, then

$$f(x) - g(x) \leq -\frac{M}{3} - g(x)$$

and

$$g(x) - f(x) \geq g(x) + \frac{M}{3}$$

and since $g(x) \in [-M/3, M/3]$,

$$|f(x) - g(x)| \leq \frac{2}{3}M$$

If $x \notin A$ and $x \notin B$, then $f(x) \in [-M/3, M/3]$, so

$$f(x) - g(x) \leq \frac{M}{3} - g(x)$$

and

$$g(x) - f(x) \leq g(x) - \frac{M}{3}$$

so again

$$|f(x) - g(x)| \leq \frac{2}{3}M$$

However, this does not work when A or B are empty. If it is the case that they are both empty, a smaller and smaller bound M can be taken until one of them is empty, or it is the case that $f(F) = \{0\}$. When this is the case, it is clear to see that defining $\tilde{f}(x) = 0$ for all $x \in X$ is a sufficient continuous extension. We will be careful to case this out in the proof of the Tietze Extension Theorem.

This lemma is getting quite lengthy, so I will not discuss what to do when just one of A or B is empty, but the function to define is covered in [9:3.9, BBT1997], and verification that it fulfills the three criteria is swift.

□

Lemma 3.3. *Let ρ denote the distance function associated with the $\|\cdot\|_\infty$ norm on functions.*

Let (X, d) be a metric space. Suppose $(f_n)_{n\geq 1}$ is a sequence of continuous functions, where $f_i : X \to \mathbb{R}$ for each $i \geq 1$.

If $f_n \underset{\rho}{\longrightarrow} f$, then f is continuous.

Proof. Let $\epsilon > 0$. Then, there is $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$, and all $z \in X$,

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

Let $x, y \in X$. Since each f_n is continuous too, there is $\delta_n \in \mathbb{R}$ so that

$$d(x, y) < \delta_n \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3}$$

Then, picking the same $n_0 \in \mathbb{N}$, and taking $x, y \in X$ so that $d(x, y) < \delta_{n_0}$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

hence completing the proof using the $\epsilon - \delta$ definition of continuous functions. □

We are now ready to prove the Theorem 3.1, the Tietze Extension Theorem.

Proof. Let $g_0 : X \to \mathbb{R}$ be defined by applying Lemma 3.2 to get a function so that

$$|f(x) - g_0(x)| \leq \frac{2}{3}M, \quad \forall\ x \in F$$

Then, for $n > 0$, recursively define $g_n : X \to \mathbb{R}$ by applying Lemma 3.2 to the function $f - \sum_{i=0}^{n-1} g_i$ with the bound $(2/3)^n M$. This yields a function g_n such that

$$(\spadesuit) \qquad |g_n(x)| \leq \frac{2^n}{3^{n-1}} M, \forall\ x \in F$$

$$(\diamondsuit) \qquad |g_n(x)| < \frac{2^n}{3^{n-1}} M, \forall\ x \in X \setminus F$$

$$(\clubsuit) \qquad \left| f(x) - \sum_{i=0}^{n-1} g_i(x) - g_n(x) \right| = \left| f(x) - \sum_{i=0}^{n} g_i(x) \right| < \frac{2^n}{3^n} M, \forall\ x \in F$$

From ($\spadesuit$), it follows that $\sum_{i=0}^n g_i \underset{\rho}{\longrightarrow} \tilde{f}$ for some continuous $\tilde{f}$ by Lemma 3.3 since we have that

$$\lim_{n\to\infty} \frac{2^n}{3^n} M = 0^+$$

we will now show the three criteria of Theorem 3.1.

(1) Again from (♣), for all $n \in \mathbb{N}$,

$$|f(x) - \tilde{f}(x)| \leq \left| f(x) - \sum_{i=0}^{n} g_i(x) \right| < \frac{2^n}{3^n} M, \forall\, x \in F$$

hence by the squeeze theorem, on F, $f = \tilde{f}$.

(2) Let $x \in F$. Then, from (♠),

$$\begin{aligned} |\tilde{f}(x)| &= \left| \sum_{i \geq 0} g_i(x) \right| \\ &\leq \sum_{i \geq 0} |g_i(x)| \\ &\leq \frac{M}{3} \sum_{i \geq 0} \frac{2^n}{3^n} \\ &= M \end{aligned}$$

$x \in X \setminus F$ will be shown in (3):

(3) Let $x \in X \setminus F$. Then, from (♢),

$$\begin{aligned} |\tilde{f}(x)| &= \left| \sum_{i \geq 0} g_i(x) \right| \\ &\leq \sum_{i \geq 0} |g_i(x)| \\ &< \frac{M}{3} \sum_{i \geq 0} \frac{2^n}{3^n} \\ &= M \end{aligned}$$

□

Remark 3.4. The Tietze Extension Theorem also holds for continuous, real-valued functions $f : F \to \mathbb{R}$ that are unbounded: that is, that there is $\tilde{f} : X \to \mathbb{R}$ that is also continuous and unbounded. This will not be proven, but the proof is essentially the same as the one given above.

The proof idea is to find an analogue to Lemma 3.2, where the conditions of (1) and (2) don't matter anymore, but that (3) still holds for input M. This way, we can still construct $\sum_{i=0}^{n} g_i \underset{\rho}{\longrightarrow} \tilde{f}$, and the rest of the proof follows similarly.

This result for unbounded functions will be used in the following application of the Tietze Extension Theorem.

4. An Application of Tietze Extension Theorem

Proposition 4.1. *Let (X, d) be a metric space. X is compact when viewed as a subset of itself if and only if for all $f : X \to \mathbb{R}$ such that f is continuous, f is bounded.*

Proof of forward direction. Suppose for contradiction that there is a continuous function $f : X \to \mathbb{R}$ that is unbounded.

Then, for each $n \in \mathbb{N}$, there is $x_n \in X$ so that

$$|f(x_n)| > n$$

Then, since X is compact, and f is continuous, $f(X)$ is compact, which implies that it is bounded. Then, suppose $M \in \mathbb{R}$ so that

$$f(X) \in [-M, M]$$

However, pick $n = \lceil M + 1 \rceil$ so that

$$|f(x_n)| > \lceil M + 1 \rceil > M$$

which is a contradiction. $\square$

Proof of backward direction. By contrapositive, suppose X is not compact, so it suffices to show the existence of a continuous, real-valued, unbounded function.

X is not sequentially compact. Then, let $(x_n)_{n \geq 1} \subseteq X$ be a sequence with no convergent subsequences.

let $D := \{x_n \mid n \in \mathbb{N}\}$. Then, D is closed because there are no convergent subsequences in D, so D vacuously contains all of its limit points.

Then, define $f : D \to \mathbb{R}$ by

$$f(x_i) = i, \quad \forall\, i \in \mathbb{N}$$

Let $[a, b] \subseteq \mathbb{R}$ be closed. Then, $f^{-1}([a, b])$ is closed because it again vacuously contains all of its limit points.

Then, f is continuous by the closed-mapping characterization of continuous function.

Now, apply the unbounded function version of the Tietze Extension Theorem to f to get $\tilde{f} : X \to \mathbb{R}$ that is continuous and unbounded. $\square$

References

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