Multi-unit Auction in Discrete Type Space

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Abstract

Bidder's behaviour will differ a lot in multi-unit auctions than in single-unit auctions. We study a multi-unit auction where there are two discrete types of bidders and each type of bidder demands two units. We find closed-form solutions for symmetric Bayesnian Nash Equilibria for different proportions of types in the population and one main feature is identical bidding behaviour. We also find that distributions for mixed strategy equilibrium from different types will have overlapping support in bidding spaces. These two features will lead to inefficient allocations. The identical bidding behaviour is also reported in empirical literature studying treasury bill auctions.

We compare expected revenue between formats of multi-unit auctions and confirm that revenue equivalence does not hold in multi-unit settings with ambiguous ranking between revenue from pay-as-bid and Vickrey auctions, while both dominated uniform-price auction in expected revenue. The identical bidding behaviour can also be extended to higher-unit settings.

1 Introduction

In auction theory literature not too much attention has been given to multi-unit pay-as-bid auctions, where the monetary payment for each unit is the winning price for that unit. But in reality multi-unit auction is not rare: the sale of treasury bill auction is an example of multi-unit auction with identical goods. Aalsmeer flower auction is an example of multi-unit auctions for indivisible objects. Unlike single-unit auctions where bidders need to propose a price higher than any other's bid for the win, a bidder does not need to outbid her opponent's highest bid in order to win her first unit in a multi-unit auction. On the contrary, a bidder will get her first unit as long as her highest bid is higher than her opponent's smallest bid when there are two bidders competing for two units. The mechanism underlying unit assignment mentioned above, is the main difference between multi-unit and single-unit auctions for indivisible goods. Complications of multi-unit auctions arise not only because we need to solve multiple optimal bidding functions at the same time but also because bidders will have incentives to decrease higher bids and increase their lower bids since all bidders understand that their higher bids are competing with opponents' lower bids and vice versa. And such behaviours usually lead to inefficient allocations in terms of auction results.

We will be looking at a particular version of pay-as-bid multi-unit auction by making the following assumptions: two identical and indivisible units are being sold and two ax-ante identical bidders with multi-unit demand are participating the auction; bidders' type spaces are binary with diminishing marginal valuations; bidders have incomplete information about each other's types. Bidders are risk-neutral and only care about monetary payoff. To be more precise, we focus on case where "high" type of bidders has private valuations $\bar{v} = (\bar{v}_1, \bar{v}_2)$ and "low" type of bidders have private valuations $\underline{v} = (\underline{v}_1, \underline{v}_2)$, with value ordering $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2 \geq 0$. We will also report mixed strategy equilibria in cases where ranking for private valuations is $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2 \geq 0$ or $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2 \geq 0$ but results in those cases are far less complicated. So we will refer to result where ordering for private valuations is $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$ our **main results**. Both bidders have a common prior that a low type opponent will appear with probability $p \in [0, 1]$ and a high type opponent will appear with probability $1 - p$.

We will construct symmetric mixed strategy equilibrium for these multi-unit auctions. Our main results are that when high type has marginal valuations $\bar{v} = (\bar{v}_1, \bar{v}_2)$ with value ordering $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2 \geq 0$:

1. as long as p is not too large (i.e. $p < \frac{\underline{v}_2}{\underline{v}_1}$), low type bidders' equilibrium bids are generally perfectly correlated (with a few exceptions);
2. when p is very small (i.e. $p < \frac{\underline{v}_2}{\bar{v}_1 + \underline{v}_2}$), we expect high type to put atoms at the lower bound of distribution of her first bid;
3. when p is large enough (i.e. $p > \frac{\underline{v}_2}{\underline{v}_1}$), we find equilibria where distributions of first bid of both high and low types are degenerate on $\underline{v}_2$ while second bid of low type is mixing strictly below $\underline{v}_2$.

One feature of our equilibrium result is that we always have a functional (conditionally deterministic) relationship between two bids of low type. In most cases, low type will be bidding identical bids so the functional relationship is identical function. But we still have a few exceptions where two bids from low type are distinct but connected by an increasing and differentiable function. Since our results are mixed strategy equilibrium, overlapping of support for bids of low and high types will be inevitable. Identical bids from bidders, together with overlapping of support of high and low type leads to the next feature of our results: equilibrium allocations tend to be inefficient. Intuitively speaking, inefficiency arises from the fact that bidders understand their higher bids are competing with opponents' lower bids and they will accordingly make their first bids lower in exchange for higher net payoff. Given that bidders understand they will face lower higher bids from opponents, they will respond by bidding higher second bids for a better chance of winning. Efficiency is guaranteed in our equilibrium when $p > \frac{\underline{v}_2}{\underline{v}_1}$ where higher bids are equivalent to $\underline{v}_2$, the marginal valuation of second unit of low type. So low type will be able to not get any positive expected payoff from her second bid and second bids of low type will mix by distributions aggressively enough to prevent first bids from deviating.

Maskin and Riley (1985) studied a single-unit first price-auction with private valuation where high type had valuation v_H and low type had valuation $v_L < v_H$. Low type bidders would bid their private valuations and high type bidder would use mixed strategy by randomizing over an interval between v_L and v_H. We treat this single-unit private value model as the single-unit benchmark to our model since we have binary types of bidders as well and we will also report mixed strategy equilibria randomizing above the smallest marginal valuation $\bar{v}_2$. The equilibrium strategy implies that our benchmark scenario will achieve efficiency.

Simultaneous auctions, where multiple single-unit auctions are run simultaneously, are comparable to multi-unit auctions. Szentes (2007) and Szentes and Rosenthal (2003) studied two identical bidders with three and two objects simultaneous auctions respectively. Both auctions were complete information auctions with discrete valuations where bidders had multi-unit demands. They also allowed for complementarities or substitutes among objects for the bidders, while our model only consider additive valuation. Szentes (2007) established conditions for symmetric mixed strategy equilibrium when goods are substitutes or complements. Szentes and Rosenthal (2003) found symmetric mixed strategy equilibrium, which was a probability measure with support being surfaces of tetrahedron describing combinations of equilibrium bids. Results in those scenarios were not necessarily efficient either since overlapping of support is inevitable when symmetric bidders are bidding the same strategy. Gentry, Komarovaz and Schiraldi (2020) studied empirical evidence of synergies in pay-as-bid simultaneous auctions. They modeled simultaneous auctions of heterogeneous objects with private valuations in Michigan Department of Transportation highway procurement auctions, and their estimation found evidence of cost reduction for highly complementary projects while increment in cost on the other end of complementarities.

It is easy to find analogies of the 2 most frequently used forms of multi-unit auctions in single-unit settings. Uniform-price auction in the multi-unit setting is analogous to second-price auction in single-unit setting where winners pay the highest rejected price as their prices, and the first-price auction in multi-unit realm is usually called pay-as-bid auction or discriminatory auction. Ausbel et al. (2014) solved equilibrium strategy for uniform-price auction and pay-as-bid auctions with divisible goods when demand is constant or downward sloping. They also compared efficiency (and revenue) of pay-as-bid and uniform-price auctions with private and interdependent valuations under many assumptions. They found conditions for pay-as-bid auctions or uniform-price auction to achieve efficiency with perfectly divisible goods and constant marginal valuations, although they also established in general ranking in terms of efficiency was ambiguous under constant marginal valuations. Ausbel et al. (2014) found that with diminishing linear demand and increasing linear supply, expected revenue from linear equilibrium of pay-as-bid auctions were strictly higher than that of uniform-price auctions, but none was able to achieve efficiency.

Branco (1996) showed that deterministic mechanism (i.e. sellers announced that she would implement a specific allocation for sure) was efficient for multi-unit demand pay-as-bid auction where (asymmetric) bidders with private and interdependent valuations were competing for homogeneous indivisible objects. Branco (1996) also proposed conditions (i.e. required minimum bids for kth unit and bid monotonically w.r.t. signals) for some common single-demand auctions (e.g. pay-as-bid, uniform price and sequential auctions) to be efficient by restricting only to homogeneous bidders. Engelbrecht-Wiggans and Kahn (1998) studied a pay-as-bid auction similar to our set-up. They assumed bidders with diminishing marginal valuations competed for two objects in a pay-as-bid multi-unit auction as well. They proposed a system of differential equations derived from first order conditions from expected payment as equilibrium bids and constructed an example of pure strategy equilibrium by using a specific marginal distribution, where the optimal bid is a function of combinations of valuations. Engelbrecht-Wiggans and Kahn (1998) established the existence of both pooling and separating equilibrium in multi-unit auction, where pooling equilibrium describes the behaviour that one bidder is bidding identically for both bids while separating equilibrium is that one bids differently. Their paper differed from ours by the following aspects: they assumed bidders' valuation come from atomless distributions while we assumed discrete distribution with binary types of bidders. Their results were more of a characterization of equilibrium properties since they only showed that there will be positive probability that the auction ends in a pooling equilibrium without solving the general model. They only showed bidding functions in terms of marginal valuation when marginal valuations are drawn from uniformly on $[0, 1]$. Anwar (2007) extended the affiliated model [1] from Milgrom and Weber (1981) to multi-unit demand environment. Anwar (2007) assumed that a bidder's valuation is a non-decreasing function of her own private information about the object, the highest information from other bidders and an additional common signal about the object. The multi-unit auction studied in Anwar (2007) is competition for $k \geq 2$ objects. Anwar (2007) solved the unique pure strategy equilibrium where bidders would bid identical bids for all objects with a simplification by restricting to case of constant marginal valuations.

One characteristic of our findings, bids from low type are identical (i.e. conditionally deterministic), can be found in literature. We can see pooling equilibrium in multi-unit auctions from both Engelbrecht-Wiggans and Kahn (1998) and Anwar (2007) as mentioned in the previous paragraphs. Empirical evidence where bidders tend to bid identically can also be found. Hortaçsu and McAdams (2010) studied bidding behaviour from Turkish Treasury auction market and modelled the auction as multi-unit auction with indivisible but identical objects. They found that bidders submitted bids as step-functions, indicating that bidders used identical prices for certain ranges of quantities. Cassola, Hortaçsu and Kastl (2013) also found out that bidders would bid by similar step-functions when studying European banks' demand for short-term funds before and after the 2007 subprime market crisis, although their model is to study multi-unit auction with divisible objects.

We may also be able to derive other implications from the pooling equilibrium. Ausbel et al. (2014) mentioned differential bid-shading where bidders shaded bids differently across units. Given that our model found pooling equilibrium for low type, we can treat our pooling equilibrium as bidders shaded more for higher marginal valuations. Besides the pooling equilibrium which prevails in majority of our results, we are able to find some separating equilibrium for low type for some small range of p. We get separating equilibrium by assuming that it is the interior solution to maximization problem where low type maximizes her expected payoff from second bid given any first bid in the support of joint bids, while we may interpret pooling equilibrium as boundary solutions since first and second bids in such equilibrium are at their extreme. We can conclude for separating equilibrium that first and second bids of low type are related by an increasing function, which is strictly smaller than the identity function.

Establishment of revenue equivalence theorem has always been a topic discussed in auction literature. In fact we can compare revenue generated from our pay-as-bid auction and a hypothetical uniform-price auction, where common monetary payment for each unit is the highest losing bid. The pay-as-bid auction will generate positive revenue by its own rule: the monetary payment for each unit in a pay-as-bid auction is the winning price for that unit and it is highly unlikely for bidders to win a unit by bidding zero. And accordingly we should expect winning prices and expected revenue in pay-as-bid auction to be strictly positive. The uniform-price auction, on the other hand, has an obvious equilibrium where bidders are bidding truthfully for their first units and 0 for their second units. Such a bidding strategy leads to zero revenue since the highest losing price is always 0. So without any computation we can conclude that pay-as-bid auction will dominate uniform-price auction in terms of revenue given our multi-unit setting and accordingly we do not have a version of revenue equivalence. Besides, we can also compare expected

[1] Each bidder has private information that is positively correlated with the bidder's value of the good.

revenue from our pay-as-bid auction with Vickrey auctions, where a bidder i who wins k_i units will pay the highest losing k_i bids from her opponent. Truthful bidding is an equilibrium for Vickrey auction and each bidder will win one unit and pay the marginal valuation of her opponent's second unit. Our comparison indicates an ambiguous relationship between expected revenue of pay-as-bid and Vickrey auctions: Vickrey auction generates higher expected revenue when p is relatively low but pay-as-bid auction will have higher revenue when p is high.

Our comparison above fits with consensus from literature: in multi-unit setting, the revenue equivalence theorem prevalent from single-unit environment does not hold in general. Revenue equivalence is possible when assignment from different auctions turn out to be identical, with Engelbrecht-Wiggans (1988) being an good example showing the result. And when different formats of auctions lead to different assignment, revenue equivalence does not apply. Theoretical and empirical literature draw different conclusions regarding revenue ranking among different auction formats. Tenorio (1999) studied a two-agent three-unit multi-unit auction where capacity of demand of identical bidders may be either two or three. Tenorio (1999) proved revenue generated from different formats of auctions were equivalent as long as bidders have the same units of demand, but revenue from auctions where bidders' demand is three-unit is higher than that where bidders' demand is two-unit. What's more, as mentioned in the previous paragraphs, Ausbel et al. (2014) showed that revenue ranking between uniform-price and pay-as-bid auctions are ambiguous: when demand is flat they provided examples where each type of auction dominated. When demand is downward sloping, they found that pay-as-bid auctions would dominate in terms of revenue. Hortaçsu and McAdams (2010) conducted counter-factual simulation to compute a hypothetical revenue if the auction were switched to the format of uniform-price. But they could not reject the hypothesis that the two formats (pays-as-bid and uniform-price) generated same level of revenue.

2 Example

We will illustrate one numerical example of our results in this section before showing any theoretical results. The auction we look into is a multi-unit auction with two identical units. Any bidder will be a high or low type with probability $\frac{1}{4}$ or $\frac{3}{4}$ respectively. We suppose high type's marginal valuation for the two units is $(3, 0)$ while low type's marginal valuation is $(2, 1)$. The format of the auction is pay-as-bid, meaning that the monetary payment for each unit is the winning price for that unit. We normalize high type's marginal valuation to be zero so that high type's bid will only be one non-negative price. On the contrary, low type will be submitting two non-negative prices.

We report a mixed strategy equilibrium where a high type will be bidding by CDF $F_H(x) = \frac{x^2}{(1-x)(3-x)}$ on support $[0, \frac{3}{4}]$. In the meanwhile, a low type will be bidding her two bids identically and mixing by a common CDF $G_L(x) = \frac{3x}{3-x}$ over the same support. A good way to understand this equilibrium is to look at low type's expected payoff from her higher bid b_{l1}, which is $\frac{1}{4}\frac{3b_{l1}}{3-b_{l1}}(2-b_{l1}) + \frac{3}{4}(2-b_{l1}) = \frac{9(2-b_{l1})}{4(3-b_{l1})}$. It is not hard to notice that derivative with respect to b_{l1} is negative, implying that the low type should pick the smallest feasible price as her higher bid, and low type's higher bid should be no smaller than her lower bid. So a low type will be bidding identically due to monotonicity of expected payoff from her marginal bid.

This numerical example highlights the main findings of our theorems: low type will be bidding identically for her two bids. Another feature arises from the bidding strategies is overlapping of support. With bidders mixing their bids in identical support, it is likely that our result leads to inefficient allocation of units. A low type may win both the units while efficient allocation is always to make each bidder get one object regardless of type. Additionally, there are some cases where over some region the low type may not choose to bid identically, but there will be a conditionally deterministic relationship between bids of low type.

3 Model

There are two identical indivisible objects being auctioned off. Each of two bidders, $i = 1, 2$, demand up to two units of the object. In particular, bidder i's valuations are given by (v_{i1}, v_{i2}), where v_{i1} indicates the bidder's value of the first unit obtained and v_{i2} indicates the bidder's value of the second unit obtained. Note $v_{i1} > v_{i2} \geq 0$.

Bidders can be one of two types: high or low. The high type has valuations $\overline{v} = (\overline{v}_1, \overline{v}_2)$ and the low type has valuations $\underline{v} = (\underline{v}_1, \underline{v}_2)$. Note, $\overline{v}_1 > \underline{v}_1 > \underline{v}_2 > \overline{v}_2 \geq 0$. So the high type has high-variance in their valuations and the low type has low-variance in their valuations. Let $V = \{\overline{v}, \underline{v}\}$ be the set of possible valuations (or types). The bidders' types are drawn independently from a common prior. And we denote $p \in (0, 1)$ for the probability that bidder i is the low type.

The objects are auctioned off in a multi-unit pay-as-bid auction: the bidders simultaneously submit bids for both units of the object. In particular, bidder i's bid is given by a vector $b_i = (b_{i1}, b_{i2})$, where $b_{i1} \geq b_{i2} \geq 0$. b_{i1} is bidder i's **first bid** (i.e., bid for the first unit) and b_{i2} is her **second bid** (i.e., bid for the second unit). So, b_{i1} denotes i's payment if she only gets one unit of the object and $b_{i1} + b_{i2}$ represents her payment if she gets both units of the object. Furthermore, we let $\mathcal{B}_i$ to be the set of possible bids of i, i.e., $\mathcal{B}_i = \{(b_{i1}, b_{i2}) : (b_{i1}, b_{i2}) \in R_+^2, b_{i1} \geq b_{i2}\}$.

The winner of the auction is determined by the profile of bids $(b_{11}, b_{12}, b_{21}, b_{22})$. If $b_{i1} > b_{-i1}$, the allocation is determined by comparing bidder i's second bid b_{i2} to bidder $-i$'s first bid b_{-i1}. Each bidder wins exactly one unit if $b_{i1} > b_{-i1}$ and $b_{-i1} > b_{i2}$. Bidder i wins both units if $b_{i1} > b_{-i1}$ and $b_{-i1} < b_{i2}$. Moreover, if $b_{i1} > b_{-i1}$ and $b_{-i1} = b_{i2}$, bidder i wins the first with probability one and the players split the second unit with $.5 : .5$ probability. Finally, if $b_{11} = b_{21}$ then each bidder i wins exactly one unit of the object.

The payoffs depend on the profile of bids and the type of the bidder. In particular, the ex-post payoff function of a bidder of type (v_{i1}, v_{i2}) is given by

$$\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_{i1}, v_{i2}) = \begin{cases} v_{i1} + v_{i2} - b_{i1} - b_{i2} & \text{if } b_{i1} > b_{-i1} \text{ and } b_{i2} > b_{-i1} \\ v_{i1} - b_{i1} + \frac{1}{2}(v_{i2} - b_{i2}) & \text{if } b_{i1} > b_{-i1} \text{ and } b_{i2} = b_{-i1} \\ \frac{1}{2}(v_{i1} - b_{i1}) & \text{if } b_{-i1} > b_{i1} \text{ and } b_{-i2} = b_{i1} \\ v_{i1} - b_{i1} & \text{if } b_{i1} = b_{-i1} \text{ or } b_{i1} > b_{-i2} \text{ and } b_{-i1} > b_{i2}. \\ 0 & \text{otherwise} \end{cases}$$

Let $\Delta(\mathcal{B}_i)$ be the set of probability distributions over $\mathcal{B}_i$. A strategy for bidder i is a mapping $\sigma_i : V \to \Delta(\mathcal{B}_i)$. So, $\sigma_i(v_i)$ is bidder i's mixed bid when she is of type $v_i = (v_{i1}, v_{i2})$. It will be convenient to denote distribution of opponent's mixed bid (b_{-i1}, b_{-i2}) as $\mathbb{P}$ and to write $\mathbb{P}_{\sigma_i(v_i)}$ as the distribution induced by mixed bid $\sigma_i(v_i)$.

Write $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)]$ for bidder i's expected payoff from (b_{i1}, b_{i2}) given that her value is v_i and the distribution induced by (b_{-i1}, b_{-i2}) is $\mathbb{P}$, i.e.,
$\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = \mathbb{P}(b_{-i2} < b_{i1}, b_{-i1} < b_{i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + \mathbb{P}(b_{-i1} < b_{i1}, b_{-i1} = b_{i2})(v_{i1} - b_{i1} + \frac{1}{2}(v_{i2} - b_{i2})) + \mathbb{P}(b_{-i1} > b_{i1}, b_{-i2} = b_{i1})\frac{1}{2}(v_{i1} - b_{i1}) + (1 - \mathbb{P}(b_{-i2} < b_{i1}, b_{-i1} < b_{i2}) - \mathbb{P}(b_{-i1} < b_{i1}, b_{-i1} = b_{i2}) - \mathbb{P}(b_{-i1} > b_{i1}, b_{-i2} = b_{i1}))(v_{i1} - b_{i1})$.

With this, bidder i's interim expected payoffs from bidding (b_{i1}, b_{i2}) given that her value is v_i and her opponent chooses σ_{-i} is given by

$$\pi_i(b_{i1}, b_{i2}, \sigma_{-i} \mid v_i) = p\mathbb{E}_{\mathbb{P}_{\sigma_{-i}(\underline{v})}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] + (1-p)\mathbb{E}_{\mathbb{P}_{\sigma_{-i}(\overline{v})}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)].$$

The paper restricts to **symmetric** Bayesian Nash equilibria, (σ_1^*, σ_2^*). So, we always have $\sigma_1^* = \sigma_2^*$ and, secondly for each i and each $v_i \in V$, $\sigma_i^*(v_i)$ maximizes $\pi_i(b_{i1}, b_{i2}, \sigma_{-i}^* \mid v_i)$.

4 Preliminary Results

4.1 Separation of Marginal Bidding Distributions

Consider a bidder i in the auction, who given his type, bids (b_{i1}, b_{i2}) with $b_{i1} \geq b_{i2} \geq 0$. Suppose (b_{-i1}, b_{-i2}) are her opponent's bids with $b_{-i1} \geq b_{-i2} \geq 0$. Recall that we let $\mathbb{P}$ be the distribution induced by (b_{-i1}, b_{-i2}) in definition of $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)]$. We invent another notations with B_{i1}, B_{i2} to be the marginal CDFs for bids b_{i1}, b_{i2} respectively, i.e. $B_{-i1}(x) = \mathbb{P}(b_{-i1} \leq x)$ and $B_{-i2}(y) = \mathbb{P}(b_{-i2} \leq y)$. We will show in later subsections (without invoking result in this subsection) that tie happens with zero probability when $p < \frac{v_2}{\underline{v}_1}$ or upper bounds of support of distributions are below $\underline{v}_2$. And when both type bids $\underline{v}_2$ with $p > \frac{v_2}{\underline{v}_1}$, there is no tie since assignment rule will simply let each bidder get one unit. So it is safe for us only to care about events $\{b_{i1} > b_{-i1}, b_{i2} > b_{-i1}\}$ and $\{b_{i1} > b_{-i2}, b_{-i1} > b_{i2}\}$, since all other events from our definition of ex-post payoff are involved in ties and will be of zero probability.

We have simplified
$\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = \mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + \mathbb{P} \leq b_{i1}, b_{-i1} > b_{i2})(v_{i1} - b_{i1})$.
Terms $\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2})$ and $\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2})$ are probabilities when one bidder wins exactly 2 and 1 units, which associates with the joint distribution of opponent's bids. Notice also that by arguing ties happen at zero probability, we are free to interchange between weak and strict inequalities for expressions in the probability notations.

If we want to look for equilibrium strategies, we could try to use first order approaches. But first order partial derivative on the joint distribution function will further complicate the computational process [2]. But if we denote $B_{-i1}(x) = \mathbb{P}(b_{-i1} \leq x)$ and $B_{-i2}(y) = \mathbb{P}(b_{-i2} \leq y)$, the following lemma essentially shows that instead of focusing on joint distributions we are able to simplify our computation by using only marginal distributions B_{-i1} and B_{-i2} in the computation of expected payoff.

[2] Actually $\frac{\partial}{\partial x} F_{X,Y}(x,y) = \int\limits_{-\infty}^{y} f_{X,Y}(x,t)dt = \int\limits_{-\infty}^{y} f_{Y|X}(t|x) f_X(x)dt = \int\limits_{-\infty}^{y} f_{Y|X}(t|x)dt \times f_X(x) = \mathbb{P}(Y \leqslant y|X = x) f_X(x)$

Lemma 1. $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = B_{-i1}(b_{i2})(v_{i2} - b_{i2}) + B_{-i2}(b_{i1})(v_{i1} - b_{i1})$.

Proof. Expected payoff of bidding $b_{i1} \geq b_{i2}$ is $\mathbb{E}_{\mathbb{P}}[\Pi_i(b_{i1}, b_{i2}, b_{-i1}, b_{-i2} \mid v_i)] = \mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + \mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2})(v_{i1} - b_{i1})$. Note that
$\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} \leq b_{i2}) = \mathbb{P}((b_{-i2} \leq b_{i1}) \cap (b_{-i1} \leq b_{i2})) = \mathbb{P}(b_{-i1} \leq b_{i2})) = B_{-i1}(b_{i2})$ by the ordering of b_{i1}, b_{i2} and b_{-i1}, b_{-i2}.
$\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2}) = \mathbb{P}((b_{-i2} \leq b_{i1}) \cap (b_{-i1} > b_{i2})) = \mathbb{P}(b_{-i2} \leq b_{i1}) - \mathbb{P}((b_{-i2} \leq b_{i1}) \cap (b_{-i1} \leq b_{i2}))$ by Carathéodory's criterion. And it can be simplified to
$\mathbb{P}(b_{-i2} \leq b_{i1}, b_{-i1} > b_{i2}) = \mathbb{P}(b_{-i2} \leq b_{i1}) - \mathbb{P}(b_{-i1} \leq b_{i2})$ or equivalently $B_{-i2}(b_{i1}) - B_{-i1}(b_{i2})$.
So we can write the expected payoff as
$\pi_i = B_{i1}(b_{-i2})(v_{i1} + v_{i2} - b_{i1} - b_{i2}) + (B_{-i2}(b_{i1}) - B_{-i1}(b_{i2}))(v_{i1} - b_{i1})$
$= B_{-i1}(b_{i2})(v_{i2} - b_{i2}) + B_{-i2}(b_{i1})(v_{i1} - b_{i1})$. □

4.2 Second Bid From High Type

In the proof of lemma 1, we are assuming no ties happen with positive probability. And we will argue that it is safe to make such an assumption by showing several results regarding atoms on distributions. But before we do the proof, we can first simplify our analysis by showing second bid of low type will never win in equilibrium. Before we do the proof, we can first simplify our analysis by showing second bid of low type will never win in equilibrium. With lemma 1 established, we always suppose F_{H1}, F_{H2} are marginal distributions of high type's first and second bids respectively while G_{L1}, G_{L2} are marginal distributions of low type's first and second bids in the following parts.

Theorem 1. *For any equilibrium distribution, a high type will win at most 1 object.*

Proof. We will show an equivalent statement in order to prove this theorem: no type will put lower bound of first bids lower than $\bar{v}_2$. So second bid of high type will not outbid any first bid and accordingly high type will at most win one object.

If second bid of high type outbids another bid, it must be that at least one of high and low types is putting positive probability on $\bar{v}_2$ or smaller values on support of F_{H1} or G_{L1}. Without loss of generality, we assume F_{H1} is putting positive probability. If a high type is using $(\bar{v}_2^+, \bar{v}_2)$ as her two bids, her expected payoff will be no smaller than $(1-p)(\bar{v}_1 - \bar{v}_2)$, since $\bar{v}_2^+$ will definitely outbid second bid of high type. And accordingly to support bids lower than $\bar{v}_2$, the expected payoff from bids in that region must be no smaller than $(1-p)(\bar{v}_1 - \bar{v}_2)$. In particular, high type's expected payoff from bidding at exactly her lower bounds, which are no greater than $\bar{v}_2$ in this scenario, should be no smaller than $(1-p)(\bar{v}_1 - \bar{v}_2) > 0$. If a high type gets positive payoff by breaking ties at $\bar{v}_2$, she will have incentive to deviate to bid slightly higher than $\bar{v}_2$ and win higher payoff. For atoms at values strictly lower than $\bar{v}_2$ and atomless distributions, there are two sources of this positive payoff for high type:

1. When lower bounds of F_{H1}, F_{H2} does not coincide and lower bound of F_{H2} is lower than $\bar{v}_2$, lowest first bid of high type can outbid second bid of high type with positive probability. But this indicates that

   (a) When lower bound of support from G_{L1} is no smaller than that from F_{H2}, high type will deviate her lower bound of F_{H2} to higher values for strictly higher payoff since the current lower bound for F_{H2} is not able to outbid any first bids.

(b) When lower bound of support from G_{L1} is smaller than that from F_{H2}, low type will deviate to higher lower bounds by a similar reason since bidding at the current lower bound will not outbid any bids.

2. When lower bounds of F_{H1}, F_{H2} coincide and lower bound of F_{H2} is lower than $\bar{v}_2$, high type may get her positive payoff at her lowest bids

   (a) by outbidding bids of low type with positive probability. But this means low type will deviate her lower bounds to values no lower than that of high type.

   (b) if both F_{H1}, F_{H2} put atoms at the lower bound of their supports. We argue this arrangement of distributions is not an equilibrium distribution since high type will have incentive to raise lower bound to break the tie and get strictly higher payoff.

   (c) or if only F_{H2} has an atom at lower bound. But high type will move the atom at bottom of support from F_{H2} to higher values since by bidding at the atom second bid of high type will outbid first bids of high and low type with zero probability.

So we conclude that there will be no equilibrium when first bid of high type is lower than $\bar{v}_2$ or F_{H1} has an atom at $\bar{v}_2$. For low type, bids no greater than $\bar{v}_2$ will be dominated by $(\bar{v}_2^+, \bar{v}_2^+)$ by a similar reason. □

The intuition is clear: with our set-up, marginal valuation of second good of high type is the lowest. So first bid of both types will have strong incentive to bid at least $\bar{v}_2$, which will guarantee a positive expected payoff as long as there is positive probability of facing high-type opponents. This behaviour will incentivize second bid from low type to put zero probability on values below $\bar{v}_2$ because otherwise she will be "wasting" probability on a unwinnable range. Such a theorem is in consistent with with our single-unit benchmark (Maskin and Riley, 1985) where high type is randomizing between v_L and v_H, which makes it impossible for low type to win.

With theorem 1 established, we will normalize $\bar{v}_2 = 0$ to simplify our analysis. With our existing tie-breaking rules, we will encounter several interesting scenarios: when one bidder is bidding $(0, 0)$, a high type can get 1.5 objects by bidding one positive bid and one zero bid. But high type's marginal valuation toward second object is 0 so we want some new rules to get rid of the possibility that high type will get more than one object. We can add a few new auction rules besides the existing tie-breaking rules. And we will call the following rules assumption 1:

Assumption 1.

1. *Bidding* $(0, 0)$ *is not allowed;*

2. *As long as some bidder submits a bid containing* 0*, she can get at most one object;*

The first rule requires a bidder to bid either a single zero or at least one positive bid. The second rule has two implications: high type will not get an extra 0.5 object by bidding one positive price and one zero price when her opponent is bidding zero. Low type will not be bidding zero when she submits two bids since it is a weakly dominated strategy: under the new rule, by bidding zero low type is essentially giving up one bid since the only object she can win is through her first bid as her total win in the auction is capped at one. Similarly, we can conclude that low type will always bid 2 prices. If she only bids one price, she is able to get weakly better payoff by adding another bidding price as long as the new bidding price is smaller than the marginal valuation of her less favoured unit. And hence we conclude that for a low type bidding only one price is a weakly dominated strategy. So high type should just submit one bid while low type should submit two bids In all, our rules will solve the problem mentioned in the previous paragraph: when two high types meet each will get one object regardless of bids and when one low type bids zero, she is guaranteed to get one object when facing a high type.

We have a direct result from introduction of assumption 1:

Lemma 2. *High type may put an atom at* 0 *while low type will never put an atom at* 0.

The intuition for this result is that high type will automatically wins one object when facing another high type, so bidding zero means high type will get a high net surplus when she faces a high type with a trade-off of losing to low type with certainty. On the other hand, low type can always get more payoff by submitting two bids and will have incentive to do so.

4.3 No Ties Happen with Positive Probability

With $\bar{v}_2$, high type's marginal valuation of second object, being normalized to 0, we can do the proof mentioned in subsection 4.1 to show that no tie will happen with positive probability when $p < \frac{\underline{v}_2}{\underline{v}_1}$ or upper bounds of support of distributions are below $\underline{v}_2$. Ties may happen when high type submits $\underline{v}_2$ and low type submits $(\underline{v}_2, \underline{v}_2)$ with $p > \frac{\underline{v}_2}{\underline{v}_1}$. But we will argue that our tie-breaking rule dictates that each unit wins only 1 unit in this scenario, so the "tie" in this scenario can be trivially resolved.

Before checking atoms at positive values, we first take a look at gaps in support of marginal distributions. It turns out we can make the following conclusions regarding gaps on marginal distributions:

Lemma 3.

1. *There can be no gaps of interval on marginal distribution of second bid for low type.*

2. *If first bids of high and low types both have gaps in the support of distributions, the gaps must have intersection with zero measure.*

Proof. For the first point, suppose the gap from support of marginal distribution of second bid of low type is interval (a, b) with $a < b$. Then first bids from high and low types will put zero probability in interval (a, b) as well since bidding those values will be dominated by bidding a while holding second bid constant. First we assume low type does not put any atom at a. Mathematically speaking, we can hold second bid constant and only compare the probability of winning any unit by bidding a or $x \in (a, b)$:
$\mathbb{P}(b_{-i2} \leq a, b_{-i1} \leq b_{i2}) = \mathbb{P}(b_{-i2} \leq x, b_{-i1} \leq b_{i2})$ and $\mathbb{P}(b_{-i2} \leq a, b_{-i1} \geq b_{i2}) = \mathbb{P}(b_{-i2} \leq x, b_{-i1} \geq b_{i2})$. In a word, bidding in interval (a, b) will give the same probability of winning as bidding a when facing second bids of high and low types but one has to pay more. If low type puts an atom at a on her marginal distribution of second bid, bidders of any type will prefer to bid (a^+, x) with $x \leq a$ than bid (a, x) or any bid (b_1, b_2) in intersection of joint support of one bidder's mixed strategy and set $\{(b_1, b_2) : b_2 \leq b_1 \leq a\}$ [3]. So putting an atom at a on marginal distribution of second bid will lead to first bids of any bidder to bid no lower than a. And hence we exclude possibility of interval $[a, b)$.

[3] To support a mixed strategy equilibrium, a necessary condition would be bids in intersection of support of bids for low type and set $\{(b_1, b_2) : b_2 \leq b_1 \leq a\}$ generate identical payoff.

If we look again at the comparison last paragraph, we know that bidding exactly at or slightly higher than b is dominated by bidding a as well: by bidding in right neighbourhood of b, when $\epsilon > 0$ is sufficiently small bidding $x \in (b, b+\epsilon)$ will give almost the same probability of winning as bidding a, but bidders have to much higher price when they win. We exclude possibility of interval $[a, b)$ (i.e. atom at $\{b\}$) for the following reason: if low type puts an atom at b on the marginal distribution of her second bid, bidders will respond by putting a gap at singleton set $\{b\}$ on marginal distributions for first bids, since bidding slightly higher than b will break the tie and generate strictly higher payoff than bidding at b. Knowing this, a bid (x, b) with $x > b$ for low type will be dominated by bidding (x, a), which means low type will not put any positive probability at b at the first place. In all we conclude that if second bid of low type is putting a positively measured gap in the support, first bids from high and low type will respond by putting a larger gap (a, b') where $b' > b$ in the support.

Knowing this, second bid of low type will not bid at b since it is dominated by bidding at a (while holding first bid constant) when she knows that distribution of first bids will respond to put a larger gap. So we conclude we can not have an equilibrium where low type puts a positively measured gap in the marginal distribution of her second bid. And hence we have our first conclusion regarding gaps in marginal distribution.

For the second point, assume that intersection of gaps in support of distributions of first bids from high and low type is interval (c, d) with $c < d$. Low type will use similar deviation method mentioned in the previous paragraphs, i.e. second bid of low type will not bid in interval (c, d) or prices slightly higher than d since all bids in that area are dominated by bidding c while holding first bid constant. Such a behaviour by low type will lead to a gap in the support of distributions of second bids from low type. □

As long as we conclude that low type will not put any gap of interval on marginal distribution of her second bid and gaps from distributions of first bids of high and low type have zero-measured intersection, we can show some results regarding ties and atoms in the multi-item auction.

Lemma 4. *No equilibrium distribution will put atoms at positive values smaller than upper bound of support.*

Proof. Suppose second bid of low type puts an atom at x, which is lower than the upper bound. Lemma 3 has established that at least one from marginal distributions of first bids of high and low types will have support containing neighbourhood of x. So we can look at deviations case by case. If support of distribution for first bid of low type contains neighbourhood of x, we may consider bids (x, b) where x is her first bid and b is her second bid. She will have incentive to deviate her first bid to x^+ while holding second bid constant. Such a deviation will lead to strictly higher payoff for low type since it breaks the tie at x, where there is an atom with positive measure. Note that to support a mixed strategy equilibrium, a necessary condition would be bids in intersection of support of bids for low type and set $\{(b_1, b_2) : b_2 \leq b_1 \leq x\}$ generates identical payoff [4]. Since bid $(x, b) \in \{(b_1, b_2) : b_2 \leq b_1 \leq x\}$ is dominated by (x^+, b), bids in set $\{(b_1, b_2) : b_2 \leq b_1 \leq x\}$ will also be dominated by (x^+, b). Similarly if distribution of high type contains neighbourhood of x, it is easy to see that bidding x^+ will generate strictly higher payoff than bidding x^- or x. And accordingly by a similar reason bids lower than x will be dominated by bid x^+.

[4] (b_1, b_2) denote first and second bids for low type respectively.

If distribution of a high type or first bid of low type puts an atom at y, which is smaller than the upper bound of bids, we will have a similar argument as the previous paragraph since lemma 3 shows that low type will not have a positively measured gap. Suppose low type is bidding (b, y) (or (b, y^-)) as her pair of bids ($b > y$), and we will see that deviating the second bid to y^+ generates higher payoff. When first bid of low type is already higher than y (i.e. $b > y$), we allow it to stay at b, but when first bid of low type is no greater than y we can also increase it to be y^+. The slight increment in first bid will not change payment from first bid. $\square$

Lemma 5. *No equilibrium distribution will put atoms at upper bounds of support when the upper bound is smaller than $\underline{v}_2$.*

Proof. When upper bound of bids is smaller than $\underline{v}_2$ and some type chooses to put an atom at the upper bound, an obvious deviation will be bidding slightly higher than the upper bound. And by a similar reason discussed in the proof of lemma 4, such a behaviour will generate strictly higher payoff. Note that this argument works for any p. $\square$

These three lemmata imply that equilibrium distributions will be atomless when upper bound of support is smaller than $\underline{v}_2$ unless high type puts an atom at 0. Since we normalize $\bar{v}_2$ to be 0, proof of lemma 1 will only be restricted to low type. And since low type is bidding positive bids and atom only happens at 0, ties will not appear with positive probability in equations used in lemma 1. What's more, distributions will have the same upper bound in this case since ties only happen with zero probability.

Lemma 6. *When $p < \frac{\underline{v}_2}{\underline{v}_1}$, no equilibrium distribution will put atoms at upper bounds of support when the upper bound is equal to $\underline{v}_2$.*

Proof. With common upper bound being equivalent to $\underline{v}_2$, a high type will get $\bar{v}_1 - \underline{v}_2$ when she bids at $\underline{v}_2$ and a low type will get $\underline{v}_1 - \underline{v}_2$ when both her bids are at upper bound. Since we normalize marginal valuation of second object of high type to zero, low type will get at least $(1-p)\underline{v}_1$ by deviating first (and second) bid to slightly above 0. Given $p < \frac{\underline{v}_2}{\underline{v}_1}$, we have $(1-p)\underline{v}_1 \geq \underline{v}_1 - \underline{v}_2$. So putting upper bound of bids at $\underline{v}_2$ will not be an equilibrium strategy at the first place when $p < \frac{\underline{v}_2}{\underline{v}_1}$. $\square$

Given the tie-breaking rules introduced previously, these four lemmas imply that when $p < \frac{\underline{v}_2}{\underline{v}_1}$ ties will not appear with positive probability in equations used in lemma 1: the only atom in this situation happens at 0 but low type is bidding positive bids and high type only bids one bid. Such results imply that we do not need to worry about ties in our proof of lemma 1 when $p < \frac{\underline{v}_2}{\underline{v}_1}$.

On the other hand, when $p > \frac{\underline{v}_2}{\underline{v}_1}$, a bidder may choose to set upper bounds on $\underline{v}_2$. If one bidder puts upper bound at $\underline{v}_2$, we expect the upper bound of the other bidder's bids to be at $\underline{v}_2$ as well since otherwise this bidder will decrease her upper bound to avoid wasting probability on a range that is too high. With this observation, we consider the case where low type bidding $(\underline{v}_2, \underline{v}_2)$ again: low type's expected payoff from second bid is 0 since $\underline{v}_2 - \underline{v}_2 = 0$. If distribution of high type or first bid of low type puts non-zero probability on values lower than $\underline{v}_2$, low type will be deviating her second bid to lower values in order for strictly higher payoff. So we have to conclude that when $p > \frac{\underline{v}_2}{\underline{v}_1}$, to support bid $(\underline{v}_2, \underline{v}_2)$ for low type, first bids of low type and high type must put zero probability on values lower than $\underline{v}_2$, i.e. the atom at $\underline{v}_2$ must be of size 1 [5].

[5] We will discuss this scenario in more detail in lemma 8 and theorem 5.

Results in the last paragraph indicate that for equation in lemma 1, we have to consider possible ties at $\underline{v}_2$ and 0 since there may be two atoms. Low type is bidding positive bids so atom at 0 will not lead to ties. For atom at $\underline{v}_2$, our tie breaking rules will dictate each bidder to get their first object when high type bids $\underline{v}_2$ and low type bids $(\underline{v}_2, \underline{v}_2)$, which seems to be in contradiction with equations in lemma 1. But note that low type's marginal valuation of second object is $\underline{v}_2$. Low type will not get any net surplus from her second bid in this scenario regardless of winning the second object or not. So our equation in lemma 1 works trivially for atoms at $\underline{v}_2$.

With lemmas established in this subsection, we may conclude common upper bound for all cases.

Corollary 1.1. *Equilibrium distributions should have the common upper bound.*

Proof. When upper bound is smaller than $\underline{v}_2$ or $p < \frac{\underline{v}_2}{\underline{v}_1}$, we know from lemmata in this subsection that no ties happen with positive probability. And hence any bidder can get the object with certainty by bidding at a common upper bound, while bidding beyond the upper bound only implies paying strictly higher and getting lower net payoff.

When upper bound is $\underline{v}_2$ and $p > \frac{\underline{v}_2}{\underline{v}_1}$, we know that ties happen when high type bids $\underline{v}_2$ and low type bids $(\underline{v}_2, \underline{v}_2)$. Tie breaking rules will assign each bidder one object in this case. If low type raises her first bid, she still gets one object but she pays more; if low type raises her both bid, she still gets two objects but she gets less net payoff Since marginal valuation of low type's second object is $\underline{v}_2$. In all, low type will not want to increase her bids. High type will not raise her bids as well since it only means she pay more as well. ☐

4.4 Pure Strategy Equilibrium

Lemma 7. *Bidding the smaller marginal valuation for both bids is a pure strategy equilibrium when $p = 0$ or 1.*

Proof. If $p = 0$ (or 1), bidding 0 (or $\underline{v}_2$) for both objects will be an equilibrium. Bidding at the marginal valuations of the second object guarantees each bidder exactly one object. Increasing bids will decrease net payoff: firstly it would be a strictly dominated strategy for low type to use a second bid higher than marginal valuation of that object, and secondly increasing first bid will only mean the bidder pays more for the only object she can win. Decreasing just one bid will not change the allocation but decreasing both bids will lead to 0 payoff since the highest two bids will both come from opponent. ☐

We have mentioned that when $p \geq \frac{\underline{v}_2}{\underline{v}_1}$, high type may bid $\underline{v}_2$ and low type will bid $(\underline{v}_2, \underline{v}_2)$, and now we will formally show this is actually a pure strategy equilibrium:

Lemma 8. *For $p \in (0, 1)$, there is a unique pure strategy symmetric equilibrium in our multi-item auction when $p \geq \frac{\underline{v}_2}{\underline{v}_1}$.*

Proof. First we suppose in this proof that high type is bidding non-negative (b_{h1}, b_{h2}) with $b_{h1} \geq b_{h2}$ and low type is bidding non-negative (b_{l1}, b_{l2}) with $b_{l1} \geq b_{l2}$. We propose $b_{h1} = b_{l1} = b_{l2} = \underline{v}_2$ and $b_{h2} = 0$ as the equilibrium strategy. Each type is getting one object by the current pure strategy under our tie-breaking rule. If high type increases her first bid, she still wins 1 object but she has to pay more. If she decreases her first bid she will win nothing when facing a low type. Range of p will indicate that her payoff will be (weakly) lower since $\bar{v}_1 - \underline{v}_2 \geq (1-p)\bar{v}_1$, where the right hand side is the highest payoff high type can get by bidding lower than $\underline{v}_2$ [6]. For a low type, decreasing only one bid does not change the allocation of auction. Decreasing both bids will strictly lower her payoff because when she faces another low type she can not win and range of p guarantees $\underline{v}_1 - \underline{v}_2 \geq (1-p)\underline{v}_1$. Second bid of low type is exactly at the marginal valuation of her second object and hence reason to eliminate increasing bids for low type is similar to the reason used in lemma 7.

We now move on to check uniqueness. We still assume the non-negative bids from high type are $b_{h1} \geq b_{h2}$ and bids from low type are $b_{l1} \geq b_{l2}$. If $b_{l2} = \underline{v}_2$, tie-breaking rule will predict b_{L1} will either be $\underline{v}_2$ (or slightly higher than $\underline{v}_2$ [7]). We can analyze all possible cases:

1. If $\bar{v}_1 - \underline{v}_2 \geq (1-p)\bar{v}_1$, or equivalently $p > \frac{\underline{v}_2}{\bar{v}_1}$, high type will bid $\underline{v}_2$ (or $\underline{v}_2 + \epsilon$)
   (a) if $p > \frac{\underline{v}_2}{\underline{v}_1}$, we are in the proposed equilibrium;
   (b) if $p \in (\frac{\underline{v}_2}{\bar{v}_1}, \frac{\underline{v}_2}{\underline{v}_1})$, high type is glad to bid $\underline{v}_2$ (or $\underline{v}_2 + \epsilon$) but low type will want to deviate to bid close to 0 since bidding just above 0 will give $(1-p)\underline{v}_1$, which is greater than $\underline{v}_1 - \underline{v}_2$ under this range of p. And high type will consequently deviate to just outbid low type so that high type could get an payoff close to $\bar{v}_1$.

2. If $p < \frac{\underline{v}_2}{\bar{v}_1}$, first bid of high type will be just above 0 since $(1-p)\bar{v}_1 > \bar{v}_1 - \underline{v}_2$. Low type will have incentive to decrease her second bid to just outbid high type. First bid of low type can either stay at $\underline{v}_2$ or be just above 0. The former choice generates payoff $\underline{v}_1 - \underline{v}_2 + (1-p)\underline{v}_2$ for low type while the second choice generates payoff of at least $(1-p)(\underline{v}_1 + \underline{v}_2)$. Given $p < \frac{\underline{v}_2}{\bar{v}_1} < \frac{\underline{v}_2}{\underline{v}_1}$, we have $\underline{v}_1 - \underline{v}_2 + (1-p)\underline{v}_2 \leq (1-p)(\underline{v}_1 + \underline{v}_2)$. So low type should also deviate her first bid to just above 0. It is easy to check the deviating payoff $(1-p)(\underline{v}_1 + \underline{v}_2) \geq \underline{v}_1 - \underline{v}_2$ when $p \leq \frac{2\underline{v}_2}{\underline{v}_1 + \underline{v}_2}$. But note that $\frac{2\underline{v}_2}{\underline{v}_1 + \underline{v}_2} > (\frac{\underline{v}_2}{\underline{v}_1} >) \frac{\underline{v}_2}{\bar{v}_1}$. So as long as first bid of high type is close to 0, it it optimal for low type to decrease her bids to slightly outbid high type.

Similarly, suppose $b_{l2} < \underline{v}_2$, we can still first conclude that $b_3 = b_{l2}$ or $b_{l2} + \epsilon$ because of the tie-breaking rules and we want a symmetric equilibrium. We can still do a case-by-case analysis:

1. When $p > \frac{b_{l2}}{\bar{v}_1}$, $b_{h1} = b_{l2}$ (or $b_{l2} + \epsilon$ as above) since $\bar{v}_1 - b_{l2} > (1-p)\bar{v}_1$. Low type has incentive to raise second bid (and hence her first bid) slightly to outbid b_{h1};

2. When $p < \frac{b_{l2}}{\bar{v}_1}$, b_{h1} is just above to 0. Low type will have incentive to decrease her second bid. She has 2 choices for her first bid now: either stay at b_{l2} or just be above 0. But bidding first bid at b_{l2} will generate lower payoff than bidding first bid at 0: $\underline{v}_1 - b_4 + (1-p)\underline{v}_2 < (1-p)(\underline{v}_1 + \underline{v}_2)$ since $p < \frac{b_{l2}}{\bar{v}_1} < \frac{b_{l2}}{\underline{v}_1}$. So low type will deviate her first bid to just above 0 as well. Note that $(1-p)(\underline{v}_1 + \underline{v}_2) \geq \underline{v}_1 - b_4$ when $p < \frac{\underline{v}_2 + b_{l2}}{\underline{v}_1 + \underline{v}_2}$. And it is easy to check $\frac{\underline{v}_2 + b_{l2}}{\underline{v}_1 + \underline{v}_2} > \frac{b_{l2}}{\bar{v}_1}$. So it is optimal for low type to decrease both bids when high type is bidding close to 0.

□

[6] If a high type bids lower than $\underline{v}_2$, she can only win when facing another high type. So a high type would rather bid 0 when she is bidding below $\underline{v}_2$.

[7] If a low type knows another low type is bidding $\underline{v}_2 + \epsilon$ and $\underline{v}_2$ ($\epsilon > 0$), she will respond by bidding $\underline{v}_2 + \epsilon$ and $\underline{v}_2$ because she can only win 0.5 objects on average if her first bid is $\underline{v}_2$. If a low type knows another low type is bidding $\underline{v}_2$ and $\underline{v}_2$, she will respond by bidding $\underline{v}_2$ and $\underline{v}_2$.

We will see later that this pure strategy equilibrium is just a specific case of the mixed strategy equilibria.

Remark. *Note that pure strategy in this subsection is efficient since high and low type each get one object.*

5 Mixed Strategy Equilibrium

In this section, we will formally show the symmetric mixed strategy Bayesian Nash Equilibrium by range of p, the probability of low type. With theorem 1, we will always assume that first bid of high type follows distribution F_H and first and second bids of low type follow distributions G_{L1} and G_{L2} respectively. High type will bid b_{h1} where $b_{h1} \geq 0$ and low type will bid by (b_{l1}, b_{l2}) with $b_{l1} \geq b_{l2} \geq 0$.

The mixed strategy equilibria will have two main features: support for bids of high is a subset of support of bids for low types (i.e. overlapping support) and low type will bid identically for two objects, where the first feature implies when high type bids low and high types will share common support. As argued in introduction, since bidders understand that their higher bids are competing with other's lower bids, they will have incentive to bid lower higher bids for higher net surplus. On the other hand, knowing first bids will be lower, bidder will be submitting higher second bids for a better chance of winning. The overlapping support and identical bids will make our mixed strategy equilibria inefficient since there will be positive probability that one low type gets both objects. An efficient allocation should let each bidder get exactly one object since we assume high type has valuation $\bar{v}_1, \bar{v}_2$ while low type has valuation $\underline{v}_1, \underline{v}_2$ with $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. The top two highest marginal valuations will always come from each bidder's highest valuation.

5.1 Case $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$

For $p < \frac{\underline{v}_2}{\underline{v}_1}$, we will introduce the mixed strategy equilibria by different range of marginal valuations and by range of p.

5.1.1 When $p \leq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$

We can summarize results in this subsection by a theorem:

Theorem 2. *Suppose $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$ and $p \leq \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$. Low type will be bidding the same price for her bids with distribution $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$ and high type is bidding according to distribution $F_H(x) = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + x)x}{(\underline{v}_2 - x)(\bar{v}_1 - x)} + \frac{\underline{v}_2 - (2\bar{v}_1 - \underline{v}_1)p}{(\underline{v}_2 - x)(1-p)}$ on common support $[0, \bar{v}_1 p]$*

This theorem implies that when p is low or when a low type appears rarely, high type will focus on getting a high net payoff when she wins. And high type will accomplish this goal by putting an atom at 0. Lemma 9 and 10 will be dealing with equilibrium distributions of low and high type respectively:

Lemma 9. *Low type will be bidding the same price for her bids with distribution $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$ on support $[0, \bar{v}_1 p]$.*

Proof. High type will be facing indifferent condition: $(1-p)[\bar{v}_1 - b_{h1}] + p[G_{L2}(b_{h1})(\bar{v}_1 - b_{h1})] = \bar{v}_1(1-p)$. Solving the indifferent condition, we can get $G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1 - x)}$. And $G_{L2}(x) = 1$ when $x = \bar{v}_1 p$.

With $G_{L2}(\cdot)$ calculated, we now compute low type's expected payoff of her first bid (denoted as b_{l1}): $\frac{(1-p)b_{l1}}{p(\bar{v}_1-b_{l1})}p(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \frac{\bar{v}_1}{\bar{v}_1-b_{l1}}(1-p)(\underline{v}_1 - b_{l1})$. Note that first order derivative of low type's expected payoff w.r.t. b_{l1} is negative, which implies that low type's expected payoff of her first bid is a decreasing function. If first and second bids for low type are (b_{l1}, b_{l2}) respectively, low type should pick the smallest $b_{l1} \geq b_{l2}$. So for any given y, we must let $b_{l1} = b_{l2}$ and hence $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\bar{v}_1-x)}$.

We now argue that low type will not deviate to bid differently: for a mixed strategy equilibrium, a bidder will be having a fixed expected payoff for all bids that she is randomizing with. We have computed that expected payoff for first bid for low type is a decreasing function. Fixed expected payoff implies expected payoff for second bid of low type must be an increasing function. If a bidder is bidding identically by $(b_{l1}, b_{l2}) = (x, x)$ currently for any positive and real x, she will not want to deviate only one bid since derivative of expected payoff of first bid is negative and derivative of expected payoff of second bid is positive. If the bidder deviates both bids to $(b_{l1}, b_{l2}) = (z_1, z_2)$ where $z_1 > x$ and $z_2 < x$, we can treat this scenario as deviating one bid from $(b_{l1}, b_{l2}) = (z_1, z_1)$ or (z_2, z_2) to $(b'_{l1}, b'_{l2}) = (z_1, z_2)$. And hence our previous argument still works since the monotone condition for expected payoff of first and second bid will give bidder incentive to decrease her first bid and increase her second bid until they are identical. No bidders will bid higher than the upper bound since biding exactly at the upper bound means that a low type will win both objects with certainty. And hence bidding above upper bound only indicates lower payoff. $\square$

We call results where the low type is bidding identical bids the **perfectly correlated equilibrium**.

We now construct the distribution for high type's first bid. Our tie-breaking rules introduced in section 2 and the new auction rules discussed in subsection 3.2 guarantees high type to get one object when facing another high type. So high type may choose to put an atom at 0 for high net surplus when there are high probability that she faces another high type in the population.

Lemma 10. *High type is bidding according to distribution* $F_H(x) = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+x)x}{(\underline{v}_2-x)(\bar{v}_1-x)} + \frac{\underline{v}_2-(2\bar{v}_1-\underline{v}_1)p}{(\underline{v}_2-x)(1-p)}$ *when* $p \leq \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ *with support* $[0, p\bar{v}_1]$.

Proof. With G_{L1}, G_{L2} computed, we look at low type's indifferent condition to compute high type's distribution when $p = \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$:
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p)$. With $b_{l1} = b_{l2}$, we have $(1-p)F_H(b_{l1})(\underline{v}_2 - b_{l1}) = (1-p)b_{l1} - p\frac{(1-p)b_{l1}}{p(\bar{v}_1-b_{l1})}(\underline{v}_1 + \underline{v}_2 - 2b_{l1})$
$= (1-p)b_{l1} - \frac{(1-p)b_{l1}}{(\bar{v}_1-b_{l1})}(\underline{v}_1 + \underline{v}_2 - 2b_{l1}) = \frac{\bar{v}_1-\underline{v}_1-\underline{v}_2+b_{l1}}{\bar{v}_1}(1-p)b_{l1}$. So $F_H(x) = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+x)x}{(\underline{v}_2-x)(\bar{v}_1-x)}$. When $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$, $F_H(x)$ is always positive. $F_H(x) = 1$ when $x = \frac{\bar{v}_1\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$. Comparing upper bounds for G_L's and F_H, we conclude that when $p = \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ F_H is an atomless distribution.

When $p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ we have to put an atom with size T on F_H. $G_{L1} = G_{L2}$ is still true since G_{L2} is computed from high type's indifferent condition. So indifferent condition for low type is
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(y)(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p) + (1-p)T\underline{v}_2$
with $b_{l1} = b_{l2}$, we have $F_H(x) = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+x)x}{(\underline{v}_2-x)(\bar{v}_1-x)} + \frac{T\underline{v}_2}{\underline{v}_2-x}$. We need to solve T. Let $x = \bar{v}_1 p$,
$1 = \frac{T\underline{v}_2}{\underline{v}_2-\bar{v}_1 p} + \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}p)\bar{v}_1 p}{(\underline{v}_2-\bar{v}p)\bar{v}_1(1-p)}$; $1 = \frac{T\underline{v}_2}{\underline{v}_2-\bar{v}_1 p} + \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1 p)p}{(\underline{v}_2-\bar{v}p)(1-p)}$; $\frac{T\underline{v}_2}{\underline{v}_2-\bar{v}_1 p} = 1 - \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1 p)p}{(\underline{v}_2-\bar{v}_1 p)(1-p)}$;
$T\underline{v}_2 = (\underline{v}_2 - \bar{v}_1 p) - \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1 p)p}{1-p}$; $T = \frac{\underline{v}_2-(2\bar{v}_1-\underline{v}_1)p}{\underline{v}_2(1-p)}$. So $F_H(x) = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2+x)x}{(\underline{v}_2-x)(\bar{v}_1-x)} + \frac{\underline{v}_2-(2\bar{v}_1-\underline{v}_1)p}{(\underline{v}_2-x)(1-p)}$. It is easy

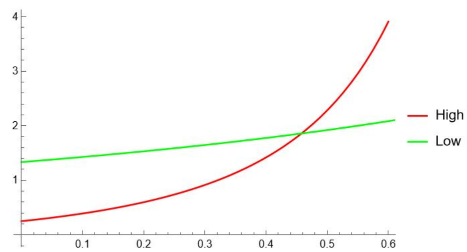
to see that when $x < p\bar{v}_1 < \underline{v}_2$, function F_H is increasing. This F_H function coincide with the distribution computed last paragraph when $p = \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$.

The high type will not bid above the upper bound since a high type can secure the object by bidding $\bar{v}_1 p$ and bidding above that value only means paying more to get the object. $\square$

Remark. *There will be a positive probability that one low type gets both objects due to common support of mixed strategy equilibrium distributions. And the equilibrium strategy is not necessarily efficient.*

Graphical Illustration

We can illustrate the theorems in this subsection by showing plots of density functions with $\bar{v}_1 = 3, \underline{v}_1 = 2, \underline{v}_2 = 1$. Let $p = \frac{1}{5}$, and probability density functions will be



with support being $[0, \frac{3}{5}]$. Note that there will be an atom of size $\frac{1}{4}$ for distribution of mixed strategy of high type at 0.

5.1.2 When $\frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$

In this range of p, we show that high type is randomizing in interval $[a_1, a_2]$ with $0 < a_1 < a_2$ since low type is now appearing with a decent probability and high type will have to bid higher to guarantee some wins. Both bids of low type are randomized in interval $[0, a_1] \cup [a_1, a_2]$. If we denote bids from low type as (b_{l1}, b_{l2}), We will have three different indifferent conditions for low type:

$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$ when $b_{l2} \leq b_{l1} \leq a_1$,

$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$ when $b_{l2} \leq a_1 \leq b_{l1}$, and

$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p)$ when $a_1 \leq b_{l2} \leq b_{l1}$.

We call support of the three indifferent conditions R_1, R_2 and R_3 respectively, i.e.

$R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$, $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \leq a_1 \leq b_{l1}\}$ and $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$.

First, we summarize results in this subsection via a theorem by ranges of bids and probability low type appears in the population:

Theorem 3. *Suppose $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$ and $\frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$.*

1. *High type will bid by distribution $F_H(x) = \frac{x}{\underline{v}_2 - x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1 - x)(\underline{v}_2 - x)}(\underline{v}_1 + \underline{v}_2 - 2x)$ with support $[a_1, a_2]$, where $a_2 = \frac{p\underline{v}_1 + \underline{v}_2}{2}$ and a_1 solves $F_H(a_1) = 0$.*

2. *In region $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$, low type will bid by distributions $G_{L1}(x) = G_{L2}(x) = \frac{p\bar{v}_1 - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ with support $[a_1, a_2]$.*

3. *In region* $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \le b_{l1} \le a_1\}$, *letting* $p^* = \frac{2\bar{v}_1^2\underline{v}_1 - \bar{v}_1(\underline{v}_1+\underline{v}_2)^2 + \underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_2^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$ $- \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_2^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_2^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$ *and* $C = \frac{(\bar{v}_1-\underline{v}_1)^2+\underline{v}_2^2+\underline{v}_1\underline{v}_2-2\bar{v}_1\underline{v}_2+p(\bar{v}_1\underline{v}_1-\underline{v}_1^2+\bar{v}_1^2-\bar{v}_1\underline{v}_2)}{2p(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)}$ $+\frac{1}{2}\frac{(\bar{v}_1-\underline{v}_1)\sqrt{\bar{v}_1^2(1+p)^2+\underline{v}_2^2(2-2p+p^2)+2\underline{v}_1\underline{v}_2(1-p+p^2)+\underline{v}_1^2(1-2p+2p^2)-2\bar{v}_1[\underline{v}_2(2-p+p^2)+\underline{v}_1(1-p+2p^2)]}}{p(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}$
   (a) *when* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_2} < p < p^*$, *low type will bid the same according to* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$;
   (b) *when* $p^* < p < \frac{\underline{v}_2}{\underline{v}_1}$, *low type will bid her 1st bid according to* $G_{L1}(x) = \frac{C}{\underline{v}_2-x}$ *and 2nd bid by* $G_{L2}(y) = \frac{(1-p)y-pC}{p(\underline{v}_1-y)}$ *on interval* $[a_3, a_1]$ *for* C *defined above. The two marginal distributions are related by* $G_{L1}(x) = G_{L2}(h(x))$ *where* $h(x) = \frac{Cp(\underline{v}_1+\underline{v}_2-x)}{Cp+(1-p)\underline{v}_2-(1-p)x}$. *Low type will bid identically by* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in* $[0, a_3]$. $a_3 < a_1$ *solve* $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$.
4. *Region* $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \le a_1 \le b_{l1}\}$ *has zero probability under distributions of bids for low type.*

The first two points are shown in lemma 11. Point 3.a comes from lemma 12 and point 3.b is dealt in lemma 13. Point 4 is a direct result of point 2. There will be a positive probability that one low type gets both objects due to common support of mixed strategy equilibrium distributions. And the equilibrium strategy is not necessarily efficient.

Lemma 11. *High type will bid by distribution* $F_H(x) = \frac{x}{\underline{v}_2-x} - \frac{\bar{v}_1p-a_2+(1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1+\underline{v}_2-2x)$ *with support* $[a_1, a_2]$. *In region* $R_3 = \{(b_{l1}, b_{l2}) : a_1 \le b_{l2} \le b_{l1}\}$, *low type will bid by distributions* $G_{L1}(x) = G_{L2}(x) = \frac{p\bar{v}_1-a_2+(1-p)x}{p(\bar{v}_1-x)}$ *with support* $[a_1, a_2]$. $a_2 = \frac{p\underline{v}_1+\underline{v}_2}{2}$ *and* $a_1 = \frac{-\bar{v}_1+\underline{v}_1+2\underline{v}_2-\bar{v}_1p-\underline{v}_2p+\sqrt{(\bar{v}_1-\underline{v}_2-2\underline{v}_2+\bar{v}_1p+\underline{v}_2p)^2-(2-2p)(\underline{v}_1\underline{v}_2+\underline{v}_2^2-2\bar{v}_1\underline{v}_2p+\underline{v}_1^2p-2\bar{v}_1\underline{v}_2p+\underline{v}_1\underline{v}_2p)}}{1-p}$.

Proof. For high type, the indifferent condition will be
$(1-p)[\bar{v}_1 - b_{h1}] + p[G_{L2}(b_{h1})(\bar{v}_1 - b_{h1})] = (\bar{v}_1 - a_1)(1-p) + p(\bar{v}_1 - a_1)G_{L2}(a_1) \iff$
$G_{L2}(x) = \frac{1}{p(\bar{v}_1-x)}[(1-p)(x-a_1) + p(\bar{v}_1-a_1)G_{L2}(a_1)]$. By $G_{L2}(a_2) = 1$ we have $G_{L2}(a_1) = 1 + \frac{a_1-a_2}{p(\bar{v}_1-a_1)}$.
Plugging $G_{L2}(a_1)$ into high type's indifferent condition and it will become
$(1-p)[\bar{v}_1 - b_{h1}] + p[G_{L2}(b_{h1})(\bar{v}_1 - b_{h1})] = \bar{v}_1 - a_2 \iff G_{L2}(x) = \frac{p\bar{v}_1-a_2+(1-p)x}{p(\bar{v}_1-x)}$.

With $G_{L2}(x) = \frac{p\bar{v}_1-a_2+(1-p)x}{p(\bar{v}_1-x)}$, expected payoff for first bid of low type on region R_3 will be
$pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \frac{p\bar{v}_1-a_2+(1-p)b_{l1}}{(\bar{v}_1-b_{l1})}(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1})$
$= \frac{(\bar{v}_1p-a_2)+(1-p)\bar{v}_1}{\bar{v}_1-b_{l1}}(\underline{v}_1 - b_{l1})$, which will have a negative derivative w.r.t b_{l1}. So for region R_3 we will still have the perfectly correlated equilibrium for low type. Plugging $G_{L1}(x) = G_{L2}(x)$ into indifferent condition for low type on region R_3, we have $(1-p)F_H(b_{l2})(\underline{v}_2 - b_{l2}) = (1-p)b_{l2} - p\frac{\bar{v}_1p-a_2+(1-p)b_{l2}}{p(\bar{v}_1-b_{l2})}(\underline{v}_1 + \underline{v}_2 - 2b_{l2})$ and hence $F_H(x) = \frac{x}{\underline{v}_2-x} - \frac{\bar{v}_1p-a_2+(1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$. We can solve $a_2 = \frac{p\underline{v}_1+\underline{v}_2}{2}$ by letting $F_H(a_2) = 1$ and $a_1 = \frac{-\bar{v}_1+\underline{v}_1+2\underline{v}_2-\bar{v}_1p-\underline{v}_2p+\sqrt{(\bar{v}_1-\underline{v}_1-2\underline{v}_2+\bar{v}_1p+\underline{v}_2p)^2-(2-2p)(\underline{v}_1\underline{v}_2+\underline{v}_2^2-2\bar{v}_1\underline{v}_1p+\underline{v}_1^2p-2\bar{v}_1\underline{v}_2p+\underline{v}_1\underline{v}_2p)}}{1-p}$ by solving $F_H(a_1) = 0$.

Computation will show that $F_H(x)$ is an increasing function when $F_H(x) \ge 0$: we need to check $F_H(x)$ is monotonically increasing when $x > a_1$, i.e. $\frac{dF_H(x)}{dx} = \frac{(2\bar{v}_1-\underline{v}_1-\underline{v}_2)[2x^2-2(p\underline{v}_1+\underline{v}_2)x+p\underline{v}_1(\bar{v}_1+\underline{v}_2)-\bar{v}_1\underline{v}_2+\underline{v}_2^2]}{2(p-1)(\underline{v}_2-x)^2(\bar{v}_1-x)^2} > 0$ when $x > a_1$. We need $2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2 < 0$ when $x > a_1$ given $p \in (0,1)$. Since $2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2$ is decreasing when $x < \frac{p\underline{v}_1+\underline{v}_2}{2}$, we only need $2x^2 - 2(p\underline{v}_1 + \underline{v}_2)x + p\underline{v}_1(\bar{v}_1 + \underline{v}_2) - \bar{v}_1\underline{v}_2 + \underline{v}_2^2 < 0$ when $x = a_1$. Computation shows the condition we need is $p \in (\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}, \frac{\underline{v}_2}{\underline{v}_1})$ for any $\bar{v}_1 > \underline{v}_1 > \underline{v}_2$. □

Since we have perfectly correlated equilibrium on region R_3, we conclude that region R_2 will be at most zero-measure. We can omit R_2 and look at R_1:

Lemma 12. *When* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p$ $< \frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_2^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} - \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$ *low type will bid according to* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in region* $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$.

Proof. There is only low type with indifferent condition $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$. If we assume $b_{l1} = b_{l2}$ (i.e. perfectly correlated), $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$. Then expected payment for first bid of low type is $pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \frac{(1-p)x(\underline{v}_1-b_{l1})}{\underline{v}_1+\underline{v}_2-2b_{l1}} + (1-p)(\underline{v}_1 - b_{l1}) = \frac{(1-p)(\underline{v}_1-b_{l1})(\underline{v}_1+\underline{v}_2-b_{l1})}{\underline{v}_1+\underline{v}_2-2b_{l1}}$. Derivative of payoff w.r.t. first bid is $\frac{(p-1)(2b_{l1}^2-2(\underline{v}_1+\underline{v}_2)b_{l1}+\underline{v}_2(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2b_{l1})^2}$. On the other hand, expected payoff of second bid from low type is $pG_{L2}(b_{l2})(\underline{v}_2 - b_{l2}) = \frac{(1-p)b_{l2}(\underline{v}_2-b_{l2})}{\underline{v}_1+\underline{v}_2-2b_{l2}}$ with derivative $\frac{(1-p)(2b_{l2}^2-2(\underline{v}_1+\underline{v}_2)b_{l2}+\underline{v}_2(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2b_{l2})^2}$, which is exactly the opposite of derivative of expected payoff of first bid. It is straight forward to check that G functions in the lemma will coincide with G functions in the previous lemma at exactly a_1 as desired.

As long as payoff from first bid is decreasing, payoff from second bid will be increasing. The common term on numerator of those derivatives is $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2)$ and $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2) > 0$ is equivalent to $x < \frac{\underline{v}_1+\underline{v}_2}{2} - \frac{\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$. If $F_H(x) = 0$ at a_1, we want $a_1 \leq \frac{\underline{v}_1+\underline{v}_2}{2} - \frac{\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ to support equilibrium bids in region R_1, which generates range of p to be $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} - \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)}$.

Given distributions on R_1 and R_3, we check high type will not deviate. If high type bids below a_1, she will get expected payoff $pG_{L2}(b_{h1})(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1}) = \frac{(1-p)b_{h1}}{(\underline{v}_1+\underline{v}_2-2xb_{h1})}(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1})$ $= \frac{(1-p)(\bar{v}_1-b_{h1})(\underline{v}_1+\underline{v}_2-b_{h1})}{\underline{v}_1+\underline{v}_2-2b_{h1}}$ with derivative $\frac{(p-1)(2b_{h1}^2-2(\underline{v}_1+\underline{v}_2)b_{h1}-(\bar{v}_1-\underline{v}_1-\underline{v}_2)(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2b_{h1})^2}$. We want $2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)$ to be negative for a positive first order derivative. Note that $2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)$ is decreasing when $b_{h1} < \frac{\underline{v}_1+\underline{v}_2}{2}$. If $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$, we have a positive derivative: if we plug $b_{h1} = 0$ into $2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2)$, it will become $-(\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2) < 0$. So we will see $(p-1)(2b_{h1}^2 - 2(\underline{v}_1 + \underline{v}_2)b_{h1} - (\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(\underline{v}_1 + \underline{v}_2))$ is always positive.

We have argued in proof of lemma 9 that a low type will not deviate within a region when she is bidding identical bids. We can now eliminate "across region deviation", which means low type deviates from bidding identically to put one bid in region R_1 and the other one in region R_3. We have established that expected payment for second bid for low type is increasing in region R_1 and R_3 and expected payment for first bid is decreasing in region R_1 and R_3. If a low type deviates to bid a (b'_{l1}, b'_{l2}) with $b'_{l2} < a_1 < b'_{l1}$, monotone condition for expected payoff for each bid will require the low type to increase her second bid and decrease first bid. □

Remark. *We assume that bids in region R_1 will be as low as 0. We can eliminate situations where lower bound of bids is strictly positive since we normalize $\bar{v}_2 = 0$. If lower bounds of both types' bids are strictly* *positive, high type will deviate to bid below b since bidding at the lower bound will generate a strictly lower payoff than bidding* 0 *given that distributions of bids from low type are atomless.*

In region R_1, so far we assumed $b_{l1} = b_{l2}$ without any justification. Now we move to check availability of non-identical bids in region R_1:

Lemma 13. *When* $\frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} - \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} < p < \frac{\underline{v}_2}{\underline{v}_1}$*, first bid of low type follows distribution* $G_{L1}(x) = \frac{C}{\underline{v}_2-x}$ *and second bid of low type follows* $G_{L2}(x) = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$ *in interval* $I = [a_3, a_4]$ *where* $C = \frac{(\bar{v}_1-\underline{v}_1)^2+\underline{v}_2^2+\underline{v}_1\underline{v}_2-2\bar{v}_1\underline{v}_2+p(\bar{v}_1\underline{v}_1-\underline{v}_1^2+\bar{v}_1^2-\bar{v}_1\underline{v}_1)}{2p(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)}$ $+\frac{1}{2}\frac{(\bar{v}_1-\underline{v}_1)\sqrt{\bar{v}_1^2(1+p)^2+\underline{v}_2^2(2-2p+p^2)+2\underline{v}_1\underline{v}_2(1-p+p^2)+\underline{v}_1^2(1-2p+2p^2)-2\bar{v}_1[\underline{v}_2(2-p+p^2)+\underline{v}_1(1-p+2p^2)]}}{p(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}$*. And low type will bid identically on* $[0, a_3]$ *by* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$*. What's more, endpoints of interval* $I = [a_3, a_4]$ *are determined by* $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$ *with* a_4 *being equivalent to* a_1 *from lemma 11.*

Proof. In the previous lemma we studied what if we assume low type bids identically. We can now assume that for any given b_{l1}, we have an optimal $b_{l2} < b_{l1}$ such that (b_{l1}, b_{l2}) optimizes expected payoff for low type. We further assume I is the first non-trivial (i.e. positive measure) interval where $b_{l2} = h(b_{l1})$ (with $(h(b_{l1}) < b_{l1})$ solves the first order condition on interior of interval I. We denote $a_3 = \inf_{x\in I} I$ and $a_4 = \sup_{x\in I} I$ so $I = [a_3, a_4]$. By construction we have $h(a_3) = a_3$ and $h(a_4) = a_4$. Indifferent condition in region R_1 is $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p)$. If we take derivative with respect to b_{l2} for any fixed b_{l1}, we get $g_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) - G_{L1}(b_{l2}) = 0$, with g being derivative of G functions. Solving the differential equation, we have $G_{L1}(x) = \frac{C}{\underline{v}_2-x}$ for some constant C, and hence $G_{L2}(y) = \frac{(1-p)y-pC}{p(\underline{v}_1-y)}$. We can formally define $a_3 < a_4$ to be solution to $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)}$.

Since $\frac{(1-p)x-pC}{p(\underline{v}_1-x)} = \frac{C}{\underline{v}_2-x}$ at a_3, a_4, we can rearrange the equation above to $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ at a_3, a_4. So we need to find out $a_3 < a_4$ such that $C = \frac{(1-p)a_3}{(\underline{v}_1+\underline{v}_2-2a_3)}(\underline{v}_2 - a_3) = \frac{(1-p)a_4}{(\underline{v}_1+\underline{v}_2-2a_4)}(\underline{v}_2 - a_4)$. But we know that function $\frac{(1-p)x}{(\underline{v}_1+\underline{v}_2-2x)}(\underline{v}_2 - x)$ is increasing when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1-\underline{v}_2^2}}{2}$ and decreasing when $x > \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ by proof in the previous lemma. So to make equation $\frac{(1-p)a_3}{(\underline{v}_1+\underline{v}_2-2a_3)}(\underline{v}_2 - a_3) = \frac{(1-p)a_4}{(\underline{v}_1+\underline{v}_2-2a_4)}(\underline{v}_2 - a_4)$ valid, we must make $a_3 < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_2^2-\underline{v}_2^2}}{2} < a_4$. What's more, for the right neighbourhood of a_4, we are in the perfectly correlated equilibrium by construction. To support such an equilibrium, our previous result requires that $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2) > 0$, which is positive when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ or $x > \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$. So right-neighbourhood of a_4 must be greater than $\frac{\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$, which is impossible since $\frac{\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$ is already greater than $\underline{v}_2$. So we conclude $a_4 = a_1$. Although we assume I to be the first interval where first and second bids of low type differ, it is actually the only interval since it ends at endpoint of region R_1.

Although we have a specific relation between first and second bid by $b_{l2} = h(b_{l1})$, it actually does not matter if low type deviates in the interval I, because payoff from first and second bid of low type are constructed to be constant at respectively $\underline{v}_1(1-p) - pC$ and pC [8]. If second bid of low type deviates downward to become smaller than a_3, the optimal deviating bid should be bidding at a_3 because we know that for values lower than a_3 low type is bidding identically. And in such a perfectly correlated equilibrium

[8] $pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) = \underline{v}_1(1-p) - pC$ and $pG_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) = pC$

expected payment from second bid is strictly increasing. Similarly if first bid of low type deviates upward to be higher than a_4, the deviating bid better be $a_4 = a_1$ since in region R_3 low type will bid identically and first bid is strictly decreasing. If high type deviates to bid below a_1 in interval I, she will get $(1-p)(\bar{v}_1 - b_{l1}) + p\frac{(1-p)b_{l1}-pC}{p(\underline{v}_1-b_{l1})}(\bar{v}_1 - b_{l1}) = (\bar{v}_1 - b_{l1})[(1-p) + \frac{(1-p)b_{l1}-pC}{\underline{v}_1-b_{l1}}]$ with derivative $\frac{(\underline{v}_1-\bar{v}_1)[\underline{v}_1(-1+p)+pC]}{(\underline{v}_1-b_{l1})^2}$. We require $\underline{v}_1(-1+p) + pC < 0$ for a positive derivative so that a high type would rather bid a_1 instead of prices lower than a_1. If a high type further deviates to bid below a_3, we use the argument in proof of lemma 12 to eliminate such a deviating possibility: derivative of high type's expected payoff will be increasing as long as her bid is lower than a_1 so high type will bid a_3 when she has to bid no greater than a_3. But high type will then immediately bid $a_4 = a_1$ since her deviating payoff is an increasing function on interval (a_3, a_4).

To make the distributions consistent, we have to let $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x-pC}{p(\underline{v}_1-x)} = \frac{\bar{v}_1 p-a_2+(1-p)x}{p(\bar{v}_1-x)}$ when $x = a_4 = a_1$. The last expression is distribution of low type's bids on region R_3 (when $b_{l1} \geq b_{l2} \geq a_1$). Interpretation of the equalities above is that since $a_4 = a_1$ and $G_{L1}(a_1) = G_{L2}(a_1)$ on R_3, we should have the distribution at a_1 on interval I to be identical to the distribution at a_1 on region R_3. Condition satisfying equations above is $\frac{2\bar{v}_1^2\underline{v}_1-\bar{v}_1(\underline{v}_1+\underline{v}_2)^2+\underline{v}_2(\underline{v}_1^2+\underline{v}_2^2)}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} - \frac{\sqrt{(\bar{v}_1-\underline{v}_2)^2(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)^2(-\underline{v}_1^2+\underline{v}_2^2)}}{-\underline{v}_1^3+2\bar{v}_1^2\underline{v}_2+\underline{v}_1^2\underline{v}_2+\underline{v}_1\underline{v}_2^2+\underline{v}_1^3+2\bar{v}_1(\underline{v}_1^2-2\underline{v}_1\underline{v}_2-\underline{v}_2^2)} < p < \frac{\underline{v}_2}{\underline{v}_1}$. We actually have an expression for constant C by solving $\frac{C}{\underline{v}_2-a_1} = \frac{(1-p)a_1-pC}{p(\underline{v}_1-a_1)} = \frac{\bar{v}_1 p-a_1+(1-p)a_1}{p(\bar{v}_1-a_1)}$:

$$C = \frac{(\bar{v}_1-\underline{v}_1)^2+\underline{v}_2^2+\underline{v}_1\underline{v}_2-2\bar{v}_1\underline{v}_2+p(\bar{v}_1\underline{v}_1-\underline{v}_1^2+\bar{v}_1^2-\bar{v}_1\underline{v}_2)}{2p(-2\bar{v}_1+\underline{v}_1+\underline{v}_2)}$$
$$+\frac{1}{2}\frac{(\bar{v}_1-\underline{v}_1)\sqrt{\bar{v}_1^2(1+p)^2+\underline{v}_2^2(2-2p+p^2)+2\underline{v}_1\underline{v}_2(1-p+p^2)+\underline{v}_1^2(1-2p+2p^2)-2\bar{v}_1[\underline{v}_2(2-p+p^2)+\underline{v}_1(1-p+2p^2)]}}{p(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}.$$

Our last task is to check condition supporting perfectly correlated equilibrium holds when $b_{l1} < a_3$ and $b_{l2} < a_3$. Recall in proof of lemma 12, we require $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2) > 0$ for a perfectly correlated equilibrium. Note $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2)$ is a decreasing function when $x < \frac{\underline{v}_1+\underline{v}_2}{2}$, and hence we need to guarantee that $2x^2 - 2(\underline{v}_1 + \underline{v}_2)x + \underline{v}_2(\underline{v}_1 + \underline{v}_2)$ is positive when $x = a_3$. Some computation will show that we need condition $\underline{v}_1 + \sqrt{\frac{(\underline{v}_1^2-\underline{v}_2^2)(1-p)^2}{p^2}} + 2C \leq \frac{\underline{v}_1}{p}$. Adding this condition into $\frac{C}{\underline{v}_2-a_1} = \frac{(1-p)a_1-pC}{p(\underline{v}_1-a_1)} = \frac{\bar{v}_1 p-a_2+(1-p)a_1}{p(\bar{v}_1-a_1)}$, we still get the same range of p and expression of C. What's more, we need $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ at a_3 to support atomless distributions. And computation will show that solutions to this equation are just a_3 computed by solving $\frac{C}{\underline{v}_2-x} = \frac{\bar{v}_1 p-a_2+(1-p)a_1}{p(\bar{v}_1-x)}$. This should not be a surprising observation since we have argued in the second paragraph of this proof that $\frac{(1-p)x-pC}{p(\underline{v}_1-x)} = \frac{C}{\underline{v}_2-x}$ can be rearranged to $\frac{C}{\underline{v}_2-x} = \frac{(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ when $x = a_3$. □

Remark.

1. *$a_3 > 0$ because otherwise G_{L1} will be just be 0.*

2. *It is easy to exclude deviations above the common upper bound: all distributions are atomless at upper bounds. So bidding $(\frac{p\underline{v}_1+\underline{v}_2}{2}, \frac{p\underline{v}_1+\underline{v}_2}{2})$ ($\frac{p\underline{v}_1+\underline{v}_2}{2}$ is the upper bound in this scenario) will give low type two objects with certainty and bidding $\frac{p\underline{v}_1+\underline{v}_2}{2}$ will give high type one object with certainty. And hence bidding above the upper bound will only decrease the expected payoff for any type.*

For the two marginal distributions G_{L1}, G_{L2} introduced in lemma 13, since we computed distribution of second bid G_{L2} by solving first order condition to maximize expected payoff for any given first bid, we are actually able to characterize a functional relationship between distributions of first bid G_{L1} and second bid

G_{L2}. We will compute a function h on interval $I = [a_3, a_4]$ introduced in lemma 13 which relates G_{L1}, G_{L2} by $G_{L2}(h(x)) = G_{L1}(x)$. We are also able to prove that $h(x) < x$ in the interior of I:

Corollary 3.1. *$h(x) < x$ in interval $I = (a_3, a_4)$ and $h(x)$ is an increasing function as long as $pC < \underline{v}_1(1-p)$.*

Proof. We assume x, y evolve according to $y = h(x)$ in interval I since we solve an optimal y for any given x to maximize the expected payoff for low type in interval $I = [a_3, a_4]$. We must have $G_{L2}(h(x)) = G_{L1}(x)$ for all x in interval I by change of variable technique. Using functional forms of G_{L1} and G_{L2}, we have $h(x) = \frac{Cp(\underline{v}_1+\underline{v}_2-x)}{Cp+(1-p)\underline{v}_2-(1-p)x}$, which will be an increasing function when $pC < \underline{v}_1(1-p)$. Note that this requirement is actually the identical condition to prevent high type from deviating below a_1 constructed in the proof of previous lemma.

If we want $h(x) < x$ in the interior of I, we must have $\frac{Cp(\underline{v}_1+\underline{v}_2-x)}{Cp+(1-p)\underline{v}_2-(1-p)x} < x$, which is equivalent to $C < \frac{(1-p)(\underline{v}_2-x)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ in the interior of I. Note that $C = \frac{(1-p)a_3}{(\underline{v}_1+\underline{v}_2-2a_3)}(\underline{v}_2 - a_3) = \frac{(1-p)a_4}{(\underline{v}_1+\underline{v}_2-2a_4)}(\underline{v}_2 - a_4)$ with $a_3 < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2} < a_4$. Function $\frac{(1-p)(\underline{v}_2-x)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ is actually decreasing when $x \in (\frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}, \frac{\underline{v}_1+\underline{v}_2+\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2})$ and increasing when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{\underline{v}_1^2-\underline{v}_2^2}}{2}$. So we conclude that $C < \frac{(1-p)(\underline{v}_2-x)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ in the interior of I as desired. □

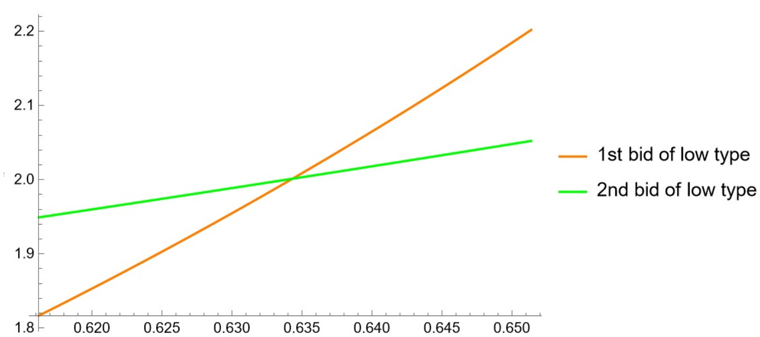
Remark. *We have two remarks to make:*

1. *By change of variable and $h(x) < x$, we now confirm that $G_{L2}(x) > G_{L2}(h(x)) = G_{L1}(x)$.*

2. *$b_{l2} = h(b_{l1})$ can be treated as an interior solution to the maximization problem $\max_{0 \le b_{l2} \le b_{l1}} p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1})$ since $b_{l2} = h(b_{l1})$ solves the first order condition: $g_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) - G_{L1}(b_{l2}) = 0$. The next question to ask is do we have an interior solution where $b_{l2} = h(b_{l1}) = b_{l1}$? If so, we must have $G_{L1}(b_{l1}) = G_{L2}(b_{l1}) = \frac{(1-p)b_{l1}}{p(\underline{v}_1+\underline{v}_2-2b_{l1})}$. First order condition to the maximization problem will become $\frac{(1-p)[2b_{l1}^2-2(\underline{v}_1+\underline{v}_2)b_{l1}+(\underline{v}_1+\underline{v}_2)\underline{v}_2)]}{p(\underline{v}_1+\underline{v}_2-2b_{l1})^2} = 0$. However, this equation only achieves 0 at 2 specific values of x, which is contradictory to our assumption of an interior solution on an interval. So there is no interior solution generating the perfectly correlated equilibrium.*

Graphical Illustration

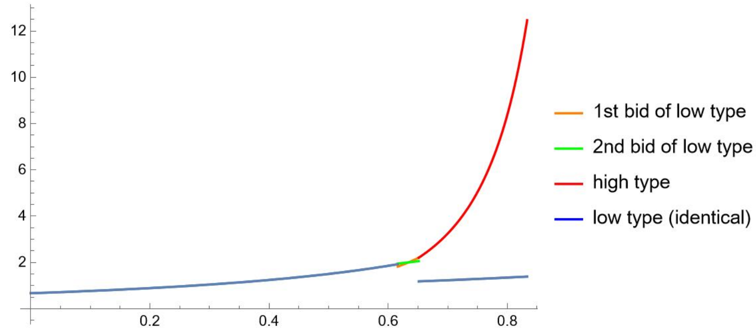
We can graphically illustrate density functions proposed in lemma 13 via:

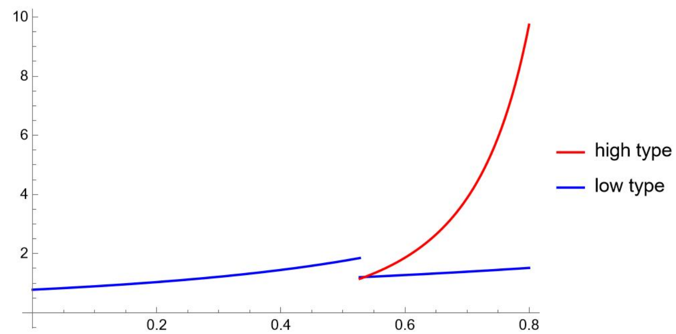


We pick $\bar{v}_1 = 3, \underline{v}_1 = 2, \underline{v}_2 = 1$ and $p = \frac{1}{3}$. Two bids from low type will be different in interval $[\frac{1}{12}(11 - \sqrt{13}), \frac{\sqrt{13}-1}{4}]$. Note that corollary 3.1 demonstrates that $G_{L1}(x) = G_{L2}(h(x))$ in interval

$[\frac{1}{12}(11-\sqrt{13}), \frac{\sqrt{13}-1}{4}]$, where $h(x) < x$ for values in $(\frac{1}{12}(11-\sqrt{13}), \frac{\sqrt{13}-1}{4})$ and $h(x) = x$ for endpoints. The graph above reflects such a property by assigning G_{L1} a flatter slope when x is small and steeper slope when x is high.

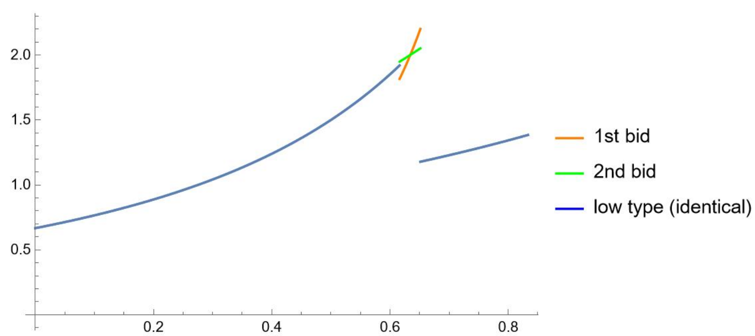
We can also illustrate density functions of equilibrium distributions graphically:





We still select $\bar{v}_1 = 3, \underline{v}_1 = 2, \underline{v}_2 = 1$ and the first graph is when $p = \frac{1}{3}$, which covers points 1,2, 3.b and 4 of theorem 3, when there is an interval where first and second bid of low type are different. Support for distributions of low type is $[0, \frac{5}{6}]$ and support for distribution of high type is $[\frac{\sqrt{13}-1}{4}, \frac{5}{6}]$. The second graph is when $p = 0.3$ where two bids of low type are always identical, as shown in points 1, 2, 3.a and 4 of theorem 3. Support for distributions of low type is $[0, \frac{4}{5}]$ and support for distribution of high type is $[0.527, \frac{4}{5}]$.

We can also illustrate density functions for only low type only:



The pdfs only differ for bids in $[\frac{1}{12}(11-\sqrt{13}), \frac{\sqrt{13}-1}{4}]$.

If we take derivative on the distribution functions and compute pdfs of low type's each bid are, we will have the following result at a_1:

Corollary 3.2. *We have the following results for pdfs of low type's distributions:*

1. *When* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < p^*$, *left derivative of* G *functions in region* R_1 *at* a_1 *is greater than right derivative of* G *functions in region* R_3 *at* a_1*;*

2. *When* $p^* < p < \frac{\underline{v}_2}{\underline{v}_1}$, *left derivative will satisfy* $\frac{dG_{L1}(x)}{dx} > \frac{dG_{L2}(x)}{dx}$ *at* a_1 *and left derivative* $\frac{dG_{L2}(x)}{dx}$ *in region* R_1 *at* a_1 *will be greater than right derivative* $\frac{dG(x)}{dx}$ *in region* R_3 *at* a_1*.*

We omit the proof since it is just direct computation.

5.2 When $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$

With $F_H(x)$ in the form in lemma 10, we should require $F_H(x)$ to be an increasing function on $(0, \bar{v}_1 p)$. $\frac{dF_H(x)}{dx} = \frac{(2\bar{v}_1-\underline{v}_1-\underline{v}_2)[x^2-2p\bar{v}_1x+p\bar{v}_1\underline{v}_2-\bar{v}_1\underline{v}_2+p\bar{v}_1^2]}{(p-1)(\bar{v}_1-x)^2(\underline{v}_2-x)^2}$. To make $F_H(x)$ an increasing function, we need $x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2$ to be negative, which means maximum of $x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2$ is negative. And $x^2 - 2p\bar{v}_1 x + p\bar{v}_1\underline{v}_2 - \bar{v}_1\underline{v}_2 + p\bar{v}_1^2$ is decreasing on $(0, p\bar{v}_1)$ so we should require $p\bar{v}_1\underline{v}_2 + p\bar{v}_1^2 < \bar{v}_1\underline{v}_2 \iff p < \frac{\bar{v}_1\underline{v}_2}{\bar{v}_1(\bar{v}_1+\underline{v}_2)} = \frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}$. When $\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2$, $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} \geq \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ and we get an increasing F_H for free. But if $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$. So computation above indicates that when $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ we are missing some range of p.

What's more, we can look at computation in lemma 12 when $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ as well. Recall in lemma 12, we compute derivative of expected payoff of high type if she deviates to bid below a_1. If $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, the derivative of high type's deviating payoff [9] is negative when $x < \frac{\underline{v}_1+\underline{v}_2-\sqrt{(2\bar{v}_1-\underline{v}_1-\underline{v}_2)(\underline{v}_1+\underline{v}_2)}}{2}$ and positive when $x > \frac{\underline{v}_1+\underline{v}_2-\sqrt{(2\bar{v}_1-\underline{v}_1-\underline{v}_2)(\underline{v}_1+\underline{v}_2)}}{2}$. Computation will show that $a_1 > \frac{\underline{v}_1+\underline{v}_2-\sqrt{(2\bar{v}_1-\underline{v}_1-\underline{v}_2)(\underline{v}_1+\underline{v}_2)}}{2}$. So candidates for potential optimizers must be at the endpoints. We compare $\bar{v}_1 - a_2 \geq (1-p)(\bar{v}_1 - 0) \iff p \geq \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$. Computation above shows that when $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$, to support results like lemma 12, we just need $p > \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$.

Combining the previous two paragraphs, we miss to characterize equilibria when $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$ with $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$.

When $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$, our conjecture is that F_H is mixture of the previous F_H distributions for high type in theorem 2 and 3, i.e. support of F_H has an atom at 0, puts no probability on interval $(0, a_1)$ (i.e. gap on interval $(0, a_1)$) and put the remaining probability on interval $[a_1, a_2]$. As with the paragraph before lemma 10, high type will choose to put an atom at 0 since range of p, the probability low type appears in the population, is still not that high. We will later show there is no incentive for high type to deviate from bidding according to F_H.

We still assume a to be the upper bound of distributions. Let R_1, R_2, R_3 still be defined as $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$, $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \leq a_1 \leq b_{l1}\}$ and $R_3 = \{(b_{l1}, b_{l2}) : a_1 \leq b_{l2} \leq b_{l1}\}$.

As usual, we summarize our results into a theorem:

Theorem 4. *Suppose* $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ *and* $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$*.*

[9] $\frac{(p-1)(2x^2-2(\underline{v}_1+\underline{v}_2)x-(\bar{v}_1-\underline{v}_1-\underline{v}_2)(\underline{v}_1+\underline{v}_2))}{(\underline{v}_1+\underline{v}_2-2x)^2}$

1. *High type will bid by* $F_H(x) = \frac{x+T\underline{v}_2}{\underline{v}_2-x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$ *with support* $\{0\} \cup [a_1, a_2]$. $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$, $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$ *and* $a_2 = \bar{v}_1 p$.

2. *In region* $R_3 = \{(b_{l1}, b_{l2}) : a_1 \le b_{l2} \le b_{l1}\}$, *low type will bid by* $G_{L1}(x) = G_{L2}(x) = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ *with support* $[a_1, a_2]$.

3. *In region* $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \le b_{l1} \le a_1\}$, *letting* $C = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2)(-\underline{v}_2+\bar{v}_1 p)}{\underline{v}_2 p}$,

   (a) *when* $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \le 2\bar{v}_1$, *low type bids according to distribution* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *for all* $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$ *in the region;*

   (b) *When* $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$,

      i. *when* $\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$, *low type will bid her first bid according to* $G_{L1}(x) = \frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T$ *and second bid by* $G_{L2}(x) = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$ *in interval* $[a_3, a_1]$ *for* C *defined above. The two marginal distributions are related by* $G_{L2}(h(x)) = G_{L1}(x)$ *where* $h(x) = \frac{-\underline{v}_2(\underline{v}_2-p(2\bar{v}_1+C))(\underline{v}_2-x)+\underline{v}_1^2 p(-\underline{v}_2+x)+\underline{v}_1(-\underline{v}_2^2(1+p)-2\bar{v}_1 px+\underline{v}_2(2\bar{v}_1 p+x+p(C+x)))}{p(2\bar{v}_1(\underline{v}_2-x)+b(-\underline{v}_2+x)+c(-\underline{v}_2+C+x))}$. *And low type will bid by* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in region* $[0, a_3]$. $a_3 < a_1$ *solve* $\frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$.

      ii. *when* $p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$, *low type will bid according to* $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ *in region* R_1.

4. *Region* $R_2 = \{(b_{l1}, b_{l2}) : b_{l2} \le a_1 \le b_{l1}\}$ *has zero probability under distributions of bids for low type.*

The first two points are illustrated in lemma 14. 3.a and 3.b.ii come from lemma 15 and 3.b.i is dealt in lemma 16. Point 4 is a direct result of point 2. Similar to previous cases, there will be a positive probability that one low type gets both objects due to common support of mixed strategy equilibrium distributions. And hence the equilibrium strategy is not necessarily efficient.

Lemma 14. *High type will bid by* $F_H(x) = \frac{x+T\underline{v}_2}{\underline{v}_2-x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$ *with support* $\{0\} \cup [a_1, a_2]$. *In region* $R_3 = \{(b_{l1}, b_{l2}) : a_1 \le b_{l2} \le b_{l1}\}$, *low type will bid by* $G_{L1}(x) = G_{L2}(x) = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1-x)}$ *with support* $[a_1, a_2]$. *We express* $a_2 = \bar{v}_1 p$ *and* $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$ *where* T *can be expressed as* $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$.

Proof. In region R_3, we still have $G_{L2}(x) = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1-x)}$ as in lemma 11 by looking at high type's indifferent condition in the region. And by similar argument in region R_3 low type must be bidding identical prices. In region R_3 we have indifferent condition for low type:
$p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)[(\underline{v}_1 - b_{l1}) + F_H(b_{l2})(\underline{v}_2 - b_{l2})] = \underline{v}_1(1-p) + (1-p)T\underline{v}_2$.
Plugging G_{L1}, G_{L2} we have
$(1-p)F_H(x)(\underline{v}_2 - b_{l1}) = (1-p)b_{l1} + (1-p)H\underline{v}_2 - \frac{\bar{v}_1 p - a_2 + (1-p)b_{l1}}{\bar{v}_1 - b_{l1}}(\underline{v}_1 + \underline{v}_2 - 2b_{l1})$. And we solve
$F_H(x) = \frac{x+T\underline{v}_2}{\underline{v}_2-x} - \frac{\bar{v}_1 p - a_2 + (1-p)x}{(1-p)(\bar{v}_1-x)(\underline{v}_2-x)}(\underline{v}_1 + \underline{v}_2 - 2x)$. Solving $F_H(a_2) = 1$ gives $a_2 = \frac{p\underline{v}_1+\underline{v}_2}{2} - \frac{T(1-p)\underline{v}_2}{2}$.

We should require $F_H(0) = T$, which is only valid when $a_2 = \bar{v}_1 p$. So $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$. T is an decreasing function of p. When $p = \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$, $T = 0$. This computation guarantees $T \in (0, \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1})$ for $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$. Another requirement is $F_H(a_1) = T$, and we have $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$. It is easy to compute expected payment for high type is $(\bar{v}_1 - a_1)(1-p) + p(\bar{v}_1 - a_1)G_{L2}(a_1) = \bar{v}_1 - \bar{v}_1 p$.

To make sure $F_H(x)$ is increasing when $x > a_1$, we need $\frac{dF_H(x)}{dx} > 0$ when $x \in (a_1, a_2)$. Plugging $a_2 = \bar{v}_1 p$, we have $\frac{dF_H(x)}{dx} = \frac{x^2(T\underline{v}_2+\underline{v}_1-2\bar{v}_1)-2(T-1)\bar{v}_1\underline{v}_2x+\bar{v}_1\underline{v}_2(T\bar{v}_1-\underline{v}_1-\underline{v}_2+\bar{v}_1)}{(\bar{v}_1-x)^2(\underline{v}_2-x)^2} > 0$ when $x \in (a_1, a_2)$. $\iff$ $x^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) + 2(1-T)\bar{v}_1\underline{v}_2 x + \bar{v}_1\underline{v}_2(T\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1) > 0$ when $x \in (a_1, a_2)$. And we have to guarantee $x^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) - 2(T-1)\bar{v}_1\underline{v}_2 x + \bar{v}_1\underline{v}_2(T\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1) > 0$ when $x = a_1$. Plugging $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$ into the equation, we have $x^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) - 2(T-1)\bar{v}_1\underline{v}_2 x + \bar{v}_1\underline{v}_2(T\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1) = a_1^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) - (T-1)\bar{v}_1\underline{v}_2 a_1$. And $a_1^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) - (T-1)\bar{v}_1\underline{v}_2 a_1 > 0$ can be achieved when $a_1 \in (0, \frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2})$. Some computation will show that $a_1 \in (0, \frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2})$ is satisfied as long as $T \in (-\frac{\bar{v}_1-\underline{v}_1}{\bar{v}_1-\underline{v}_2}, \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1})$, which contains $(0, \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1})$. So we confirm that as long as $0 < a_1 < \frac{-\bar{v}_1\underline{v}_2+H\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+H\underline{v}_2}$, derivative is positive when $x = a_1$.

Note that $x^2(T\underline{v}_2 + \underline{v}_1 - 2\bar{v}_1) + 2(1-T)\bar{v}_1\underline{v}_2 x + \bar{v}_1\underline{v}_2(T\bar{v}_1 - \underline{v}_1 - \underline{v}_2 + \bar{v}_1)$ is a quadratic function with a negative coefficient on x^2 term and positive coefficient on x term. So such an expression will be increasing when $x < \frac{(T-1)\bar{v}_1\underline{v}_2}{T\underline{v}_2-2\bar{v}_1+\underline{v}_1}$. Another fact is that if we solve $F_H(a_2) = 1$ by plugging into $a_2 = \bar{v}_1 p$, we can get $a_2 = \frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2}$ [10]. So results above imply that derivative at a_1 is positive and it is the minimal value $\frac{dF_H(x)}{dx}$ will achieve. And hence we can prove that $\frac{dF_H(x)}{dx} > 0$ when $x > a_1$ by showing $\frac{dF_H(x)}{dx} > 0$ when $x = a_1$. □

When low type is bidding the same in region R_3, we are able to conclude that R_2 will at most be a zero-measure region. And hence we move on to look at region R_1 and we propose similar solutions to lemma 12. But condition to support the lemma will be more complicated:

Lemma 15. *In region $R_1 = \{(b_{l1}, b_{l2}) : b_{l2} \leq b_{l1} \leq a_1\}$,*

1. *If $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$ there is a perfectly correlated equilibrium where low type bids according to distribution $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ with support $[0, a_1]$ when $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$.*
2. *If $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1$, perfectly correlated equilibrium can be supported by all $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$ with $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ on support $[0, a_1]$.*

Proof. On region R_1 we have indifferent condition for low type: $p[G_{L2}(b_{l1})(\underline{v}_1 - x) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) + (1-p)T(\underline{v}_2 - b_{l2}) = \underline{v}_1(1-p) + (1-p)T\underline{v}_2$. Perfectly correlated equilibrium will give a result $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$. Note that $G_{L1}(0) = G_{L2}(0) = 0$. G functions on R_1 and G functions on R_3 will coincide when $x = a_1$. Expected payoff for first bid of low type is $p\frac{(1-p)x+T(1-p)b_{l1}}{p(\underline{v}_1+\underline{v}_2-2b_{l1})}(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1})$ with derivative $-\frac{(p-1)[2(T-1)b_{l1}^2-2(T-1)(\underline{v}_1+\underline{v}_2)b_{l1}+(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)]}{(\underline{v}_1+\underline{v}_2-2b_{l1})^2}$. Expected payoff for second bid of low type is $p\frac{(1-p)b_{l2}+T(1-p)b_{l2}}{p(\underline{v}_1+\underline{v}_2-2b_{l2})}(\underline{v}_2 - b_{l2}) + (1-p)T(\underline{v}_2 - b_{l2})$ with derivative $\frac{(p-1)[2(T-1)b_{l2}^2-2(T-1)(\underline{v}_1+\underline{v}_2)b_{l2}+(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)]}{(\underline{v}_1+\underline{v}_2-2b_{l2})^2}$ which is exactly the negative of derivative of expected payoff for first bid of low type.

So condition to make payment from first bid to be decreasing is still the same condition to make payment from second bid to be increasing: $(p-1)[2(T-1)x^2 - 2(T-1)(\underline{v}_1 + \underline{v}_2)x + (\underline{v}_1 + \underline{v}_2)(T\underline{v}_1 - \underline{v}_2)] > 0$, which is equivalent to $2(1-T)x^2 - 2(1-T)(\underline{v}_1 + \underline{v}_2)x - (\underline{v}_1 + \underline{v}_2)(T\underline{v}_1 - \underline{v}_2) > 0$. We know that

[10] Expressions $a_2 = \bar{v}_1 p = \frac{-\bar{v}_1\underline{v}_2+T\bar{v}_1\underline{v}_2}{-2\bar{v}_1+\underline{v}_1+T\underline{v}_2} = \frac{p\underline{v}_1+\underline{v}_2}{2} - \frac{T(1-p)\underline{v}_2}{2}$ are equivalent as long as $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$.

$2(1-T)x^2 - 2(1-T)(\underline{v}_1 + \underline{v}_2)x - (\underline{v}_1 + \underline{v}_2)(T\underline{v}_1 - \underline{v}_2)$ is decreasing when $x < a_1 < \frac{\underline{v}_1+\underline{v}_2}{2}$. We can compute that when $x = a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$, the expression above is positive for 2 conditions: if $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1$, we have perfectly correlated equilibrium for all $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$; and if $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$ perfectly correlated equilibrium exists as long as $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$.

If high type bids below a_1, she will get $p\frac{(1-p)b_{h1}+T(1-p)b_{h1}}{p(\underline{v}_1+\underline{v}_2-2b_{h1})}(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1})$ $= \frac{(1-p)(\bar{v}_1-b_{h1})(\underline{v}_1+\underline{v}_2-b_{h1})}{\underline{v}_1+\underline{v}_2-2b_{h1}} + \frac{T(1-p)b_{h1}}{\underline{v}_1+\underline{v}_2-2b_{h1}}(\bar{v}_1 - b_{h1}) = \frac{\underline{v}_1+\underline{v}_2-b_{h1}+Tb_{h1}}{\underline{v}_1+\underline{v}_2-2b_{h1}}(1-p)(\bar{v}_1 - b_{h1})$ with derivative $\frac{(1-p)[-2(1-T)b_{h1}^2+2(\underline{v}_1+\underline{v}_2)(1-T)x+(\underline{v}_1+\underline{v}_2)(T+1)\bar{v}_1-(\underline{v}_1+\underline{v}_2)^2]}{(\underline{v}_1+\underline{v}_2-2b_{h1})^2}$. When $b_{h1} = 0$, the numerator is $(\underline{v}_1 + \underline{v}_2)(T+1)\bar{v}_1 - (\underline{v}_1 + \underline{v}_2)^2 < (\underline{v}_1 + \underline{v}_2)(\frac{\underline{v}_1+\underline{v}_2-\bar{v}_1}{\bar{v}_1} + 1)\bar{v}_1 - (\underline{v}_1 + \underline{v}_2)^2 = 0$. So the numerator of derivative (a quadratic function) of deviating payoff will be negative and may turn to positive afterwards since coefficient for term x^2 is negative while coefficient for term x is positive. In fact, if we plug $a_1 = \frac{\underline{v}_1+\underline{v}_2-\bar{v}_1(1+T)}{1-T}$ into the derivative, the quadratic function in numerator is $[\frac{2\bar{v}_1(1+T)}{1-T} - (\underline{v}_1 + \underline{v}_2)\frac{1+T}{1-T}][\underline{v}_1 + \underline{v}_2 - (1+T)\bar{v}_1] > 0$. So we just need to compare deviating payments when high type bids 0 since computation above reveals that derivative below a_1 is initially negative and will eventually turn positive. We need $\bar{v}_1 - a_2 \geq (1-p)(\bar{v}_1 - 0)$. And it is satisfied by an equality since $a_2 = \bar{v}_1 p$. So we know that high type will not deviate to bid anything below a_1 unless she is bidding 0. $\square$

We propose lemma 15 by simply assuming low type is bidding the same in region R_1. And we can check we have a result similar to lemma 13 when $p \in (\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$.

Lemma 16. *When $\frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ and $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$, low type will bid according to $G_{L1}(x) = \frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T$ and $G_{L2}(x) = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$ in interval $I = [a_3, a_4]$, with $G_{L2}(h(x)) = G_{L1}(x)$ where $h(x) = \frac{-\underline{v}_2(\underline{v}_2-p(2\bar{v}_1+C))(\underline{v}_2-x)+\underline{v}_1^2 p(-\underline{v}_2+x)+\underline{v}_1(-\underline{v}_2^2(1+p)-2\bar{v}_1 px+\underline{v}_2(2\bar{v}_1 p+x+p(C+x)))}{p(2\bar{v}_1(\underline{v}_2-x)+b(-\underline{v}_2+x)+c(-\underline{v}_2+C+x))}$. And low type will bid by $G_{L1}(x) = G_{L2}(x) = \frac{(1-p)x+T(1-p)x}{p(\underline{v}_1+\underline{v}_2-2x)}$ in interval $[0, a_3]$. We express $T = \frac{p\underline{v}_1+\underline{v}_2-2p\bar{v}_1}{(1-p)\underline{v}_2}$ and $C = \frac{(\bar{v}_1-\underline{v}_1-\underline{v}_2)(-\underline{v}_2+\bar{v}_1 p)}{\underline{v}_2 p}$. What's more, $a_4 = a_1$ introduced in lemma 14.*

Proof. Similar to lemma 13, we still define I as the first non-trivial (i.e. positive measure) interval where $b_{l2} = h(b_{l1})$ solves the first order condition and $h(b_{l1}) < b_{l1}$ on interior of interval I. We denote $a_3 = \inf_{x \in I} I$ and $a_4 = \sup_{x \in I} I$. By construction we have $h(a_3) = a_3$ and $h(a_4) = a_4$.

Recall expected payoff for low type in region R_1 is $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) + (1-p)T(\underline{v}_2 - b_{l2})$ and first order condition with respect to b_{l2} will be $p[g_{L1}(b_{l2})(\underline{v}_2 - b_{l2}) - G_{L1}(b_{l2})] - (1-p)T = 0$. We express $G_{L1}(x) = \frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T$ with some constant C to be determined. Plugging $G_{L1}(b_{l2}) = \frac{C}{\underline{v}_2-b_{l2}} - \frac{1-p}{p}T$ into the indifferent condition of low type, which is $p[G_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + G_{L1}(b_{l2})(\underline{v}_2 - b_{l2})] + (1-p)(\underline{v}_1 - b_{l1}) + (1-p)T(\underline{v}_2 - b_{l2}) = \underline{v}_1(1-p) + (1-p)T\underline{v}_2$, we can solve $G_{L2}(x) = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$. Similar to lemma 13, $G_{L1}(x) = G_{L2}(x)$ at $a_3 < a_4$ so we have $\frac{C}{\underline{v}_2-x} - \frac{1-p}{p}T = \frac{(1-p)x-pC+(1-p)T\underline{v}_2}{p(\underline{v}_1-x)}$ when $x = a_3$ and $x = a_4$. Rearranging equation above presents $C = \frac{(1-p)x+(1-p)H\underline{v}_2}{p(\underline{v}_1+\underline{v}_2-2x)}(\underline{v}_2 - x) + \frac{(1-p)T(\underline{v}_1-x)(\underline{v}_2-x)}{p(\underline{v}_1+\underline{v}_2-2x)} = \frac{(1-p)x(\underline{v}_2-x)}{p(\underline{v}_1+\underline{v}_2-2x)} + \frac{(1-p)T(\underline{v}_2-x)(\underline{v}_1+\underline{v}_2-x)}{p(\underline{v}_1+\underline{v}_2-2x)}$ when $x = a_3$ or a_4. Taking derivative on C with respect to x will give us $(1-p)\frac{2(1-T)x^2-2(\underline{v}_1+\underline{v}_2)(1-T)x-(\underline{v}_1+\underline{v}_2)(T\underline{v}_1-\underline{v}_2)}{p(\underline{v}_1+\underline{v}_2-2x)^2}$. The derivative is positive when $x < \frac{1}{2}[\underline{v}_1 + \underline{v}_2 - \sqrt{\frac{(1+T)(\underline{v}_1^2-\underline{v}_2^2)}{1-T}}]$ and $x > \frac{1}{2}[\underline{v}_1 + \underline{v}_2 + \sqrt{\frac{(1+T)(\underline{v}_1^2-\underline{v}_2^2)}{1-T}}]$. By a similar argument from lemma 13, if we still denote a_3 as left endpoint of I and a_4 as right endpoint of I, we must have $a_3 < \frac{1}{2}[\underline{v}_1 + \underline{v}_2 - \sqrt{\frac{(1+T)(\underline{v}_1^2-\underline{v}_2^2)}{1-T}}] < a_4$ and that $a_4 = a_1$. So we have equations

$\frac{C}{\underline{v}_2 - x} - \frac{1-p}{p} H = \frac{(1-p)x - pC + (1-p)H\underline{v}_2}{p(\underline{v}_1 - x)} = \frac{\bar{v}_1 p - a_2 + (1-p)x}{p(\bar{v}_1 - x)}$ when $x = a_4$, similar to lemma 13. Solving this equation we get $\frac{\underline{v}_2(2\bar{v}_1 - \underline{v}_1 - \underline{v}_2)}{2\bar{v}_1^2 - 2\bar{v}_1\underline{v}_1 + \underline{v}_1^2 - \underline{v}_2^2} < p < \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$ and $C = \frac{(\bar{v}_1 - \underline{v}_1 - \underline{v}_2)(-\underline{v}_2 + \bar{v}_1 p)}{\underline{v}_2 p}$ as long as
$2\bar{v}_1 < \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2}$.

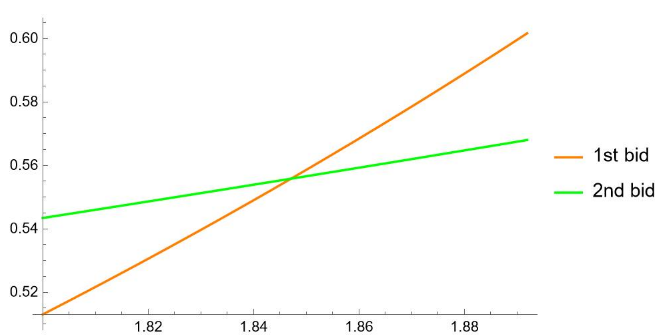
We can also confirm that $2(1-T)x^2 - 2(1-T)(\underline{v}_1 + \underline{v}_2)x - (\underline{v}_1 + \underline{v}_2)(T\underline{v}_1 - \underline{v}_2)$ is positive when $x \leq a_3$ given range of p and expression of C provided in the last paragraph, which shows existence of perfectly correlated equilibrium for low type when $x \in R_1 \setminus I = (0, a_3)$. What's more, solving $G_{L2}(h(x)) = G_{L1}(x)$ gives us $h(x) = \frac{-\underline{v}_2(\underline{v}_2 - p(2\bar{v}_1 + C))(\underline{v}_2 - x) + \underline{v}_1^2 p(-\underline{v}_2 + x) + \underline{v}_1(-\underline{v}_2^2(1+p) - 2\bar{v}_1 p x + \underline{v}_2(2\bar{v}_1 p + x + p(C+x)))}{p(2\bar{v}_1(\underline{v}_2 - x) + b(-\underline{v}_2 + x) + c(-\underline{v}_2 + C + x))}$ with derivative $\frac{\underline{v}_2^2 C(\underline{v}_1 + \underline{v}_2 - p(2\bar{v}_1 + C))}{p((2\bar{v}_1 - \underline{v}_1)(\underline{v}_2 - x) + \underline{v}_2(-\underline{v}_2 + C + x))^2}$. We need $\underline{v}_1 + \underline{v}_2 - p(2\bar{v}_1 + C) > 0$ for an increasing function $h(x)$. And this condition is consistent with the p, C expressions we computed last paragraph.

Similar to argument in lemma 13, payoff from first and second bid of low type are constructed to be constant in interval I. If second bid of low type deviates downward to become smaller than a_3, the optimal deviating bid should be bidding at a_3 because we know that in perfectly correlated equilibrium payment from second bid is strictly increasing. Similarly if first bid of low type deviates upward to be higher than a_4, the deviating bid better be bidding $a_4 = a_1$ since in perfectly correlated equilibrium payment from first bid is strictly decreasing. If high type deviates to bid below a_1 and chooses to bid in I, her expected payoff will be $(1-p)(\bar{v}_1 - b_{h1}) + p\frac{(1-p)b_{h1} - pC + (1-p)T\underline{v}_2}{p(\underline{v}_1 - b_{h1})}(\bar{v}_1 - b_{h1})$ with derivative $-\frac{\bar{v}_1(\bar{v}_1 - \underline{v}_1)[\frac{\bar{v}_1 - \underline{v}_1 + \underline{v}_2}{\underline{v}_2}p - 1]}{(\underline{v}_1 - b_{h1})^2}$. Given range of p, $1 - \frac{\bar{v}_1 - \underline{v}_1 + \underline{v}_2}{\underline{v}_2} p > 1 - \frac{\bar{v}_1 - \underline{v}_1 + \underline{v}_2}{\underline{v}_2} \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1} = \frac{\bar{v}_1 - \underline{v}_2}{2\bar{v}_1 - \underline{v}_1} > 0$ and hence the derivative is positive. So bidding in interior of I will be dominated by bidding at a_1. Proof of lemma 15 can be used to show that high type should not be bidding below a_3.

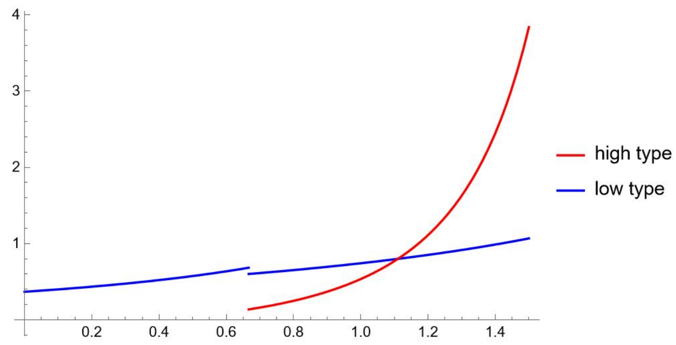
It is easy to exclude deviations above the common upper bound: all distributions are atomless at upper bounds. So bidding $(\bar{v}_1 p, \bar{v}_1 p)$ ($\bar{v}_1 p$ is the upper bound in this scenario) will give low type two objects with certainty and bidding $\bar{v}_1 p$ will give high type one object with certainty. And hence bidding above the upper bound will only decrease the expected payoff for any type. □

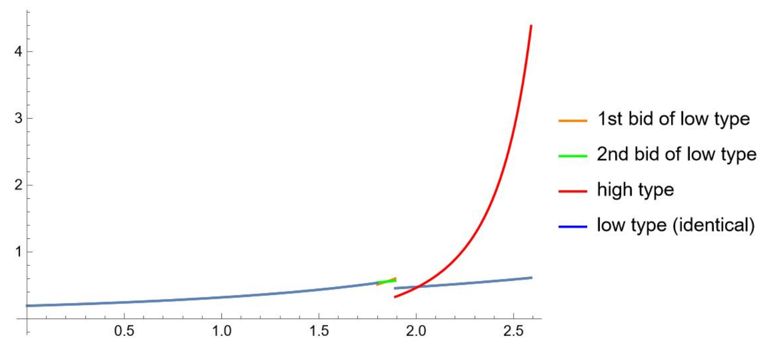
Graphical Illustration

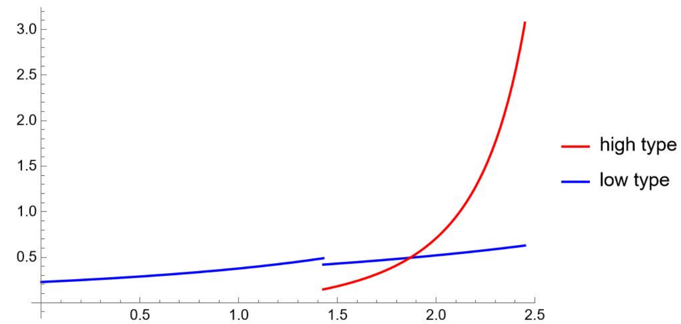
We will also demonstrate lemma 16 separately since it shows situation where bids from low type are distinct. We pick $\bar{v}_1 = 7$, $\underline{v}_1 = 6$, $\underline{v}_2 = 3$ and $p = 0.37$. Two bids from low type will be different in interval $[1.8, 1.892]$. We still know that $G_{L1}(x) = G_{L2}(h(x))$ in interval $[1.8, 1.892]$ with $h(x) < x$ for $x \in (1.8, 1.892)$ and $h(x) = x$ at endpoints as in the graphical illustration in the last subsection.



We illustrate density functions of equilibrium distributions graphically:



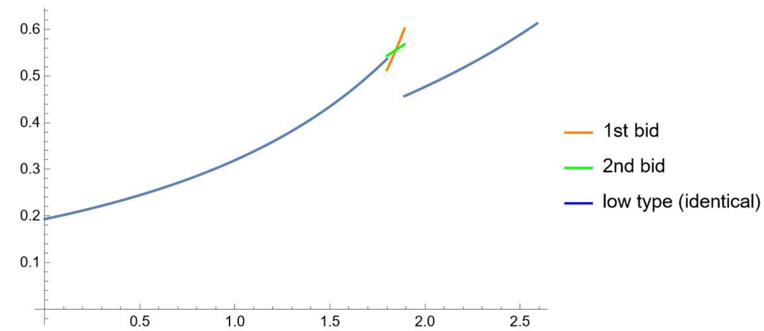




The first graph is when $\bar{v}_1 = 4, \underline{v}_1 = 3, \underline{v}_2 = 2$ and $p = \frac{3}{8}$, which covers points 1,2, 3.a and 4 of theorem 4. Support for distribution of high type is $[\frac{2}{3}, \frac{3}{2}]$ and support for distributions of low type is $[0, \frac{3}{2}]$. Note that there will be an atom of size $\frac{1}{10}$ for distribution of mixed strategy of high type at 0.

We continue to select $\bar{v}_1 = 7, \underline{v}_1 = 6, \underline{v}_2 = 3$ for the second and third graphs and the second graph is when $p = 0.37$, which covers points 1,2, 3.b.i and 4 of theorem 4, when there is an interval where first and second bid of low type are different. Support for distributions of low type is $[0, 2.6]$ and support for distribution of high type is $[1.89, 2.6]$. The third graph is when $p = 0.35$ where two bids of low type are always identical, as shown in points 1,2 , 3.b.ii and 4 of theorem 4. Support for distributions of low type is $[0, 2.45]$ and support for distribution of high type is $[1.43, 2.45]$. Note that there will be an atom of size 0.021 for distribution of mixed strategy of high type at 0.

We can also illustrate density functions for low type only:



The pdfs only differ with bids in $[1.8, 1.89]$.

Analogous to corollary 3.2, we have a similar result regarding pdf at a_1:

Corollary 4.1. *For* $p \in (\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}, \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1})$*, we have the following results:*

1. *When* $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1$ *or* $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$ *but* $p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$*, left derivative of low type's distribution at* a_1 *will be greater than right derivative of low type at* a_1

2. *When* $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} > 2\bar{v}_1$ *but* $p > \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$*, left derivative will satisfy* $\frac{dG_{L1}(x)}{dx} > \frac{dG_{L2}(x)}{dx}$ *at* a_1 *and left derivative* $\frac{dG_{L2}(x)}{dx}$ *in region* R_1 *at* a_1 *will be greater than right derivative* $\frac{dG(x)}{dx}$ *in region* R_3 *at* a_1*.*

Corollary 4.2. *With* $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$*, results in theorem 2 are valid when* $p < \frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2}$*; results in theorem 3 are valid when* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$*. What's more, corollary 3.2 holds for* $\bar{v}_1 < \underline{v}_1 + \underline{v}_2$ *when* $\frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1} < p < \frac{\underline{v}_2}{\underline{v}_1}$*.*

5.3 When $p \geq \frac{\underline{v}_2}{\underline{v}_1}$

Theorem 5. *When* $p \geq \frac{\underline{v}_2}{\underline{v}_1}$ *, high type will bid* $\underline{v}_2$ *and low type will be bidding* $\underline{v}_2$ *and by distribution* $G_{L2}(x) = \frac{(1-p)x+p\underline{v}_1-\underline{v}_2}{p(\underline{v}_1-x)}$ *in interval* $[0, \underline{v}_2)$ *for her first and second bid respectively.*

Proof. Abusing notation, high type will not deviate to bid $b_{h1} < \underline{v}_2$ if
$pG_{L2}(b_{h1})(\bar{v}_1 - b_{h1}) + (1-p)(\bar{v}_1 - b_{h1}) \leq \bar{v}_1 - \underline{v}_2 \iff G_{L2}(x) \leq \frac{(1-p)x+p\bar{v}_1-\underline{v}_2}{p(\bar{v}_1-x)}$ and low type will not deviate to make her first bid $b_{l1} < \underline{v}_2$ for her first bid if $pG_{L2}(b_{l1})(\underline{v}_1 - b_{l1}) + (1-p)(\underline{v}_1 - b_{l1}) \leq \underline{v}_1 - \underline{v}_2$ $\iff G_{L2}(x) \leq \frac{(1-p)x+p\underline{v}_1-\underline{v}_2}{p(\underline{v}_1-x)}$.
Some computation will show that $\frac{(1-p)x+p\underline{v}_1-\underline{v}_2}{p(\underline{v}_1-x)} < \frac{(1-p)x+p\bar{v}_1-\underline{v}_2}{p(\bar{v}_1-x)}$ as long as $x < \underline{v}_2$.
If we let $G_{L2}(x) = \frac{(1-p)x+p\underline{v}_1-\underline{v}_2}{p(\underline{v}_1-x)}$, with $G_{L2}(x) = 1$ when $x = \underline{v}_2$, we can successfully support bids from high type and first bid of low type to be degenerated on $\underline{v}_2$. When $x = 0$, $G_{L2}(x) = \frac{p\underline{v}_1-\underline{v}_2}{p\underline{v}_1}$. So as long as $p \geq \frac{\underline{v}_2}{\underline{v}_1}$, distribution $G_{L2}(x)$ is valid. When $p > \frac{\underline{v}_2}{\underline{v}_1}$, the low type will put an atom with size $\frac{p\underline{v}_1-\underline{v}_2}{p\underline{v}_1}$ for G_{L2} when $x = 0$.

The indifferent condition of first bid from low type is binding only when $p = \frac{\underline{v}_2}{\underline{v}_1}$, which implies that both types strictly prefer bidding $\underline{v}_2$ with higher p. What's more, when $p > \frac{\underline{v}_2}{\underline{v}_1}$ and high (low) type unilaterally deviates to bid 0, high (low) type will only get half object on average when second bid of low type is 0, (because deviating to bid 0 means first bid ties with opponent's second bid which is strictly smaller than opponent's first bid), which leads to payoff strictly smaller than $\bar{v}_1 - \underline{v}_2$ ($\underline{v}_1 - \underline{v}_2$). When high type and first bid of low type bid a pure strategy on $\underline{v}_2$, second bid of low type will not win and will not generate any positive expected payoff. A low type will be indifferent to any distribution on her second bid.

When $p \to 1$, $G_{L2}(x)$ will also be degenerated on $\underline{v}_2$. And the equilibrium result will converge to low type bidding 1 for both bids, and each bidder get exactly 1 object, i.e. the pure strategy equilibrium found in lemma 7 $\square$

Remark. *This equilibrium is obviously not unique. We can pick any $x > 0$ on support of G_{L2} and construct an atom at x by truncate the probability for values strictly below x to be exactly at x.*

6 Other Cases

We studied case where high type has marginal valuation $(\bar{v}_1, \bar{v}_2)$ and low type has marginal valuation $(\underline{v}_1, \underline{v}_2)$ where $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. In this section, we will show mixed strategy equilibrium of the other two cases. We continue to assume that probability a low type appears in the population is p so probability a high type appears in the population is $1-p$. Additionally, we still assume that F_{H1}, F_{H2} are marginal distributions of high type's first and second bids respectively while G_{L1}, G_{L2} are marginal distributions of low type's first and second bids.

6.1 Value Ordering $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$

We can state the mixed strategy equilibrium when value ordering is $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$:

Theorem 6. *When $p \leq 1 - \frac{\bar{v}_2}{\underline{v}_1}$, first bids of both type will be pure strategy at $\underline{v}_2$ and second bid of high type will follow $F_{H2}(x) = \frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)}$.*

Proof. High type will not deviate her first bid to $b_{h1} < \bar{v}_2$ when $(1-p)F_{H2}(x)(\bar{v}_1 - b_{h1}) + p(\bar{v}_1 - b_{h1}) \leq \bar{v}_1 - \bar{v}_2$, which makes $F_{H2}(x) \leq \frac{(\bar{v}_1 - \bar{v}_2) - p(\bar{v}_1 - x)}{(1-p)(\bar{v}_1 - x)}$. Low type will not deviate her first bid to $b_{l1} < \bar{v}_2$ when $(1-p)F_2(b_{l1})(\underline{v}_1 - b_{l1}) + p(\underline{v}_1 - b_{l1}) \leq \underline{v}_1 - \bar{v}_2$, which makes $F_{H2}(x) \leq \frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)}$. Computation will show that $\frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)} \leq \frac{(\bar{v}_1 - \bar{v}_2) - p(\bar{v}_1 - x)}{(1-p)(\bar{v}_1 - x)}$. So similar to the previous result, we have a mixed strategy equilibrium where first bids of both type are degenerate at $\bar{v}_2$ and second bid of high type follows distribution $F_{H2}(x) = \frac{(\underline{v}_1 - \bar{v}_2) - p(\underline{v}_1 - x)}{(1-p)(\underline{v}_1 - x)}$ with support $[0, \bar{v}_2]$. When $x = 0$, $F_{H2}(x) = \frac{(1-p)\underline{v}_1 - \bar{v}_2}{(1-p)\underline{v}_1}$. So we should require $(1-p)\underline{v}_1 - \bar{v}_2 \geq 0$ or equivalently $p \leq 1 - \frac{\bar{v}_2}{\underline{v}_1}$.

We construct the distributions by making sure that first bids will not deviate to lower values. If first bids deviate to higher values, bidders just pay more to get lower payoff. When first bids are both at $\bar{v}_2$, second bid of high type will never get positive payoff. So high type will have no incentive to deviate her first bid. $\square$

Theorem 7. *When $p > 1 - \frac{\bar{v}_2}{\underline{v}_1}$, low type will mix by distribution $G_{L1}(x) = \frac{T\bar{v}_2}{\bar{v}_2 - x}$ where $T = \frac{-\bar{v}_1 + \bar{v}_2 + \bar{v}_2 p + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(-1+p)(\bar{v}_2 + 2\underline{v}_1 p)}}{2\bar{v}_2 p}$ with support $[0, a_1]$. In interval $[0, a_1]$ second bid of high type will follow distribution $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - x)}$. In interval $[a_1, a_2]$ first and second bid of high type will mix by distribution $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ and $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ respectively. We are able to express $C = \frac{-\bar{v}_1 + \bar{v}_2 + \bar{v}_2 p + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(-1+p)(\bar{v}_2 + 2\underline{v}_1 p)}}{2(1-p)}$, $a_1 = \bar{v}_2(1-H)$ and $a_2 = \bar{v}_2 - (1-p)C$.*

Proof. We suppose that G_{L1} will have support $[0, a_1]$, F_{H1} will have support $[a_1, a_2]$ and F_{H2} will have support $[0, a_2]$. To be more precise, we require an $a_3 \in (a_1, a_2)$ so that when first bid of high type is bidding in interval (a_1, a_3), second bid of high type will be bidding in interval $(0, a_1)$. And when first bid of high type is in interval (a_3, a_2), second bid of high type will be in interval (a_1, a_2).

Consider indifferent condition for low type, which is $p(\underline{v}_1 - b_{l1}) + (1-p)F_{H2}(b_{l1})(\underline{v}_1 - b_{l1}) = p\underline{v}_1$. So we have $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - b_{l1})}$ on $(0, a_1)$. Note that $\frac{px}{(1-p)(\underline{v}_1 - x)} = 1$ when $x = (1-p)\underline{v}_1$. We require $a_1 < (1-p)\underline{v}_1$ since upper bound of support for F_{H2} is $a_2 > a_1$.

Consider indifferent condition for high type when first bid of high type is in (a_3, a_2) and second bid is in (a_1, a_2): $p(\bar{v}_1 + \bar{v}_2 - b_{l1} - b_{l2}) + (1-p)[F_{H2}(b_{l1})(\bar{v}_1 - b_{l1}) + F_{H1}(b_{l2})(\bar{v}_2 - b_{l2})] = \bar{v}_1 + \bar{v}_2 - 2a_2$. By construction it is unlikely that bids from high type are perfectly correlated (supports for first and second bids have different measure) and we need to construct separating equilibrium. We assume a bidder is maximizing her expected payoff by choosing the optimal second bid b_{l2} given any first bid b_{l1}, so we have $-p + (1-p)[f_{H1}(b_{l2})(\bar{v}_2 - b_{l2}) - F_{H1}(b_{l2})] = 0$ by taking first order derivative with respect to y. Solving the differential equation, we have $F_{H1}(y) = \frac{C}{\bar{v}_2 - y} - \frac{p}{1-p}$ on interval (a_1, a_2) with some constant C to be determined. (Note that we are looking at symmetric mixed strategies.) Note that expected payoff for second bid of high type is $C(1-p)$, which is constant. And we solve $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_1, a_2). The two distribution functions should match when $F_{H1}(a_2) = F_{H2}(a_2) = 1$ by construction. And we solve $a_2 = \bar{v}_2 + (-1+p)C$.

When first bid of high type is in (a_1, a_3) and second bid of high type is in $(0, a_1)$, indifferent condition for high type will become $p[\bar{v}_1 - b_{l1} + G_{L1}(b_{l2})(\bar{v}_2 - b_{l2})] + (1-p)F_{H2}(x)(\bar{v}_1 - b_{l1}) = \bar{v}_1 + \bar{v}_2 - 2a_2$. Note that expected payoff for second bid is $pG_{L1}(b_{l2})(\bar{v}_2 - b_{l2})$, which indicates that second bid of high type is only possible to win from low type. We argue high type is getting constant payoff from her second bid: if $pG_{L1}(b_{l2})(\bar{v}_2 - b_{l2})$ is not constant and high type can get higher payoff by bidding at b^*_{l2} than any other bids, high type will be always bidding such b^*_{l2} regardless of how she bids her first bid when she faces the indifferent condition mentioned above. And hence high type will not be randomizing in interval $(0, a_1)$. An additional requirement is that the constant payoff high type is getting for her second bid is positive, which requires low type to put an atom with size T at 0. So high type will get $pG_{L1}(b_{h2})(\bar{v}_2 - b_{h2}) = pT\bar{v}_2$ when her second bid b_{h2} is below a_1 and we better require $pT\bar{v}_2 = C(1-p)$ so that high type will not want to deviate her second bid. So $G_{L1}(x) = \frac{T\bar{v}_2}{\bar{v}_2 - x}$, which reaches 1 when $x = \bar{v}_2(1-T)$. And we require $a_1 = \bar{v}_2(1-T) < \underline{v}_1(1-p)$. We continue to solve $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_1, a_3) since $pT\bar{v}_2 = C(1-p)$. So we have same expressions for F_{H2} on interval (a_1, a_3) and (a_3, a_2). When two expressions of F_{H2} match at a_1, we are able to solve another expression of a_1, i.e. $a_1 = \frac{\bar{v}_1\underline{v}_1 - \underline{v}_1\bar{v}_2 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 - \bar{v}_2 - \underline{v}_1 p + C - pC}$ should also solve $\frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p} = \frac{px}{(1-p)(\underline{v}_1 - x)}$.

Additionally, since we know from indifferent condition of high type that $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ on interval (a_1, a_2), we are able to generate another version of a_1, which is $a_1 = \frac{\bar{v}_2 p - C + pC}{p}$ by solving $F_{H1}(x) = 0$. So letting $\frac{\bar{v}_2 p - C + pC}{p} = \frac{\bar{v}_1\underline{v}_1 - \underline{v}_1\bar{v}_2 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 - \bar{v}_2 - \underline{v}_1 p + C - pC}$, we solve an expression of the constant parameter $C = \frac{-\bar{v}_1 + \bar{v}_2 + \bar{v}_2 p + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(-1+p)(\bar{v}_2 + 2\underline{v}_1 p)}}{2(1-p)}$, which is positive given $p > 1 - \frac{\bar{v}_2}{\underline{v}_1}$. We solve T via $pH\bar{v}_2 = C(1-p)$ and $T = \frac{-\bar{v}_1 + \bar{v}_2 + \bar{v}_2 p + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(-1+p)(\bar{v}_2 + 2\underline{v}_1 p)}}{2\bar{v}_2 p}$.

Similarly, as long as $p > 1 - \frac{\bar{v}_2}{\underline{v}_1}$, T is guaranteed to be positive but smaller than 1 [11]. Note that we are also able to express $a_1 = \bar{v}_2(1-T)$. Some computation will show that $\bar{v}_2(1-T) = \frac{\bar{v}_2 p - C + pC}{p}$ is equivalent to $pH\bar{v}_2 = C(1-p)$ and hence all expressions of a_1 are consistent. What's more, it is easy to see $a_1 = \bar{v}_2(1-T) < b = \bar{v}_2 + (-1+p)C < \bar{v}_2$ since $pT\bar{v}_2 = C(1-p)$ for $p \in (0,1)$. Note that we need $a_2 < \bar{v}_2$ in this scenario since b is constructed to be upper bound of support for F_{H2} and a high type can not bid beyond $\bar{v}_2$ for her second bid.

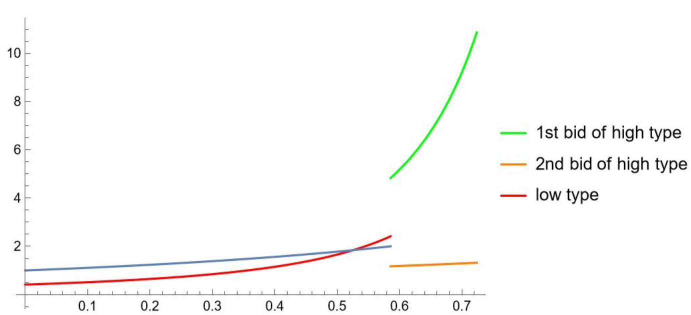
By construction, expected payoffs for each bid of high type are constant, that is to say, first and second bid of high type are always getting expected payoffs $\bar{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C$ and $C(1-p)$ respectively. So second bid of low type will not have particular incentive to deviate. Similarly, expected payoff for first bid of high type is constant, which makes a high type indifferent between bidding her first bid above or below a as long as her first bid is higher than a_1. If a high type puts both her bids b_{h1} below a_1 (i.e. first bid of low type is also below a_1), expected payoff for higher bid of high type is $p(\bar{v}_1 - b_{h1}) + (1-p)\frac{pb_{h1}}{(1-p)(\underline{v}_1 - b_{h1})}(\bar{v}_1 - x) = p(\bar{v}_1 - x)\frac{\underline{v}_1}{\underline{v}_1 - b_{h1}}$, which is an increasing function of b_{h1}. So high type would rather make her first bid at a_1. And we conclude high type will not deviate her bids. If low type deviates and bids b_{l1} higher than a_1, she gets expected payoff $p(\underline{v}_1 - b_{l1}) + (1-p)[\frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\underline{v}_1 - b_{l1})} - \frac{p}{1-p}](\underline{v}_1 - b_{l1}) = \frac{\underline{v}_1 - b_{l1}}{\underline{v}_1 - b_{l1}}[\underline{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C]$, which is decreasing in b_{l1}. So for a low type bidding above a_1 is dominated by bidding exactly at a_1. What's more, no type will bid higher than a_2 since all marginal distributions contain no atoms at upper bound of support and bidding (a_2, a_2) will guarantee high type two objects.

We introduce an a_3 in indifferent conditions of high type and our last task is to figure out what a_3 should be. It turns out we only need to place $a_3 \in (a_1, a_2)$ since both bids of high type are making the same constant payoffs under both indifferent conditions. So it actually does not matter which value we select as a_3 as long as it is strictly smaller than a_2 and strictly greater than a_1. In other words, we do not have the conditionally deterministic relationship between bids of the same type as in case when ordering valuation is $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$.

$\square$

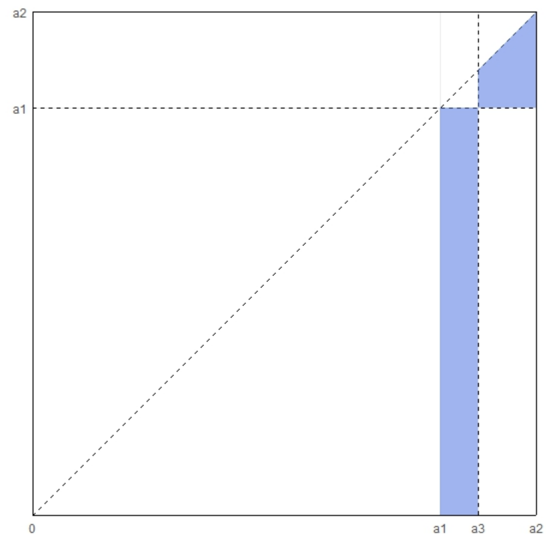
Graphical Illustration

We demonstrate theorem 7 by picking $\bar{v}_1 = 3$, $\underline{v}_1 = 2$, $\bar{v}_2 = 1$ and $p = \frac{2}{3}$:



[11] $H \to 1$ when $p \to 1$ and $T \to 0$ when $p \to 1 - \frac{\bar{v}_2}{\underline{v}_1}$.

Pdf of low type's mixed strategy is displayed in red, with support being $[0, 0.586]$. Second bid of high type has two parts: the blue curve in interval $[0, 0.586]$ and orange curve on interval $[0.586, 0.724]$. Pdf of first bid of high type is displayed by the green curve with support being $[0.586, 0.724]$. Note that there will be an atom of size 0.414 for distribution of mixed strategy of low type at 0.



We plot an illustration of joint support of bids of high type in the graph above with the same numerical values of $\bar{v}_1, \bar{v}_2, \underline{v}_1$ and p. We pick a_3 at roughly 0.65 since by our theorem the intermediate cutoff value a_3 can be any real number between 0.586 and 0.724. When first bid is between a_1 and a_2, the second bid is below a_1, which is represented by the shaded rectangle in the plot. When first bid is between a_3 and a_2, second bid of high type will be in (a_1, a_2). But we always have an implicit condition that first bid should be no lower than second bid, so we introduce the dashed 45-degree line and denote the trapezoid as the joint support.

6.2 Value Ordering $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2$

We will move on to show mixed strategy equilibria when value ordering becomes $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2$. And we have a result similar to theorem 7.

Theorem 8. *When $p \in (0,1)$, low type will mix by distribution $G_{L1}(x) = \frac{T\underline{v}_1}{\bar{v}_2 - x}$ where $T = -\frac{-\bar{v}_1+\bar{v}_2(1+p)+\sqrt{\bar{v}_1^2+(-1+p)(\bar{v}_2(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(\bar{v}_2+2\underline{v}_1 p)}}{2\bar{v}_2 p}$ with support $[0, a_1]$. In interval $[0, a_1]$, second bid of high type will follow distribution $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - x)}$. In interval $[a_1, a_2]$, first and second bid of high type will mix by distribution $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ and $F_{H2}(x) = \frac{\bar{v}_1+\underline{v}_1-2b-(1-p)C}{(1-p)(\bar{v}_1-x)} - \frac{p}{1-p}$ respectively. We are able to express $C = -\frac{-\bar{v}_1+\bar{v}_2(1+p)+\sqrt{\bar{v}_1^2+(-1+p)(\bar{v}_2(\bar{v}_2(-1+p)-4\underline{v}_1 p)+2\bar{v}_1(\bar{v}_2+2\underline{v}_1 p)}}{2(-1+p)}$, $a_1 = \bar{v}_2(1-H)$ and $a_2 = \bar{v}_2 - (1-p)C$.*

Proof. We continue to assume the same structure of support as marginal distributions in theorem 7. So low type will face indifferent condition $p(\underline{v}_1 - b_{l1}) + (1-p)F_{H2}(b_{l1})(\underline{v}_1 - x) = p\underline{v}_1$. So we solve $F_{H2}(x) = \frac{px}{(1-p)(\underline{v}_1 - x)}$ on interval $[0, a_1]$ with $a_1 < (1-p)\underline{v}_1$.

Consider high type's indifferent condition when her first bid b_{h1} is in (a, b) and second bid b_{h2} is in (a_1, a_2): $p(\bar{v}_1 + \bar{v}_2 - b_{h1} - b_{h2}) + (1-p)[F_{H2}(b_{h1})(\bar{v}_1 - b_{h1}) + F_{H1}(b_{h2})(\bar{v}_2 - b_{h2})] = \bar{v}_1 + \bar{v}_2 - 2a_2$. Similar to the previous theorem, we check separating equilibrium for high type: if we assume a bidder is maximizing her expected payoff by choosing the optimal second bid b_{h2} for any first bid b_{h1}, so we have $-p + (1-p)[f_{H1}(b_{h2})(\bar{v}_2 - b_{h2}) - F_{H1}(b_{h2})] = 0$ by taking first order derivative with respect to b_{h2}. Solving the differential equation, we have $F_{H1}(x) = \frac{C}{\bar{v}_2 - x} - \frac{p}{1-p}$ on interval (a_1, a_2) with some constant C to be determined. Expected payoff for second bid of high type is $p(\bar{v}_2 - b_{h2}) + (1-p)F_{H1}(b_{h2})(\bar{v}_2 - b_{h2}) = (1-p)C$ and we can solve $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_3, a_2). The two distribution functions should match when $x = a_2$ since we have $F_{H1}(a_2) = F_{H2}(a_2) = 1$ by construction. And we solve $a_2 = \bar{v}_2 + (-1+p)C$.

When second bid of high type b_{h2} is below a_1 and first bid b_{h1} is in interval (a_1, a_3), indifferent condition for high type will become $p[\bar{v}_1 - b_{h1} + G_{L1}(b_{h2})(\bar{v}_2 - b_{h2})] + (1-p)F_{H2}(b_{h1})(\bar{v}_1 - b_{h1}) = \bar{v}_1 + \bar{v}_2 - 2a_2$. We still conclude the constant payoff for second bid of high type is positive, which requires low type to put an atom with size T at 0. So high type will get $pG_{L1}(b_{h2})(\bar{v}_2 - b_{h2}) = pH\bar{v}_2$ when her second bid is below a_1 and we better require $pT\bar{v}_2 = C(1-p)$ so that high type will not want to deviate her second bid. So $G_{L1}(x) = \frac{T\bar{v}_2}{\bar{v}_2 - x}$, which reaches 1 when $x = \bar{v}_2(1-T)$. And we require $a_1 = \bar{v}_2(1-T) < \underline{v}_1(1-p)$. We continue to solve $F_{H2}(x) = \frac{\bar{v}_1 + \bar{v}_2 - 2b - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p}$ on interval (a_1, a_3) since $pT\bar{v}_2 = C(1-p)$. So we have same expressions for F_{H2} on interval (a_1, a_3) and (a_3, a_2). When two expressions of F_{H2} match, we solve another expression of a_1, i.e. $a_1 = \frac{\bar{v}_1\underline{v}_1 - \bar{v}_2\underline{v}_1 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 + \bar{v}_2 - \underline{v}_1 p + C - pC}$ should also solve $\frac{\bar{v}_1 + \bar{v}_2 - 2a_2 - (1-p)C}{(1-p)(\bar{v}_1 - x)} - \frac{p}{1-p} = \frac{px}{(1-p)(\underline{v}_1 - x)}$.

Additionally, solving $F_{H1}(x) = 0$ we are able to generate another version of a_1, where $a_1 = \frac{\bar{v}_2 p - C + pC}{p}$. So letting $\frac{\bar{v}_2 p - C + pC}{p} = \frac{\bar{v}_1\underline{v}_1 - \bar{v}_2\underline{v}_1 - \bar{v}_1\underline{v}_1 p + \underline{v}_1 C - \underline{v}_1 pC}{\bar{v}_1 + \bar{v}_2 - \underline{v}_1 p + C - pC}$, we solve $C = -\frac{-\bar{v}_1 + \bar{v}_2(1+p) + \sqrt{\bar{v}_1^2 + \bar{v}_2(-1+p)((\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(\bar{v}_2 + 2\underline{v}_1 p)}}{2(-1+p)}$, which is positive as long as $p \in (0,1)$. We solve T via $pT\bar{v}_2 = C(1-p)$ and $T = -\frac{-\bar{v}_1 + \bar{v}_2(1+p) + \sqrt{\bar{v}_1^2 + (-1+p)(\bar{v}_2(\bar{v}_2(-1+p) - 4\underline{v}_1 p) + 2\bar{v}_1(\bar{v}_2 + 2\underline{v}_1 p)}}{2\bar{v}_2 p}$. Similarly to C, T is guaranteed to be positive but smaller than 1 for all $p \in (0,1)$ [12].. What differs this theorem from theorem 7 is that in this scenario we only require $a_2 = \bar{v}_2 + (-1+p)C < \bar{v}_2$ because marginal valuation of second object of high type is now $\bar{v}_2$ and low type never bids above a_1. (We will argue $a_1 < \underline{v}_1$ later.) We may move on to compute that the upper bound of support b is smaller than $\underline{v}_1$ *if and only if* $p \in (\frac{1}{2}[\frac{\bar{v}_1(-\bar{v}_2 + \underline{v}_1)}{(\bar{v}_1 - \bar{v}_2)\underline{v}_1} + \frac{\sqrt{(\bar{v}_2 - \underline{v}_1)(4\bar{v}_2\underline{v}_1^2 - 4\bar{v}_1\underline{v}_1(\bar{v}_2 + \underline{v}_1) + \bar{v}_1^2(\bar{v}_2 + 3\underline{v}_1)}}{(\bar{v}_1 - \bar{v}_2)\underline{v}_1}], 1)$, which indicates a very intuitive result: when p, the probability of low type appearing in the population is relatively large, high type will focus on outbidding low type and hence a high type will not bid above $\underline{v}_1$, the highest marginal valuation a low type will have; when probability of low type appearing in the population is relatively small, a high type will focus on outbidding another high type, which indicates that bids for high type will surpass $\underline{v}_1$ but not $\bar{v}_2$ since second bid of high type will never be higher than $\bar{v}_2$. At the same time, to make sure that $a_1 = \bar{v}_2(1-T) < \underline{v}_1(1-p)$, we only need $p \in (0,1)$. Additionally, it is easy to see $a_1 < \underline{v}_1(1-p) < \underline{v}_1$ and $a_1 = \bar{v}_2(1-T) < b = \bar{v}_2 + (-1+p)C < \bar{v}_2$ since we construct $pT\bar{v}_2 = C(1-p)$.

By construction, expected payoffs for each bid of high type are constant, that is to say, first and second bid of high type are always getting expected payoffs $\bar{v}_1 + \underline{v}_1 - 2b - (1-p)C$ and $C(1-p)$ respectively. So

[12] $T \to 1$ when $p \to 1$ and $H \to 0$ when $p \to 0$.

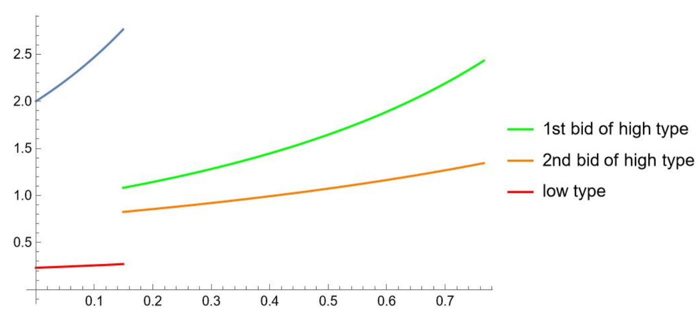
second bid of low type will not have particular incentive to deviate. Similarly, expected payoff for first bid of high type is constant, which makes a high type between indifferent bidding first bid above or below a as long as her first bid is higher than a_1. If a high type puts both her bids below a_1 (i.e. first bid of low type is below a_1), expected payoff for higher bid of high type is
$p(\bar{v}_1 - b_{h1}) + (1-p)\frac{pb_{h1}}{(1-p)(\underline{v}_1 - b_{h1})}(\bar{v}_1 - x) = p(\bar{v}_1 - b_{h1})\frac{\bar{v}_2}{\underline{v}_1 - b_{h1}}$, which is an increasing function of b_{h1}. So high type would rather make her first bid at a_1. And we conclude high type will not deviate her bids. If low type deviates and bid higher than a_1, she gets
$p(\underline{v}_1 - b_{l1}) + (1-p)[\frac{\bar{v}_1 + \underline{v}_1 - 2a_2 - (1-p)C}{(1-p)(\bar{v}_1 - b_{l1})} - \frac{p}{1-p}](\underline{v}_1 - b_{l1}) = \frac{\underline{v}_1 - b_{l1}}{\bar{v}_1 - b_{l1}}[\bar{v}_1 + \underline{v}_1 - 2a_2 - (1-p)C]$, which is decreasing in b_{l1}. So for a low type, bidding above a_1 is dominated by bidding exactly at a_1. What's more, no type will bid higher than a_2 since the marginal distributions will have no atoms at upper bound of support and bidding (a_2, a_2) will guarantee high type two objects.

Note that our two indifferent conditions for high type indicate that when first bid of high type is bidding below a threshold $a_3 < a_2$, second bid of high type will be no greater than a_1. But since we have established that F_{H1}, the marginal distribution of first bid of high type, will be following the same functional form in both scenarios and that both bids of high type are making the same constant payoffs in both scenarios, it actually does not matter which value we select as a_3 as long as it is strictly smaller than b and greater than a_1. □

In this subsection, we do not have an analogous result to theorem 6, since theorem 6 in this scenario would require high type to bid $(\underline{v}_1, \underline{v}_1)$ for both bids and low type to mix in interval $[0, \underline{v}_1]$. However, in this scenario when a low type faces another low type, she will want to bid 0 instead of mixing in any interval.

Graphical Illustration

We demonstrate theorem 8 by picking $\bar{v}_1 = 3$, $\bar{v}_2 = 2$, $\underline{v}_1 = 1$ and $p = \frac{2}{3}$:



Pdf of low type's mixed strategy is displayed in red, with support being $[0, 0.149]$. Second bid of high type has two parts: the blue curve in interval $[0, 0.586]$ and orange curve on interval $[0.149, 0.766]$. Pdf of first bid of high type is displayed by the green curve with support being $[0.149, 0.766]$. Note that there will be an atom of size 0.925 for distribution of mixed strategy of low type at 0, which is understandable since low type's marginal valuation is only $\underline{v}_1$.

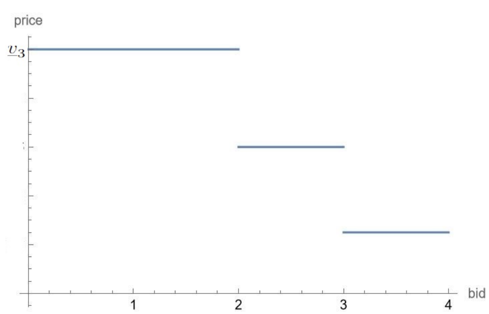
7 More Than Two Units

In the Introduction, we highlighted that bidding behavior in the Turkish Treasury auction takes the form of a step-function. That is, there are quantities $q_1 < \cdots < q_K$, so that bids jump downward at each quantity q_k but bids are constant at all quantities between q_k and q_{k+1}. (See Hortaçsu and McAdams, 2010.) This section shows, by way of example, that bidding behavior in a multi-unit auction can take the form of a step function.

To do so, we focus on the minimal environment that can distinguish a step function from either a cutoff rule or a strictly negatively sloped bid function: an environment with four units. We provide two examples in which the low type's bidding behavior is consistent with a step function. (The focus on the low type is only for tractability.) The two examples differ in the qualitative nature of the step functions: In the first example, the bids are same for the first two units; in the second example, the bids are the same for the middle two units.

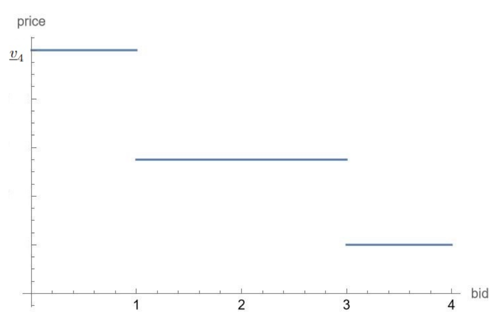
There are four identical units and two bidders that are ex-ante identical. The high type ($\bar{v}$) has marginal valuations $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)$ with $\bar{v}_1 > \bar{v}_2 > 0$ and $\bar{v}_3 = \bar{v}_4 = 0$. The low type has marginal valuations $(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$, $\underline{v}_1 > \underline{v}_2 > \underline{v}_3 \geq \underline{v}_4 \geq 0$. The probability of the low type ($\underline{v}$) is p and the probability of the high type ($\bar{v}$) is $1-p$. Now a bid for i is a profile $(b_{i1}, b_{i2}, b_{i3}, b_{i4}) \in R^4_+$ with $b_{i1} \geq b_{i2} \geq b_{i3} \geq b_{i4}$. We refer to b_{in} as bidder i's n^{th} bid. For a given strategy of bidder i, the marginal distribution of the high type's ($\bar{v}$'s) n^{th} bid is F_{Hn} and the marginal distribution of the low type's ($\underline{v}$'s) n^{th} bid is G_{Ln}.

Assume $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2 > \underline{v}_3 = \underline{v}_4 > 0$. Moreover, assume that the probability of type $\underline{v}$ is some $p = \frac{\underline{v}_3}{\underline{v}_2}$. Then there exists an equilibrium that takes the following form: Each type bids $\underline{v}_3$ for their first and second bids. The high type bids 0 for their third and fourth bids. But, the low type mixes on the interval $(0, \underline{v}_3)$ for their third and fourth bids; in particular, the low type's mixture is different for the third and fourth bids. Under this equilibrium, if the pure-strategy (b_1, b_2, b_3, b_4) is in the support of the equilibrium for the low type, then the pure strategy is a step function that is constant on units 1 and 2, lower for the third unit and even lower for the fourth unit. Such a realized pure-strategy is illustrated in the following plot:



To understand why this is an equilibrium, note that under this strategy profile both bidders receive 2 units for sure. There is no incentive to bid higher, since a higher bid can only serve to pay a higher price and potentially get a third unit at that higher price. But since $\underline{v}_3 > \bar{v}_3$, no bidder would want to get a third unit at a higher price. Likewise, no bidder has an incentive to bid lower. This is because the low type bids aggressively on the third and fourth bids—sufficiently aggressively to ensure that the high type would not deviate downward. In particular, choosing $G_{L3}(x) = \frac{(1-p)x}{p(\underline{v}_2 - x)}$ and $G_{L4}(y) = \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)y}{p(\underline{v}_1 - y)}$ ensures that this constitutes an equilibrium. What's more, by restricting probability p to $p=\frac{\underline{v}_3}{\underline{v}_2}$, we are able to construct a correlation (functional relationship) between third and forth bids b_{l3}, b_{l4} of low type by $b_{l4}=h(b_{l3})=\frac{\underline{v}_3(\underline{v}_2-b_{l3})+\underline{v}_1(-\underline{v}_3+b_{l3})}{\underline{v}_2-\underline{v}_3}$. This correlation guarantees realization of bids for low type will be of the shape shown in the previous paragraph with probability 1.

For the second set of examples, we assume the ordering of private valuation is $\underline{v}_1 > \bar{v}_1 > \bar{v}_2 > \underline{v}_2 > \underline{v}_3 > \underline{v}_4 > 0$. We will show an example when $p=\frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2} > \frac{\underline{v}_4}{\bar{v}_1}$, $\bar{v}_2 > \underline{v}_2+\underline{v}_3$ and $\bar{v}_2=\frac{\underline{v}_2\underline{v}_4}{2\underline{v}_4-\underline{v}_3}$. In this example, each bidder will use pure strategy at $\underline{v}_4$ for first bids and accordingly is guaranteed to win her first unit. Observing this, a type-$\underline{v}$ bidder knows that she can not win the fourth unit and hence the fourth bid of type-$\underline{v}$ bidder will only mix to prevent first bids from deviating downward. Type-$\underline{v}$ bidders understand that their second (third) bids are competing with opponents' third (second) bids, and hence bidders will have incentive to bid lower (higher) prices for their second (third) bids, which makes second and third bids identical since the lowest feasible bids for bidders' to pick for second bids will be the third bids and vice versa. First and fourth bids in this example is similar to our result from theorem 5, and argument for second and third bids are similar to scenario described in theorem 2. Mathematically speaking, first bid of type-$\bar{v}$ will be a pure strategy at $\underline{v}_4$, and second bid of type-$\bar{v}$ will be mixing in interval $(0,\underline{v}_4)$ by distribution $F_{H2}(x)=\frac{x(x+\bar{v}_2-\underline{v}_2-\underline{v}_3)}{(\bar{v}_2-x)(\underline{v}_3-x)}$. First bid of type-$\underline{v}$ will be a pure strategy at $\underline{v}_4$, second and third bids of type-$\underline{v}$ will be identical in interval $(0,\underline{v}_4)$ by distribution $G_{L2}(x)=G_{L3}(x)=\frac{(1-p)x}{p(\bar{v}_2-x)}$ and fourth bid of type-$\underline{v}$ will follow distribution $G_{L4}(x)=\frac{p\bar{v}_1+(1-p)x-\underline{v}_4}{p(\bar{v}_1-x)}$ in interval $(0,\underline{v}_4)$. We particularly require $p=\frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}$ and $\bar{v}_2=\frac{\underline{v}_2\underline{v}_4}{2\underline{v}_4-\underline{v}_3}$ to make the support of distributions above to have identical endpoints. What's more, to make $G_{L4}(x)$ and $F_{H2}(x)$ non-negative over support $(0,\underline{v}_4)$, we impose conditions $p(=\frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}) > \frac{\underline{v}_4}{\bar{v}_1}$ and $\bar{v}_2 > \underline{v}_2+\underline{v}_3$ [13]. A feasible example of private valuations can be $\bar{v}=(6,4,0,0)$ and $\underline{v}=(7,2,1.5,1)$. We can plot a possible realization of bids for type-$\underline{v}$ as well [14]:



8 Comparison of Expected Revenue

We are interested in comparing expected revenue for three common formats of multi-unit auctions: pay-as-bid auction, uniform-price auction and Vickrey auction. For uniform-price auction, winners in the auction will pay the highest losing price. It is straightforward to check that the strategy where bidders bid truthfully for their first units and bid 0 for their second units forms an equilibrium. So each bidder wins exactly one unit but always pays zero, which leads to an expected revenue of zero.

[13] We pick the precise probability at $\frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}$ for simplicity since it leads to an atomless $F_{H2}(x)$ distribution. and we may allow $\frac{\underline{v}_4}{\bar{v}_1} < p < \frac{\underline{v}_3}{2\bar{v}_2-\underline{v}_2}$ by putting an atom at 0 with $F_{H2}(x)=\frac{x(x+\bar{v}_2-\underline{v}_2-\underline{v}_3)}{(\bar{v}_2-x)(\underline{v}_3-x)}+\frac{\underline{v}_3-2\bar{v}_2p+\underline{v}_2p}{(1-p)(\underline{v}_3-x)}$, as with theorem 2.

[14] There is a functional relationship h between third and fourth bids b_{l3}, b_{l4} of type-$\underline{v}$ by $b_{l4}=h(b_{l3})=\frac{\underline{v}_4(\bar{v}_2-b_{l3})+\bar{v}_1(-\underline{v}_4+b_{l3})}{\bar{v}_2-\underline{v}_4}$.

For Vickrey auction, any bidder i who wins k_i units will be paying the highest k_i losing bids among her rivals. And accordingly, one equilibrium for Vickrey auction in our multi-unit setting is that each bidder is bidding the marginal valuations truthfully for every unit [15]. So in this equilibrium, each bidder will win exactly one unit and be paying 0 when facing a high type and $\underline{v}_2$ when facing a low type. Expected payment for each bidder is $p\underline{v}_2$, which makes total revenue equivalent to $2p\underline{v}_2$.

For pay-as-bid auction, we restrict to situation where private valuation satisfies $\bar{v}_1 > \underline{v}_1 + \underline{v}_2$ and check all range of p. When $p < \frac{\underline{v}_2}{2\bar{v}_1 - \underline{v}_1}$, both bidders will mix in interval $(0, \bar{v}_1 p)$, as summarized by theorem 2. There will be probability p^2 two low types meet, probability $2p(1-p)$ a high and a low type meet and probability $(1-p)^2$ two high types meet. Our approach for expected revenue is to establish is the order statistics for highest and second highest bid and compute the expected value.

First we consider the scenario when two high types meet. We denote B_1, B_2 as non-zero bids for those two bidders and B_1, B_2 are independent. Since high type's valuation toward second unit is normalized to 0, $F_{(2)}(x) = \mathbb{P}(B_1 \leq x, B_2 \leq x) = F_H^2(x)$ and
$F_{(1)}(x) = 1 - \mathbb{P}(B_1 > x, B_2 > x) = 1 - (1 - F_H(x))^2 = 2F_H(x) - F_H^2(x)$. Expected value of highest two bids in this scenario will be $\pi_{HH} = \int_0^{\bar{v}_1 p} x\, dF_H^2(x) + \int_0^{\bar{v}_1 p} y\, d[2F_H(y) - F_H^2(y)]]$. We now consider scenario when two low types meet. In this scenario, bidders will propose four bids. We assume the bids are $B_{11}, B_{12}, B_{21}, B_{22}$ with $B_{11} = B_{12}$ and $B_{21} = B_{22}$. What's more, B_1's and B_2's are independent. So $F_{(3)}(x) = F_{(4)}(x) = \mathbb{P}(B_{11} \leq x, B_{12} \leq x, B_{21} \leq x, B_{22} \leq x) = \mathbb{P}(B_{11} \leq x, B_{21} \leq x) = G_L^2(x)$ since $B_{11} = B_{12}$ and $B_{21} = B_{22}$. Expected value of highest two bids in this scenario will be
$\pi_{LL} = 2\int_0^{\bar{v}_1 p} x\, d[G_L^2(x)]$.

Now we consider the scenario when a high and a low type meet. Assume that high type's bid is B_H, and low type's bids are B_{L1}, B_{L2} with $B_{L1} = B_{L2}$. It is clear that B_H and B_L's are independent since they comes from different bidders who bid independently. So the order statistics will be
$\mathbb{P}(B_{(1)} \leq x) = 1 - \mathbb{P}(B_H \geq x, B_{L1} \geq x, B_{L2} \geq x) = 1 - \mathbb{P}(B_H \geq x, B_{L1} \geq x)$ since we have $B_{L1} = B_{L2}$.
$\mathbb{P}(B_{(2)} \leq x) = \mathbb{P}(B_H \leq x, B_{L1} \leq x, B_{L2} \leq x) + \mathbb{P}(B_H > x, B_{L1} \leq x, B_{L2} \leq x)$
$= \mathbb{P}(B_{(3)} \leq x) + \mathbb{P}(B_{L1} \leq x < B_H)$. $\mathbb{P}(B_{(3)} \leq x) = \mathbb{P}(B_H \leq x, B_{L1} \leq x, B_{L2} \leq x) = \mathbb{P}(B_H \leq x, B_{L1} \leq x)$ by $B_{L1} = B_{L2}$. Note that $1 - \mathbb{P}(B_H \geq x, B_{L1} \geq x), \mathbb{P}(B_H \leq x, B_{L1} \leq x)$ are just expressions for order statistics when there are only two bids B_H and B_{L1}. And $\mathbb{P}(B_{(2)} \leq x)$ happens when all bids are smaller than x or when only bids from low type are smaller than x.

We can invoke the Bapat–Beg Theorem [16] to compute CDF of order statistics of non-identical distributions when we only have B_H, B_{L1}. If we use $F_{X_{(i)}}$ to denote distributions of order statistics when we have three bids B_{L1}, B_{L2}, B_H and $F_{(i)}$ to denote distributions of order statistics when we have two bids B_{L1}, B_H, our argument above shows that $F_{X_{(3)}}(x) = F_{(2)}(x) = \frac{\text{per}\begin{bmatrix} F_H(x) & F_H(x) \\ G_L(x) & G_L(x) \end{bmatrix}}{2!(2-2)!} = F_H(x)G_L(x)$ with per being permanent of the given block matrix. And accordingly,

[15] Actually, equilibrium strategy mentioned in the previous paragraph also forms an equilibrium in Vickrey auction, but it is traditional to look at the truthful reporting equilibrium for Vickrey auction.

[16] Theorem 4.2 from Bapat, Beg (1989), proved in Hande (1994).

$F_{X_{(2)}}(x) = F_H(x)G_L(x) + G_L(x)[1 - F_H(x)] = G_L(x)$. Expected value of highest two bids in this scenario will be $\pi_{HL} = \int_0^{\bar{v}_1 p} x\, d[F_H(x)G_L(x)] + \int_0^{\bar{v}_1 p} y\, d[G_L(y)]]$.

With order statistics established, we may compute expected valuation of the two distributions which is also the monetary payment for the first and second unit respectively, i.e. expected revenue of the pay-as-bid auction is $p^2\pi_{LL} + 2p(1-p)\pi_{HL} + (1-p)^2\pi_{HH}$. If we assume $\bar{v}_1 = 3, \underline{v}_1 = 2$, and $\underline{v}_2 = 1$, expected revenue from pay-as-bid auction will dominate Vickrey auction when $p \in [0.125, \frac{1}{4}]$ [17]. So our example indicates that revenue ranking between pay-as-bid auction and Vickrey auction is ambiguous. We conclude that pay-as-bid and Vickrey auction dominates uniform-price auction in terms of expected revenue but ranking between pay-as-bid and Vickrey auction is ambiguous.

If we raise probability of p to range covered by theorem 3, where distribution function gets more complicated since for some subset of p low type may bid differently, we can instead compute expected value of bids from high and low type. Summation of any such two expected values should be no greater than the summation of expected value of highest and second highest bids by construction. However, we can report that expected value of any single bid from either type is greater than p, which makes summation of expected values of any two bids greater than $2p$, the expected revenue of Vickrey auction. When $p \in [\frac{\underline{v}_2}{\underline{v}_1}, 1]$, theorem 4 indicates that each bidder will always bid $\underline{v}_2$ and accordingly expected revenue of auction under theorem 4 will be $2\underline{v}_2$, which is higher than the expected revenue of Vickrey auction as well. In all, we conclude that if we assume $\bar{v}_1 = 3, \underline{v}_1 = 2$, and $\underline{v}_2 = 1$, Vickrey auction generates higher expected revenue when $p < 0.125$ and pay-as-bid auction generates higher expected revenue when $p > 0.125$.

The last interesting result to notice is that despite having identical allocations where each bidder wins one unit, our hypothetical uniform-price auction and Vickrey auction generate different expected revenue. To validate the revenue equivalence theorem for single-unit auction, one necessary condition is some type should get same expected payoff from different formats of auctions. But payment from Vickrey and uniform-price auctions are not identical as shown in the previous paragraph. Another obvious violation in the establishment of revenue equivalence theorem is that we need to integrate over the range from lowest type to some type to construct payment, but we do not have such integration due to our discrete type space.

9 Conclusion and Discussion

We study a multi-item auction where there are two discrete types of bidders and each type of bidder demands two objects. We always assume a high type will have marginal valuations $\bar{v}_1, \bar{v}_2$ and low type will have marginal valuations $\underline{v}_1, \underline{v}_2$. But we focus on case with ordering $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$. After normalizing the smallest marginal valuation (i.e. $\bar{v}_2$) to 0, we look at symmetric pure and mixed strategy equilibria for different proportions of high and low types in the population. We find out that high type may put an atom at 0 for distribution of first bid when probability low type appears in the auction is small, and low type will bid identically for both units in most mixed strategy equilibria (i.e. perfectly correlated equilibrium). We find out empirical evidence which is consistent with the identical bidding behaviour from our theoretical results and are able to extend some of our results into higher-unit environment to show bidders would still

[17] 0.125 is a decimal approximation of an irrational number starting with 0.1249595, not $\frac{1}{8}$.

bid identically for several units when they bid for more units. We will have pure strategy equilibrium when probability low type appears is relatively large and bidders are just bidding $\underline{v}_2$, the marginal valuation of low type's second object.

Given that private valuations in our auction are $\bar{v}_1 > \underline{v}_1 > \underline{v}_2 > \bar{v}_2$, an efficient allocation should let each bidder get one object since whenever a high/low type meets another high/low type, the highest two valuations always come from valuations of first marginal valuation from different bidder. But in majority of our results, we propose perfectly correlated equilibrium where low type bids identically. What's more, we have overlapping of supports when high and low types bid mixed strategy. All the features above indicate that our equilibrium allocation is likely to be inefficient by assigning both objects to one low-type bidder with positive probability (i.e. misallocation). This inefficiency arises from the fact that bidders understand their higher bids are competing with opponents' lower bids and they will have incentive to make their first bids lower for higher net payoff. But knowing first bids will be generally low in price, bidders will consequently bid higher second bids for a better chance of winning.

Using a terminology from auction literature, we conclude low types in our model are displaying *differential bid-shading* behaviour: when two bids from a low type are identical it must be that a low type is bid-shading more on her first bid. The differential bid-shading behaviour in our multi-item auction makes it impossible to know the true types of bidders from auction results when both high and low types share the identical support for their bids, as situations described in lemma 11 and 14.

Besides this inefficient allocation feature, our analysis finds a conditionally deterministic relationship between two bids for low type, i.e. if we know the range of p and what low type bids for her first bid b_{l1}, we can compute her second bid b_{l2}. The most common case in our result is when low type bids identically i.e. $b_{l2} = b_{l1}$. Previous literature like Anwar (2007) and Engelbrecht-Wiggans and Kahn (1998) also reported such type of pooling equilibrium. We also find out cases where first (b_{l2}) and second bid (b_{l2}) of low type follow a functional relationship $b_{l2} = h(b_{l1}) \leq b_{l1}$ for all (b_{l1}, b_{l2}) in support $[a_3, a_4]^2$, as displayed in lemma 13 and 16. We may treat low type's bids x, y as solution to an optimization problem where low type is trying to compute her optimal second bid b_{l2} given every possible first bid b_{l1} in the joint support of bids (b_{l1}, b_{l2}). And consequently situations where $b_{l2} = b_{l1}$ can be treated as corner solution to the optimization problem while $b_{l2} = h(b_{l1}) \leq b_{l1}$ is an interior solution.

We know that inefficiency comes from misallocation of objects since our symmetric equilibria propose identical bids for low type and overlapping of support for different types. To achieve efficiency under the private valuations in our model, each bidder should just get one object. Our results always imply a positive probability of inefficient allocations, although we checked all possible combinations of high and low types. However, we do not establish uniqueness of our mixed strategies, and hence we can not exclude possibilities of efficient allocations through mixed strategy equilibrium. Other potential solutions to this issue and future questions to answer may include whether we can have efficient allocation if we implement simultaneous auctions with the same valuation distribution introduced in multi-item auction. It may be that in a simultaneous auction bidders propose their higher bids toward different objects and each ends up getting one object.

We also compared expected revenue of several common formats of multi-unit auction: pay-as-bid auction, uniform-price auction and Vickrey auction. We found that uniform-price auction would give the lowest expected revenue among the three while ranking between pay-as-bid and Vickrey auction was ambiguous. Our numerical example comparing revenue from pay-as-bid and Vickrey auction was weakly monotone in p, which indicates that there is a cutoff p^* so that pay-as-bid auction dominated Vickrey auction if and only if $p > p^*$.

Our results when valuation ordering is $\bar{v}_1 > \underline{v}_1 > \bar{v}_2 > \underline{v}_2$ or $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2$ differ from the main results discussed above in two features: we do not find perfectly correlated equilibrium for any type and we do not have the conditional deterministic relationship between two bids from same type. One common feature is that we are not guaranteed to have efficient allocations in these cases either since overlapping of support persists. which will lead to misallocation of objects.

A Proof in Lemma 15

Computation shows when $\underline{v}_1 + \sqrt{2}\underline{v}_1 \leq 2\bar{v}_1$, $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ is enough;

On the other hand, if $\underline{v}_1 + \sqrt{2}\underline{v}_1 \geq 2\bar{v}_1$, computation generates
$(\underline{v}_1 + \sqrt{-4\bar{v}_1^2 + 4\bar{v}_1\underline{v}_1 + \underline{v}_1^2} + 2\underline{v}_2 \leq 2\bar{v}_1) \cup (2\bar{v}_1 + \sqrt{-4\bar{v}_1^2 + 4\bar{v}_1\underline{v}_1 + \underline{v}_1^2} \leq \underline{v}_1 + 2\underline{v}_2)$
$\cup[(2\bar{v}_1 + \sqrt{-4\bar{v}_1^2 + 4\bar{v}_1\underline{v}_1 + \underline{v}_1^2} > \underline{v}_1 + 2\underline{v}_2) \cap (\underline{v}_1 + \sqrt{-4\bar{v}_1^2 + 4\bar{v}_1\underline{v}_1 + \underline{v}_1^2} + 2\underline{v}_2 > 2\bar{v}_1) \cap (p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2})]$.
To interpret this result, we denote $\mathbb{P}_1 = (\underline{v}_1 + \sqrt{-4\bar{v}_1^2 + 4\bar{v}_1\underline{v}_1 + \underline{v}_1^2} + 2\underline{v}_2 \leq 2\bar{v}_1)$,
$\mathbb{P}_2 = (2\bar{v}_1 + \sqrt{-4\bar{v}_1^2 + 4\bar{v}_1\underline{v}_1 + \underline{v}_1^2} \leq \underline{v}_1 + 2\underline{v}_2)$ and $A = (p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2})$. So expression above is actually $\mathbb{P}_1 \cup \mathbb{P}_2 \cup [\mathbb{P}_1^c \cap \mathbb{P}_2^c \cap A]$.

$\mathbb{P}_1 \cup \mathbb{P}_2 \cup [\mathbb{P}_1^c \cap \mathbb{P}_2^c \cap A] = [(\mathbb{P}_1 \cup \mathbb{P}_2) \cup (\mathbb{P}_1^c \cap \mathbb{P}_2^c)] \cap [(\mathbb{P}_1 \cup \mathbb{P}_2) \cup A]$ by distributive law of set operations. Note that complement of $\mathbb{P}_1 \cup \mathbb{P}_2$ is $\mathbb{P}_1^c \cap \mathbb{P}_2^c$, and accordingly $\mathbb{P}_1 \cup \mathbb{P}_2 \cup [\mathbb{P}_1^c \cap \mathbb{P}_2^c \cap A] = (\mathbb{P}_1 \cup \mathbb{P}_2) \cup A$. To see what is $\mathbb{P}_1 \cup \mathbb{P}_2$, we still compute its complement and it turns out complement of $\mathbb{P}_1 \cup \mathbb{P}_2$ is
$(2\bar{v}_1 < \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2})$. So $\mathbb{P}_1 \cup \mathbb{P}_2 = (2\bar{v}_1 \geq \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2})$. But it is straightforward to check $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq \underline{v}_1 + \sqrt{2}\underline{v}_1$ since $\underline{v}_2 < \underline{v}_1$, which indicates that $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ when $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1 < \underline{v}_1 + \sqrt{2}\underline{v}_1$ and $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$ when $2\bar{v}_1 < \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2}$ are both feasible solutions. (To be precise, the 2nd result should be $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$ when $2\bar{v}_1 < \underline{v}_1 + \sqrt{2}\underline{v}_1$, but $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq \underline{v}_1 + \sqrt{2}\underline{v}_1$ implies $2\bar{v}_1 < \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2}$ is a subset of condition $2\bar{v}_1 < \underline{v}_1 + \sqrt{2}\underline{v}_1$).

In conclusion, when $\underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2} \leq 2\bar{v}_1$, perfectly correlated equilibrium exists for $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2}{2\bar{v}_1-\underline{v}_1}$ while when $2\bar{v}_1 < \underline{v}_1 + \underline{v}_2 + \sqrt{\underline{v}_1^2 - \underline{v}_2^2}$, perfectly correlated equilibrium exists when $\frac{\underline{v}_2}{\bar{v}_1+\underline{v}_2} < p < \frac{\underline{v}_2(2\bar{v}_1-\underline{v}_1-\underline{v}_2)}{2\bar{v}_1^2-2\bar{v}_1\underline{v}_1+\underline{v}_1^2-\underline{v}_2^2}$.

B Computation in Section 7

B.1 Example 1

Consider a new case where high type has private valuation $\bar{v}_1 > \bar{v}_2 > v_7 = v_8 = 0$ and low type has private valuation $\underline{v}_1 > \underline{v}_2 > \underline{v}_3 = \underline{v}_4$. We assume that bidders are competing four identical objects where $\bar{v}_1 > \bar{v}_2 > \underline{v}_1 > \underline{v}_2 > \underline{v}_3$.

High type will not make one of her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as x if $\bar{v}_1 + \bar{v}_2 - 2\underline{v}_3 \geq (1-p)(\bar{v}_1 - \underline{v}_3 + \bar{v}_2 - x) + p[(\bar{v}_1 - \underline{v}_3) + (\bar{v}_2 - x)G_{L3}(x)] \iff G_{L3}(x) \leq \frac{p\bar{v}_2+(1-p)x-\underline{v}_3}{p(\bar{v}_2-x)}$.
Similarly, low type will not make one of her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as x if $\underline{v}_1 + \underline{v}_2 - 2\underline{v}_3 \geq (1-p)(\underline{v}_1 - \underline{v}_3 + \underline{v}_2 - x) + p[(\underline{v}_1 - \underline{v}_3) + (\underline{v}_2 - x)G_{L3}(x)]$
$\iff G_{L3}(x) \leq \frac{p\underline{v}_2+(1-p)x-\underline{v}_3}{p(\underline{v}_2-x)}$. We conclude $G_{L3}(x) \leq \frac{p\underline{v}_2+(1-p)x-\underline{v}_3}{p(\underline{v}_2-x)}$ since $\frac{p\underline{v}_2+(1-p)x-\underline{v}_3}{p(\underline{v}_2-x)} \leq \frac{p\bar{v}_2+(1-p)x-\underline{v}_3}{p(\bar{v}_2-x)}$.

High type will not make both her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as $x \geq y$ if $\bar{v}_1 + \bar{v}_2 - 2\underline{v}_3 \geq (1-p)(\bar{v}_1 - x + \bar{v}_2 - y) + p[G_{L4}(x)(\bar{v}_1 - x) + (\bar{v}_2 - y)G_{L3}(y)]$
$\iff G_{L4}(x) \leq \frac{p\bar{v}_1+p\bar{v}_2+(1-p)x-\underline{v}_3-p\underline{v}_2-(\bar{v}_2-\underline{v}_2)\frac{p\underline{v}_2+(1-p)y-\underline{v}_3}{\underline{v}_2-y}}{p(\bar{v}_1-x)}$.

Abusing notation, low type will not make both her highest two bids lower than $\underline{v}_3$ with her deviating bids denoted as $x \geq y$ if $\underline{v}_1 + \underline{v}_2 - 2\underline{v}_3 \geq (1-p)(\underline{v}_1 - x + \underline{v}_2 - y) + p[G_{L4}(x)(\underline{v}_1 - x) + (\underline{v}_2 - y)G_{L3}(y)]$ $\Longleftrightarrow G_{L4}(x) \leq \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)}$. Note that we can rewrite $\frac{p\bar{v}_1 + p\bar{v}_2 + (1-p)x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)} =$ $\frac{p(\bar{v}_1 - x) + p\bar{v}_2 + x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)} = 1 + \frac{p\bar{v}_2 + x - \underline{v}_3 - p\underline{v}_2 - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}}{p(\bar{v}_1 - x)}$ $= 1 + \frac{p(\bar{v}_2 - \underline{v}_2) - (\bar{v}_2 - \underline{v}_2)\frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y} + x - \underline{v}_3}{p(\bar{v}_1 - x)} = 1 + \frac{(\bar{v}_2 - \underline{v}_2)[p - \frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}] + x - \underline{v}_3}{p(\bar{v}_1 - x)}$ $= 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} + \frac{(\bar{v}_2 - \underline{v}_2)[p - \frac{p\underline{v}_2 + (1-p)y - \underline{v}_3}{\underline{v}_2 - y}]}{p(\bar{v}_1 - x)} = 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} + \frac{(\bar{v}_2 - \underline{v}_2)(\underline{v}_3 - y)}{p(\bar{v}_1 - x)(\underline{v}_2 - y)}$. Since both x, y satisfy $y \leq x \leq \underline{v}_3 \leq \underline{v}_2 \leq \bar{v}_2 \leq \bar{v}_1$, we conclude that $\frac{(\bar{v}_2 - \underline{v}_2)(\underline{v}_3 - y)}{p(\bar{v}_1 - x)(\underline{v}_2 - y)}$ is positive. We have $\frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)} = 1 + \frac{x - \underline{v}_3}{p(\underline{v}_1 - x)} \leq 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} \leq 1 + \frac{x - \underline{v}_3}{p(\bar{v}_1 - x)} + \frac{(\bar{v}_2 - \underline{v}_2)(\underline{v}_3 - y)}{p(\bar{v}_1 - x)(\underline{v}_2 - y)}$. So we conclude that $G_{L4}(x) \leq \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)}$. And it is not hard to check $\frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)} \leq \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)x}{p(\underline{v}_1 - x)}$. So if $G_{L3}(x) = \frac{p\underline{v}_2 + (1-p)x - \underline{v}_3}{p(\underline{v}_2 - x)}$, it is feasible to pick $G_{L4}(y) = \frac{p\underline{v}_1 - \underline{v}_3 + (1-p)y}{p(\underline{v}_1 - y)}$.

B.2 Example 2

We suppose high type is bidding $\underline{v}_4$ for her first bid so she will be getting constant payoff. We need to look at her indifferent condition: $pG_{L3}(x)(\bar{v}_2 - x) + (1-p)(\bar{v}_2 - x) = (1-p)\bar{v}_2$. And hence we have $G_{L3}(x) = \frac{(1-p)x}{p(\bar{v}_2 - x)}$. A high type will not deviate to bid $\underline{v}_4 > x \geq y$ if $pG_{L3}(y)(\bar{v}_2 - y) + (1-p)(\bar{v}_2 - y) + pG_{L4}(x)(\bar{v}_1 - x) + (1-p)(\bar{v}_1 - x) \leq pG_{L3}(x)(\bar{v}_2 - x) + (1-p)(\bar{v}_2 - x) + \bar{v}_1 - \underline{v}_4$. Note that $pG_{L3}(y)(\bar{v}_2 - y) + (1-p)(\bar{v}_2 - y) = \frac{(1-p)y}{(\bar{v}_2 - y)}(\bar{v}_2 - y) + (1-p)(\bar{v}_2 - y) = (1-p)\bar{v}_2$, so we should have $G_{L4}(x) \leq \frac{\bar{v}_1 - \underline{v}_4 - (1-p)(\bar{v}_1 - x)}{p(\bar{v}_1 - x)} = \frac{p\bar{v}_1 - \underline{v}_4 + (1-p)x}{p(\bar{v}_1 - x)}$.

For a low type, we still assume her first bid is at $\underline{v}_4$. And a low type will not deviate her first bid downward if expected payoff for her first bid is non-increasing: i.e. $(1-p)(\underline{v}_1 - x) + pG_{L4}(x)(\underline{v}_1 - x)$ needs to have a non-decreasing derivative. $\frac{d}{dx}[(1-p)(\underline{v}_1 - x) + pG_{L4}(x)(\underline{v}_1 - x)] = -\frac{(\bar{v}_1 - \underline{v}_1)(\bar{v}_1 - \underline{v}_4)}{(\bar{v}_1 - x)^2}$ and hence we need $\underline{v}_1 > \bar{v}_1$. With first bids bidding a pure strategy at $\underline{v}_4$, fourth bid of low type will never win positive expected payoff and hence indifferent condition for low type can be simplified to $(1-p)[(\underline{v}_2 - x) + F_{H2}(y)(\underline{v}_3 - y)] + p[G_{L3}(x)(\underline{v}_2 - x) + G_{L2}(y)(\underline{v}_3 - y)] = (1-p)\underline{v}_2$. Expected payoff for second bid for low type is $(1-p)(\underline{v}_2 - x) + pG_3(x)(\underline{v}_2 - x)$ and will be a decreasing function for x. So second and third bid for low type should be identical. And we solve $F_{H2}(y) = \frac{y(y + \bar{v}_2 - \underline{v}_2 - \underline{v}_3)}{(\bar{v}_2 - y)(\underline{v}_3 - y)}$. To make right endpoints of the distributions established above identical, we need $p\bar{v}_2 = \underline{v}_4 = \frac{\bar{v}_2 \underline{v}_3}{2\bar{v}_2 - \underline{v}_2}$. We need $p = \frac{\underline{v}_3}{2\bar{v}_2 - \underline{v}_2}$ and $\bar{v}_2 = \frac{\underline{v}_2 \underline{v}_4}{2\underline{v}_4 - \underline{v}_3}$. With identical endpoints, it is not hard to check $G_2(x) \leq G_4(x)$. A low type will not deviate any single bid by construction. Expected payoff for first bid of low type is an increasing function so making first bid at $\underline{v}_4$ is always a best response. Monotone conditions on second and third bids will imply identical bids. At last, a low type is willing to mix her fourth bid in interval $(0, \underline{v}_2)$ since first bids of any bidder will be $\underline{v}_4$ so fourth bid of low type will never win.

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