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CONTENTS

INWARD SPIRALS IN TURTLE GEOMETRY*

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PROBLEM BACKGROUND

As a Mathematics major and a Data Processing minor, I am currently preparing to engage in secondary mathematics and computer science education. A major portion of that preparation involves a teaching internship. A component of my internship is teaching a beginning computer course to junior high school students. A significant part of that course deals with turtle geometry in the programming language LOGO.

In order to understand the concept of turtle geometry, think of the turtle as a computer controlled animal that lives on the display screen and responds to LOGO commands that make it do the following:

1. Move forward and backward.
2. Rotate left and right.

* A paper presented at the 1987 National Convention of KME and awarded second place by the Awards Committee.

As the turtle moves, it leaves a trace of its path, and in this way, can be used to make drawings on the display screen. For example, the turtle will draw a square of side 100 units if it repeats the following commands four times:

```
FORWARD 100
RIGHT 90.
```

A procedure to produce this square is:

```
TO SQUARE
    REPEAT 4 [FORWARD 100 RIGHT 90]
END.
```

In order to help better prepare to teach turtle geometry, I recently undertook an independent study that focused on turtle geometry. The subject of this paper is an interesting finding I've made in my study of a particular LOGO procedure which Abelson and diSessa[1] call INSPI. INSPI is an abbreviation for inward spiral. The procedure is:

```
TO INSPI :SIDE :ANGLE :INC
    FD :SIDE
    RT :ANGLE
    INSPI :SIDE (:ANGLE + :INC) :INC
END
```

The diagram in Figure 1 illustrates what INSPI does. The diagram is constructed with side input = 10, angle input = 30, and increment = 10. For purposes of illustration, only the first six sides generated by INSPI are shown. From its initial position, the turtle goes forward 10 units and turns right 30 degrees. On its second step, it procedes forward 10 units and turns 30 + 10 = 40 degrees. From that position, it travels 10 units and turns 40 + 10 = 50 degrees. As this process continues, the figure begins the characteristic inward spiral from which the name INSPI is derived.

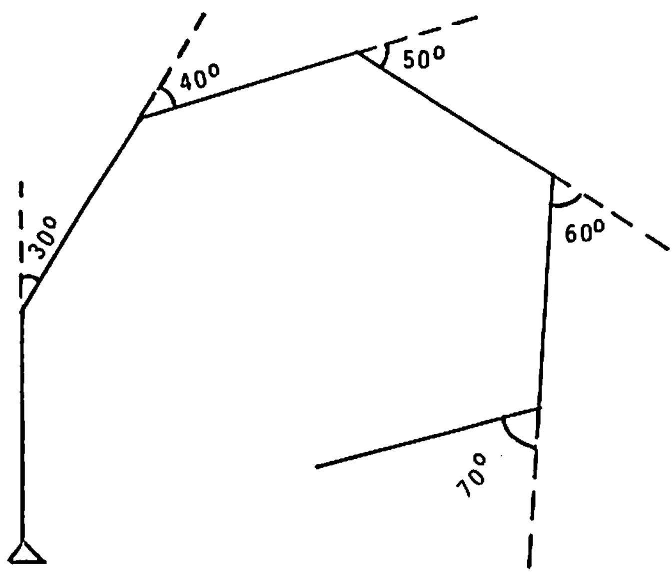
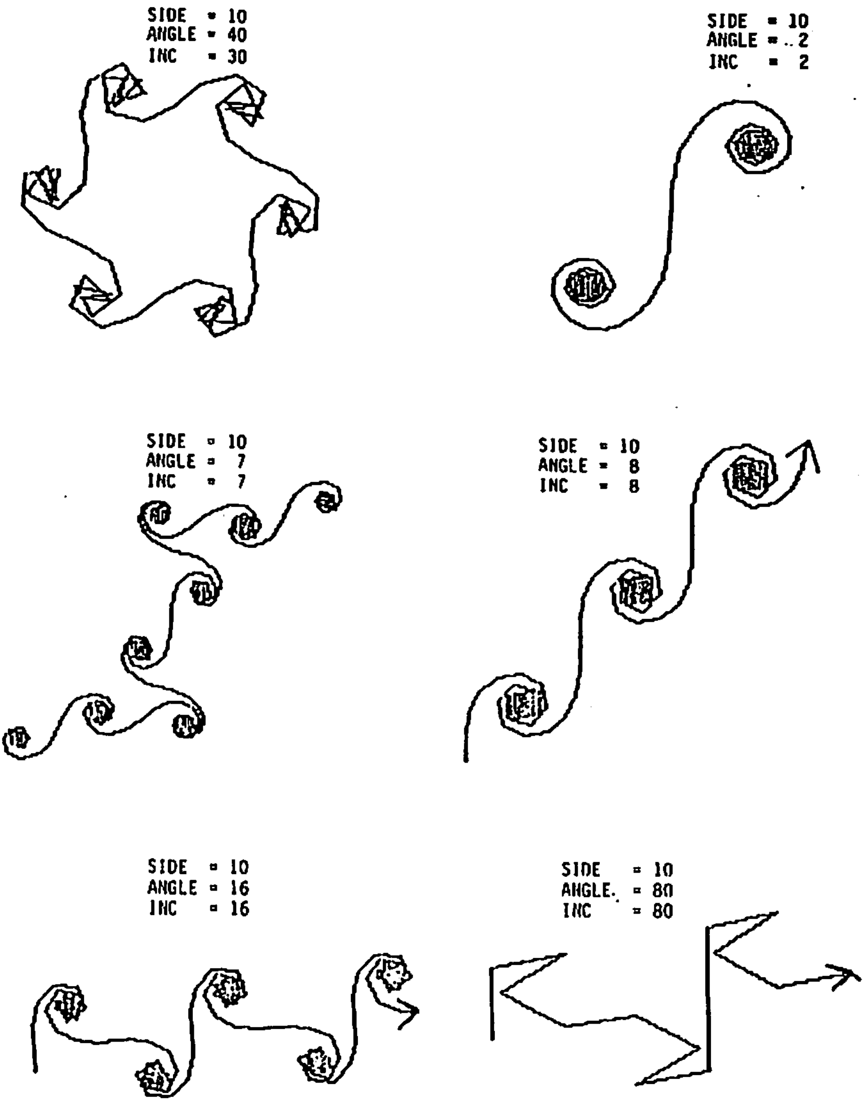


Figure 1



(Examples of INSPI with a variety of innuts.)

FIGURE 2

Careful analysis of these figures reveals that some of the figures are closed and some are not. In other words, some figures return to their initial position and heading while others do not.

PROBLEM DEFINITION

To determine which angle inputs to the INSPI procedure produce closed figures, and which do not.

An examination of the following figure will help in determining which figures are closed, and which ones are not. The figure was produced with the INSPI procedure with the following inputs: SIDE = 10, ANGLE = 60, INC = 60.

TURN	ANGLE
1	60
2	120
3	180
4	240
5	300
6	360
7	60
8	120
9	180
10	240
11	300
12	360

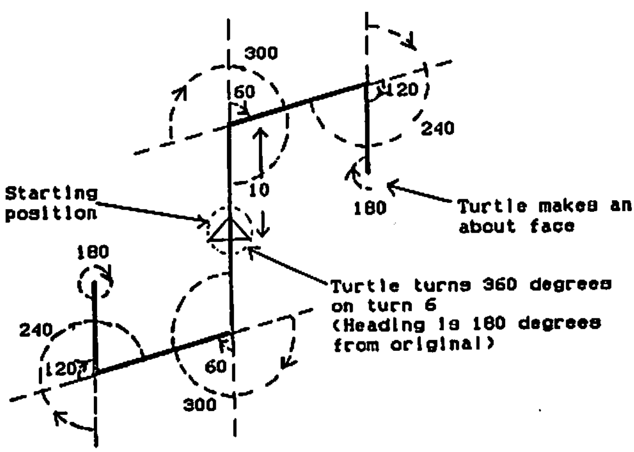


FIGURE 3

The figure is closed because the turtle returns at some point to its initial position and heading. An essential clue to the determination of closure or non-closure of the figure can be seen when the turtle makes its first turn which is a multiple of 360 degrees. At that point, the turtle's heading is 180 degrees from its original heading. This characteristic was seen in the above figure and any other closed figure that I generated. Any non-closed figure that I generated did not possess this characteristic.

LOOKING AT A SPECIAL CASE

In the original problem, any combination of positive integers for the angle input and increment could be used. This makes it very difficult to determine which angle inputs produce closed, and which produce non-closed figures because the outcome depends not only on the angle input, but on the increment as well. Therefore, I've decided to look at a special case of the problem at hand. The restriction for this special case is that the angle input (A) must equal the increment (I). This means the turn is always nA for some integer n.

A PROGRAM TO AID CALCULATIONS

I stated earlier that closed figures seem to have the unique characteristic in which the heading is 180 degrees from the original heading when the turn reaches a multiple of 360 degrees for the first time. The calculation of the heading when the turn reaches a multiple of 360 degrees can be very time consuming. Therefore, I created a computer program to aid the number crunching process. The program, which is written in the LOGO language, follows:

```
TO INSPICALC :ANGLE :INC
  MAKE "TOTALTURN 0
  MAKE "TURN :ANGLE
  MAKE "TOTALTURN :TOTALTURN + :TURN
  IF NOT((REMAINDER :TURN 360) = 0 )
        THEN INSPILOOP :TURN :INC
        ELSE (PRINT :TURN :TOTALTURN)
END
TO INSPILOOP :TURN :INC
  MAKE "TURN :TURN + :INC
  MAKE "TOTALTURN :TOTALTURN + :TURN
  IF NOT ((REMAINDER :TURN 360) = 0)
        THEN INSPILOOP :TURN :INC
        ELSE (PRINT :TURN :TOTALTURN)
END.
```

The above program follows the steps of an INSPI procedure, calculating the heading and turn that the turtle is about to make. It terminates when the turtle turns through an angle of 360n, where n is a positive integer. At this point, it prints out the last turn made by the turtle and its heading at that point. As stated earlier, if the heading is 180 degrees from the original heading at that point, then the figure is a closed figure. The table below shows results using a variety of angle inputs. For ease of viewing, the angle inputs are grouped according to closed figures and non-closed figures.

CLOSED FIGURES

ANGLE	INCREMENT	TURN	HEADING
1	1	360	64,980 = 180 * 361
2	2	360	32,580 = 180 * 181
3	3	360	21,780 = 180 * 121
7	7	2,520	454,860 = 180 * 2,527
11	11	3,960	714,780 = 180 * 3,971
12	12	360	5,580 = 180 * 31
14	14	2,520	228,060 = 180 * 1,267
17	17	6,120	1,104,660 = 180 * 6,137
19	19	6,840	1,234,620 = 180 * 6,859
21	21	2,520	152,460 = 180 * 847
22	22	3,960	358,380 = 180 * 1,991
23	23	8,280	1,494,540 = 180 * 8,303
25	25	1,800	65,700 = 180 * 365
27	27	1,080	22,140 = 180 * 123

NON-CLOSED FIGURES

ANGLE	INCREMENT	TURN	HEADING
8	8	360	8,280 = 360 * 23
16	16	760	16,560 = 360 * 46
24	24	360	2,880 = 360 * 8
32	32	1,440	33,120 = 360 * 92
40	40	360	1,800 = 360 * 5
48	48	720	5,760 = 360 * 16

These experiments lead to the following two conjectures:

1. All closed figures have a heading of 180 degrees from the original heading when the turn reaches a multiple of 360 degrees.
2. The only angle inputs that produce non-closed figures are multiples of 8.

I've stated the second conjecture as my main theorem and the first as a lemma to the theorem. The lemma and theorem and proofs for each follow.

<u>LEMMA</u>: For the INSPI procedure with angle input = increment, the generated figure is closed if and only if, at some point the turn = 180k, where k is an odd integer.

Let A = the initial angle input to INSPI.

Any odd multiple of 180 degrees results in a heading change of 180 degrees. Since INSPI produces turns of nA, where n is a positive integer, then the turn just before the 180 degree turn is 180 - A, and the turn just after the 180 degree turn is 180 + A.

Let us call the original heading prior to the 180 - A turn 0 degrees.

Then, the heading just after the turn 180 + A is (180k - A) + (180k) + (180k + A) = 540K = 180 degrees (mod 360).

Since the forward distance traveled is constant, the turtle is at the same point, P2, just after the 180 - A turn and just after the 180 + A turn, except that the heading is now 180 degrees from the original heading.

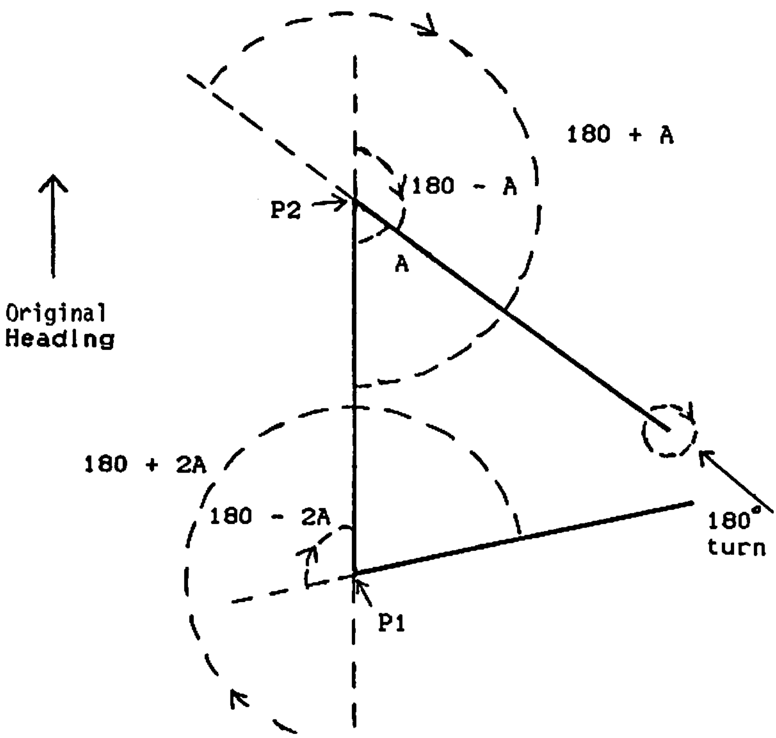


FIGURE 4

Since the angle = increment, the turn just before the 180 - A turn is 180 - 2A. Then, the heading just after the 180 + 2A turn is (180k - 2A) + (180k - A) + (180k) + (180k + A) + (180k + 2A) = 900k = 180 degrees(mod 360). Likewise, at each point where a turn occurs, the new heading for departing the point will differ by 180 degrees (mod 360) from the previous heading upon arriving at the point. Thus, the turtle will begin to trace the previous portion of the figure in reverse. Hence, the figure continues to retrace itself.

To prove the second part of the two part lemma, I will prove the contrapositive. If the turn is never = 180k, where k is an odd integer, then the figure is never closed.

If the turn is never an odd multiple of 180, then it must be, at some point, an even multiple of 180. This means that it is a multiple of 360. If that is the case, the turn will never make an about face which allows it to retrace itself. Therefore, it will never close.

<u>THEOREM</u>: For the INSPI procedure with the angle input = the increment, all angle inputs will produce closed figures except inputs of 8m, where m is a positive integer.

<u>PROOF</u>: From the previous lemma, a turn of $nA = 180k$ produces a closed figure, if k is an odd integer, and n is an integer. In the proof of of the theorem, it can be assumed that n and k are the smallest n and k for which $nA = 180k$. Then, k and n have no common factor except 1.

A. Angle inputs of 8m produce non-closed figures.

$(8m)n = 180k$

$(2m)n = 45k$

This implies that k is even.

By the previous lemma, k must be odd to produce closed figures.

Therefore, multiples of 8 produce non-closed figures.

B. Inputs other than 8m produce closed figures.

1. If A is odd then n must be even.

For $nA = 180k$:

Since n and k have no factor in common, k must be odd.

Therefore, odd angle inputs produce closed figures.

2. If A is an even number not equal to 8m then A cannot have a factor of 8, so it must have a factor of 2 or 4.

   a. If A has a factor of 2. Then $n(A/2) = 90k$. The integer A/2 is not even, therefore n is even and so k is odd.

   b. If A has a factor of 4, then $n(A/4) = 45k$. The integer A/4 is not even, therefore both n and k must be odd.

Therefore, inputs other than 8m produce closed figures.

REFERENCES

1. Harold Abelson and diSessa, Turtle Geometry, London: MIT Press, 1982.