MTH362 Spring 2016

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Contents

CHAPTER 1

Review

1.1. Preliminaries and Definitions

Definition 1.1.1 (binary operation). *A **binary operation** on a set S is a function from $S \times S$ into S.*

Examples of binary operations:

- $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *Addition of natural numbers*
- $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *Multiplication of natural numbers*

Definition 1.1.2 (group). *A **group** consists of:*

- *A set G*
- *A binary operation $+ : G \times G \to G$ with the following properties:*

$x + (y + z) = (x + y) + z \ \forall x, y, z \in G$ *(associativity)*
$\exists 0 \in G$ *such that* $a + 0 = 0 + a = a \ \forall a \in G$ *(identity)*
$\forall a \in G \ \exists \ a^{-1}$ *such that* $a + a^{-1} = a^{-1} + a = 0$ *(inverse)*

Definition 1.1.3 (field). *A **field** consists of:*

- *A set F*
- *A binary operation $+ : F \times F \to F$ with the following properties:*

$x + y = y + x \ \forall x, y \in F$ *(additive commutativity)*
$x + (y + z) = (x + y) + z \ \forall x, y, z \in F$ *(additive associativity)*
$\exists 0 \in F$ *such that* $a + 0 = 0 + a = a \ \forall a \in F$ *(additive identity)*
$\forall a \in F \ \exists \ a^{-1}$ *such that* $a + a^{-1} = a^{-1} + a = 0$ *(additive inverse)*

- *A binary operation $: F \times F \to F$ with the following properties:*

$xy = yx \ \forall x, y \in F$ *(multiplicative commutativity)*
$x(yz) = (xy)z \ \forall x, y, z \in F$ *(multiplicative associativity)*
$\exists 1 \in F$ *such that* $a1 = 1a = a \ \forall a \in F$ *(multiplicative identity)*
$\forall a \in F \setminus 0 \ \exists \ a^{-1}$ *such that* $aa^{-1} = a^{-1}a = 1$ *(multiplicative inverse)*
$x(y + z) = xy + xz \quad \forall x, y, z \in F$ *(distributive property)*

DEFINITION 1.1.4 (vector space). *A **vector space** or **linear space** consists of:*

- *A field F of elements called **scalars***
- *A commutative group V of elements called **vectors** with respect to a binary operation $+$*
- *A binary operation $: F \times V \to V$ called **scalar multiplication** that associates with each scalar $\alpha \in F$ and vector $v \in V$ a vector αv in such a way that:*
  $1v = v \quad \forall v \in V$
  $(\alpha\beta)v = \alpha(\beta v) \quad \forall \alpha, \beta \in F,\ v \in V$
  $\alpha(v + w) = \alpha v + \alpha w \quad \forall \alpha \in F,\ v, w \in V$
  $(\alpha + \beta)v = \alpha v + \beta v \quad \forall \alpha, \beta \in F,\ v \in V$

DEFINITION 1.1.5 (norm). *A nonnegative real-valued function $\| \ \| : V \to \mathbb{R}$ is called a **norm** if:*

- $\|v\| \geq 0$ *and* $\|v\| = 0 \Leftrightarrow v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality)*
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F,\ v \in V$

DEFINITION 1.1.6 (normed linear space). *A linear space V together with a norm $\| \cdot \|$, denoted by the pair $(V, \| \cdot \|)$, is called a **normed linear space***

DEFINITION 1.1.7 (inner product). *Let the field F be either $\mathbb{R}$ or $\mathbb{C}$ and a set V of vectors which together with F form a vector space. An **inner product** on V is a map*

$$\cdot : V \times V \to \mathbb{F}$$

with the following properties:

$$\begin{array}{ll} (u + v) \cdot w = u \cdot w \ + \ v \cdot w & \forall u, v, w \in V \\ (\alpha u) \cdot v = \alpha(u \cdot v) & \forall \alpha \in F,\ u, v \in V \\ u \cdot v = (\overline{v \cdot u}) & \forall u, v \in V \\ u \cdot u \geq 0 & \forall u \in V \text{ with equality when } u = \vec{0} \end{array}$$

If the underlying field is $\mathbb{R}$, the fourth condition can be replaced by

$u \cdot v = v \cdot u \quad \forall u, v \in V$

since a real number is its own conjugate. In this case, the condition just says the inner product is commutative.

DEFINITION 1.1.8 (metric). *A **metric** on a set S is a function*

$$\rho : S \times S \to \mathbb{R}$$

where ρ has the following three properties for any $x, y, z \in S$:

$$\begin{array}{l} \rho(x,y) \geq 0 \quad \text{and } \rho(x,y) = 0 \Leftrightarrow x = y \\ \rho(x,y) = \rho(y,x) \\ \rho(x,y) \leq \rho(x,z) + \rho(z,y) \end{array}$$

DEFINITION 1.1.9 (metric space). *A **metric space** is a pair $\{S, \rho\}$ where S is a set and ρ is a metric defined on S.*

DEFINITION 1.1.10 (topology). *A **topology** is a set X and a collection $\mathcal{J}$ of subsets of X having the following properties:*

- *$\emptyset$ and X are in $\mathcal{J}$*
- *The union of any subcollection of elements of $\mathcal{J}$ belongs to $\mathbb{J}$*
- *The intersection of any finite subcollection of $\mathcal{J}$ belongs to $\mathcal{J}$*

Convergence

DEFINITION 1.1.11 (sequence). *A sequence is a function whose domain is $\mathbb{N}$.*

DEFINITION 1.1.12 (convergent sequence). *A sequence x_n in $\mathbb{R}$ converges to x if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$|x_n - x|, \epsilon \quad \text{whenever} \quad n \geq N$$

DEFINITION 1.1.13 (Cauchy sequence). *A sequence x_n in $\mathbb{R}$ is said to be a Cauchy sequence if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$|x_n - x_m|, \epsilon \quad \text{whenever} \quad n, m \geq N$$

Topology Review

Definition 1.1.14 (ϵ-neighborhood). *If $x \in \mathbb{R}$ and $\epsilon > 0$, the ϵ-neighborhood $V_\epsilon(x)$ is defined by*

$$V_\epsilon = \{y \in \mathbb{R} \, : \, |x - y| < \epsilon\}$$

Definition 1.1.15 (limit point). *A real number $x \in A$ is a limit point of A if for every $\epsilon > 0$, $V_\epsilon(x)$ contains elements of A other than x.*

Definition 1.1.16 (isolated point). *A element $x \in A$ is an isolated point of A if $x \in A$ and x is not a limit point of A.*

Definition 1.1.17 (open set). *A set $A \subseteq \mathbb{R}$ is open if, for every $a \in A$, there is an $\epsilon > 0$ such that*

$$V_\epsilon(a) \subseteq A$$

That is, there is an ϵ-neighborhood of every element of A that is contained entirely in A.

Definition 1.1.18 (closed set). *A set $A \subseteq \mathbb{R}$ is closed if it contains all of its limit points.*

Definition 1.1.19 (closure). *The closure of a set $A \subseteq \mathbb{R}$ denoted by $\overline{A}$ is the union of A and its limit points.*

Definition 1.1.20 (compact set). *A set $A \subseteq \mathbb{R}$ is compact if every sequence in A has a convergent subsequence whose limit is in A.*

DEFINITION 1.1.21 (perfect set). *A set $A \subseteq \mathbb{R}$ is perfect if it is closed and has no isolated points.*

DEFINITION 1.1.22 (bounded set). *A set $A \subseteq \mathbb{R}$ is bounded if there exists an $M > 0$ such that*

$$|a| \leq M \quad \textit{for all} \quad a \in A$$

DEFINITION 1.1.23 (separated sets). *Two nonempty sets $A, B \subseteq \mathbb{R}$ are separated if*

$$A \cap \overline{B} = \emptyset = \overline{A} \cap B$$

DEFINITION 1.1.24 (disconnected set). *A set $E \subseteq \mathbb{R}$ is disconnected if it can be written as*

$$E = A \cup B$$

where A and B are nonempty separated sets.

DEFINITION 1.1.25 (connected set). *A set $A \subseteq \mathbb{R}$ is connected if it is not disconnected.*

THEOREM 1.1.1. *A set $A \subseteq \mathbb{R}$ is open if and only if its compliment A^c is closed.*

PROOF. ($\Rightarrow$) Let $A \subseteq \mathbb{R}$ be open. Suppose for the sake of contradiction that $a \in A$ is a limit point of A^c. By definition, for every $\epsilon > 0$, $V_\epsilon(a)$ contains points of A^c. This means that no ϵ-neighborhood of a is entirely contained in A, contradicting the assumption that A is open.

($\Leftarrow$) Now suppose $A \subseteq \mathbb{R}$ with A^c closed. Let a be an element of $(A^c)^c = A$. By hypothesis, A^c is closed, so a cannot be a limit point of A^c because $a \notin A$. By definition, this means there must be an $\epsilon > 0$ such that $V_\epsilon(a)$ contains no points of A^c. Therefore, $V_\epsilon(a) \subseteq A$. Since a was arbitrarily chosen, we can find an ϵ-neighborhood of every element of A that is entirely contained in A, so by definition A is open. $\square$

CHAPTER 2

Homework Problems

2.1. Week 1 Homework (assignment 1)

Team 1 (Ali, Emily, Frank).

PROBLEM 2.1.1. *Let S be the set of ordered n-tuples in $\mathbb{R}$:*

$$S = \{\ (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \quad 1 \le i \le n$$

- *a) Define a binary operation on $S \times S$, an identity element, and an inverse that makes S into an Abelian group.*
- *b) Define a multiplication operation on $\mathbb{R} \times S$ that makes S into a vector space over $\mathbb{R}$.*

PROBLEM 2.1.2. *Let S be set of real-valued functions on $[-1, 1]$:*

$$S = \{\ f : [-1, 1] \to \mathbb{R}\}$$

- *a) Define a binary operation on $S \times S$, an identity element, and an inverse that makes S into an Abelian group.*
- *b) Define a multiplication operation on $\mathbb{R} \times S$ that makes S into a vector space over $\mathbb{R}$.*

PROBLEM 2.1.3. *Let V be $\mathbb{R}^3$, the set of ordered triples of real numbers, with addition defined in the usual way:*

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \forall u, v \in V$$

*Define the **sum norm** on this vector space as:*

$$\|v\|_1 = |v_1| + |v_2| + |v_3|$$

Show that the sum norm satisfies the triangle inequality.

Presenters:

- 2.1.1 Ali

- 2.1.2 Frank
- 2.1.3 Emily

Team 2 (Blaine, Katie, Siobhan, Rachel).

PROBLEM 2.1.4. *Let $S^{m\times n}$ be the set of all $m \times n$ matrices over $\mathbb{R}$.*

$$S = \{\ x_{i,j} : 1 \le i \le m,\ 1 \le j \le n, x_{i,j} \in \mathbb{R}\}$$

- a) Define a binary operation on $S \times S$, an identity element, and an inverse that makes S into an Abelian group.
- b) Define a multiplication operation on $\mathbb{R} \times S$ that makes S into a vector space over $\mathbb{R}$.

PROBLEM 2.1.5. *Let P be set of polynomial functions on $\mathbb{R}$:*

$$P = \{\ p : \mathbb{R} \to \mathbb{R} \quad \textit{such that for } x \in \mathbb{R},\ p(x) = a_0+a_1x+a_2x^2+\cdots+a_nx^n\}$$

where $a_1, a_1, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$.

- a) Define a binary operation on $P \times P$, an identity element, and an inverse that makes P into an Abelian group.
- b) Define a multiplication operation on $\mathbb{R} \times P$ that makes P into a vector space over $\mathbb{R}$.

PROBLEM 2.1.6. *Let V be $\mathbb{R}^3$, the set of ordered triples of real numbers, with addition defined in the usual way:*

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \forall u, v \in V$$

*Define the **max norm** on this vector space as:*

$$\|v\|_\infty = \max(|v_1|, |v_2|, |v_3|)$$

Show that the max norm satisfies the triangle inequality.

PROBLEM 2.1.7. *Let V be $\mathbb{R}^3$, the set of ordered triples of real numbers, with zero element:*

$$\vec{0} = (0, 0, 0)$$

*Define the **sum norm** on this vector space as:*

$$\|v\|_1 = |v_1| + |v_2| + |v_3|$$

Show that the sum norm satisfies the required condition

$$\|v\| = 0 \quad \textit{if and only if} \quad v = \vec{0}$$

Presenters:

- 2.1.4 Blaine
- 2.1.5 Katie
- 2.1.6 Rachel
- 2.1.7 Siobhan

2.2. Week 1 Homework (assignment 2)

Team 1 (Ali, Emily, Frank).

PROBLEM 2.2.1. *Prove that arbitrary unions of open sets are open.*

PROBLEM 2.2.2. *Prove that a finite union of closed sets is closed.*

PROBLEM 2.2.3. *Prove that a point x is a limit point of A if and only if $x = \lim a_n$ for some sequence (a_n) in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Presenters:

- 2.2.1 Ali
- 2.2.2 Frank
- 2.2.3 Emily

Team 2 (Blaine, Katie, Siobhan, Rachel).

PROBLEM 2.2.4. *Prove that finite intersections of open sets are open.*

PROBLEM 2.2.5. *Prove that a set O is open if and only if its compliment, O^c, is closed.*

PROBLEM 2.2.6. *Prove that a set F is closed if and only if its compliment F^c is open.*

PROBLEM 2.2.7. *Prove that arbitrary intersections of closed sets are closed.*

Presenters:

- 2.2.4 Blaine
- 2.2.5 Katie
- 2.2.6 Rachel
- 2.2.7 Siobhan