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Contents

CHAPTER 1

Review

1.1. Preliminaries and Definitions

Definition 1.1.1 (binary operation). *A* ***binary operation*** *on a set S is a function from $S \times S$ into S.*

Examples of binary operations:

- $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *Addition of natural numbers*
- $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *Multiplication of natural numbers*

Definition 1.1.2 (group). *A* ***group*** *consists of:*

- *A set G*
- *A binary operation $+ : G \times G \to G$ with the following properties:*

$$x + (y + z) = (x + y) + z \; \forall x, y, z \in G \qquad \textit{(associativity)}$$
$$\exists 0 \in G \text{ such that } a + 0 = 0 + a = a \; \forall a \in G \qquad \textit{(identity)}$$
$$\forall a \in G \; \exists \, a^{-1} \text{ such that } a + a^{-1} = a^{-1} + a = 0 \qquad \textit{(inverse)}$$

Definition 1.1.3 (field). *A* ***field*** *consists of:*

- *A set F*
- *A binary operation $+ : F \times F \to F$ with the following properties:*

$$x + y = y + x \; \forall x, y \in F \qquad \textit{(additive commutativity)}$$
$$x + (y + z) = (x + y) + z \; \forall x, y, z \in F \qquad \textit{(additive associativity)}$$
$$\exists 0 \in F \text{ such that } a + 0 = 0 + a = a \; \forall a \in F \qquad \textit{(additive identity)}$$
$$\forall a \in F \; \exists \, a^{-1} \text{ such that } a + a^{-1} = a^{-1} + a = 0 \qquad \textit{(additive inverse)}$$

- *A binary operation $: F \times F \to F$ with the following properties:*

$$xy = yx \; \forall x, y \in F \qquad \textit{(multiplicative commutativity)}$$
$$x(yz) = (xy)z \; \forall x, y, z \in F \qquad \textit{(multiplicative associativity)}$$
$$\exists 1 \in F \text{ such that } a1 = 1a = a \; \forall a \in F \qquad \textit{(multiplicative identity)}$$
$$\forall a \in F \setminus 0 \; \exists \, a^{-1} \text{ such that } aa^{-1} = a^{-1}a = 1 \qquad \textit{(multiplicative inverse)}$$
$$x(y + z) = xy + xz \quad \forall x, y, z \in F \qquad \textit{(distributive property)}$$

Definition 1.1.4 (vector space). *A **vector space** or **linear space** consists of:*

- *A field F of elements called **scalars***
- *A commutative group V of elements called **vectors** with respect to a binary operation $+$*
- *A binary operation $: F \times V \to V$ called **scalar multiplication** that associates with each scalar $\alpha \in F$ and vector $v \in V$ a vector αv in such a way that:*
  $1v = v \quad \forall v \in V$
  $(\alpha\beta)v = \alpha(\beta v) \quad \forall \alpha, \beta \in F,\ v \in V$
  $\alpha(v + w) = \alpha v + \alpha w \quad \forall \alpha \in F,\ v, w \in V$
  $(\alpha + \beta)v = \alpha v + \beta v \quad \forall \alpha, \beta \in F,\ v \in V$

Definition 1.1.5 (norm). *A nonnegative real-valued function $\| \, \| : V \to \mathbb{R}$ is called a **norm** if:*

- $\|v\| \geq 0$ *and* $\|v\| = 0 \ \Leftrightarrow\ v = \vec{0}$
- $\|v + w\| \leq \|v\| + \|w\|$ *(triangle inequality)*
- $\|\alpha v\| = |\alpha| \, \|x\| \quad \forall \alpha \in F,\ v \in V$

Definition 1.1.6 (normed linear space). *A linear space V together with a norm $\| \cdot \|$, denoted by the pair $(V, \| \cdot \|)$, is called a **normed linear space***

Definition 1.1.7 (inner product). *Let the field F be either $\mathbb{R}$ or $\mathbb{C}$ and a set V of vectors which together with F form a vector space. An **inner product** on V is a map*

$$\cdot : V \times V \to \mathbb{F}$$

with the following properties:

$$\begin{array}{ll} (u + v) \cdot w = u \cdot w \ + \ v \cdot w & \forall u, v, w \in V \\ (\alpha u) \cdot v = \alpha(u \cdot v) & \forall \alpha \in F,\ u, v \in V \\ u \cdot v = (\overline{v \cdot u}) & \forall u, v \in V \\ u \cdot u \geq 0 & \forall u \in V \text{ with equality when } u = \vec{0} \end{array}$$

If the underlying field is $\mathbb{R}$, the fourth condition can be replaced by

$u \cdot v = v \cdot u \quad \forall u, v \in V$

since a real number is its own conjugate. In this case, the condition just says the inner product is commutative.

DEFINITION 1.1.8 (metric). *A **metric** on a set S is a function*

$$\rho : S \times S \to \mathbb{R}$$

where ρ has the following three properties for any $x, y, z \in S$:

$$\begin{array}{l} \rho(x,y) \geq 0 \ \ \text{and } \rho(x,y) = 0 \Leftrightarrow x = y \\ \rho(x,y) = \rho(y,x) \\ \rho(x,y) \leq \rho(x,z) + \rho(z,y) \end{array}$$

DEFINITION 1.1.9 (metric space). *A **metric space** is a pair $\{S, \rho\}$ where S is a set and ρ is a metric defined on S.*

DEFINITION 1.1.10 (topology). *A **topology** is a set X and a collection $\mathcal{J}$ of subsets of X having the following properties:*

- *$\emptyset$ and X are in $\mathcal{J}$*
- *The union of any subcollection of elements of $\mathcal{J}$ belongs to $\mathbb{J}$*
- *The intersection of any finite subcollection of $\mathcal{J}$ belongs to $\mathcal{J}$*

Convergence

DEFINITION 1.1.11 (sequence). *A sequence is a function whose domain is $\mathbb{N}$.*

DEFINITION 1.1.12 (convergent sequence). *A sequence x_n in $\mathbb{R}$ converges to x if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$|x_n - x|, \epsilon \quad \text{whenever} \quad n \geq N$$

DEFINITION 1.1.13 (Cauchy sequence). *A sequence x_n in $\mathbb{R}$ is said to be a Cauchy sequence if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$|x_n - x_m|, \epsilon \quad \text{whenever} \quad n, m \geq N$$

Topology Review

DEFINITION 1.1.14 (ϵ-neighborhood). *If $x \in \mathbb{R}$ and $\epsilon > 0$, the ϵ-neighborhood $V_\epsilon(x)$ is defined by*

$$V_\epsilon = \{y \in \mathbb{R} \,:\, |x - y| < \epsilon\}$$

DEFINITION 1.1.15 (limit point). *A real number $x \in A$ is a limit point of A if for every $\epsilon > 0$, $V_\epsilon(x)$ contains elements of A other than x.*

DEFINITION 1.1.16 (isolated point). *A element $x \in A$ is an isolated point of A if $x \in A$ and x is not a limit point of A.*

DEFINITION 1.1.17 (open set). *A set $A \subseteq \mathbb{R}$ is open if, for every $a \in A$, there is an $\epsilon > 0$ such that*

$$V_\epsilon(a) \subseteq A$$

That is, there is an ϵ-neighborhood of every element of A that is contained entirely in A.

DEFINITION 1.1.18 (closed set). *A set $A \subseteq \mathbb{R}$ is closed if it contains all of its limit points.*

DEFINITION 1.1.19 (closure). *The closure of a set $A \subseteq \mathbb{R}$ denoted by $\overline{A}$ is the union of A and its limit points.*

DEFINITION 1.1.20 (compact set). *A set $A \subseteq \mathbb{R}$ is compact if every sequence in A has a convergent subsequence whose limit is in A.*

Definition 1.1.21 (perfect set). *A set $A \subseteq \mathbb{R}$ is perfect if it is closed and has no isolated points.*

Definition 1.1.22 (bounded set). *A set $A \subseteq \mathbb{R}$ is bounded if there exists an $M > 0$ such that*

$$|a| \leq M \quad \textit{for all} \quad a \in A$$

Definition 1.1.23 (separated sets). *Two nonempty sets $A, B \subseteq \mathbb{R}$ are separated if*

$$A \cap \overline{B} = \emptyset = \overline{A} \cap B$$

Definition 1.1.24 (disconnected set). *A set $E \subseteq \mathbb{R}$ is disconnected if it can be written as*

$$E = A \cup B$$

where A and B are nonempty separated sets.

Definition 1.1.25 (connected set). *A set $A \subseteq \mathbb{R}$ is connected if it is not disconnected.*

Theorem 1.1.1. *A set $A \subseteq \mathbb{R}$ is open if and only if its compliment A^c is closed.*

Proof. ($\Rightarrow$) Let $A \subseteq \mathbb{R}$ be open. Suppose for the sake of contradiction that $a \in A$ is a limit point of A^c. By definition, for every $\epsilon > 0$, $V_\epsilon(a)$ contains points of A^c. This means that no ϵ-neighborhood of a is entirely contained in A, contradicting the assumption that A is open.

($\Leftarrow$) Now suppose $A \subseteq \mathbb{R}$ with A^c closed. Let a be an element of $(A^c)^c = A$. By hypothesis, A^c is closed, so a cannot be a limit point of A^c because $a \notin A$. By definition, this means there must be an $\epsilon > 0$ such that $V_\epsilon(a)$ contains no points of A^c. Therefore, $V_\epsilon(a) \subseteq A$. Since a was arbitrarily chosen, we can find an ϵ-neighborhood of every element of A that is entirely contained in A, so by definition A is open. $\square$

THEOREM 1.1.2 (Heine-Borel Theorem). *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

PROOF. ($\Rightarrow$) Let $K \subseteq \mathbb{R}$ be compact. Suppose for the sake of contradiction that K is not bounded. By definition, for any $M \in \mathbb{R}$, there exists an element x_M of K with $|x_M| > M$. We will use this fact to construct a sequence in K that has no convergent subsequence. If K is unbounded, there must be an $x_1 \in K$ with $|x_1| > 1$. By the same argument, there must be an element x_2 in K with $|x_2| > 2$. Continuing in this fashion, we may construct a sequence

$$x_1, x_2, x_3, \ldots, x_n, \ldots \quad \text{with} \quad |x_n| > n \quad \text{for every} \quad n \in \mathbb{N}$$

Now suppose (x_{n_k}) is a subsequence of (x_n). Since $|x_{n_k}| > n_k$ and n_k is unbounded, every subsequence is unbounded. Since all convergent sequences are bounded, this means that no subsequence is convergent, contradicting the assumption that K is compact.

This establishes that K is bounded. To show that K is closed, suppose x is a limit point of K. Then there is a sequence (x_n) in K that converges to x. By hypothesis, K is compact, so by definition (x_n) has a convergent subsequence whose limit is in K. However, by an earlier theorem, any subsequence of a convergent sequence is also convergent, and has the same limit x. Therefore both (x_n) and its subsequence have the same limit, which must belong to K. Since x was arbitrarily chosen, this is true of any limit point, so K must contain all of its limit points and therefore is closed.

($\Leftarrow$) Now suppose K is closed and bounded. Let (x_n) be an arbitrary sequence in K. Then (x_n) must be bounded, and by the Bolzano-Weierstrass theorem, it must have a convergent subsequence (x_{n_k}). The limit of this subsequence is, by definition, a limit point of K, and by hypothesis K is closed and therefore contains its limit points. Therefore, every sequence in K has a subsequence that converges to a point in K, and by definition, K is compact.

□

THEOREM 1.1.3. *A set $E \subseteq \mathbb{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$ there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A and B, and x an element of the other.*

PROOF. ($\Rightarrow$) Let E be a connected set. Suppose $E = A \cup B$ where A and B are disjoint, nonempty sets. By hypothesis, E is connected, so A and B are not separated. This means that one of $A \cap \overline{B}$ and $\overline{A} \cap B$ is not empty. Without loss of generality, assume that $x \in \overline{A} \cap B$. Then $x \in \overline{A}$ and $x \in B$. But A and B are disjoint, so $x \notin A$. By definition, $\overline{A}$ is the union of A and its limit points, and since $x \notin A$, x must be a limit point of A. By an earlier theorem, there is a sequence in A that converges to x.

($\Leftarrow$) (contrapositive argument) Suppose $E \subseteq \mathbb{R}$ is disconnected. We need to find two nonempty, disjoint sets A and B such that $E = A \cap B$ and there does not exist a convergent sequence (x_n) in A with its limit x in B, or vice-versa. By hypothesis, E is separated, so there exist separated sets A and B with $E = A \cup B$. Now suppose (x_n) is a convergent sequence contained in A whose limit is x. By definition, since x is a limit of a sequence in A, x belongs to $\overline{A}$. Because A and B are disconnected, by definition $\overline{A} \cap B$ is empty, so $x \notin B$. Since (x_n) was an arbitrary sequence, no convergent sequence in A has its limit in B. A similar argument shows that no convergent sequence in B has its limit in A. $\square$

DEFINITION 1.1.26 (continuous function). *In a metric space* (X, ρ) *a function* $f : D \subseteq X \to X$ *is continuous if, for every* $\epsilon > 0$*, there is a* $\delta > 0$ *such that*

$$\rho(f(x), f(y)) < \epsilon \quad \textit{whenever} \quad \rho(x, y) < \delta$$

LEMMA 1.1.1. *If* (X, ρ) *is a metric space and* $f : D \subseteq X \to X$ *is continuous, and* (x_n) *is a convergent sequence in* D *whose limit* x *is also in* D*, then* $\lim f(x_n) = f(x)$.

PROOF. Let $\epsilon > 0$ be given. We need to show that there is an $N \in \mathbb{N}$ such that

$$\rho(f(x_n), f(x)) < \epsilon \quad \text{whenever} \quad n \geq N$$

By hypothesis, f is continuous, so by definition for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\rho(f(x_n), f(x)) < \epsilon \quad \text{whenever} \quad \rho(x, x_n) < \delta$$

Also by hypothesis, (x_n) is convergent with limit x, so by definition there is an $N \in \mathbb{N}$ such that

$$\rho(x_n, x) < \delta \quad \text{whenever} \quad n \geq N$$

But by the continuity of f, this is equivalent to saying that $\rho(f(x_n), f(x)) < \epsilon$ whenever $n \geq \mathbb{N}$. □

THEOREM 1.1.4. *If $f : \mathbb{R} \to \mathbb{R}$ and $K \subseteq \mathbb{R}$ is compact, then $f[K]$, the image of K under f, is also compact.*

PROOF. Let y_n be a sequence in $f[K]$. By hypothesis, $y_n \in f[K]$ for each $n \in \mathbb{N}$ so there is at least one $x \in K$, call it x_n, with $f(x_n) = y_n$. Because K is compact, the sequence (x_n) in K must have a convergent subsequence (x_{n_k}) whose limit x is in K. By the previous lemma, with the hypothesis that f is continuous it must be true that that (y_{n_k}) is convergent with limit $y = f(x)$. Since (y_n) was an arbitrary sequence in $f(K)$, it follows that every sequence in $f(K)$ has a convergent subsequence whose limit is in $f(K)$, and therefore $f(K)$ is compact. □

1.2. Metric Spaces

The following definitions are straightforward generalizations of definitions we have previously encountered.

DEFINITION 1.2.1 (convergence). *A sequence (x_n) in a metric space (X, d) is said to converge to a limit x if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$d((x_n, x) < \epsilon \quad \textit{whenever} \quad n \geq N$$

DEFINITION 1.2.2 (Cauchy sequence). *A sequence (x_n) in a metric space (X, d) is said to be Cauchy if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$d((x_n, x_m) < \epsilon \quad \textit{whenever} \quad m, n \geq N$$

DEFINITION 1.2.3 (discrete metric). *If X is a set, the function $d : X \times X \to \mathbb{R}^+$ defined by:*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on X.

THEOREM 1.2.1. *The metric space consisting of $\mathbb{R}^2$ with the discrete metric is complete.*

PROOF. To show that $X = (\mathbb{R}^2, d)$ is complete, we must show that every Cauchy sequence in X converges to a limit in X. Suppose (x_n) is a Caucy sequence in X. By definition, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever} \quad m, n \geq N$$

Let $\epsilon = 1/2$. Then there is an $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \frac{1}{2} \quad \text{whenever} \quad n, m \geq N$$

But $d(x_n, x_m) < 1/2$ can only be true if $x_n = x_m$, so we can say that $x_n = x_m$ whenever $n, m \geq N$, and therefore all terms (x_n) with $n \geq N$ are equal, and therefore the sequence is eventually constant and therefore convergent. □

1.3. Sequences and Series of Functions

We will consider how to extend the idea of a sequence of numbers to a sequence of functions $(f_1, f_2, f_3, \ldots)$.

To simplify things, we'll let the domain of each function f_n is the same set A.

DEFINITION 1.3.1 (pointwise convergence). *For each $n \in \mathbb{N}$ let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of functions converges pointwise on A to a function f if, for all $x \in A$, the sequence of real numbers $f_n(x)$ converges to $f(x)$.*

EXAMPLE 1.3.1. *Let*

$$f_n(x) = (x^2 + nx)/n \quad x \in \mathbb{R}$$

Then

$$f_1(x) = x^2 + x, \quad f_2(x) = \frac{x^2 + 2x}{2}, \quad f_3(x) = \frac{x^2 + 3x}{3}, \ldots$$

and

$$f_n(x) = \frac{x^2 + nx}{n} = \frac{x^2}{n} + x \to x \quad as \quad n \to \infty$$

Therefore, (f_n) converges pointwise to $f(x) = x$.

EXAMPLE 1.3.2. *Let $g_n(x) = x^n$ for $x \in [0, 1]$. For $x \in [0, 1)$,*

$$g_n(x) \to 0 \quad as \quad n \to 0$$

but if $x = 1$, $g_n(1) = 1$ for all $n \in \mathbb{N}$, so

$$\lim_{n\to\infty} g_n(x) = g(x) = \begin{cases} 0 & if \quad 0 \le x < 1 \\ 1 & if \quad x = 1 \end{cases}$$

Note that each g_n is continuous on $[0, 1]$, but g is not. This illustrates the fact that the pointwise limit of a sequence of continuous functions may not be continuous.

EXAMPLE 1.3.3. *Let*

$$h_n(x) = x^{1+\frac{1}{2n-1}} \quad x \in [-1, 1]$$

Then

$$h_1(x) = x \cdot x, \quad h_2(x) = x \cdot x^{\frac{1}{3}}, \quad h_2(x) = x \cdot x^{\frac{1}{5}},$$

For a fixed x,

$$\lim_{n\to\infty} = x \lim_{n\to\infty} x^{\frac{1}{2n-1}} = |x|$$

In this example, for each n, $g_n(x)$ is differentiable on $[-1, 1]$, but the pointwise limit $g(x)$ is not.

THEOREM 1.3.1 (Cauchy criterion for uniform convergence). *A sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$ converges uniformly on A if and only if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that*

$$|f_n(x) - f_m(x)| < \epsilon \quad whenever \quad n, m \ge N \quad and \quad x \in A$$

PROOF. ($\Leftarrow$) Suppose (f_n) is a series of functions on $A \subseteq \mathbb{R}$ that converges uniformly on A to f. Let $\epsilon > 0$ be given. Then by the definition of uniform convergence there exists an $N \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \frac{\epsilon}{2} \quad \text{whenever} \quad k \geq N \quad \text{and} \quad x \in A$$

and, for $n, m \geq N$,

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n, m \geq N$ and $x \in A$.

($\Rightarrow$) Now suppose (f_n) is a sequence of functions defined on $A \subseteq \mathbb{R}$ with the property that, for any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{whenever} \quad m, n \geq N \quad \text{and} \quad x \in A$$

Let $\epsilon > 0$ be given. By hypothesis there is an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \quad \text{whenever} \quad n \geq N \quad \text{and} \quad x \in A$$

For any $x \in A$, we can say that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$, and by the Cauchy criterion for real sequences, for any fixed $x \in A$, the sequence $(f_n(x))$ converges. For every $x \in A$, define $f(x)$ to be the limit of the sequence $f_n(x)$. Then because $f(x)$ is the limit of the sequence $(f_n(x))$, there is an $N_x \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \epsilon \quad \text{whenever} \quad k \geq N_x \quad \text{with} \quad N_x \geq N$$

(note: we can assume without loss of generality that $N_x \geq N$ because even if a smaller value will work, so will any larger value, so we can always increase the original N_x to N)

Then for $n \geq N$, for any $x \in A$,

$$|f_n(x) - f(x)| = |f_n(x) - f_{N_x}(x) + f_{N_x}(x) - f(x)|$$
$$\leq |f_n(x) - f_{N_x}(x)| + |f_{N_x} - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since this is true for any $x \in A$, (f_n) converges uniformly to f. (Note: the N_x values may depend on x, but they are always greater than or equal to N, and for this reason no matter what N_x is,

$$|f(x_n) - f_{N_x}| < \frac{\epsilon}{2}$$

for any $N_x > N$. The particular choice of N_x does not matter, and we can always choose N_x large enough to make the other term smaller than $\epsilon/2$.

$\square$

CONJECTURE 1.3.1. *If $f_n \to f$ pointwise on a compact set K, then $f_n \to f$ uniformly on K.*

REMARK 1. *The conjecture is false. Consider the sequence from an earlier example:*

$$f_n(x) = x^n \quad x \in [0,1]$$

In this case $K = [0,1]$ is compact, and $f_n \to f$ pointwise on K with

$$f(x) = \left\{ \begin{array}{ccc} 0 & \textit{if} & x < 1 \\ 1 & \textit{if} & x = 1 \end{array} \right\}$$

In this case the limit of a sequence of continuous functions on a compact set is not continuous. We have a theorem stating that the uniform limit of a sequence of continuous functions is continuous, so the convergence cannot be uniform.

CONJECTURE 1.3.2. *If $f_n \to f$ on A and g is bounded on A, then $f_n g \to fg$ on A.*

PROOF. Let $\epsilon > 0$ be given. By hypothesis, g is bounded on A, so there is an $M \in (0, \infty)$ such that

$$|g(x)| \leq M \quad \text{for all} \quad x \in A$$

We have to show that there exists an $N \in \mathbb{N}$ such that when $n \geq N$,

$$|f_n g - fg| < \epsilon$$

By hypothesis, $f_n \to f$ uniformly on A, so there is an $N \in \mathbb{N}$ such that

$$|f_n - f| < \frac{\epsilon}{M} \quad \text{whenever} \quad n \geq N$$

Then for $n \geq N$,

$$|f_n g - fg| = |g||f_n - f| \leq M|f_n - f| < M\frac{\epsilon}{M} = \epsilon$$

$\square$

CONJECTURE 1.3.3. *If $f_n \to f$ uniformly on A and each f_n is bounded, then f is bounded.*

PROOF. By hypothesis, $f_n \to f$ uniformly on A. By definition, for $\epsilon = 1$, there exists an $N \in \mathbb{N}$ such that

$$|f_N - f| < 1$$

(this is a special case of the statement than $|f_n - f| < \epsilon$ when $n \geq N$)
By hypothesis, f_N is bounded, so there is an $M \in (0, \infty)$ such that

$$|f_N(x)| < M \quad \text{for all} \quad x \in A$$

Then for all $x \in A$,

$$|f(x)| = |-f(x)| = |f_N(x)-f(x)+f_N(x)| \leq |f_N(x)-f(x)|+|f_N(x)| < 1+M$$

so f is bounded. ☐

CONJECTURE 1.3.4. *If $f_n \to f$ uniformly on A and $f_n \to f$ uniformly on B, then $f_n \to f$ uniformly on $A \cup B$.*

PROOF. Let $\epsilon > 0$ be given. By hypothesis, $f_n \to f$ uniformly on A, so there is an $N_A \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all $x \in A$,

$$|f_n - f| < \epsilon \quad \text{whenever} \quad n \geq N_A$$

Also by hypothesis, $f_n \to f$ uniformly on B, so there is an $N_B \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and all $x \in B$,

$$|f_n - f| < \epsilon \quad \text{whenever} \quad n \geq N_B$$

Let N be the larger of N_A and N_B. Then for all $x \in A \cup B$,

$$|f_n(x) - f(x)| < \epsilon \quad \text{whenever} \quad n \geq N$$

☐

CONJECTURE 1.3.5. *If $f_n \to f$ uniformly on an interval A and if each f_n is increasing, then f is increasing.*

PROOF. Let a, b be arbitrary points in A with $a < b$. By hypothesis, f_n is increasing for every $n \in \mathbb{N}$, so we can write

$$f_n(a) \leq f_n(b) \quad \text{for all} \quad n$$

By the order limit theorem, this means

$$\lim f_n(a) \leq \lim f_n(b)$$

But $\lim f_n = f$, so $f_n(a) \to f(a)$ and $f_n(b) \to f(b)$, and therefore

$$f(a) \leq f(b)$$

☐

CONJECTURE 1.3.6. *If $f_n \to f$ pointwise on an interval A and if each f_n is increasing, then f is increasing.*

PROOF. The proof in the case of uniform convergence did not use uniform convergence, only pointwise convergence, so this is true by the same proof. ☐

CONJECTURE 1.3.7. *Let $f_0(x) = x$ for $x \in [0,1]$.*

Now let

$$f_1(x) = \begin{cases} (3/2)x & \text{if } 0 \le x \le 1/3 \\ 1/2 & \text{if } 1/3 < x < 1/3 \\ (3/2)x - 1/2 & \text{if } 2/3 \le x \le 1 \end{cases}$$

Then f is continuous and increasing on $[0,1]$, and constant on the middle third.

CONJECTURE 1.3.8. *Let*

$$f_2(x) = \begin{cases} f_1(3x)/2 & \text{if } 0 \le x \le 1/3 \\ f_1(x) & \text{if } 1/3 < x < 1/3 \\ f_1(3x-2)/2 - 1/2 & \text{if } 2/3 \le x \le 1 \end{cases}$$

If we continue this process, $f_n \to f$ uniformly on $[0,1]$.

1.4. The Differentiable Limit Theorem

In this section we will show that if we start with a pointwise-convergent sequence of differentiable functions $f_n \to f$, and the sequence of derivatives (f_n') converges uniformly to a function g, then g is differentiable and $f' = g$.

THEOREM 1.4.1 (Differentiable Limit Theorem). *Let $f_n \to f$ pointwise on the closed interval $[a,b]$, and assume that each f_n is differentiable. If the sequence $(f_n') \to g$ uniformly on $[a,b]$, then f is differentiable and $f' = g$.*

PROOF. Choose a fixed $c \in [a,b]$. To show that $f'(c)$ exists and is equal to $g(c)$, where

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

To show this, we need to show that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = g(c)$$

(Note that this statement implies that $f'(c)$ exists and is equal to $g(c)$.)

Let $\epsilon > 0$ be given. Then we need to show there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

As usual we employ a clever substitution and the triangle inequality:

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| =$$

$$\left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c} + \frac{f_n(x)-f_n(c)}{x-c} - f_n'(c) + f_n'(c) - g(c)\right|$$

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| \leq \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| +$$

$$\left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| + |f_n'(c) - g(c)|$$

This equality holds no matter what the value of $n \in \mathbb{N}$ is, i.e, it holds for every $n \in \mathbb{N}$.

We will show that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ and a $\delta > 0$ that makes each of the three terms on the right hand side less than $\epsilon/3$ when $0 < |x-c| < \delta$

By hypothesis, $(f_n') \to g$ uniformly on $[a,b]$, so by the definition of uniform convergence for sequences of functions, there is an $N_1 \in \mathbb{N}$ such that

$$|f_n'(c) - g(c)| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \geq N_1 \quad \text{for any} \quad c \in [a,b]$$

Furthermore, the Cauchy criterion for uniform convergence, there is an $N_2 \in \mathbb{N}$ such that

$$|f_m'(x) - f_n'(x)| < \frac{\epsilon}{3} \quad \text{when} \quad m,n \geq N_2 \quad \text{for every} \quad x \in [a,b]$$

Let N be the larger of N_1 and N_2. By hypothesis, f_n is differentiable on $[a,b]$ for each n, so f_N is differentiable, and so by definition

$$\lim_{x\to c} \frac{f_N(c) - f_N(x)}{x-c} \to f_N'(c)$$

for any $c \in [a,b]$, so for any c there is a $\delta > 0$ such that

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\epsilon}{3} \quad \text{whenever} \quad 0 < |x - c| < \delta$$

It remains to show that for this δ and this N,

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| < \frac{\epsilon}{3}$$

Let x be any value for which $0 < |x - c| < \delta$. Without loss of generality, assume $x > c$. By hypothesis, each f_n is differentiable on $[a, b]$, so f_N is differentiable on $[x, c]$, and we can apply the mean value theorem to the function $f_m - f_N$, where $m > N$, to establish that there is an $\alpha \in (x, c)$ such that

$$f'_n(\alpha) - f'_N(\alpha) = \frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c}$$

Our choice of m and N together with the Cauchy criterion for uniform convergence guarantees that

$$|f'_n(\alpha) - f'_N(\alpha)| < \frac{\epsilon}{3}$$

so by substitution

$$\left| \frac{(f_m(x) - f_N(x)) - (f_m(c) - f_N(c))}{x - c} \right| < \frac{\epsilon}{3}$$

which we can write as

$$\left| \frac{(f_m(x) - f_N(x))}{x - c} - \frac{(f_m(c) - f_N(c))}{x - c} \right| < \frac{\epsilon}{3}$$

By hypothesis $f_n \to f$ so can apply the Order Limit Theorem to write

$$\left|\frac{(f(x) - f_N(x))}{x - c} - \frac{(f(c) - f_N(c))}{x - c}\right| \leq \frac{\epsilon}{3}$$

Now we combine these results to write

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| \leq \left|\frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c}\right| +$$

$$\left|\frac{f_N(x) - f_N(c)}{x - c} - f_n'(c)\right| + |f_N'(c) - g(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

□

1.5. Series of Functions

DEFINITION 1.5.1 (pointwise convergence of a series). *Let f_n for each $n \in \mathbb{N}$ and f be functions defined on a set $a \subseteq \mathbb{R}$. The infinite series*

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $s_k(x)$ of partial sums,

$$s_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$.

The series converges uniformly to f on A if the sequence $s_k(x)$ converges uniformly to $f(x)$ on A.

Adapting the Cauchy criterion for sequences of functions to series:

THEOREM 1.5.1 (Cauchy Criterion for Uniform Convergence of Series). *A series*

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on $A \subseteq \mathbb{R}$ if and only if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon \quad \textit{whenever} \quad m, n \geq N \quad \textit{for all} \quad x \in A$$

Theorem 1.5.2 (Weierstrass M-Test). *For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying*

$$|f_n(x)| \leq M_n$$

for all $x \in A$. If

$$\sum_{n=1}^{\infty} M_n$$

converges, then

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on A.

Proof. Suppose

$$\sum_{n=1}^{\infty} M_n$$

converges. Let $\epsilon > 0$ be given. We need to show that there exists an $N \in \mathbb{N}$ such that

$$|f_{m+1}(x)+f_{m+2}(x)+\cdots+f_n(x)| < \epsilon \quad \text{whenever} \quad m, n \geq N \quad \text{for all} \quad x \in A$$

By hypothesis, $\sum M_n$ converges, so by the Cauchy Criterion there is an $N \in \mathbb{N}$ such that

$$M_{m+1} + M_{m+2} + \cdots + M_n < \epsilon \quad \text{whenever} \quad m, n \geq N$$

Then by the triangle inequality,

$$\begin{aligned} |f_{m+1}(x)+f_{m+2}(x)+\cdots+f_n(x)| &\leq |f_{m+1}(x)+f_{m+2}(x)+\cdots+f_{n-1}(x)|+|f_n(x)| \\ &\leq |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_{n-1}(x)| + M_n \end{aligned}$$

Applying the triangle inequality again gives

$$|f_{m+1}(x)+f_{m+2}(x)+\cdots+f_n(x)| \leq |f_{m+1}(x)+f_{m+2}(x)+\cdots+f_{n-2}(x)|+M_{n-1}+M_n$$

And eventually

$$|f_{m+1}(x)+f_{m+2}(x)+\cdots+f_n(x)| \leq M_{m+1}+M_{m+2}+\cdots+M_n < \epsilon \quad \text{whenever} \quad m, n \geq N$$

for all $x \in A$. □

Definition 1.5.1 (power series). *A function of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called a power series.

THEOREM 1.5.3. *The power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

converges to a_0 *if* $x = 0$.

PROOF. The result follows by substitution of $x = 0$ into the power series:

$$f(0) = \sum_{n=0}^{\infty} a_n x^n = a_0 + 0 + 0 + 0 + \cdots$$

□

THEOREM 1.5.4. *If a power series*

$$\sum_{n=0}^{\infty} a_n x^n$$

converges at some point $x_0 \in \mathbb{R}$ *then it converges uniformly on the closed interval* $[-c, c]$ *where*

$$c = |x_0|$$

PROOF. Set

$$M_n = |a_n x_0^n|$$

By hypothesis, the series converges absolutely, so we can write

$$\sum_{n=0}^{\infty} |a_n x_0^n| = \sum_{n=0}^{\infty} M_n$$

so $\sum M_n$ converges. For any $x \in [-c, c]$,

$$|a_n x^n| \leq |a_n x_0^n| = M_n$$

so the Weierstrass M-test guarantees that $\sum_{x=0}^{\infty} a_n x^n$ converges uniformly on $[-c, c]$. □

CHAPTER 2

Homework Problems

2.1. Week 1 Homework (assignment 1)

Team 1 (Ali, Emily, Frank).

PROBLEM 2.1.1. *Let S be the set of ordered n-tuples in $\mathbb{R}$:*

$$S = \{\ (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \quad 1 \leq i \leq n$$

- *a) Define a binary operation on $S \times S$, an identity element, and an inverse that makes S into an Abelian group.*
- *b) Define a multiplication operation on $\mathbb{R} \times S$ that makes S into a vector space over $\mathbb{R}$.*

PROBLEM 2.1.2. *Let S be set of real-valued functions on $[-1, 1]$:*

$$S = \{\ f : [-1, 1] \to \mathbb{R}\}$$

- *a) Define a binary operation on $S \times S$, an identity element, and an inverse that makes S into an Abelian group.*
- *b) Define a multiplication operation on $\mathbb{R} \times S$ that makes S into a vector space over $\mathbb{R}$.*

PROBLEM 2.1.3. *Let V be $\mathbb{R}^3$, the set of ordered triples of real numbers, with addition defined in the usual way:*

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \forall u, v \in V$$

*Define the **sum norm** on this vector space as:*

$$\|v\|_1 = |v_1| + |v_2| + |v_3|$$

Show that the sum norm satisfies the triangle inequality.

Presenters:

- 2.1.1 Ali

- 2.1.2 Frank
- 2.1.3 Emily

Team 2 (Blaine, Katie, Siobhan, Rachel).

PROBLEM 2.1.4. *Let $S^{m \times n}$ be the set of all $m \times n$ matrices over $\mathbb{R}$.*

$$S = \{\ x_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n, x_{i,j} \in \mathbb{R}\}$$

- a) Define a binary operation on $S \times S$, an identity element, and an inverse that makes S into an Abelian group.
- b) Define a multiplication operation on $\mathbb{R} \times S$ that makes S into a vector space over $\mathbb{R}$.

PROBLEM 2.1.5. *Let P be set of polynomial functions on $\mathbb{R}$:*

$$P = \{\ p : \mathbb{R} \to \mathbb{R} \quad \textit{such that for } x \in \mathbb{R},\ p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\}$$

where $a_1, a_1, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$.

- a) Define a binary operation on $P \times P$, an identity element, and an inverse that makes P into an Abelian group.
- b) Define a multiplication operation on $\mathbb{R} \times P$ that makes P into a vector space over $\mathbb{R}$.

PROBLEM 2.1.6. *Let V be $\mathbb{R}^3$, the set of ordered triples of real numbers, with addition defined in the usual way:*

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \quad \forall u, v \in V$$

Define the ***max norm*** *on this vector space as:*

$$\|v\|_\infty = \max(|v_1|, |v_2|, |v_3|)$$

Show that the max norm satisfies the triangle inequality.

PROBLEM 2.1.7. *Let V be $\mathbb{R}^3$, the set of ordered triples of real numbers, with zero element:*

$$\vec{0} = (0, 0, 0)$$

Define the ***sum norm*** *on this vector space as:*

$$\|v\|_1 = |v_1| + |v_2| + |v_3|$$

Show that the sum norm satisfies the required condition

$$\|v\| = 0 \quad \textit{if and only if} \quad v = \vec{0}$$

Presenters:

- 2.1.4 Blaine
- 2.1.5 Katie
- 2.1.6 Rachel
- 2.1.7 Siobhan

2.2. Week 1 Homework (assignment 2)

Team 1 (Ali, Emily, Frank).

PROBLEM 2.2.1. *Prove that arbitrary unions of open sets are open.*

PROBLEM 2.2.2. *Prove that a finite union of closed sets is closed.*

PROBLEM 2.2.3. *Prove that a point x is a limit point of A if and only if $x = \lim a_n$ for some sequence (a_n) in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Presenters:

- 2.2.1 Ali
- 2.2.2 Frank
- 2.2.3 Emily

Team 2 (Blaine, Katie, Siobhan, Rachel).

PROBLEM 2.2.4. *Prove that finite intersections of open sets are open.*

PROBLEM 2.2.5. *Prove that a set O is open if and only if its compliment, O^c, is closed.*

Problem 2.2.6. *Prove that a set F is closed if and only if its compliment F^c is open.*

Problem 2.2.7. *Prove that arbitrary intersections of closed sets are closed.*

Presenters:

- 2.2.4 Blaine
- 2.2.5 Katie
- 2.2.6 Rachel
- 2.2.7 Siobhan

2.3. Homework (assignment 3)

Team 1 (Ali, Emily, Frank).

Problem 2.3.1. *Prove that in a metric space (X, ρ) arbitrary unions of open sets are open.*

Problem 2.3.2. *Prove that in a metric space (x, ρ) a finite union of closed sets is closed.*

Problem 2.3.3. *Prove that a point x is a limit point of A if and only if $x = \lim a_n$ for some sequence (a_n) in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.*

Presenters:

- 2.3.1 Ali
- 2.3.2 Frank
- 2.3.3 Emily

Team 2 (Blaine, Katie, Siobhan, Rachel).

Problem 2.3.4. *Prove that in a metric space (X, ρ) finite intersections of open sets are open.*

PROBLEM 2.3.5. *Prove that in a metric space (X, ρ) a set O is open if and only if its compliment, O^c, is closed.*

PROBLEM 2.3.6. *Prove that in a metric space (X, ρ) a set F is closed if and only if its compliment F^c is open.*

PROBLEM 2.3.7. *Prove that in a metric space (X, ρ) arbitrary intersections of closed sets are closed.*

Presenters:

- 2.3.4 Blaine
- 2.3.5 Katie
- 2.3.6 Rachel
- 2.3.7 Siobhan

2.4. Homework (assignment 4)

Team 1 (Ali, Emily, Frank).

PROBLEM 2.4.1. *Define a sequence of functions on $\mathbb{R}$ by*

$$f_n(x) = \begin{cases} 1 & \textit{if} \quad x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \\ 0 & \textit{otherwise} \end{cases}$$

a) Is each f_n continuous at zero?

b) Does $f_n \to f$ uniformly on $\mathbb{R}$?

c) Is f continuous at zero?

Team 2 (Blaine, Katie, Siobhan, Rachel).

PROBLEM 2.4.2. *Define a sequence of functions on $\mathbb{R}$ by*

$$f_n(x) = \begin{cases} x & \textit{if} \quad x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \\ 0 & \textit{otherwise} \end{cases}$$

a) Is each f_n continuous at zero?

b) Does $f_n \to f$ uniformly on $\mathbb{R}$?

c) Is f continuous at zero?

2.5. Homework (assignment 5)

Teams 1 and 2.

PROBLEM 2.5.1 (Arzela-Ascoli Theorem). *For each $n \in \mathbb{N}$, let f_n be a function defined on $[0,1]$. If (f_n) is bounded on $[0,1]$, i.e., if there exists an $M > 0$ such that*

$$|f_n(x)| \leq M \quad \text{for all} \quad n \in \mathbb{N} \quad \text{ane} \quad x \in [0,1]$$

and if the collection (f_n) is equicontinuous, show that (f_n) contains a uniformly convergent subsequence.

Part a): Assume the results of problem 6.2.13 (2nd edition), which states the following:

Let A be a countable set $A = \{x_1, x_2, x_3, \ldots\}$. For each $n \in \mathbb{N}$, let h_n be a bounded, real-valued function defined on A. Then there exists a subsequence of (h_n) that converges pointwise on A.

Show that there exists a subsequence of (f_n) that converges pointwise at every rational point in $[0,1]$. To simplify te notation, let

$$g_k = f_{n_k}$$

Part b): Let $\epsilon > 0$ be given. By hypothesis, (f_n) is equicontinuous. Argue that there exists a $\delta > 0$ such that

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |x - y| < \delta \quad \text{for all} \quad k \in \mathbb{N}$$

Part c): Using this delta, let $r_1, r_2, r_3, \ldots, r_n$ be a finite collection of rational points with the property that

$$[0,1] \subseteq \bigcup_{i=1}^{n} V_\delta(r_i)$$

(i.e., the union of the δ-neighborhoods of the r_i contains the entire interval $[0,1]$.

Argue that there must exist an $N \in \mathbb{N}$ such that

$$|g_s(r_i) - g_t(r_i)| < \frac{\epsilon}{3} \quad \text{for all} \quad s, t \geq N \quad \text{and} \quad r_i, \quad i = 1, 2, /ldots, n$$

Part d): Complete the argument by showing that for arbitrary $x, y \in [0, 1]$,

$$|g_s(x) - g_t(y)| < \epsilon \quad \textit{for all} \quad s, t \in \mathbb{N}$$