Notes on Baire's Category Theory

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1 Basics

Definition 1.1. Let X be a topological space and S be a set in X.

1. The interior of S is defined as $\mathrm{int}(S) = \bigcup\{U : U \subset S \text{ and } U \text{ is open in } X\}$.
2. The closure of S is defined as $\mathrm{cl}(S) = \bigcap\{G : G \supset S \text{ and } G \text{ is closed in } X\}$.

Proposition 1.1. *Let A and B be two sets in a topological space X.*

1. *Duality between interior and closure:* $\mathrm{int}(A) = \mathrm{cl}(A^{\mathrm{c}})^{\mathrm{c}}$, $\mathrm{cl}(A) = \mathrm{int}(A^{\mathrm{c}})^{\mathrm{c}}$.
2. *For any $x \in X$, $x \in \mathrm{int}(A)$ if and only if there exists an open set U such that $x \in U \subset A$.*
3. *For any $x \in X$, $x \in \mathrm{cl}(A)$ if and only if $U \cap A \neq \emptyset$ for any open set U such that $x \in U$.*
4. $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.
5. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.

Proposition 1.2. *Let X be a topological space and Y be its subspace*

1. *For any $S \subset Y$, $\mathrm{int}_Y(S) \supset \mathrm{int}_X(S)$; if Y is open in X, then $\mathrm{int}_Y(S) = \mathrm{int}_X(S)$.*
2. *For any $S \subset Y$, $\mathrm{cl}_Y(S) = \mathrm{cl}_X(S) \cap Y$.*

In Proposition 1.2, we use a subscript to indicate the topological space which the interior or closure is defined with respect to. We will always do this if the context does not suffice to avoid confusion.

Proof. 1. $\mathrm{int}_X(S)$ is open in X and $\mathrm{int}_X(S) \subset S \subset Y$. Thus $\mathrm{int}_X(S)$ is an open subset of S in Y, leading to $\mathrm{int}_X(S) \subset \mathrm{int}_Y(S)$. If Y is open, then $\mathrm{int}_Y(S)$ is an open subset of S in X, implying that $\mathrm{int}_Y(S) \subset \mathrm{int}_X(S)$ and hence $\mathrm{int}_Y(S) = \mathrm{int}_X(S)$.

2. Since $\mathrm{cl}_Y(S)$ is closed in Y, there exists a closed set G in X such that $\mathrm{cl}_Y(S) = G \cap Y$. Thus $S \subset G$, implying that $\mathrm{cl}_X(S) \subset G$. Therefore, $\mathrm{cl}_X(S) \cap Y \subset G \cap Y = \mathrm{cl}_Y(S)$. On the other hand, $\mathrm{cl}_X(S) \cap Y$ is a closed set in Y and it contains S, implying that $\mathrm{cl}_X(S) \cap Y \supset \mathrm{cl}_Y(S)$. Hence $\mathrm{cl}_X(S) \cap Y = \mathrm{cl}_Y(S)$. □

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Definition 1.2. Let A and B be two sets in a topological space X

1. If $\mathrm{cl}(A) = X$, then A is said to be dense in X.

2. If $A \subset B$ and A is dense in B with respect to the subspace topology on B, then we will simply say that A is dense in B.

Proposition 1.3. *Let A and B be two sets in a topological space X such that $A \subset B$. Then A is dense in B if and only if* $\mathrm{cl}(A) \supset B$.

Proof. A is dense in $B \Leftrightarrow \mathrm{cl}_B(A) = B \Leftrightarrow \mathrm{cl}(A) \cap B = B \Leftrightarrow \mathrm{cl}(A) \supset B$. □

Proposition 1.4. *Let A and B be two sets in a topological space X.*

1. *$A \subset \mathrm{cl}(B)$ if and only if* $\mathrm{int}(B^{\mathrm{c}}) \subset \mathrm{int}(A^{\mathrm{c}})$.

2. *$A \subset \mathrm{cl}(B)$ if and only if $U \cap B \neq \emptyset$ for any open set U such that $U \cap A \neq \emptyset$*

3. *If A is open, then $A \subset \mathrm{cl}(B)$ if and only if $U \cap B \neq \emptyset$ for any nonempty open set $U \subset A$.*

4. *If $A \subset \mathrm{cl}(B)$, then $A \cap U \subset \mathrm{cl}(B \cap U)$ for any open set U.*

5. *If A is open and $A \subset \mathrm{cl}(B)$, then $A \cap B$ is dense in A.*

6. $\mathrm{int}(\mathrm{cl}(A)) \cap A$ *is dense in* $\mathrm{int}(\mathrm{cl}(A))$.

Proof. 1. $A \subset \mathrm{cl}(B) \Leftrightarrow (\mathrm{cl}(B))^{\mathrm{c}} \subset A^{\mathrm{c}} \Leftrightarrow \mathrm{int}(B^{\mathrm{c}}) \subset A^{\mathrm{c}} \Leftrightarrow \mathrm{int}(B^{\mathrm{c}}) \subset \mathrm{int}(A^{\mathrm{c}})$.

2. Suppose that $A \subset \mathrm{cl}(B)$. For any open set U such that $U \cap A \neq \emptyset$, we have $U \cap \mathrm{cl}(B) \neq \emptyset$, and hence $U \cap B \neq \emptyset$. If $A \not\subset \mathrm{cl}(B)$, then $U = (\mathrm{cl}(B))^{\mathrm{c}}$ is an open set such that $U \cap A \neq \emptyset$ while $U \cap B = \emptyset$.

3. Suppose that $A \subset \mathrm{cl}(B)$. For any nonempty open set $U \subset A$, we have $U \cap A = U \neq \emptyset$, and hence $U \cap B \neq \emptyset$ according to 2. If $A \not\subset \mathrm{cl}(B)$, then $U = A \cap (\mathrm{cl}(B))^{\mathrm{c}}$ is a nonempty open set such that $U \subset A$ while $U \cap B = \emptyset$.

4. Since $A \subset \mathrm{cl}(B)$, for any open set V such that $A \cap U \cap V \neq \emptyset$, we have according to 3 that $B \cap U \cap V \neq \emptyset$. This implies $A \cap U \subset \mathrm{cl}(B \cap U)$ according to 3.

5. According to 4, $A = A \cap A \subset \mathrm{cl}(A \cap B)$ since A is open.

6. Since $\mathrm{int}(\mathrm{cl}(A))$ is open, and $\mathrm{int}(\mathrm{cl}(A)) \subset \mathrm{cl}(A)$, we know from 5 that $\mathrm{int}(\mathrm{cl}(A)) \cap A$ is dense in $\mathrm{int}(\mathrm{cl}(A))$. □

2 Cantor's theorem and its consequences

Theorem 2.1 (Cantor's theorem)**.** *Let X be a topological space, and $\{C_n\}$ be a sequence of closed sets such that $C_{n+1} \subset C_n$ for each $n \geq 1$.*

1. *If X is complete metric space and* $\mathrm{diam}(C_n) \to 0$, *then $\bigcap_{n=1}^{\infty} C_n$ is a singleton.*

2. *If each C_n is compact, then $\bigcap_{n=1}^{\infty} C_n$ is nonempty.*

Theorem 2.2 ([3, Theorems 1.4.5–1.4.6])**.** *Let X be a complete metric space and $\{S_n\}_{n=1}^{\infty}$ be a sequence of sets in X.*

1. *If each S_n is open, then* $\mathrm{cl}(\bigcap_{n=1}^{\infty} S_n)$ *and* $\bigcap_{n=1}^{\infty} \mathrm{cl}(S_n)$ *have the same interior.*

2. *If each S_n is closed, then* $\mathrm{int}(\bigcup_{n=1}^{\infty} S_n)$ *and* $\bigcup_{n=1}^{\infty} \mathrm{int}(S_n)$ *have the same closure.*

Proof. Due to the duality between closure and interior, we only prove 1, for which it suffices to show that $\mathrm{int}(\bigcap_{n=1}^{\infty} \mathrm{cl}(S_n)) \subset \mathrm{cl}(\bigcap_{n=1}^{\infty} S_n)$. By item 3 of Proposition 1.4, we only need to prove for a given nonempty open set $U \subset \bigcap_{n=1}^{\infty} \mathrm{cl}(S_n)$ that

$$U \bigcap \left(\bigcap_{n=1}^{\infty} S_n \right) \neq \emptyset. \tag{2.1}$$

To this end, we will define a sequence of closed balls $\{B_n\}_{n=0}^{\infty}$ such that

$$0 < \mathrm{diam}(B_n) < 2^{-n}, \quad B_{n+1} \subset B_n \subset U \bigcap \left(\bigcap_{k=1}^{n} S_k \right) \quad \text{for each} \quad n \geq 0, \tag{2.2}$$

where $\bigcap_{k=1}^{0} S_k = X$. We will obtain (2.1) once this is done, as Cantor's theorem will yield

$$\emptyset \neq \bigcap_{n=1}^{\infty} B_n \subset U \bigcap \left(\bigcap_{n=1}^{\infty} S_n \right). \tag{2.3}$$

We define $\{B_n\}$ inductively. As U is a nonempty open set, we can take a closed ball $B_0 \subset U$ such that $0 < \mathrm{diam}(B_0) < 1$. Assume that the closed ball B_n is already defined for an $n \geq 0$ so that $0 < \mathrm{diam}(B_n) < 2^{-n}$ and $B_n \subset U \cap (\bigcap_{k=1}^{n} S_k)$. Recalling that $U \subset \mathrm{cl}(S_{n+1})$, we have $\mathrm{int}(B_n) \subset \mathrm{cl}(S_{n+1})$, which implies that $\mathrm{int}(B_n) \cap S_{n+1} \neq \emptyset$. Since $\mathrm{int}(B_n) \cap S_{n+1}$ is open, we can take a closed ball $B_{n+1} \subset \mathrm{int}(B_n) \cap S_{n+1}$ such that $0 < \mathrm{diam}(B_{n+1}) < 2^{-(n+1)}$. It is easy to see that $B_{n+1} \subset B_n$ and $B_{n+1} \subset B_n \cap S_{n+1} \subset U \cap (\bigcap_{k=1}^{n+1} S_k)$. This finishes the induction and completes the proof. □

Theorem 2.3. *Theorem 2.2 still holds if X is a locally compact Hausdorff space.*

Proof. The proof duplicates that of Theorem 2.2, except that $\{B_n\}$ is now a sequence of compact sets with nonempty interior such that $B_{n+1} \subset B_n \subset U \cap (\bigcap_{k=1}^{n} S_k)$, the existence of which can be established by the local compactness of X. Since X is a Hausdorff space, each B_n is closed, and hence $\bigcap_{n=1}^{\infty} B_n$ is nonempty by Cantor's theorem. □

3 Baire's category theorem

Definition 3.1. Let X be a topological space.

1. A set in X is said to be rare (or *nowhere dense*) if its closure has empty interior.

2. A set in X is said to be meager if it is a countable union of rare sets.

3. The complement of a meager set is called a comeager (or *residual*) set.

4. A meager set is also said to be of the first category; other sets are of the second category.

Proposition 3.1. *Rare sets, subsets of meager sets, and countable unions of meager sets are all meager sets.*

Example 3.1. *Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all the rational numbers. Define*

$$S = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{m\,2^n},\ r_n + \frac{1}{m\,2^n} \right).$$

It is easy to check that S is comeager in $\mathbb{R}$. It is dense in $\mathbb{R}$ but with zero Lebesgue measure.

Definition 3.2. A topological space X is called a Baire space if every meager set in X has empty interior.

Proposition 3.2. *Let X be a topological space. The following statements are equivalent.*

1. *X is a Baire space.*
2. *Any comeager set in X is dense.*
3. *Any nonempty open set in X is not meager.*
4. *Any countable intersection of dense open sets in X is still dense.*
5. *Any countable union of rare closed sets in X has empty interior.*

Proof. $1 \Rightarrow 2$. Let S be a comeager set in X. Then S^{c} is a meager set in X. Thus S has empty interior, which implies that S is dense in X.

$2 \Rightarrow 3$. Let U be an open set in X. If U is meager, then U^{c} is dense in X. Thus $U = \mathrm{int}(U) = \emptyset$.

$3 \Rightarrow 4$. Let $\{U_n\}$ be a sequence of dense open sets in X. Then $(\bigcap_{n=1}^{\infty} U_n)^{\mathrm{c}}$ is meager. Its interior is an open meager set in X, and hence empty. Thus $\bigcap_{n=1}^{\infty} U_n$ is dense.

$4 \Rightarrow 5$. Let $\{C_n\}$ be a sequence of rare closed sets in X. Then $\{C_n^{\mathrm{c}}\}$ is a seqeunce of dense open sets. Thus $\bigcap_{n=1}^{\infty} C_n^{\mathrm{c}}$ is dense in X. Taking the complement, we know that $\bigcup_{n=1}^{\infty} C_n$ has empty interior.

$5 \Rightarrow 1$. Let S be a meager set in X. Then there exists a sequence of rare sets $\{S_n\}$ such that $S = \bigcup_{n=1}^{\infty} S_n$. Thus $S \subset \bigcup_{n=1}^{\infty} \mathrm{cl}(S_n)$, the latter of which has empty interior because $\{\mathrm{cl}(S_n)\}$ is a sequence of rare closed sets. Hence S has empty interior. □

Proposition 3.3. *Let X be a topological space and S be a set in X. Then S is rare in X if and only if* $\mathrm{int}(\mathrm{cl}(S)) \cap S = \emptyset$.

Proof. By item 6 of Proposition 1.4, $\mathrm{int}(\mathrm{cl}(S)) \cap S$ is dense in $\mathrm{int}(\mathrm{cl}(S))$. Thus $\mathrm{int}(\mathrm{cl}(S)) = \emptyset$ if and only if $\mathrm{int}(\mathrm{cl}(S)) \cap S = \emptyset$. □

Lemma 3.1. *Let X be a topological space, Y be its open subspace, and S be a subset of Y.*

1. *S is rare in X if and only if S is rare in Y.*
2. *S is meager in X if and only if S is meager in Y.*

Proof. 1. Since Y is open in X, we have

$$\mathrm{int}_Y(\mathrm{cl}_Y(S)) = \mathrm{int}_X(\mathrm{cl}_Y(S)) \;=\; \mathrm{int}_X(\mathrm{cl}_X(S) \cap Y) = \mathrm{int}_X(\mathrm{cl}_X(S)) \cap Y. \tag{3.1}$$

If S is rare in X, then $\mathrm{int}_X(\mathrm{cl}_X(S)) = \emptyset$, and hence $\mathrm{int}_Y(\mathrm{cl}_Y(S)) = \emptyset$ by (3.1), implying that S is rare in Y. If S is rare in Y, then (3.1) leads to $\mathrm{int}_X(\mathrm{cl}_X(S)) \cap Y = \emptyset$, which implies $\mathrm{int}_X(\mathrm{cl}_X(S)) \cap S = \emptyset$, ensuring that S is rare in X by Proposition 3.3.

2. If S is meager in X, then $S = \bigcup_{n=1}^{\infty} S_n$ with a sequence of rare sets $\{S_n\}$ in X. Each S_n is a subset of S, and hence of Y, and it is rare in Y according to 1. Thus S is meager in Y. If S is meager in Y, then $S = \bigcup_{n=1}^{\infty} S_n$ with a sequence of rare sets $\{S_n\}$ in Y. Each S_n is rare in in X according to 1. Thus S is meager in X. $\square$

Theorem 3.1. *Let X be a topological space. The following statements are equivalent.*

1. *X is a Baire space.*
2. *Any open subspace of X is a Baire space.*
3. *For any sequence $\{S_n\}$ of open sets in X, $\operatorname{cl}(\bigcap_{n=1}^{\infty} S_n)$ and $\bigcap_{n=1}^{\infty} \operatorname{cl}(S_n)$ have the same interior.*
4. *For any sequence $\{S_n\}$ of closed sets in X, $\operatorname{int}(\bigcup_{n=1}^{\infty} S_n)$ and $\bigcup_{n=1}^{\infty} \operatorname{int}(S_n)$ have the same closure.*

Proof. $1 \Rightarrow 2$. Let Y be an open subspace of X, and S be a meager set in Y. Then S is meager in X by Lemma 3.1. Since X is a Baire space, we have $\operatorname{int}_X(S) = \emptyset$, which implies $\operatorname{int}_Y(S) = \emptyset$ as Y is open in X. Hence Y is a Baire space.

$2 \Rightarrow 3$. It suffices to prove that $\operatorname{int}(\bigcap_{n=1}^{\infty} \operatorname{cl}(S_n)) \subset \operatorname{cl}(\bigcap_{n=1}^{\infty} S_n)$. Let $Y = \operatorname{int}(\bigcap_{n=1}^{\infty} \operatorname{cl}(S_n))$. Then Y is a Baire space. Define $T_n = S_n \cap Y$ for each $n \geq 1$. Then each T_n is open in Y. Since Y is open and $Y \subset \operatorname{cl}(S_n)$, we know from item 5 of Proposition 1.4 that T_n is dense in Y. Hence $\bigcap_{n=1}^{\infty} T_n$ is dense in Y. Therefore, $Y \subset \operatorname{cl}(\bigcap_{n=1}^{\infty} T_n) \subset \operatorname{cl}(\bigcap_{n=1}^{\infty} S_n)$ as desired.

$3 \Rightarrow 4$. Obvious by the duality between interior and closure.

$4 \Rightarrow 5$. Let $\{S_n\}$ be a sequence of closed sets in X with empty interior. Then we have $\operatorname{cl}(\operatorname{int}(\bigcup_{n=1}^{\infty} S_n)) = \operatorname{cl}(\bigcup_{n=1}^{\infty} \operatorname{int}(S_n)) = \emptyset$. Thus $\operatorname{int}(\bigcup_{n=1}^{\infty} S_n) = \emptyset$. Hence X is a Baire space. $\square$

Theorems 2.2, 2.3, and 3.1 lead us to Baire's Category Theorem.

Theorem 3.2 (Baire's Category Theorem, [1, Theorem 48.2])**.** *Complete metric spaces and locally compact Hausdorff spaces are Baire spaces.*

4 Baire's category in topological vector spaces

Definition 4.1. A topological vector space X is a vector space over a topological field $\mathbb{K}$ (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that the vector addition $(x, y) \mapsto x + y$ and scalar multiplication $(\lambda, x) \mapsto \lambda x$ are continuous functions, where the domains of these functions are endowed with product topologies.

Proposition 4.1. *Suppose that X be a topological vector space and U is a neighborhood of 0. Then $X = \bigcup_{n=1}^{\infty}(nU)$.*

Proof. For any $x \in X$, $n^{-1}x \to 0$ due to the continuity of the scalar multiplication. Since U is a neighborhood of 0, $n^{-1}x$ falls inside U when n is sufficiently large, which implies $x \in nU$. $\square$

Theorem 4.1 ([2, Theorem 11.6.7])**.** *If X is a topological vector space, then X is a Baire space if and only if it is not meager.*

Proof. If X is a Baire space, then it is not meager as an open subset of itself. If X is not a Baire space, then there exists a nonempty open set U that is meager. Take $x \in U$ and set $V = U - x$. Then V is a meager neighborhood of 0. Thus $X = \bigcup_{n=1}^{\infty}(nV)$ is meager. $\square$

5 Examples

Proposition 5.1. *Let $f : (0, +\infty) \to \mathbb{R}$ be a continuous function. If $\lim_{n\to\infty} f(nx) = 0$ for all $x > 0$, then $\lim_{x\to+\infty} f(x) = 0$.*

Proof. We give two proofs.

1. Proof by Cantor's theorem. Assume that $f(x) \nrightarrow 0$ when $x \to +\infty$. Then there exists a constant $\epsilon > 0$ such that there exists $x \geq M$ with $|f(x)| > \epsilon$ for any $M > 0$. We will define a strictly increasing sequence of integers $\{n_k\}$ and a decreasing nested sequence of closed intervals $\{[a_k, b_k]\}$ such that $b_k > a_k > 0$ and

$$|f(n_k x)| > \epsilon \quad \text{for all} \quad x \in [a_k, b_k] \quad \text{and} \quad k \geq 1. \tag{5.1}$$

Once this is done, then there exists $y \in \bigcap_{k=1}^{\infty} [a_k, b_k]$, and $f(n_k y) > \epsilon$ for each $k \geq 1$, contradicting the assumption.

We define the aforementioned $\{[a_k, b_k]\}$ and $\{n_k\}$ inductively. Let $n_1 = 1$. By the continuity of f, there exists an interval $[a_1, b_1]$ such that $b_1 > a_1 > 0$ and $f(n_1 x) > \epsilon$ for all $x \in [a_1, b_1]$. Assume that n_k and $[a_k, b_k]$ has been defined for a $k \geq 1$ so that $b_k > a_k > 0$. Note that there exists $M > 0$ such that

$$[M, +\infty) \subset \bigcup_{n > n_k} n[a_k, b_k]. \tag{5.2}$$

Take $a \geq M$ such that $|f(a)| > \epsilon$. By the continuity of f, there exists $b > a$ such that $|f(x)| > \epsilon$ for all $x \in [a, b]$. According to (5.2), there exists $n_{k+1} > n_k$ such that $[a, b] \subset n_{k+1}[a_k, b_k]$. Let $a_{k+1} = a/n_{k+1}$ and $b_{k+1} = b/n_{k+1}$. Then $b_{k+1} > a_{k+1} > 0$, $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$, and $f(n_{k+1} x) > \epsilon$ for all $x \in [a_{k+1}, b_{k+1}]$. This finishes the induction and completes the proof.

2. Proof by Baire's Category Theorem. Let ϵ be a positive constant. Since $f(nx) \to 0$ for all $x > 0$, we have

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x > 0 : |f(nx)| \leq \epsilon\} \;=\; (0, +\infty). \tag{5.3}$$

By the continuity of f, $\bigcap_{n=N}^{\infty} \{x > 0 : |f(nx) \leq \epsilon|\}$ is closed for each $N \geq 1$. As $\mathbb{R}$ is a Baire space, $(0, +\infty)$ is not meager. Thus there exists N and an open interval (a, b) such that

$$(a, b) \subset \bigcap_{n=N}^{\infty} \{x > 0 : |f(nx)| \leq \epsilon\}. \tag{5.4}$$

Therefore

$$n(a, b) \subset \{x > 0 : |f(x)| \leq \epsilon\} \qquad \text{for each} \qquad n \geq N. \tag{5.5}$$

Note that there exists $M > 0$ such that

$$[M, \infty) \subset \bigcup_{n=N}^{\infty} n(a, b). \tag{5.6}$$

Combining (5.5)–(5.6), we have

$$[M, \infty) \subset \{x > 0 : |f(x)| \leq \epsilon\}, \tag{5.7}$$

which completes the proof.

□

Proposition 5.2. *Let $f : (0, +\infty) \to \mathbb{R}$ be a continuous function. If $\lim_{n\to\infty} f(nx)$ exists for all $x > 0$, then $\lim_{x\to+\infty} f(x)$ exists.*

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