

# Notes on Baire's Category Theory

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## 1 Basics

**Definition 1.1.** Let  $X$  be a topological space and  $S$  be a set in  $X$ .

1. The interior of  $S$  is defined as  $\text{int}(S) = \bigcup \{U : U \subset S \text{ and } U \text{ is open in } X\}$ .
2. The closure of  $S$  is defined as  $\text{cl}(S) = \bigcap \{G : G \supset S \text{ and } G \text{ is closed in } X\}$ .

**Proposition 1.1.** Let  $A$  and  $B$  be two sets in a topological space  $X$ .

1. Duality between interior and closure:  $\text{int}(A) = \text{cl}(A^c)^c$ ,  $\text{cl}(A) = \text{int}(A^c)^c$ .
2. For any  $x \in X$ ,  $x \in \text{int}(A)$  if and only if there exists an open set  $U$  such that  $x \in U \subset A$ .
3. For any  $x \in X$ ,  $x \in \text{cl}(A)$  if and only if  $U \cap A \neq \emptyset$  for any open set  $U$  such that  $x \in U$ .
4.  $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$ .
5.  $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ .

**Proposition 1.2.** Let  $X$  be a topological space and  $Y$  be its subspace

1. For any  $S \subset Y$ ,  $\text{int}_Y(S) \supset \text{int}_X(S)$ ; if  $Y$  is open in  $X$ , then  $\text{int}_Y(S) = \text{int}_X(S)$ .
2. For any  $S \subset Y$ ,  $\text{cl}_Y(S) = \text{cl}_X(S) \cap Y$ .

In Proposition 1.2, we use a subscript to indicate the topological space which the interior or closure is defined with respect to. We will always do this if the context does not suffice to avoid confusion.

**Proof.** 1.  $\text{int}_X(S)$  is open in  $X$  and  $\text{int}_X(S) \subset S \subset Y$ . Thus  $\text{int}_X(S)$  is an open subset of  $S$  in  $Y$ , leading to  $\text{int}_X(S) \subset \text{int}_Y(S)$ . If  $Y$  is open, then  $\text{int}_Y(S)$  is an open subset of  $S$  in  $X$ , implying that  $\text{int}_Y(S) \subset \text{int}_X(S)$  and hence  $\text{int}_Y(S) = \text{int}_X(S)$ .

2. Since  $\text{cl}_Y(S)$  is closed in  $Y$ , there exists a closed set  $G$  in  $X$  such that  $\text{cl}_Y(S) = G \cap Y$ . Thus  $S \subset G$ , implying that  $\text{cl}_X(S) \subset G$ . Therefore,  $\text{cl}_X(S) \cap Y \subset G \cap Y = \text{cl}_Y(S)$ . On the other hand,  $\text{cl}_X(S) \cap Y$  is a closed set in  $Y$  and it contains  $S$ , implying that  $\text{cl}_X(S) \cap Y \supset \text{cl}_Y(S)$ . Hence  $\text{cl}_X(S) \cap Y = \text{cl}_Y(S)$ .  $\square$

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**Definition 1.2.** Let  $A$  and  $B$  be two sets in a topological space  $X$

1. If  $\text{cl}(A) = X$ , then  $A$  is said to be dense in  $X$ .
2. If  $A \subset B$  and  $A$  is dense in  $B$  with respect to the subspace topology on  $B$ , then we will simply say that  $A$  is dense in  $B$ .

**Proposition 1.3.** Let  $A$  and  $B$  be two sets in a topological space  $X$  such that  $A \subset B$ . Then  $A$  is dense in  $B$  if and only if  $\text{cl}(A) \supset B$ .

**Proof.**  $A$  is dense in  $B \Leftrightarrow \text{cl}_B(A) = B \Leftrightarrow \text{cl}(A) \cap B = B \Leftrightarrow \text{cl}(A) \supset B$ .  $\square$

**Proposition 1.4.** Let  $A$  and  $B$  be two sets in a topological space  $X$ .

1.  $A \subset \text{cl}(B)$  if and only if  $\text{int}(B^c) \subset \text{int}(A^c)$ .
2.  $A \subset \text{cl}(B)$  if and only if  $U \cap B \neq \emptyset$  for any open set  $U$  such that  $U \cap A \neq \emptyset$
3. If  $A$  is open, then  $A \subset \text{cl}(B)$  if and only if  $U \cap B \neq \emptyset$  for any nonempty open set  $U \subset A$ .
4. If  $A \subset \text{cl}(B)$ , then  $A \cap U \subset \text{cl}(B \cap U)$  for any open set  $U$ .
5. If  $A$  is open and  $A \subset \text{cl}(B)$ , then  $A \cap B$  is dense in  $A$ .
6.  $\text{int}(\text{cl}(A)) \cap A$  is dense in  $\text{int}(\text{cl}(A))$ .

**Proof.** 1.  $A \subset \text{cl}(B) \Leftrightarrow (\text{cl}(B))^c \subset A^c \Leftrightarrow \text{int}(B^c) \subset A^c \Leftrightarrow \text{int}(B^c) \subset \text{int}(A^c)$ .

2. Suppose that  $A \subset \text{cl}(B)$ . For any open set  $U$  such that  $U \cap A \neq \emptyset$ , we have  $U \cap \text{cl}(B) \neq \emptyset$ , and hence  $U \cap B \neq \emptyset$ . If  $A \not\subset \text{cl}(B)$ , then  $U = (\text{cl}(B))^c$  is an open set such that  $U \cap A \neq \emptyset$  while  $U \cap B = \emptyset$ .

3. Suppose that  $A \subset \text{cl}(B)$ . For any nonempty open set  $U \subset A$ , we have  $U \cap A = U \neq \emptyset$ , and hence  $U \cap B \neq \emptyset$ . If  $A \not\subset \text{cl}(B)$ , then  $U = A \cap (\text{cl}(B))^c$  is a nonempty open set such that  $U \subset A$  while  $U \cap B = \emptyset$ .

4. Since  $A \subset \text{cl}(B)$ , for any open set  $V$  such that  $A \cap U \cap V \neq \emptyset$ , we have according to 3 that  $B \cap U \cap V \neq \emptyset$ . This implies  $A \cap U \subset \text{cl}(B \cap U)$  according to 3.

5. According to 4,  $A = A \cap A \subset \text{cl}(A \cap B)$  since  $A$  is open.

6. Since  $\text{int}(\text{cl}(A))$  is open, and  $\text{int}(\text{cl}(A)) \subset \text{cl}(A)$ , we know from 5 that  $\text{int}(\text{cl}(A)) \cap A$  is dense in  $\text{int}(\text{cl}(A))$ .  $\square$

## 2 Cantor's theorem and its consequences

**Theorem 2.1** (Cantor's theorem). Let  $X$  be a topological space, and  $\{C_n\}$  be a sequence of closed sets such that  $C_{n+1} \subset C_n$  for each  $n \geq 1$ .

1. If  $X$  is complete metric space and  $\text{diam}(C_n) \rightarrow 0$ , then  $\bigcap_{n=1}^{\infty} C_n$  is a singleton.
2. If each  $C_n$  is compact, then  $\bigcap_{n=1}^{\infty} C_n$  is nonempty.

**Theorem 2.2** ([3, Theorems 1.4.5–1.4.6]). Let  $X$  be a complete metric space and  $\{S_n\}_{n=1}^{\infty}$  be a sequence of sets in  $X$ .

1. If each  $S_n$  is open, then  $\text{cl}(\bigcap_{n=1}^{\infty} S_n)$  and  $\bigcap_{n=1}^{\infty} \text{cl}(S_n)$  have the same interior.
2. If each  $S_n$  is closed, then  $\text{int}(\bigcup_{n=1}^{\infty} S_n)$  and  $\bigcup_{n=1}^{\infty} \text{int}(S_n)$  have the same closure.

**Proof.** Due to the duality between closure and interior, we only prove 1, for which it suffices to show that  $\text{int}(\bigcap_{n=1}^{\infty} \text{cl}(S_n)) \subset \text{cl}(\bigcap_{n=1}^{\infty} S_n)$ . By item 3 of Proposition 1.4, we only need to prove for a given nonempty open set  $U \subset \bigcap_{n=1}^{\infty} \text{cl}(S_n)$  that

$$U \cap \left( \bigcap_{n=1}^{\infty} S_n \right) \neq \emptyset. \quad (2.1)$$

To this end, we will define a sequence of closed balls  $\{B_n\}_{n=0}^{\infty}$  such that

$$0 < \text{diam}(B_n) < 2^{-n}, \quad B_{n+1} \subset B_n \subset U \cap \left( \bigcap_{k=1}^n S_k \right) \quad \text{for each } n \geq 0, \quad (2.2)$$

where  $\bigcap_{k=1}^0 S_k = X$ . We obtain (2.1) once this is done, as Cantor's theorem will yield

$$\emptyset \neq \bigcap_{n=1}^{\infty} B_n \subset U \cap \left( \bigcap_{n=1}^{\infty} S_n \right). \quad (2.3)$$

We define  $\{B_n\}$  inductively. As  $U$  is a nonempty open set, we can take a closed ball  $B_0 \subset U$  such that  $0 < \text{diam}(B_0) < 1$ . Assume that the closed ball  $B_n$  is already defined for an  $n \geq 0$  so that  $0 < \text{diam}(B_n) < 2^{-n}$  and  $B_n \subset U \cap (\bigcap_{k=1}^n S_k)$ . Recalling that  $U \subset \text{cl}(S_{n+1})$ , we have  $\text{int}(B_n) \subset \text{cl}(S_{n+1})$ , which implies that  $\text{int}(B_n) \cap S_{n+1} \neq \emptyset$ . Since  $\text{int}(B_n) \cap S_{n+1}$  is open, we can take a closed ball  $B_{n+1} \subset \text{int}(B_n) \cap S_{n+1}$  such that  $0 < \text{diam}(B_{n+1}) < 2^{-(n+1)}$ . It is easy to see that  $B_{n+1} \subset B_n$  and  $B_{n+1} \subset B_n \cap S_{n+1} \subset U \cap (\bigcap_{k=1}^{n+1} S_k)$ . This finishes the induction and completes the proof.  $\square$

**Theorem 2.3.** *Theorem 2.2 still holds if  $X$  is a locally compact Hausdorff space.*

**Proof.** The proof duplicates that of Theorem 2.2, except that  $\{B_n\}$  is now a sequence of compact sets with nonempty interior such that  $B_{n+1} \subset B_n \subset U \cap (\bigcap_{k=1}^n S_k)$ , the existence of which can be established by the local compactness of  $X$ . Since  $X$  is a Hausdorff space, each  $B_n$  is closed, and hence  $\bigcap_{n=1}^{\infty} B_n$  is nonempty by Cantor's theorem.  $\square$

### 3 Baire's category theorem

**Definition 3.1.** Let  $X$  be a topological space.

1. A set in  $X$  is said to be rare (or *nowhere dense*) if its closure has empty interior.
2. A set in  $X$  is said to be meager if it is a countable union of rare sets.
3. The complement of a meager set is called a comeager (or *residual*) set.
4. A meager set is also said to be of the first category; other sets are of the second category.

**Proposition 3.1.** *Rare sets, subsets of meager sets, and countable unions of meager sets are all meager sets.*

**Definition 3.2.** A topological space  $X$  is called a Baire space if every meager set in  $X$  has empty interior.

**Proposition 3.2.** Let  $X$  be a topological space. The following statements are equivalent.

1.  $X$  is a Baire space.
2. Any comeager set in  $X$  is dense.
3. Any nonempty open set in  $X$  is not meager.
4. Any countable intersection of dense open sets in  $X$  is still dense.
5. Any countable union of rare closed sets in  $X$  has empty interior.

**Proof.**

$1 \Rightarrow 2$ . Let  $S$  be a comeager set in  $X$ . Then  $S^c$  is a meager set in  $X$ . Thus  $S$  has empty interior, which implies that  $S$  is dense in  $X$ .

$2 \Rightarrow 3$ . Let  $U$  be an open set in  $X$ . If  $U$  is meager, then  $U^c$  is dense in  $X$ . Thus  $U = \text{int}(U) = \emptyset$ .

$3 \Rightarrow 4$ . Let  $\{U_n\}$  be a sequence of dense open sets in  $X$ . Then  $(\bigcap_{n=1}^{\infty} U_n)^c$  is meager. Its interior is an open meager set in  $X$ , and hence empty. Thus  $\bigcap_{n=1}^{\infty} U_n$  is dense.

$4 \Rightarrow 5$ . Let  $\{C_n\}$  be a sequence of rare closed sets in  $X$ . Then  $\{C_n^c\}$  is a sequence of dense open sets. Thus  $\bigcap_{n=1}^{\infty} C_n^c$  is dense in  $X$ . Taking the complement, we know that  $\bigcup_{n=1}^{\infty} C_n$  has empty interior.

$5 \Rightarrow 1$ . Let  $S$  be a meager set in  $X$ . Then there exists a sequence of rare sets  $\{S_n\}$  such that  $S = \bigcup_{n=1}^{\infty} S_n$ . Thus  $S \subset \bigcup_{n=1}^{\infty} \text{cl}(S_n)$ , the latter of which has empty interior because  $\{\text{cl}(S_n)\}$  is a sequence of rare closed sets. Hence  $S$  has empty interior.  $\square$

**Proposition 3.3.** Let  $X$  be a topological space and  $S$  be a set in  $X$ . Then  $S$  is rare in  $X$  if and only if  $\text{int}(\text{cl}(S)) \cap S = \emptyset$ .

**Proof.** By item 6 of Proposition 1.4,  $\text{int}(\text{cl}(S)) \cap S$  is dense in  $\text{int}(\text{cl}(S))$ . Thus  $\text{int}(\text{cl}(S)) = \emptyset$  if and only if  $\text{int}(\text{cl}(S)) \cap S = \emptyset$ .  $\square$

**Lemma 3.1.** Let  $X$  be a topological space,  $Y$  be its open subspace, and  $S$  be a subset of  $Y$ .

1.  $S$  is rare in  $X$  if and only if  $S$  is rare in  $Y$ .
2.  $S$  is meager in  $X$  if and only if  $S$  is meager in  $Y$ .

**Proof.** 1. Since  $Y$  is open in  $X$ , we have

$$\text{int}_Y(\text{cl}_Y(S)) = \text{int}_X(\text{cl}_Y(S)) = \text{int}_X(\text{cl}_X(S) \cap Y) = \text{int}_X(\text{cl}_X(S)) \cap Y. \quad (3.1)$$

If  $S$  is rare in  $X$ , then  $\text{int}_X(\text{cl}_X(S)) = \emptyset$ , and hence  $\text{int}_Y(\text{cl}_Y(S)) = \emptyset$  by (3.1), implying that  $S$  is rare in  $Y$ . If  $S$  is rare in  $Y$ , then (3.1) leads to  $\text{int}_X(\text{cl}_X(S)) \cap Y = \emptyset$ , which implies  $\text{int}_X(\text{cl}_X(S)) \cap S = \emptyset$ , ensuring that  $S$  is rare in  $X$  by Proposition 3.3.

2. If  $S$  is meager in  $X$ , then  $S = \bigcup_{n=1}^{\infty} S_n$  with a sequence of rare sets  $\{S_n\}$  in  $X$ . Each  $S_n$  is a subset of  $S$ , and hence of  $Y$ , and it is rare in  $Y$  according to 1. Thus  $S$  is meager in  $Y$ . If  $S$  is meager in  $Y$ , then  $S = \bigcup_{n=1}^{\infty} S_n$  with a sequence of rare sets  $\{S_n\}$  in  $Y$ . Each  $S_n$  is rare in  $Y$  according to 1. Thus  $S$  is meager in  $X$ .  $\square$

**Theorem 3.1.** *Let  $X$  be a topological space. The following statements are equivalent.*

1.  $X$  is a Baire space.
2. Any open subspace of  $X$  is a Baire space.
3. For any sequence  $\{S_n\}$  of open sets in  $X$ ,  $\text{cl}(\bigcap_{n=1}^{\infty} S_n)$  and  $\bigcap_{n=1}^{\infty} \text{cl}(S_n)$  have the same interior.
4. For any sequence  $\{S_n\}$  of closed sets in  $X$ ,  $\text{int}(\bigcup_{n=1}^{\infty} S_n)$  and  $\bigcup_{n=1}^{\infty} \text{int}(S_n)$  have the same closure.

**Proof.**

$1 \Rightarrow 2$ . Let  $Y$  be an open subspace of  $X$ , and  $S$  be a meager set in  $Y$ . Then  $S$  is meager in  $X$  by Lemma 3.1. Since  $X$  is a Baire space,  $S$  has empty interior in  $X$ , which implies that  $S$  has empty interior in  $Y$ . Hence  $Y$  is a Baire space.

$2 \Rightarrow 3$ . It suffices to prove that  $\text{int}(\bigcap_{n=1}^{\infty} \text{cl}(S_n)) \subset \text{cl}(\bigcap_{n=1}^{\infty} S_n)$ . Let  $Y = \text{int}(\bigcap_{n=1}^{\infty} \text{cl}(S_n))$ . Then  $Y$  is a Baire space. Define  $T_n = S_n \cap Y$  for each  $n \geq 1$ . Then each  $T_n$  is open in  $Y$ . Since  $Y$  is open and  $Y \subset \text{cl}(S_n)$ , we know from item 5 of Proposition 1.4 that  $T_n$  is dense in  $Y$ . Hence  $\bigcap_{n=1}^{\infty} T_n$  is dense in  $Y$ . Therefore,  $Y \subset \text{cl}(\bigcap_{n=1}^{\infty} T_n) \subset \text{cl}(\bigcap_{n=1}^{\infty} S_n)$  as desired.

$3 \Rightarrow 4$ . Obvious by the duality between interior and closure.

$4 \Rightarrow 5$ . Let  $\{S_n\}$  be a sequence of closed sets in  $X$  with empty interior. Then we have  $\text{cl}(\text{int}(\bigcup_{n=1}^{\infty} S_n)) = \text{cl}(\bigcup_{n=1}^{\infty} \text{int}(S_n)) = \emptyset$ . Thus  $\text{int}(\bigcup_{n=1}^{\infty} S_n) = \emptyset$ . Hence  $X$  is a Baire space.  $\square$

**Theorem 3.2** (Baire's Category Theorem, [1, Theorem 48.2]). *Complete metric spaces and locally compact Hausdorff spaces are Baire spaces.*

## 4 Baire's category in topological vector spaces

**Definition 4.1.** A topological vector space  $X$  is a vector space over a topological field  $\mathbb{K}$  (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that the vector addition  $(x, y) \mapsto x + y$  and scalar multiplication  $(\lambda, x) \mapsto \lambda x$  are continuous functions, where the domains of these functions are endowed with product topologies.

**Proposition 4.1.** *Suppose that  $X$  be a topological vector space and  $U$  is a neighborhood of 0. Then  $X = \bigcup_{n=1}^{\infty} (nU)$ .*

**Proof.** For any  $x \in X$ ,  $n^{-1}x \rightarrow 0$  due to the continuity of the scalar multiplication. Since  $U$  is a neighborhood of 0,  $n^{-1}x$  falls inside  $U$  when  $n$  is sufficiently large, which implies  $x \in nU$ .  $\square$

**Theorem 4.1** ([2, Theorem 11.6.7]). *If  $X$  is a topological vector space, then  $X$  is a Baire space if and only if it is not meager.*

**Proof.** If  $X$  is a Baire space, then it is not meager as an open subset of itself. If  $X$  is not a Baire space, then there exists a nonempty open set  $U$  that is meager. Take  $x \in U$  and set  $V = U - x$ . Then  $V$  is a meager neighborhood of 0. Thus  $X = \bigcup_{n=1}^{\infty} (nV)$  is meager.  $\square$

## 5 Examples

**Proposition 5.1.** *Let  $f : (0, +\infty) \rightarrow \mathbb{R}$  be a continuous function. If  $\lim_{n \rightarrow \infty} f(nx) = 0$  for all  $x > 0$ , then  $\lim_{x \rightarrow +\infty} f(x) = 0$ .*

**Proof.** We give two proofs.

1. Proof by Cantor's theorem. Assume that  $f(x) \not\rightarrow 0$  when  $x \rightarrow +\infty$ . Then there exists a constant  $\epsilon > 0$  such that there exists  $x \geq M$  with  $|f(x)| > \epsilon$  for any  $M > 0$ . We will define a strictly increasing sequence of integers  $\{n_k\}$  and a decreasing nested sequence of closed intervals  $\{[a_k, b_k]\}$  such that  $b_k > a_k > 0$  and

$$|f(n_k x)| > \epsilon \quad \text{for all } x \in [a_k, b_k] \quad \text{and } k \geq 1. \quad (5.1)$$

Once this is done, then there exists  $y \in \bigcap_{k=1}^{\infty} [a_k, b_k]$ , and  $f(n_k y) > \epsilon$  for each  $k \geq 1$ , contradicting the assumption.

We define the aforementioned  $\{[a_k, b_k]\}$  and  $\{n_k\}$  inductively. Let  $n_1 = 1$ . By the continuity of  $f$ , there exists an interval  $[a_1, b_1]$  such that  $b_1 > a_1 > 0$  and  $f(n_1 x) > \epsilon$  for all  $x \in [a_1, b_1]$ . Assume that  $n_k$  and  $[a_k, b_k]$  has been defined for a  $k \geq 1$  so that  $b_k > a_k > 0$ . Note that there exists  $M > 0$  such that

$$[M, +\infty) \subset \bigcup_{n > n_k} n[a_k, b_k]. \quad (5.2)$$

Take  $a \geq M$  such that  $|f(a)| > \epsilon$ . By the continuity of  $f$ , there exists  $b > a$  such that  $|f(x)| > \epsilon$  for all  $x \in [a, b]$ . According to (5.2), there exists  $n_{k+1} > n_k$  such that  $[a, b] \subset n_{k+1}[a_k, b_k]$ . Let  $a_{k+1} = a/n_{k+1}$  and  $b_{k+1} = b/n_{k+1}$ . Then  $b_{k+1} > a_{k+1} > 0$ ,  $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$ , and  $f(n_{k+1}x) > \epsilon$  for all  $x \in [a_{k+1}, b_{k+1}]$ . This finishes the induction and completes the proof.

2. Proof by Baire's Category Theorem. Let  $\epsilon$  be a positive constant. Since  $f(nx) \rightarrow 0$  for all  $x > 0$ , we have

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x > 0 : |f(nx)| \leq \epsilon\} = (0, +\infty). \quad (5.3)$$

By the continuity of  $f$ ,  $\bigcap_{n=N}^{\infty} \{x > 0 : |f(nx)| \leq \epsilon\}$  is closed for each  $N \geq 1$ . As  $\mathbb{R}$  is a Baire space,  $(0, +\infty)$  is not meager. Thus there exists  $N$  and an open interval  $(a, b)$  such that

$$(a, b) \subset \bigcap_{n=N}^{\infty} \{x > 0 : |f(nx)| \leq \epsilon\}. \quad (5.4)$$

Therefore

$$n(a, b) \subset \{x > 0 : |f(x)| \leq \epsilon\} \quad \text{for each } n \geq N. \quad (5.5)$$

Note that there exists  $M > 0$  such that

$$[M, \infty) \subset \bigcup_{n=N}^{\infty} n(a, b). \quad (5.6)$$

Combining (5.5)–(5.6), we have

$$[M, \infty) \subset \{x > 0 : |f(x)| \leq \epsilon\}, \quad (5.7)$$

which completes the proof.  $\square$

**Proposition 5.2.** *Let  $f : (0, +\infty) \rightarrow \mathbb{R}$  be a continuous function. If  $\lim_{n \rightarrow \infty} f(nx)$  exists for all  $x > 0$ , then  $\lim_{x \rightarrow +\infty} f(x)$  exists.*

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