

# Notes on the Law of Large Numbers and the Borel-Cantelli Lemmas

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For a sequence of random variables  $\{X_k\}$ , we will say it is nonnegative if  $X_k \geq 0$  a.s. for each  $k$ , and uniformly bounded if there exists a constant  $M$  such that  $|X_k| \leq M$  a.s. for each  $k$ . In addition, we define  $\bar{X}_k = k^{-1} \sum_{\ell=1}^k X_\ell$  for each  $k \geq 1$ .

## 1 The strong law of large numbers

### 1.1 The i.i.d. case

**Theorem 1.1.** *Let  $\{X_k\}$  be a sequence of i.i.d. random variables.*

- (a)  *$\{\bar{X}_k\}$  converges a.s. to a finite random variable if and only if  $\mathbb{E}(|X_1|) < \infty$ . When it converges a.s., the limit is  $\mathbb{E}(X_1)$ .*
- (b) *If  $\{X_k\}$  is nonnegative, then  $\{\bar{X}_k\}$  converges to  $\mathbb{E}(X_1)$  a.s.*

**Proof.** (a) See [10, Theorem 5.23].

(b) If  $\mathbb{E}(X_1) < \infty$ , then the convergence holds according to (a). Otherwise, for each integer  $n \geq 1$ , we have

$$\liminf_{k \rightarrow \infty} \bar{X}_k \geq \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{\ell=1}^k \min\{X_\ell, n\} = \mathbb{E}(\min\{X_1, n\}).$$

Letting  $n \rightarrow \infty$ , we have  $\liminf_k \bar{X}_k = \infty$  a.s., because  $\mathbb{E}(\min\{X_1, n\}) \rightarrow \mathbb{E}(X_1) = \infty$  according to Levi's monotone convergence theorem. Hence  $\bar{X}_k \rightarrow \infty = \mathbb{E}(X_1)$  a.s.  $\square$

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**Remark 1.1.** For the “if” part of Theorem 1.1, see also [7, Theorem 1] and [6, Theorem 2.4.1], which show that the mutual independence assumption can be weakened to pairwise independence.

**Remark 1.2.** Suppose that  $\{X_k\}$  is a sequence of mutually independent random variables. If  $\{\bar{X}_k\}$  converges a.s., then the limit is a constant a.s. according to Kolmogorov’s zero-one law. For the same reason, if  $\{X_k\}$  converges a.s., then its limit is a constant a.s..

Item (b) of Theorem 1.1 can be generalized to the following, with a very similar proof.

**Theorem 1.2.** *Let  $\{X_k\}$  be a sequence of i.i.d. random variables. If  $\mathbb{E}(X_1^+) < \infty$  or  $\mathbb{E}(X_1^-) < \infty$ , then  $\{\bar{X}_k\}$  converges to  $\mathbb{E}(X_1)$  a.s.*

**Proof.** If  $\mathbb{E}(X_1^+) < \infty$  and  $\mathbb{E}(X_1^-) < \infty$ , then  $\mathbb{E}(|X_1|) < \infty$ . Thus the convergence holds. If  $\mathbb{E}(X_1^+) = \infty$  and  $\mathbb{E}(X_1^-) < \infty$ , then  $\mathbb{E}(X_1) = -\infty$  and the convergence can be established by considering the truncated sequence  $\{\min\{X_k, n\}\}$  for each integer  $n \geq 1$ .  $\square$

## 1.2 The martingale-difference case

**Theorem 1.3** (Martingale convergence theorem [6, Theorem 4.2.11]). *If  $\{X_k\}$  is a submartingale with  $\sup_k \mathbb{E}(X_k^+) < \infty$ , then  $X_k \rightarrow X$  a.s. for some random variable  $X$  with  $\mathbb{E}(|X|) < \infty$ .*

**Corollary 1.1** ([6, Theorem 4.2.12]). *If  $\{X_k\}$  is a nonnegative supermartingale, then  $X_k \rightarrow X$  a.s. for some random variable  $X$  with  $\mathbb{E}(X) \leq \mathbb{E}(X_1)$ .*

By Theorem 1.3, if  $\{X_k\}$  is an  $L^1$ -bounded martingale, then  $X_k \rightarrow X$  a.s. for some random variable  $X \in L^1$ . However, the convergence may not hold in  $L^1$ . See [6, Example 4.2.13]. In contrast, the following theorem ensures the convergence in  $L^p$  for  $p > 1$ .

**Theorem 1.4** (Martingale  $L^p$  convergence theorem [6, Theorem 4.4.6]). *If  $p > 1$  and  $\{X_k\}$  is a martingale with  $\sup_{k \geq 1} \|X_k\|_p < \infty$ , then there exists a random variable  $X \in L^p$  such that  $X_k \rightarrow X$  a.s. and in  $L^p$ .*

**Lemma 1.1** (Kronecker’s Lemma). *Let  $\{a_k\}$  be a sequence of nonnegative numbers and  $\{b_k\}$  be a non-decreasing sequence with  $b_k \rightarrow \infty$ .*

- (a) *If  $\sum_{k=1}^{\infty} b_k^{-1} a_k < \infty$ , then  $b_k^{-1} \sum_{\ell=1}^k a_{\ell} \rightarrow 0$ .*
- (b) *If  $\sum_{k=1}^{\infty} a_k < \infty$ , then  $b_k^{-1} \sum_{\ell=1}^k b_{\ell} a_{\ell} \rightarrow 0$ .*

**Remark 1.3.** In item (a) of Lemma 1.1, we trivially have  $b_k^{-1}a_k \rightarrow 0$ , and Kronecker's Lemma strengthens it to  $b_k^{-1} \sum_{\ell=1}^k a_\ell \rightarrow 0$ . In item (b), we trivially have  $b_k^{-1} \sum_{\ell=1}^k a_\ell \rightarrow 0$ , and Kronecker's Lemma strengthens it to  $b_k^{-1} \sum_{\ell=1}^k b_\ell a_\ell \rightarrow 0$ .

**Definition 1.1.** Let  $\{X_k\}$  be a sequence of random variables and  $\{\mathcal{F}_k\}$  be a filtration. Then  $\{X_k\}$  is called a martingale difference sequence adapted to  $\{\mathcal{F}_k\}$  if there exists a martingale  $\{S_k\}_{k \geq 0}$  adapted to  $\{\mathcal{F}_k\}$  such that  $X_k = S_k - S_{k-1}$  for each  $k \geq 1$ , or, equivalently, if  $\mathbb{E}(|X_k|) < \infty$  and  $\mathbb{E}(X_k | \mathcal{F}_{k-1}) = 0$  a.s. for each  $k \geq 1$ .

**Lemma 1.2** (Orthogonality). *If  $\{X_k\}$  is a martingale difference sequence, then  $\mathbb{E}(X_i X_j) = 0$  and  $\text{Cov}(X_i, X_j) = 0$  for any distinct  $i$  and  $j$ .*

**Proof.** For any  $i > j \geq 1$ , we have

$$\mathbb{E}(X_i X_j) = \mathbb{E}(X_j \mathbb{E}(X_i | \mathcal{F}_{i-1})) = 0.$$

In addition,  $\mathbb{E}(X_i) = \mathbb{E}(\mathbb{E}(X_i | \mathcal{F}_{i-1})) = 0$  and similarly  $\mathbb{E}(X_j) = 0$ . Hence  $\text{Cov}(X_i, X_j) = 0$ .  $\square$

**Theorem 1.5** (Chow [4]). *Let  $\{X_k\}$  be a martingale difference sequence. If there exists a constant  $p \geq 1$  such that*

$$\sum_{k=1}^{\infty} k^{-p-1} \mathbb{E}(|X_k|^{2p}) < \infty,$$

*then  $\bar{X}_k \rightarrow 0$  a.s.*

**Remark 1.4.** Chow's theorem is a consequence of [2, Theorem 2] and [1, Theorem 9]. Chow [4] mentions these references without specifying the theorems. See also

<https://math.stackexchange.com/questions/4965300/>.

**Theorem 1.6** ([8, Theorem 2.18]). *If  $1 \leq p \leq 2$  and  $\{X_k\}$  is a martingale difference sequence adapted to a filtration  $\{\mathcal{F}_k\}$ , then  $\bar{X}_k \rightarrow 0$  a.s. on the set*

$$\left\{ \sum_{k=1}^{\infty} k^{-p} \mathbb{E}(|X_k|^p | \mathcal{F}_{k-1}) < \infty \right\}.$$

*In particular, if  $\sum_{k=1}^{\infty} k^{-p} \mathbb{E}(|X_k|^p) < \infty$ , then  $\bar{X}_k \rightarrow 0$  a.s.*

Theorem 1.6 follows from Kronecker's Lemma and the following theorem due to Chow [3]. It generalizes the Kolmogorov's two-series theorem (Theorem 1.10) to martingale difference sequences.

**Theorem 1.7** ([3, Corollary 5]). *If  $1 \leq p \leq 2$  and  $\{X_k\}$  is a martingale difference sequence adapted to a filtration  $\{\mathcal{F}_k\}$ , then  $\sum_{k=1}^{\infty} X_k$  converges a.s. on the set*

$$\left\{ \sum_{k=1}^{\infty} \mathbb{E}(|X_k|^p \mid \mathcal{F}_{k-1}) < \infty \right\}.$$

*In particular, if  $\sum_{k=1}^{\infty} \mathbb{E}(|X_k|^p) < \infty$ , then  $\sum_{k=1}^{\infty} X_k$  converges a.s.*

Theorem 1.7 in turn is a consequence of the following lemma, which is a conditional version of Kolmogorov's three-series theorem (Theorem 1.11). See [8, pages 33-36] for more details.

**Theorem 1.8** ([8, Theorem 2.16]). *Let  $\{X_k\}$  be a sequence of random variables adapted to a filtration  $\{\mathcal{F}_k\}$ ,  $c$  be a positive constant, and  $Y_k = X_k \mathbb{1}(X_k \leq c)$  for each  $k \geq 1$ . Then  $\sum_{k=1}^{\infty} X_k$  converges a.s. on the set*

$$\left\{ \sum_{k=1}^{\infty} \mathbb{1}(X_k > c \mid \mathcal{F}_{k-1}) < \infty, \sum_{k=1}^{\infty} \mathbb{E}(Y_k \mid \mathcal{F}_{k-1}) \text{ converges}, \sum_{k=1}^{\infty} \text{Var}(|Y_k| \mid \mathcal{F}_{k-1}) < \infty \right\}.$$

**Corollary 1.2.** *Let  $\{X_k\}$  be a martingale difference sequence. If  $\sum_{k=1}^{\infty} k^{-2} \text{Var}(X_k) < \infty$ , then  $\bar{X}_k \rightarrow 0$  a.s.*

**Proof.** This is the special case of Theorem 1.5 with  $p = 1$  or that of Theorem 1.6 with  $p = 2$ . We now provide an alternative proof using the martingale  $L^p$  convergence theorem and Kronecker's Lemma. Define  $Y_k = \sum_{\ell=1}^k \ell^{-1} X_{\ell}$  for each  $k \geq 1$ . Then  $\{Y_k\}$  is a martingale. In addition, due to Lemma 1.2, we have

$$\mathbb{E}(Y_k^2) = \sum_{\ell=1}^k \ell^{-2} \mathbb{E}(X_{\ell}^2) \leq \sum_{\ell=1}^{\infty} \ell^{-2} \mathbb{E}(X_{\ell}^2) < \infty.$$

Thus  $\{Y_k\}$  is bounded in  $L^2$  and hence converges a.s. Invoking Kronecker's Lemma, we have  $\bar{X}_k \rightarrow 0$  a.s.  $\square$

### 1.3 The independent case with moment conditions

**Lemma 1.3.** *Let  $X$  be a random and  $p \geq 1$  be a constant.*

- (a)  $\mathbb{E}(|X|) \leq \mathbb{E}(|X|^p)^{\frac{1}{p}}.$
- (b) *If  $\mathbb{E}(|X|) < \infty$ , then  $\mathbb{E}(|X - \mathbb{E}(X)|^p) \leq 2^p \mathbb{E}(|X|^p).$*

**Proof.** (a) Jensen's inequality (or Hölder's inequality).

(b) By the triangle inequality, we have  $\|X - \mathbb{E}(X)\|_p \leq \|X\|_p + |\mathbb{E}(X)| \leq 2\|X\|_p$ .  $\square$

**Theorem 1.9.** *Let  $\{X_k\}$  be a sequence of independent random variables. If either*

$$\sum_{k=1}^{\infty} k^{-p-1} \mathbb{E}(|X_k|^{2p}) < \infty \quad (1.1)$$

*for a constant  $p \geq 1$ , or*

$$\sum_{k=1}^{\infty} k^{-p} \mathbb{E}(|X_k|^p) < \infty \quad (1.2)$$

*for a constant  $p \in [1, 2]$ , then  $\bar{X}_k - \mathbb{E}(\bar{X}_k) \rightarrow 0$  a.s.*

**Proof.** According to Lemma 1.3, we have  $\mathbb{E}(|X_k|) < \infty$  for each  $k \geq 1$  in both cases. Define  $Y_k = X_k - \mathbb{E}(X_k)$  for each  $k \geq 1$ . Then  $\{Y_k\}$  is a martingale difference sequence. In addition,

$$\mathbb{E}(|Y_k|^p) \leq 2^p \mathbb{E}(|X_k|^p)$$

for all  $k \geq 1$  and  $p \geq 1$ . Hence (1.3) implies

$$\sum_{k=1}^{\infty} k^{-p-1} \mathbb{E}(|Y_k|^{2p}) < \infty, \quad (1.3)$$

and (1.4) implies

$$\sum_{k=1}^{\infty} k^{-p} \mathbb{E}(|Y_k|^p) < \infty. \quad (1.4)$$

In both cases,  $\bar{Y}_k \rightarrow 0$  a.s. by Theorems 1.5 and 1.6. Thus  $\bar{X}_k - \mathbb{E}(\bar{X}_k) \rightarrow 0$  a.s.  $\square$

**Theorem 1.10** (Kolmogorov's two-series theorem [6, Theorems 2.5.6]). *Let  $\{X_k\}$  be a sequence of independent random variables. Then  $\sum_{k=1}^{\infty} X_k$  converges a.s. if the two series  $\sum_{k=1}^{\infty} \mathbb{E}(X_k)$  and  $\sum_{k=1}^{\infty} \text{Var}(X_k)$  both converge.*

**Remark 1.5.** Theorem 1.10 is a special case of Theorem 1.4 with  $p = 2$ . To see this, define  $Y_k = \sum_{\ell=1}^k [X_\ell - \mathbb{E}(X_\ell)]$  for each  $k \geq 1$ . Then  $\{Y_k\}$  is a martingale. In addition, it is bounded in  $L^2$ , since

$$\mathbb{E}(Y_k^2) = \sum_{\ell=1}^k \text{Var}(X_\ell) \leq \sum_{\ell=1}^{\infty} \text{Var}(X_\ell) < \infty.$$

Thus  $\{Y_k\}$  converges a.s. Combining this with the convergence of  $\sum_{k=1}^{\infty} \mathbb{E}(X_k)$ , we obtain the almost sure convergence of  $\sum_{k=1}^{\infty} X_k$ .

**Theorem 1.11** (Kolmogorov's three-series theorem [6, Theorems 2.5.8]). *Let  $\{X_k\}$  be a sequence of independent random variables,  $c$  be a positive constant, and  $Y_k = X_k \mathbb{1}(|X_k| \leq c)$ . Then  $\sum_{k=1}^{\infty} X_k$  converges a.s. **if and only if** the three series  $\sum_{k=1}^{\infty} \mathbb{P}(|X_k| > c)$ ,  $\sum_{k=1}^{\infty} \mathbb{E}(Y_k)$ , and  $\sum_{k=1}^{\infty} \text{Var}(Y_k)$  all converge.*

The following proposition is sometimes known as ‘‘Kolmogorov's Criterion of SSLN’’. It follows from Theorem 1.10 and Kronecker's Lemma (consider the sequence  $\{\sum_{\ell=1}^k \ell^{-1} X_\ell\}$ ). It is also a special case of Corollary 1.2 and Theorem 1.9.

**Corollary 1.3.** *Let  $\{X_k\}$  be a sequence of independent random variables with  $\mathbb{E}(X_k) = 0$  for each  $k \geq 0$ . If  $\sum_{k=1}^{\infty} k^{-2} \text{Var}(X_k) < \infty$ , then  $\bar{X}_k \rightarrow 0$  a.s.*

As a side note, we have the following result due to Lévy.

**Theorem 1.12** (Lévy [5, Theorem 5.3.4]). *If  $\{X_k\}$  is a sequence of independent random variables then  $\sum_{k=1}^{\infty} X_k$  converges a.s. if and only if it converges in probability.*

**Question 1.1.** *What if  $\sum_{k=1}^{\infty} X_k$  diverges to infinity in probability or a.s.?*

Note that Theorem 1.12 cannot be extended to the case with  $\{X_k\}$  being a martingale difference sequence. In other words, for a martingale, the convergence in probability does not imply the almost sure convergence, as illustrated by the following example.

**Example 1.1** ([6, Example 4.2.14]). *Let  $\{A_k\}$  and  $\{B_k\}$  be two sequences of i.i.d. random variables with*

$$\mathbb{P}(A_k = 1) = \frac{1}{k}, \quad \mathbb{P}(A_k = 0) = 1 - \frac{1}{k}, \quad \text{and} \quad \mathbb{P}(B_k = 1) = \frac{1}{2} = \mathbb{P}(B_k = -1).$$

*Additionally, we require that  $\{A_k\}$  and  $\{B_k\}$  are independent of each other. Define  $X_1 = 0$  and*

$$X_{k+1} = A_k[B_k \mathbb{1}(X_k = 0) + kX_k], \quad k \geq 1.$$

*Then  $X_k$  is independent of  $\{A_\ell\}_{\ell \geq k}$ , and  $\{B_\ell\}_{\ell \geq k}$  for each  $k \geq 1$ . Thus*

$$\mathbb{E}(X_{k+1} \mid X_1, \dots, X_k) = \mathbb{E}(A_k)[\mathbb{E}(B_k) \mathbb{1}(X_k = 0) + kX_k] = X_k, \quad k \geq 1.$$

*Hence  $\{X_k\}$  is a martingale. Noting that  $B_k \mathbb{1}(X_k = 0) + kX_k \neq 0$ , we have*

$$\{X_{k+1} = 0\} = \{A_k = 0\},$$

*implying that the events  $\{X_k = 0\}$  are mutually independent of each other. In addition,*

$$\mathbb{P}(X_{k+1} = 0) = \mathbb{P}(A_k = 0) = 1 - \frac{1}{k}.$$

Thus  $X_k \rightarrow 0$  in probability. However, the Borel-Cantelli Lemma (Theorem 2.1) ensures that  $\mathbb{P}(X_k \neq 0 \text{ i.o.}) = 1$ , preventing the almost sure convergence, because  $\{X_k\}$  takes only integer values.

**Question 1.2.** What if  $\{X_k\}$  is a martingale with bounded increments?

For a sequence of independent random variables, the convergence in probability does not imply the almost sure convergence either. This can be seen by considering a sequence  $\{X_k\}$  with  $\mathbb{P}(X_k = 1) = 1/k$  and  $\mathbb{P}(X_k = 0) = 1 - 1/k$  independently for each  $k \geq 1$  and applying the same argument as the end of Example 1.1.

## 2 The Borel-Cantelli Lemmas

**Theorem 2.1** (The Borel-Cantelli Lemmas). *Let  $\{E_k\}$  be a sequence of events.*

- (a) *If  $\sum_{k=1}^{\infty} \mathbb{P}(E_k) < \infty$ , then  $\mathbb{P}(E_k \text{ i.o.}) = 0$ .*
- (b) *If  $\{E_k\}$  is mutually independent and  $\sum_{k=1}^{\infty} \mathbb{P}(E_k) = \infty$ , then  $\mathbb{P}(E_k \text{ i.o.}) = 1$ .*

**Remark 2.1.** The Borel-Cantelli Lemmas lead to a 0-1 law: For a sequence of mutually independent events  $\{E_k\}$ , the probability  $\mathbb{P}(\{E_k \text{ i.o.}\})$  is either 0 or 1, corresponding to  $\sum_{k=1}^{\infty} \mathbb{P}(E_k) < \infty$  or  $\sum_{k=1}^{\infty} \mathbb{P}(E_k) = \infty$ , respectively.

**Lemma 2.1.** *For a sequence  $\{x_k\} \subset [0, 1)$ ,  $\sum_{k=1}^{\infty} x_k = \infty$  if and only if  $\prod_{k=1}^{\infty} (1 - x_k) = 0$ .*

**Proof.** If  $\sum_{k=1}^{\infty} x_k = \infty$ , then  $\prod_{k=1}^{\infty} (1 - x_k) \leq \exp(-\sum_{k=1}^{\infty} x_k) = 0$ . If  $\sum_{k=1}^{\infty} x_k < \infty$ , then there exists an integer  $K$  such that, for each  $k \geq K$ , we have  $x_k \leq 1/2$  and hence  $1 - x_k \geq \exp(-2x_k)$ . Thus  $\prod_{k=K}^{\infty} (1 - x_k) \geq \exp(-2 \sum_{k=K}^{\infty} x_k) > 0$ , implying that  $\prod_{k=1}^{\infty} (1 - x_k) > 0$ .  $\square$

**Theorem 2.2.** *Let  $\{X_k\}$  be a sequence of nonnegative random variables.*

- (a) *If  $\sum_{k=1}^{\infty} \mathbb{E}(X_k) < \infty$ , then  $\sum_{k=1}^{\infty} X_k < \infty$  a.s.*
- (b) *If  $\sum_{k=1}^{\infty} \mathbb{E}(X_k) = \infty$ , then  $\sum_{k=1}^{\infty} X_k = \infty$  a.s. provided that the random variables  $\{X_k\}$  are i.i.d. or mutually independent and uniformly bounded.*

**Proof.** (a) By the Fubini-Tonelli theorem (or Levi's monotone convergence theorem), we have

$$\mathbb{E} \left( \sum_{k=1}^{\infty} X_k \right) = \sum_{k=1}^{\infty} \mathbb{E}(X_k) < \infty.$$

Hence  $\sum_{k=1}^{\infty} X_k < \infty$  a.s.

(b) Suppose that  $\{X_k\}$  is i.i.d.. By the strong law of large numbers (Theorem 1.1), we have  $k^{-1} \sum_{\ell=1}^k X_{\ell} \rightarrow \mathbb{E}(X_1)$  a.s., which is positive (possibly infinite) since  $\sum_{k=1}^{\infty} \mathbb{E}(X_k) = \infty$ . Hence  $\sum_{k=1}^{\infty} X_k = \infty$  a.s.

Now suppose that  $\{X_k\}$  is a sequence of mutually independent and uniformly bounded random variables. Without loss of generality, we assume that  $X_k < 1$  a.s. for each  $k \geq 1$ . Then  $\mathbb{E}(X_k) < 1$  for each  $k \geq 1$ . According to Lemma 2.1,  $\sum_{k=1}^{\infty} \mathbb{E}(X_k) = \infty$  implies that  $\prod_{k=1}^{\infty} [1 - \mathbb{E}(X_k)] = 0$ . Due to the independence of  $\{X_k\}$ , we have

$$\mathbb{E} \left[ \prod_{k=1}^{\infty} (1 - X_k) \right] = \prod_{k=1}^{\infty} \mathbb{E}(1 - X_k) = \prod_{k=1}^{\infty} [1 - \mathbb{E}(X_k)] = 0.$$

Thus  $\prod_{k=1}^{\infty} (1 - X_k) = 0$  a.s. since this is a nonnegative random variable, implying that  $\sum_{k=1}^{\infty} X_k = \infty$  a.s. according to Lemma 2.1.  $\square$

**Remark 2.2.** The Borel-Cantelli Lemmas are special cases of Theorem 2.3 with  $X_k = \mathbb{1}(E_k)$ . Note that  $\{E_k \text{ i.o.}\} = \{\sum_{k=1}^{\infty} X_k = \infty\}$ .

The following theorem generalizes item (b) of Theorem 2.2 in the uniformly bounded case.

**Theorem 2.3.** *Let  $\{X_k\}$  be a sequence of uniformly bounded random variables adapted to a filtration  $\{\mathcal{F}_k\}$ . Define*

$$A = \left\{ \sum_{k=1}^{\infty} X_k = \infty \right\} \quad \text{and} \quad B = \left\{ \sum_{k=1}^{\infty} \mathbb{E}(X_k \mid \mathcal{F}_{k-1}) = \infty \right\}.$$

(a) *If  $\{\mathbb{E}(X_k \mid \mathcal{F}_{k-1})\}$  is nonnegative, then  $\mathbb{P}(A \cap B^c) = 0$ .*

(b) *If  $\{X_k\}$  is nonnegative, then  $\mathbb{P}((A \cap B^c) \cup (B \cap A^c)) = 0$ .*

**Proof.** This proof is essentially the same as that of [6, Theorem 4.3.4]. Define

$$Y_k = \sum_{\ell=1}^k X_{\ell} - \sum_{\ell=1}^k \mathbb{E}(X_{\ell} \mid \mathcal{F}_{\ell-1}).$$

Then  $\{Y_k\}$  is a martingale with respect to the filtration  $\{\mathcal{F}_k\}$ , and it has bounded increments if  $\{X_k\}$  is uniformly bounded. Hence  $\mathbb{P}(C \cup D) = 1$  with

$$C = \left\{ \lim_{k \rightarrow \infty} Y_k \text{ exists and is finite} \right\},$$

$$D = \left\{ \liminf_{k \rightarrow \infty} Y_k = -\infty \text{ and } \limsup_{k \rightarrow \infty} Y_k = \infty \right\}.$$



It is clear that

$$A \cap C = B \cap C. \quad (2.1)$$

Suppose that  $\{\mathbb{E}(X_k \mid \mathcal{F}_{k-1})\}$  is nonnegative. Then

$$B^c = \left\{ \sum_{k=1}^{\infty} \mathbb{E}(X_k \mid \mathcal{F}_{k-1}) < \infty \right\}.$$

Therefore, we can check that

$$A \cap B^c \cap D = \emptyset. \quad (2.2)$$

Combining (2.1) and (2.2), we have

$$(A \cap B^c) \cap (C \cup D) = \emptyset,$$

which implies that  $\mathbb{P}(A \cap B^c) = 0$  since  $\mathbb{P}(C \cap D) = 1$ .

If  $\{X_k\}$  is nonnegative, then so is  $\{\mathbb{E}(X_k \mid \mathcal{F}_{k-1})\}$ , and hence  $\mathbb{P}(A \cap B^c) = 0$ . In addition, we have

$$A^c = \left\{ \sum_{k=1}^{\infty} X_k < \infty \right\},$$

which implies  $\mathbb{P}(B \cap A^c) = 0$  in a way similar to the above. Thus  $\mathbb{P}((A \cap B^c) \cup (B \cap A^c)) = 0$ .  $\square$

**Question 2.1.** *What if  $\{X_k\}$  is a sequence of identically distributed but not necessarily independent variables? What if  $\{\mathbb{E}(X_k \mid \mathcal{F}_{k-1})\}$  is identically distributed?*

**Remark 2.3.**  $\mathbb{P}((A \cap B^c) \cup (B \cap A^c)) = 0$  means that the symmetric difference between the events  $A$  and  $B$  is a null event, or the two events are equal except for a null event.

**Remark 2.4.** [6, Theorem 4.3.4] is a special case of Theorem 2.3, where  $X_k = \mathbb{1}(E_k)$  with an event  $E_k \in \mathcal{F}_k$  for each  $k \geq 1$ . However, the conclusion of [6, Theorem 4.3.4] is

$$\{E_k \text{ i.o.}\} = \left\{ \sum_{k=1}^{\infty} \mathbb{P}(E_k \mid \mathcal{F}_{k-1}) = \infty \right\}, \quad (2.3)$$

which seems not completely rigorous. The correct conclusion should be that the events on the left and right sides of (2.3) are equal except for a null event. In addition, in the second line of the proof of [6, Theorem 4.3.4], the  $A$  and  $B$  seem to be reversed. Finally, the proof refers to Doob's decomposition theorem ([6, Theorem 4.3.2]), which seems not necessary.

The following proposition shows that  $\mathbb{P}(B \cap A^c)$  is not necessarily zero in item (a) of Theorem 2.3.

**Proposition 2.1.** *Let  $\{X_k\}$  be a sequence of mutually independent random variables taking either 1 or  $-1$  with  $\mathbb{E}(X_k) = Ck^{-\alpha}$  for each  $k \geq 1$ , where  $C$  and  $\alpha$  are positive constants. Then, almost surely,*

$$\liminf_{k \rightarrow \infty} \sum_{\ell=1}^k X_{\ell} = \begin{cases} +\infty & \text{if } \alpha < \frac{1}{2}, \\ -\infty & \text{if } \alpha \geq \frac{1}{2}. \end{cases}$$

**Proof.** According to Kolmogorov's iterated law of logarithm [11] (see also [12, Theorems 7.1–7.3]), we have

$$\liminf_{k \rightarrow \infty} \frac{\sum_{\ell=1}^k [X_{\ell} - \mathbb{E}(X_{\ell})]}{\sqrt{2B_k \log \log B_k}} = -1, \quad (2.4)$$

where  $B_k = \sum_{\ell=1}^k \text{Var}(X_{\ell})$ . The law is applicable because  $B_k \rightarrow \infty$  and

$$\|X_k - \mathbb{E}(X_k)\|_{\infty} = o(\sqrt{B_k / \log \log B_k}).$$

Since  $B_k = \sum_{\ell=1}^k (1 - 1/\ell)$ , it is clear that  $(B_k \log \log B_k)/(k \log \log k) \rightarrow 1$ . Hence (2.4) implies

$$\liminf_{k \rightarrow \infty} \frac{\sum_{\ell=1}^k [X_{\ell} - \mathbb{E}(X_{\ell})]}{\sqrt{2k \log \log k}} = -1. \quad (2.5)$$

Meanwhile, since  $\mathbb{E}(X_k) = Ck^{-\alpha}$ , we have

$$\lim_{k \rightarrow \infty} \frac{\sum_{\ell=1}^{\infty} \mathbb{E}(X_{\ell})}{\sqrt{2k \log \log k}} = \begin{cases} +\infty & \text{if } \alpha < \frac{1}{2}, \\ 0 & \text{if } \alpha \geq \frac{1}{2}. \end{cases} \quad (2.6)$$

Combining (2.5) and (2.6), we have

$$\liminf_{k \rightarrow \infty} \frac{\sum_{\ell=1}^k X_{\ell}}{\sqrt{2k \log \log k}} = \begin{cases} +\infty & \text{if } \alpha < \frac{1}{2}, \\ -1 & \text{if } \alpha \geq \frac{1}{2}, \end{cases}$$

which implies the desired conclusion.  $\square$

**Remark 2.5.** For more discussions on Proposition 2.1, including a perspective from the Kakutani's Dichotomy theorem [9], see

<https://math.stackexchange.com/questions/4968996/>.

## References

- [1] D. L. Burkholder. Martingale transforms. *Ann. Math. Statist.*, 37:1494–1504, 1966.
- [2] Y. S. Chow. A martingale inequality and the law of large numbers. *Proc. Amer. Math. Soc.*, 11:107–111, 1960.
- [3] Y. S. Chow. Local convergence of martingales and the law of large numbers. *Ann. Math. Statist.*, 36:552–558, 1965.
- [4] Y. S. Chow. On a strong law of large numbers for martingales. *Ann. Math. Statist.*, 38, 1967.
- [5] K. L. Chung. *A Course in Probability Theory*. Academic Press, San Diego, CA, third edition, 2001.
- [6] R. Durrett. *Probability: Theory and Examples*. Camb. Ser. Stat. Probab. Math. Cambridge University Press, Cambridge, fifth edition, 2019.
- [7] N. Etemadi. An elementary proof of the strong law of large numbers. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 55:119–122, 1981.
- [8] P. Hall and C. C. Heyde. *Martingale Limit Theory and Its Application*. Academic Press, New York, 1980.
- [9] S. Kakutani. On equivalence of infinite product measures. *Ann. of Math.*, 49(1):214–224, 1948.
- [10] O. Kallenberg. *Foundations of Modern Probability*. Springer Nature Switzerland AG, Gewerbestrasse, third edition, 2021.
- [11] A. Kolmogoroff. Über das Gesetz des iterierten Logarithmus. *Math. Ann.*, 101:126–135, 1929.
- [12] V. V. Petrov. *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*. Clarendon Press, Oxford, 1995.