Spectral Mapping Theorem and Beyond

An Introduction to Functional Calculus

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June 24, 2008

Dedicated to
Wenchuan, Sichuan,
and
Great China.

Beauty is the first test: there is no permanent place in the world for ugly mathematics.

– G. H. Hardy

Wir müssen wissen. Wir werden wissen.

– David Hilbert

One should always generalize.

– Carl Gustav Jacobi

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An Introduction to Functional Calculus

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Abstract. An introduction to Riesz functional calculus and related spectral theorems is given.
Keywords: Banach algebra, Riesz functional calculus, spectral theorem, eigenvalue, eigenspace.

1 Motivation and Introduction

Given a matrix $A \in \mathbb{C}^{m\times m}$ and a polynomial

$$f(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^k,$$

the symbol $f(A)$ has a "natural" definition, namely

$$f(A) = \sum_{k=0}^{n} \alpha_k A^k.$$

Under this definition, we have the following important result.

Proposition 1.1. *Suppose that $A \in \mathbb{C}^{m\times m}$ and f is a polynomial, then*

$$\sigma[f(A)] = f[\sigma(A)].$$

More exactly, if the eigenvalues of A are

$$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_m,$$

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then the eigenvalues of $f(A)$ are

$$f(\lambda_1),\ f(\lambda_2),\ \ldots,\ f(\lambda_m).$$

Here the eigenvalues are counted according to their multiplicities.

Proof. Suppose that the Schur decomposition of A is

$$A = U^*TU,$$

where U is unitary and T is upper triangular, and

$$t_{ii} = \lambda_i.$$

Since U is unitary, we have that

$$f(A) = U^* f(T)U.$$

Besides, it is easy to check that $f(T)$ is upper triangular and its i-th diagonal entry is $f(t_{ii})$. Hence the proposition holds. ∎

One should always generalize[1]. To generalize Proposition 1.1, suppose that f is a rational function, i.e.,

$$f(\lambda) = p(\lambda)/q(\lambda),$$

where p and q are polynomials. If $q(A)$ is nonsingular, then $f(A)$ can be "naturally" defined as follows:

$$f(A) = p(A)[q(A)]^{-1}.$$

In this case, the conclusion of Proposition 1.1 holds as well, and its proof is precisely the same as the preceding one.

Can we generalize the conclusion of Proposition 1.1 further? Certainly we can, as has been done in a classical subject of functional analysis, namely the *functional calculus*[2].

Consider a complex algebra $\mathfrak{A}$. The functional calculus refers to, roughly speaking, a specific process which enables the expression $f(x)$ to make sense as an element of $\mathfrak{A}$, for certain functions $f : \mathbb{C} \to \mathbb{C}$ and certain elements

[1] Carl Gustav Jacobi (1804-1851).

[2] The terms *operational calculus* and *symbolic calculus* are also frequently used.

$x \in \mathfrak{A}$. The generalization of Proposition 1.1 in such a context is the *spectral mapping theorem*, which is an important result of the functional calculus[3].

This report is oriented to present an introduction to the theory of functional calculus, especially spectral mapping theorem. It is organized as follows. First, we present some purely algebraic generalizations of Proposition 1.1 in Section 2; in Section 3, we introduce the functional calculus in Banach algebras and generalize Proposition 1.1 in such a context; the spectral mapping theorems about linear operators and matrices are dealt with in Section 4; after that, an appendix is given to provide preliminaries so that our discussions are self-contained.

Since functional calculus is a mature subject, this report is nothing but an introduction. There is little originality in our discussions. Most of the proofs can be found in the references.

We will proceed our discussions in complex field. All the algebras considered will be supposed to have an identity element referred to as e.

2 A Purely Algebraic Discussion

Since spectrum is a purely algebraic concept, spectral mapping theorem deserves a purely algebraic discussion. That is what we will do in this section. Due to the limitation of our premise, here we only study rational functions. Our central results are Theorem 2.3 and 2.4.

Definition 2.1. *Consider an algebra $\mathfrak{A}$ and an element $x \in \mathfrak{A}$.*
1.) Suppose that f is a polynomial with $f(\lambda) = \sum_{k=1}^{n} \alpha_k \lambda^k$, then $f(x)$ is define to be $\sum_{k=1}^{n} \alpha_k x^k$.
2.) Suppose that f is a rational function with $f(\lambda) = p(\lambda)/q(\lambda)$, where p and q are polynomials. If $q(x)$ is invertible, then $f(x)$ is define to be $p(x)[q(x)]^{-1}$.

For part 2.) of Definition 2.1, we give two remarks.

1.) It is easy to check that $p(x)$ and $[q(x)]^{-1}$ commute, thus we also have $f(x) = [q(x)]^{-1}p(x)$.

2.)In the following discussion, the condition that $q(x)$ *is invertible* will be referred to as $f(x)$ *is well defined*.

The following lemma is needed.

Lemma 2.2. *Suppose that $\mathfrak{A}$ is an algebra, $x \in \mathfrak{A}$, and p is a polynomial. Then*

[3]Usually $\mathfrak{A}$ is supposed to have topological structure on it. For example, we frequently suppose $\mathfrak{A}$ to be a Banach algebra. In a purely algebraic context, there are still something, though very limited, can be done. See Section 2.

1.) $p(x)$ *is invertible iff* $p(\lambda) \neq 0$ *for all* $\lambda \in \sigma(x)$.
2.) $p(x)$ *is left invertible iff* $p(\lambda) \neq 0$ *for all* $\lambda \in \sigma_p(x)$.

Proof. We only consider part 1.). The proof of part 2.) is very similar.

Suppose that

$$p(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_k),$$

then we can show that

$$p(x) = \prod_{k=1}^{n} (x - \lambda_k e).$$

Since $x - \lambda_i e$ commutes with $x - \lambda_j e$ $(i,\ j \in \{1,\ 2,\ ...,\ n\})$, we have that

$$\begin{aligned} & p(x) \text{ is invertible} \\ \Longleftrightarrow\ & x - \lambda_k e \text{ is invertible for all } k \in \{1,\ 2,\ ...,\ n\} \qquad (2.1) \\ \Longleftrightarrow\ & \lambda_k \notin \sigma(x) \text{ for all } k \in \{1,\ 2,\ ...,\ n\} \\ \Longleftrightarrow\ & p(\lambda) \neq 0 \text{ for all } \lambda \in \sigma(x). \end{aligned}$$

For the validity of (2.1), see Lemma A.5 in Appendix. ∎

Theorem 2.3. *Suppose that* $\mathfrak{A}$ *is an algebra,* $x \in \mathfrak{A}$*, and* f *is a rational function. If* $f(x)$ *is well defined, then*

$$\sigma[f(x)] = f[\sigma(x)],$$
$$\sigma_p[f(x)] = f[\sigma_p(x)].$$

Proof. We only prove the first equation.

Suppose that $f(\lambda) = p(\lambda)/q(\lambda)$, where p and q are polynomials. Since $f(x)$ is well defined, we know that $q(x)$ is invertible, and accordingly, $q(\lambda) \neq 0$ for any $\lambda \in \sigma(x)$ (Lemma 2.2). Thus for any $\mu \in \mathbb{C}$ we have that

$$\begin{aligned} & \mu \in \sigma[f(x)] \\ \Longleftrightarrow\ & \mu e - p(x)[q(x)]^{-1} \text{ is not invertible} \\ \Longleftrightarrow\ & \mu q(x) - p(x) \text{ is not invertible} \\ \Longleftrightarrow\ & \exists \lambda \in \sigma(x) \text{ s.t. } \mu q(\lambda) - p(\lambda) = 0 \\ \Longleftrightarrow\ & \exists \lambda \in \sigma(x) \text{ s.t. } \mu = p(\lambda)/q(\lambda) \\ \Longleftrightarrow\ & \mu \in f[\sigma(x)]. \end{aligned}$$

∎

Theorem 2.4. *Suppose that X is a linear space, $T : X \to X$ is a linear operator and f is a rational function. If $f(T)$ is well defined, then*
1.) if λ_0 is an eigenvalue of T with x as an eigenvector, then $f(\lambda_0)$ is an eigenvalue of $f(T)$ with x as an eigenvector;
2.) $\sigma_p[f(T)] = f[\sigma_p(T)]$.

Proof. Only part 1.) needs proving.

It is clear that there exists a rational function g such that

$$f(\lambda) - f(\lambda_0) = g(\lambda)(\lambda - \lambda_0)$$

and $g(T)$ is well defined. Hence we can show that

$$f(T) - f(\lambda_0)I = g(T)(T - \lambda_0 I).$$

Thus

$$[f(T) - f(\lambda_0)I]x = 0.$$

■

3 Functional Calculus in Banach Algebras

3.1 Introduction

In this section, we will always suppose that $\mathfrak{B}$ is a Banach algebra, Ω is a domain in $\mathbb{C}$, and $H(\Omega)$ is the set of all complex holomorphic functions in Ω. Besides, we define

$$\mathfrak{B}_\Omega = \{x \in \mathfrak{B};\ \sigma(x) \subset \Omega\}.$$

It can be shown that $\mathfrak{B}_\Omega$ is an open subset of $\mathfrak{B}$ (See [1]). Our purpose is to give a reasonable definition of $f(x)$ for some x in $\mathfrak{B}$ and discuss corresponding generalization of Theorem 1.1.

Some conclusions about vector-valued holomorphic functions and integration are needed. They can be found in subsections A.2 and A.3 of Appendix.

3.2 Riesz Functional Calculus

Definition 3.1. *For any $f \in H(\Omega)$, define $\tilde{f} : \mathfrak{B}_\Omega \to \mathfrak{B}$ as follows:*

$$\tilde{f}(x) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda)(\lambda e - x)^{-1}\, \mathrm{d}\lambda, \tag{3.1}$$

where Γ is any contour that surrounds $\sigma(x)$ in Ω.

Note that the notation $\tilde{f}(x)$ has been written instead of the expected $f(x)$ in the definition above. We will always apply this notation henceforth. This is to make our discussions more exact and avoid certain ambiguities that might cause misunderstandings. Besides, the transformation of notations is reasonable in logical aspect: since $f : \Omega \to \mathbb{C}$ and $\tilde{f} : \mathfrak{B}_\Omega \to \mathfrak{B}$ are different mappings, they deserve different notations[4].

Definition 3.1 calls for some comments.

1.) For the exact meaning of the statement that Γ *is any contour that surrounds* $\sigma(x)$ *in* Ω, see [1]. We should note that, by the definition, Γ never insects $\sigma(x)$ and satisfies

$$\mathrm{Ind}_\Gamma(\zeta) = \begin{cases} 1, & \text{if } \zeta \in \sigma(x), \qquad (3.2a) \\ 0, & \text{if } \zeta \notin \Omega, \qquad (3.2b) \end{cases}$$

where $\mathrm{Ind}_\Gamma(\zeta)$ refers to the *index* of ζ with respect to Γ (or *winding number* of Γ about ζ. See [1] and [2]). Such a Γ exists because $\sigma(x)$ is compact, Ω is open and $\sigma(x) \subset \Omega$.

2.) Since Γ stays away from $\sigma(x)$ and since inversion is continuous in $\mathfrak{B}$, the integrand is continuous in (3.1), so that the integral exists and defines $\tilde{f}(x)$ as an element of $\mathfrak{B}$.

3.) The integrand is actually a holomorphic $\mathfrak{B}$-valued function in the complement of $\sigma(x)$ (See Lemma A.10 in Appendix). The Cauchy Theorem A.8 implies that $\tilde{f}(x)$ is independent of the choice of Γ, provided only that Γ surrounds $\sigma(x)$ in Ω.

3.3 Some Discussions about the Definition

Now we check some cases in which $\tilde{f}(x)$ has "natural" definitions. It will be shown that Definition 3.1 coincides with such definitions. First is a lemma.

Lemma 3.2. *Suppose that $\lambda_0 \in \mathbb{C}$, n is an integer and $f(\lambda) = (\lambda - \lambda_0)^n$.*
1.) If n is nonnegative and $\Omega = \mathbb{C}$, then $\mathfrak{B}_\Omega = \mathfrak{B}$ and

$$\tilde{f}(x) = (x - \lambda_0 e)^n, \quad \forall x \in \mathfrak{B}_\Omega. \tag{3.3}$$

2.) If n is negative and $\Omega = \mathbb{C}\backslash\{\lambda_0\}$, then $\mathfrak{B}_\Omega = \{x \in \mathfrak{B};\ \lambda_0 \notin \sigma(x)\}$ and

$$\tilde{f}(x) = (x - \lambda_0 e)^n, \quad \forall x \in \mathfrak{B}_\Omega. \tag{3.4}$$

[4]In most treatments of this topic, however, $f(x)$ is written in place of our $\tilde{f}(x)$. The notation $f(x)$ is luminous.

Proof. We only justify part 2.). The proof of part 1.) is very similar and even easier.

It is obvious that $\mathfrak{B}_\Omega = \{x \in \mathfrak{B};\ \lambda_0 \notin \sigma(x)\}$. Fix an element x of $\mathfrak{B}_\Omega$, suppose that Γ is a contour surrounding $\sigma(x)$ in Ω, and define

$$y_n = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - \lambda_0)^n (\lambda e - x)^{-1}\, \mathrm{d}\lambda, \quad n \in \mathbb{Z}.$$

For any integer n and any $\lambda \in \Gamma$, we have

$$(\lambda - \lambda_0)^{n+1} (\lambda e - x)^{-1} = (\lambda - \lambda_0)^n e + (\lambda - \lambda_0)^n (\lambda e - x)^{-1} (x - \lambda_0 e). \quad (3.5)$$

Since $(\lambda - \lambda_0)^n e$ is holomorphic on Ω and $\mathrm{Ind}_\Gamma(\lambda_0) = 0$ (See (3.2b)), we have

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda - \lambda_0)^n e\, \mathrm{d}\lambda = 0.$$

Hence (3.5) implies that

$$y_{n+1} = y_n (x - \lambda_0 e), \quad n \in \mathbb{Z}.$$

This recursion formula shows that (3.4) holds if we can show $y_0 = e$. To do this, let γ be a positively oriented circle, centered at 0, with radius $\|x\| + 1$. On γ, $(\lambda e - x)^{-1} = \sum \lambda^{-n-1} x^n$. Termwise integration of this series gives

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda e - x)^{-1}\, \mathrm{d}\lambda = e. \quad (3.6)$$

Since $(\lambda e - x)^{-1}$ is holomorphic in the complement of $\sigma(x)$ (See Lemma A.10 of Appendix), and since

$$\mathrm{Ind}_\gamma(\zeta) = 1 = \mathrm{Ind}_\Gamma(\zeta)$$

for every $\zeta \in \sigma(x)$ (See (3.2a)), the Cauchy theorem shows that the integral (3.6) is unaffected if γ is replaced by Γ. This completes the proof. ∎

According to Lemma 3.2 and applying termwise integration, we obtain the following two theorems.

Theorem 3.3. *If f is a rational function, and $\Omega = \{\lambda;\ \lambda$ is not a pole of $f\}$, then for any $x \in \mathfrak{B}$, $x \in \mathfrak{B}_\Omega$ iff $f(x)$ is well defined by Definition 2.1, and*

$$\tilde{f}(x) = f(x), \quad \forall x \in \mathfrak{B}_\Omega.$$

Theorem 3.4. *If $\Omega = B(\lambda_0, \rho)$ $(0 < \rho \leq \infty)$ and*

$$f(\lambda) = \sum_{k=0}^{\infty} \alpha_k (\lambda - \lambda_0)^k, \quad \forall \lambda \in B(\lambda_0, \rho),$$

then $B(\lambda_0 e, \rho) \subset \mathfrak{B}_\Omega$ and

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \alpha_k (x - \lambda_0 e)^k, \quad \forall x \in B(\lambda_0 e, \rho).$$

3.4 Riesz Functional Calculus as an Extension

Now we point out that the Riesz functional calculus can be reckoned as an extension of functions in $H(\Omega)$. The following lemma is obvious and we omit its proof.

Lemma 3.5. *For a scalar $\lambda \in \mathbb{C}$, $\lambda e \in \mathfrak{B}_\Omega$ iff $\lambda \in \Omega$. Besides, we have*

$$\tilde{f}(\lambda e) = f(\lambda) e, \quad \forall \lambda \in \Omega.$$

Note that $\mathbb{C}$ is a subalgebra of $\mathfrak{B}$, if we identify λ with λe for every $\lambda \in \mathbb{C}$. In this sense, Ω is a subset of $\mathfrak{B}_\Omega$, and Lemma 3.5 shows that $\tilde{f}$ is an extension of f. If we define

$$E : H(\Omega) \to H(\mathfrak{B}_\Omega)$$

with

$$E(f) = \tilde{f},$$

then E is an extension operator. In order to discuss the property of this operator, we consider the structure of $H(\Omega)$ and $H(\mathfrak{B}_\Omega)$ first.

If we denote

$$H(\mathfrak{B}_\Omega) = \{\tilde{f};\ f \in H(\Omega)\}$$

and define

$$(\alpha f + \beta g)(\lambda) = \alpha f(\lambda) + \beta g(\lambda), \quad (fg)(\lambda) = f(\lambda) g(\lambda),$$
$$(\alpha \tilde{f} + \beta \tilde{g})(x) = \alpha \tilde{f}(x) + \beta \tilde{g}(x), \quad (\tilde{f}\tilde{g})(x) = \tilde{f}(\lambda) \tilde{g}(x),$$

for all α, $\beta \in \mathbb{C}$, f, $g \in H(\Omega)$, $\lambda \in \Omega$ and $x \in \mathfrak{B}_\Omega$, then both $H(\Omega)$ and $H(\mathfrak{B}_\Omega)$ make algebras.

Considering such structures of $H(\Omega)$ and $H(\mathfrak{B}_\Omega)$, the extension operator E is actually an isomorphism.

Theorem 3.6. *The extension $E : H(\Omega) \to H(\mathfrak{B}_\Omega)$ is an isomorphism, i.e.,*
1.) E is bijective;
2.) for any $\alpha,\ \beta \in \mathbb{C}$ and any $f,\ g \in H(\Omega)$, we have

$$\begin{aligned} E(\alpha f + \beta g) &= \alpha E(f) + \beta E(g), \\ E(fg) &= E(f)E(g); \end{aligned}$$

Proof. Part 1.) is trivial. For the proof of part 2.), see Theorem 10.27 of [1] or Theorem 4.7 of [2]. ∎

Corollary 3.7. *$H(\mathfrak{B}_\Omega)$ is commutative, i.e., if $f,\ g \in H(\Omega)$, and $x \in \mathfrak{B}_\Omega$, we have*

$$\tilde{f}(x)\tilde{g}(x) = \tilde{g}(x)\tilde{f}(x).$$

3.5 Spectral Mapping Theorem

In this subsection, we will give the generalization of Proposition 1.1. First is a travail lemma.

Lemma 3.8. *Consider a function $f \in H(\Omega)$ and a domain $\hat{\Omega} \subset \Omega$. If $f|_{\hat{\Omega}}$ denotes the restriction of f on $\hat{\Omega}$, then*

$$\widetilde{f|_{\hat{\Omega}}}(x) = \tilde{f}(x),\ \forall x \in \mathfrak{B}_{\hat{\Omega}}.$$

In other words,

$$\widetilde{f|_{\hat{\Omega}}} = \tilde{f}|_{\mathfrak{B}_{\hat{\Omega}}}.$$

Lemma 3.9. *Suppose that $f \in H(\Omega)$ and $x \in \mathfrak{B}_\Omega$, then*
1.) $\tilde{f}(x)$ is invertible iff $f(\lambda) \neq 0$ for any $\lambda \in \sigma(x)$;
2.) $\tilde{f}(x)$ is left invertible iff $f(\lambda) \neq 0$ for any $\lambda \in \sigma_p(x)$.

Proof. 1.) If f has no zero in $\sigma(x)$, then there exists a domain $\hat{\Omega}$ such that $\sigma(x) \subset \hat{\Omega} \subset \Omega$ and $f|_{\hat{\Omega}}$ has no zeros. Set $g = 1/(f|_{\hat{\Omega}})$, then g is the inverse of $(f|_{\hat{\Omega}})$ in $H(\hat{\Omega})$. Theorem 3.6 shows that $\widetilde{f|_{\hat{\Omega}}}(x)\tilde{g}(x) = e$. Thus $\widetilde{f_{\hat{\Omega}}}(x)$ is invertible, for it commutes with $\tilde{g}(x)$ (See Lemma A.5 in Appendix). So is $\tilde{f}(x)$.

Conversely, if $f(\lambda_0) = 0$ for a certain $\lambda_0 \in \sigma(x)$, then there exists a function $h \in H(\Omega)$ such that

$$f(\lambda) = (\lambda - \lambda_0)h(\lambda).$$

Applying Theorem 3.6 again, we obtain

$$\tilde{f}(x) = (x - \lambda_0 e)\tilde{h}(x).$$

Since $x - \lambda_0 e$ is not invertible ($\lambda_0 \in \sigma(x)$) and commuting with $\tilde{h}(x)$, $\tilde{f}(x)$ is not invertible (See Lemma A.5 in Appendix).

2.) We prove the following equivalent of part 2.).

$\tilde{f}(x)$ is not left invertible iff $f(\lambda_0) = 0$ for some $\lambda_0 \in \sigma_p(x)$.

If $\lambda_0 \in \sigma_p$ is a zero of f, then there exists a function $g \in H(\Omega)$ such that

$$f(\lambda) = g(\lambda)(\lambda - \lambda_0), \quad \forall \lambda \in \Omega.$$

Hence, according to Theorem 3.6, we have

$$\tilde{f}(x) = \tilde{g}(x)(x - \lambda_0 e). \tag{3.7}$$

Since $\tilde{g}(x)$ and $x - \lambda_0 e$ commute, (3.7) implies that $\tilde{f}(x)$ is not left invertible, for $\lambda_0 \in \sigma_p(x)$.

Conversely, if $\tilde{f}(x)$ is not left invertible, then

$$f^{-1}(0) \cap \sigma(x) \neq \emptyset. \tag{3.8}$$

If the set (3.8) is infinite, then the compactness of $\sigma(x)$ implies that it has some cluster points in Ω, and consequently, $f(\lambda) = 0$ for all $\lambda \in \Omega$. Thus there is nothing needs proving. Now suppose that the set (3.8) is finite. Let λ_1, λ_2, ..., λ_n be the zeros of f in $\sigma(x)$, counting according to their multiplicities. Then

$$f(\lambda) = h(\lambda) \prod_{k=1}^{n} (\lambda - \lambda_k), \quad \forall \lambda \in \Omega,$$

where $h \in H(\Omega)$ and has no zero in $\sigma(x)$. Hence, according to Theorem 3.6, we have

$$\tilde{f}(x) = \tilde{h}(x) \prod_{k=1}^{n} (x - \lambda_k e). \tag{3.9}$$

As has been shown in part 1.), $\tilde{h}(x)$ is invertible. Since $\tilde{f}(x)$ is not left invertible, and $x - \lambda_1 e$, $x - \lambda_2 e$, ..., $x - \lambda_n e$ are pairwise commuting, (3.9) implies that there exists a certain $k_0 \in \{1, 2, ..., n\}$ such that $x - \lambda_{k_0}$ fails to be left invertible, i.e., $\lambda_{k_0} \in f^{-1}(0) \cap \sigma_p(x)$. ■

Now we give the spectral mapping theorem about Banach algebras.

Theorem 3.10. *Suppose that $f \in H(\Omega)$ and $x \in \mathfrak{B}_\Omega$, then*

$$\sigma[\tilde{f}(x)] = f[\sigma(x)],$$
$$\sigma_p[\tilde{f}(x)] = f[\sigma_p(x)].$$

Proof. We only prove the first equation.

For any $\mu \in \mathbb{C}$, we have

$$\begin{aligned} & \mu \in \sigma[\tilde{f}(x)] \\ \Longleftrightarrow\ & \mu e - \tilde{f}(x) \text{ is not invertible} \\ \Longleftrightarrow\ & \mu - f(\lambda) \text{ have some zero on } \sigma(x) \\ \Longleftrightarrow\ & \mu \in f[\sigma(x)]. \end{aligned}$$

■

The spectral mapping theorem makes it possible to include composition of functions among the operations of the functional calculus. Explicitly, we have the following theorem, whose proof can be found in [1].

Theorem 3.11. *If $f \in H(\Omega)$ and $g \in H[f(\Omega)]$, then for any $x \in \mathfrak{B}_\Omega$, we have $f(x) \in \mathfrak{B}_{f(\Omega)}$ and*

$$\widetilde{(g \circ f)}(x) = \tilde{g}[\tilde{f}(x)].$$

In other words,

$$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}.$$

Note that $f(\Omega)$ is also a domain in $\mathbb{C}$.

4 Spectral Mapping Theorems of Operators and Matrices

In this section, we will refine spectral mapping theorem for linear operators and matrices. We still assume that Ω is a domain in $\mathbb{C}$, and $H(\Omega)$ is the algebra of all complex holomorphic functions in Ω.

4.1 Spectral Mapping Theorem of Operators

For the algebra of bounded linear operators on a Banach space, all the conclusions of Section 3 hold, including the spectral mapping theorem 3.10. But, for bounded linear operators, we are more concerned with its eigenvalues and eigenspaces. Now we give a version of spectral mapping theorem about this.

Theorem 4.1. *Suppose that X is a Banach space, $T \in \mathscr{B}(X)$, $\sigma(T) \subset \Omega$ and $f \in H(\Omega)$. Then*
1.) if λ_0 is an eigenvalue of T with x as an eigenvector, then $f(\lambda_0)$ is an eigenvalue of $\tilde{f}(T)$ with x as an eigenvector;
2.) $\sigma_p[\tilde{f}(T)] = f[\sigma_p(T)]$.

Proof. 1.) There exists a function $g \in H(\Omega)$ such that

$$f(\lambda) - f(\lambda_0) = g(\lambda)(\lambda - \lambda_0), \quad \forall \lambda \in \Omega.$$

According to Theorem 3.6, we have

$$\tilde{f}(T) - f(\lambda_0)T = \tilde{g}(T)(T - \lambda_0 T).$$

Thus part 1.) holds.

2.) We can conclude from part 1.) that $f[\sigma_p(T)] \subset \sigma_p[\tilde{f}(T)]$. Now we suppose that $\mu_0 \in \sigma_p[\tilde{f}(T)]$. By the spectral mapping theorem 3.10,

$$f^{-1}(\mu_0) \cap \sigma(T) \neq \emptyset. \tag{4.1}$$

If the set (4.1) is infinite, then the compactness of $\sigma(T)$ implies that it has a cluster point $\lambda_0 \in \Omega$, and consequently, $f(\lambda) = \mu_0$ for all $\lambda \in \Omega$. Thus there is nothing needs proving. Now suppose that the set (4.1) is finite. Let λ_1, λ_2, ..., λ_n be the zeros of $f(\lambda) - \mu_0$, counting according to their multiplicities. Then

$$f(\lambda) - \mu_0 = h(\lambda)(\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \quad \forall \lambda \in \Omega,$$

where $h \in H(\Omega)$ and has no zero in $\sigma(T)$. Hence

$$\tilde{f}(T) - \mu_0 I = \tilde{h}(T)(T - \lambda_1 I) \cdots (T - \lambda_n I). \tag{4.2}$$

By Lemma 3.9, $\tilde{h}(T)$ is invertible. Since $\tilde{f}(T) - \mu_0 I$ is not injective, (4.2) implies that at least one of the operators $T - \lambda_i$ ($i = 1, 2, ..., n$) fails to be injective. The corresponding λ_i is in $\sigma_p(T)$, and hence $\mu_0 = f(\lambda_i) \in f[\sigma_p(T)]$. ∎

In Theorem 4.1, the converse of part 1.) does not hold in general. But we have the following.

Proposition 4.2. *Suppose that X is a Banach space, $T \in \mathscr{B}(X)$, $\sigma(T) \subset \Omega$ and $f \in H(\Omega)$. For $\mu \in \mathbb{C}$, if*

$$\dim[\ker(\mu I - \tilde{f}(T))] \leq \sum_{f(\lambda)=\mu} \dim[\ker(\lambda I - T)] < \aleph_0$$

then

$$\ker(\mu I - \tilde{f}(T)) = \bigoplus_{f(\lambda)=\mu} \ker(\lambda I - T).$$

Here we take $\bigoplus_{f(\lambda)=\mu} \ker(\lambda I - T)$ as $\{0\}$ if $f(\lambda) = \mu$ never holds.

Proof. Note that $\bigoplus_{f(\lambda)=\mu}\ker(\lambda I - T)$ is a linear subspace of $\ker(\mu I - \tilde{f}(T))$ (If $\mu \notin \sigma_p[\tilde{f}(T)]$, this holds trivially; if $\mu \in \sigma_p[\tilde{f}(T)]$, this follows from part 1.) of Theorem 4.1). ∎

Corollary 4.3. *Suppose that X is a Banach space, $T \in \mathscr{B}(X)$, $\sigma(T) \subset \Omega$, $f \in H(\Omega)$ and $\mu \in \mathbb{C}$. If*

$$\dim[\ker(\mu I - \tilde{f}(T))] = 1,$$

then there exists a unique $\lambda \in \sigma_p(T)$ such that $f(\lambda) = \mu$. Besides,

$$\ker(\mu I - \tilde{f}(T)) = \ker(\lambda I - T).$$

For compact normal operators on a Hilbert space, we can go further.

Theorem 4.4. *Suppose that H is a Hilbert space, $T \in \mathscr{B}(H)$, $\sigma(T) \subset \Omega$ and $f \in H(\Omega)$. If T is compact and normal, then for any $\mu \in \mathbb{C}$, we have*

$$\ker(\mu I - \tilde{f}(T)) = \overline{\bigoplus_{f(\lambda)=\mu}\ker(\lambda I - T)}.$$

Proof. For any $\lambda \in \sigma_p(T)$, let $\mathcal{B}_\lambda$ be an orthonormal basis of $\ker(\lambda I - T)$. Since T is compact and normal, $\bigcup_{\lambda\in\sigma_p(T)}\mathcal{B}_\lambda$ makes an orthonormal basis of H. Thus

$$x = \sum_{\lambda\in\sigma_p(T)}\sum_{\varphi\in\mathcal{B}_\lambda}\langle x, \varphi\rangle\varphi, \quad \forall x \in H,$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the inner product on H. Hence, for any $x \in H$, we have

$$\begin{aligned}
&x \in \ker(\mu I - \tilde{f}(T))\\
\Longleftrightarrow\ &(\mu I - \tilde{f}(T))x = 0\\
\Longleftrightarrow\ &(\mu I - \tilde{f}(T))\sum_{\lambda\in\sigma_p(T)}\sum_{\varphi\in\mathcal{B}_\lambda}\langle x, \varphi\rangle\varphi = 0\\
\Longleftrightarrow\ &\sum_{\lambda\in\sigma_p(T)}\sum_{\varphi\in\mathcal{B}_\lambda}\langle x, \varphi\rangle(\mu - f(\lambda))\varphi = 0\\
\Longleftrightarrow\ &\langle x, \varphi\rangle(\mu - f(\lambda)) = 0, \quad \forall\varphi \in \mathcal{B}_\lambda,\ \lambda \in \sigma_p(T)\\
\Longleftrightarrow\ &\langle x, \varphi\rangle = 0, \quad \forall\varphi \in \mathcal{B}_\lambda,\ \lambda \in \sigma_p(T)\backslash f^{-1}(\mu)\\
\Longleftrightarrow\ &x \in \overline{\bigoplus_{\substack{\lambda\in\sigma_p(T)\\ f(\lambda)=\mu}}\ker(\lambda I - T)}\\
\Longleftrightarrow\ &x \in \overline{\bigoplus_{f(\lambda)=\mu}\ker(\lambda I - T)}.
\end{aligned}$$

∎

Proposition 4.2 and Theorem 4.4 are essentially statements about the eigenspaces of $\tilde{f}(T)$. They can be taken as supplements of Theorem 4.1.

4.2 Spectral Mapping Theorem of Matrices

Now we present the spectral mapping theorem of matrices.

Theorem 4.5. *Suppose that $A \in \mathbb{C}^{m\times m}$, $\sigma(A) \subset \Omega$, and $f \in H(\Omega)$. Then*
1.) if the eigenvalues of A are $\lambda_1,\ \lambda_2,\ ...,\ \lambda_m$, counted according to their multiplicities, then the eigenvalues of $\tilde{f}(A)$ are $f(\lambda_1),\ f(\lambda_2),\ ...,\ f(\lambda_m)$, counted according to their multiplicities;
2.) if x is an eigenvector of A for eigenvalue λ, then it is also an eigenvector of $\tilde{f}(A)$ for eigenvalue $f(\lambda)$;
3.) if A is diagonalizable, then for any $\mu \in \mathbb{C}$, we have

$$\ker(\mu I - \tilde{f}(A)) = \bigoplus_{f(\lambda)=\mu} \ker(\lambda I - A).$$

Proof. 1.) Suppose that the Schur decomposition of A is

$$A = U^*TU,$$

where U is unitary and T is upper triangular, and

$$t_{ii} = \lambda_i.$$

Then

$$\begin{aligned}
\tilde{f}(A) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda)(\lambda I - A)^{-1}\,\mathrm{d}\lambda \\
&= \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda)(\lambda I - U^*TU)^{-1}\,\mathrm{d}\lambda \\
&= \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda)U^*(\lambda I - T)^{-1}U\,\mathrm{d}\lambda \\
&= U^*\tilde{f}(T)U.
\end{aligned}$$

According to Theorem A.6, $\tilde{f}(T)$ is an upper triangular matrix, whose i-th diagonal element is $f(t_{ii})$. This implies part 1.).

2.) Apply part 1.) of Theorem 4.1

3.) Suppose that the eigenvalues of A are $\lambda_1,\ \lambda_2,\ ...,\ \lambda_m$, counted according to their multiplicities. By the assumption, there exists an invertible matrix P such that

$$A = P\Lambda P^{-1},$$

where $\Lambda = \operatorname{diag}(\lambda_1,\ ...,\ \lambda_n)$. As has been done in part 1.), we can show that

$$\tilde{f}(A) = P\tilde{f}(\Lambda)P^{-1}.$$

According to Theorem A.6, $\tilde{f}(\Lambda)$ is a diagonal matrix, whose i-th diagonal element is $f(\lambda_i)$. This implies part 3.). ■

5 Remarks

With stronger assumptions, the domain of the functional calculus can be broadened. The theory of functional calculus concerning Borel functions has been developed. For details, see [1], [2] and [3].

To conclude our discussions, we propose an unsettled question.

Question 5.1. *Suppose that Ω is a domain in $\mathbb{C}$, X is a Banach space, $T \in \mathscr{B}(X)$, $\sigma(T) \subset \Omega$ and $f \in H(\Omega)$. Does*

$$\overline{\bigoplus_{\lambda\in\mathbb{C}} \ker(\lambda I - T)} = X$$

implies

$$\ker(\mu I - \tilde{f}(T)) = \overline{\bigoplus_{f(\lambda)=\mu} \ker(\lambda I - T)}$$

for every $\mu \in \mathbb{C}$?

In general case, we guess that the answer of Question 5.1 is negative. It is interesting to find reasonable assumptions to make the answer be positive.

Appendix

A.1 Algebra

First is the definition of spectrum.

Definition A.2. *Consider an algebra $\mathfrak{A}$ and an element $x \in \mathfrak{A}$.*
1.) The set

$$\{\lambda \in \mathbb{C};\ \lambda e - x \textit{ is not invertible}\}$$

is called the spectrum of x and is denoted by $\sigma(x)$.
2.) The set

$$\{\lambda \in \mathbb{C};\ \lambda e - x \textit{ is not left invertible}\}$$

is called the point spectrum[5] *of x and is denoted by $\sigma_p(x)$.*

Note that the concept of spectrum is purely algebraic.

Now we give some lemmas concerning invertibility in the context of monoid.

Lemma A.3. *In a monoid, an element is invertible iff it is both left invertible and right invertible.*

Proof. Only the *if* part needs proving. Suppose that $\mathfrak{M}$ is a monoid with an identity element e and $a \in \mathfrak{M}$. If there exist elements $b,\ c \in \mathfrak{M}$ such that

$$ab = e = ca,$$

then

$$b = eb = cab = ce = c.$$

Thus a is invertible. ■

Lemma A.4. *Suppose that $\mathfrak{M}$ is a monoid and $a,\ b \in \mathfrak{M}$, then*
1.) ab and ba are both left (right) invertible iff a and b are both left (right) invertible;
2.) ab and ba are both invertible iff a and b are both invertible.

Proof. 1.) We only consider the *left* case. If $\tilde{a}$ and $\tilde{b}$ are the left inverses of a and b respectively, then $\tilde{b}\tilde{a}$ and $\tilde{a}\tilde{b}$ are the left inverses of ab and ba respectively. Conversely, if c and d are the left inverses of ab and ba respectively, then db and ca are the left inverses of a and b respectively.

2.) Apply Lemma A.3 and part 1.). ■

The following is a direct corollary of Lemma A.4.

Lemma A.5. *Suppose that $\mathfrak{M}$ is a monoid and $\{a_k;\ k = 1,\ 2,\ ...,\ n\}$ is a collection of pairwise commuting elements of $\mathfrak{M}$, then*
1.) $\prod_{k=1}^{n} a_k$ is left (right) invertible iff a_i is left (right) invertible for every $i \in \{1,\ 2,\ ...,\ n\}$;
2.) $\prod_{k=1}^{n} a_k$ is invertible iff a_i is invertible for every $i \in \{1,\ 2,\ ...,\ n\}$.

[5] Pay attention to the following fact: suppose that X is a nonempty set, $\mathfrak{M}$ is the monoid consist of all the transformations on X, and $f \in \mathfrak{M}$, then f is injective iff it is left invertible (applying *Axiom of Choice*, we can show that f is surjective iff it is right invertible.). Hence the definition of point spectrum is reasonable.

A.2 Vector-valued Integration

Suppose that X is a Banach space, $\Gamma : [0,1] \to \mathbb{C}$ is a rectifiable curve, and f is a continuous function defined in a neighborhood of Γ with values in X, then $\int_\Gamma f(\lambda)\,\mathrm{d}\lambda$ can be defined as the limit in X of sums of the form

$$\sum_{j=1}^{n} [\Gamma(t_j) - \Gamma(t_{j-1})] f[\Gamma(t_j)],$$

where $\{t_0,\ t_1,\ ...,\ t_n\}$ is a partition of $[0,1]$.

It is easy to check the following conclusion about matrix-valued integration.

Theorem A.6. *Consider the case when $X = \mathbb{C}^{m\times m}$. If $f = (f_{ij})_{m\times m}$, where f_{ij} $(i,\ j = 1,\ 2,\ ...,\ m)$ are continuous scalar-valued functions, then*

$$\int_\Gamma f(\lambda)\,\mathrm{d}\lambda = (\int_\Gamma f_{ij}(\lambda)\,\mathrm{d}\lambda)_{m\times m}.$$

A.3 Vector-valued Holomorphic Functions

The definition of vector-valued holomorphic function is similar to that of scalar-valued one.

Definition A.7. *Suppose that Ω is an open set in $\mathbb{C}$ and X is a Banach Space. A function $f : \Omega \to X$ is said to be holomorphic in Ω if*

$$\lim_{\zeta\to\lambda} \frac{f(\zeta) - f(\lambda)}{\zeta - \lambda}$$

exists (in the topology of X) for every $\lambda \in \Omega$.

The most important conclusions about holomorphic functions are Cauchy theorem and Cauchy formula.

Theorem A.8. (**Cauchy Theorem**) *Suppose that Ω is an open set in $\mathbb{C}$, X is a Banach Space and $f : \Omega \to X$ is holomorphic in Ω.*
1.) If Γ is a closed curve in Ω such that $\mathrm{Ind}_\Gamma(\omega) = 0$ for any $\omega \notin \Omega$, then

$$\int_\Gamma f(\lambda)\,\mathrm{d}\lambda = 0.$$

2.) If Γ_1 and Γ_2 are closed curves in Ω such that $\mathrm{Ind}_{\Gamma_1}(\omega) = \mathrm{Ind}_{\Gamma_2}(\omega)$ for any $\omega \notin \Omega$, then

$$\int_{\Gamma_1} f(\lambda)\,\mathrm{d}\lambda = \int_{\Gamma_2} f(\lambda)\,\mathrm{d}\lambda.$$

Theorem A.9. (**Cauchy Formula**) *Suppose that* Ω *is an open set in* $\mathbb{C}$, X *is a Banach Space and* $f : \Omega \to X$ *is holomorphic in* Ω. *If* Γ *is a closed curve in* Ω *such that* $\mathrm{Ind}_\Gamma(\omega) = 0$ *for any* $\omega \notin \Omega$, *then*

$$f(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\zeta - \lambda)^{-1} f(\lambda)\, \mathrm{d}\lambda$$

if $\lambda \in \Omega$ *and* $\mathrm{Ind}_\Gamma(\lambda) = 1$.

The proof of Theorem A.8 and A.9 can be found in [1].

Now we give a lemma which has been cited several times in Section 3.

Lemma A.10. *Suppose that* $\mathfrak{B}$ *is a Banach algebra and* $x \in \mathfrak{B}$. *Define the function* $f : [\sigma(x)]^{\mathrm{c}} \to \mathfrak{B}$ *by*

$$f(\lambda) = (\lambda e - x)^{-1},$$

then f *is holomorphic in* $[\sigma(x)]^{\mathrm{c}}$.

Proof. Fix a point $\lambda \in [\sigma(x)]^{\mathrm{c}}$, we have

$$\frac{f(\zeta) - f(\lambda)}{\zeta - \lambda} = -f(\lambda)f(\zeta), \quad \forall \zeta \in \Omega.$$

Hence, by the continuity of f, we have

$$\lim_{\zeta \to \lambda} \frac{f(\zeta) - f(\lambda)}{\zeta - \lambda} = -f^2(\lambda).$$

Thus f is holomorphic in Ω. ∎

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