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The Gamma Distribution

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$$\begin{aligned} F(x) &= P(X \leq x) = 1 - P(\text{fewer than } \alpha \text{ occurrences before time } x) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{x^k e^{-x/\theta}}{\theta^k k!}, \qquad x \geq 0 \end{aligned}$$

The expression inside the summation represents the probability of exactly k occurrences in time x in the Poisson distribution with parameter θ.

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While this is fine for many purposes, we might sometimes want to allow α to be a non-integer. In this case, we can replace the factorial with the more general *gamma function*, which has the property that $\Gamma(n) = (n-1)!$ when n is an integer.

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the version usually cited in books.

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Since θ is the average time between occurrences, λ represents the average number of occurrences per unit time. For this reason, it's referred to as the *rate* parameter.

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Each of these facts can be proved using the moment-generating function,

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}}.$$

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$$P(X \le 10) \;=\; \int_0^{10} \frac{x^{3-1} e^{-x/4}}{(3-1)!4^3}\, dx$$

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$$\begin{aligned} P(X \leq 10) &= \int_0^{10} \frac{x^{3-1} e^{-x/4}}{(3-1)!4^3}\, dx = \int_0^{10} \frac{x^2 e^{-x/4}}{128}\, dx \\ &\approx 0.456 \end{aligned}$$

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$$P(X = 15) \; = \; \frac{15^{15} e^{-15}}{15!} \; \approx \; 0.102$$

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- When $\alpha = 1$, we have the *exponential distribution*, which models the waiting time between occurrences in a Poisson process. The pdf of that distribution is

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$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad x \geq 0.$$

- When instead $\theta = 2$ and $\alpha = r/2$, where r is a positive integer, we have the *χ-squared distribution with r degrees of freedom*, which models the distribution of the sum of the squares of r random variables, each with the standard normal distribution. The pdf of that distribution is

$$f(x) = \frac{x^{r/2-1} e^{-x/2}}{\Gamma(r/2) 2^{r/2}}, \qquad x \geq 0.$$