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Discrete Bivariate Distributions

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The probability $P(X = x, Y = y)$, where x and y are particular values, is called the **joint probability mass function**, often abbreviated $f(x, y)$.

Example. Suppose the random variables X and Y have the following joint probability distribution.

$f(x, y)$	y			
x	1	2	3	4
1	-	0.12	0.10	0.09
2	-	0.18	0.07	-
3	0.02	0.14	0.11	-
4	0.05	0.12	-	-

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$$= 0.18 + 0.07 + 0.10 + .0.11 = 0.46.$$

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- $\sum_{(x,y)\in S} f(x, y) = 1$.
- For any event $A \subset S$,

$$P\big[(X, Y) \in A\big] \; = \sum_{(x,y)\in A} f(x, y)$$

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$$
\begin{aligned}
f_X(x) &= P(X = x) = \sum_y f(x, y), \qquad x \in S_x \\
f_Y(y) &= P(Y = y) = \sum_x f(x, y), \qquad y \in S_y
\end{aligned}
$$

Here S_x and S_y are the sample spaces (or *supports*) of X and Y, respectively.

It's natural to add marginal probabilities to tables like the one in the first example.

$f(x,y)$	y				
x	1	2	3	4	$f_X(x)$
1	-	0.12	0.10	0.09	0.31
2	-	0.18	0.07	-	0.25
3	0.02	0.14	0.11	-	0.27
4	0.05	0.12	-	-	0.17
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Note that $\sum_x f_X(x) = \sum_y f_Y(y) = 1$.

Two random variables X and Y are said to be **independent** if and only if

$$f(x, y) \; = \; f_X(x) \cdot f_Y(y)$$

for every $x \in S_x$ and $y \in S_y$.

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$$f(x, y) \;=\; f_X(x) \cdot f_Y(y)$$

for every $x \in S_x$ and $y \in S_y$. If this equality fails for even one (x, y) pair, we say the random variables are **dependent**.

The random variables in the previous example are not independent.

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For instance, $f(1, 2) = 0.12$ while $f_X(1) \cdot f_Y(2) = (0.31)(0.56)$
$= 0.1736$.

Example. Suppose X and Y have the following joint pmf.

$$f(x, y) = Cx^2y, \qquad (x, y) \in \Big\{(1, 1), (1, 2), (2, 2), (3, 1), (3, 2)\Big\}$$

First find C, then compute the marginal probability mass functions $f_X(x)$ and $f_Y(y)$. Finally, determine whether X and Y are independent or not.

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$$1 \;=\; f(1,1) + f(1,2) + f(2,2) + f(3,1) + f(3,2)$$

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$$\begin{aligned} 1 &= f(1,1) + f(1,2) + f(2,2) + f(3,1) + f(3,2) \\ &= C + 2C + 8C + 9C + 18C \ = \ 38C \end{aligned}$$

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&= C + 2C + 8C + 9C + 18C \;=\; 38C \\
C &= 1/38 \\
f(x, y) &= \frac{1}{38}x^2y
\end{aligned}$$

We've determined that

$$f(x,y) = \frac{1}{38}x^2 y, \qquad (x,y) = (1,1),(1,2),(2,2),(3,1),(3,2)$$

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\end{aligned}$$

Notice that $\sum_x f_X(x) = \frac{3}{38} + \frac{8}{38} + \frac{27}{38} = 1.$

$$f(x,y) = \frac{1}{38}x^2 y, \qquad (x,y) = (1,1),(1,2),(2,2),(3,1),(3,2)$$

Similarly,

$$f_Y(1) = \sum_x f(x,1) = f(1,1) + f(3,1) = \frac{1}{38} + \frac{9}{38} = \frac{10}{38}$$

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Again, the marginal probabilities sum to one.

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We've determined that

$$f_X(x) = \begin{cases} 3/38 & \text{if} \quad x = 1 \\ 8/38 & \text{if} \quad x = 2 \\ 27/38 & \text{if} \quad x = 3 \end{cases}, \qquad f_Y(y) = \begin{cases} 10/38 & \text{if} \quad y = 1 \\ 28/38 & \text{if} \quad y = 2 \end{cases}.$$

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If X and Y are independent, then $f(x, y) = f_X(x)f_Y(y)$ for all (x, y).

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$$\begin{aligned} f_X(1)f_Y(1) &= \frac{3}{38} \cdot \frac{10}{38} = \frac{30}{1444} \approx 0.021 \\ f(1,1) &= \frac{1}{38} \approx 0.026 \end{aligned}$$

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Since $f(1,1) \neq f_X(1)f_Y(1)$, the random variables are not independent.