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The Poisson Distribution

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The number of occurrences of an event in a specified period can be viewed as a discrete random variable X whose support (list of possible values) is the non-negative integers. If the events are independent, occur at a constant average rate, and satisfy a few technical axioms which I'll list at the end of the vid, then X is said to have a **Poisson distribution.**

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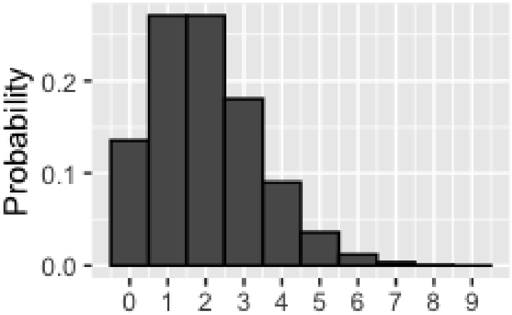

Notice the strong right skew and the mode centered just to the left of the mean $x = 2$.

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We need to find a value x such that $P(X > x) < .10$, or equivalently, such that $P(X \leq x) \geq .90$. We can do this *inverse Poisson calculation* using technology or by manually constructing the cumulative probability distribution for X.

When $\lambda = 4$, we have the following probabilities:

x	0	1	2	3	4	5	6	7	8
$P(X \leq x)$	.018	.092	.238	.433	.629	.785	.889	.949	.979

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In *R*, we would solve this problem with *qpois()* command.

```
qpois(.9, 4)
```

```
## [1] 7
```

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For instance, we compute the moment-generating function (mgf) of a random variable with a Poisson distribution as follows:

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The mgf $M(t)$ immediately gives us the expected value and variance of the Poisson distribution with parameter λ:

$$\mu = M'(0) = \lambda, \qquad \sigma^2 = M''(0) = \lambda^2$$

Both the mean and standard deviation are equal to the average number of occurrences per unit of time.

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$$P(12 < X < 20) = \sum_{x=13}^{19} \frac{16^x e^{-16}}{x!} = .619$$

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- The **binomial distribution** models the the number of occurrences of an event over a finite number of trials when the event has a fixed probability in each trial. By contrast, there are no individual trials for random variables with Poisson distributions.

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In practice, these assumptions are never quite fully met. The phrase "approximate Poisson process" is commonly used to describe situations in which they are reasonable simplifying assumptions.