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Order Statistics

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Order statistics can be used to perform nonparametric statistical inference, for instance when an assumption of normality wouldn't be justified.

Example. Let $X_1, \dots, X_5$ be a simple random sample from $U(0,1)$, the uniform distribution on the interval $(0,1)$. Compute the cumulative distribution functions for the order statistics $Y_1, \dots, Y_5$.

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$$\begin{aligned} G_4(y) &= P(Y_4 \leq y) = P(\text{at least } 4 \ X_i \leq y) \\ &= P(\text{exactly } 4 \ X_i \leq y) + P(\text{exactly } 5 \ X_i \leq y) \end{aligned}$$

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G_4(y) &= P(Y_4 \leq y) = P(\text{at least } 4\ X_i \leq y) \\
&= P(\text{exactly } 4\ X_i \leq y) + P(\text{exactly } 5\ X_i \leq y) \\
&= 5[P(X \leq y)]^4 P(X > y] + [P(X \leq y)]^5
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\begin{aligned}
G_5(y) &= y^5 \\
G_4(y) &= 5y^4(1-y) + y^5 \\
G_3(y) &= 10y^3(1-y)^2 + 5y^4(1-y) + y^5 \\
G_2(y) &= 10y^2(1-y)^3 + 10y^3(1-y)^2 + 5y^4(1-y) + y^5 \\
G_1(y) &= 5y(1-y)^4 + 10y^2(1-y)^3 + 10y^3(1-y)^2 + 5y^4(1-y) + y^5
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\end{aligned}
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For any given $y \in (0,1)$, $G_r(y)$ gives the probability that the r^{th} smallest value in a sample of size 5 will be less than or equal to y.

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$$\begin{aligned} g_5(y) &= 5y^4 \\ g_4(y) &= 20y^3(1-y) \\ g_3(y) &= 30y^2(1-y)^2 \\ g_2(y) &= 20y(1-y)^3 \\ g_1(y) &= 5(1-y)^4 \end{aligned}$$

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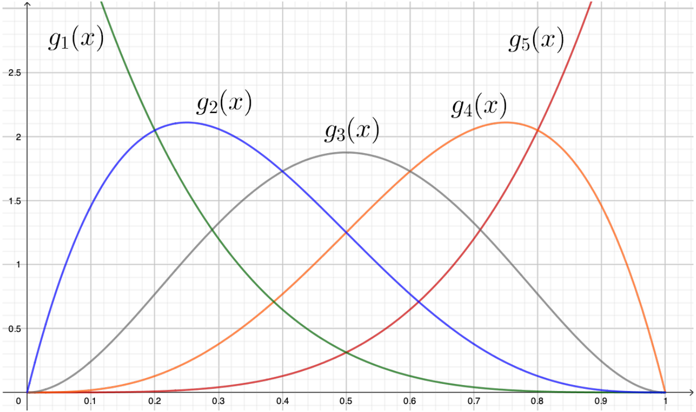
All of these are defined on the support of the original random variable, namely $(0, 1)$.

The graphs of these pdfs are typical:



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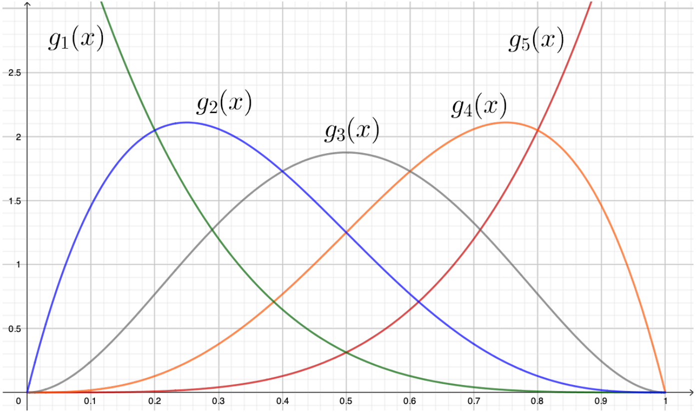

As we'd expect, the probability densities are concentrated further to the right for higher order statistics. In light of this, the cdfs will satisfy the relation $F_1(y) < F_2(y) < \cdots < F_5(y)$.

The calculations in this example are completely typical, so let's generalize them.

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Theorem. When sampling from a continuous distribution, the cdf of the r^{th} order statistic Y_r is

$$G_r(y) = \sum_{k=r}^{n} \binom{n}{k} [F(y)]^k [1 - F(y)]^{n-k}$$

where F is the cdf of the distribution being sampled from and n is the size of the sample.

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$$g_r(y) = \frac{n!}{(r-1)!(n-r)!} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y)$$

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Memorizing these formulas isn't the best idea. Generally, it's easier to compute cdfs directly using the process in the example.

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The cdf of the sample median Y_4 is given by

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\begin{aligned}
G_4(y) &= P(\text{at least } 4 \; X_i \leq y) \\
&= \binom{7}{4}[F(y)]^4[1 - F(y)]^3 + \binom{7}{5}[F(y)]^5[1 - F(y)]^2 + \\
&\quad \binom{7}{6}[F(y)]^6[1 - F(y)] + [F(y)]^7
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The cdf of this distribution is $F(x) = x^2$. All we ned to do is substitute.

The algebra isn't pretty, but it isn't complicated either..

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&= 35y^8(1-y^2)^3 + 21y^{10}(1-y^2)^2 + 7y^{12}(1-y^2) + y^{14}
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&= 35y^8(1-y^2)^3 + 21y^{10}(1-y^2)^2 + 7y^{12}(1-y^2) + y^{14} \\
G_4(.5) &= 35(.5)^8(1-(.5)^2)^3 + 21(.5)^{10}(1-(.5)^2)^2 + \\
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G_4(.5) &= 35(.5)^8(1-(.5)^2)^3+21(.5)^{10}(1-(.5)^2)^2+ \\
&\quad 7(.5)^{12}(1-(.5)^2)+(.5)^{14} \\
&\approx .071
\end{aligned}
$$

There is a 7.1% chance that the median will be less than .5 in a sample of size 7 from this distribution.