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The log-normal distribution

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If X has a log-normal distribution, then $\log(X)$ is normally-distributed. While the base of the logarithm doesn't matter, the natural logarithm is standard.

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- Household incomes
- Population sizes

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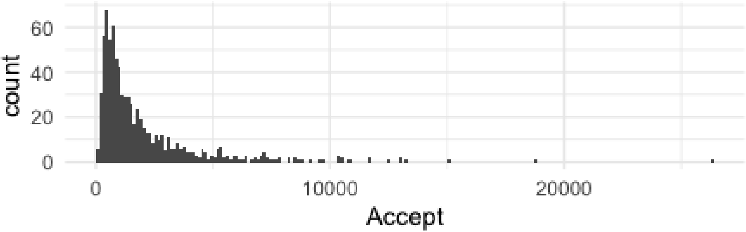
Example. In the U.S. News `College` data set, included with the `ISLR2` R package, many of the variables can be modeled with a log-normal distribution. For instance, the `Accept` variable, which represents the number of students accepted, looks like this:



While acceptance at most schools is in the hundreds or thousands, a few accepted less than 100 or more than 10,000.

The following table shows a breakdown of the number of digits in each college's acceptance count.

Enroll: number of digits	number of schools
2	2
3	346
4	413
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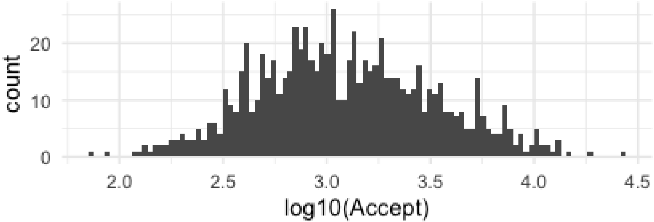
Here's a histogram showing counts of the base-10 logarithm of the acceptance counts.



Notice how bell-shaped it looks!

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We write $X \sim \text{Lognormal}(\mu, \sigma^2)$ for a random variable with this distribution.

It turns out that the mean and variance of such an X are given by

$$\mu_X = e^{\mu+\sigma^2/2} \quad \text{and} \quad \sigma_X^2 = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$$

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where F is the cumulative distribution function (cdf) of $N(\mu, \sigma^2)$.

$$F(x) = P(Y \le x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) dt$$

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We get the cdf of the log-normal distribution by replacing x with $\ln x$.

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$$P(a \leq X \leq b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b \frac{1}{x} \cdot \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) dx$$

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Typically we use technology such as R to do this.

$$P(a \leq X \leq b) = \texttt{plnorm}(\texttt{b}, \mu, \sigma) - \texttt{plnorm}(\texttt{a}, \mu, \sigma)$$

Estimating μ and σ^2 from data

The maximum likelihood estimators for μ and σ^2 in the log-normal distribution are

$$\hat{\mu} = \frac{1}{n}\sum \ln x_i \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum(\ln x_i - \hat{\mu})^2$$

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The latter is biased, however. On average, it will slightly underestimate the parameter σ^2. Similarly to the normal distribution, an unbiased estimator of population variance is given by

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum (\ln x_i - \hat{\mu})^2$$

Example. Freshman undergraduate enrollments at 8 randomly-selected schools in the `College` data set are given below.

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The actual parameters in the `College` data set are $\mu = 6.18$ and $\sigma = 0.91$.