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The Central Limit Theorem

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$$\begin{aligned} \mu_{\overline{x}} &= \mu \\ \sigma_{\overline{x}} &= \sigma/\sqrt{n} \end{aligned}$$

So the Central Limit Theorem says that the distribution of $\overline{x}$ is always approximately $N\left(\mu, \sigma^2/n\right)$.

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$$P(\overline{x} < 2.5) \text{ in } N(2, 9/40) \quad = \quad P\left(z < \frac{2.5 - 2}{3/\sqrt{40}}\right) \text{ in } N(0,1)$$

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&= P(z < 1.05) \\
&= 85.3\%
\end{aligned}$$

Normal probabilities like this should be calculated using technology. In *R*, for instance, the commands pnorm(1.05) and pnorm(2.5, 2, 3/sqrt(40)) both do the job.

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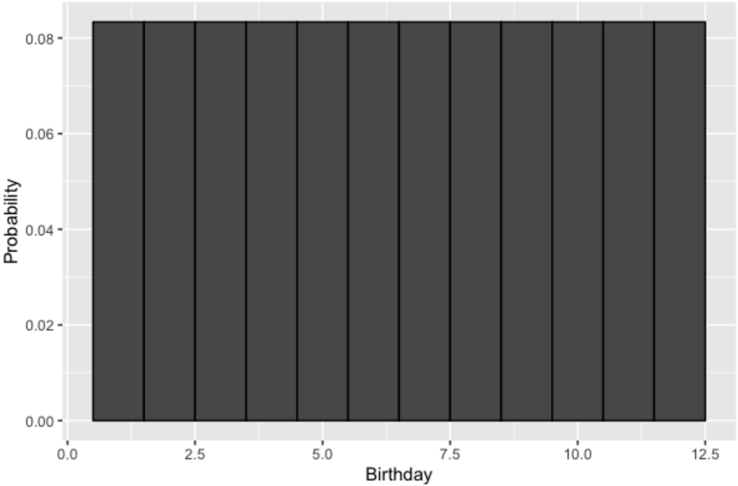

Next, we use the random number generator twice and take the mean, $\overline{x}$. The result is again a number between 1 and 12.

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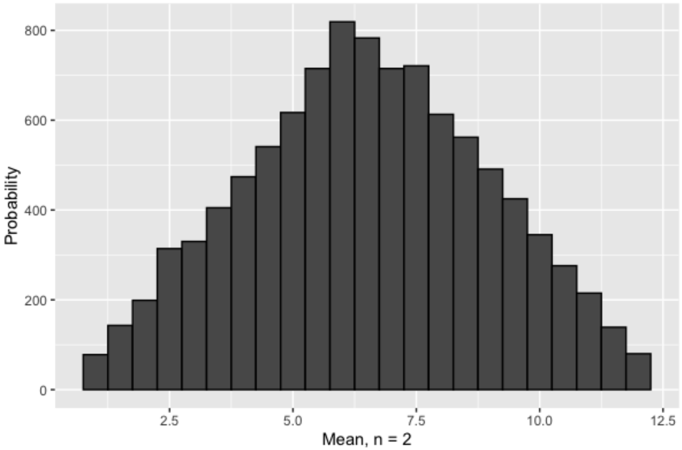
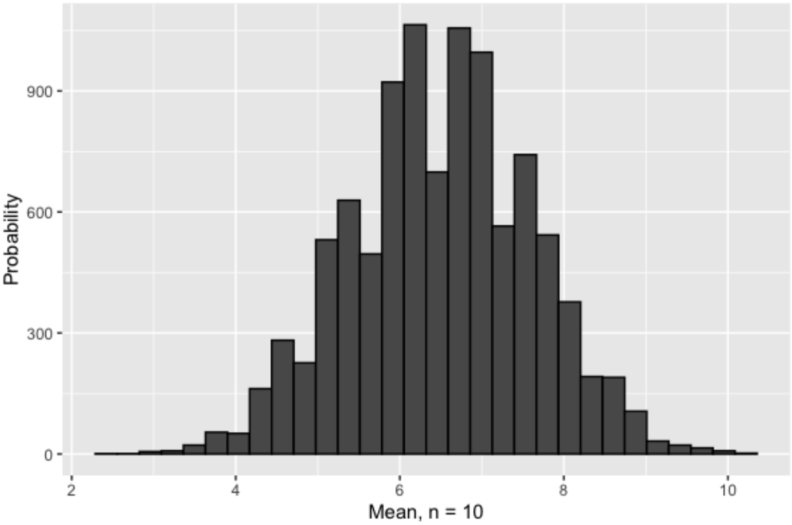


Results near 6.5 are more likely, while 1's and 12's are rare.

Now take 1000 samples of size $n = 10$.



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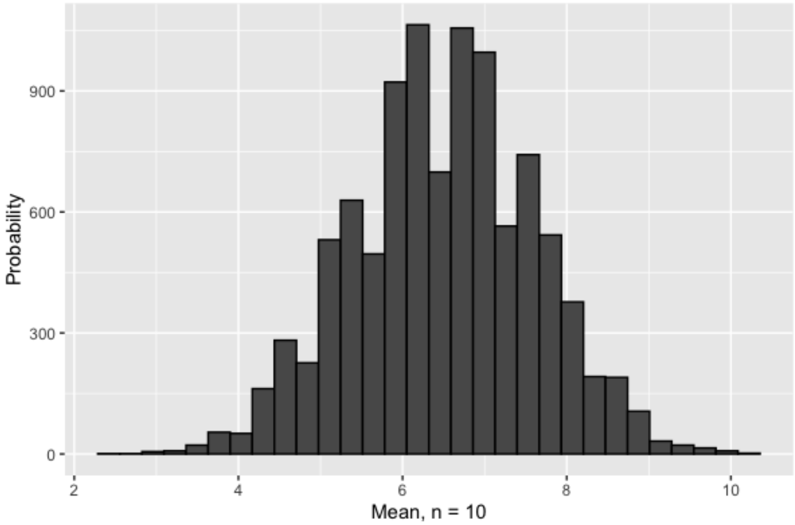
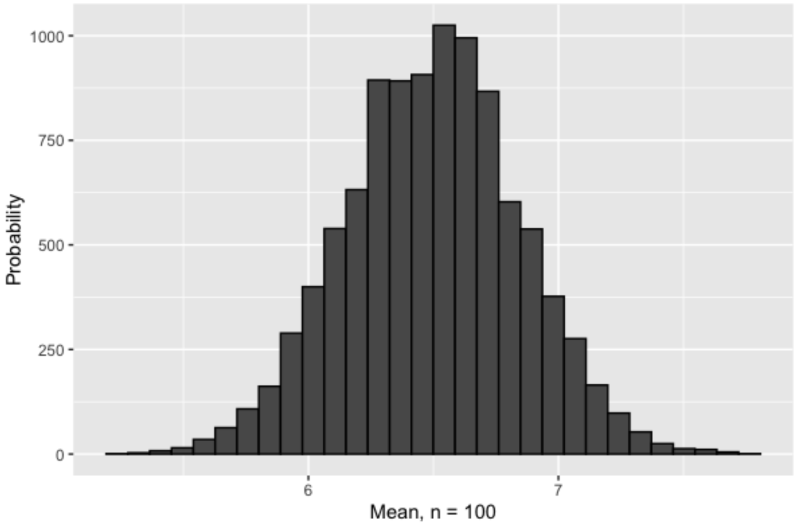


The histogram is beginning to have a vaguely bell shape. Note also that nearly all of the results lie between 4 and 9.

Using 1000 samples of size 100:



The bell shape is clear.

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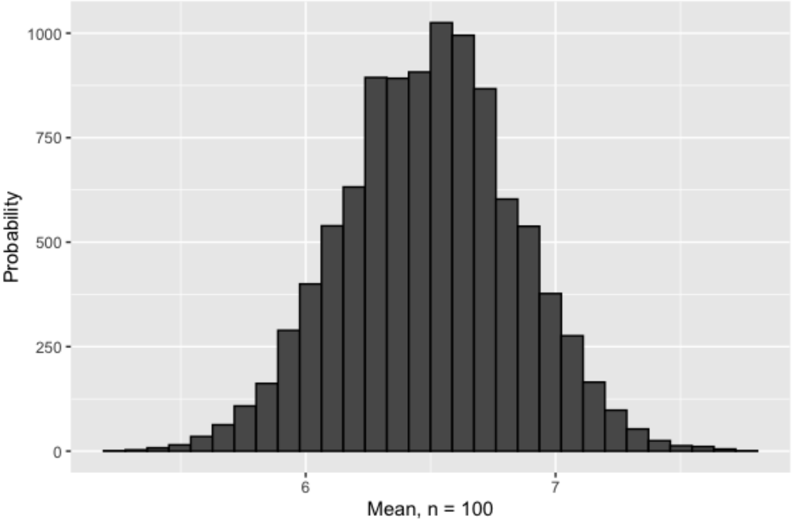
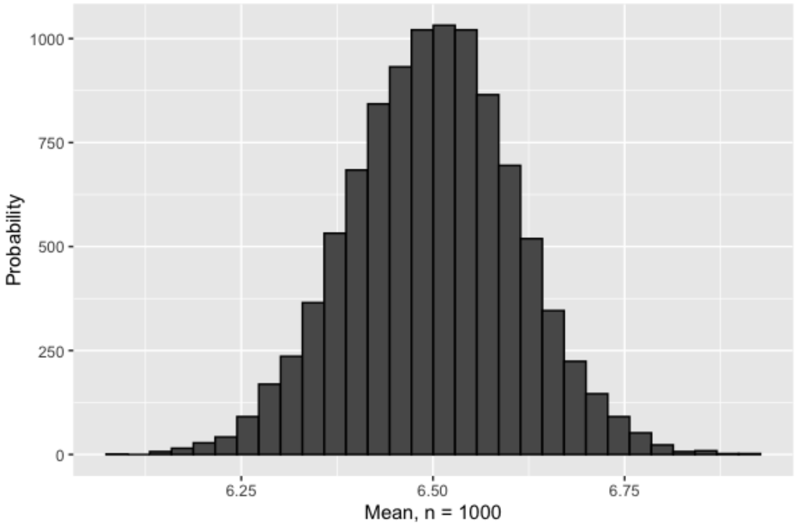


The bell shape is clear. Now, most of the data lies between 6 and 7.

Finally, let's use 1000 samples of size 1000:



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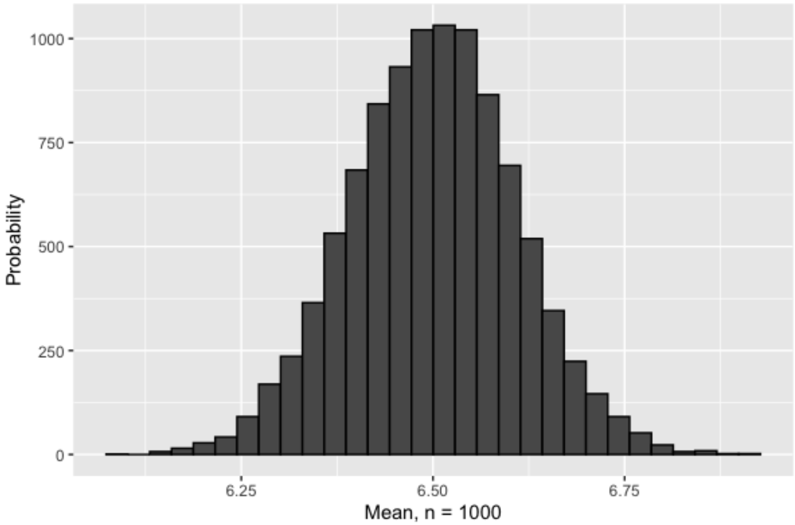

A normal curve fits these results exceptionally well. Most of the results lie between 6.25 and 6.75

Let's finish this example by actually computing the mean and standard deviation of each of these sampling distributions. If we randomly select a single number between 1 and 12, each value has a 1/12 chance of being picked. The mean and standard deviation of this probability distribution are

$$\mu = 6.50 \quad \text{and} \quad \sigma = 3.45$$

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Roughly speaking, as n increases, the sample mean $\overline{x}$ becomes a more reliable estimator of μ.

Example 3. A distilled-water dispenser vends 'gallons' of water according to a normal distribution with mean 1.03 gallons and standard deviation .02 gallons. What is the probability that a single 'gallon' dispensed is actually under 1 gallon? What is the probability that 10 dispensed 'gallons' average less than 1 gallon each?

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For an individual 'gallon' x,

$$P(x < 1.00) \text{ in } N\left(1.03, .02^2\right) \quad = \quad P\left(z < \frac{1 - 1.03}{.02}\right) \text{ in } N\left(0, 1\right)$$

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On the other hand, when $n = 10$, the sample mean has the distribution $N(1.03, .02^2/10)$, by the Central Limit Theorem.

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&= P(z < -4.74) \\
&= .0001\%
\end{aligned}
$$

We conclude that while it's somewhat unlikely (6.68% chance) that a single 'gallon' will be underfilled, it would be extremely unusual if the mean of 10 'gallons' were less than 1.

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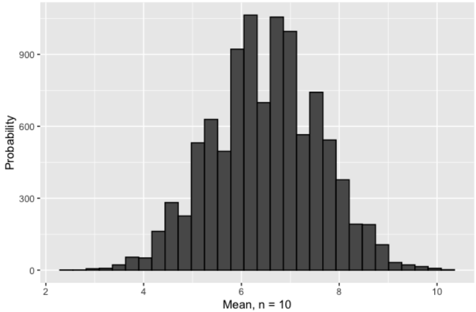
In example 2, for instance, where we selected numbers between 1 and 12 at random, the normal shape was clear even for $n = 10$.



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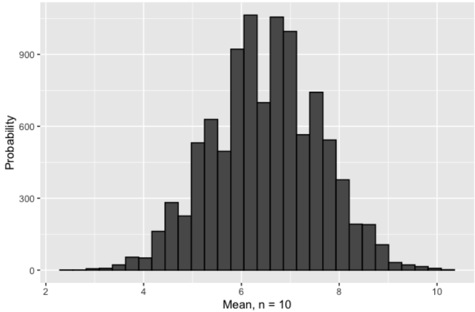

The more skewed the distribution, the larger the sample size needed.