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Bivariate Distributions of Continuous Random Variables

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The set of points (x, y) where $f(x, y) > 0$ is called the **support** of the distribution.

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Most simply, if the region D is rectangular, given by $a \leq x \leq b$ and $c \leq y \leq d$, then

$$\begin{aligned}
\iint_D f(x,y)\,dA &= \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\,dy\,dx \\
&= \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\,dx\,dy
\end{aligned}$$

Example. Let X and Y be continuous random variables with probability density function $f(x,y) = \frac{1}{16}(x+3y)$ on $[0,2]\times[0,2]$. Compute $P\big((x,y)\in D\big)$ where $D = [0,1]\times[1,2]$.

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This amounts to computing

$$\iint_D \frac{1}{16}(x+3y)\,dA$$

For $D = [0,1] \times [1,2]$,

$$\iint_D \frac{1}{16}(x+3y)\,dA \;=\; \int_{x=0}^{x=1} \int_{y=1}^{y=2} \frac{1}{16}(x+3y)\,dy\,dx$$

For $D = [0,1] \times [1,2]$,

$$\begin{aligned}
\iint_D \frac{1}{16}(x+3y)\,dA &= \int_{x=0}^{x=1} \int_{y=1}^{y=2} \frac{1}{16}(x+3y)\,dy\,dx \\
&= \frac{1}{16} \int_{x=0}^{x=1} \left[yx + \frac{3}{2}y^2 \right]_{y=1}^{y=2} dx
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&= \frac{1}{16}\int_{x=0}^{x=1}\left(x + \frac{9}{2}\right) dx \\
&= \frac{1}{16}\left[\frac{1}{2}x^2 + \frac{9}{2}x\right]_0^1 \\
P\big((x,y)\in D\big) &= \frac{5}{16}
\end{aligned}$$

Notice that the region D is 1/4 the area of the region where $f(x, y)$ is nonzero, but includes 5/16 of the probability. The joint pdf $f(x, y) = \frac{1}{16}(x + 3y)$ is larger for larger values of x and y, so the probability is more concentrated in the upper-right of the support.

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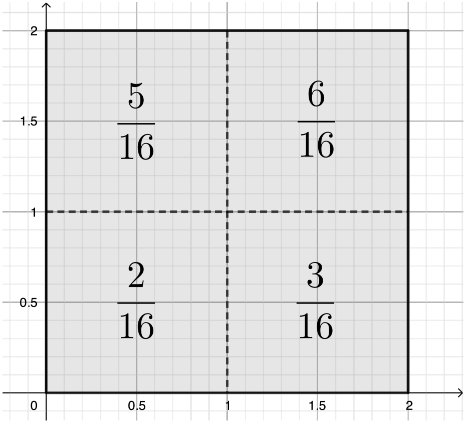

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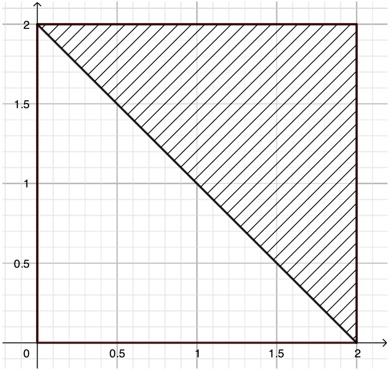

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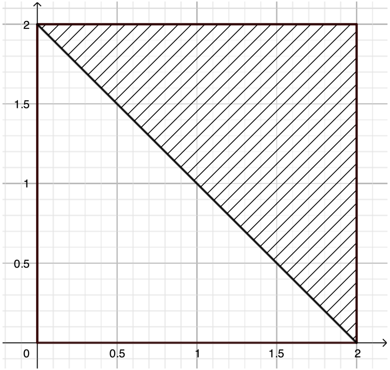
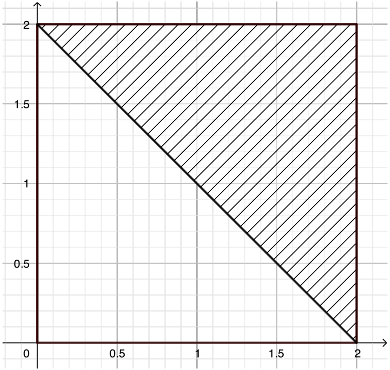


In order to write an iterated integral, we need to come up with inequalities describing the region.

In order to describe this region, we need two inequalities, one for x and one for y.



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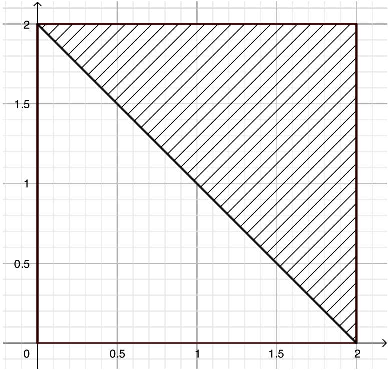
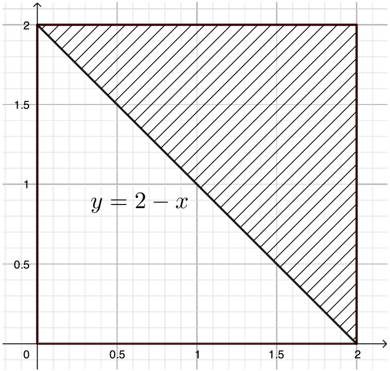


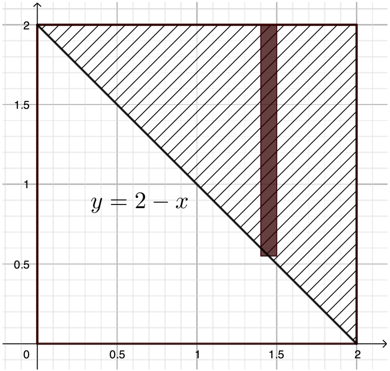
Whichever we write first should just capture the range of possible values of that variable. For instance, $0 \leq x \leq 2$.

The second inequality bounds the second variable between functions of the first.



When $0 \leq x \leq 2$, the shaded region is defined by $2 - x \leq y \leq 2$.

When writing the second inequality, it can be helpful to draw a rectangle as if setting up an integral for the area between curves.



This makes it more clear that the y-values are bounded by $y = 2 - x$ and $y = 2$.

We've managed to parameterize the region D.

$$\begin{aligned} 0 &\leq x \leq 2 \\ 2 - x &\leq y \leq 2 \end{aligned}$$

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$$\begin{array}{rcccl} 0 & \leq & x & \leq & 2 \\ 2-x & \leq & y & \leq & 2 \end{array}$$

We're ready to set up an integral.

$$\int_{x=0}^{x=2} \int_{y=2-x}^{y=2} \frac{1}{16}(x+3y)\, dy\, dx$$

Notice that the variable with the constant limits goes on the outside and so will be evaluated second.

Here's the calculation:

$$\begin{aligned}
\int_{x=0}^{x=2}\int_{y=2-x}^{y=2}\frac{1}{16}(x+3y)\,dy\,dx &= \frac{1}{16}\int_{x=0}^{x=2}\left(xy+\frac{3}{2}y^2\right)_{y=2-x}^{y=2}dx \\
&= \frac{1}{16}\int_{x=0}^{x=2}\left(6x-\frac{1}{2}x^2\right)dx \\
&= \frac{1}{16}\left(3x^2-\frac{1}{6}x^3\right)_{x=0}^{x=2} \\
&= \frac{2}{3}
\end{aligned}$$

In general, this technique can be used whenever D can be represented as the region between two curves $y = g_1(x)$ and $y = g_2(x)$.

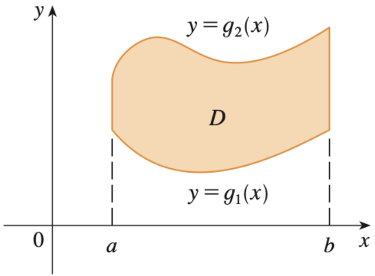
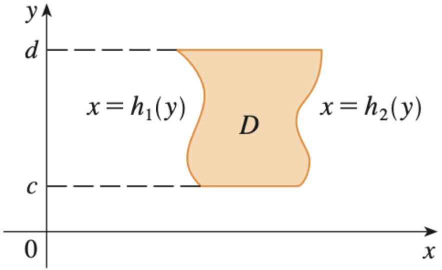


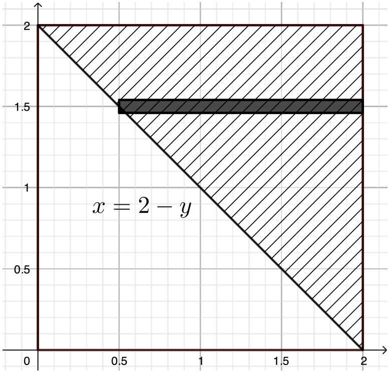
Image source: Stewart's *Calculus* 10e

The same technique works if D can be described as the region between two function of y.



In such a case, we parameterize the region as $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$ and integrate with respect to y first.

The region in the last example can also be described by bounding x between functions of y. You should try this for practice!



You answer should be the same as before: $P\big((x, y) \in A\big) = \frac{2}{3}$.