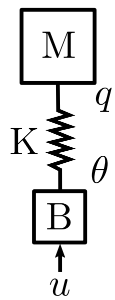
Should we really use $\bar{\boldsymbol{q}}$?

1 Spring-mass

1.1 The system



Dynamics equation:

$$u - K(\theta - q) = B\ddot{\theta} \tag{1}$$
$$K(\theta - q) = M\ddot{q} + g(q) \tag{2}$$

Defining desired motor side position according to link side position:

$$K(\theta_d - q_d) = M\ddot{q}_d \tag{3}$$

Control input:

$$u = -K_q\tilde{q} - D_q\dot{\tilde{q}} + M\ddot{q}_d + B\ddot{\theta}_d + \hat{g}(q) \tag{4}$$
$$\tilde{q} = q - q_d$$

1.2 Problem formulation

Closed loop dynamics:

$$M\ddot{q}_d + B\ddot{\theta}_d - B\ddot{\theta} - K_q\tilde{q} - D_q\dot{\tilde{q}} + \hat{g}(q) = M\ddot{q} + g(q) \tag{5}$$

Power balance equation with flow $\dot\theta$:

$$\underbrace{\dot\theta \underbrace{M\ddot q_d}_{K(\theta_d-q_d)}}_{\dot q \underbrace{K(\theta_d-q_d)}_{M\ddot q_d} + (\dot\theta-\dot q)K(\theta_d-q_d)} + \dot\theta B\ddot\theta_d - \dot\theta B\ddot\theta - \dot\theta(K_q\tilde q + D_q\dot{\tilde q}) + \dot\theta\hat g(q) = \underbrace{\dot\theta \underbrace{(M\ddot q + g(q))}_{K(\theta-q)}}_{\dot q \underbrace{K(\theta-q)}_{M\ddot q+g(q)} + (\dot\theta-\dot q)K(\theta-q)} \tag{6}$$

Power balance equation with flow $\dot\theta_d$:

$$\underbrace{\dot\theta_d \underbrace{M\ddot q_d}_{K(\theta_d-q_d)}}_{\dot q_d \underbrace{K(\theta_d-q_d)}_{M\ddot q_d} + (\dot\theta_d-\dot q_d)K(\theta_d-q_d)} + \dot\theta_d B\ddot\theta_d - \dot\theta_d B\ddot\theta - \dot\theta_d(K_q\tilde q + D_q\dot{\tilde q}) + \dot\theta_d\hat g(q) = \underbrace{\dot\theta_d \underbrace{(M\ddot q + g(q))}_{K(\theta-q)}}_{\dot q_d \underbrace{K(\theta-q)}_{M\ddot q+g(q)} + (\dot\theta_d-\dot q_d)K(\theta-q)} \tag{7}$$

Subtracting (6) from (7):

$$\dot{\tilde q} M\ddot{\tilde q} + \dot{\tilde\theta} B\ddot{\tilde\theta} + \underbrace{\dot{\tilde\theta} K_q\tilde q}_{\overset{?}{=}\dot S} \underbrace{-\dot{\tilde\theta}\hat g(q) + \dot{\tilde q} g(q)}_{\overset{?}{=}\dot S/0} = -\underbrace{\dot{\tilde\theta} D_q\dot{\tilde q}}_{\overset{?}{=}P_D} \tag{8}$$

1.3 Only regulation and Albu-schaeffer idea

Closed loop dynamics (8) will boil down to:

$$\dot q M\ddot q + \dot\theta B\ddot\theta + \dot\theta K_q\tilde q - \dot\theta\hat g(q) + \dot q g(q) = -\dot\theta D_q\dot q \tag{9}$$

According to [1] the storage function for the closed loop system is:

$$S_{tot} = \underbrace{\frac{1}{2}\dot\theta B\dot\theta + \frac{1}{2}(\theta-q)K(\theta-q) + \underbrace{\underbrace{\frac{1}{2}\tilde{\bar q}K_q\tilde{\bar q} - V_g(\bar q)}_{-V_l} \underbrace{-\frac{1}{2}l(\bar q(\theta))K^{-1}l(\bar q(\theta))}_{S_l}}_{-V_{\bar l}}}_{S_\theta} + \frac{1}{2}\dot q M\dot q + V_g(q) \tag{10}$$

where:

$$l(\bar q(\theta)) = -K_q\tilde{\bar q}(\theta) + g(\bar q(\theta)) \tag{11}$$

$$\tilde{\bar q}(\theta) = \bar q(\theta) - q_d \tag{12}$$

$$\bar q(\theta) = h^{-1}(\theta) \tag{13}$$

and $h^{-1}(\theta)$ is corresponding q to this θ in static case. Thus, for this static q we have:

$$l(\bar{q}(\theta)) = K(\theta - \bar{q}(\theta)) \tag{14}$$

and as a result S_l in equation (10) will be:

$$S_l = -\frac{1}{2}(\theta - \bar{q}(\theta))K(\theta - \bar{q}(\theta)) \tag{15}$$

why should we have this storage function in S_{tot}? Before proceeding, we have:

$$\frac{\partial V_l}{\partial \bar{q}} = -K_q\tilde{\bar{q}} + g(\bar{q}) = l(\bar{q}) \tag{16}$$

which makes sense because $\dot{q}l(q)$ is the power going out of controller, and thus for the overall storage function the notion of $-V_l$ is used. Now, we also have:

$$\begin{aligned}\frac{\partial V_{\bar{l}}}{\partial \bar{\theta}} &= -\frac{\partial \bar{q}}{\partial \theta}K_q\tilde{\bar{q}} + \frac{\partial \bar{q}}{\partial \theta}g(\bar{q}) + (1 - \frac{\partial \bar{q}}{\partial \theta})\underbrace{K(\theta - \bar{q}(\theta))}_{l(\bar{q})} \\ &= \frac{\partial \bar{q}}{\partial \theta}\underbrace{(-K_q\tilde{\bar{q}} + g(\bar{q}))}_{l(\bar{q})} + l(\bar{q}) - \frac{\partial \bar{q}}{\partial \theta}l(\bar{q}) \\ &= l(\bar{q})\end{aligned} \tag{17}$$

References

[1] A. Albu-Schäffer, C. Ott, and G. Hirzinger. Passivity based cartesian impedance control for flexible joint manipulators. *IFAC Proceedings Volumes*, 37(13):901–906, 2004.