Cognitive Task Analysis for ProveIt

1.Introduction

ProveIt is a website-based educational app, aiming to help students write mathematical proof. In order to design this app, we conducted cognitive task analysis (CTA) on experts who are students proficient in mathematics and have experience in tutoring. The reason we chose these experts is because we want to get both a student perspective and also a teacher perspective. We are focusing on mathematical induction in this CTA, but later on we will also include other mathematics proof methods to the app. We designed one control question to help our expert get accustomed to think-aloud during solving the problem and two questions to extract knowledge components from the expert. After the expert solves the questions, we ask some follow-up questions below.

1. What is the most common problem when writing proof with mathematical induction?
2. What is the knowledge or mathematical concept I should have in order to have flexibility on ideas of writing proof?
3. How do you decide whether a problem is suitable for mathematical induction?"
4. When you get stuck, what strategies do you use to move forward?"
5. "How do you check that your proof is complete and correct?"

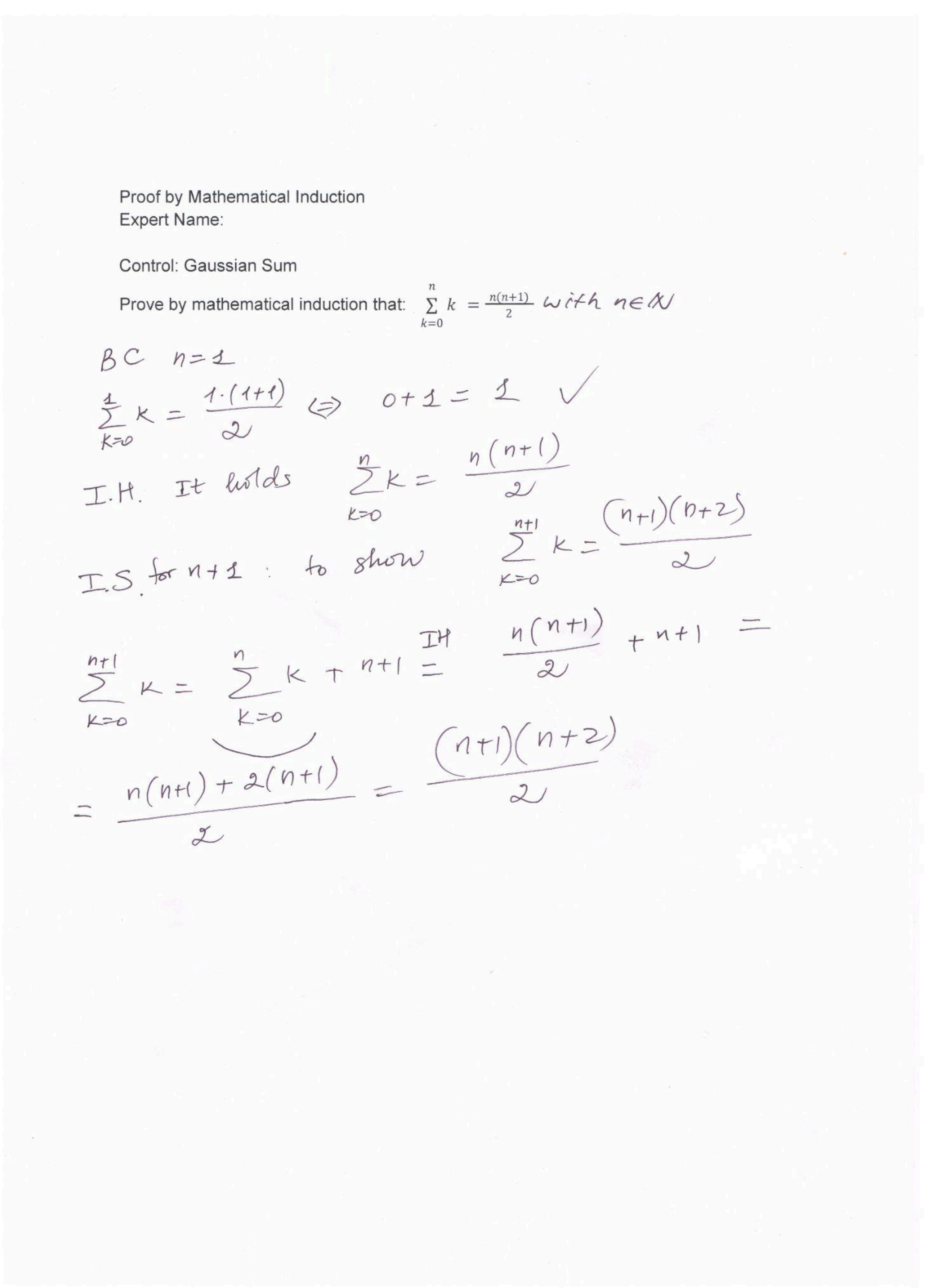
We found unclear parts and formalities from the first expert's CTA , in this case the induction hypothesis. In order to verify our understanding about the domain, we decided to clarify this by doing another CTA with another expert. We recorded the CTA and analyzed them closely by listening to the recordings. In the following section, we will present our findings from both of the experts.

2.CTA Analysis

Expert 1 CTA Analysis

Our expert is a mathematics tutor for the course *Mathematics for Computer Scientists*. We chose her due to her subject expertise and familiarity with the type of problems our interactive tool aims to support. The session was attended by three team members: Rishabh, Evan, and Muhammadsultonbek. She was instructed to think aloud and explain each step clearly. The objective was to understand her decision-making process, highlight expert strategies, and identify areas where learners might struggle.

We asked our expert to solve three proof problems given below using only mathematical induction.



Proof by Mathematical Induction
Expert Name:

Control: Gaussian Sum

Prove by mathematical induction that: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ with $n \in \mathbb{N}$

BC $n = 1$

$$\sum_{k=0}^{1} k = \frac{1 \cdot (1+1)}{2} \Leftrightarrow 0 + 1 = 1 \checkmark$$

I.H. It holds $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

I.S. for $n+1$: to show $\sum_{k=0}^{n+1} k = \frac{(n+1)(n+2)}{2}$

$$\sum_{k=0}^{n+1} k = \underbrace{\sum_{k=0}^{n} k}_{} + n+1 \overset{IH}{=} \frac{n(n+1)}{2} + n+1 =$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Figure 1.1: Control Question from CTA of Expert 2

$$(a+b)^3 =$$
$$= a^3+3a^2b+3ab^2+b^3$$

Prove that n³ - n is divisible by 3 for all positive integers

$$\cancel{a^3-b^3=(a-b)(a^2+ab+b^2)}$$

$n \in \mathbb{Z}_{>0}$

BC. $n=0$ $\quad n^3-n=0 \vdots 3$

IH. $(n^3-n) \vdots 3$

IS. $((n+1)^3-(n+1)) \vdots 3 \leftarrow$ to show

$$(n+1)^3-(n+1) = \cancel{n^3+3n^2+3n+1-n^3} =$$
$$\cancel{(n^3-n)+(n-n^3)+}$$

$$= n^3+3n^2+3n+\cancel{1}-n-\cancel{1} =$$
$$= \underbrace{(n^3-n)}_{\vdots 3 \text{ by IH}} + \underbrace{3(n^2+n)}_{\vdots 3} \vdots 3$$

Figure 1.2: Question 1 from CTA of Expert 2

Prove that the sum of the first n odd numbers is n².

$2k-1 \qquad k \in \mathbb{N}$

$$\sum_{k=1}^{n} (2k-1) = n^2 \qquad n \in \mathbb{N}.$$

BC. $n = 1$.

$$\sum_{k=1}^{1} (2k-1) = 1^2 \quad \checkmark$$

$$1 = 1$$

IH. $\sum_{k=1}^{n} (2k-1) = n^2$

IS. $\sum_{k=1}^{n+1} (2k-1) = (n+1)^2$

$$\sum_{k=1}^{n+1} (2k-1) = \underbrace{\sum_{k=1}^{n} (2k-1)} + 2(n+1) - 1 \overset{IH}{=}$$

$$= n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n+1)^2$$

Figure 1.3: Question 2 from CTA of Expert 2

Key Insights into How the Expert Performed the Task:

For each problem, the expert adhered to the standard structure of Mathematical induction:

1. Base Case: Prove the statement for the initial value (usually $n=1$).
2. Induction Hypothesis: Assumed the statement is true for some arbitrary $n=k$.
3. Induction Step: Used the hypothesis to prove the case for $n=k+1$.

Before beginning the first problem, the expert immediately identified a foundational issue: the range of n was not clearly defined. She emphasized the importance of specifying that $n \in N$ and checking whether the base case should start at $n=0$ or $n=1$. This step set the stage for a structurally sound proof.

The interesting thing about the second question our expert tried to solve was that her works lacked explanations in the written format. This might be one of the reasons she got confused and got stuck in this problem. In the induction step she forgot to replace one of the n with n+1 that led to getting stuck . In that situation she tried to redo all the steps of mathematical induction to see where she made an error. Apart from that she asked us whether the problem must be proven with mathematical induction saying that it is easier to do it with direct proof which is another method for the proof problems.

The expert solved the third problem without much difficulty, reinforcing the effectiveness of a well-structured approach. She reformulates the statement into mathematical notation, which helps her to use the definition of sigma notation in the proof.

Follow-up:

The expert emphasized that the most common mistake students make is not understanding the structure of induction. She reiterated that in weak induction, each step depends only on the one before it. Recognizing this helps students stay grounded. She also pointed out that some problems require multiple base cases when steps depend on more than one previous term.

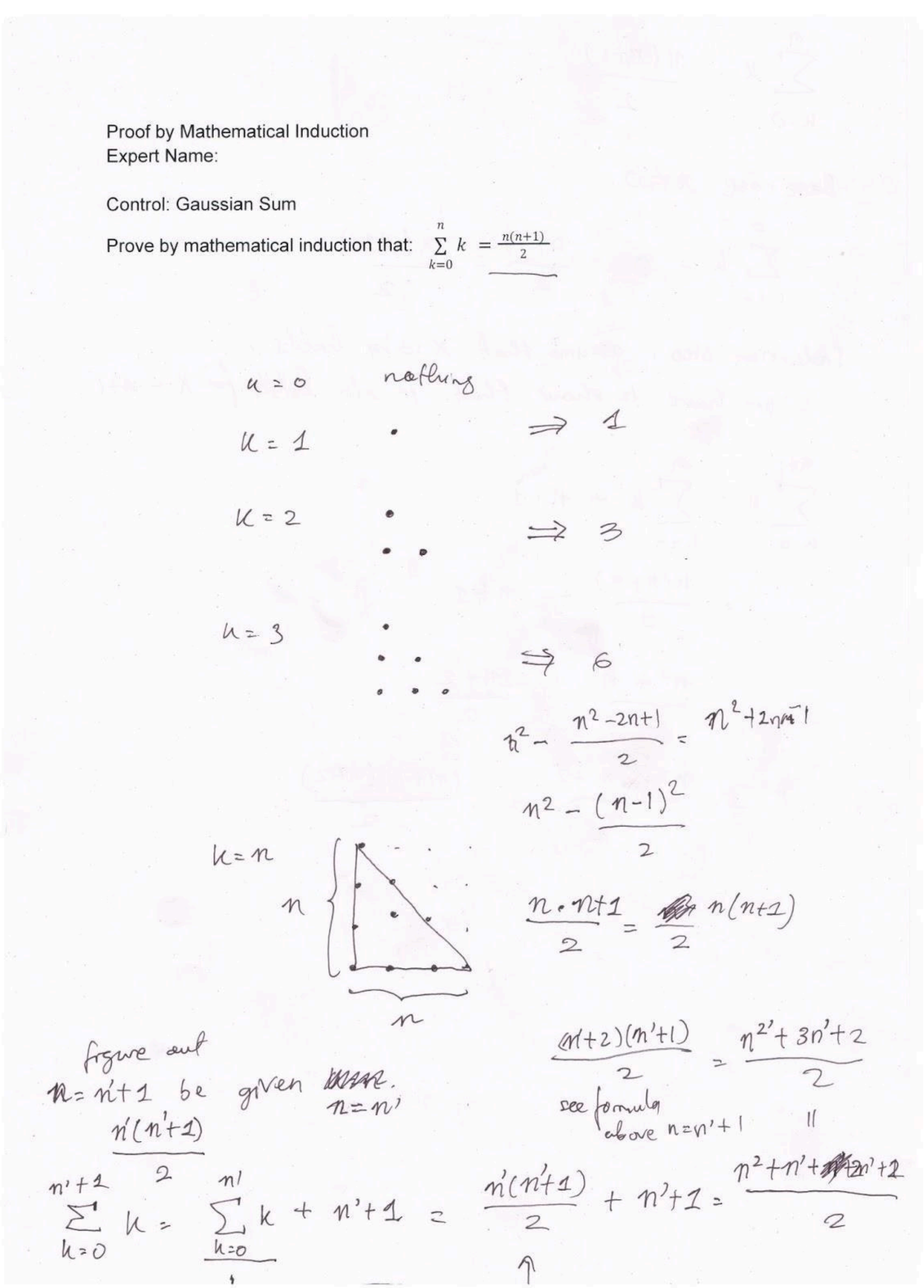
She noted that larger or more complex formulas often confuse students, so she encourages them to think ahead about what the n+1 step should look like before diving into algebra. According to her, a solid grasp of high school mathematics and comfort with working around different notations like sigma notation are necessary to succeed in these proofs.

She said that mathematical induction is most suitable for problems involving summation formulas or universally quantified statements over natural numbers. The induction step must be explicitly built on the hypothesis, and the best way to check correctness is by strictly following the induction structure.

This method of proof is usually done just by following all the steps correctly even though you don't understand what it actually does. For a student to fully understand what is achieved with this, solving a lot of different problems with mathematical induction will be necessary.

Expert 2 CTA Analysis

The second Expert is a bachelor student with high proficiency in mathematics with experience of tutoring *Element of Machine Learning* and *Statistics Lab*. The session was attended by one of our team members: Evan. He interviewed him at first by asking him to solve the control problem by thinking aloud. The expert explained the problem first and immediately went into the analogy of the triangle area, which isn't part of the proof that we expected. This is how he thinks mathematical induction looks like. After that, we asked him to do formal mathematical proof and stick with the structure that we used to. He proves all the problems with the structure of mathematical induction. The following images are the CTA that the Expert 2 did.



Proof by Mathematical Induction
Expert Name:

Control: Gaussian Sum

Prove by mathematical induction that: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$

$k=0$ nothing

$k=1$ $\Rightarrow 1$

$k=2$ $\Rightarrow 3$

$k=3$ $\Rightarrow 6$

$n^2 - \frac{n^2-2n+1}{2} = n^2+2n-1$

$n^2 - \frac{(n-1)^2}{2}$

$k=n$

$\frac{n \cdot n+1}{2} = \frac{n(n+1)}{2}$

$\frac{(n'+2)(n'+1)}{2} = \frac{n'^2+3n'+2}{2}$

see formula above $n=n'+1$

figure out

$n = n'+1$ be given $n=n'$.

$\frac{n'(n'+1)}{2}$

$\sum_{k=0}^{n'+1} k = \sum_{k=0}^{n'} k + n'+1 = \frac{n'(n'+1)}{2} + n'+1 = \frac{n'^2+n'+2n'+2}{2}$

Figure 2.1: Control question from CTA of Expert 2

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Base case $x := 0$,

$$\sum_{k=0}^{0} k = 0 = \frac{0(1)}{2} = \frac{x(x+1)}{2}$$

Induction step, assume that $x := n$ holds,
we have to show that it also holds for $x := n+1$.

$$\sum_{k=0}^{n+1} k = \sum_{k=0}^{n} k + n+1$$

$$= \frac{n(n+1)}{2} + n+1$$

$$= \frac{n^2+n}{2} + \frac{2n+2}{2}$$

$$= \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

Figure 2.2: Control question from CTA of Expert 2

$\underline{n^3 - n}$

Prove that $\underline{n^3 - n}$ is divisible by 3 for all positive integers $\underline{n \in \mathbb{N}}$

$$n^3 - n = n(n^2-1) = n(n-1)(n+1).$$

~~Base case~~

Base case:

$1^3 - 1 = 0$, $3 \mid 0$ ✓.

Induction case:

suppose it holds for $x^3 - x$

we'd like to show $\underbrace{(x+1)^3 - x+1}_{\|}$.

$$x(x+1)(x+2)$$

$$x^3 + 3x^2 + 3\cancel{x} + \cancel{1} - \cancel{x} + \cancel{1}.$$

$$x^3 + 3x^2 + 2x$$

$$\cancel{x^3 + 4x + 4}.$$

$$x^3 + 3x^2 + \underline{3x - x},$$

$$\underbrace{x^3 - x}_{\text{div by 3}} + \underbrace{3x^2 + 3x}_{3(x^2+x)}$$

$\uparrow$

Figure 2.3: Question 1 from CTA of Expert 2

Prove that the sum of the first n odd numbers is n^2.

Base case

$1 = 1.$

For first n odd numbers we assume it holds that the sum is n^2.

$n+1:$

$$+2n-1+$$

$$1+3+\dots\ 2n+1$$

$$\underbrace{1+3+\dots+2n-1}_{n^2} + \underbrace{2n+1}_{n+1^{th}}$$

$$n^2+2n+1=(n+1)^2.$$

Figure 2.4: Question 2 from CTA of Expert 2

Key Insights into How the Expert Performed the Task:

From the CTA, we can see that Expert 2 has a similar way to approach the problem. It gave us some key insight and we noticed some differences from Expert 1’s proof. Here is the key differences of the proof with some picture below it:

1. The use of another variable to execute mathematical induction
2. The statement of induction hypothesis
3. The idea of graphical representation of mathematical induction

Base case $X = 0$,

$$\sum_{k=0}^{0} k = 0 = \frac{0(1)}{2} = \frac{X(X+1)}{2}$$

Figure 3.1: The use of X for executing mathematical induction from Expert 2

Induction step, assume that $X := n$ holds,
we have to show that it also holds for $X := n+1$.

Figure 3.2: The statement of induction hypothesis from Expert 2

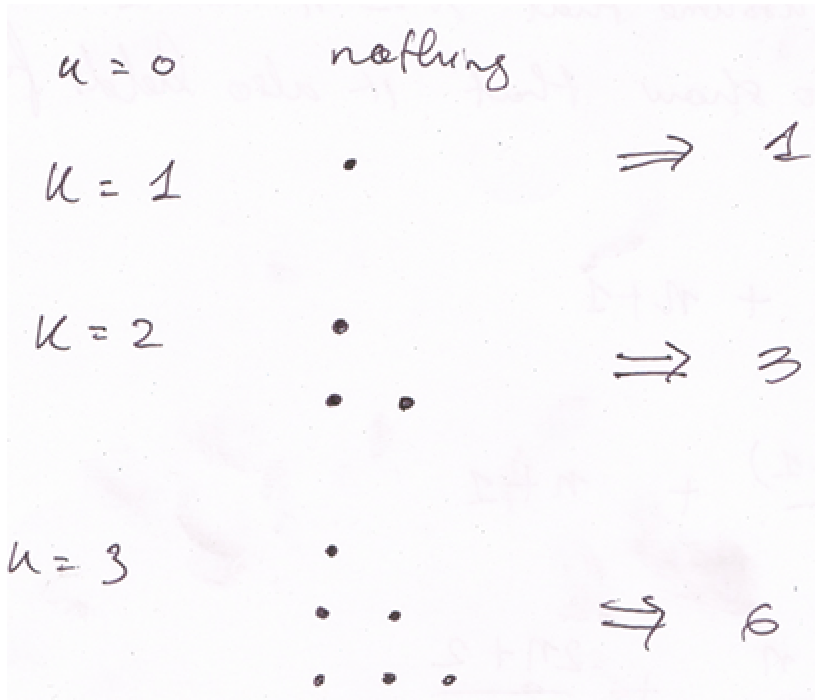


Figure 3.3: Graphical representations of mathematical induction

From these differences, we got the structural differences of mathematical induction that we are looking for. The induction hypothesis from Expert 2 is aligned with what we see in many sources. This is important because we want to show the reason we do the induction step to people who read the proof. Another distinction is the variable that we use in mathematical induction proof. This is just minor details but in order to show that any variable that represents a natural number works, we need to define this. We also found that Expert 2 didn't formulate the statement from the second question. This shows that reformulation of statements is not necessary, but helpful since it avoids misinterpretation of the statement most of the time.

The next one is the use of graphical representations to do mathematical induction. Some proof might need this because it is not always the case that we do induction over natural numbers only. So we use graphical representation to show mathematical induction works over the structure of the given problem. This is the gateway to structural induction and this could give users more insight in the future, especially learning other types of induction method.

Follow-up:

After the CTA, we asked Expert 2 some follow-up questions, which were also asked to

Expert 1. He mentioned that there is generally no issue in learning mathematical induction, except that students often lack intuition for the induction step. Once that intuition develops, the process becomes much easier. He also noted that a common mistake students make in writing these proofs is failing to justify every step in the induction process.

Regarding the prior knowledge needed to learn this type of proof, he stated that virtually none is required for students with a high school diploma.

Expert 2 believes that mathematical induction can be applied to almost anything. However, he emphasized the importance of learning other proof techniques to gain flexibility in proof writing. If he encounters a problem he cannot solve using induction, he tries other methods. If induction is necessary, he revisits the problem for a new perspective. He did not have any specific advice in this regard.

Expert 2 also mentioned that he has never questioned the validity of his mathematical induction proofs.

3. Results and Implementation of Our Findings from CTA for Design of Instruction in ProveIt

After analyzing the CTA, we have concluded that our findings align with our initial design of the app. We discovered from both CTAs the following points:

1. Structure in mathematical induction is very helpful for users to know the goal of the proof and what step they should do.
2. Statement of Induction Hypothesis is often forgotten even by experts but helps the user to remember and to know what they need to show and can use.
3. Every small step in the induction step must have reasoning, and these reasonings need to be specified.
4. Both experts agree that the hardest part, in which people often produce mistakes, is the induction step.
5. Graphical representation is a way to visualize mathematical induction and can help to detect some patterns to apply mathematical induction to other problems.
6. In order to master using mathematical induction, it is important to get used to the intuition of using the induction hypothesis and planning the induction step.
7. Reformulating the statement with mathematical notation can help users to find ideas or intuitions for the proof.
8. Correctness of the proof is implied by the structure and showing the n+1 case.

From the points above, our initial idea of pedagogical principles that we want to use can be implemented in the app. In the project description, we wrote our intention to use constructivism, fading scaffolding, step-by-step scaffolding, and emotional encouragement.

Constructivism will be implemented in the app by giving users the explanation with some small interactivity and exercises in the app. We will explain the concept of mathematical induction and ask the user to participate in finding out the way to engage them in the material. After that, we will give them some exercises to help them master the concept and structure. At the end, there will be a final exercise where they will write the proof by themselves.

Since mathematical induction proof has structure to it, we will use step-by-step scaffolding to help them get used to the structure. We will dissect the proof into 3 parts based on the proof structure: base case, induction hypothesis, and induction step. We will gradually remove this structure. At the final exercises, the user will write their own proof.

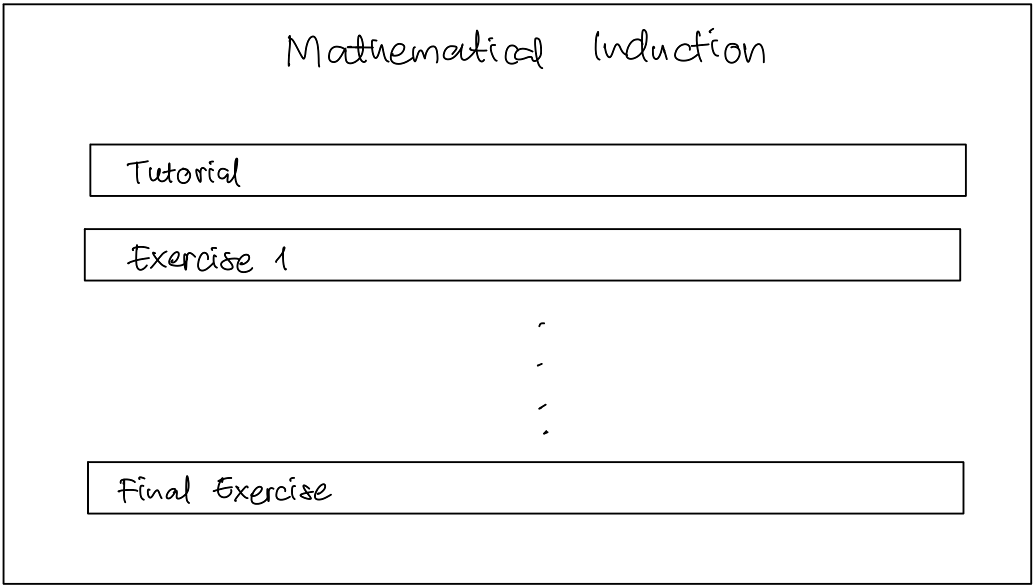
In our implementation of fading scaffolding, we will use this principle for the induction step. Because of the 4th point of our findings, we intended to reduce the cognitive load of the user by providing them with possible options on each step of the induction step. We will change the form of the option slightly in order to induce reasoning on each step they made in the induction step part.

To provide emotional encouragement, we recognize that many students struggle to learn this new mathematical concept. In order to support them, we introduce our box character to show emotional encouragement and companionship to guide them during solving exercises that we provided. We are also going to give hints from this character that encourage users to keep moving forward with the exercises.

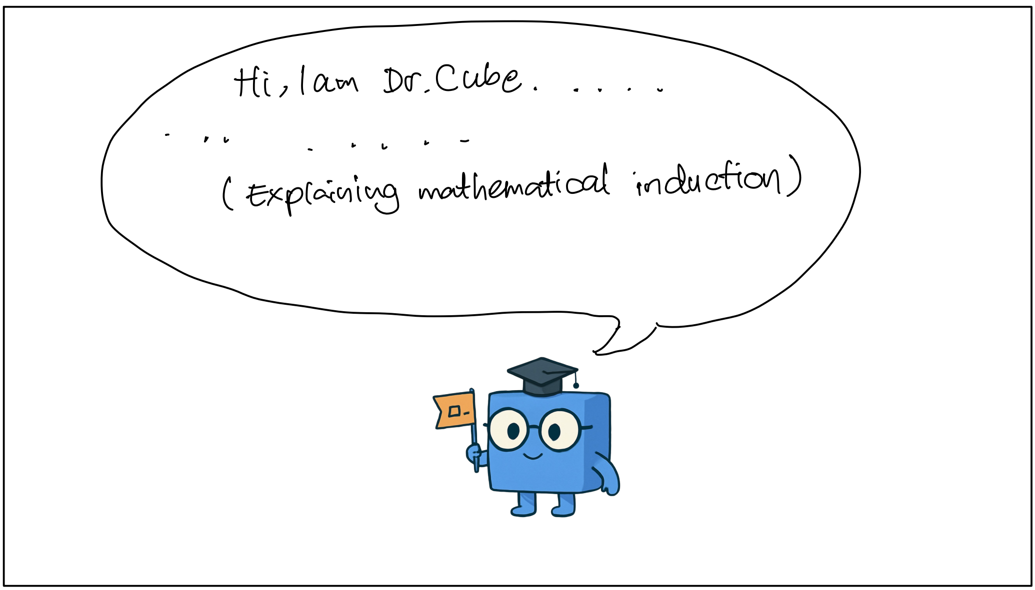
Some of our findings cannot be implemented due to the time constraint of this project. Graphical representations of proof concepts can be a visual helper for users to understand these concepts. This will be included in the future development of the app.

4. Low Fi Prototype for ProveIt

We designed our app based on the findings and present on the following pages our low-fi prototype. Some remarks about the prototype: The box agent is designed with ChatGPT, and we will try to design something similar to this character. The green box or text is user input. We will explain our design features after the images.



Page 1 : Mathematical Induction Main Page



Page 2.1 : Tutorial Introduction by Dr. Cube

> Tutorial

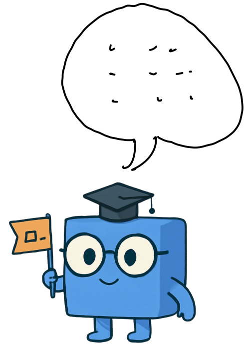
Q.E.D

We proof the statement by mathematical induction

Base Case:

$n = 1$

$\sum_{k=1}^{1} k = 1 = \ldots$



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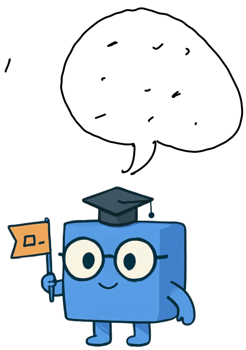
Page 2.2 : Base Case

Q.E.D

Induction hypothesis:

We assume the statement holds for $k = n$,
now we show it holds also for $k = n+1$



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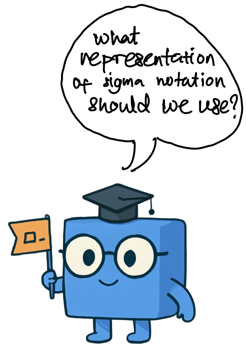
Page 2.3 : Induction Hypothesis

Induction Step :

Q.E.D

$$\sum_{k=1}^{n+1} k = \boxed{}$$



$\sum_{k=1}^{n} k + n+1$	$n+1 + n + \ldots + 1$
option 1	option 2

continue ▷

Page 2.4 : Induction Step

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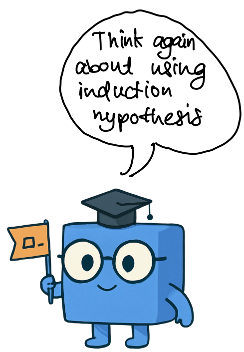
induction Hypothesis : assume . . . .

Q.E.D

Induction Step :

$$\sum_{k=1}^{n+1} k = \boxed{}$$



$\sum_{k=1}^{n} k + n+1$	$n+1 + n + \ldots + 1$
option 1	option 2

continue ▷

Page 2.4a : Induction Step Wrong Input

Induction Hypothesis : assume . . . . . . ,

Q.E.D

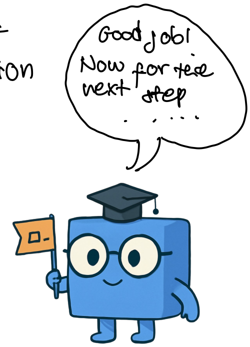
Induction Step :

$$\sum_{k=1}^{n+1} k = \boxed{\sum_{k=1}^{n} k + n+1}$$ | Definition of Sigma notation

$$= \boxed{}$$

$\boxed{\frac{(n+1).n}{2} + n+1}$ option 1

$\boxed{\sum_{k=1}^{n-1} k + n + n+1}$ option 2



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Page 2.4b: Induction step right input

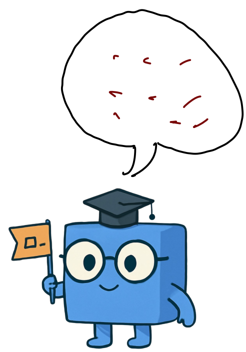
Induction Step :

Q.E.D

$$\sum_{k=1}^{n+1} k = ...$$

$$= . . . .$$

$$= \frac{(n+1)(n+2)}{2}$$



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Page 2.3 : End of the proof

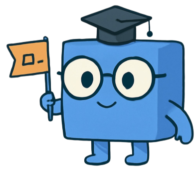
Induction Step :

Q.E.D

$$\sum_{k=1}^{n+1} k = \ldots\ldots$$

$$= \ldots\ldots$$

$$= \frac{(n+1)(n+2)}{2}$$



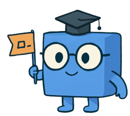
Page 2.2b: End of the proof but user chose to continue

Induction Step :

$$\sum_{k=1}^{n+1} k = \ldots\ldots$$

$$= \ldots\ldots$$

$$= \frac{(n+1)(n+2)}{2} \quad \square$$



Page 2.3 : End of the proof right inputs

> Exercise 1

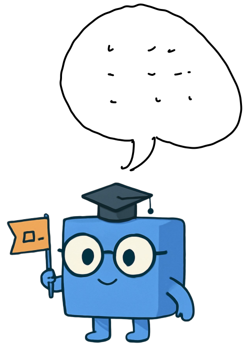
Q.E.D

We proof the statement by mathematical induction

Base Case:

$n = 1$

$$\sum_{k=1}^{1} k = 1 = \ldots$$



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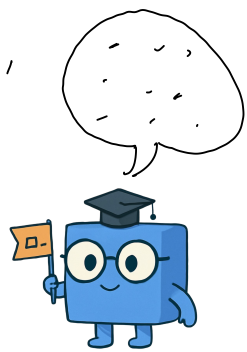
Page 3.1: Base Case

Q.E.D

Induction hypothesis:

We assume the statement holds for $k=n$,
now we show it holds also for $k=n+1$



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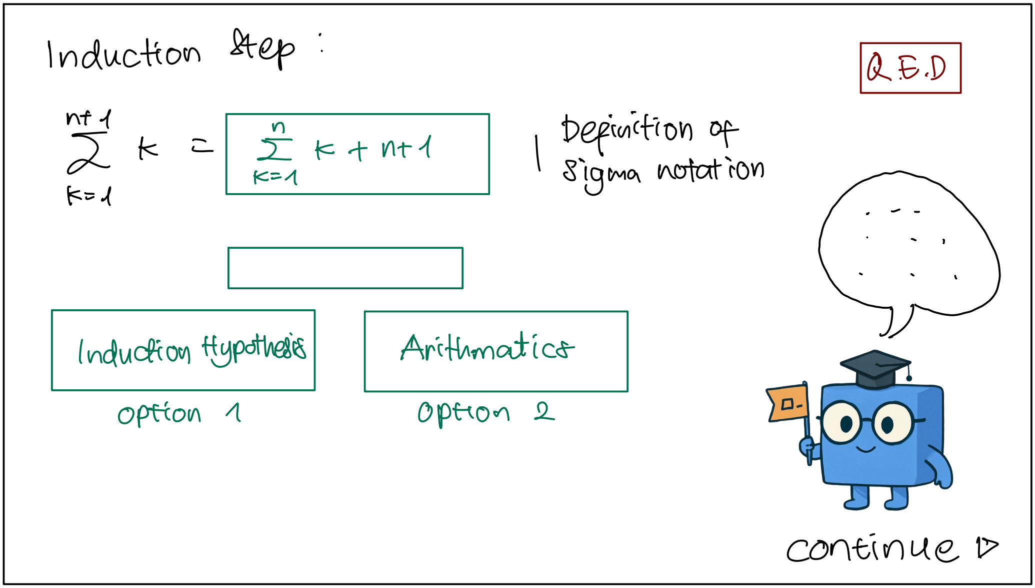
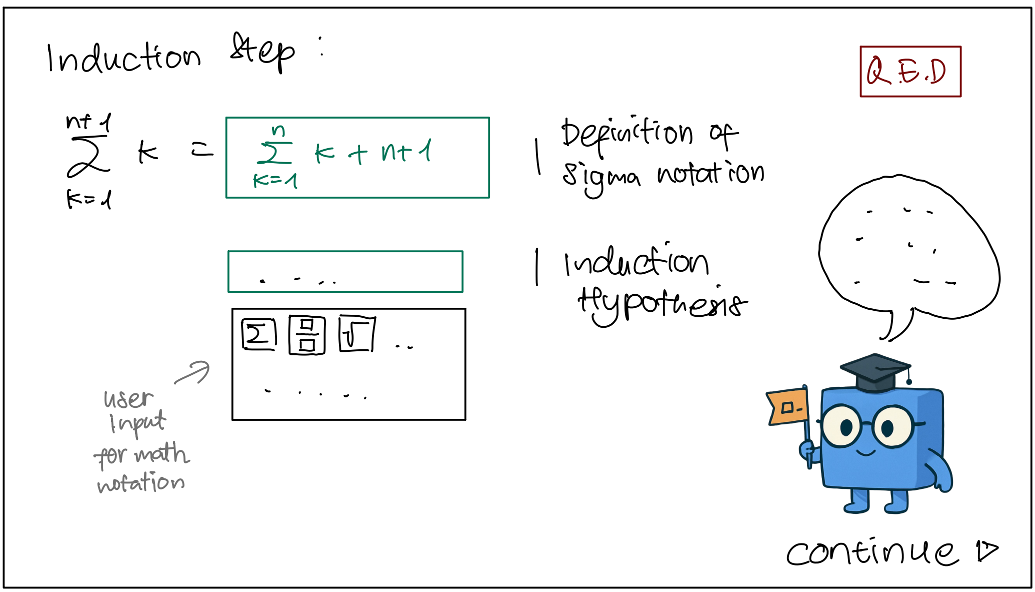
Page 3.2: Induction Hypothesis



Page 3.3 : Induction step with scaffolded option



Page 3.4 : Induction step with user own input and correct option chosen

The pages with 2.x numbering represent the tutorial section, where users are introduced to the concept of mathematical induction. In this section, we do not require users to make many decisions. At each step, users are given the option to decide whether the proof is complete (by clicking Q.E.D.) or whether it requires additional steps (by clicking Continue). This design encourages users to independently determine when a proof is complete.

We also ask users to choose the first step of the induction process. While they may already have an idea of what it is, this interaction serves to familiarize them with the app and promote the strategy of using the induction hypothesis in the inductive step.

The pages with 3.x numbering present the layout for the exercises. These pages retain the same core functionality as the tutorial but incorporate fading scaffolding to guide users through the reasoning of each step, as previously mentioned. This is not the only scaffolding method we use; we adapt the options based on the specific problem set.

For example, on page 3.4, users are tasked with writing a transformation based on the option they selected earlier. This becomes more prominent in the final exercises, where users are required to write the entire proof without assistance from Dr. Cube.

The goal of this design is to gradually reduce the cognitive load associated with writing proofs independently while enhancing users' understanding of each reasoning step. Ultimately, we want users to become independent of Dr. Cube's hints and the app's scaffolding. Our target audience is university students, so we focus on the app's effectiveness in transforming knowledge and training users to become independent problem solvers.

Acknowledgement

We wrote this report with the help of ChatGPT to fix grammatical errors and word choices. We thank both of our experts also for their time and participation on CTA.