Optimization HW 1 Sept 19 Due

er2978

September 2022

1 Question 1

a $\|Cy + d\|_2^2 = (Cy + d)^T(Cy + d) = y^T(C^TC)y + y^TC^Td + d^TCy + d^Td$.
Let the symmetric matrix $C^TC = B$ with $b_{ij} \in B$.

b $y^T(C^TC)y + y^TC^Td + d^TCy + d^Td =:$
$\sum_{i=1}^{n}\sum_{j=1}^{n} y_j b_{ij} y_i + \sum_{j=1}^{n}\sum_{i=1}^{m} y_j c_{ji} d_i + \sum_{j=1}^{m}\sum_{i=1}^{n} d_j c_{ij} y_i + \sum_{i=1}^{m}\sum_{j=1}^{m} d_j d_i$.
$\frac{\partial g}{\partial y_i} =:$
$\frac{\partial}{\partial y_i}\sum_{i=1}^{n}\sum_{j=1}^{n} y_j b_{ji} y_i = \sum_{j=1}^{n} y_j b_{ji} + \sum_{i=1}^{n} b_{ij} y_j =$

$$\begin{bmatrix} \sum_{j=1}^{n} y_j b_{j1} \\ \sum_{j=1}^{n} y_j b_{j2} \\ \vdots \\ \sum_{j=1}^{n} y_j b_{jn} \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n} b_{1j} y_j \\ \sum_{j=1}^{n} b_{2j} y_j \\ \vdots \\ \sum_{j=1}^{n} b_{nj} y_j \end{bmatrix} = 2By = 2C^TCy$$

$\frac{\partial}{\partial y_i}\sum_{j=1}^{n}\sum_{i=1}^{m} y_j c_{ji} d_i =$

$$\begin{bmatrix} \sum_{j=1}^{n}\sum_{i=1}^{m} c_{1i}d_1 \\ \sum_{j=1}^{n}\sum_{i=1}^{m} c_{2i}d_2 \\ \vdots \\ \sum_{j=1}^{n}\sum_{i=1}^{m} c_{mi}d_m \end{bmatrix} = C^T d$$

$$\sum_{j=1}^{m}\sum_{i=1}^{n} d_j c_{ij} y_i =$$

$$\begin{bmatrix} \sum_{j=1}^{n}\sum_{i=1}^{m} c_{1i}d_1 \\ \sum_{j=1}^{n}\sum_{i=1}^{m} c_{2i}d_2 \\ \vdots \\ \sum_{j=1}^{n}\sum_{i=1}^{m} c_{mi}d_m \end{bmatrix} = C^T d$$

$\frac{\partial^2 g}{\partial y_i \partial y_j} g(y) = \frac{\partial}{\partial y_j}(2By) = \frac{\partial}{\partial y_j}(2C^T Cy) =$ partial with respect to y_j of:

$$\begin{bmatrix} \sum_{j=1}^{n} y_j b_{j1} \\ \sum_{j=1}^{n} y_j b_{j2} \\ \vdots \\ \sum_{j=1}^{n} y_j b_{jn} \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n} b_{1j} y_j \\ \sum_{j=1}^{n} b_{2j} y_j \\ \vdots \\ \sum_{j=1}^{n} b_{nj} y_j \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} b_{j1} \\ \sum_{j=1}^{n} b_{j2} \\ \vdots \\ \sum_{j=1}^{n} b_{jn} \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{n} b_{1j} \\ \sum_{j=1}^{n} b_{2j} \\ \vdots \\ \sum_{j=1}^{n} b_{nj} \end{bmatrix}$$

$= BB^T = 2B = 2C^T C$ since B is a symmetric matrix.

c $\nabla(g(y)) = 2By + 2C^T d = 2C^T(Cy + d)$.
$\nabla^2(g(y)) = 2B = 2C^T C$

d $g(y+h) = (y^T + h^T)(C^T C)(y+h) + (y^T + h^T)C^T d + d^T C(y+h) + d^T d =$
$[\mathbf{y^T(C^T C)y} + y^T(C^T C)h + h^T(C^T C)y + h^T(C^T C)h]$
$+[\mathbf{y^T C^T d} + h^T C^T d]$
$+[\mathbf{d^T C y} + d^T C h]$
$+[\mathbf{d^T d}]$

$\Rightarrow \mathbf{y^T(C^TC)y + y^TC^Td + d^TCy + d^Td} = g(y)$
$\Rightarrow y^T(C^TC)h + h^T(C^TC)y + h^TC^Td + d^TCh = E(y)$
$\Rightarrow h^T(C^TC)h = F(y)$
$E(y) = y^T(C^TC)h + h^T(C^TC)y + h^TC^Td + d^TCh = 2C^T(Cy + d) = \nabla(g(y))$
$F(y) = h^T(C^TC)h = \frac{1}{2}\nabla^2 g(y) \implies \nabla^2 g(y) = 2C^TC$

2 Question 2

a $f(x, y)$ achieves a minimum each time $cos(\frac{\pi x}{2}) = 1$ since the y term is always positive (since it's squared.) Therefore $\beta = \{(0, 1), (0, -1)\}$ is the set of minimizers of function $f(x, y)$ on B and $f(x, y) = -1$ at those points.

b (Check Jupyter Doc)

c (Check Jupyter Doc)

d The trajectories look like linear versions of the paths a ball takes to settle in the middle of a a bin when it's rolled down the side; they jump from one side to the next, but they inevitably settle somewhere down between the 4 corners.

e (Check Jupyter Doc) I notice that all the trajectories make a beeline to the center of the contour map, between the 4 quarter-circles in the corner.