NUMERICAL ANALYSIS

MATLAB Practicals (Odd Semester, 2022-23)
B.E. III Semester
Thapar Institute of Engineering & Technology
Patiala

Name:................
Roll No:................
Group:..................
Instructor:................

Contents

Experiment 1: Bisection and Fixed-Point Iteration Methods

1. **Algorithm of Bisection method:** To determine a root of $f(x) = 0$ that is accurate within a specified tolerance value ϵ, given values a and b such that $f(a)\ f(b) < 0$.
   Define $c = \dfrac{a+b}{2}$.
   If $f(a)\ f(c) < 0$, then set $b = c$, otherwise $a = c$.
   End if.
   Until $|a - b| \leq \epsilon$ (tolerance value).
   Print root as c.
   Stopping Criteria: Since this is an iterative method, we must determine some stopping criteria that will allow the iteration to stop. Criterion $|f(c_k)|$ very small can be misleading since it is possible to have $|f(c_k)|$ very small, even if c_k is not close to the root.
   The interval length after N iterations is $\dfrac{b-a}{2^N}$. So, to obtain an accuracy of ϵ, we must have
   $$N \geq \frac{\log(b-a) - \log \epsilon}{\log 2}.$$

2. **Algorithm of fixed-point iterations:** To find a solution to $x = g(x)$ given an initial approximation x_0.
   Input: Initial approximation x_0, tolerance value ϵ, maximum number of iterations N.
   Output: Approximate solution α or message of failure.
   Step 1: Set $i = 1$.
   Step 2: While $i \leq N$ do Steps 3 to 6.
   Step 3: Set $x_1 = g(x_0)$. (Compute x_i.)
   Step 4: If $|x_1 - x_0| \leq \epsilon$ or $\dfrac{|x_1 - x_0|}{|x_1|} < \epsilon$ then OUTPUT x_1; (The procedure was successful.)
   STOP.
   Step 5: Set $i = i + 1$.
   Step 6: Set $x_0 = x_1$. (Update x_0.)
   Step 7: Print the output and STOP.

3. Students are required to write both the programs and implement them on the following examples:
   (i) Use bisection method for computing the value of $\sqrt{29}$.
   (ii) Find smallest positive root of $\cos x = 1/2 + \sin x$ with $\epsilon = 0.00$.
   (iii) Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using $a = 1$ and $b = 2$ and hence find the root with desired accuracy.
   (iv) Find smallest and second smallest positive roots of the equation $\tan x = 4x$, with an accuracy of 10^{-3} using fixed-point iterations.
   (v) Use fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2\sin \pi x + x = 0$ in $[1, 2]$. Take initial guess as $x_0 = 1$.

   Sol.

4. The equation $f(x) = x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation. Let g_1, g_2, g_3, g_4, g_5 are iteration functions obtained by the given function, then check which of the following iteration functions will converge to the fixed point?
(a) $g_1(x) = x - x^3 - 4x^2 + 10$
(b) $g_2(x) = \sqrt{\frac{10}{x} - 4x}$
(c) $g_3(x) = 0.5\sqrt{10 - x^3}$
(d) $g_4(x) = \sqrt{\dfrac{10}{4 + x}}$
(e) $g_5(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$.

Sol.

Experiment 2: Newton's and Secant Methods

1. Write an algorithm for Newton's and Secant method to compute the roots of a given non-linear equation.
   Algorithm for Newton's method: To find a solution to $f(x) = 0$, given an initial approximation x_0.
   Input: Initial approximation x_0, tolerance value ϵ, maximum number of iterations N.
   Output: Approximate solution x_1 or message of failure.
   Step 1: Set $i = 1$.
   Step 2: While $i \leq N$ do Steps 3 to 6.
   Step 3: Set $x_1 = x_0 - \dfrac{f(x_0)}{df(x_0)}$. (Compute x_i.)
   Step 4: If $|x_1 - x_0| \leq \epsilon$ or $\dfrac{|x_1 - x_0|}{|x_1|} < \epsilon$ then OUTPUT x_1; (The procedure was successful.) STOP.
   Step 5: Set $i = i + 1$.
   Step 6: Set $x_0 = x_1$. (Update x_0.)
   Step 7: Output ('The method failed after Niterations, N=', N); (The procedure was unsuccessful.) STOP.
   Algorithm for secant method:
   1. Give inputs and take two initial guesses x_0 and x_1.
   2. Start iterations
   $$x_2 = x_1 - \frac{x_1 - x_0}{f_1 - f_0} f_0.$$
   3. If
   $$|x_2 - x_1| < \epsilon \text{ or } \frac{|x_2 - x_1|}{|x_2|} < \epsilon$$
   then stop and print the root.
   4. Repeat the iterations (step 2). Also check if the number of iterations has exceeded the maximum number of iterations.

2. Students are required to write a program and implement it on the following examples:
   (i) Use secant method for computing $\sqrt{17}$.
   (ii) Use Newton's method to find the indicated root of the following equations. Use an error tolerance of $\epsilon = 0.00001$.
   (a) Find the root of $\cos x - x\exp(x) = 0$
   (b) The smallest positive root of $\cos x = 1/2 + \sin x$.
   (c) The root of $\exp(-x)(x^2 + 5x + 2) + 1 = 0$. Take initial guess -2.0 and -1.0.
   (iii) Using Newton's Method to find a non-zero solution of $x = 2\sin x$ with an accuracy 10^{-3}.
   (d) Solve the equation $4x^2 - e^x - e^{-x} = 0$ which has two positive solutions x_1 and x_2. Use Newton's method to approximate the solution to within 10^{-5} with the following values of x_0:
   $$x_0 = -10, -5, -3, 0, 1, 3, 5, 10.$$
   Sol.

3. The accumulated value of a savings account based on regular periodic payments can be determined from the annuity due equation,

$$A = \frac{P}{i}[(1+i)^n - 1].$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

Sol.

Experiment 3: Gauss Elimination

1. Write an algorithm for Gauss elimination method to obtain a solution of system of linear equations.
Algorithm for Gauss elimination method: Input: number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \leq i \leq n$ and $1 \leq j \leq n+1$.
Output: solution $x_1, x_2, \cdots, x_n$ or message that the linear system has no unique solution.
Step 1: For $i = 1, \cdots, n-1$ do Steps 2-4. (Elimination process.)
Step 2: Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$.
If no integer p can be found
then OUTPUT ('no unique solution exists');
STOP.
Step 3: If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$.
Step 4: For $j = i+1, \cdots, n$ do Steps 5 and 6.
Step 5: Set $m_{ji} = a_{ji}/a_{ii}$.
Step 6: Perform $(E_j - m_{ji}E_i) \rightarrow (E_j)$;
Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists');
STOP.
Step 8: Set $x_n = a_{n,n+1}/a_{nn}$.(Start backward substitution.)
Step 9: For $i = n-1, \cdots, 1$ set $x_i = [a_{i,n+1} - \sum\limits_{j=i+1}^{n} a_{ij}x_j]/a_{ii}$.
Step 10: OUTPUT $(x_1, \cdots, x_n)$; (Procedure completed successfully.)
STOP.

2. Use Gauss elimination method to find the solution of the following system of equations:

$$\begin{aligned} 10x + 8y - 3z + u &= 16 \\ 2x + 10y + z - 4u &= 9 \\ 3x - 4y + 10z + u &= 10 \\ 2x + 2y - 3z + 10u &= 11 \end{aligned}$$

Sol.

3. Solve the following linear system:

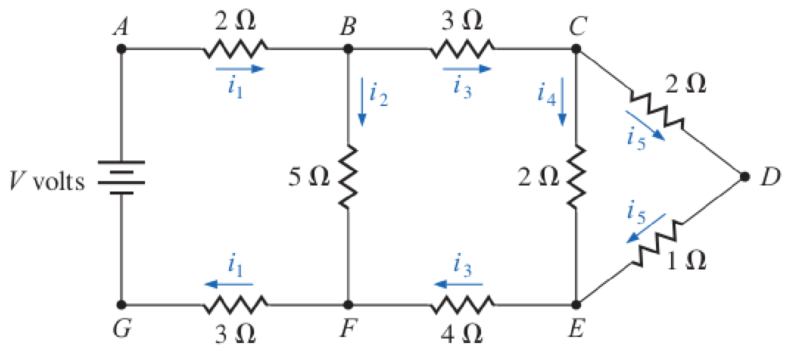
$$\begin{aligned} \pi x_1 + \sqrt{2}x_2 - x_3 + x_4 &= 0 \\ e x_1 - x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - \sqrt{3}x_3 + x_4 &= 2 \\ -x_1 - x_2 + x_3 - \sqrt{5}x_4 &= 3. \end{aligned}$$

Sol.

4. Kirchhoff's laws of electrical circuits state that both the net flow of current through each junction and the net voltage drop around each closed loop of a circuit are zero. Suppose that a potential of V volts is applied between the points A and G in the circuit and that i_1, i_2, i_3, i_4, and i_5 represent current flow as shown in the diagram. Using G as a reference point, Kirchhoff's laws imply that the currents satisfy the following system of linear equations:

$$\begin{aligned} 5i_1 + 5i_2 &= V, \\ i_3 - i_4 - i_5 &= 0, \\ 2i_4 - 3i_5 &= 0, \\ i_1 - i_2 - i_3 &= 0, \\ 5i_2 - 7i_3 - 2i_4 &= 0. \end{aligned}$$

By taking $V = 5.5$, solve the system.



Sol.

Experiment 4: LU Factorization

1. Write an algorithm for LU Factorization using Gauss elimination method.
Algorithm for LU Factorization: To factor the $n \times n$ matrix $A = [a_{ij}]$ into the product of the lower-triangular matrix $L = [l_{ij}]$ and the upper-triangular matrix $U = [u_{ij}]$; that is, $A = LU$, where the main diagonal of either L or U consists of all ones:
INPUT: dimension n; the entries $a_{ij}, 1 \leq i, j \leq n$ of A; the diagonal $l_{11} = \cdots = l_{nn} = 1$ of L or the diagonal $u_{11} = \cdots = u_{nn} = 1$ of U.
OUTPUT: the entries $l_{ij}, 1 \leq j \leq i, 1 \leq i \leq n$ of L and the entries, $u_{ij}, i \leq j \leq n$, $1 \leq i \leq n$ of U.
Step 1: Select l_{11} and u_{11} satisfying $l_{11}u_{11} = a_{11}$.
If $l_{11}u_{11} = 0$ then OUTPUT ('Factorization impossible');
STOP. Step 2: For $j = 2, \cdots, n$ set $u_{1j} = a_{1j}/l_{11}$; (First row of U.)

$$l_{j1} = a_{j1}/u_{11}. \quad \text{(First column of L.)}$$

Step 3: For $i = 2, \cdots, n-1$ do Steps 4 and 5.
Step 4: Select l_{ii} and u_{ii} satisfying $l_{ii}u_{ii} = a_{ii} - \sum_{k=1}^{i-1} l_{ik}u_{ki}$.
If $l_{ii}u_{ii} = 0$ then OUTPUT ('Factorization impossible');
STOP.
Step 5: For $j = i+1, \cdots, n$ set $u_{ij} = \frac{1}{l_{ii}}[a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj}]$ (ith row of U.)
$l_{ji} = \frac{1}{l_{ui}}[a_{ji} - \sum_{k=1}^{i-1} l_{jk}u_{ki}]$ (ith column of L.)
Step 6: Select l_{nn} and u_{nn} satisfying $l_{nn}u_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk}u_{kn}$.
(Note: If $l_{nn}u_{nn} = 0$, then $A = LU$ but A is singular.)
Step 7: OUTPUT (l_{ij} for $j = 1, \cdots, i$ and $i = 1, \cdots, n$);
OUTPUT (u_{ij} for $j = i, \cdots, n$ and $i = 1, \cdots, n$);
STOP.

2. Factor the following matrices using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

   (a)
$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}.$$

   (b)
$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{5} & \frac{2}{3} & \frac{3}{8} \\ \frac{2}{3} & -\frac{2}{3} & \frac{5}{8} \end{bmatrix}.$$

   Sol.

Experiment 5: Gauss-Seidel and Successive-Over-Relaxation Methods

1. **Algorithm:** Given a system of linear equations:

$$Ax = b.$$

In order to obtain the solution of the above system of equations by Gauss-Seidel method, we use the following algorithm:

1 Input matrix $A = [a_{ij}], b, x0$, tolerance TOL, maximum number of iterations

2 Set $k = 1$

3 while $(k \leq N)$ do step 4-7

4 For $i = 1, 2, \cdots, n$

$$x_i = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j) - \sum_{j=i+1}^{n}(a_{ij}x0_j) + b_i)\right]$$

5 If $||x - x0|| < TOL$, then OUTPUT $(x_1, x_2, \cdots, x_n)$
STOP

6 $k = k + 1$

7 For $i = 1, 2, \cdots, n$
Set $x0_i = x_i$

8 OUTPUT $(x_1, x_2, \cdots, x_n)$
STOP.

2. Write an algorithm for Successive-Over-Relaxation (SOR) method.

3. Solve this system of equations by Gauss-Seidel starting with the initial vector [0,0,0]:

$$\begin{aligned} 4.63x_1 - 1.21x_2 + 3.22x_3 &= 2.22 \\ -3.07x_1 + 5.48x_2 + 2.11x_3 &= -3.17 \\ 1.26x_1 + 3.11x_2 + 4.57x_3 &= 5.11. \end{aligned}$$

   Sol.

4. Use the SOR method with $\omega = 1.2$ to solve the linear system with a tolerance 10^{-3} in the $\|.\|_\infty$ norm.

$$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2 \\ x_1 + 4x_2 - x_3 - x_4 &= -1 \\ -x_1 - x_2 + 5x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + 3x_4 &= 1. \end{aligned}$$

   Sol.

Experiment 6: Power Method

1. **Algorithm:**
   (a) Start
   (b) Define matrix A and initial guess x
   (c) Calculate $y = Ax$
   (d) Find the largest element in magnitude of matrix y and assign it to K.
   (e) Calculate fresh value $x = (1/K) * y$
   (f) If $|K(n) - K(n-1)| >$ error, goto step c.
   (g) Stop

2. Determine the largest eigenvalue and the corresponding eigenvector of the matrix

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 20 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

   using the power method. Use $x^0 = [1, 1, 1]^T$.

   Sol.

3. Find the first three iterations obtained by the Power method applied to the following matrix.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Use $x^0 = [1, 1, 0, 1]^T$.

Solution.

Experiment 7: Lagrange Interpolation

1. **Algorithm:** Given a set of function values:

x	x_1	x_2	$\cdots$	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

To approximate the value of a function $f(x)$ at $x = p$ using Lagrange's interpolating polynomial $P_n(x)$ of degree n, given by

$$P_n(x) = l_1(x)f(x_1) + l_2(x)f(x_2) + \ldots + l_n(x)f(x_n)$$

where $l_i(x) = \prod_{j=1;i\neq j}^{n} \frac{(p - x_j)}{(x_i - x_j)}$.
We write the following algorithm by taking n points and thus we will obtain a polynomial of degree $\leq n - 1$:

- Input: The degree of the polynomial, the values $x(i)$ and $f(i)$, $i = 1, 2, \ldots, n$, and the point of interpolation p.
- Calculate the Lagrange's fundamental polynomials $l_i(x)$ using the following loop:
  `for i=1 to n`
  $l(i) = 1.0$
  `for j=1 to n`
  `if` $j \neq i$
  $l(i) = \dfrac{p - x(j)}{x(i) - x(j)} l(i)$
  `end j`
  `end i`
- Calculate the approximate value of the function at $x = p$ using the following loop:
  `sum=0.0`
  `for i=1 to n`
  `sum = sum +` $l(i) * f(i)$
  `end i`
- Print `sum`.

2. Use Lagrange's interpolation formula to approximate the value of $f(0.43)$, given that

x	0	0.25	0.5	0.75
$f(x)$	1	1.64872	2.71828	4.48169

Sol.

3. A census of the population of the United States is taken every 10 years. The following table lists the population, in thousands of people, from 1950 to 2000, and the data are also represented in the figure.

Year	1950	1960	1970	1980	1990	2000
Population(in thousands)	151326	179323	203302	226542	249633	281422

Use Lagrange interpolation to approximate the population in the years 1965, 1975, and 1995.

Sol.

Experiment 8: Newton's divided difference Interpolation

1. **Algorithm** Write the algorithm for Newton's divided difference interpolation and then apply for the given examples on next page.

2. The following data represents the function $f(x) = e^x$.

x	1	1.5	2.0	2.5
$f(x)$	2.7183	4.4817	7.3891	12.1825

Estimate the value of $f(2.25)$ using the Newton's divided difference interpolation. Compare with the exact value.

Sol.

3. Using Newton's divided difference interpolation, construct interpolating polynomials for the following data. Approximate the specified value using each of the polynomials.

$$f(0.43) \text{ if } f(0) = 1, f(0.25) = 1.64872, f(0.5) = 2.71828, f(0.75) = 4.4816.$$

Sol.

Experiment 9: Numerical Quadrature

1. **Algorithm (Composite trapezoidal rule):**
   Step 1 : Inputs: function $f(x)$; end points a and b; and N number of subintervals.
   Step 2 : Set $h = (b - a)/N$.
   Step 3 : Set sum $= 0$
   Step 4 : For $i = 1$ to $N - 1$
   Step 5 : Set $x = a + h * i$
   Step 6 : Set sum $=$ sum$+2 * f(x)$
   end
   Step 7 : Set sum $=$ sum$+f(a) + f(b)$
   Step 8 : Set ans $=$ sum$*(h/2)$
   End

2. **Algorithm (Composite Simpson's rule):**
   Step 1 : Inputs: function $f(x)$; end points a and b; and N number of subintervals (even).
   Step 2 : Set $h = (b - a)/N$.
   Step 3 : Set sum $= 0$
   Step 4 : For $i = 1$ to $N - 1$
   Step 5 : Set $x = a + h * i$
   Step 6 : If rem$(i, 2) == 0$
   sum $=$ sum$+2 * f(x)$
   else
   sum $=$ sum$+4 * f(x)$
   end
   Step 7 : Set sum $=$ sum$+f(a) + f(b)$
   Step 8 : Set ans $=$ sum$*(h/3)$
   End

3. Approximate the following integrals using the composite trapezoidal and Simpson rule by taking different subintervals (e.g. 4, 6, 10, 20). a. $I = \int\limits_{-0.25}^{0.25} (\cos x)^2 \; dx$ b. $\int\limits_{e}^{e+1} \frac{1}{x \ln x} \; dx$. c. $\int_{-1}^{1} e^{-x^2} \cos x \, dx$.
   Sol.

4. The length of the curve represented by a function $y = f(x)$ on an interval $[a, b]$ is given by the integral

$$I = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Use the trapezoidal rule and Simpson's rule with 4 and 8 subintervals compute the length of the curve $y = \tan^{-1}(1 + x^2),\ 0 \le x \le 2$.
Sol.

Experiment 10

1. **Algorithm (Modified Euler's method):**

2. **Algorithm (Runge-Kutta fourth-order method):**

3. Solve the following differential equation by the modified Euler's method

$$y' = -y + 2\cos t,\ y(0) = 1.$$

Compute solution in the interval $[0, 1]$ with mesh length 0.2.
Sol.

4. Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2 \sqrt{2g}\frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft$/s^2$, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)\text{ft}^3$. Use the Runge-Kutta method of order four to find the following.
a. The water level after 10 min with $h = 20$ s.
b. When the tank will be empty, to within 1 min.
Sol.