MCKEAN-SINGER VIA EQUIVARIANT QUANTIZATION

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1. INTRODUCTION

The aim of this note is to present a proof, using the language of factorization algebras, and in particular the index theorem in Chapter 7 of [G12], of the following

Theorem 1.1 (McKean-Singer). *Let V be a Hermitian, $\mathbb{Z}_2$-graded vector bundle on a closed Riemannian manifold M, with $|dx|$ the Riemannian volume form on M. Let D be a self-adjoint Dirac operator on V, with k_t the heat kernel of D^2. Then*

$$\operatorname{ind}(D) = \int_M \operatorname{Str}(k_t(x,x))|dx|. \tag{1.1}$$

The actual McKean-Singer theorem works for non-self-adjoint Dirac operators as well, but our proof will require D to be self-adjoint. We will give definitions of all of the objects in the theorem shortly, but first a bit of philosophy. This theorem gives us a relationship between a global, analytic quantity (the index of a Dirac operator) and a local, physical quantity (the super-trace of a heat kernel). This is what the index theorem is most famous for. We will see that the theorem of Gwilliam is similar in nature: it describes two ways to compute the obstruction to quantizing a field theory equivariantly with respect to the action of an L_∞ algebra. One involves Feynman diagrams (which involve heat kernels), and the other is a global characterization (which will give us the index). This is, very roughly speaking, why we are able to use the theorem relating to field theory to prove an index-type theorem.

2. GENERALIZED LAPLACIANS, HEAT KERNELS, AND DIRAC OPERATORS

We present here a list of definitions and results relevant to the result. Throughout, M is a Riemannian manifold with Riemannian volume form $|dx|$. We let $V \to M$ be a vector bundle, which we will eventually specialize to be $\mathbb{Z}_2$-graded. We let $\mathscr{V}$ be the sheaf of smooth sections of V. We always use normally-fonted letters for vector bundles and scripty letters for the sheaves of sections of the corresponding vector bundles.

Definition 2.1. *A **generalized Laplacian** is a differential operator*

$$H : \mathscr{V}(M) \to \mathscr{V}(M)$$

such that

$$[[H, f], f] = -2|df|^2,$$

where we are thinking of C^∞ functions as operators given multiplication by those functions.

Now we let V be $\mathbb{Z}_2$-graded, and we denote by $V^{\pm}$ the plus or minus graded components of V.

Definition 2.2. *A **Dirac operator** on V is a grading-reversing operator*

$$D : \mathscr{V}^{\pm} \to \mathscr{V}^{\mp}$$

*such that D^2 is a generalized Laplacian. If V is a Hermitian bundle with inner product $(,)$, then we say that D is **self-adjoint** if for all $s, r \in \mathscr{V}$,*

$$\int_M (s, Dr)|dx| = \int_M (Ds, r)|dx|$$

Theorem 2.3 (The Heat Kernel)**.** *Let V be a $\mathbb{Z}_2$-graded vector bundle with Dirac operator D. Write $H := D^2$ for the generalized Laplacian corresponding to D. Then there is a unique **heat kernel** $k \in \Gamma(M \times M \times \mathbb{R}_{>0}, V \boxtimes V^{\vee})$ satisfying:*

(1)
$$\frac{d}{dt}k_t + (H \otimes 1)k_t = 0$$

(2) *For $s \in \Gamma(M, E)$,*
$$\lim_{t\to 0} \int_{y\in M} k_t(x, y)s(y)|dx| = s(x),$$
where the limit is uniform over M and is taken with respect to some norm on V.

The heat kernel is the kernel of the operator e^{-tH} in the sense that

$$\int_{y\in M} k_t(x, y)s(y) = (e^{-tH}s)(x).$$

Definition 2.4. *Let D^+ denote the restriction of a self-adjoint Dirac operator D to the space of positively-graded sections, and similarly for D^-. Then, the **index** $\operatorname{ind}(D)$ of D is $\dim(\ker(D^+)) - \dim(\operatorname{coker}(D^+))$.*

The last definition we need to understand this theorem as stated is

Definition 2.5. *If $\phi : V \to V$ is a grading-preserving endomorphism of the super-vector space V, then the **supertrace** $\operatorname{Str}(\phi)$ is defined to be*

$$\operatorname{Str}(\phi) = \operatorname{Tr}(\phi\mid_{V^+}) - \operatorname{Tr}(\phi\mid_{V^-})$$

With these definitions in place, the statement of the McKean-Singer formula should be comprehensible.

3. Equivariant Quantization of Free Theories

In this section, we assume familiarity with chapter 7 of [G12]. All our notation will match that section, except that we will use Q and Q^{GF} to denote the differential and gauge-fixing operators of the full, cotangent theory. We will apply the general theory there to a specific example, which we now describe. In our context, since we are dealing with a Riemannian manifold, we will always use the Riemannian density to trivialize the bundle of densities.

3.1. **Motivation.** Our goal is to prove formula 1.1 using the techniques of chapter 7 of [G12], but it is worth commenting on the physical setup from which it arises. Given a Hermitian, $\mathbb{Z}_2$-graded vector bundle V and self-adjoint Dirac operator D on $\mathscr{V}$, we can think of these data as specifying a free field theory with space of fields $\mathscr{V}$ and action

$$S = \int (s, Ds)\, |dx|.$$

This action corresponds to the equation of motion $Ds = 0$, which is obviously still satisfied if s is replaced with $(1+\lambda)s$, where $\lambda \in \mathbb{R}$. Thus, the classical theory possesses a scaling symmetry, and we would like to see whether it persists at the quantum level. The obstruction to quantizing this symmetry is called the *scaling anomaly*. We expect to be able to find two ways of computing the scaling anomaly; comparing these two ways should give us formula 1.1.

So, we have an action of the abelian Lie algebra $\mathbb{R}$ (which we may think of as an L_∞-algebra, if we like) on $\mathscr{V}$, given by $\lambda \cdot s = \lambda s$. As it stands, however, we do not have a setup which matches [**?**], where we need a *local* action of an *elliptic* L_∞-algebra. The resolution of this conundrum is provided to us by Lemma 11.1.3.2 in [CG16], which tells us that there is a homotopy equivalence between

(1) the simplicial set of actions of an L_∞-algebra $\mathfrak{g}$ on an elliptic L_∞-algebra $\mathscr{M}$, and
(2) the simplicial set of local actions of $\mathfrak{g} \otimes \Omega^\bullet$ on $\mathscr{M}$.

Thus, we can choose any extension of the action of $\mathbb{R}$ on $\mathscr{V}$ to an action of $\Omega^\bullet$ on $\mathscr{V}$ and the choice will not matter homotopically. We therefore proceed to give one such extension next.

3.2. **The Main Example.** We continue to assume that V and D are as above. Based on the above discussion, we wish to find a local action of $\mathscr{L} = \Omega^\bullet$ on $\mathscr{E}$, where $\mathscr{E}$ is the complex $\mathscr{V}^+ \xrightarrow{D^+} \mathscr{V}^-$ with $\mathscr{V}^+$ in degree 0, and we are thinking of $\Omega^\bullet$ and $\mathscr{E}$ as abelian dg Lie algebras.

To give an action of $\mathscr{L}$ on $\mathscr{E}$, it will suffice to define a map

$$[\cdot,\cdot] : \mathscr{L} \otimes \mathscr{E} \to \mathscr{E},$$

with the elements of $\mathscr{L}$ acting by differential operators and satisfying

(1) The derivation property:

$$Q([X,\phi]) = [dX, \phi] + (-1)^{|X|}[X, Q\phi]$$

(2) The Jacobi identity:

$$[X,[Y,\phi]] = (-1)^{|X||Y|}[Y,[X,\phi]],$$

for all $X, Y \in \Omega^\bullet$ and $\phi \in \mathscr{E}$. This is enough to define an elliptic dg Lie algebra structure on $\mathscr{L} \oplus \mathscr{E}$ fitting into a short exact sequence of dglas

$$0 \to \mathscr{E} \to \mathscr{L} \oplus \mathscr{E} \to \mathscr{L} \to 0.$$

Finally, we will look for an action that is C^∞-linear. In fact, we have the following

Proposition 3.1. *There is a unique C^∞-module map*

$$[\cdot,\cdot] : \mathscr{L} \otimes_{C^\infty} \mathscr{E} \to \mathscr{E}$$

satisfying the above two properties.

Remark: $\Diamond$

Proof. For $f \in C^\infty(M)$, we must have

$$[f, \phi] = f\phi$$

by C^∞-linearity. By the derivation property, we must have

$$[df, \phi] = D^+(f\phi) - f(D^+\phi).$$

By $C^\infty(M)$-linearity, this fixes the action of all 1-forms on $\mathscr{E}$, since all 1-forms can be written locally as sums of forms like fdg for $f, g \in C^\infty(M)$. The brackets of all higher forms on elements of $\mathscr{E}$ must vanish for degree reasons.

For two smooth functions $f, g \in C^\infty(M)$, the Jacobi identity requires just that the actions of f and g on $\mathscr{E}$ commute, which is obvious. For $f \in C^\infty(M), gdh \in \Omega^1(M)$, the Jacobi identity is also satisfied because it is easily verified that the operator $[D, h]$ is $C^\infty(M)$-linear. $\square$

Thus, $\mathscr{E}$ provides a local representation for $\mathscr{L}$. Continuing to transcribe the general setup of [G12] to this situation, we note that we have a single interaction term

$$I(X, \varphi, \psi) = \langle \psi, [X, \varphi] \rangle,$$

where $X \in \Omega^\bullet$, $\psi \in \mathscr{E}$, and $\varphi \in \mathscr{E}^![-1]$.

Another way to think about this setup is to think of X as providing a deformation of the complex $(\mathscr{V}, D^+)$ with "differential" $D^+ + [X, \_]$. This operator will be degree +1 if X lives in degree 1 in $\mathscr{L}$ and will square to zero if

$$(D^+)^2\phi + D^+[X, \phi] + [X, D^+\phi] + [X, [X, \phi]] = [dX, \phi] = 0,$$

where we have used both properties of a local representation in the penultimate equality. Thus, for every closed degree 1 element X of $\mathscr{L}$, we have another elliptic complex $(\mathscr{E}, D^+ + [X, \_])$.

Now is the right time to say something about the Feynman diagrammatic way to describe the situation. We should think of the term

$$\langle \psi, [X, \varphi] \rangle$$

as corresponding to a trivalent vertex that we can put in graphs, with one half-edge corresponding to an element of $\mathscr{L}$, one to $\mathscr{E}^![-1]$, and one to $\mathscr{E}$. (see figure 3.2).

Remark (Going Under the Hood)*:* Now that we have an action of $\mathscr{L}$ on $\mathscr{E}$, we would actually like to study the action of $\mathscr{L}$ on $T^*[-1]\mathscr{E}$, which has a natural structure of a free theory. We want to be explicit about the pairing $\langle \cdot, \cdot \rangle$, as well as the operators Q

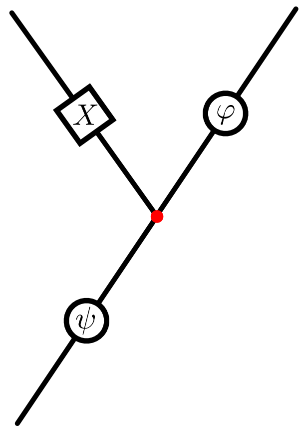

FIGURE 1. The single vertex in our theory. It corresponds to the interaction $\langle \psi, [X, \varphi] \rangle$.

and Q^{GF} in the cotangent theory to $\mathscr{E}$. The cotangent theory has space of fields

$$\begin{array}{ccc} \mathscr{V}^{+,0} & \xrightarrow{D^+} & \mathscr{V}^{-,1} \\ \oplus & & \oplus \\ \mathscr{V}^{-,0} & \xrightarrow{T} & \mathscr{V}^{+,1}, \end{array}$$

where we have used the metric $(\cdot, \cdot)$ to identify V with $V^\vee$, and the Riemannian density to trivialize the density bundle. Once we specify the pairing $\langle , \rangle$ for our cotangent theory, T will be determined by the requirement that Q be skew-self-adjoint.

If $\psi \in \mathscr{V}^!$, $\varphi \in \mathscr{V}$, we let $\langle \psi, \varphi \rangle$ be the natural "pair and integrate" pairing. We note that this pairing comes with a minus sign when $\psi \in \mathscr{V}^{!-}$ and $\varphi \in \mathscr{V}^-$. The anti-symmetry of $\langle , \rangle$ requires that $\langle \varphi, \psi \rangle = -\langle \psi, \varphi \rangle$. To understand $D^{+!}$, we look at the requirement that the operator $Q = D^+ + D^{+!}$ be skew self-adjoint for $\langle \cdot, \cdot \rangle$. So, we let $\varphi \in \mathscr{V}^+$, and $\psi \in \mathscr{V}^{-!}$; then, we must have

$$\langle \psi, D^+ \phi \rangle = -\langle D^{+!} \psi, \phi \rangle,$$

and in fact this serves as a definition of $D^{+!}$. Similarly, we would like to define $Q^{GF} = D^- + D^{-!}$; since Q^{GF} must be self-adjoint for the invariant pairing, we have, assuming $\varphi \in \mathscr{V}^-$ and $\psi \in \mathscr{V}^{+!}$,

$$\langle \psi, D^- \phi \rangle = -\langle D^{-!} \psi, \phi \rangle,$$

where the minus sign appears because ψ has cohomological degree 1. Again, this is sufficient to define $D^{-!}$. Finally, we want to define the action of $\mathscr{L}$ on the fiber directions of $T^*[-1]\mathscr{E}$. Here, we require that the action preserve the pairing in the sense of Chapter 11 of [CG16]:

$$\langle [X, \psi], \varphi \rangle = \langle [X, \varphi], \psi \rangle = -\langle \psi, [X, \varphi] \rangle.$$

In other words, whenever we switch one of the operators $[X, \cdot]$, Q, Q^{GF} from one side to the other, we always get a minus sign. This will be important to remember below. $\Diamond$

4. McKean-Singer

Our main tool in proving formula 1.1 is the following theorem

Theorem 4.1 (Gwilliam)**.**
(1) *The obstruction to the $\mathscr{L}$-equivariant quantization of the cotangent theory to $\mathscr{E}$ is given by a well-defined cohomology class $\mathcal{O} \in H^\bullet(\widehat{Sym}(\mathscr{L}[1]^\vee))$.*
(2) *If the gauge-fixing is* ***positive*** *(in the sense of* [?, ref: othesis]*, which is the case if in our key example D is self-adjoint with respect to some Hermitian metric on V and M is compact, then for a closed form $\alpha \in \Omega^\bullet$, $\mathcal{O}(M)(\alpha)$ is given by the trace of the action of $H^\bullet(\mathscr{L}(M))$ on the determinant of $H^*(\mathscr{E}(M))$. Here we mean the graded determinant: if V is a Z-graded vector space,*

$$\det(V) = \bigotimes_i \left(\bigwedge^{\dim V_i} V_i \right)^{(-1)^i},$$

with W^{-1} defined as $W^\vee$.

Now, we can describe how to compute the obstruction $\mathcal{O}(U)$. We make the following

Definition 4.2. *The* ***tree-level, scale t interaction*** *is the element $I_{tr}[t]$ of $\mathrm{Obs}^q(U)[t]$ given by taking a sum over all connected tree-graphs with trivalent vertices as described above, with the propagator $Q^{GF} \int_0^t e^{-[Q,Q^{GF}]}$ inserted at each internal edge. In the language of* [Cos11]*, this is the $\varepsilon \to 0$ limit of the mod $\hbar$ term of $W(P(\varepsilon, L), I)$.*

Figure 4 gives a diagrammatic depiction of $I_{tr}[t]$. As an example, the second diagram corresponds to the term

$$(X, Y, \psi, \varphi) \mapsto \left\langle \psi, \left[Y, Q^{GF} \int_0^t \exp\left(-s[Q, Q^{GF}]\right)[X, \varphi] \right] \right\rangle,$$

which we symmetrize over X, Y to get the final contribution to $I_{tr}[t]$. Notice that for simple combinatorial reasons, all of the trees contributing to I_{tr} have only two external $T^*[-1]\mathscr{E}$ edges. Thus, $\Delta_t I_{tr}$ belongs to $\widehat{Sym}(\mathscr{L}[1]^\vee)$.

Now that we have this definition in place, we can state, in addition to Theorem 4.1, the following

Lemma 4.3. *A representative of the obstruction class is given by $\Delta_t I_{tr}[t]$. The cohomology class of this obstruction is independent of t.*

Notice that part (2) of Theorem 4.1 and Lemma 4.3 give us two ways to evaluate the obstruction $\mathcal{O}$ on a closed form α. Indeed, using these two ways, we will recover the McKean-Singer formula, to the proof of which we now turn.

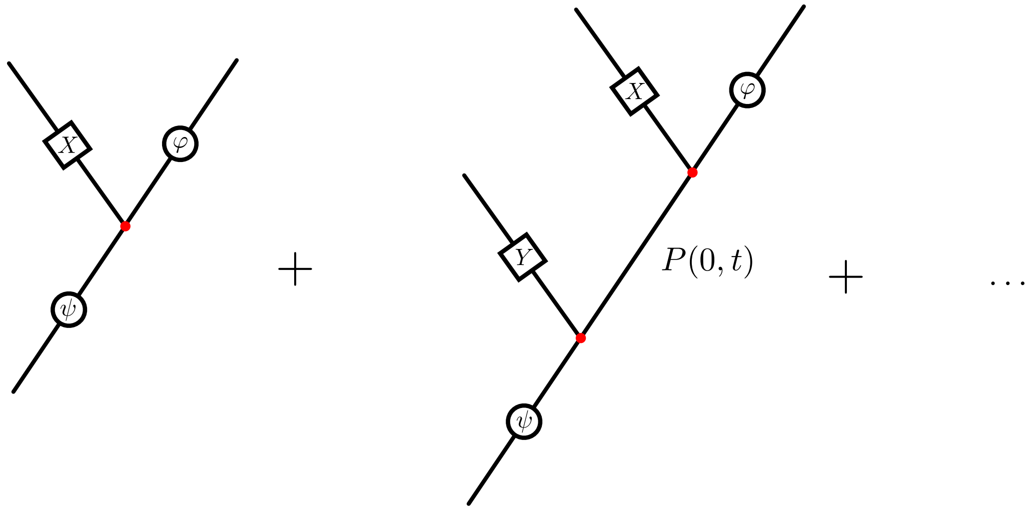

FIGURE 2. The tree-level diagrams contributing to $I_{tr}[t]$.

Proof of Equation 1.1. We will use part (2) of Theorem 4.1 and Lemma 4.3 to evaluate $\mathcal{O}(M)$ on a constant function $\lambda \in \mathbb{R}$. We should check that the gauge-fixing $D^- + D^{-!}$ is positive, but we defer this to the end of the proof.

Let us first work out what part (2) of Theorem 4.1 tells us: in our case $H^\bullet(\mathscr{E})$ has $\ker(D^+)$ in degree 0 and $\operatorname{coker}(D^+)$ in degree 1. Thus,

$$\det(H^\bullet(\mathscr{E}) \cong \bigwedge^{\dim\ker D^+} \ker D^+ \otimes \bigwedge^{\dim\operatorname{coker} D^+} (\operatorname{coker} D^+)^\vee.$$

λ acts on this one-dimensional space by multiplication by

$$\lambda\left(\dim\ker(D^+) - \dim\operatorname{coker}(D^+)\right) = \lambda \operatorname{ind}(D).$$

In other words, $\mathcal{O}(M)(\lambda) = \operatorname{ind}(D)$. This is the left-hand side of 1.1.

On the other hand, let us study the Feynman diagrams appearing in $I_{tr}[t]$ with a λ on each $\mathscr{L}$ edge. A tree diagram with n vertices gives, before symmetrization over $\mathscr{L}$ inputs

$$\lambda^n \left\langle \psi, \left(Q^{GF}\int_0^t e^{-s[Q,Q^{GF}]}ds\right)^{n-1} \varphi \right\rangle.$$

However, since $I_{tr}[t] \in C^\bullet_{loc}(\mathscr{L}) \otimes C^\bullet_{loc}(\mathscr{E}[-1])$, it needs to be graded-symmetric over its $\mathscr{L}$ inputs, and since $C^\bullet_{loc}(\mathscr{L}) = \widehat{Sym}^\bullet(\mathscr{L}[[1]^\vee)$, when we have a zero-form input λ into $I_{tr}[t]$, all terms non-linear in λ are anti-symmetrized away to zero. Thus, after applying Δ_t to $I_{tr}[t]$, we find that the only non-zero contribution to $\mathscr{O}(M)(\lambda)$ is represented by the "tadpole" diagram depicted in figure 4.

The tadpole diagram, evaluated on λ, gives

$$-\partial_{K_t} I_{tr}[t] = -\lambda\langle\cdot,\cdot\rangle(K_t \mid_{\mathscr{E}}),$$

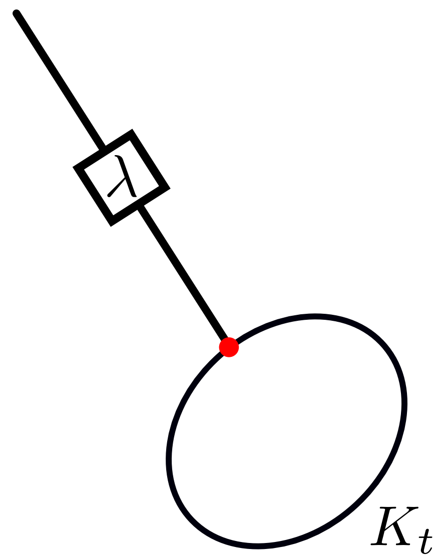

FIGURE 3. The single diagram appearing in the Feynman diagram expansion for $\mathcal{O}(\lambda)$.

where we are restricting K_t just to $\mathscr{E}$ from all of $T^*[-1]\mathscr{E}$ because the interaction takes only an $\mathscr{E}$ input into the first slot of $\langle,\rangle$, and a $\mathscr{E}^!$ input into the second. Thus, only the components of K_t lying in $\mathscr{E}\otimes\mathscr{E}^!$ survive, and these are precisely the components we are considering when we take $K_t\mid_{\mathscr{E}}$.

This is a good opportunity to review Costello's convention for K_t, which is subtly different from k_t as defined and discussed above. By definition, K_t is defined so that

$$-1\otimes\langle\cdot,\cdot\rangle(K_t\otimes s)=e^{-t[Q,Q^{GF}]}s,$$

where $s\in T^*[-1]\mathscr{E}$. Let us denote by k_t^+ the heat kernel for the generalized Laplace operator D^-D^+, and similarly for the pairs k_t^- and D^+D^-. Table 4 shows the bundles of which each is a section. It therefore follows that

Heat Kernel	Section of
k_t^+	$V^+\boxtimes V^{+\vee}$
k_t^-	$V^-\boxtimes V^{-\vee}$

FIGURE 4. The two heat kernels and the bundles of which they are sections.

$$K_t\mid_{\mathscr{E}}=(-k_t^+-k_t^-)|dx|\in T^*[-1]\mathscr{E}\otimes T^*[-1]\mathscr{E}.$$

Finally, we wish to calculate

$$\langle,\rangle(K_t\mid_{\mathscr{E}}).$$

In words, at each point $x\in M$, we take $K_t\mid_{\mathscr{E}}(x)$, which is an element of the fiber of $V\otimes V^!$ over x, we pair using the rules for $\langle,\rangle$, and we integrate the resulting density over M. But since we have V on the left and $V^!$ on the right, we use the evaluation pairing with a minus sign, and we remember that the evaluation pairing comes with a minus sign for sections of $V^- \otimes V^{!-}$. Taking into account all of these factors, we have

$$\langle , \rangle (K_t \mid_{\mathscr{E}}) = \int_M \left(\mathrm{Tr}(k_t^+) - \mathrm{Tr}(k_t^-)\right) |dx| = \int_m \mathrm{Str}(k_t)|dx|,$$

which gives the right hand side of equation **??**.

The last thing we have to check is that Q^{GF} is a *positive* gauge fix. A positive gauge fix, by definition, needs to satisfy:

(1) The operator $H := [Q, Q^{GF}]$ is symmetric for some Hermitian metric on the vector bundle $T^*[-1]E$.
(2) The eigenvalues of H are non-negative.
(3) We have a direct sum decomposition
$$T^*[-1]\mathscr{E} = \ker H \oplus \mathrm{im} Q \oplus \mathrm{im} Q^{GF}.$$
This decomposition is as topological vector spaces.

Item 1 is satisfied by assumption, item 2 is satisfied because H is the square of a self-adjoint operator, and item 3 follows from the statement and proof of proposition 3.48 in [BGV04]. □

5. More Feynman Diagrams

We have our desired result; however, we have only evaluated the obstruction $\mathcal{O}$ on a constant function λ. What happens when we evaluate this obstruction on an element $\lambda + \alpha$, where α is a closed 1-form? Well, since α acts off-diagonally on $\mathscr{E}$, part (2) of 4.1 still gives us $2\lambda\, \mathrm{ind}(D)$. However, the Feynman diagrammatics look *a priori* very different. There is now no reason to rule out more complicated tree diagrams from appearing. However, we do still have the following

Proposition 5.1. *For $\lambda \in \mathbb{R}$, $\alpha \in \Omega^1(M)$, the Feynman diagram expansion of $\Delta_t I_{tr}[t](\lambda + \alpha)$ gives $\int_M \mathrm{Str}\, k_t |dx|$.*

Proof. It is easy to see for degree reasons that the only tree diagrams giving a non-zero contribution to $I_{tr}[t]$ are those with λ on one of the external $\mathscr{L}$ edges (which we will now call "legs") and α on all the rest. Now, the diagram with one leg gives the term $\lambda \int_M \mathrm{Str}\, k_t(x,x)|dx|$, as we saw above. So, we need to show that all the other diagrams give zero. Let's focus first on the two-leg diagrams; the higher-leg diagrams will vanish for very similar reasons. One of the two two-leg tree diagrams is depicted in Figure 5; the other is given by exchanging λ and α.

The contribution from these two diagrams is given by the term

$$\lambda \left(\langle \psi, [\alpha, P\varphi]\rangle + \langle \psi, P[\alpha, \varphi]\rangle\right),$$

where $\psi \in \mathscr{E}$ and $\varphi \in \mathscr{E}^!$. Recall that whenever we move $[\alpha, \cdot]$, Q, and Q^{GF} from one side of the pairing to the other, we pick up a minus sign. Since $P = Q^{GF} \int_0^t e^{-s[Q,Q^{GF}]} ds$, and since Q^{GF} and $[Q, Q^{GF}]$ commute, we also pick up a minus sign in moving P from one side of the pairing to the other. Thus, the two-leg tree diagrams contribute

$$\lambda \left(\langle P[\alpha, \psi], \varphi\rangle + \langle \psi, P[\alpha, \varphi]\rangle\right) = \lambda \left(-\langle \varphi, P[\alpha, \psi]\rangle + \langle \psi, P[\alpha, \varphi]\rangle\right).$$

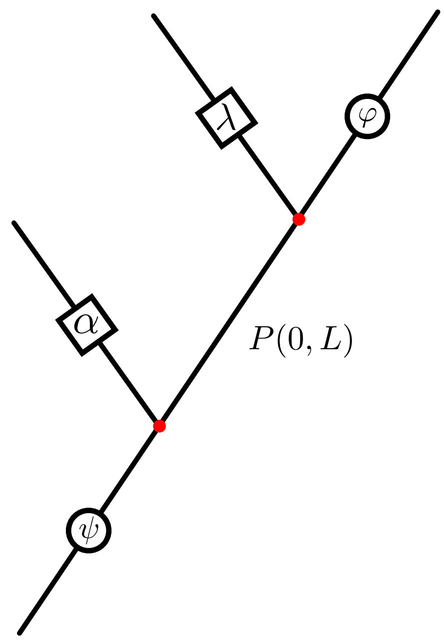

FIGURE 5. One of the two two-leg, tree-level diagrams contributing to $I_{tr}[t]$. The other one is given by exchanging λ and α.

In other words, this contribution is anti-symmetric under the transposition of its two inputs. We need to show that when evaluated on K_t, this contribution gives 0.

It is not hard to extend this argument to the higher-leg terms; there, the n-leg diagrams give the following contribution

$$\lambda \left(\langle \psi, ([\alpha, \cdot]P)^{n-1}\varphi \rangle + \langle \psi, ([\alpha, \cdot]P)(P[\alpha, \cdot])^{n-2}\varphi \rangle + \cdots + \langle \psi, (P[\alpha, \cdot])^{n-1}\varphi \rangle \right).$$

Since all but the first and last terms involve $P^2 = 0$, only the first and last terms need to be shown to be zero. But this argument is entirely analogous to the one for two-leg diagrams. $\square$

6. EQUIVARIANT MCKEAN-SINGER

We would like to generalize the set-up of our problem in the following way: suppose we have a Lie algebra $\mathfrak{g}$ acting on V by (self-adjoint) even operators and that this action commutes both with the C^∞ action and with D. In other words, we have a Lie algebra map

$$\rho : \mathfrak{g} \to \Gamma(\mathrm{End}^0_{(s.a.)}(V))$$

such that $\rho(\gamma)(Ds) = D\rho(\gamma)(s)$ for all $s \in \mathscr{V}$, $f \in C^\infty(M)$, and $\gamma \in \mathfrak{g}$.

Remark: Since the equation of motion of the free theory we are considering is essentially $D\varphi = 0$, we are to think of $\mathfrak{g}$ as a Lie algebra of symmetries of our theory. After all, if $D\varphi = 0$, then $D((1+\gamma)\varphi) = D\varphi + D\gamma\varphi = 0 + \gamma D\varphi = 0$, so that γ infinitesimally preserves the equation of motion. In physics lingo, the obstruction $\mathcal{O}$ is the *anomaly*, the measure of the violation of the symmetry at the quantum level. $\Diamond$

We will usually just denote the action of an element $\gamma \in \mathfrak{g}$ on a section $s \in \mathscr{V}$ by $\gamma.s$. Since γ commutes with D, it preserves $\ker D^+$ and $\operatorname{coker} D^+$, so we are free to make the following

Definition 6.1. *Let $\mathfrak{g}$ act on V as above. Then, the* ***equivariant index*** *of D is the following function on $\mathfrak{g}$:*

$$\operatorname{ind}(\gamma, D) = \operatorname{Tr}(\gamma\mid_{\ker D^+}) - \operatorname{Tr}(\gamma\mid_{\operatorname{coker} D^+}).$$

With this definition in hand, we can now state the **equivariant McKean-Singer theorem**:

Theorem 6.2.

$$\operatorname{ind}(\gamma, D) = \int_M \operatorname{Str}(\gamma k_t(x,x))|dx|. \tag{6.1}$$

Note that in the special case where $\mathfrak{g} = \mathbb{R}$ and the action is given by scalar multiplication, this theorem just reproduces the regular McKean-Singer formula.

We will see that our proof of the regular McKean-Singer formula carries over *mutatis mutandis* to the equivariant case, once we carry out the due diligence of constructing a local action of $\mathfrak{g} \otimes \Omega^\bullet$ on $\mathscr{E}$. Since this is elliptic L_∞ algebra with which we are now pre-occupied, we will use $\mathscr{L}$ to denote its sheaf of sections. This is the work to which we now turn:

Proposition 6.3. *There is a unique central extension of elliptic dglas*

$$0 \to \mathscr{E} \to \mathscr{L} \oplus \mathscr{E} \to \mathscr{L} \to 0$$

satisfying

(1) $[x,y]_{\mathscr{L}\oplus\mathscr{E}} = [x,y]$ *if x and y are both in $\mathscr{L}$ or both in $\mathscr{E}$. We are viewing $\mathscr{E}$ as an abelian elliptic dgla.*
(2) *For $\gamma \in \mathfrak{g}$ and $\varphi \in \mathscr{E}$, $[\gamma \otimes 1, \varphi] = \gamma.\varphi$, so that the action of $\mathscr{L}$ on $\mathscr{E}$ coincides with the action of $\mathfrak{g}$ for elements of $\mathfrak{g} \otimes 1$.*
(3) *The map*

$$[,] : \mathscr{L} \otimes \mathscr{E} \to \mathscr{E}$$

is C^∞-bilinear.

Proof. For degree reasons (as before), we need only to specify a degree zero action of functions and a degree 1 action of one-forms satisfying

$$D^+([X,\varphi]) = [dX,\varphi] + (-1)^{|X|}[X, D^+\varphi]$$
$$[X,[Y,\varphi]] = [[X,Y],\varphi] + (-1)^{|X||Y|}[Y,[X,\varphi]].$$

for all $X, Y \in \mathscr{L}$ and $\varphi \in \mathscr{E}$. Items 2) and 3) in the statement of the proposition tell us that for elements of $\mathscr{L}$ of the form $\gamma \otimes f$ with $f \in C^\infty$, we must have

$$[\gamma \otimes f, \varphi] = f\gamma.\varphi.$$

On the other hand, for elements of $\mathscr{L}$ of the form $\gamma \otimes df$, the derivation property requires that

$$[\gamma \otimes df, \varphi] = D^+(f\gamma.\varphi) - f\gamma.(D^+\varphi) = \gamma.([D^+, f]\varphi), \tag{6.2}$$

where we have used the fact that γ commutes with C^∞ functions and D^+. Just as above, this fixes the action of one-forms on $\mathscr{E}$. More precisely, if we denote by $c(\alpha)$ the action of the one-form α on $\mathscr{E}$ defined in Proposition 3.1, then the action of an element $\gamma \otimes \alpha$ of $\mathscr{L}$ on an element φ of $\mathscr{E}$ is given by

$$[\gamma \otimes \alpha, \varphi] = \gamma.(c(\alpha)\varphi). \tag{6.3}$$

Equations 6.2 and 6.3 define the bracket $\mathscr{L} \otimes \mathscr{E} \to \mathscr{E}$, and we have just argued for its uniqueness. It remains to verify the Jacobi identity, since we have already extracted all of the non-trivial information from the derivation property. For degree reasons, the Jacobi identity is trivially satisfied as long as both X and Y are of non-zero degree. So it suffices to check the Jacobi identity for $X = \gamma_1 \otimes f$ and $Y = \gamma_2 \otimes g$ or $\gamma_2 \otimes \alpha$ where $f, g \in C^\infty$ and $\alpha \in \Omega^1$. In the first case, we have

$$\begin{gathered}
[\gamma_1 \otimes f, [\gamma_2 \otimes g, \varphi]] = fg\gamma_1.\gamma_2.\varphi \\
[[\gamma_1 \otimes f, \gamma_2 \otimes g], \varphi] + [\gamma_2 \otimes g, [\gamma_1 \otimes f, \varphi]] = fg[\gamma_1, \gamma_2].\varphi + fg\gamma_2.\gamma_1.\varphi,
\end{gathered}$$

and we see that the Jacobi identity is satisfied because $\mathfrak{g} \to \Gamma(\mathrm{End}^0(V))$ is a Lie algebra homomorphism. In the second case, we have

$$\begin{gathered}
[\gamma_1 \otimes f, [\gamma_2 \otimes \alpha, \varphi]] = fg\gamma_1.\gamma_2.(c(\alpha)\varphi) \\
[[\gamma_1 \otimes f, \gamma_2 \otimes \alpha], \varphi] + [\gamma_2 \otimes \alpha, [\gamma_1 \otimes f, \varphi]] = f[\gamma_1, \gamma_2].(c(\alpha)\varphi) + f\gamma_2.(c(\alpha)\gamma_1.\varphi).
\end{gathered}$$

Because $c(\alpha)$ is defined using only the actions of D^+ and C^∞ on $\mathscr{E}$, $c(\alpha)$ commutes with γ_1, and so the last term in the above equation is $\gamma_2.\gamma_1(c(\alpha)\varphi)$. Just as for the first case, then, the Jacobi identity is satisfied. □

Now, we proceed just as above to prove Equation 6.1:

Proof of Equation 6.1. We evaluate the obstruction class on an element $\gamma \otimes 1$ using Lemma 4.3 and Theorem 4.1. Just as for the non-equivariant formula, for symmetry reasons, only one diagram contributes to the Feynman diagrammatics, and that diagram gives a contribution

$$\int_M \mathrm{Str}(\gamma k_t(x,x))|dx|.$$

On the other hand, $\gamma \otimes 1$ acts on $\det\ker(D^+)$ by $\mathrm{Tr}(\gamma\ |_{\ker D^+})$ and similarly for $\det(\mathrm{coker}(D^+)^\vee)$, so that the application of Theorem 4.1 gives us

$$\mathcal{O}(M)(\gamma \otimes 1) = \mathrm{ind}(\gamma, D).$$

□

7. Maturer Statements

There is a much cleaner theoretical language in which to understand these results, based on the following nice characterization of the obstruction complex $C^\bullet_{loc}(\Omega^\bullet_{dR}(M))$. We will assume in this section that M is orientable, so that there is no difference between densities and volume forms.

Lemma 7.1 (Non-Equivariant Obstruction Complex)**.** *The map*

$$\Omega^\bullet(M)[n-1] \to J(\Omega^\bullet_{dR}(M))^\vee[-1] \hookrightarrow C^\bullet_{loc}(\Omega^\bullet_{dR}(M))$$

given by

$$\alpha \mapsto \left(\beta \mapsto \int_M \alpha \wedge \beta\right)$$

is a quasi-isomorphism.

Proof. We first recall that one definition of $C^\bullet_{loc}$ is

$$C^\bullet_{loc}(\Omega^\bullet_{dR}(M)) = \mathrm{Dens}_M \otimes_{D_M} C^\bullet_{red}(J(\Omega^\bullet_{dR})(M)),$$

where $J(\Omega^\bullet_{dR}(M))$ is the space of global sections of the (infinite) jet bundle of $\Omega^\bullet_{dR}$ and the subscript *red* means that we quotient out by the Sym^0 piece of $C^\bullet$. In our case, since the L_∞-algebra we're considering is abelian, $C^\bullet_{red}(J(\Omega^\bullet_{dR})$ is just $\mathrm{Sym}^{>0}(J(\Omega^\bullet_{dR})^\vee[-1])$ with the differential induced by the de Rham differential.

We will replace both terms in the (derived) tensor product defining the obstruction complex with quasi-isomorphic complexes. Let's start with the map of D_X modules

$$C^\infty_X \hookrightarrow J(\Omega^\bullet_{dR})$$

The Poincaré lemma can be used to show that this is a quasi-isomorphism of D_X modules. Namely, letting (α, U) be a representative of the germ of a closed cohomological degree d section of the bundle $J(\Omega^\bullet_{dR}$ at a point p, and assuming that U is homeomorphic to $\mathbb{R}^n$, the Poincaré Lemma tells us that α is exact on U unless $d = 0$, in which case α is cohomologous to the jet of a constant function. Thus, the above map is a quasiisomorphism of D_X modules, and it follows that we have a quasi-isomorphism □

The main result of this section is

Theorem 7.2. *Under the quasi-isomorphism of 7.1, the obstruction $\mathcal{O}(M)$ corresponds to* $\mathrm{Str}\, k_t(x,x)|dx|$.

Proof. content... □

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