On the interplay between Lorentzian Causality and Finsler metrics of Randers type

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Universidad de Granada

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Interplay between Randers metrics and stationary spacetimes

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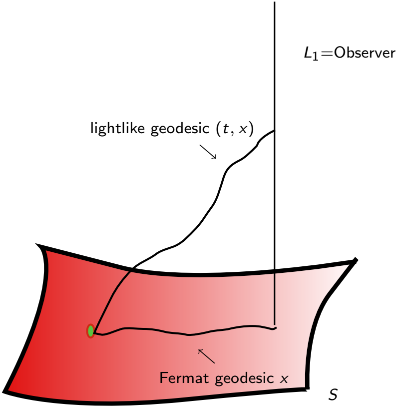
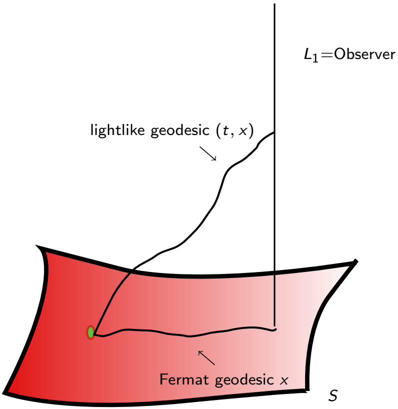
$(\mathbb{R} \times S, l)$ is a standard stationary spacetime



S is naturally endowed with a Randers metric F called the Fermat metric

Interplay between Randers metrics and stationary spacetimes

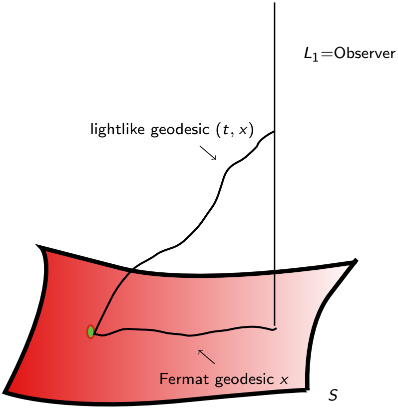


Causal properties of $(\mathbb{R} \times S, l)$

$\Updownarrow$

Hopf-Rinow properties of (S, F)

Interplay between Randers metrics and stationary spacetimes

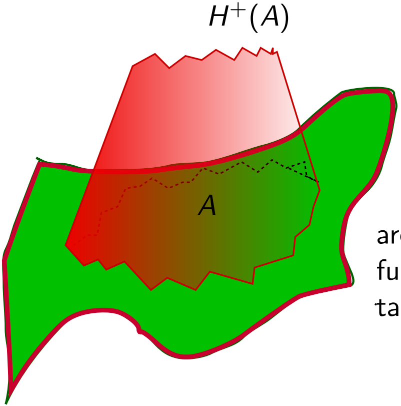


Global hyperbolicity of $(\mathbb{R} \times S, l)$

$\Updownarrow$

$\bar{B}^+(p, r) \cap \bar{B}^-(p, r)$ compact $\forall p \in S$ and $\forall r > 0$ in (S, F)

Interplay between Randers metrics and stationary spacetimes

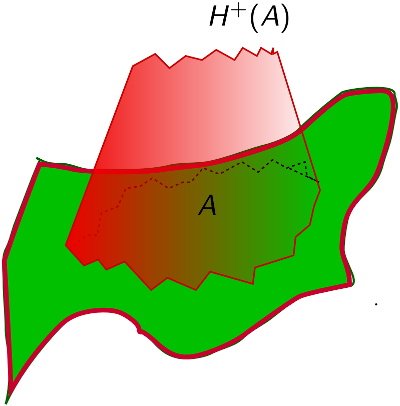


Cauchy horizons of a subset A contained in a slice $\{t_0\} \times S$

$\Downarrow$

are the graph of the distance function to the complementary A^c in (S, F)

Interplay between Randers metrics and stationary spacetimes



Differential properities of the Cauchy horizons in $(\mathbb{R}\times S, l)$

$\Updownarrow$

Differential properties of the distance function to a subset in (S, F)

Program of the talk

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- First application of the Interplay: Causal properties in terms of Hopf-Rinow properties of the Fermat metric

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- Preliminaries:
  - Causality (the causal ladder)
  - Standard stationary spacetimes and Fermat metrics
  - Randers and Finsler metrics
- First application of the Interplay: Causal properties in terms of Hopf-Rinow properties of the Fermat metric
- Second application: equivalence of differentiability of Cauchy horizons and the distance function to a subset.

The causal ladder

Causal properties classify spacetimes depending on the behaviour of causal cones. A spacetime is:

Globally hyperbolic

$\Downarrow$

Causally simple

$\Downarrow$

Causally continuous

$\Downarrow$

Stably causal

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Strongly causal

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Distinguishing

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- Globally hyperbolic if it admits a Cauchy hypersurface (a subset S that meets exactly once every inextendible timelike curve)

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M. A. J. and M. Sánchez, *A note on the existence of standard splittings for conformally stationary spacetimes*, Classical Quantum Gravity, 25 (2008), pp. 168001, 7.

Causal condition to have a standard splitting

Theorem (M. A. J.- M. Sánchez)

If a stationary spacetime L is distinguishing and the timelike Killing field is complete, then it is causally continuous and standard

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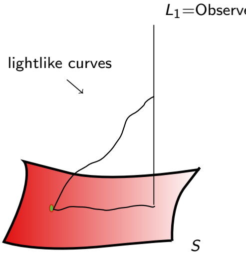
Fermat principle in standard stationary spacetimes

- Relativistic Fermat Principle: lightlike pregeodesics are critical points of the arrival time function corresponding to an *observer* in a suitable class of lightlike curves



Pierre de Fermat (1601-1665)

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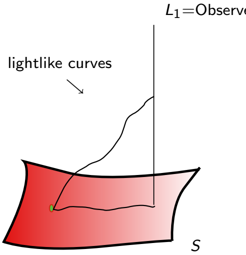

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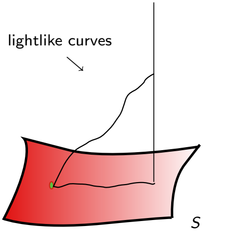

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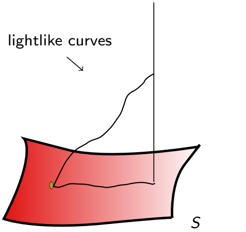
- Let us define the Fermat (Finslerian) metric in S as

$$F(x,v)=\tfrac{1}{\beta} g_0(v,\delta)+\sqrt{\tfrac{1}{\beta} g_0(v,v)+\tfrac{1}{\beta^2} g_0(v,\delta)^2},$$



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Fermat metric and lightlike geodesics

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Theorem

A curve $s \to \gamma(s) = (s, x(s))$ *is a lightlike pregeodesic of* $(\mathbb{R} \times S, g)$ *iff* $s \to x(s)$ *is a Fermat geodesic with unit speed.*

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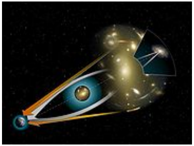
- Consequences:
  - Gravitational lensing can be studied from geodesic connectedness in Fermat metric

Einstein Cross



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  - Existence of *t*-periodic lightlike geodesics is equivalent to existence of Fermat closed geodesics (Biliotti-M.A.J. to appear in Houston J. Math.)

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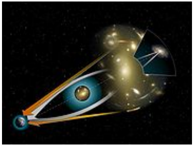

Gravitational lensing

Randers metrics

- Randers metrics in a manifold M is a function $R : TM \to \mathbb{R}$ defined as:

$$R(x, v) = \sqrt{h(v, v)} + \omega_x[v]$$

where h is Riemannian and ω a 1-form with $\|\omega_x\|_h < 1 \ \forall x \in M$,
are basic examples of non-reversible Finsler metrics: $R(x, -v) \neq R(x, v)$.

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- Named after the norwegian physicist Gunnar Randers (1914-1992):

Randers, G.: On an asymmetrical metric in the fourspace of General Relativity.
Phys. Rev. (2) **59**, 195–199 (1941)

Gunnar Randers with Albert Einstein

Finsler metrics

Main reference:

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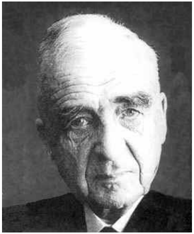

Paul Finsler (1894-1970)

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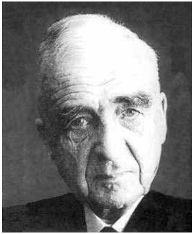

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DEFINITION: $F : TM \to [0, +\infty)$ continuous and

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2. Positively homogeneous of degree one
   $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$

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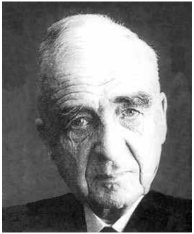
3. Fiberwise strictly convex square:
   $g_{ij}(x, y) = \left[\frac{1}{2} \frac{\partial^2 (F^2)}{\partial y^i \partial y^j}(x, y)\right]$ is positively defined.



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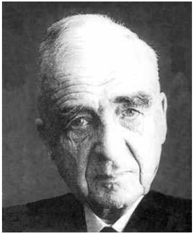
It can be showed that this implies:

- F is positive in $TM \setminus \{0\}$



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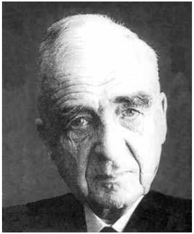

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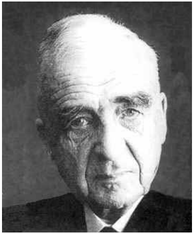
Finsler metrics

Main reference:

Bao, D., Chern, S.S., Shen, Z.: An Introduction to Riemann-Finsler geometry.



PAUL FINSLER (1894-1970)

DEFINITION: $F : TM \to [0, +\infty)$ continuous and

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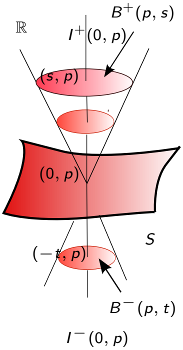

Causality through the Fermat metric

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Globally hyperbolic

$\Downarrow$

Causally simple

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(c) *a slice $\{t_0\} \times S$, $t_0 \in \mathbb{R}$, is a Cauchy hypersurface if and only if the Fermat metric F on S is forward and backward complete.*

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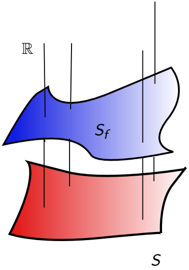

$$S_f = \{(f(x), x) : x \in S\}$$

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- Moreover, if $\mathbb{R} \times S$ is the splitting associated to R, the splitting associated to R' is $\mathbb{R} \times S_f$, where

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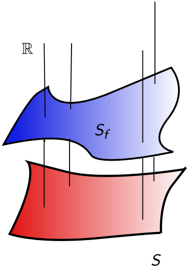

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Theorem (Accurate Hopf-Rinow for Randers metrics)

Let (S, R) *a Randers manifold and given a function* $f : S \to \mathbb{R}$ *define* $R_f(x, v) = R(x, v) - \mathrm{d}f_x(v)$. *The following conditions are equivalent:*

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In such a case, (S, R) is convex.

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- As an application we obtain Morse theory for lightlike geodesics and timelike geodesics with fixed proper time from a point to a vertical line.

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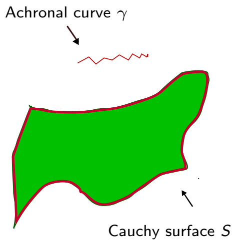

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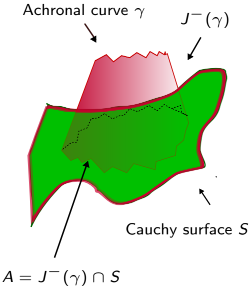

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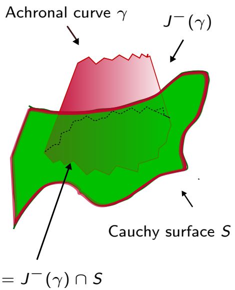

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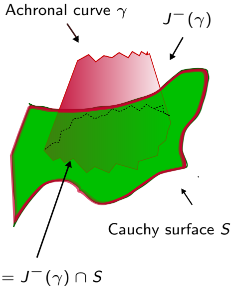

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Cauchy developments and distance function to a subset

Theorem

Let $(\mathbb{R} \times S, g)$ be a standard stationary spacetime such that $\{t_0\} \times S$ is Cauchy, and $A_{t_0} = \{t_0\} \times A$. Then

- $$D^+(A_{t_0}) = \{(t, y) : \ \mathrm{d}(x, y) > t - t_0 \ \ \forall x \notin A \text{ and } t \geq t_0\},$$
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Cauchy horizons can be seen as the graph of the distance function to a subset!!!!

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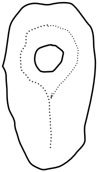

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Photograph courtesy of Yan Yan Li.

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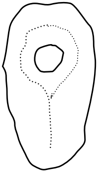
Theorem (Li-Nirenberg)

The function $\partial\Omega \ni y \to \min(N, \ell(y)) \in \mathbb{R}^+$ is Lipschitz-continuous on any compact subset. As a consequence $\mathfrak{h}^{n-1}(\Sigma \cap B) < +\infty$, being B bounded.



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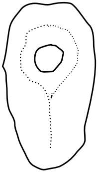

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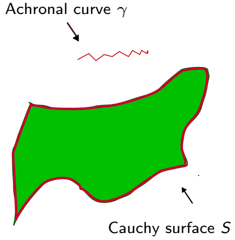

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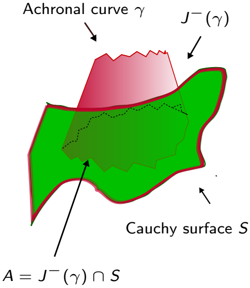

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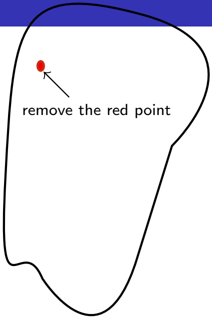

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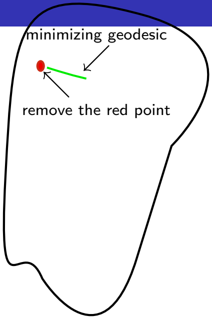

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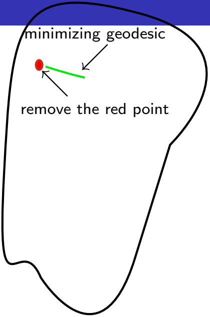

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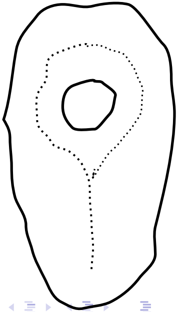

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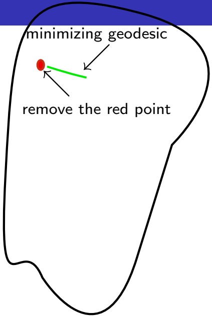

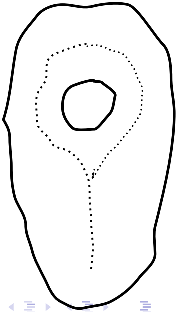

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- There are several results for the differentiability of future horizons:

J. K. Beem and A. Królak, *Cauchy horizon end points and differentiability*,
J. Math. Phys., 39 (1998), pp. 6001–6010.

P. T. Chruściel, J. H. G. Fu, G. J. Galloway, and R. Howard, *On fine differentiability properties of horizons and applications to Riemannian geometry*,
J. Geom. Phys., 41 (2002), pp. 1–12.

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Corollary

The n-dimensional Haussdorf measure of Cut_C is zero.

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More information in:

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