Lorentzian G-manifolds and cohomogenity one manifolds

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Abstract

*Keywords:*Isometry group of Lorentzian manifolds, isometric actions, orbit space, slice, cohomogeneity one Lorentzian manifolds.

Contents

1 Survey of results about Lorentzian G-manifolds

Let (M, g) be a Lorentzian G- manifold. The orbit space $\Omega = M/G$ (with the quotient topology) may be non-Hausdorf (see the Minkowski space $\mathbb{R}^{n-1}$ with the action of the Lorentz group $SO_{1,n-1}$.)

Definition 1 *A G-manifold M is called a proper manifold if the action of G on M is proper that is the map*

$$G \times M \to M,\ (g, x) \mapsto gx$$

is proper.

This implies that the stabilizer of any point is compact.

1.1 Isometric action of non compact semisimple Lie group G on a Lorentzian manifold M

Let G be a group of isometries of a Lorentzian manifold M.

Theorem 1 *(Zimmer) If M is compact and G is non compact simple Lie group, then it is locally isomorphic to $SL_2(\mathbb{R})$.*

The classification of Lie groups which can acts isometrically on a compact Lorentz manifold is obtained by Adams and Stuck.

Hernandez-Zamova obtained some splitting theorem for G-manifold M under assumption that M has finite volume and G is a non compact simple Lie group , which acts analytically on M and has generically non degenerate orbits.

1.2 N. Kowalsky theorem and its generalizations

N. Kowalsky had proven the following remarkable result.

Theorem 2 *Let G be a simple connected Lie group with finite center acts non proper and isometrically on a Lorentzian manifold M, then it is locally isomorphic to $SO(1,n)$ or $SO(2,n)$ with the non-proper orbits*
$S = SO(1,n)/SO(1,n), SO(1,n)/SO(1,n-1), SO(1,n)/SO(n-1)\cdot\mathbb{R}^{n-1}$, *or* $SO(2,n)/SO(1,n)$.

Any such orbit S with semisimple stabilizer is a space of constant curvature and locally the manifold M is a warped product $M = N \times_f S$ of S and a Riemannian manifold N.

Alternative proof of this results are given by S.Adam and D.Witte (in homogeneous case). Deffaf, Melnik and Zeghib get a generalization of Kowalsky result to the case of semisimple Lie group G without $SL_2(\mathbb{R})$ factor.

1.3 Other results

Let G be a connected isometry group of a Lorentz manifold which has an unproper orbit. S. Adam proved that under the assumption that the nilradical N of G is simply connected, at least one of the following conditions must be met:
(1) either there exists a closed connected subgroup H of G such that the standard action of G on G/H is locally faithful and preserves a Lorentz metric or
(2) G admits a locally faithful action for which the connected component of the stabilizer of some point of the manifold within the center of N is noncompact.

Adams and Stuck study the locally faithful action of the group $SL_n(\mathbb{R}) \times \mathbb{R}^n$, $n > 3$ on a Lorentz manifold and prove that it is always proper action.

Adams gives sufficient conditions for existence of a locally free orbit nonproper action of a Lie group G on a Lorentz manifold : noncompactness of the center of G, non closeness of the adjoint group Ad_G, existence of a one-dimensional ideal in $\mathfrak{g}$, some direct summand of $\mathfrak{g}$ being isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

1.4 Lorentzian manifold with big group G of isometries

Patrangenaru proved that any Lorentzian n-dimensional manifold which admits isometry group of dimension m is homogeneous if $m > (1/2)n(n-1)+1$ and if $m = (1/2)n(n-1)+1$ then M is the Egorov space with the metric of the form

$$g = f(x_n) \sum_{i=1}^{n-2} dx_i^2 + 2dx_{n-1}dx_n.$$

1.5 Lorentzian manifold with big group G of isotropy

Let (M, g) be a G-manifold and $H = G_p$ is the stbilizer of a point $p \in M$. Since the isotropy representation $j : H \to O(T_pM)$ is exact, we can identify H with a subgroup $j(H)$ of the Lorentz group and its Lie algebra $\mathfrak{h}$ with a subalgebra of the Lorentz Lie algebra $\mathfrak{so}_n$.

Lemma 1 *There is three (up to a conjugation) connected maximal subgroups of the Lorentz group $SO_{1,n-1}$:*
group $(SO_{1,n-1})_t = SO_{n-1}$ which preserves a which preserves a time-like vector t,
group $(SO_{1,n-1})_s = SO_{1,n-2}$ which preserves a space-like vector s,
and the group $(SO_{1,n-1})_{\mathbb{R}p} = Sim(E^{n-2})$ which preserves an isotropic line Rp.

Theorem 3 *i) Let G be an isometry group of a Lorentzian manifold $(M,g))$ which preserves a point p. If the isotropy group $j(G)$ is isomorphic to the Lorentz group or to the group $Sim(E^{n-2})$, then it is a cohomogeneity one Lorentzian G-manifold.*

ii) If it is isomorphic to SO_{n-1} or $SO_{1,n-2}$, the group G has codimension two orbits near p.

iii) If the stabilizer of two sufficiently closed points p,q of a Lorentzian G-manifold is isomorphic to the Lorentz group than the manifold M is a maximally homogeneous manifold of constant curvature.

iv) If the stabilizer of two closed points is isomorphic to SO_{n-1} or $SO_{1,n-2}$, then M is cohomogeneity one manifold.

2 Lorentzian homogeneous manifolds with completely reducible isotropy

2.1 Infinitezimal description

Let $(M = G/H, g)$ *be a homogeneous pseudo-Riemannian manifold. We will assume that the isotropy group* $j(H)$ *is a connected completely reducible linear group.*
Then there is a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p},\ [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{m}.$$

We identify $\mathfrak{m}$ *with the tangent space* T_oM *at the point* $o = eH$ *and denote by* g *the Minkowski metric on* $\mathfrak{p}$ *induces by* g^M*. We will call the pair (* $\mathfrak{g}= \mathfrak{h}+ \mathfrak{p},\ g$*) the* **metric reductive decomposition** *associated with homogeneous pseudo-Riemannian manifold* $(M = G/H, g^M)$*.*

Conversely, let $(\mathfrak{g} = \mathfrak{h} + \mathfrak{p}, g)$ *be a reductive decomposition of a Lie algebra* $\mathfrak{g}$ *together with* $\mathrm{ad}_{\mathfrak{h}}$*-invariant pseudo-Euclidean metric* g *on* $\mathfrak{p}$ *and* G *is a Lie group with the Lie algebra* $\mathfrak{g}$ *such that the connected subgroup* $H \subset G$ *generated by* $\mathfrak{h}$ *is closed.*
Then we can identify the tangent space $T_o(G/H)$ *of the homogeneous manifold* $M = G/H$ *at point* $o = eH$ *with* $\mathfrak{p}$ *and the metric* g *induces an invariant pseudo-Riemannian metric on* $M = G/H$*. Moreover, if the representation* $\mathrm{ad}_{\mathfrak{h}}|_{\mathfrak{p}}$ *is exact, then the action of* G *on* M *is almost effective (i.e. has a discrete kernel).*

2.2 D-extension of a homogeneous pseudo-Riemannian manifolds and D-product of homogeneous pseudo-Riemannian manifolds

Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ *be a reductive decomposition of a Lie algebra. A derivation* D *of the Lie algebra* $\mathfrak{g}$ *is called a* **derivation of the reductive decomposition** *if it preserves the reductive decomposition and acts trivially on* $\mathfrak{h}$ *i.e.* $D\mathfrak{h} = 0$.

We denote by $\mathfrak{g}(D) = \mathbb{R}D + \mathfrak{g}$ *the corresponding extended Lie algebra with* $[D, x] := Dx,\ x \in \mathfrak{g}$. *It has the natural reductive decomposition*

$$\mathfrak{g}(D) = \tilde{\mathfrak{g}} = \mathfrak{h} + (\mathbb{R}D + \mathfrak{p}) = \mathfrak{h} + \tilde{\mathfrak{p}}. \tag{2.1}$$

If g *is an* $\mathrm{ad}_{\mathfrak{h}}$*-invariant pseudo-Euclidean metric on* $\mathfrak{p}$, *we extend it to* $\mathrm{ad}_{\mathfrak{h}}$*-invariant metric* $\tilde{g}$ *on* $\tilde{\mathfrak{p}}$ *such that* $\tilde{g}(D, \mathfrak{p}) = 0$. *We normalize the metric by the condition* $\tilde{g}(D, D) = 1$. *The metric reductive decomposition*

$$(\tilde{\mathfrak{g}} = \mathfrak{h} + \tilde{\mathfrak{p}}, \tilde{g}) \tag{2.2}$$

is called the D**-extension** *of the metric reductive decomposition associated with derivation* D. *We associate with a metric reductive decomposition a homogeneous pseudo-Riemannian manifold* $(M = G/H, g^M)$. *We denote by* $\tilde{G} = G \cdot \exp\,(\mathbb{R}D)$ *the Lie group which is a semidirect product of the normal subgroup* G *and the 1-parameter group generated by* D. *Since* G *is a closed normal subgroup of* $\tilde{G}$, *the subgroup* $H \subset G \subset \tilde{G}$ *is closed in* $\tilde{G}$. *The metric* $\tilde{g}$ *induces an invariant pseudo-Riemannian metric on* $\tilde{M} = \tilde{G}/H$. *The homogeneous pseudo-Riemannian manifold* $(\tilde{M} = \tilde{G}/H, g^{\tilde{M}})$ *is called the* D**-extension** *of a homogeneous pseudo-Riemannian manifold* (M, g^M) *associated with derivation* D.

2.3 Completely reducible subalgebras of the Lorentz algebra

Theorem 4 *Let V be the Minkowski vector space of a signature $(1,n)$ and $\mathfrak{h}$ a proper completely reducible subalgebra of the Lorentz algebra $\mathfrak{so}(V)$. Then there is an orthogonal decomposition $V=\mathfrak{m}+\mathfrak{e}$ where $\mathfrak{m}$ is a subspace of a signature $(1,m-1)$, $1\leq m\leq n$ and $\mathfrak{e}$ is an $(n-m+1)$-dimensional Euclidean subspace such that $\mathfrak{h}=\mathfrak{so}(\mathfrak{m})+\mathfrak{k}$ where $\mathfrak{k}$ is a subalgebra of the orthogonal Lie algebra $\mathfrak{so}(\mathfrak{e})$.*

Note that $\mathfrak{so}(\mathfrak{m})$ is maximal non compact ideal of $\mathfrak{h}$ and $\mathfrak{so}(\mathfrak{e})$ is the maximal compact ideal of $\mathfrak{h}$. We will call $\mathfrak{so}(\mathfrak{m})$ the **non compact part of** $\mathfrak{h}$.

Corollary 1 *The subalgebra $\mathfrak{h}$ is compact if and only if $m=1$.*
If $m=2$, then $\mathfrak{m}\wedge\mathfrak{m}\approx\mathfrak{so}(\mathfrak{m})$ is 1-dimensional trivial $\mathfrak{h}$-module.
If $m=3$, then $\mathfrak{m}\wedge\mathfrak{m}\approx\mathfrak{so}(\mathfrak{m})$ is a simple $\mathfrak{h}$-module isomorphic to $\mathfrak{m}$.
If $m>3$, then $\mathfrak{h}$-module $\mathfrak{m}\wedge\mathfrak{m}$ is simple and is not isomorphic to $\mathfrak{m}$.

2.4 Homogeneous Lorentzian manifolds with completely reductive non-compact isotropy group

Let $(M = G/H, g^M)$ *be a homogeneous Lorentzian manifold and* $(\mathfrak{g} = \mathfrak{h} + \mathfrak{p}, g)$ *the corresponding metric reductive decomposition. There is an orthogonal decomposition*

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{e}$$

such thatthe isotropy Lie algebra $j(\mathfrak{h}) = \mathfrak{so}(\mathfrak{m}) \oplus \mathfrak{k}$, *where* $\mathfrak{k} \subset \mathfrak{so}(\mathfrak{e})$. *We will identify the stability subalgebra* $\mathfrak{h}$ *with the isotropy Lie algebra* $j(\mathfrak{h})$ *and put*

$$\mathfrak{g}_0 := \mathfrak{so}(\mathfrak{m}) + \mathfrak{m}, \quad \mathfrak{g}_1 := \mathfrak{k} + \mathfrak{e}.$$

Lemma 2 *If* $m = \dim \mathfrak{m} > 2$, *then the decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$$

is a semidirect decomposition into a sum of the subalgebra

$$\mathfrak{g}_1 = \mathfrak{k} + \mathfrak{e}$$

and the ideal

$$\mathfrak{g}_0 = \mathfrak{so}(\mathfrak{m}) + \mathfrak{m}$$

isomorphic either to the Poincare algebra $\mathfrak{so}(1, m-1) + \mathbb{R}^m$ *or to the algebras* $\mathfrak{so}(1, m)$ *or* $\mathfrak{so}(2, m-1)$. *Moreover,*
the orbit $M_1 := G_1 o$ *of the subgroup* $G_1 \subset G$, *generated by* $\mathfrak{g}_1$, *is a totally geodesic Riemannian submanifold of* M *and the orbit* $S := G_0 o$ *of the subgroup* G_0 *generated by* $\mathfrak{g}_0$ *is an* m*-dimensional Lorentzian submanifold of constant curvature.*

We call $S = G_0 o$ *the* **orbit of constant curvature**.

2.5 Main theorem in the case when $m = \dim \mathfrak{m} > 2$

Theorem 5 *Let $(M = G/H, g^M)$ be a homogeneous Lorentzian manifold with completely reducible non compact isotropy group $j(H)$. Assume that $\dim \mathfrak{m} > 2$. If the orbit of constant curvature $S = G_0 o$ is not flat, then $G = G_0 \times G_1$ and the manifold M is a Riemannian direct product of the Lorentzian space of constant curvature $S = S_m(k)$ and a homogeneous Riemannian manifold $M_1 = G_1/K$.*
If the orbit of constant curvature S is flat,(i.e $S = S_m(0)$ is the Minkowski space) then either M is a Riemannian direct product of $S_m(0)$ and a homogeneous Riemannian manifold or it is the D-product of $S_m(0)$ and a homogeneous Riemannian manifold M_1.
Here $D = E \oplus D_1$ where $E = \mathrm{Id}\,|\mathfrak{m}$ the identity endomorphism considered as a derivation of the Poincare algebra $\mathfrak{g}_0 = \mathfrak{so}(\mathfrak{m}) + \mathfrak{m}$ and D_1 is a derivation of the reductive decomposition $\mathfrak{g}' = \mathfrak{k} + \mathfrak{e}'$ associated with M_1.

3 Lorentzian G-manifolds. General theory

3.1 Proper G-manifolds

Let G be a closed subgroup os the isometry group of a Riemannian manifold M. Then the Riemannian G-manifold N is proper. Conversely we have the following classical result by Montgomery, Palais, Bredon.

Theorem 6 *A proper G-manifold M admits a G-invariant complete metric g. Hence, the orbit space Ω is a metric space. Moreover, an appropriate G-invariant neighborhood $M(P)$ of any orbit $P = Gx = G/H$ is G-diffeomorphic to the normal bundle*

$$T^{\perp}P) = G \times_H V = (G \times V)/H.$$

Here $V = T_x^{\perp}M$ and H acts on V by the isotropy representation.

" A Riemannian G-manifold is a collection of vector bundles" (Davis).
The identification of $M(P)$ with $T^{\perp}P$ is called the slice representation and it is defined by the normal exponential map

$$exp : T^{\perp}P \to M, \; n_x \mapsto exp_x(n).$$

3.2 Normal bundle and quasi-slice representation

We will discuss the structure of a neighborhood of an orbit $P = Gx$ of a Lorentzian G-manifold M.

We say that an orbit $P = Gx = G/H$ of the isometry group G of a Lorentzian G-manifold is time-like (T-orbit), respectively, space-like (S-orbit) or isotropic (N-orbit) if the tangent space T_xP at any point is time-like, resp., space-like or isotropic.
We denote by M^T,resp.,M^S the open submanifolds which consist of T-orbits, resp., S-orbits and by
$M^N = M \setminus (M^T \cup M^S)$
the closed subset on N-orbits.

For an N-orbit $P = G/H$ we will assume that the tangent space T_xM has a direct sum decomposition $T_xM = T_xP + N_x$ invariant under the isotropy group $j(H)$ and we denote by τ, ν the restriction of the isotropy representation to the tangent space T_xP and the "normal space" $T_x^{\perp}P := N_x$.

The vector bundle

$$T^{\perp}P = \{gn,\ g \in G,\, n \in N_x\} = G \times_H N_x$$

is called the normal bundle of the orbit P.

As in Riemannian case, we consider the normal exponential map

$$exp : T^{\perp}P \to M,\ n_y \mapsto exp_y n,\ y \in P.$$

Lemma 3 *The normal exponential map is a G-equivariant local diffeomorphism of some G-invariant neighborhood $N(P) \subset T^{\perp}P$ of zero section of the normal bundle onto a neighborhood $M(P)$ of the orbit P in M.*

Unfortunately, it is not clear that it is a diffeomorphism, in general.

The metric $g|_{M(P)}$ induces a G-invariant Lorentz metric $g^N = exp^* g_{M(P)}$ in $N(P)$ such that $exp : N(P) \to M(P)$ becomes a local isometry. The G-equivariant map exp is called a quasi-slice representation of a neighborhood $M(P)$ of P.

3.2.1 Slice representation and the standard model of a Lorentzian G-manifold near an orbit

Definition 2 *1. Lorentzian G-manifold $(N(P), g^N)$ is called the standard model of the Lorentzian manifold (M, g) in a neighborhood of the orbit P.*

2. We say that the manifold (M, g) admits a slice representation near an orbit $P = G/H$ is there exist a standard model $(N(P), g^N)$ such that $exp : N(P) \to M(P)$ is a diffeomorphism of $N(P)$ on a neighborhood $M(P)$ of the orbit P in M.

3.2.2 Sufficient conditions that a Lorentzian G-manifold admits a slice representation near an orbit P

Proposition 1 *A Lorentzian G-manifold (M, g) admits a slice representation near an orbit $P = G/H \subset M$ if one of the following conditions holds:*

1. *The action of G is proper in some neighborhood of P;*
2. *The stability subgroup H is (relatively) compact ;*
3. *The normal isotropy group $\nu(H) \subset O(T_x^{\perp} P)$ is (relatively) compact;*
4. *P is a T-orbit;*
5. *There is a G-invariant neighborhood $M(P) \subset M$ of the orbit P such that there is unique perpendicular from a point $yz \in M(P)$ onto P.*

Note that there is a neighborhood of P with the above property, but it is not clear that it can be chosen to be G-invariant.

3.3 A description of invariant Lorentzian metrics on a homogeneous vector bundle

We describe the structure of standard models of a Lorentzian G-manifolds near an orbit $P = G/H$. The problem can be formulated as follows.
Given a homogeneous manifold $P = G/H$ and an H-module V. We describe the construction of G-invariant Lorentzian metrics g^M on the homogeneous vector bundle
$\pi : M = G \times_H V \to P = G/H$
(defined in a G-invariant neighborhood of zero section) in terms of a connection in π , G-invariant metric g^P in the base $P = G/H$ and H-invariant metric g^V in the fibre V (defined in an invariant neighborhood of the origin).

Proposition 2 *Let ∇ be a G-invariant connection in π with the horizontal subdistribution $\mathcal{H} \subset TM$, g^P is an invariant Riemannian (resp. Lorentzian) metric in $P = G/H$ and g^V an H-invariant Lorentzian (resp., Riemannian) metric in V.*
We can consider g^V as a G-invariant metric on the vertical subbundle T^vM. The metric g^P defines a G-invariant metric $g^{\mathcal{H}}$ on the horizontal subbundle $\mathcal{H}$.
Then $g^M := g^V \oplus g^{\mathcal{H}}$ is a G-invariant Lorentzian metric in M and P is S-orbit (resp. T-orbit). Any invariant metric on M such that $\pi : M \to P$ is a pseudo-Riemannian submersion can be obtained by this construction.

Similar we can describe a class on invariant metrics on M such that P is an N-orbit.

3.4 Structure of the standard homogeneous vector bundle

Let $M = G \times_H V$ *be a standard model near a non-degenerate orbit* $P = G/H$. *We give a description of the associated triple* $(\mathfrak{g}, \mathfrak{h}, V)$ *as follows. We denote by* $\mathfrak{k}_\tau = \mathrm{Ker}\ \tau$ *and* $\mathfrak{k}_\nu = \mathrm{Ker}\ \nu$ *the kernel of the isotropy representation of the stability subalgebra* $\mathfrak{h}$ *in the tangent space* $T_x P = \mathfrak{p}$ *and in the normal space* $V = T_x^\perp$.

Lemma 4 *There is a direct sum decomposition of the stability subalgebra*

$$\mathfrak{h} = \mathfrak{k}_\tau \oplus \mathfrak{k}_\nu \oplus \mathfrak{h}'.$$

where $\mathfrak{h}'$ *is a compact ideal of* $\mathfrak{h}$ *and* $\mathfrak{k}_\tau$ *is an ideal of* $\mathfrak{g}$.

Theorem 7 *Let*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$$

and

$$\mathfrak{h} = \mathfrak{k}_\tau \oplus \mathfrak{k}_\nu \oplus \mathfrak{h}'$$

where $[\mathfrak{k}_\tau, \mathfrak{p}] \subset \mathfrak{k}_\tau$.
1) Assume that
i) the restriction of the adjoint representation of $\mathfrak{k}_\nu \oplus \mathfrak{h}'$ *to* $\mathfrak{m}$ *is exact and preserves an Euclidean metric* $g^{\mathfrak{m}}$ *and*
ii) there is an isomorphism $\nu : \mathfrak{k}_\tau + \mathfrak{h}' \to \nu(\mathfrak{h}) \subset \mathfrak{so}(V) = \mathfrak{so}_{1,m-1}$. *Then the homogeneous vector bundle* $M = G \times_H V$ *associated with the triple* $(\mathfrak{g}, \mathfrak{h}, V)$ *admits an invariant Lorentzian metric such that* $P = G/H$ *is a T-orbit.*
2) Assume that
i) the restriction of the adjoint representation of $\mathfrak{k}_\nu \oplus \mathfrak{h}'$ *to* $\mathfrak{m}$ *is exact and preserves a Lorentz metric* $g^{\mathfrak{m}}$ *and*
ii) there is an isomorphism $\nu : \mathfrak{k}_\tau + \mathfrak{h}' \to \nu(\mathfrak{h}) \subset \mathfrak{so}(V) = \mathfrak{so}_m$ *onto a subalgebra of the orthogonal Lie algebra. Then the homogeneous vector bundle* $M = G \times_H V$ *associated with the triple* $(\mathfrak{g}, \mathfrak{h}, V)$ *admits an invariant Lorentzian metric such that* $P = G/H$ *is an S-orbit.*

For example, a direct sum of the constant H*-invariant metric* g^V *and the invariant metric on the horozontal bundle defined by* $g^{\mathfrak{m}}$ *will be an invariant Lorentz metric in* M.

Remark *Similar description of a class of standard model near an N-orbit can be given.*

4 Cohomogeneity one Lorentzian manifolds

We will apply this general construction to a description of standard models which have codimension one orbit. We need to describe subgroups of the orthogonal group and the Lorentz group which have codimension one orbits.

Subgroups H of $SO(V)$ with codimension one orbit and H-invariant metrics

4.0.1 Sphere transitive subgroups of the orthogonal group $SO(V)$

Let $V = \mathbb{R}^m$ be the Euclidean vector space and $SO(V)$ the special orthogonal group.

Theorem 8 *(A. Borel) Connected subgroups $H \subset SO(V)$ which acts transitively on the sphere S^{n-1} belong to the following Borel list*

$$SO_m, SU_{m/2}, U_{m/2}, Sp_{m/4}, Sp_1 \cdot Sp_{m/4}, G_2\ (m=7), Spin_7,\ (m=8), Spin_9,\ m=16.$$

An invariant Riemannian metric on the unit sphere S^{m-1} invariant under the group H depends on one parameter (a scaling) for $H = SO_m, G_2, Spin_7$, depends on two parameter for $H = SU_{m/2,U_{m/2}}, Sp_1 \cdot Sp_{m/4}, Spin_9$ and depends on 3 parameters for $Sp_{m/4}$.
If the metric of S^m depends on one parameter, then any H-invariant Riemannian metric g on V is given by

$$g_e = f^2(r)dt^2 + h^2(r)g_0|_{e^\perp},\ e \in S^{m-1}$$

where $f(t), h(t)$ are even positive smooth functions on $\mathbb{R}$ with $f(0) = h(0)$. Similar, but more complicated description is given by L. Verdiani in all other cases.

4.0.2 Cohomogeneity one subgroups of the Lorentz group $SO(1,n)$

Let $V = \mathbb{R}^{1,m}$ *be the Minkowski space and*

$$V = \mathbb{R}p + \mathbb{R}q + E$$

a decomposition where p, q *are isotropic vectors with* $< p, q >= 1$ *and* $E = \text{span}\ \{p, q\}^{\perp} = \mathbb{R}^{m-1}$ *the Euclidean space.*

Proposition 3 *Any proper connected subgroup* $H \subset SO(1,n)$ *of the Lorentz group with codimension one orbit in* V *is conjugated to the subgroup associated with Lie algebra of the form*

$$\mathfrak{h} = \mathbb{R}h_0 + p \wedge E + \mathfrak{k}$$

where $\mathfrak{k} \subset \mathfrak{so}(E)$ *is a subalgebra of the orthogonal algebra and* $h_0 = p \wedge q + k$ *where* $k \in Z(\mathfrak{k})$ *is an element of the centre of* $\mathfrak{k}$*.*

The orbit of a point $x = \lambda p + \mu q + e_0,\ |e_0| = 1$ *has codimension one if* $\mu \neq 0$ *and can be space-like, time-like or isotropic.*

If $\mu = 0$ *, it is an isotropic orbit of cosimension 2.*

In the case $H = Sim(E)$*, the codimension one orbits are isomorphic to* $V_t = Sim(E)/SO(E) = R^* \cdot SO(E) \cdot E/SO(E)$ *with the reductive decomposition*

$$\mathfrak{sim}(E) = \mathfrak{so}(E) + (\mathbb{R}p \wedge q + p \wedge E).$$

There is a two parametric family of invariant Lorentzian (and also Riemannian) metrics $g_{u,v}$ *on* V_t*. A* $Sim(E)$*-invariant metric on* V *is given by*

$$g = dt^2 + g_{u(t),v(t)}.$$

4.1 Cohomogeneity one Lorentzian manifolds. Case of time-like orbit

Recall that a G-invariant Lorentzian manifold (M, g) admits a slice representation near a T-orbit. So to describe the structure of Cohomogeneity one Lorentzian manifold near such orbit P we may assume that $M = G \times_H V$ is a standard model.

Case of proper action

Assume that the orbit $P = G/H$ is a proper G-manifold. Then H is compact and there is a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

and the isotropy representation $\tau(H) = \mathrm{Ad}_H|_{\mathfrak{m}}$ has an invariant vector $t \in \mathfrak{m}$. Any epimorphism $\nu : H \to B \subset SO(V)$ of H onto a group from Borel list defines a standard model $M = G \times_H V$ with invariant Lorentz metric with T-orbit P.

Case when $P = G/H$ is a Lorentz manifold with completely reducible non compact isotropy group $\tau(H)$ *If $\dim P > 2$ and it has no flat factor, then it is a direct product of a space of non zero constant curvature S/L and a homogeneous Riemannian manifold G'/K. Then the corresponding standard model has the form $M = S/L \times G' \times_K V$ where the action of K in V is defined by a epimorphism $\nu : K \to B \subset SO(V)$ onto a Borel group B.*

Case, when $\tau(H)$ preserves an isotropic line $\mathbb{R}p$

Assume that $\mathfrak{k}_\tau = 0$. Then the group G acts almost effectively on P and the Lie algebra $\mathfrak{h}$ must admit a decomposition into a direct sum $\mathfrak{h} = \mathfrak{h}_{nc} \oplus \mathfrak{k}$ where $\mathfrak{k}$ is the maximal compact ideal.

Then an epimorphism $K \to B \subset SO(V)$ of the corresponding normal subgroup K onto a Borel subgroup B defines a standard model $M = G \times_H V$.

In general case, we have a decomposition

$$\mathfrak{g} = \mathfrak{k}_\tau + \mathfrak{h}_{nc} + \mathfrak{k} + \mathfrak{m}$$

and a standard model is define by an epimorphism $K_\tau \cdot K$ onto a Borel subgroup B with kernel in K. In the case of simple Borel subgroup, the corresponding standard model is a direct product of $P = G/H$ and the Euclidean space V with the action of the group $K_\tau \approx B$.

4.2 Cohomogeneity one Lorentzian manifold. Case of space-like orbit

Proposition 4 *Let $(M = G \times_H V, g)$ be the standard Cohomogeneity one Lorentzian manifold with space-like orbit $P = G/H$. Then*
i) there is a connected normal subgroup $L \subset \mathrm{Ker}\,\tau \subset H$ with the Lie algebra of the form $\mathfrak{l} = \mathbb{R}h_0 + \mathfrak{a}$ where $\mathrm{ad}_{h_0}|_{\mathfrak{a}} = 1,\ [\mathfrak{a}, \mathfrak{a}] = 0$, such that the group $\bar{G} = G/L$ acts transitively on P with compact stability subgroup $\bar{K}$ and a reductive decomposition

$$\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{m}}.$$

ii) Moreover one can choose a lift $\mathfrak{k} + \mathfrak{m}$ of $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{m}}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathbb{R}h_0 + \mathfrak{a} + \mathfrak{k}) + \mathfrak{m}$$

and $\mathfrak{k}$ is a compact subalgebra with $[\mathfrak{l}, \mathfrak{k}] = 0$.

iii) Assume moreover that either ad_{h_0} is a semisimple endomorphism or $\bar{G}$ has no invariant vector field on P (or , equivalently, $[\bar{\mathfrak{k}}, \bar{\mathfrak{m}}] = \bar{\mathfrak{m}}$).
Then the centralizer

$$\mathfrak{g}^0 := C_{\mathfrak{g}}(h_0) = \mathbb{R}h_0 + \mathfrak{k} + m$$

and $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{g}^1 = (\mathbb{R}h_0 + \mathfrak{k} + \mathfrak{m}) + \mathfrak{a}$ is a graded Lie algebra.

Corollary 2 *Under assumption of Proposition, $\mathfrak{g}_0$ is a central extension of the Lie algebra $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \mathfrak{m}$ defined by a closed 2-form ω on $\bar{\mathfrak{g}}$ with $\mathrm{Ker}\,\omega \supset \bar{\mathfrak{k}}$, that is the Lie bracket $[.,.]_\omega$ in $\mathfrak{g}^0$ is given by*

$$[k, k']_\omega = [k, k'],\ [k, x]_\omega = [k, x],\ [x, y]_\omega = [x, y] + \omega(x, y)h_0,\ k, k' \in \mathfrak{k},\ x, y \in \mathfrak{m}.$$

If the subalgebra $\mathfrak{g}^0$ is given, the Lie algebra $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{g}^1$ is defined by a representation ρ of $\mathfrak{g}^0$ on the vector space $\mathfrak{g}^1 = \mathfrak{a}$ with $\rho(h_0) = 1$ such that $\rho(\mathfrak{k})$ preserves an Euclidean metric $g^{\mathfrak{a}}$ on $\mathfrak{a}$.

This leads to the following construction of standard cohomogeneity one Lorentzian manifold with space-like orbit $P = G/H$ under assumption that G does not preserve any vector field on P.

Theorem 9 *Let $(P = \bar{G}/K, g^P)$ be a simply connected homogeneous Riemannian manifold with a compact stabilizer K and*

$$\bar{\mathfrak{g}} = \mathfrak{k} + \mathfrak{m}$$

the corresponding reductive decomposition.
Let ω be a closed (may be zero) 2-form on $\bar{\mathfrak{g}}$ with support in $\mathfrak{m}$ and

$$\mathfrak{g}^0 = \mathbb{R}h_0 + \mathfrak{k} + \mathfrak{m}$$

the central extension of $\bar{\mathfrak{g}}$ defined by the cocycle ω.
Let $\rho : \mathfrak{g}^0 \to \mathfrak{gl}(\mathfrak{a})$ be a representation in an Euclidean vector space E, g^E such that $\rho(h_0) = 1$ and $\rho(\mathfrak{k})$ preserves g^E.
Let

$$\mathfrak{g} = \mathfrak{g}^0 + E = \mathbb{R}h_0 + \mathfrak{k} + \mathfrak{m} + E$$

be the Lie algebra with the commutative ideal E and the action ρ of $\mathfrak{g}^0$ on E and G connected simply connected Lie group with the Lie algebra $\mathfrak{g}$ and H the connected subgroup of G generated by $\mathfrak{h} := \mathbb{R}h_0 + \mathfrak{k} + E$. Then the group G acts transitively as a group of isometries on the manifold (P, g^P) such that $P = G/H$.

Consider the representation of H in the Minkowski space $V = \mathbb{R}p + \mathbb{R}q + E$ generated by the following representation $\hat{\rho}$ of the Lie algebra $\mathfrak{h} = \mathbb{R}h_0 + E + \mathfrak{k}$:

$$\hat{\rho}(h_0) = p \wedge q, \;\; \hat{\rho}(e) = p \wedge e, \;\; \hat{\rho}(k) = \rho(k), \;\; \text{for } e \in E, \, k \in \mathfrak{k}$$

where we assume that $\rho(k)p = \rho(k)q = 0$. It preserves the Lorentzian metric g^V in V defined by $g^V|_E = g^E, g^V(p, p + E) = g^V(q, q + E) = 0, \; g^V(p, q) = 1$. Then the cohomogeneity one manifold $M = G \times_{\hat{\rho}(H)} V$ has an invariant metric $g^P \oplus g^V$.
Moreover, any standard cohomogeneity one Lorentzian manifold with space-like orbit $P = \bar{G}/K$ can be obtained by this construction.

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