Einstein-like Walker manifolds and generalized locally symmetric spaces

Sandra Gavino Fernández

Department of Geometry and Topology
University of Santiago de Compostela

July 11th, 2009

Work in progress with Miguel Brozos Vázquez

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Aim: To investigate some generalizations of locally symmetric spaces.

Different approaches:

1. Einstein-like manifolds (Algebraic conditions)
2. $\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces (Geometry of geodesics)
3. $\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces (Geometry of unit circles)

Three-dimensional Walker metrics

Definition

A manifold $(\mathcal{M}, g)$ is Walker if it admits a null parallel distribution $\mathcal{D}$ of dimension r.

Definition

A manifold $(\mathcal{M}, g)$ is Walker if it admits a null parallel distribution $\mathcal{D}$ of dimension r.

- In dimension 3 there exists coordinates such that:

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ where } f = f(t, x, y) \text{ and } \varepsilon = \pm 1.$$

Definition

A manifold $(\mathcal{M}, g)$ is Walker if it admits a null parallel distribution $\mathcal{D}$ of dimension r.

- In dimension 3 there exists coordinates such that:
$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ where } f = f(t, x, y) \text{ and } \varepsilon = \pm 1.$$
- $\mathcal{D}$ is locally generated by ∂_t.

Definition

A manifold $(\mathcal{M}, g)$ is Walker if it admits a null parallel distribution $\mathcal{D}$ of dimension r.

- In dimension 3 there exists coordinates such that:
$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ where } f = f(t, x, y) \text{ and } \varepsilon = \pm 1.$$
- $\mathcal{D}$ is locally generated by ∂_t.
- $(\mathcal{M}, g)$ is strictly Walker if $\mathcal{D}$ admits a null parallel vector field.

Definition

A manifold $(\mathcal{M}, g)$ is Walker if it admits a null parallel distribution $\mathcal{D}$ of dimension r.

- In dimension 3 there exists coordinates such that:
$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ where } f = f(t, x, y) \text{ and } \varepsilon = \pm 1.$$
- $\mathcal{D}$ is locally generated by ∂_t.
- $(\mathcal{M}, g)$ is strictly Walker if $\mathcal{D}$ admits a null parallel vector field.

$$- > (\mathcal{M}, g) \text{ is strictly Walker} \Leftrightarrow f = f(x, y)$$

Definition

A manifold $(\mathcal{M}, g)$ is Walker if it admits a null parallel distribution $\mathcal{D}$ of dimension r.

- In dimension 3 there exists coordinates such that:
  $g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}$ where $f = f(t, x, y)$ and $\varepsilon = \pm 1$.
- $\mathcal{D}$ is locally generated by ∂_t.
- $(\mathcal{M}, g)$ is strictly Walker if $\mathcal{D}$ admits a null parallel vector field.

  $->$ $(\mathcal{M}, g)$ is strictly Walker $\Leftrightarrow$ $f = f(x, y)$
  $->$ $(\mathcal{M}, g)$ is strictly Walker $\Leftrightarrow$ $Ric^2 = 0$

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x,y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Ricci tensor

$\rho(x,y) = \sum_i \varepsilon_i R(e_i, x, y, e_i)$

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x,y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Ricci tensor

$\rho(x,y) = \sum_i \varepsilon_i R(e_i, x, y, e_i)$

Scalar curvature

$\tau = \sum_i \varepsilon_i \rho(e_i, e_i)$

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Ricci tensor

$\rho(x, y) = \sum_i \varepsilon_i R(e_i, x, y, e_i)$

Scalar curvature

$\tau = \sum_i \varepsilon_i \rho(e_i, e_i)$

Ricci operator

$g(\rho(x), y) := \rho(x, y)$ for a unit x and y.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Ricci tensor

$\rho(x, y) = \sum_i \varepsilon_i R(e_i, x, y, e_i)$

Scalar curvature

$\tau = \sum_i \varepsilon_i \rho(e_i, e_i)$

Ricci operator

$g(\rho(x), y) := \rho(x, y)$ for a unit x and y.

Skew-symmetric operator

$g(R(x, y)z, w) = R(x, y, z, w)$ for $x, y \in T_p\mathcal{M}$.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Ricci tensor

$\rho(x, y) = \sum_i \varepsilon_i R(e_i, x, y, e_i)$

Scalar curvature

$\tau = \sum_i \varepsilon_i \rho(e_i, e_i)$

Ricci operator

$g(\rho(x), y) := \rho(x, y)$ for a unit x and y.

Skew-symmetric operator

$g(R(x, y)z, w) = R(x, y, z, w)$ for $x, y \in T_p\mathcal{M}$.

Jacobi operator

$R_x = R(\cdot, x)x$ for a unit x.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension n.

1. ∇ denotes the Levi-Civita connection.
2. $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ is the curvature operator.

Ricci tensor

$\rho(x, y) = \sum_i \varepsilon_i R(e_i, x, y, e_i)$

Scalar curvature

$\tau = \sum_i \varepsilon_i \rho(e_i, e_i)$

Ricci operator

$g(\rho(x), y) := \rho(x, y)$ for a unit x and y.

Skew-symmetric operator

$g(R(x, y)z, w) = R(x, y, z, w)$ for $x, y \in T_p\mathcal{M}$.

Jacobi operator

$R_x = R(\cdot, x)x$ for a unit x.

Szabó operator

$R'_x = \nabla_x R(\cdot, x)x$ for a unit x.

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries

  $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries
  - $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$
  - $->$ dim $\mathcal{M} = 3$
    $(\mathcal{M}, g)$ locally symmetric $\Leftrightarrow \nabla \rho = 0$

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries
  - $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$
  - $->$ dim $\mathcal{M} = 3$
    $(\mathcal{M}, g)$ locally symmetric $\Leftrightarrow \nabla \rho = 0$

$$W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}^{\perp}$$

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries
  - $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$
  - $->$ $\dim \mathcal{M} = 3$
    $(\mathcal{M}, g)$ locally symmetric $\Leftrightarrow \nabla \rho = 0$

$$W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}^{\perp}$$

Einstein-like manifolds (Gray, 1978)

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries
  - $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$
  - $->$ $\dim \mathcal{M} = 3$
    $(\mathcal{M}, g)$ locally symmetric $\Leftrightarrow \nabla \rho = 0$

$$W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}^{\perp}$$

Einstein-like manifolds (Gray, 1978)

- $\mathcal{A}$: $(\nabla_X \rho)(X, X) = 0$,

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries
  - $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$
  - $->$ $\dim \mathcal{M} = 3$
    $(\mathcal{M}, g)$ locally symmetric $\Leftrightarrow \nabla \rho = 0$

$$W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}^{\perp}$$

Einstein-like manifolds (Gray, 1978)

- $\mathcal{A}$: $(\nabla_X \rho)(X, X) = 0$,
- $\mathcal{B}$: $(\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z)$,

- $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ local geodesic symmetries are isometries
  - $->$ $(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow \nabla R = 0$
  - $->$ $\dim \mathcal{M} = 3$
    $(\mathcal{M}, g)$ locally symmetric $\Leftrightarrow \nabla \rho = 0$

$$W = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}^{\perp}$$

Einstein-like manifolds (Gray, 1978)

- $\mathcal{A}$: $(\nabla_X \rho)(X, X) = 0$,
- $\mathcal{B}$: $(\nabla_X \rho)(Y, Z) = (\nabla_Y \rho)(X, Z)$,
- $\mathcal{C}^{\perp}$: $\nabla_X(\rho)(Y, Z) = \frac{1}{(n+2)(n-1)} \{nX(\tau)g(Y, Z) + \frac{n-2}{2}[Y(\tau)g(X, Z) + Z(\tau)g(X, Y)]\}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{C}^{\perp}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{C}^{\perp}$.
4. $(\mathcal{M}, g)$ is in class $\mathcal{A} \oplus \mathcal{C}^{\perp}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{C}^{\perp}$.
4. $(\mathcal{M}, g)$ is in class $\mathcal{A} \oplus \mathcal{C}^{\perp}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{C}^{\perp}$.
4. $(\mathcal{M}, g)$ is in class $\mathcal{A} \oplus \mathcal{C}^{\perp}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally conformally flat.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{C}^{\perp}$.
4. $(\mathcal{M}, g)$ is in class $\mathcal{A} \oplus \mathcal{C}^{\perp}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally conformally flat.
2. $(\mathcal{M}, g)$ is in class $\mathcal{B}$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally symmetric.
2. $(\mathcal{M}, g)$ is in class $\mathcal{A}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{C}^\perp$.
4. $(\mathcal{M}, g)$ is in class $\mathcal{A} \oplus \mathcal{C}^\perp$.

Theorem

For a 3-dimensional Walker manifold $(\mathcal{M}, g)$ are equivalent:

1. $(\mathcal{M}, g)$ is locally conformally flat.
2. $(\mathcal{M}, g)$ is in class $\mathcal{B}$.
3. $(\mathcal{M}, g)$ is in class $\mathcal{B} \oplus \mathcal{C}^\perp$.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Theorem (Berndt-Vanhecke, 1992)

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Theorem (Berndt-Vanhecke, 1992)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Theorem (Berndt-Vanhecke, 1992)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is simultaneously $\mathfrak{C}$-space and $\mathfrak{P}$-space.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Theorem (Berndt-Vanhecke, 1992)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is simultaneously $\mathfrak{C}$-space and $\mathfrak{P}$-space.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Theorem (Berndt-Vanhecke, 1992)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is simultaneously $\mathfrak{C}$-space and $\mathfrak{P}$-space.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.

1. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space $(T\mathfrak{C})$ if the Jacobi operator has constant eigenvalues along every timelike geodesic.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{C}$-space if the Jacobi operator has constant eigenvalues along every geodesic.
2. $(\mathcal{M}, g)$ is a $\mathfrak{P}$-space if the Jacobi operator has parallel eigenspaces along every geodesic.

Theorem (Berndt-Vanhecke, 1992)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is simultaneously $\mathfrak{C}$-space and $\mathfrak{P}$-space.

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.

1. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space ($T\mathfrak{C}$) if the Jacobi operator has constant eigenvalues along every timelike geodesic.
2. $(\mathcal{M}, g)$ is a timelike $\mathfrak{P}$-space ($T\mathfrak{P}$) if the Jacobi operator has parallel eigenspaces along every timelike geodesic.

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a timelike $\mathfrak{C}$-space and a timelike $\mathfrak{P}$-space.

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a timelike $\mathfrak{C}$-space and a timelike $\mathfrak{P}$-space.

$T\mathfrak{C} \cap T\mathfrak{P}$ is equivalent to

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a timelike $\mathfrak{C}$-space and a timelike $\mathfrak{P}$-space.

$T\mathfrak{C} \cap T\mathfrak{P}$ is equivalent to

$$\nabla_x R_{xyyx} = 0, \ \forall x \text{ timelike and } y \perp x$$

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a timelike $\mathfrak{C}$-space and a timelike $\mathfrak{P}$-space.

$T\mathfrak{C} \cap T\mathfrak{P}$ is equivalent to

$$\nabla_x R_{xyyx} = 0, \ \forall x \text{ timelike and } y \perp x$$

This is equivalent to $\nabla R = 0$ (García-Río, Vázquez-Abal, 1993).

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a timelike $\mathfrak{C}$-space and a timelike $\mathfrak{P}$-space.

$T\mathfrak{C} \cap T\mathfrak{P}$ is equivalent to

$$\nabla_x R_{xyyx} = 0, \ \forall x \text{ timelike and } y \perp x$$

This is equivalent to $\nabla R = 0$ (García-Río, Vázquez-Abal, 1993).

Remark

Previous theorem is false in other signatures.

Theorem

Let $(\mathcal{M}, g)$ be a Lorentzian manifold.
$(\mathcal{M}, g)$ is locally symmetric $\Leftrightarrow$ $(\mathcal{M}, g)$ is simultaneously a timelike $\mathfrak{C}$-space and a timelike $\mathfrak{P}$-space.

$T\mathfrak{C} \cap T\mathfrak{P}$ is equivalent to

$$\nabla_x R_{xyyx} = 0, \ \forall x \text{ timelike and } y \perp x$$

This is equivalent to $\nabla R = 0$ (García-Río, Vázquez-Abal, 1993).

Example

A strict walker 4-dimensional manifold of signature $(2, 2)$:
$g = 2dx^1 \circ dx^3 + 2dx^2 \circ dx^4 + b(x^3)dx^4 \circ dx^4$, is always $T\mathfrak{C}$ and $T\mathfrak{P}$ but not locally symmetric ($\nabla R = 0 \Leftrightarrow b^{(3)}(x^3) = 0$).

$(\mathcal{M}, g)$ timelike $\mathfrak{C}$-space $\Leftrightarrow$ R_γ has constant eigenvalues along γ for any timelike γ

$(\mathcal{M}, g)$ timelike $\mathfrak{C}$-space $\Leftrightarrow$ R_γ has constant eigenvalues along γ for any timelike γ

$\Leftrightarrow$ $\nabla_{\gamma'} TrR_\gamma^{(k)} = 0$ for any timelike γ

$(\mathcal{M}, g)$ timelike $\mathfrak{C}$-space $\Leftrightarrow$ R_γ has constant eigenvalues along γ for any timelike γ

$\Leftrightarrow \nabla_{\gamma'} TrR_\gamma^{(k)} = 0$ for any timelike γ

$\Leftrightarrow \nabla_X TrR_X^{(k)} = 0$, for all X

$$
\begin{aligned}
(\mathcal{M}, g) \text{ timelike } \mathfrak{C}\text{-space} \Leftrightarrow\ & R_\gamma \text{ has constant eigenvalues along } \gamma \\
& \text{for any timelike } \gamma \\
\Leftrightarrow\ & \nabla_{\gamma'} TrR_\gamma^{(k)} = 0 \text{ for any timelike } \gamma \\
\Leftrightarrow\ & \nabla_X TrR_X^{(k)} = 0, \text{ for all } X
\end{aligned}
$$

Lemma (Characterization)

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension 3. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space if and only if:

$$
\begin{aligned}
(\mathcal{M}, g) \text{ timelike } \mathfrak{C}\text{-space} \Leftrightarrow\ & R_\gamma \text{ has constant eigenvalues along } \gamma \\
& \text{for any timelike } \gamma \\
\Leftrightarrow\ & \nabla_{\gamma'} Tr R_\gamma^{(k)} = 0 \text{ for any timelike } \gamma \\
\Leftrightarrow\ & \nabla_X Tr R_X^{(k)} = 0, \text{ for all } X
\end{aligned}
$$

Lemma (Characterization)

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension 3. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space if and only if:

$$(1)\quad \nabla_X \rho(X, X) = 0,$$

$(\mathcal{M}, g)$ timelike $\mathfrak{C}$-space $\Leftrightarrow$ R_γ has constant eigenvalues along γ for any timelike γ

$\Leftrightarrow \nabla_{\gamma'} TrR_\gamma^{(k)} = 0$ for any timelike γ

$\Leftrightarrow \nabla_X TrR_X^{(k)} = 0$, for all X

Lemma (Characterization)

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension 3. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space if and only if:

(1) $\nabla_X \rho(X, X) = 0$,

(2) $\sum_i \varepsilon_i R(e_i, X, X, e_i) \nabla_X R(e_i, X, X, e_i) = 0$,

$(\mathcal{M}, g)$ timelike $\mathfrak{C}$-space $\Leftrightarrow$ R_γ has constant eigenvalues along γ for any timelike γ
$\Leftrightarrow \nabla_{\gamma'} TrR_\gamma^{(k)} = 0$ for any timelike γ
$\Leftrightarrow \nabla_X TrR_X^{(k)} = 0$, for all X

Lemma (Characterization)

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension 3. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space if and only if:

(1) $\nabla_X \rho(X, X) = 0$,

(2) $\sum_i \varepsilon_i R(e_i, X, X, e_i) \nabla_X R(e_i, X, X, e_i) = 0$,

$\forall X \in T\mathcal{M}$, where $\{e_i\}$ is an orthonormal basis.

$(\mathcal{M}, g)$ timelike $\mathfrak{C}$-space $\Leftrightarrow$ R_γ has constant eigenvalues along γ for any timelike γ

$$\Leftrightarrow \nabla_{\gamma'} Tr R_\gamma^{(k)} = 0 \text{ for any timelike } \gamma$$

$$\Leftrightarrow \nabla_X Tr R_X^{(k)} = 0, \text{ for all } X$$

Lemma (Characterization)

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of dimension 3. $(\mathcal{M}, g)$ is a timelike $\mathfrak{C}$-space if and only if:

(1) $\nabla_X \rho(X, X) = 0$,

(2) $\sum_i \varepsilon_i R(e_i, X, X, e_i) \nabla_X R(e_i, X, X, e_i) = 0$,

$\forall X \in T\mathcal{M}$, where $\{e_i\}$ is an orthonormal basis.

(1) and (2) are the Ledger conditions of order 3 and 5.

Lemma

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. $(\mathcal{M}, g)$ is a timelike $\mathfrak{P}$-space if and only if:

Lemma

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. $(\mathcal{M}, g)$ is a timelike $\mathfrak{P}$-space if and only if:

$$R_\gamma R'_\gamma - R'_\gamma R_\gamma = 0$$

Lemma

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. $(\mathcal{M}, g)$ is a timelike $\mathfrak{P}$-space if and only if:

$$R_\gamma R'_\gamma - R'_\gamma R_\gamma = 0$$

for all timelike geodesic γ.

Lemma

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. $(\mathcal{M}, g)$ is a timelike $\mathfrak{P}$-space if and only if:

$$R_\gamma R'_\gamma - R'_\gamma R_\gamma = 0$$

for all timelike geodesic γ.

The Jacobi and the Szabó operators commute for timelike geodesics if and only $R_X R'_X - R'_X R_X = 0$ for all X.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a timelike $\mathfrak{C}$-space

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a timelike $\mathfrak{C}$-space if and only if it is locally symmetric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a timelike $\mathfrak{C}$-space if and only if it is locally symmetric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a timelike $\mathfrak{P}$-space

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a timelike $\mathfrak{C}$-space if and only if it is locally symmetric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a timelike $\mathfrak{P}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a timelike $\mathfrak{C}$-space if and only if it is locally symmetric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a timelike $\mathfrak{P}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a timelike $\mathfrak{C}$-space if and only if it is locally symmetric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a timelike $\mathfrak{P}$-space if and only if it is locally symmetric or a strictly Walker metric.

$->$ A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a timelike $\mathfrak{P}$-space if and only if it is curvature recurrent.

Generalized locally symmetric spaces

Sandra Gavino Fernández

Algebraic conditions

Geometry of the Jacobi operator

$\mathfrak{C}$-spaces and $\mathfrak{P}$-spaces

Geometry of the skew-symmetric curvature operator

$\mathfrak{O}$-spaces and $\mathfrak{T}$-spaces

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if the skew-symmetric curvature operator has constant eigenvalues along each unit circle.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if the skew-symmetric curvature operator has constant eigenvalues along each unit circle.
2. $(\mathcal{M}, g)$ is a $\mathfrak{T}$-space if there is a parallel Jordan basis for the skew-symmetric curvature operator along each unit circle.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if the skew-symmetric curvature operator has constant eigenvalues along each unit circle.
2. $(\mathcal{M}, g)$ is a $\mathfrak{T}$-space if there is a parallel Jordan basis for the skew-symmetric curvature operator along each unit circle.

Theorem (Ivanov-Petrova, 1997)

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if the skew-symmetric curvature operator has constant eigenvalues along each unit circle.
2. $(\mathcal{M}, g)$ is a $\mathfrak{T}$-space if there is a parallel Jordan basis for the skew-symmetric curvature operator along each unit circle.

Theorem (Ivanov-Petrova, 1997)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if the skew-symmetric curvature operator has constant eigenvalues along each unit circle.
2. $(\mathcal{M}, g)$ is a $\mathfrak{T}$-space if there is a parallel Jordan basis for the skew-symmetric curvature operator along each unit circle.

Theorem (Ivanov-Petrova, 1997)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is simultaneously $\mathfrak{O}$-space and $\mathfrak{T}$-space.

Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. $(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if the skew-symmetric curvature operator has constant eigenvalues along each unit circle.
2. $(\mathcal{M}, g)$ is a $\mathfrak{T}$-space if there is a parallel Jordan basis for the skew-symmetric curvature operator along each unit circle.

Theorem (Ivanov-Petrova, 1997)

$(\mathcal{M}, g)$ is locally symmetric if and only if $(\mathcal{M}, g)$ is simultaneously $\mathfrak{O}$-space and $\mathfrak{T}$-space.

$(\mathcal{M}, g)$ is a $\mathfrak{O}$-space if and only if $\nabla_{c'} TrR(c', \nabla_{c'} c')^{(k)} = 0$ for all unit circle c.
$(\mathcal{M}, g)$ is a $\mathfrak{T}$-space if and only if $\nabla_{c'} R(c', \nabla_{c'} c')$ and $R(c', \nabla_{c'} c')$ commute for all unit circle c.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a $\mathfrak{T}$-space

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a $\mathfrak{T}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a $\mathfrak{T}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ is a $\mathfrak{O}$-space if and only if it is locally symmetric or a strictly Walker metric.

Theorem

A 3-dimensional Walker manifold $\mathcal{M}_f$ of constant scalar curvature is a $\mathfrak{T}$-space if and only if it is locally symmetric or a strictly Walker metric.

$$\{\mathfrak{O} - \text{space} \cap \mathfrak{T} - \text{space}\} \supsetneq \{\nabla R = 0\}.$$

Einstein-like Walker manifolds and generalized locally symmetric spaces

Sandra Gavino Fernández

Department of Geometry and Topology
University of Santiago de Compostela

July 11th, 2009

Work in progress with Miguel Brozos Vázquez