Periodic Geodesics in Compact Lorentzian Manifolds

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- Partially based on a work in collaboration with M.A. Javaloyes and P. Piccione.

Plan of the Lecture

- **Results for Compact Riemannian Manifolds:**
  - Non-simply connected case
  - Simply connected case
- **Results for Compact Lorentzian Manifolds:**
  - Classical results (Tipler, Galloway)
  - Recent results...
- **Compact Lorentzian Manifolds with a Killing Field:**
  - Existence and multiplicity of periodic geodesics
  - Topological structure
  - Isometry group

Compact Riemannian Manifolds

Non-Simply Connected Case

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There exists some periodic geodesic in any compact Riemannian manifold (M, g) *with non-trivial fundamental group.*

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Sketch of Proof.

- Let $\mathcal{C} \in \pi_1(M) \setminus \{0\}$ be a non-trivial free homotopy class.
- Then $m = \mathrm{Inf}\{\mathrm{leng}(\gamma) : \gamma \in \mathcal{C}$ is piecewise smooth$\} > 0$.
- Let $\{\gamma_n\}_n \subset \mathcal{C}$ be a sequence of piecewise smooth geodesics with $\mathrm{leng}(\gamma_n) \to m$.
- Since $\{\gamma_n\}_n$ is equicontinuous and M compact, there exists some subsequence approaching to a closed curve γ_0.
- A standard local minimization argument implies that γ_0 must be a periodic geodesic. $\square$

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Any closed curve non-freely homotopic to a constant curve can be deformed into a periodic geodesic with minimal length in its free homotopy class.

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Remarks:

- $\Rightarrow$ Every non-trivial conjugacy class of $\pi_1(M)$ admits a periodic geodesic (which minimizes the length at every class).
- $\not\Rightarrow$ Existence of infinitely many periodic geodesics "geometrically distinct" if $\pi_1(M)$ admits infinitely many conjugacy classes.

Simply Connected Case

Variational Principle:

$$\text{Periodic Geodesic} \Longleftrightarrow \text{Critical Point of Action Functional}$$

Action functional: Smooth function given by

$$A : \Lambda^1(M) \subset H^1(I, M) \to \mathbb{R}, \quad A(y) = \int_0^1 g(\dot{y}, \dot{y})$$

Domain: Riemannian Hilbert manifold of free loops $\Lambda^1(M)$,

$$\Lambda^1(M) = \{y \in H^1(I, M) : y(0) = y(1)\}$$

(recall $H^1(I, M) = \{y \in H^1(I, \mathbb{R}^N) : y(I) \subset M, i^{-1} \circ y \ \text{ cont.}\}$)

Simply Connected Case

Theorem: Let F be a smooth function defined on a Hilbert manifold $\mathcal{H}$. Assume F is lower bounded, satisfies PS and $\mathcal{H}$ is complete. Then, F admits a minimum.

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- F satisfies *PS* if any sequence $\{x_m\}$ such that $\{F(x_m)\}$ is bounded and $\nabla F(x_m) \to 0$, has a convergent subsequence.

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Application to our problem:

- If (M, g) is compact then A (which is lower bounded) satisfies PS.
- Moreover, $\Lambda^1(M)$ is complete.
- From the theorem above, A admits a minimum.
- But, again, the corresponding geodesic is constant.

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- Moreover, $\Lambda^1(M)$ is complete.
- From the theorem above, A admits a minimum.
- But, again, the corresponding geodesic is constant.

$\star$ Therefore, we need more sophisticated critical point theorems!

Simply Connected Case

- **Powerful Methods (Morse$'34$, Ljusternik-Schnirelmann$'30$):** Non-trivial topology for $\Lambda^1(M)$ (homology group, L-S category) forces the existence of more critical points for A.

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  Existence of certain number of distinct periodic geodesics for special topological types of manifolds, partly under restrictive metric conditions...

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$\star$ *Does any compact Riemannian manifold admit infinitely many periodic geodesics?*

Compact Lorentzian Manifolds

Classical Results

Tipler'79 *(Proc. AMS,* **76***, 1979, 145-147)*

There exists a periodic timelike geodesic in every compact Lorentzian manifold with a regular covering containing a compact Cauchy surface.

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Remarks:

- $\exists$ compact flat Lorentz space forms without periodic timelike geodesics (Guediri: *Math. Z.* **239** *277-291 (2002)*).
- Examples extended in (Guediri: *Trans. AMS* **355** *775-786 (2003)*).
- A compact flat spacetime (M, g) contains a periodic timelike geodesic iff $\pi_1(M)$ contains a nontrivial timelike translation.
  (Guediri: *Math. Z.* **244** *577-585 (2003)*)

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Sketch of Proof.

- There exists a periodic timelike curve α on (M, g).
- Let $\tilde{\alpha}$ be a lift of α to the regular covering space $(\tilde{M}, \tilde{g})$.
- The endpoints of $\tilde{\alpha}$ are timelike related, and so, lie on two disjoint compact Cauchy surfaces S_1, S_2 of $(\tilde{M}, \tilde{g})$.
- The Lorentzian distance function d between points of S_1, S_2 is finite and continuous.
- Let $q_1 \in S_1$, $q_2 \in S_2$ corresponding points such that d attains its maximum.

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$\vdots$

- By Avez-Seifert, there exits a maximizing timelike geodesic $\tilde{\gamma}$ joining them.
- $\tilde{\gamma}$ projects into a timelike geodesic γ in (M, g), with the same extreme points, but perhaps with different extreme velocities.
- Standard maximizing argument show that γ is a periodic geodesic. $\square$

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* A free t-homotopy class $\mathcal{C}$ of closed timelike curves is *stable* if $\exists$ another Lorentzian metric $g > g_0$ such that $\sup_{\gamma\in\mathcal{C}}\mathrm{leng}_g(\gamma) < \infty$.

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Sketch of Proof.

- Let $\mathcal{C}$ be a stable free t-homotopy class.
- Then, (1) $L_0 = \sup_{\gamma \in \mathcal{C}} \mathrm{leng}_{g_0}(\gamma) < \infty$; (2) $\sup_{\gamma \in \mathcal{C}} \mathrm{leng}_{g_r}(\gamma) < \infty$.
- From (1), $\exists$ $\{\gamma_n\} \subset \mathcal{C}$ with $\mathrm{leng}_{g_0}(\gamma_n) \to L_0$.
- From (2), the curves γ_n approach to a limiting curve γ in $\mathcal{C}$ which is a longest timelike periodic geodesic. $\square$

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- There exists a 3-dimensional Lorentzian manifold without any periodic causal geodesic (but having a spacelike one).

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- There exists a 2-dimensional Lorentzian manifold without any periodic timelike geodesic (but having a causal one).
- There exists a 3-dimensional Lorentzian manifold without any periodic causal geodesic (but having a spacelike one).
- The 3-dimensional example can be taken geodesically complete and admitting a regular globally hyperbolic covering.
  (Guediri: *Geom. Dedicata* **126**, *2007, 126-185*).

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Sketch of Proof.

- Up to certain finite covering, $M^2 \cong \mathbb{T}^2$ orientable, t-orientable.
- Let α be some closed timelike for $(M^2, -g)$.
- (M^2, g) admits a smooth compact spacelike hypersurface $\Sigma \cong \alpha$, which is the image of a smooth embedding of $\mathbb{S}^1$.
- Let $p : M_\Sigma \to M^2$ be Lorentzian regular covering.
- Every component $\tilde{\Sigma}$ of $p^{-1}(\Sigma)$ is a compact spacelike hypersurface which separates M_Σ. In particular, $\tilde{\Sigma}$ is acausal.

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$\vdots$

- If $\tilde{\Sigma}$ is a Cauchy hypersurface for M_{Σ}, apply Tipler's result.
- Otherwise, we can assume $H^{+}(\tilde{\Sigma}) \neq \emptyset$.
- Let η be the past inextendible null geodesic generator of $H^{+}(\tilde{\Sigma})$.
- The projection $\eta \circ p$ is a periodic null geodesic on M^{2}. $\square$

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  Then, each free timelike homotopy class determined by a central deck transformation must contain a periodic timelike geodesic.

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- **Theorem:** *Any compact flat Lorentzian manifold contains some periodic causal geodesic.*

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- **Theorem:** *Any compact static spacetime admits a periodic timelike geodesic.*

$\star$ Additional hypothesis: $\pi_1(M)$ has some finite conjugacy class.
Caponio, Masiello, Piccione: *Math. Z.*, **244**, *2003, 457-468*

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Guediri'07 (*Trans. AMS*, **359**, *2007, 2663-2673*)

Let M be a compact Lorentzian manifold admitting a globally hyperbolic regular covering.

A non-trivial free timelike homotopy class $\mathcal{C}$ of closed timelike curves contains a longest periodic timelike geodesic iff the timelike injectivity radius of $\mathcal{C}$ is finite.

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* The timelike injectivity radius of $\mathcal{C}$ is $Tinj(\mathcal{C}) = \frac{1}{2}\sup_{\gamma\in\mathcal{C}}\mathrm{leng}_g(\gamma)$.

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Sketch of Proof.

- Let $\mathcal{C}$ be a non-trivial free homotopy class such that $Tinj(\mathcal{C}) < \infty$, and ϕ a deck transformation corresponding to $\mathcal{C}$.
- Then, $h_\phi(x) = \sup\{\tilde{d}(x, \psi^{-1} \circ \phi \circ \psi(x)) : \psi \in \Gamma\} < \infty \; \forall x \in \tilde{M}$.
- For each $x \in \tilde{M}$, $\exists \, \tilde{\psi}$ such that $h_\phi(x) = \tilde{d}(x, \tilde{\psi}^{-1} \circ \phi \circ \tilde{\psi}(x))$.
- $\exists \, x_0 \in \tilde{M}$ such that $h_\phi(x_0) = \sup\{h_\phi(x) : x \in \tilde{M}\} > 0$.

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- Let $\tilde{\psi}_0 \in \Gamma$ such that $h_\phi(x_0) = \tilde{d}(x_0, \tilde{\psi}_0^{-1} \circ \phi \circ \tilde{\psi}_0(x_0))$ and let $\tilde{\gamma}$ tmlk. geod. from x_0 to $x_1 = \tilde{\psi}_0^{-1} \circ \phi \circ \tilde{\psi}_0(x_0)$, $h_\phi(x_0) = \mathrm{leng}_{\tilde{g}}(\tilde{\gamma})$.
- $\pi \circ \tilde{\gamma}$ is closed (perhaps broken) timelike geodesic in $\mathcal{C}$.
- Moreover, $\pi \circ \tilde{\gamma}$ is the longest curve in $\mathcal{C}$, and so, it is periodic. $\square$

Our Result

F, Javaloyes, Piccione: math.DG:0812.1163v2; to appear in *Math. Z.*

Preliminaries

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- The following map is an anti-isomorphism of Lie algebras:

$$\mathfrak{Iso}(M, g) \to \mathrm{Kill}(M, g), \quad \tau \mapsto K^\tau := d\beta_p(1)\tau, \;\; p \in M,$$

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- The following map is an anti-isomorphism of Lie algebras:

$$\mathfrak{Iso}(M, g) \longrightarrow \mathrm{Kill}(M, g), \quad \tau \mapsto K^{\tau} := d\beta_p(1)\tau, \ \ p \in M,$$

  where $\beta p : \mathrm{Iso}(M, g) \longrightarrow M$, $\beta_p(\Phi) = \Phi(p)$.
- **Lemma:** *The following formula holds:*

$$\Phi_* K^{\tau} = K^{\mathrm{Ad}_{\Phi}(\tau)} \quad \forall \Phi \in \mathrm{Iso}(M, g), \ \ \tau \in \mathfrak{Iso}(M, g)$$

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- If K is a o-closed Killing vector field then their integral curves γ are circles or points (since $\gamma(t) = \exp(t\tau)p$, $\gamma(0) = p$).
- The converse of this result is true when (M, g) is a compact Riemannian manifold.

Main Technical Result

- **Theorem:** *Let (M, g) be a compact Lorentzian manifold with a Killing vector field K that is timelike at some point. Then, every $H \leq \mathrm{Iso}(M, g)$ satisfying the property below is precompact:*

$$(\Phi_* K)_p = \pm K_p \quad \forall \Phi \in H \;\; \forall p \in M. \qquad (\star)$$

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- **Corollary:** *Under the hypotheses above, the 1-parametric subgroup of isometries $H = \{exp(t \cdot \tau) : t \in \mathbb{R}\}$ generated by $K = K^\tau$ is precompact.*

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Sketch of Proof.

- Let $p \in M$ such that $g(K_p, K_p) < 0$.
- Consider the following compact subsets of TM:

$$V = \{\pm K_q : q \;\text{ s.t. }\; g(K_q, K_q) = g(K_p, K_p)\}$$
$$V^\perp = \{v \in K_q^\perp : q \;\text{ s.t. }\; g(K_q, K_q) = g(K_p, K_p), \; |v| = 1\}.$$

- Let $b = \{v_1, \ldots, v_n\} \subset T_p M$ ortn. base with $v_1 = K_p$.

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Sketch of Proof.

$\vdots$

- From $(\star)$, every vector v of every base in the H-orbit of b belongs to $V \cup V^\perp$, which is compact.
- The subset $\mathcal{A} \subset \mathcal{F}(M)$ formed by the ortn. basis which have all its vectors in $V \cup V^\perp$ is also compact.
- $\mathcal{A}$ intersected with $\mathrm{Iso}(M, g)$-orbit is compact in $\mathcal{F}(M)$.

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Sketch of Proof.

$\vdots$

- The H-orbit of b in $\mathcal{F}(M)$ is contained in this intersection.
- Therefore, the H-orbit of b is precompact in $\mathcal{F}(M)$.
- But H is homeomorph to the H-orbit of b in $\mathcal{F}(M)$.
- Hence, H is precompact in $\mathcal{F}(M)$. $\square$

Periodic Geodesics

- **Lemma:** *Let (M, g) be a semi-Riemannian manifold with a Killing vector field K. The integral curve of K through a critical point p_0 of $f(p) = g(K_p, K_p)$ is a geodesic.*

Periodic Geodesics

- **Lemma:** *Let (M, g) be a semi-Riemannian manifold with a Killing vector field K. The integral curve of K through a critical point p_0 of $f(p) = g(K_p, K_p)$ is a geodesic.*
- **Corollary:** *Let (M, g) a compact semi-Riemannian manifold with a Killing vector field $K \not\equiv 0$ whose integral curves are periodic. Then (M, g) admits some (non-trivial) periodic geodesic.*

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Sketch of Proof:

- Integral curves of K through critical points of f are (perhaps constant) periodic geodesics.
- If $f \equiv 0 \Rightarrow$ any point is critical; since $K \not\equiv 0$, $\exists\, \infty$ many non-trivial periodic geodesics.
- If $f \not\equiv 0 \Rightarrow$ either $\min f$ or $\max f \neq 0$, and the corresponding integral curve is a non-trivial periodic geodesic. $\square$

Periodic Geodesics

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- **Corollary:** *Let (M, g) a compact semi-Riemannian manifold with a Killing vector field $K \not\equiv 0$ whose integral curves are periodic. Then (M, g) admits some (non-trivial) periodic geodesic.*

  Moreover, if

  (a) either $\min g(K, K), \max g(K, K) \neq 0$, or

  (b) K does not vanish anywhere,

  then there exist at least two (non-trivial) periodic geodesics.

Periodic Geodesics

- **Theorem:** *Let* (M,g) *be a compact Lorentzian manifold with a Killing vector field* K *timelike at some point. Then* (M,g) *admits some periodic timelike geodesic. Moreover, if*

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- **Corollary** *Any compact stationary Lorentzian manifold admits at least two (non-trivial) timelike periodic geodesics.*

Periodic Geodesics

- **Theorem:** *Let (M, g) be a compact Lorentzian manifold with a Killing vector field K timelike at some point. Then (M, g) admits some periodic timelike geodesic. Moreover, if*

  (a) *either* $\min g(K, K), \max g(K, K) \neq 0$, *or*

  (b) K *does not vanish anywhere,*

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  Sketch of Proof:

  - $G = \{exp(t \cdot \tau) : K = K^{\tau}, t \in \mathbb{R}\}$ is precompact.
  - $\overline{G} \leq \mathrm{Iso}(M, g)$ is abelian and compact, hence $\overline{G} \cong \mathbb{T}^n$ torus.
  - Let $\mathfrak{g}$ be the Lie algebra associated to $\overline{G} \cong \mathbb{T}^n$.
  - $\exists \{\tau_n\} \subset \mathfrak{g}$, $\tau_n \to \tau$, with τ_n generating closed 1-parametric subgroup in $\overline{G}$. In particular, K^{τ_n} are o-closed Killing.

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  Sketch of Proof:

  $\vdots$

  - Since $K_p^\tau = (d\beta_p)_1(\tau)$, $K_p^{\tau_n} = (d\beta_p)_1(\tau_n)$ and M is compact, $K_p^{\tau_n} \to K_p$ uniformly in $p \in M$.
  - Therefore, K^{τ_n} are timelike at some point for n big enough.
  - Apply previous corollary to K_n for n big enough. $\square$

Periodic Geodesics

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  then there are at least two (non-trivial) periodic geodesics.
  Finally, if K has at most one periodic integral curve, then there exist infinitely many (non-trivial) periodic geodesics.

Periodic Geodesics

Some Consequences:

- Any Lorentzian torus $(\mathbb{T}^2, g)$ with a Killing vector field $K \not\equiv 0$ contains at least two *geometrically distinct* (non-trivial) periodic non self-intersecting geodesics.
- Any simply connected compact real-analytic Lorentzian manifold (M, g) with $\dim\big(\mathrm{Iso}(M, g)\big) > 0$ contains some (non-trivial) periodic geodesic.
- Compact semi-Riemannian manifolds of index $k > 1$ that admits m Killing vector fields $K^1, \ldots, K^m$ which generate a negative definite subspace for g of dimension m at some point p, have infinitely many distinct periodic geodesics.

Topological Structure

- **Proposition:** *A compact manifold M admits a Lorentizian metric with a nowhere vanishing Killing vector field that is timelike somewhere iff it is diffeomorphic to a generalized Seifert fibered space. In this case, the metric can be chosen to have a timelike Killing vector field.*

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- The connection between the topology of compact stationary Lorentzian manifolds and Seifert manifolds already appears in:
A. Romero, M. Sánchez: *Proc. AMS* **123**, 1995, 2831-2833.

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  Sketch of Proof:

  $\Rightarrow$

  - There exists a nowhere vanishing o-closed Killing field K.
  - The 1-parameter group of isometries generated by K gives a smooth $\mathbb{S}^1$-action without fixed points.
  - K is tangent to the fibers of this action.

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Sketch of Proof:

$\Leftarrow$

- Consider a smooth action of $\mathbb{S}^1$ on M without fixed points.
- By a standard averaging argument, there exists a Riemannian metric g_R which makes the action $\mathbb{S}^1$ isometric
- Consider the Lorentzian metric g associated to g_R and K.
- Then K is timelike and g-Killing. $\square$

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- **Corollary:** *Let (M, g) be a compact Lorentzian manifold with a nowhere vanishing o-closed Killing vector field on M. Consider the $\mathbb{S}^1$-action on M determined by K. Then, there are at least $cat_{\mathbb{S}^1}(M)$ distinct periodic non self-intersecting geodesics in M.*

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* Let χ be a G-space, i.e. a topological space on which a compact group G is acting continuously. Then, $cat_G(\chi)$ is the cardinality of minimal family of G-categorical (defined by using G-invariant homotopy) open sets covering χ.

Isometry Group

- **Proposition:** *Let (M, g) be a compact Lorentzian manifold with a nowhere vanishing o-closed Killing vector field which is timelike at some point. Let $\pi : M \to M_0$ canonical projection with*

$$M_0 := M/\mathbb{S}^1 \quad \textit{space of orbits of the } \mathbb{S}^1\textit{-action.}$$

Then, the group $\mathrm{Iso}(M, g; \pi)$ of the isometries of (M, g) preserving the fibration is compact.

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Sketch of Proof:

- $\phi \in \mathrm{Iso}(M, g; \pi)$ iff $\phi_* K = \lambda K$, $\lambda : M \to \mathbb{R}$ smooth function.
- ϕ isometry $\Rightarrow$ $\phi_* K = \lambda K$ Killing (and K Killing) $\Rightarrow$ $\lambda \equiv$cte.
- $min_p |K_p|^2 = \min_p |K_{\phi^{-1}(p)}|^2 = \min_p |(\phi_* K)_p|^2 = \lambda^2 \min_p |K_p|^2$.
- Hence $\lambda \equiv \pm 1$, and so, $\phi_* K = \pm K$ for all $\phi \in \mathrm{Iso}(M, g; \pi)$.

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Then, the group $\mathrm{Iso}(M, g; \pi)$ *of the isometries of* (M, g) *preserving the fibration is compact.*

Sketch of Proof:

$\vdots$

- In particular, $H = \mathrm{Iso}(M, g; \pi)$ verifies criterium ($\star$).
- From our technical result, $\mathrm{Iso}(M, g; \pi)$ is precompact.
- Since $\mathrm{Iso}(M, g; \pi)$ is closed in $\mathrm{Iso}(M, g)$, it is compact. $\square$

Isometry Group

- **Corollary:** *Let (M, g) be a compact Lorentzian manifold with a Killing vector field that is timelike somewhere and assume that $\mathrm{Iso}(M, g)$ is abelian. Then, $\mathrm{Iso}(M, g)$ is compact.*

Sketch of Proof:

- Since $\mathrm{Iso}(M, g)$ is abelian, $Ad_\phi = Id$ for all $\phi \in \mathrm{Iso}(M, g)$.
- From previous formula,

$$\phi_* K^\tau = K^{Ad_{\phi}(\tau)} = K^\tau \quad \forall \phi \in \mathrm{Iso}(M, g).$$

  In particular, $H = \mathrm{Iso}(M, g)$ verifies the criterium ($\star$)
- Therefore, $\mathrm{Iso}(M, g)$ must be compact. $\square$

Fundamental Open Question

Does any compact Lorentzian manifold admit some periodic geodesic?

GRAZIE!!