Causal Boundary and its relations with the conformal boundary

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(joint work with Jose Luis Flores and Miguel Sánchez, in preparation)

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UNIVERSIDAD DE MÁLAGA

INDEX

Motivation

- Physical motivation.
  - Study of singularities.
  - Asymptotic properties of fields.
  - Global properties of spacetimes.
- Mathematical motivation.
  - Classical problem.
  - Interest of the mathematical tools involved.

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Problem

- Any timelike curve (future and past) has an endpoint.
- Some causal structures are preserved.

- Multiple solutions:
  - A-boundary.
  - B-boundary.
  - Geroch boundary.
  - **Conformal boundary**. (Most common in physics)
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- Construction:
  - For any inextendible future or past timelike curves, attach an ideal point.
  - Two inextendible future (past) timelike curves are attached the same ideal point if they have the same past (resp. future).
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  - The (total) causal boundary is defined as the union of the future and past causal boundaries under “certain identifications”.

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- Past Set: $P \subset V$ such that $I^{-}[P] = P$.
- IP: Past set which cannot be expressed by the union of two proper past sets.
- PIP: IP, $P \subset V$ such that $P = I^{-}(p)$ for $p \in V$.
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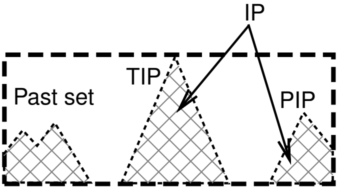
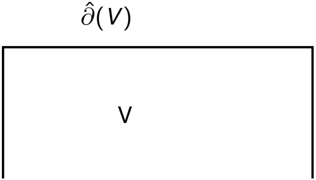
$$\hat{\partial}(V) \equiv TIPs,\ V \equiv PIPs,\ \hat{V} \equiv IPs$$





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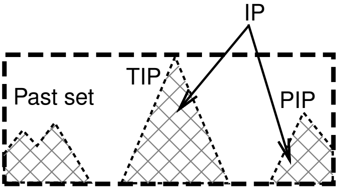

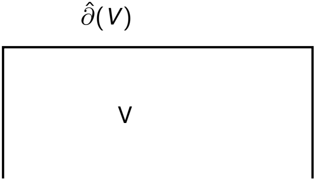

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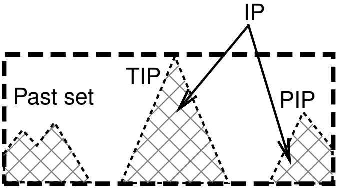

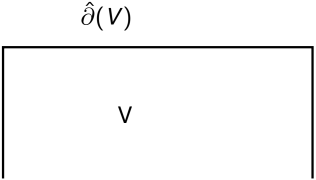

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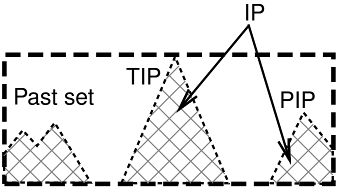

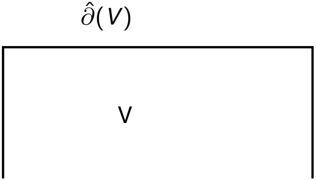

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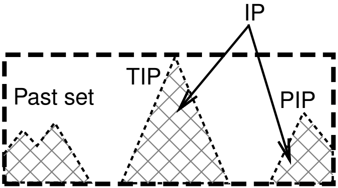

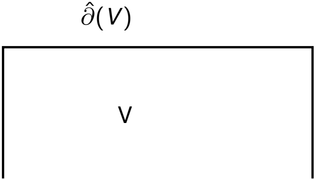

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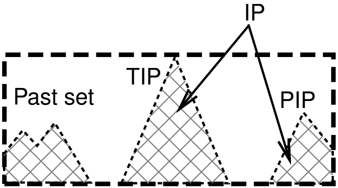
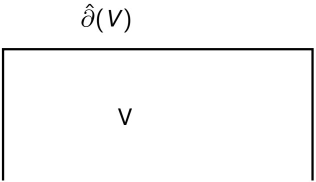
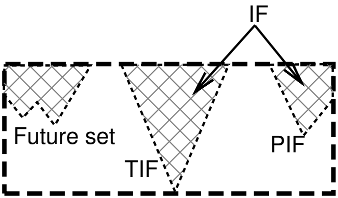


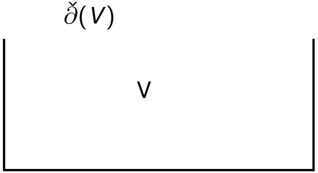


Past Causal Boundary

- Future Set: $F \subset V$ such that $I^+[F] = F$.
- IF: Future set which cannot be expressed by the union of two proper past sets.
- PIF: IP, $F \subset V$ such that $F = I^+(p)$ for $p \in V$.
- TIF: IP, $F \subset V$ such that $F \neq I^+(p) \qquad \forall p \in V$.

$$\check{\partial}(V) \equiv TIFs, V \equiv PIFs, \check{V} \equiv IFs$$



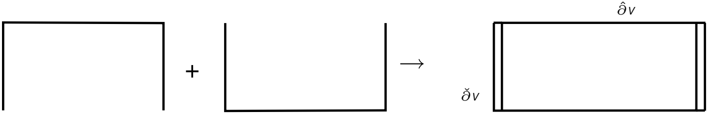


(Total) Causal Boundary

- In a first approach, we can consider that the (total) causal boundary defined by:

$$\partial(V) = \hat{\partial}(V) \cup \check{\partial}(V).$$

In our example, this definition provides:



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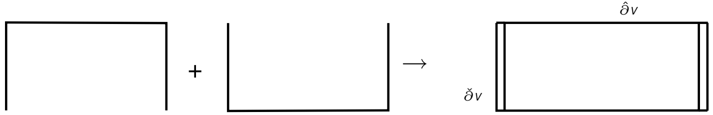

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- We will follow an approach based on the Marolf-Ross and Flores approach.

Definition (Szabados relation)

$P \sim_S F$ if

$$\left\{ \begin{array}{l} P \text{ is a maximal IP in } \downarrow F := I^-[\{q \in V : q \ll p, \forall p \in F\}] \\ F \text{ is a maximal IF in } \uparrow P := I^+[\{p \in V : q \ll p, \forall q \in P\}] \end{array} \right|$$

Definition (Total Causal Boundary)

$$\partial(V) := \{(P, F) \in \hat{\partial}(V) \times \check{\partial}(V) : P \sim_S F\}$$

If P (F) is not S-related with nobody, then $(P, \emptyset) \in \partial(V)$ $((\emptyset, F) \in \partial(V))$.

$$V \equiv \{(I^-(p), I^+(p)) : p \in V\}$$

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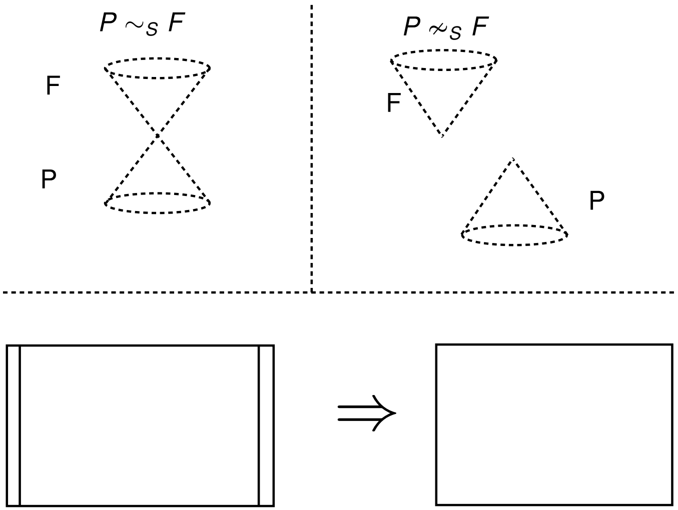
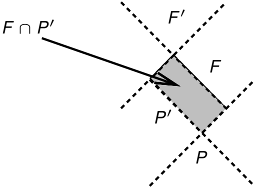


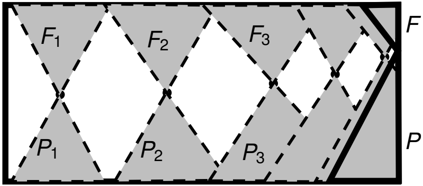
Causal Structure of the Causal Completion

- $(P,F) \ll (P',F')$ if, and only if, $F \cap P' \neq \emptyset$.



Topology

- *Limit-Operator*:$(P,F)\in L(\sigma)$, with $\sigma=\{(P_n,F_n)\}_n\subset\overline{V}$ if:
$$\left\{\begin{array}{l} P\in\hat{L}(P_n):=\{P'\in\hat{V}:P'\subset LI(P_n) \text{ maximal in } LS(P_n)\} \\ F\in\check{L}(F_n):=\{F'\in\check{V}:P'\subset LI(F_n) \text{ maximal in } LS(F_n)\}\end{array}\right|$$



- The *closed sets* of $\overline{V}$ are the subsets $C\subset\overline{V}$ such that $L(\sigma)\in C$ for all sequence $\sigma\subset C$.

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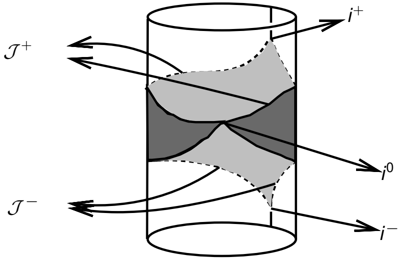
Asymptotically-flat spacetimes

- Generalize the structure of Minkowski spacetime seen inside the Einstein spacetime.



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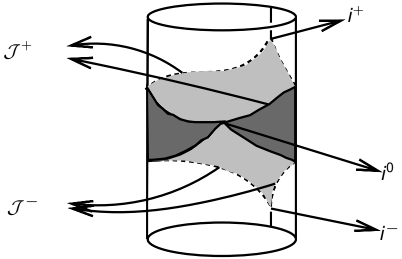

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Conformal Boundary

- Consider a conformal embedding $i : V \to V_0$. (Envelopment)
- Define:
  - $\overline{i(V)}$ (Conformal Completion).
  - $\partial_i V = \overline{i(V)} \setminus V$ (Conformal Boundary).
  - $\partial_i^* V$ set of point in $\partial_i V$ reachable by timelike curves in V.
- Topology: Induced from V_0.
- Causality: $x \ll_i y$, $\exists \gamma : [a, b] \to \overline{i(V)}$ future-directed timelike curve with $\gamma((a, b)) \subset V, \gamma(a) = x, \gamma(b) = y$.

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Comparation

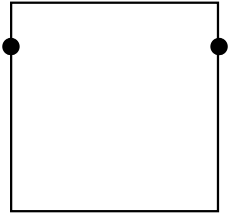
Causal Boundary	Conformal Boundary
Conformal invariant	No conformal invariant
Uniqueness	Dependence on the conformal embedding
Systematic	No systematic
Lack of physical spacetimes with known causal boundary	Most common boundary in physics

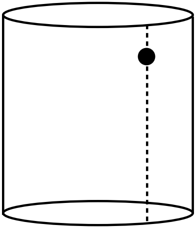
Comparation

Due to the bad behaviour of the conformal boundary, we cannot expect in general any possible relation:

- As a point set

(V, g) (V', g')





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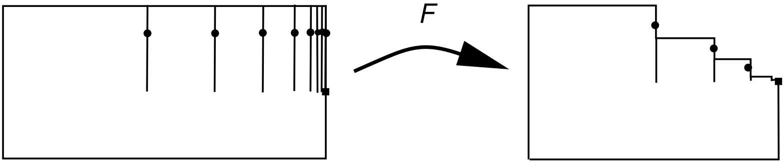
- As a point set
- As a topological structure



Comparation

- There exists conformal embeddings where conformal and causal boundary coincide.
- How can we know when the conformal embedding is good enough?
- Under certain technical conditions for the conformal boundary, it is possible to compare them.

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Technical Definitions

Consider a future (resp. past) timelike curve $\gamma : [a, b] \to \overline{i(M)}$.

Deformally timelike

γ is *deformally timelike* if there exist a neighbourhood $\gamma(b) \in U = U_0 \cap \overline{i(M)}$ (where U_0 is an open set of M_0) such that $w \in U, \gamma(a) \ll w$.

Transitively timelike

γ is *transitively timelike* if there exist a neighbourhood $\gamma(b) \in V = V_0 \cap \overline{i(M)}$ such that for all $w \in V$ with $\gamma(b) \leq w$, then $\gamma(a) \ll_i w$.

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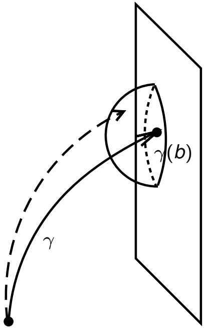

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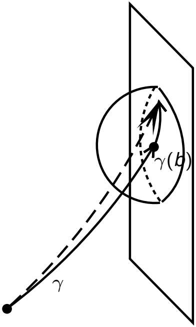

Technical Definitions

Regular Border Curve

γ is a *regular border curve* if it is *deformally timelike* and *transitively timelike* .

Chronologically complete

An envelopment $i : V \longrightarrow V_0$ is chronologically complete if all the timelike curves contained in $i(V)$ has an endpoint in V_0.

Regular accesible

An envelopment $i : V \longrightarrow V_0$ is regular accesible if all TIPs and TIFs in V can be defined by regular border curves.

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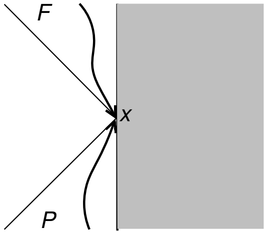
Technical definitions

Projection

$$\begin{array}{rccl} \pi : & \partial V & \rightarrow & \partial_i^* V \\ & (P, F) & \rightarrow & \pi(P, F) = x \end{array}$$

where x is the endpoint of a timelike curve such that:

- $I^-[\gamma] = P$ if γ is future-directed.
- $I^+[\gamma] = F$ if γ is past-directed.



Principal Result

Theorem

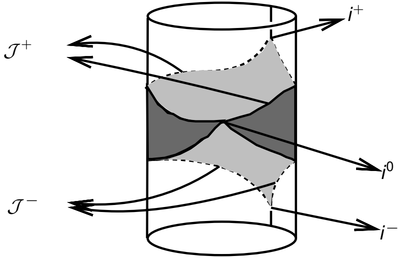
Consider $i : V \to V_0$ an envelopment chronologically complete and regular accesible. Then, the following statements hold:

- *π is well defined.*
- *The extension $\pi : \overline{V} \to \overline{V}_i^*$ is an homeomorphism and*
- *is a chronological isomorphism.*

Applications

Thanks to this study, it is straightforward to compute the causal boundary in the following cases:

- Minkowski spacetime.



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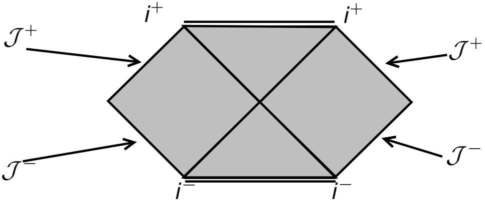

Thank you for your attention.