Cuvature homogeneous Lorentzian three-manifolds

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- Curvature homogeneity up to order one.
- Einstein-like examples.
- Explicit examples for all admissible forms of Q.

Curvature homogeneity

A p.R. manifold (M, g) is said to be curvature homogeneous up to order k if, for any points $p, q \in M$, there exists a linear isometry $\phi : T_pM \to T_qM$ such that

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When dim$M \geq 3$, a curvature homogeneous space needs not to be locally homogeneous.

3D Lorentzian manifolds

(M, g) connected 3D Lorentzian manifold. Its curvature tensor is completely determined by the Ricci tensor

$$\varrho(X, Y)_p = \sum_{i=1}^{3} \varepsilon_i g(R(X, e_i)Y, e_i),$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of T_pM and $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all i.

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where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of T_pM and $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all i.
ϱ is symmetric $\Rightarrow$ the *Ricci operator* $g(QX, Y) = \varrho(X, Y)$ is self-adjoint.
In the Riemannian case, there always exists an orthonormal basis diagonalizing Q, in the Lorentzian case four different cases can occur, called *Segre types*.

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- *Segre type* $\{3\}$: Q has three equal eigenvalues, associated to a one-dimensional eigenspace.

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In particular, at each point of a 3D Lorentzian manifold there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 timelike, such that Q takes one of the following forms:

$$\mathrm{S.\,type}\,\{11,1\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \qquad \mathrm{S.\,type}\,\{1z\bar{z}\}: \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix},$$

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The first examples of 3D Lorentzian manifolds, curvature homogeneous up to order one but not locally homogeneous, were given by Bueken and Vanhecke [Class. Quantum Grav., 1997].

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(Riemannian examples: only space forms and direct products $\mathbb{R}\times\mathbb{S}^2$, $\mathbb{R}\times\mathbb{H}^2$.)

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Theorem [GC, J.G.P., 2007] *A 3D connected, simply connected, complete homogeneous Lorentzian manifold (M, g) is either symmetric, or isometric to a Lorentzian Lie group, equipped with a left-invariant Lorentzian metric.*

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Riemannian counterpart: just one form for the unimodular Lie algebra and one form for the non-unimodular one [Milnor, Adv. Math., 1976].

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(1) either the Ricci form has any signature different from $(+,+,0)$, or
(2) the Ricci form has signature $(+,+,0)$ and, up to a re-numeration,

$$\frac{\lambda_2}{2} < \lambda_3 \leq \lambda_2.$$

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They also proved that a 3D Lorentzian manifold, curvature homogeneous up to order two, is locally homogeneous.

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" *we do not know, however, if there exist non-homogeneous curvature homogeneous three-dimensional Lorentzian manifolds whose Ricci operator is of this type or curvature homogeneity is sufficient to guarantee local homogeneity of the manifolds of this type.*" [Bueken-Djorić, 2000]

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Class $\mathcal{A}$: $(M,g) \in \mathcal{A}$ if and only if its Ricci tensor is *cyclic-parallel*:

$$(\nabla_X \varrho)(Y,Z) + (\nabla_Y \varrho)(Z,X) + (\nabla_Z \varrho)(X,Y) = 0, \quad \forall X,Y,Z.$$

This is equivalent to requiring that ϱ is a *Killing tensor*, that is, $(\nabla_X \varrho)(X,X) = 0$ for all X.

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$$\mathcal{A} \cup \mathcal{B} \subset \mathcal{C} \text{ (constant scalar curvature)},$$

$$\mathcal{E} \subset \mathcal{A} \cap \mathcal{B} = \mathcal{P} \text{ (Einstein and Ricci-parallel manifolds)}.$$

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What about Einstein-like Lorentzian metrics?

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Locally homogeneous conformally flat Riemannian manifolds are locally symmetric. Thus, conformal flatness is a weaker assumption in Lorentzian geometry than in the Riemannian framework.

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• *A curvature homogeneous Lorentzian three-manifold with Ricci operator of degenerate Segre type $\{21\}$, belonging to class $\mathcal{A}$, is curvature homogeneous up to order one.*

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Theorem [GC, Res. Math.] *Einstein-like curvature homogeneous Lorentzian three-manifolds, with Ricci operator NOT of degenerate Segre type* $\{21\}$*, are locally homogeneous.*

• *A curvature homogeneous Lorentzian three-manifold with Ricci operator of degenerate Segre type* $\{21\}$*, belonging to class* $\mathcal{A}$*, is curvature homogeneous up to order one.* (**Remark: ANY** 3D Riemannian manifold in class $\mathcal{A}$ is locally homogeneous! [Pedersen and Tod, D.G.A., 1999])

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• There is a **large class** of proper curvature homogeneous Lorentzian three-manifolds with Ricci operator of degenerate Segre type $\{21\}$, belonging to class $\mathcal{B}$.

Explicit metrics

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De Turk [Bull. A.M.S., 1980 and Invent. Math., 1981] obtained local existence theorems under very general hypotheses. In particular, if $\mathcal{R}$ is analytic in a neighborhood of $x_0 \in \mathbb{R}^n$ and $\mathcal{R}^{-1}(x_0)$ exists, then there exists an analytic metric g (of any desired signature) such that $\varrho = \mathcal{R}$ in a neighborhood of x_0.

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The second problem remains **open**, even for 3D manifolds and for particularly simple symmetric $(0, 2)$-tensors $\mathcal{R}$.

Explicit metrics

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Main Theorem: *Three-dimensional proper curvature homogeneous Lorentzian metrics exist for all different Segre types of Q (except in the degenerate diagonal case with three equal Ricci eigenvalues, when the manifold has necessarily constant sectional curvature).*

Previous contributions

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Again Bueken [J.G.P., 1997] described and classified curvature homogeneous examples with Q of degenerate Segre type $\{21\}$.

manifolds with a parallel degenerate line field

A parallel degenerate line field $\mathcal{D}$ on a Lorentzian manifold (M, g) is one spanned by a locally defined null vector U satisfying $\nabla U = \omega \otimes u$.

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A parallel degenerate line field $\mathcal{D}$ on a Lorentzian manifold (M, g) is one spanned by a locally defined null vector U satisfying $\nabla U = \omega \otimes u$.
A 3D Lorentzian manifold (M, g_f) admitting a parallel degenerate line field admits local coordinates (t, x, y) such that w.r. to $\{\partial_t, \partial_x, \partial_y\}$,

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{pmatrix}$$

for some function $f(t, x, y)$, where $\varepsilon = \pm 1$. Here we fix $\varepsilon = 1$, so that the Lorentzian metric tensor will have signature $(+, +, -)$. When $U = \partial_t$ is a parallel null vector field, then $f = f(x, y)$. These manifolds were studied by Chaichi, García-Río and Vázquez-Abal [J. Phys. A, 2005].

With respect to $\{\partial_t, \partial_x, \partial_y\}$, the Ricci operator Q of (M, g_f) is given by:

$$Q = \begin{pmatrix} \frac{1}{2}f''_{tt} & \frac{1}{2}f''_{tx} & -\frac{1}{2}f''_{xx} \\ 0 & 0 & \frac{1}{2}f''_{tx} \\ 0 & 0 & \frac{1}{2}f''_{tt} \end{pmatrix}.$$

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We now find conditions so that (M, g_f) is curvature homogeneous with Q either of Segre type $\{3\}$ or of nondegenerate Segre type $\{21\}$.

Segre type $\{3\}$

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In particular, constructing a p.o. frame field for (M, g_f), we see that if the defining function f satisfies

$$f''_{tt} = 0, \quad f''_{tx} = a_1, \quad f''_{xx} = a_2,$$

where $a_1 \neq 0$ and a_2 are two real constants, then (M, g) is curvature homogeneous and Q is of Segre type $\{3\}$.

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Theorem [GC, A.G.A.G.] *For any defining function f satisfying*

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$$f(t, x, y) = a_1 x t + \frac{a_2}{2} x^2 + p(y) t + q(y) x + s(y),$$

where p, q, s are three arbitrary one-variable functions, (M, g_f) is curvature homogeneous and has Ricci operator of Segre type $\{3\}$.

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where $\bar{p}, \bar{q}, \bar{s}$ are arbitrary one-variable functions, (M, g_f) is curvature homogeneous and has Ricci operator of nondeg. Segre type $\{21\}$.

non-isometric examples

We also studied isometries between curvature homogeneous Lorentzian three-manifolds (M, g) and (M', g'), having the Ricci operators of Segre type $\{3\}$.

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As a consequence, there are infinitely many curvature homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$, with the same Ricci operator of Segre type $\{3\}$, not locally isometric to one another.

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As a consequence, there are infinitely many curvature homogeneous Lorentzian metrics on $\mathbb{R}^3[w, x, y]$, with the same Ricci operator of Segre type $\{3\}$, not locally isometric to one another.

A similar argument applies to Lorentzian metrics with the same Ricci operator of nondegenerate Segre type $\{21\}$.

A general description

(M, g) a 3D curvature homogeneous Lorentzian manifold, $\{e_i\}$ a local pseudo-orthonormal frame field on (M, g), with respect to which the Ricci components are constant.

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The Levi Civita connection ∇ of (M, g) is completely determined by

$$\begin{array}{lll}
\nabla_{e_1} e_1 = \alpha\, e_2 + \beta\, e_3, & \nabla_{e_2} e_1 = \kappa\, e_2 + \mu\, e_3, & \nabla_{e_3} e_1 = \sigma\, e_2 + \tau\, e_3, \\
\nabla_{e_1} e_2 = -\alpha\, e_1 + \gamma\, e_3, & \nabla_{e_2} e_2 = -\kappa\, e_1 + \nu\, e_3, & \nabla_{e_3} e_2 = -\sigma\, e_1 + \psi\, e_3, \\
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for nine smooth functions $\alpha, .., \psi$.

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\end{array}$$

for nine smooth functions $\alpha, .., \psi$. These functions are not all independent. In fact, since the scalar curvature r is constant, the divergence formula $dr = 2\mathrm{div}\varrho$ implies

$$\sum_j \varepsilon_j \nabla_j \varrho_{ij} = 0 \quad \text{for all } i,$$

which gives some restrictions for the connection functions.

A general description

In function of $\alpha, .., \psi$, the Ricci components are given by

$$\begin{aligned}
\varrho_{11} &= e_2(\alpha) + e_3(\beta) - e_1(\kappa) - e_1(\tau) - \alpha^2 + \beta^2 - \kappa^2 - \tau^2 \\
&\quad +\alpha\psi + \beta\nu - 2\mu\sigma, \\
\varrho_{22} &= e_2(\alpha) - e_1(\kappa) + e_3(\nu) - e_2(\psi) - \alpha^2 - \kappa^2 + \nu^2 - \psi^2 \\
&\quad +\beta\nu - \kappa\tau + 2\gamma\sigma, \\
\varrho_{33} &= e_1(\tau) - e_3(\beta) - e_3(\nu) + e_2(\psi) - \beta^2 + \tau^2 - \nu^2 + \psi^2 \\
&\quad -\alpha\psi + \kappa\tau - 2\gamma\mu, \\
\varrho_{12} &= e_3(\gamma) - e_1(\psi) + \gamma(\beta + \nu) + \sigma(\beta - \nu) - \tau(\alpha + \psi), \\
\varrho_{13} &= e_2(\sigma) - e_3(\kappa) - \alpha(\mu + \sigma) - \nu(\kappa - \tau) - \psi(\mu - \sigma), \\
\varrho_{23} &= e_1(\mu) - e_2(\beta) + \alpha(\beta - \nu) + \gamma(\kappa - \tau) + \mu(\kappa + \tau).
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\end{aligned}
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We require ϱ_{ij} to be constant when we look for curvature homogeneous examples.

Explicit metrics

Fix $p \in M$ and consider a p.o. frame field $\{e_1, e_2, e_3\}$ as before in a neighborhood of p.

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Fix $p \in M$ and consider a p.o. frame field $\{e_1, e_2, e_3\}$ as before in a neighborhood of p.
Choose a surface S through p transversal to the lines generated by e_3, a local coordinates system (w, x) on S and a neighborhood U_p of p, sufficiently small that each $q \in U_p$ is situated on exactly one line generated by e_3 and passing through one point $\bar{q} \in S$. We then choose an orientation of S and put

$$y(q) = \mathrm{dist}(q, \pi(q)), \quad w(q) = w(\pi(q)), \quad x(q) = x(\pi(q)),$$

where $\pi : U_p \longrightarrow S$ is the corresponding projection. In this way, a local coordinate system (w, x, y) is introduced in U_p.

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where $\pi : U_p \longrightarrow S$ is the corresponding projection. In this way, a local coordinate system (w, x, y) is introduced in U_p.
The coframe $\{\omega_1, \omega_2, \omega_3\}$ of $\{e_1, e_2, e_3\}$ takes the form

$$\omega^1 = A dw + B dx, \quad \omega^2 = C dw + D dx, \quad \omega^3 = G dw + H dx + dy,$$

for some functions A, B, C, D, G, H.

Basic system of PDE

The connection forms ω^i_j are given by $d\omega^i + \sum_j \omega^i_j \wedge \omega^j = 0, \quad \forall i.$
Connection equations above are equivalent to

$$A'_y = \beta A + (\mu + \sigma)C, \qquad B'_y = \beta B + (\mu + \sigma)D,$$

$$C'_y = (\gamma - \sigma)A + \nu C, \qquad D'_y = (\gamma - \sigma)B + \nu D,$$

$$G'_y = -\tau A - \psi C, \qquad H'_y = -\tau B - \psi D,$$

$$B'_w - A'_x = \alpha \mathcal{D} - \beta \mathcal{E} - (\mu + \sigma)\mathcal{F},$$

$$D'_w - C'_x = \kappa \mathcal{D} - (\gamma - \sigma)\mathcal{E} - \nu \mathcal{F},$$

$$H'_w - G'_x = -(\gamma - \mu)\mathcal{D} + \tau \mathcal{E} + \psi \mathcal{F},$$

where $\mathcal{D} = AD - BC, \mathcal{E} = AH - BG, \mathcal{F} = CH - DG$. $\mathcal{D} \neq 0$ is a necessary and sufficient condition for linear independence of ω^i.

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The curvature forms Ω^i_j, depending on the Ricci curvature (ϱ_{ij}), are determined by

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Theorem *Let A, B, C, D, G, H be smooth functions on the three variables w, x, y, satisfying partial differential equations above. Then, the coframe $\{\omega^1, \omega^2, \omega^3\}$ describes a curvature homogeneous Lorentzian metric g on $\mathbb{R}^3$, whose Ricci tensor has constant (local) components (ϱ_{ij}).*

(REF: [Kowalski and Prüfer, Math. Ann., 1994], [GC, J.M.P., 2007, D.G.A., 2008 and A.G.A.G.])

The remaining Segre types

Theorem *For any choice of distinct real numbers q_1, q_2, q_3, there exists a family of curvature homogeneous p.R. metrics on $\mathbb{R}^3$, with diagonalizable Ricci tensor and Ricci eigenvalues q_i, depending on two functions of two variables and two more functions of one variable.*

Theorem *For any real constant $c \neq 0$, let Q be the linear operator of Segre type $\{1z\bar{z}\}$ described by*

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}.$$

Then, there exists a family of curvature homogeneous Lorentzian metrics on $\mathbb{R}^3$, having this Ricci operator, depending on two arbitray functions of two variables and two arbitrary functions of one variable.