Affine homogeneous geodesics and the invariants of $SL(2, \mathbb{R})$

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Homogeneous geodesics in homogeneous affine manifolds

Definition

Let (M, ∇) be a homogeneous affine manifold.
A geodesic is homogeneous if it is an orbit of an one-parameter group of affine diffeomorphisms. (Here the canonical parameter of the group need not be the affine parameter of the geodesic.)
An affine g.o. manifold is a homogeneous affine manifold (M, ∇) such that each geodesic is homogeneous.

Lemma

Let $M = G/H$ be a homogeneous space with a left-invariant affine connection ∇. Then each regular curve which is an orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.

Definition

Let (M, ∇) be a manifold with an affine connection.
A vector field X on M is called a Killing vector field if

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0$$

is satisfied for arbitrary vector fields Y, Z.

Lemma

(M, ∇) is homogeneous if it admits at least $n = \dim M$ complete affine Killing vector fields which are independent at each point.
If (M, ∇) admits an n-dimensional vector space of complete geodesic Killing vector fields, then it is an affine g.o. space.

Definition

A nonvanishing smooth vector field Z on M is geodesic along its regular integral curve γ if $\gamma(t)$ is geodesic up to a possible reparametrization. If all regular integral curves of Z are geodesics up to a reparametrization, then the vector field Z is called a geodesic vector field.

For example, a round two-sphere with the corresponding Levi-Civita connection does *not* admit any geodesic affine Killing vector field. Still, all geodesics are homogeneous.

Lemma

Let Z be a nonvanishing Killing vector field on $M = (G/H, \nabla)$.
1) *Z is geodesic along its integral curve γ if and only if*

$$\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)}$$

holds along γ. Here $k_\gamma \in \mathbb{R}$ is a constant.
2) *Z is a geodesic vector field if and only if*

$$\nabla_Z Z = k \cdot Z$$

holds on M. Here k is a smooth function on M which is constant along integral curves of Z.

$\dim(M) = 2$

Theorem (Opozda; Arias-Marco, Kowalski)

Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold $\mathcal{M}$. Then, either ∇ is locally a Levi-Civita connection of the unit sphere or, in a neighbourhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants A, B, C, D, E, F, G, H such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:

$$\begin{array}{lll}
\underline{\mathit{TypeA}}: & \nabla_{\partial_u}\partial_u = A\,\partial_u + B\,\partial_v, & \nabla_{\partial_u}\partial_v = C\,\partial_u + D\,\partial_v, \\
& \nabla_{\partial_v}\partial_u = E\,\partial_u + F\,\partial_v, & \nabla_{\partial_v}\partial_v = G\,\partial_u + H\,\partial_v, \\
\underline{\mathit{TypeB}}: & \nabla_{\partial_u}\partial_u = \dfrac{A}{u}\,\partial_u + \dfrac{B}{u}\,\partial_v, & \nabla_{\partial_u}\partial_v = \dfrac{C}{u}\,\partial_u + \dfrac{D}{u}\,\partial_v, \\
& \nabla_{\partial_v}\partial_u = \dfrac{E}{u}\,\partial_u + \dfrac{F}{u}\,\partial_v, & \nabla_{\partial_v}\partial_v = \dfrac{G}{u}\,\partial_u + \dfrac{H}{u}\,\partial_v,
\end{array}$$

where not all A, B, C, D, E, F, G, H are zero.

Connections of type A

- Let us have a connection ∇ with constant Christoffel symbols. The operators ∂_u, ∂_v are affine Killing vector fields.
- A general vector field $X = x\,\partial_u + y\,\partial_v$ satisfies the condition $\nabla_X X = kX$ if it holds

$$\begin{aligned} Ax^2 + (C+E)xy + Gy^2 &= k\,x, \\ Bx^2 + (D+F)xy + Hy^2 &= k\,y. \end{aligned} \tag{1}$$

- By the elimination of the factor k we obtain

$$Bx^3 - (A-D-F)x^2y - (C+E-H)xy^2 - Gy^3 = 0.$$

- A sufficient condition for a vector field $X = x\,\partial_u + y\,\partial_v$ to be geodesic is that the pair (x, y) satisfies this condition.
- For any connection of type A, a geodesic Killing field (and at least one homogeneous geodesic) exist.

Affine g.o. manifold

Theorem

For $(\mathbb{R}^2, \nabla)$ to be an affine g.o. manifold, it is sufficient that

$$B = 0, \; A = D + F, \; G = 0, \; H = C + E. \tag{2}$$

If $(\mathbb{R}^2, \nabla)$ admits only two affine Killing vector fields, then these formulas are also necessary conditions for the g.o. property.

- In this case, the equations (1) give

$$Ax + Hy = k,$$

<u>k is nonzero</u> in general and geodesics must be reparametrized.

Connections of type B

We consider the globally homogeneous manifold
$\mathcal{U} = \{\mathbb{R}(u, v) \mid u > 0\}$.

- The general Killing vector field is $X = x\partial_v + y(u\partial_u + v\partial_v)$.
- The equality $\nabla_X X = k\,X$ gives

$$\begin{aligned} Gx^2 + \big[(C+E)u + 2Gv\big]xy & \\ + \big[(A+1)u^2 + (C+E)uv + Gv^2\big]y^2 &= ku^2 y, \\ Hx^2 + \big[(D+F+1)u + 2Hv\big]xy & \\ + \big[Bu^2 + (D+F+1)uv + Hv^2\big]y^2 &= ku(x+vy). \end{aligned}$$

- The coordinate components $u(t), v(t)$ of its integral curve $\gamma(t)$ are

$$u(t) = c_1\,\mathrm{e}^{yt},\ v(t) = c_2\,\mathrm{e}^{yt} - x/y,\ c_1 > 0, \tag{3}$$

where c_1, c_2 are integration constants.

If we substitute for $u(t), v(t)$ from (3), we get

$$\begin{aligned} ((A+1)c_1^2 + (C+E)c_1c_2 + Gc_2^2)y &= k_\gamma c_1^2, \\ (Bc_1^2 + (D+F+1)c_1c_2 + Hc_2^2)y &= k_\gamma c_1 c_2. \end{aligned} \quad (4)$$

By the elimination of k_γ we obtain

$$B\, c_1^3 - (A-D-F)\, c_1^2\, c_2 - (C+E-H)\, c_1\, c_2^2 - G\, c_2^3 = 0.$$

- $(\mathcal{U}, \nabla)$ admits at least one homogeneous geodesic through each point.
- Homogeneous geodesics are the integral curves of Killing vector fields which are not geodesic.
- In general, connections of type B do not admit any geodesic Killing vector fields.

Affine g.o. manifold

- If it holds

$$B = 0,\ A = D + F,\ G = 0,\ H = C + E, \tag{5}$$

then the equations (4) give us

$$((A+1)c_1 + Hc_2)y \quad = \quad k_\gamma c_1.$$

Theorem

If the conditions (5) hold, then for any $(x, y) \neq (0, 0)$ the corresponding Killing vector field is geodesic. The manifold $(\mathcal{U}, \nabla)$ is an affine g.o. manifold and any homogeneous geodesic is the integral curve of a geodesic Killing vector field.

- For a given geodesic Killing field, different geodesics must be reparametrized by <u>different k_γ</u>.

Invariants of $SL(2,\mathbb{R})$ in $\mathbb{R}^6$

$\mathcal{H}$ ... set of torsion-free connections with constant CS on $\mathbb{R}^2$

$$\begin{array}{lll} \Gamma^1_{11} = A_1, & \Gamma^1_{12} = \Gamma^1_{21} = E_1, & \Gamma^1_{22} = B_1, \\ \Gamma^2_{11} = A_2, & \Gamma^1_{12} = \Gamma^2_{21} = E_2, & \Gamma^2_{22} = B_2. \end{array} \tag{6}$$

Vector field $X = x\,\partial_u + y\,\partial_v$ satisfies
the condition $\nabla_X X = 0$ if it holds

$$\begin{aligned} x^2 A_1 + y^2 B_1 + 2xyE_1 &= 0, \\ x^2 A_2 + y^2 B_2 + 2xyE_2 &= 0. \end{aligned}$$

We calculate the resultant

$$R_1 = 4\,(A_1\,E_2 - E_1\,A_2)\,(B_1\,E_2 - E_1\,B_2) + (A_1\,B_2 - A_2\,B_1)^2\,.$$

- R_1 is an invariant with respect to $SL(2,\mathbb{R})$ acting on $\mathcal{H}$.

Hilbert basis of invariants

- Consider the action of $SL(2, \mathbb{R})$ on $\mathcal{H}$.
- There are 3 independent invariants:

$$
\begin{aligned}
R_1 &= 4\,(A_1\,E_2 - E_1\,A_2)\,(B_1\,E_2 - E_1\,B_2) + (A_1\,B_2 - A_2\,B_1)^2\,, \\
R_2 &= (A_1 E_2 + A_2 B_2 - A_2 E_1 - E_2^2)(A_1 B_1 - B_1 E_2 + B_2 E_1 - E_1^2) \\
&\quad -(A_2 B_1 - E_1 E_2)^2, \\
R_3 &= (A_1^2 + A_1 E_2 + A_2 B_2 + A_2 E_1)(A_1 B_1 + B_1 E_2 + B_2^2 + B_2 E_1) \\
&\quad -(A_1 E_1 + B_2 E_2 + 2 E_1 E_2)^2.
\end{aligned}
$$

Invariant of $SL(2,\mathbb{R})$ in $\mathbb{R}^9$

$\mathcal{H}'$ ... set of torsion-free connections with constant CS on $\mathbb{R}^3$
$\mathbb{R}^3$, vector field $X = x\,\partial_u + y\,\partial_v + z\,\partial_w$ satisfies the condition $\nabla_X X = kX$ if it holds

$$
\begin{aligned}
x^2 A_1 + y^2 B_1 + z^2 C_1 + 2\,xy\,E_1 + 2\,xz\,F_1 + 2\,yz\,G_1 &= k\,x,\\
x^2 A_2 + y^2 B_2 + z^2 C_2 + 2\,xy\,E_2 + 2\,xz\,F_2 + 2\,yz\,G_2 &= k\,y,\\
x^2 A_3 + y^2 B_3 + z^2 C_3 + 2\,xy\,E_3 + 2\,xz\,F_3 + 2\,yz\,G_3 &= k\,z.
\end{aligned}
$$

$z = 0, y \neq 0\,(y = 1)$:

$$
\begin{aligned}
A_2\,x^3 + (2\,E_2 - A_1)\,x^2 + (B_2 - 2\,E_1)\,x - B_1 &= 0,\\
A_3\,x^2 + 2\,E_3\,x + B_3 &= 0.
\end{aligned}
$$

The resultant of this system is an invariant with respect to $SL(2,\mathbb{R})$ acting on $\mathcal{H}'$.

Invariant of $SL(2,\mathbb{R})$ in $\mathbb{R}^9$

$$
\begin{aligned}
R' \;=\;& {A_1}^2 A_3\, {B_3}^2 + {A_2}^2 {B_3}^3 + {A_3}^3 {B_1}^2 \\
&+ \big({B_2}^2 B_3 - 4\, B_2\, B_3\, E_1 + 4\, B_3\, {E_1}^2 \\
&\quad + (2\, B_2\, E_3 + 4\, B_3\, E_2 - 4\, E_1\, E_3)\, B_1\big)\, {A_3}^2 \\
&+ \big(2\, A_2\, {B_3}^2 E_3 - 2\, {A_3}^2 B_1\, B_3 \\
&\quad + \big(4\, B_1\, {E_3}^2 + 2\, B_2\, B_3\, E_3 - 4\, {B_3}^2 E_2 - 4\, B_3\, E_1\, E_3\big)\, A_3\big)\, A_1 \\
&+ \big(8\, B_1\, {E_3}^3 + 4\, B_2\, B_3\, {E_3}^2 - 4\, {B_3}^2 E_2\, E_3 - 8\, B_3\, E_1\, {E_3}^2 \\
&\quad + \big(-6\, B_1\, B_3\, E_3 - 2\, B_2\, {B_3}^2 + 4\, {B_3}^2 E_1\big)\, A_3\big)\, A_2 \\
&+ \big(-8\, B_1\, E_2\, {E_3}^2 - 4\, B_2\, B_3\, E_2\, E_3 + 4\, {B_3}^2 {E_2}^2 + 8\, B_3\, E_1\, E_2\, E_3\big)\, A_3.
\end{aligned}
$$