On Generalised Jacobi Equations on Lorentzian Manifolds

Volker Perlick (Lancaster University)

- Jacobi equation (equation of geodesic deviation):
  - linearise geodesic equation about a chosen reference geodesic

- Generalised Jacobi equation:
  - go beyond linearisation
    - Hodgkinson (1972)
    - Bażański (1977)

1. Jacobi equation

   a) in affine manifolds

   b) for timelike geodesics in Lorentzian manifolds
      (linear deformation of test bodies by tidal forces)

   c) for lightlike geodesics in Lorentzian manifolds
      (linear deformation of light bundles)

2. Generalised Jacobi equation of Hodgkinson

   a) in affine manifolds

   b) for timelike geodesics in Lorentzian manifolds
      (non-linear deformation of test bodies by tidal forces)

   c) for lightlike geodesics in Lorentzian manifolds
      (non-linear deformation of light bundles)

3. Generalised Jacobi equation of Bażański

1.Jacobi equation

a) in affine manifolds

Consider n-dimensional affine manifold (M, ∇)

Use Einstein's summation convention for coordinate indices $\mu, \nu, \sigma : 1, \ldots, n$

Introduce connection coefficients: $\nabla_{\partial_\nu} \partial_\sigma = \Gamma^\mu_{\nu\sigma} \partial_\mu$

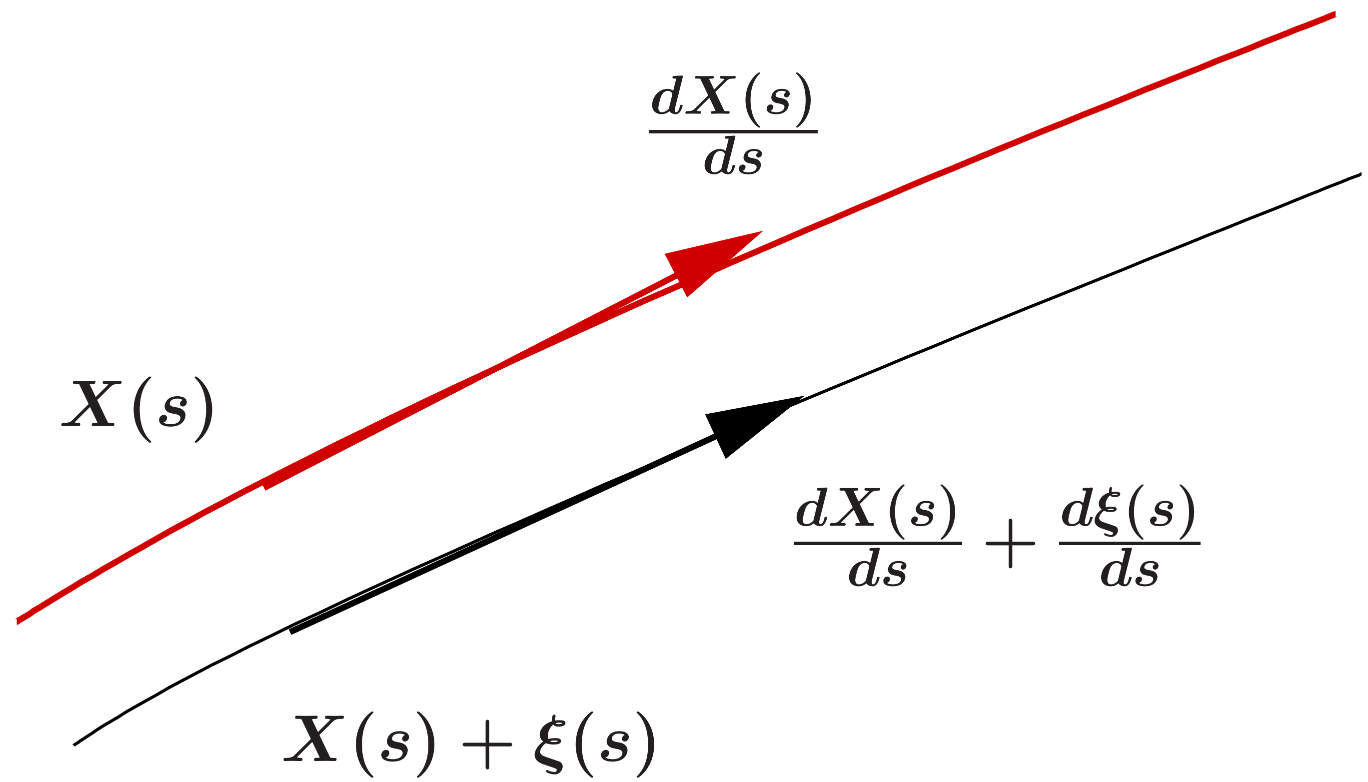
Geodesic equation: $\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(x) \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0$

Without restriction of generality: $T^\mu_{\nu\sigma} := \Gamma^\mu_{\nu\sigma} - \Gamma^\mu_{\sigma\nu} = 0$

[for non-vanishing torsion see

Swaminarayan, N.S., Safko, J.L.: "A coordinate-free derivation of a generalized geodesic deviation equation" J. Math. Phys. 24, 883–885 (1983)]

Consider two neighbouring curves $X(s)$ and $X(s) + \xi(s)$



Assume $X(s)$ is a geodesic, $\frac{d^2X^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(X)\frac{dX^\nu}{ds}\frac{dX^\sigma}{ds} = 0$

Then $X(s) + \xi(s)$ is a geodesic if and only if

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(X+\xi)\left(\frac{dX^\nu}{ds} + \frac{d\xi^\nu}{ds}\right)\left(\frac{dX^\sigma}{ds} + \frac{d\xi^\sigma}{ds}\right) - \Gamma^\mu_{\nu\sigma}(X)\frac{dX^\nu}{ds}\frac{dX^\sigma}{ds} = 0$$

Linearisation with respect to ξ and $\frac{d\xi}{ds}$ gives Jacobi equation:

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(X)\,2\,\frac{d\xi^\nu}{ds}\frac{dX^\sigma}{ds} + \partial_\tau\Gamma^\mu_{\nu\sigma}(X)\,\xi^\tau\,\frac{dX^\nu}{ds}\frac{dX^\sigma}{ds} = 0$$

With covariant derivative $\frac{D\eta^\mu}{ds} = \frac{d\eta^\mu}{ds} + \Gamma^\mu_{\rho\tau}(X)\,\eta^\rho\,\frac{dX^\tau}{ds}$

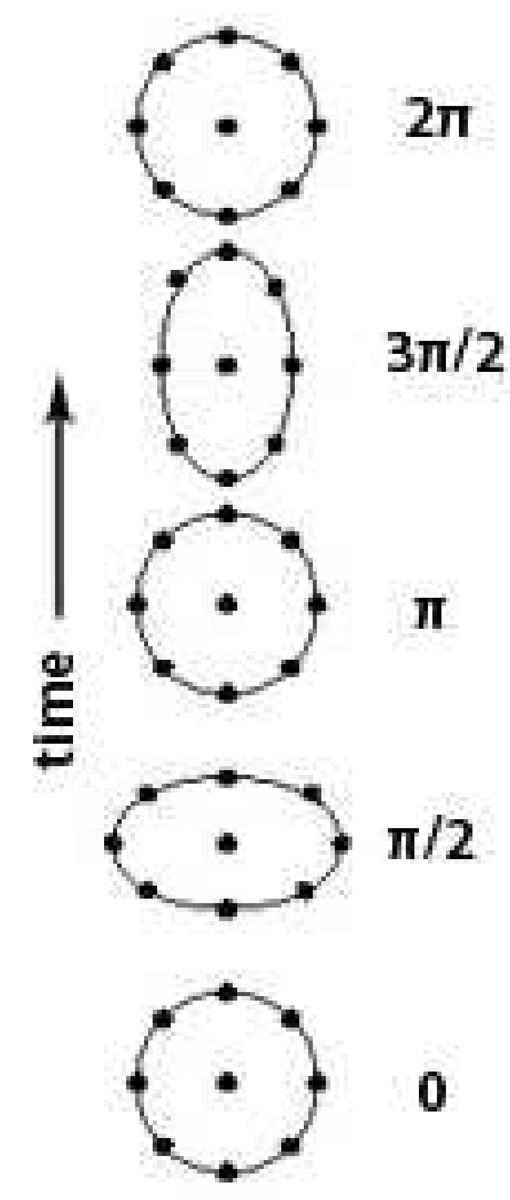
and curvature tensor $R^\mu_{\tau\nu\sigma} = \partial_\nu\Gamma^\mu_{\tau\sigma} - \partial_\tau\Gamma^\mu_{\nu\sigma} + \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\tau\sigma} - \Gamma^\mu_{\tau\lambda}\Gamma^\lambda_{\nu\sigma}$

$$\frac{D^2\xi^\mu}{ds^2} + R^\mu_{\tau\nu\sigma}(X)\,\xi^\nu\,\frac{dX^\tau}{ds}\frac{dX^\sigma}{ds} = 0$$

b) for timelike geodesics in Lorentzian manifolds

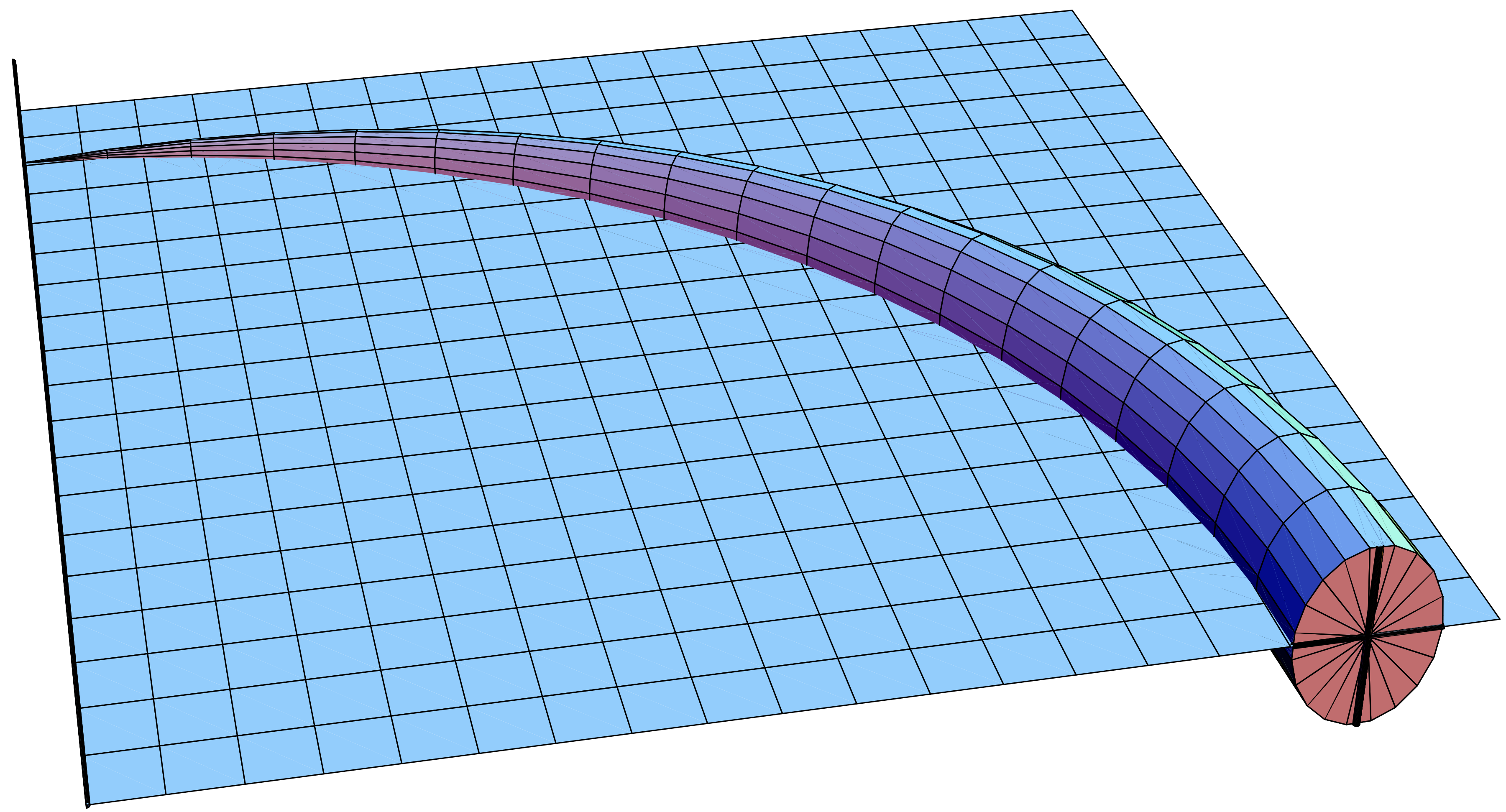
Jacobi equation describes linear deformation of test bodies by tidal forces

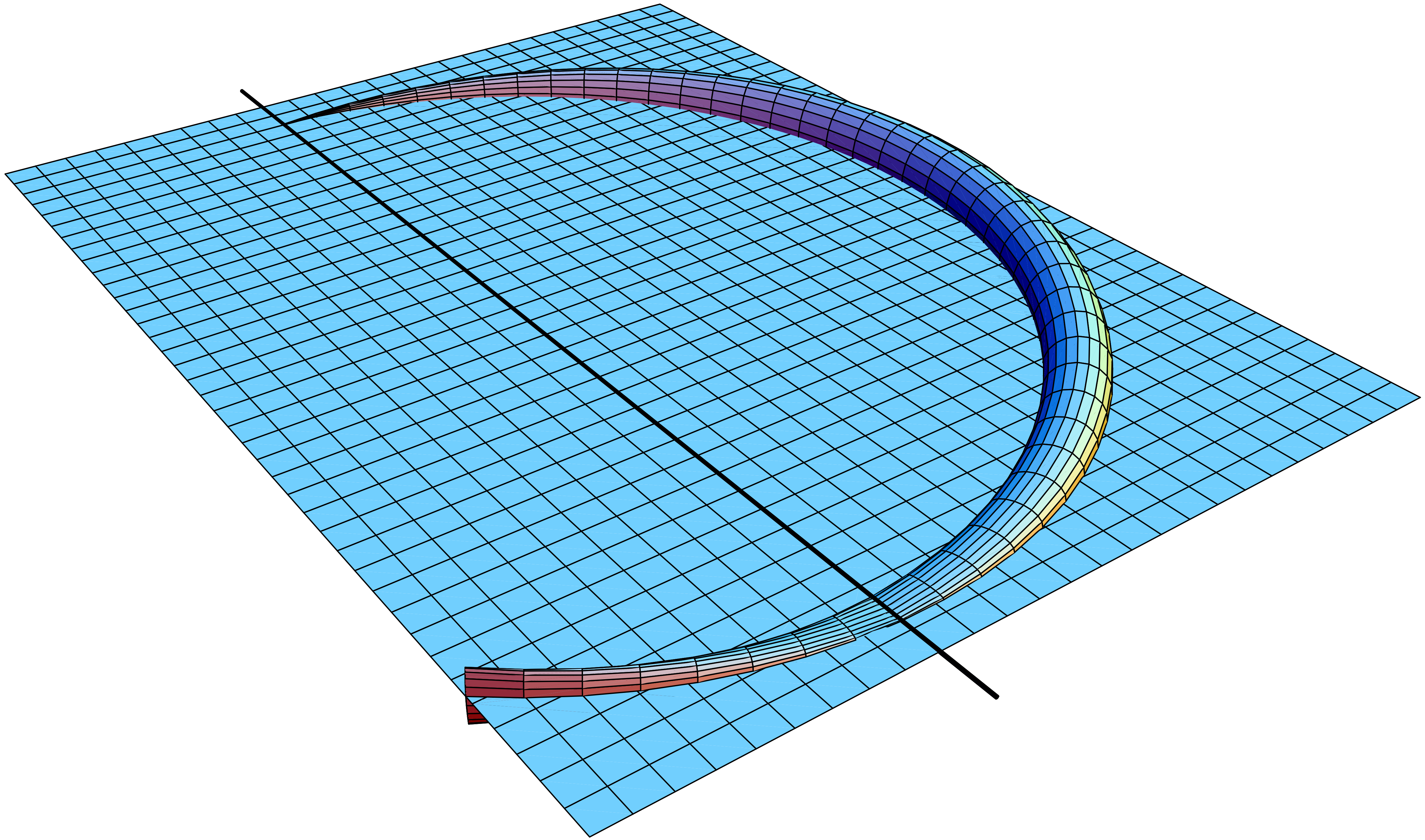
e.g. in the field of a gravitational wave



c) for lightlike geodesics in Lorentzian manifolds

Jacobi equation describes linear deformation of light bundles





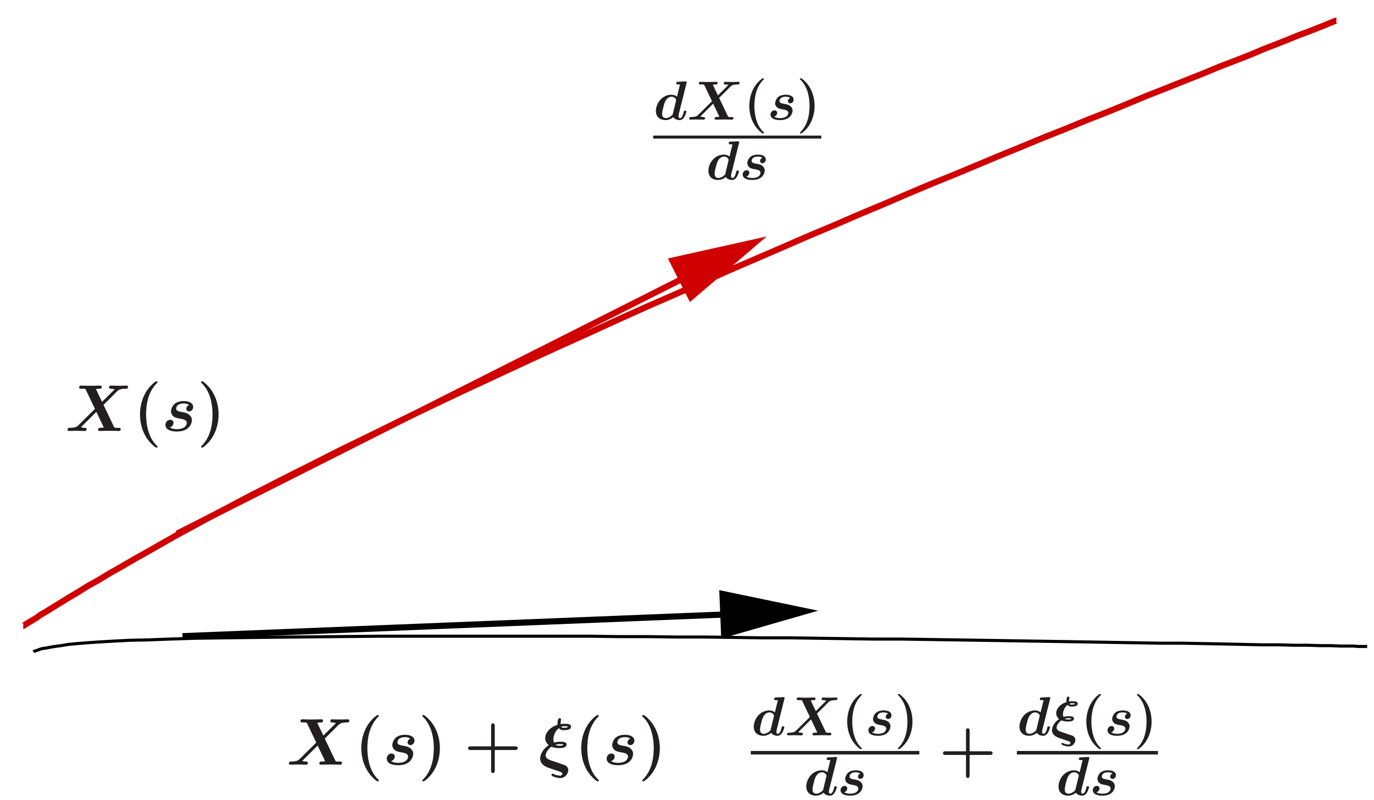
2. Generalised Jacobi equation of Hodgkinson

a) on affine manifolds

D. E. Hodgkinson: "A modified equation of geodesic deviation" Gen. Relativ. Gravit. 3, 351–375 (1972)

VP: "On the generalized Jacobi equation" Gen. Relativ. Gravit. 40, 1029–1045 (2008)

As before:



$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(X+\xi)\left(\frac{dX^\nu}{ds} + \frac{d\xi^\nu}{ds}\right)\left(\frac{dX^\sigma}{ds} + \frac{d\xi^\sigma}{ds}\right) - \Gamma^\mu_{\nu\sigma}(X)\frac{dX^\nu}{ds}\frac{dX^\sigma}{ds} = 0$$

Linearisation with respect to ξ, but NOT with respect to $\frac{d\xi}{ds}$, gives the

generalised Jacobi equation:

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(X)\left(2\,\frac{d\xi^\nu}{ds}\frac{dX^\sigma}{ds} + \frac{d\xi^\nu}{ds}\frac{d\xi^\sigma}{ds}\right)$$

$$+\,\partial_\tau\Gamma^\mu_{\nu\sigma}(X)\,\xi^\tau\left(\frac{dX^\nu}{ds} + \frac{d\xi^\nu}{ds}\right)\left(\frac{dX^\sigma}{ds} + \frac{d\xi^\sigma}{ds}\right) = 0$$

Note: For large $\frac{d\xi}{ds}$ the approximation will be good, in general, only on a short parameter interval.

Geometric interpretation in the tangent bundle TM:

Jacobi equation is valid on a tubular neighbourhood of a geodesic in TM

Generalised Jacobi equation is valid on a neighbourhood in TM whose projection to M is tubular; the neighbourhood is unbounded with respect to the fibre.

Change from affine parametrisation to parametrisation by one of the coordinates, $X^n(u) = u$ and $\xi^n(u) = 0$

With $i, j, k = 1, \ldots, (n-1)$

$$\frac{d^2\xi^i}{du^2} + \Gamma^i_{\nu\sigma}(X)\left(2\,\frac{dX^\nu}{du}\frac{d\xi^\sigma}{du} + \frac{d\xi^\nu}{du}\frac{d\xi^\sigma}{du}\right)$$

$$-\,\Gamma^n_{\nu\sigma}(X)\left(2\,\frac{dX^\nu}{du}\frac{d\xi^\sigma}{du} + \frac{d\xi^\nu}{du}\frac{d\xi^\sigma}{du}\right)\frac{dX^i}{du}$$

$$-\,\Gamma^n_{\nu\sigma}(X)\left(\frac{dX^\nu}{du} + \frac{d\xi^\nu}{du}\right)\left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du}\right)\frac{d\xi^i}{du}$$

$$+\,\partial_\tau\Gamma^i_{\nu\sigma}(X)\,\xi^\tau\left(\frac{dX^\nu}{du} + \frac{d\xi^\nu}{du}\right)\left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du}\right)$$

$$-\,\partial_\tau\Gamma^n_{\nu\sigma}(X)\,\xi^\tau\left(\frac{dX^\nu}{du} + \frac{d\xi^\nu}{du}\right)\left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du}\right)\left(\frac{dX^i}{du} + \frac{d\xi^i}{du}\right) = 0$$

Fermi coordinates: $\Gamma^{\mu}_{\nu\sigma}(X) = 0$

$$\frac{d^2\xi^i}{du^2} + \partial_\tau \Gamma^i_{\nu\sigma}(X)\,\xi^\tau \left(\frac{dX^\nu}{du} + \frac{d\xi^\nu}{du}\right)\left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du}\right)$$

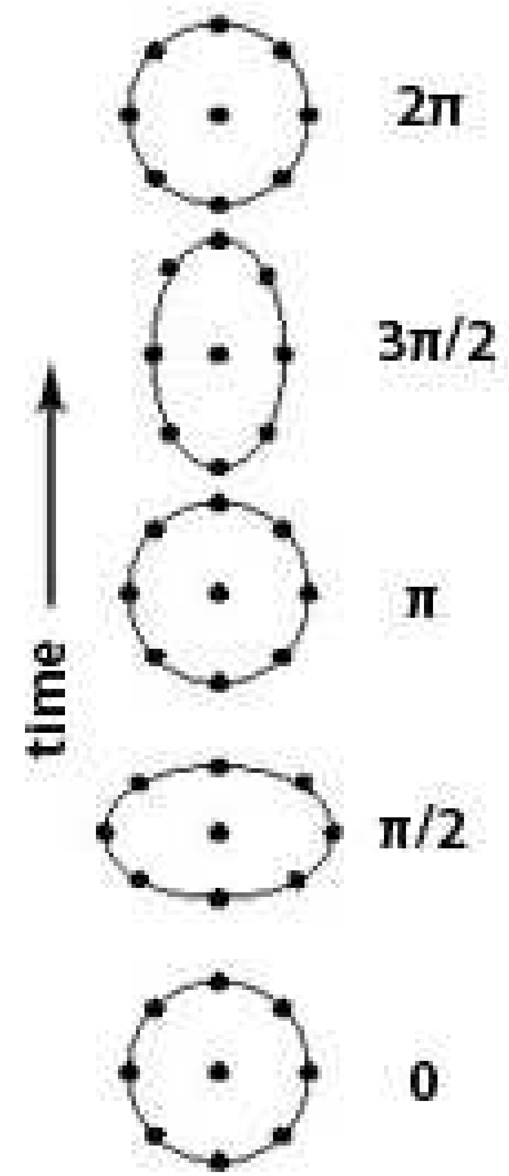
$$- \partial_\tau \Gamma^n_{\nu\sigma}(X)\,\xi^\tau \left(\frac{dX^\nu}{du} + \frac{d\xi^\nu}{du}\right)\left(\frac{dX^\sigma}{du} + \frac{d\xi^\sigma}{du}\right)\left(\frac{dX^i}{du} + \frac{d\xi^i}{du}\right) = 0$$

Where $X^n(u) = u$ and $\xi^n(u) = 0$.

This is a non-autonomous system of second-order ordinary differential equations for the $n-1$ functions $\xi^1(u), \ldots, \xi^{n-1}(u)$.

b) for timelike geodesics in Lorentzian manifolds

Generalised Jacobi equation describes non-linear deformation of test bodies by tidal forces



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- Mashhoon, B.: “On tidal phenomena in a strong gravitational field” Astrophys. J. 197, 705–716 (1975)
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- Chicone, C., Mashhoon, B.: “The generalized Jacobi equation” Class. Quant. Grav. 19, 4231–4248 (2002)
- Chicone, C., Mashhoon, B.: “Ultrarelativistic motion: inertial and tidal effects in Fermi coordinates” Class. Quant. Grav. 22, 195–205 (2005)
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Chicone & Mashhoon (2002):

- **Hamiltonian formalism**

- **Plane Gravitational Wave Spacetime**

  (Reduced) generalised Jacobi equation is chaotic

- **Kerr Spacetime**

$$\frac{d^2R}{dT^2} = -\,k(T)\left(1 \,-\, 2\,(\frac{dR}{dT})^2\right)R$$

Critical Speed $\frac{dR}{dT} = \frac{1}{\sqrt{2}}$ **for particles on axis (attractor)**

c) for lightlike geodesics in Lorentzian manifolds

- VP: "On the generalized Jacobi equation" Gen. Relativ. Gravit. 40, 1029–1045 (2008)

The generalised Jacobi equation describes deformation of light bundles with arbitrarily large opening angles but, in general, only for short parameter intervals.

Example 1:

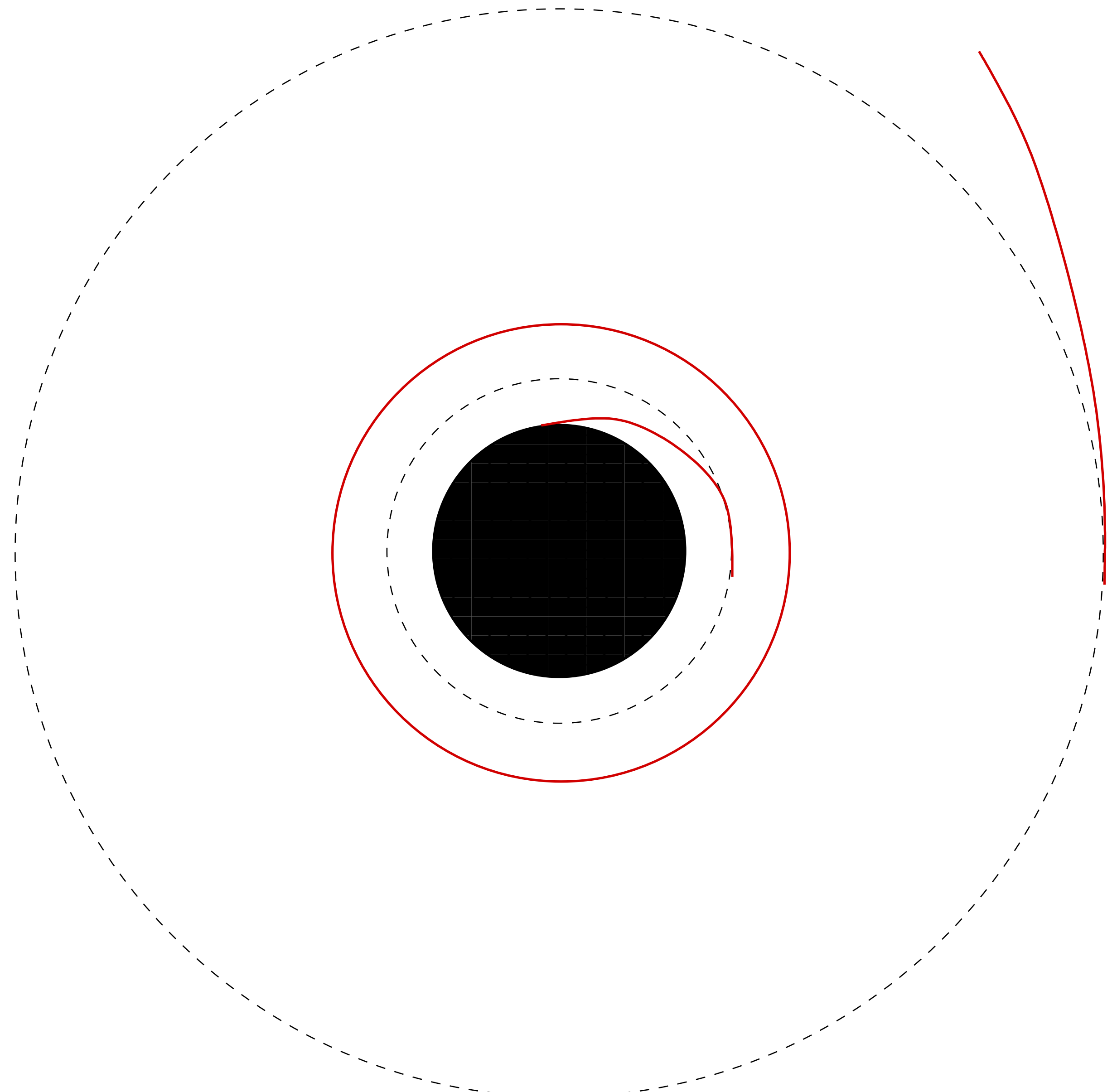
Generalised Jacobi equation for lightlike geodesics in Schwarzschild spacetime

Metric in Schwarzschild coordinates:

$$g_{\mu\nu}\, dx^\mu\, dx^\nu \, = \, - \big(1 - \frac{2m}{r}\big)\, dt^2 \, + \frac{dr^2}{1 - \frac{2m}{r}} \, + \, r^2\, d\vartheta^2 \, + \, r^2 \sin^2\!\vartheta\, d\varphi^2 \, .$$

$X =$ **lightlike geodesic at** $r = 3m$

Parametrisation by $x^n = \varphi$



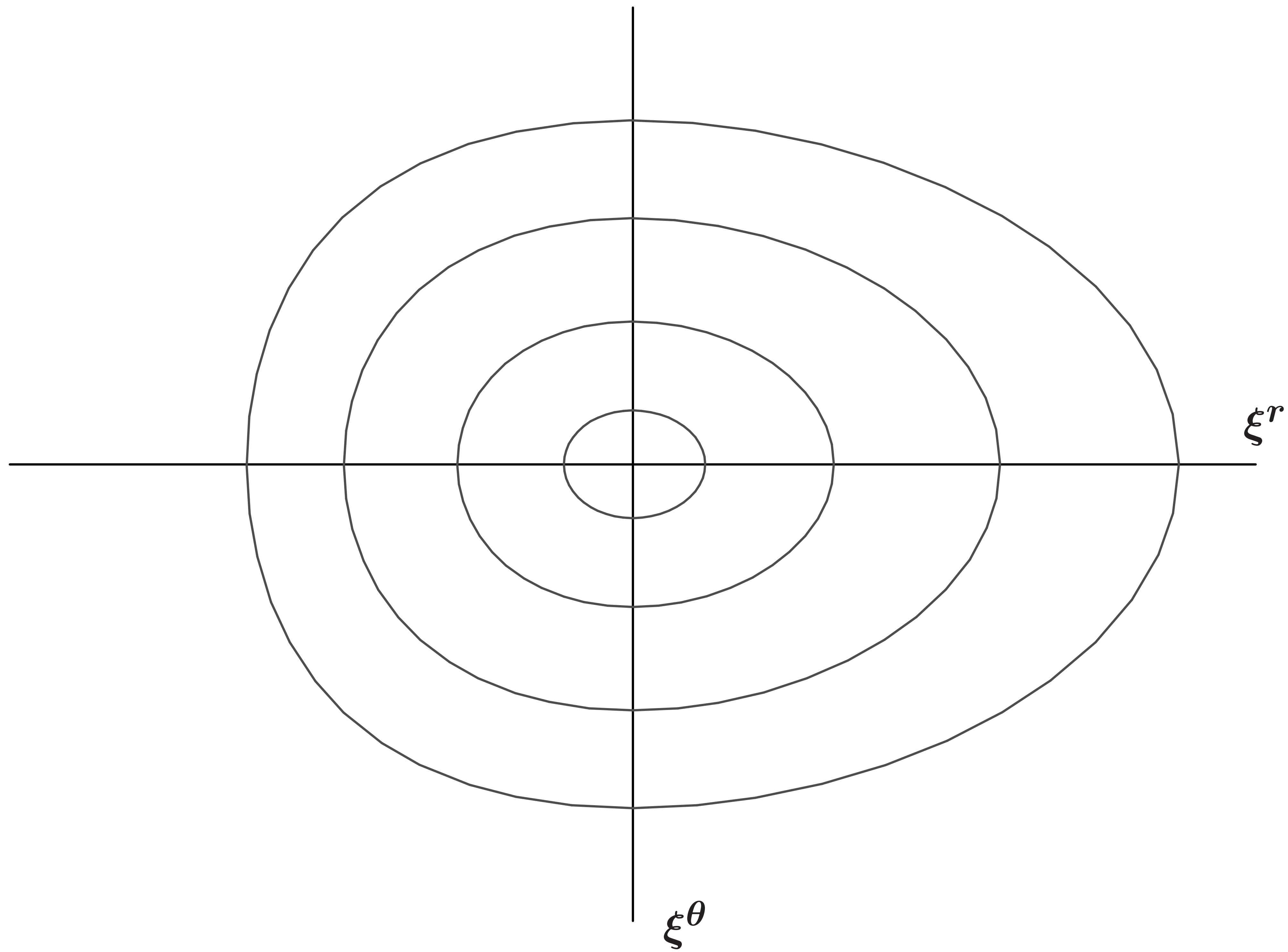
Generalised Jacobi equation for transverse coordinates:

$$\frac{d^2\xi^\vartheta}{d\varphi^2} = -\,\xi^\vartheta - \frac{2\,\xi^\vartheta}{9\,m^2}\left(\frac{d\xi^\vartheta}{d\varphi}\right)^2,$$

$$\frac{d^2\xi^r}{d\varphi^2} = \xi^r + \frac{\xi^r}{9\,m^2}\left(\left(\frac{d\xi^\vartheta}{d\varphi}\right)^2 - \frac{2}{3}\left(\frac{d\xi^r}{d\varphi}\right)^2\right)$$

$$-\frac{2\,\xi^\vartheta}{9\,m^2}\frac{d\xi^\vartheta}{d\varphi}\frac{d\xi^r}{d\varphi} + \frac{2}{3\sqrt{3}\,m}\left(\frac{d\xi^r}{d\varphi}\right)^2$$

Cross section of light bundle:



Example 2:

Generalised Jacobi equation for lightlike geodesics in spacetime of a "plane wave" (gravitational plus electromagnetic wave):

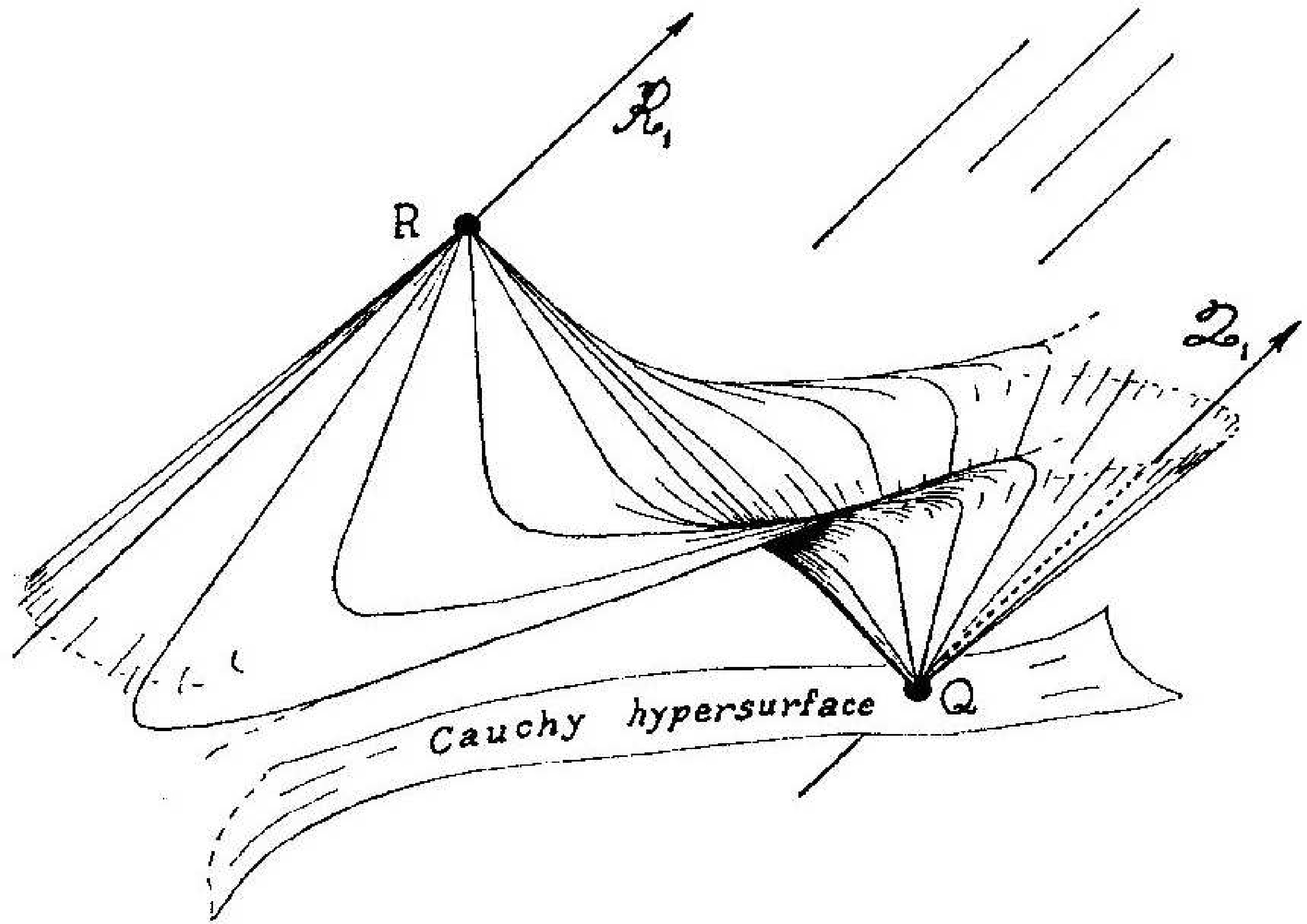
Metric in Brinkmann coordinates $u, v, x^1, \ldots, x^{n-2}$:

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -2\, du\, dv - h_{AB}(u)\, x^A\, x^B\, du^2 + \delta_{AB}\, dx^A\, dx^B$$

$$\delta^{AB}\, h_{AB}(u) \geq 0$$

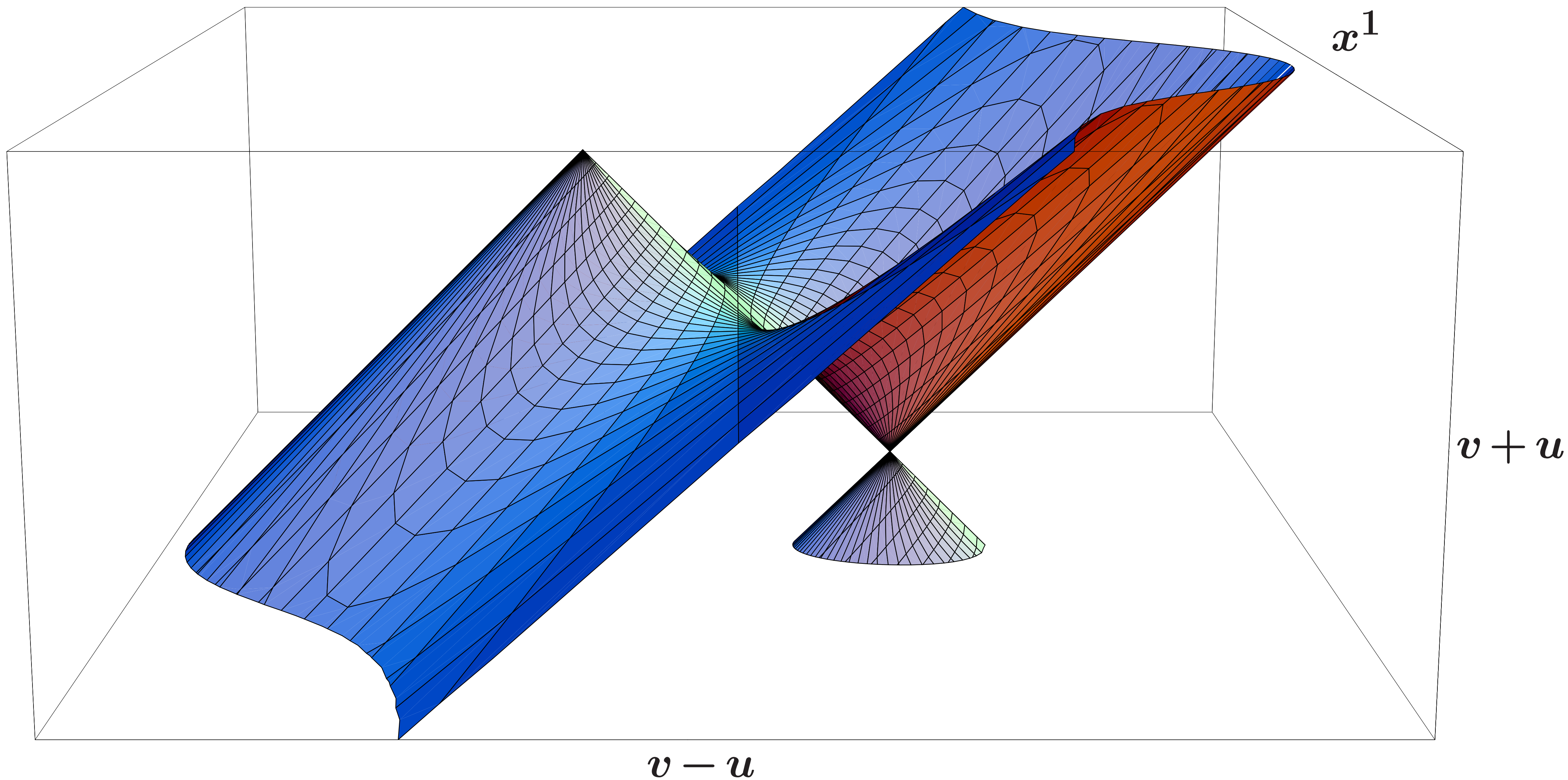
Pure gravitational wave: $\delta^{AB}\, h_{AB}(u) = 0$

Light Cone



Penrose, R.: "A remarkable property of plane waves in general relativity"
Rev. Modern Phys. 37, 215–220 (1965)

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \;=\; -\,2\, du\, dv \;-\; h_{AB}(u)\, x^{A}\, x^{B}\, du^{2} \;+\; \delta_{AB}\, dx^{A}\, dx^{B}$$



$X(u) =$ lichtartige Geodäte mit $v = 0$, $x^A = 0$

Brinkmann coordinates are Fermi coordinates

Generalised Jacobi equation for transverse coordinates

$$\frac{d^2\xi^A}{du^2} = -\,\delta^{AB}\,h_{BC}\,\xi^C$$

coincides with Jacobi equation!

Penrose Limit:

Every spacetime coincides in a neighbourhood of a lightlike geodesic with a "plane wave spacetime"

Penrose, R.: "Any space-time has a plane wave as limit" In: M. Cahen, M. Flato (eds.) Differential geometry and relativity, pp. 271–275. Reidel, Dordrecht (1976)

Difference between Jacobi equation and generalised Jacobi equation vanishes in the Penrose limit.

3. Generalised Jacobi equation of Bażański

Bażański, S.: "Kinematics of relative motion of test particles in general relativity" Ann. Inst. H. Poincaré (A) 27, 115–144 (1977)

Expansion of geodesic equation

$$\frac{d^2\xi^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma}(X+\xi)\left(\frac{dX^\nu}{ds} + \frac{d\xi^\nu}{ds}\right)\left(\frac{dX^\sigma}{ds} + \frac{d\xi^\sigma}{ds}\right) - \Gamma^\mu_{\nu\sigma}(X)\frac{dX^\nu}{ds}\frac{dX^\sigma}{ds} = 0$$

with respect to ξ and $\frac{d\xi}{ds}$ up to order N