On Gödel-type Metrics

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joined work with

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Along the following line:

1. Present (a part of) a toolkit for the construction of particular Lorentzian manifolds,
2. apply this toolkit in order to construct Gödel-type spacetimes as an interesting laboratory (our methods/ideas are *not* restricted to these spacetimes),
3. analyze the behavior of geodesics and light cones for these spacetimes as an example of the visualization techniques.

Setting

We consider $(N+1)$-dimensional time-oriented smooth Lorentzian manifolds (M, g) of signature $(+, -, \dots, -)$.

Additionally we fix a timelike unit vector field X^i (observer field).

Setting

For our analysis the usual decomposition of the covariant derivative of X_i into irreducible parts will be used:

$$\boxed{\nabla_k X_i = \omega_{ik} + \sigma_{ik} + \frac{1}{N}\Theta P_{ik} - \dot{X}_i X_k}$$

with the antisymmetric part ω_{ik} (rotation), the symmetric traceless part σ_{ik} (shear) and the trace Θ (expansion) itself.

Setting

Using the observer field we are able to give particular PDEs relating the metric tensor with the parts of the decomposition before, for example we have

$$\boxed{\omega_{ik} = \frac{1}{2}\left(g_{0k}\partial_0 g_{0i} - g_{0i}\partial_0 g_{0k} - \partial_k g_{0i} + \partial_i g_{0k}\right).}$$

Shear-free spacetimes

	Spatial metric $g_{\alpha\beta}(t) = g_{\alpha\beta}(t, x^\gamma)$
General	$S^2(t) g_{\alpha\beta}(0) + g_{0\alpha}(t) g_{0\beta}(t) - S^2(t) g_{0\alpha}(0) g_{0\beta}(0)$
$\dot{X}_i = 0$	$S^2(t) g_{\alpha\beta}(0) + \left(1 - S^2(t)\right) g_{0\alpha}(0) g_{0\beta}(0)$
$\Theta = 0$	$g_{\alpha\beta}(0) + g_{0\alpha}(t) g_{0\alpha}(t) - g_{0\alpha}(0) g_{0\alpha}(0)$
$\Theta = 0, \dot{X}_i = 0$	$g_{\alpha\beta}(0)$

Starting with the Gödel metric

$$(g_{ik}(0, x^\gamma)) = \begin{pmatrix} 1 & 0 & e^{\sqrt{2}\omega_0 x^1} & 0 \\ 0 & -1 & 0 & 0 \\ e^{\sqrt{2}\omega_0 x^1} & 0 & \frac{1}{2}e^{2\sqrt{2}\omega_0 x^1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Gödel (1949)

Starting with the Gödel metric

Some properties:

- 5-dim group of isometries which is transitive,
- no imbedded hypersurfaces without boundary which are spacelike everywhere,
- simply connected,
- geodesically complete,
- $\omega \neq 0$, $\sigma = \Theta = 0$,
- CTCs, ...

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Two friends

Generalized Gödel metric

$$(g_{ik}) = \begin{pmatrix} 1 & 0 & e^{\sqrt{2}\omega_0 x^1} & 0 \\ 0 & -S^2(t) & 0 & 0 \\ e^{\sqrt{2}\omega_0 x^1} & 0 & \frac{1}{2} e^{2\sqrt{2}\omega_0 x^1}\left(2 - S^2(t)\right) & 0 \\ 0 & 0 & 0 & -S^2(t) \end{pmatrix}$$

M. Plaue and M. S. (2008)

Here: $\sigma = 0$, $\omega \neq 0 \neq \Theta$, particular behavior of CTCs.

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What to do?

- Find geodesics γ, solving $\ddot{\gamma} = 0$ (with initial position and direction),
- 3-dim representation,
- chose convenient coordinates,
- find a parametric representation for the light cone,
- .... use the package *GeodesicGeometry*, T. Schönfeld (2009).

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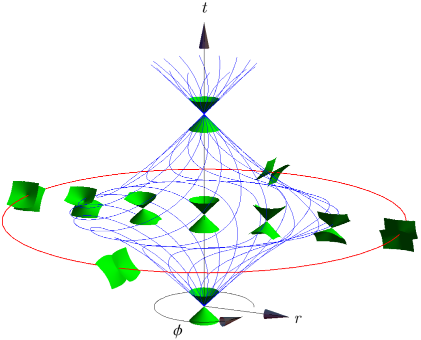
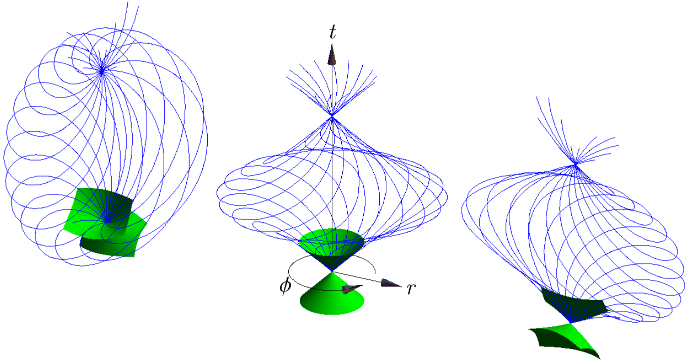
Hawking–Ellis (1973)

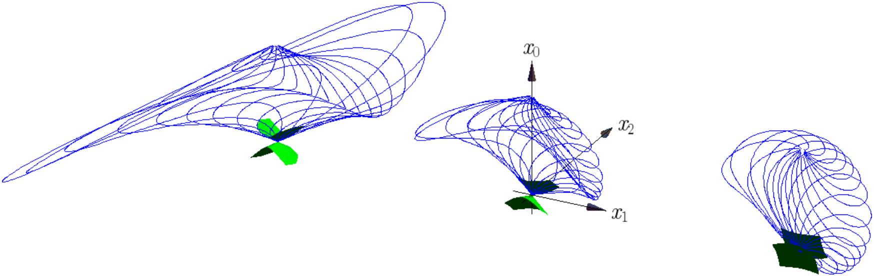


Gödel's spacetime in cylindrical coordinates: A set of geodesics starting from very close to the coordinate origin is shown along with a few light cones which exhibit the tipping effect, showing the existence of CTCs.

Some more details



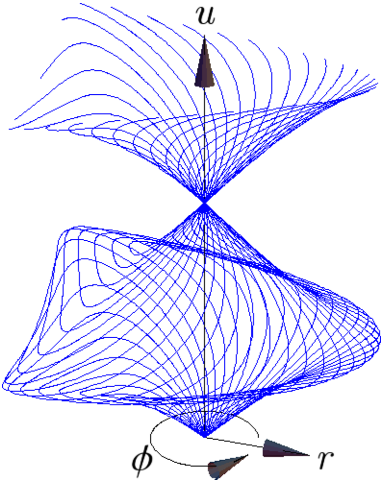
Sets of geodesics originating from points with different radial coordinates in Gödel's spacetime. Just as the light cones, the geodesics tip over; the refocusing is not affected.



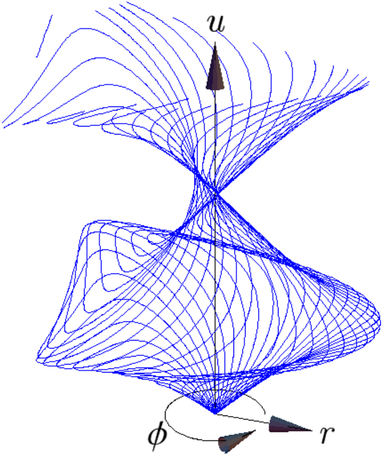
Three sets of geodesics originating from points with different x_1 values in Gödel's spacetime. Just like the light cones, they tip over as one moves to positive x_1 values, and they open up in the x_2 direction as one moves to negative x_1 values.

Expanding Gödel model (by M. Plaue and M. S.)

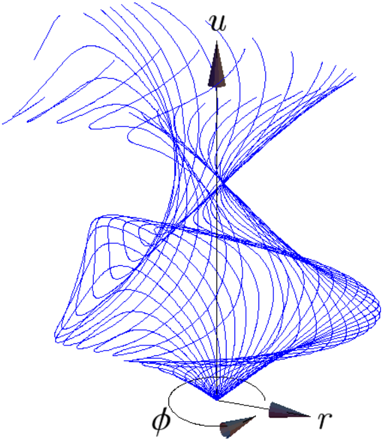
In the following: Six sets of geodesics in the expanding spacetime with identical initial conditions but with different values for the expansion Θ.



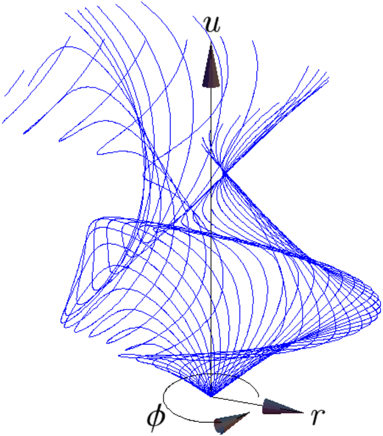
$\Theta = 0.00$



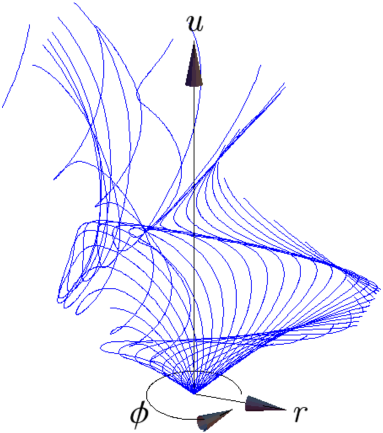
$\Theta = 0.02$



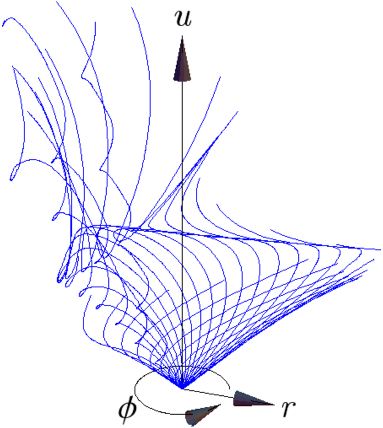
$\Theta = 0.05$



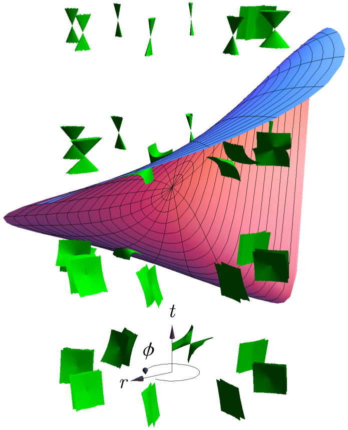
$\Theta = 0.10$



$\Theta = 0.20$



$\Theta = 0.40$



Causality change in the expanding spacetime with $\Theta = 0.4$. The area above the surface is known to be free of closed timelike or null curves.

Another metric: M. Gürses, M. Plaue and M. S. (2009)

$$g_{ik} = X_i X_k - S^2(t) P_{ik}$$