WILLMORE ACTIONS IN SEMI RIEMANNIAN GEOMETRY WITH APPLICATIONS IN PHYSICS

Manuel Barros (Granada)

V INTERNATIONAL MEETING ON

LORENTZIAN GEOMETRY

MARTINA FRANCA, TARANTO (ITALY)

JULY, 8-11, 2009

★ The Lorentzian World ★

♠ From a formal point of view, the **Lorentzian partner** of the two dimensional $\mathbf{O}(3)$ nonlinear sigmamodel, $\mathbf{NLSM}_2$, is a two dimensional nonlinear sigmamodel **where the symmetry is** $\mathbf{O}(2,1)$.

. **Andreas Bredthauer**, Tensionless Strings and Supersymmetric Sigma Models, Aspects of the Target Space Symmetry, **PhD. Uppsala Universitet**, September 2006.

♠ This model has **applications in a wide variety of physical contexts**, including Gauge and String Theories, Quantum Gravity, Quantum Mechanics and General Relativity, see the above mentioned reference and references therein.

♠ The study of **moduli spaces of solutions** (field configurations) of the two dimensional $\mathbf{O}(2,1)$ nonlinear sigma model constitutes **an ambitious program** which was inicied by **Magdalena Caballero** in her PhD.

. **Magdalena Caballero**, Superficies Willmore con Borde y Sigma Modelos no Lineales, **PhD. Universidad de Granada**, May 2008.

. M.Barros, **M.Caballero and M.Ortega**, Communications in Mathematical Physics, 2009 (will appear).

$\bigstar$ Elementary Fields after Setting Up $\bigstar$

♠ Non degenerate Surfaces ♠

As it is usual along this mini-course, $\mathbf{S}$ is a surface with or without boundary and $\phi : \mathbf{S} \rightarrow \mathbb{L}^3$ an immersion in the Lorentz-Minkowski three space with metric $g = \langle , \rangle$.

♣ If $\phi^*(g)$ is non-degenerate on the whole $\mathbf{S}$, then we have a **non-degenerate surface**.

♣ Non-degenerate surfaces can be, at least locally, **oriented** through a **unitary normal vector field** which defines the **Gauss map**, $\mathbf{N}_\phi$ with

$$\boxed{\langle \mathbf{N}_\phi, \mathbf{N}_\phi \rangle = \varepsilon}$$

♣ Consequently, we have just two possibilities:

$\bigstar$ Lorentzian or Time-like Surfaces $\bigstar$ This is the case where $\varepsilon = 1$ so $\mathbf{N}_\phi$ is space-like and $(\mathbf{S}, \phi^*(g))$ is a Lorentzian surface.

$\bigstar$ Riemannian or Space-like Surfaces $\bigstar$ This is the case where $\varepsilon = -1$ so $\mathbf{N}_\phi$ is time-like and $(\mathbf{S}, \phi^*(g))$ is a Riemannian surface.

★ In the former case, the unitary normal vector field, $\mathbf{N}_\phi$, viewed as the **Gauss map takes values in a de-Sitter plane**

$$\boxed{\mathbf{N}_\phi : \mathbf{S} \rightarrow \mathbb{S}^2_1} = \{v \in \mathbb{L}^3 \,:\, \langle v, v\rangle = 1\}$$

★ In the latter case, the unitary normal vector field, $\mathbf{N}_\phi$, viewed as the **Gauss map takes values in a hyperbolic plane**

$$\boxed{\mathbf{N}_\phi : \mathbf{S} \rightarrow \mathbb{H}^2} = \{v \in \mathbf{C}^\uparrow \subset \mathbb{L}^3 \,:\, \langle v, v\rangle = -1\}$$

where $\mathbf{C}^\uparrow$ stands for the **future cone**.

★ The **Lorentz group**, $\mathbf{O}(2,1)$, is the **isometry group of both** $\mathbb{S}^2_1$ and $\mathbb{H}^2$. Consequently, the **Gauss map of any non-degenerate surface is an elementary field in the two dimensional $\mathbf{O}(2,1)$ non linear sigma model**, from now on the Lorentzian model.

★ Therefore, the **geometrical approach identifies the dynamical variables of the model with the space of Gauss maps of non-degenerate surfaces** in $\mathbb{L}^3$.

★ First Order Boundary Conditions ★

★ We consider the boundary conditions (Γ, N_o), where

- **Prescribing Boundary:** $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ is a set of **non-null** regular curves in $\mathbb{L}^3$ with $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$.

- **Prescribing the Gauss Map along the Boundary:** N_o is a unit normal vector field along Γ, orthogonal to Γ and with constant causal character on the whole Γ

$$\boxed{\langle N_o, \Gamma' \rangle = 0} \quad \boxed{\langle N_o, N_o \rangle = \varepsilon}$$

► It is important to notice that in the case where N_o is **space-like**, $\varepsilon = 1$, then the **boundary**, Γ, **could consist of both space-like and time-like curves at the same time**.

► Furthermore, the prescribed boundary conditions determine a **third unit vector field along the boundary** by

$$\boxed{\Gamma' \wedge \nu = N_o}$$

The Boundary Value Problem

★ Let $\mathbf{S}$ be a surface with boundary $\partial\,\mathbf{S} = c_1 \cup c_2 \cdots \cup c_n$. Denote by $\boxed{\mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3)}$ the space of, non-degenerate, immersions, $\phi : \mathbf{S} \to \mathbb{L}^3$, that satisfy the following conditions:

- $\langle \mathbf{N}_\phi, \mathbf{N}_\phi \rangle = \varepsilon$

- $\phi(\partial\,\mathbf{S}) = \Gamma, \quad$ or $\phi(c_j) = \gamma_j$, $1 \leq j \leq n$

- $\mathbf{N}_\phi/_\Gamma = N_o$

★ Roughly speaking, if we **identify** each immersion $\phi \in \mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3)$ with its **graph**, $\phi(\mathbf{S})$, viewed as a surface with boundary in $\mathbb{L}^3$, then $\mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3)$ can be regarded as **the space of immersed surfaces in $\mathbb{L}^3$ having the same causal character, the same boundary and the same Gauss map along the common boundary**.

The Problem

★ Study the dynamics of the model

$$\boxed{\mathcal{E} : \mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3) \to \mathbb{R}, \qquad \mathcal{E}(\phi) = \int_{\mathbf{S}} \|d\mathbf{N}_\phi\|^2 \, dA_\phi}$$

★ Roughly speaking, with the above identification, the problem we are proposing is the study of the Lagrangian, $\mathcal{E}$, in the class of surfaces **with the same causal character, the same boundary and tangent**, the same tangent plane, **along that common boundary**.

★ Now, the **solutions** or the **field configurations** of the **Lorentzian model are the critical points** of the above stated problem.

★ However, the **concept of critical point** needs some **technical**, and on the other hand usual and standard, **considerations**.

★ **Critical point of the problem means critical point of the restricted problem on reasonable compact pieces**.

Reasonable Compact Pieces or Non-null Polygons

★ A connected, simply connected with non-empty interior, **compact domain**

$$\mathbf{K} \subset \mathbf{S}$$

is said to be a **non-null polygon** if it has a piecewise smooth boundary, $\partial\mathbf{K}$, which is **made up of a finite number of non-null curves**.

★ Now, an immersion $\phi \in \mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3)$ **defines first order boundary conditions on the boundary of each non-null polygon**, namely $(\phi(\partial\mathbf{K}), \mathbf{N}_\phi/_{\phi(\partial\mathbf{K})})$. Therefore, we can consider the space, $\mathbf{I}^{\varepsilon}_{\phi(\partial\mathbf{K})}(\mathbf{K}, \mathbb{L}^3)$, of non-degenerate immersions, $\psi : \mathbf{K} \to \mathbb{L}^3$, which satisfy these first order boundary conditions.

★ Furthermore, we have the corresponding Lagrangian,

$$\mathcal{E}^{\mathbf{K}} : \mathbf{I}^{\varepsilon}_{\phi(\partial\mathbf{K})}(\mathbf{K}, \mathbb{L}^3) \to \mathbb{R}$$

$$\boxed{\mathcal{E}^{\mathbf{K}}(\psi) = \int_{\mathbf{K}} \|d\mathbf{N}_\psi\|^2 \, dA_\psi}$$

Consequently

$$\boxed{\phi \in \mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3) \text{ critical point of } \mathcal{E}} \qquad \Leftrightarrow$$

$$\boxed{\phi/_{\mathbf{K}} \in \mathbf{I}^{\varepsilon}_{\phi(\partial \mathbf{K})}(\mathbf{K}, \mathbb{L}^3) \text{ critical point of } \mathcal{E}^{\mathbf{K}}}$$

for any non-null polygon $\mathbf{K} \subset \mathbf{S}$.

Willmore, Once More

★ The **Willmore functional** can also be considered for non-degenerate surfaces in $\mathbb{L}^3$. It is also **extended to non-null boundary surfaces**

$$\boxed{\mathcal{W} : \mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3) \to \mathbb{R}, \qquad \mathcal{W}(\phi) = \int_{\mathbf{S}} \mathbf{H}^2_{\phi}\, dA_{\phi} + \int_{\partial \mathbf{S}} \kappa_{\phi}\, ds}$$

with the same ingredients that in the Euclidean setting. This energy is still invariant under conformal transformations in $\mathbb{L}^3$, **extrinsic conformal invariant**. Critical points for this action are also defined, as above for Lorentzian model, and they are still named **Willmore surfaces**.

$\boxed{\textbf{Willmore versus The Lorentzian Model}}$

$$\boxed{\left(\mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3); \mathcal{E}\right)}$$

$$\Leftrightarrow \Leftrightarrow \Leftrightarrow$$

$$\boxed{\left(\mathbf{I}^{\varepsilon}_{\Gamma}(\mathbf{S}, \mathbb{L}^3); \mathcal{W}\right)}$$

★ This equivalence has important consequences:

- The **field configurations** of the two dimensional $\mathbf{O}(2,1)$ nonlinear sigma model **are** nothing but **the Willmore surfaces**, both with the same prescribed boundary conditions.
- The two dimensional $\mathbf{O}(2,1)$ nonlinear sigma model is **invariant under conformal transformations in** $\mathbb{L}^3$.
- Since the Willmore functional is essentially the Polyakov action, the model can be regarded as a bosonic string theory in $\mathbb{L}^3$ that is governed by the **Willmore-Polyakov action**. In this sense, the **solutions of the model provide the string worldsheet configurations**.

Proving that Equivalence

★ **First Step: Use the Gauss Equation**

$$\boxed{\|d\mathbf{N}_\phi^2\| = 4\,\mathbf{H}_\phi^2 - 2\,\varepsilon\,\mathbf{G}_\phi}$$

★ **Second Step: Gauss-Bonnet to Control Total Gauss Curvature**

- If $\varepsilon = -1$, the surface is Riemannian and we use the classical Gauss-Bonnet formula.
- If the boundary is made up of time-like pieces, then the Gauss-Bonnet formula was proved by **G.S.Birman and K.Nomizu**, Michigan Math.J. 31 (1984), 77.
- For non-null arbitrary polygons, even if the pieces of the boundary are of different causal character, the formula was proved by **Magdalena Caballero** in her PhD. To do it she needed to solve a couple of problems.

  1. First, to define the **notion of hyperbolic angle** between two arbitrary unit vectors in a Lorentzian plane.
  2. Second, to obtain a **characterization**, a la Euler, of the **curvature of a curve in a Lorentzian surface as the variation of the hyperbolic angle**.

♦ For any simply connected non-null polygon in a Lorentzian surface, $\mathbf{K} \subset \mathbf{S}$, one has

$$\boxed{-\int_{\mathbf{K}} \mathbf{G}\, dA + \int_{\partial \mathbf{K}} \kappa\, ds + \sum_{j=1}^{r} \theta_j = 0}$$

where θ_j are the hyperbolic angles at vertices.

★ Third Step: Using Boundary Conditions

One combines, on one hand the **nature of the hyperbolic angle** and on the other hand the **interpretation of the boundary curvature as a variation of a hyperbolic angle function**, with the stated **boundary conditions** to see that the **total curvature of the boundary** (including hyperbolic angles) actually **does not depend on the immersion**.

This Completes the Equivalence

Rotational Field Configurations

From now on, we will discuss the following natural problem

★★★★★ **Determine the moduli space of field configurations of the two dimensional $\mathbf{O}(2,1)$ sigma model which admit a rotational symmetry. Equivalently, classify, up to congruences, Willmore rotational surfaces in $\mathbb{L}^3$** ★★★★★

♠ This **problem** is much **more difficult and subtle that its Riemannian partner**, which we have seen.

♠ The first **main difficulty is strongly related with the couple symmetry-causality**. In contrast with the Riemannian setting, now, we have **rotational axis which can be space-like, time-like and light-like**.

♠ **The complete classification** of non-degenerate **surfaces** in $\mathbb{L}^3$ which admit a **rotational symmetry**, has been a classical problem with ample literature and for a long time. This class of surfaces admits an **obvious sub-class**, whose members we call fundamental rotational surfaces, which has been wrongly used as the whole class. This is due to the case of **space-like axis** where besides the fundamental rotational surfaces, appear a wide class of **non-fundamental rotational surfaces** whose study is difficult and so avoided in the literature. Any way, **the complete classification was obtained by Magdalena Caballero in her PhD**.

Rotational Surfaces in $\mathbb{L}^3$

★ Denote by $\mathbf{Iso}(\mathbb{L}^3)$ the group of Lorentz transformations which correspond with $\mathbf{O}(2,1)$. Those Lorentz transformations, f, with

$$\boxed{\det(f) = 1} \quad \text{and} \quad \boxed{f(\mathbf{C}^{\uparrow}) = \mathbf{C}^{\uparrow}}$$

constitute a subgroup denoted by $\mathbf{Iso}^{+\uparrow}(\mathbb{L}^3)$ and its partner in $\mathbf{O}(2,1)$ is named by $\mathbf{SO}^{\uparrow}(2,1)$.

★ For $\vec{x} \in \mathbb{L}^3$, consider the subgroup

$$\boxed{\mathbf{A} = \{f \in \mathbf{Iso}^{+\uparrow}(\mathbb{L}^3) \quad \text{with} \quad f(\vec{x}) = \vec{x}\}}$$

this is called the **group of rotations** with axis $<\vec{x}>=$ Span$\{\vec{x}\}$. Certainly, $\mathbf{A}$ acts naturally on the whole $\mathbb{L}^3$ producing orbits. However, these **orbits are quite different according to the causal character of the axis**.

(1) Time-like Axis

★ If $\vec{x}$ is time-like, we choose an orthonormal basis $\mathcal{B} = \{\vec{x}, \vec{y}, \vec{z}\}$ in $\mathbb{L}^3$ so $\{\vec{y}, \vec{z}\}$ determine an Euclidean plane. In this case, the group $\mathbf{A}$ is identified with the following subgroup of $\mathbf{SO}^{\uparrow}(2,1)$

$$\mathbf{A}_1 = \{1\} \times \mathbf{SO}(2) = \left\{ \mu_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \; : \; t \in \mathbb{R} \right\}$$

The orbits under the $\mathbf{A}_1$-action are circles in Euclidean planes orthogonal to the axis and center moving on the axis. Therefore, sometimes we call transformations in $\mathbf{A}_1$ elliptic motions or pure rotations.

★ In this case, the **description of rotational surfaces is similar to that of revolution surfaces in Euclidean space**. Choose the Lorentzian half plane

$$\mathbf{H} = \{\vec{v} \in \mathbb{L}^3 \; : \; \langle \vec{v}, \vec{z} \rangle = 0 \;\text{ and }\; \langle \vec{v}, \vec{y} \rangle > 0\}$$

For any unitary curve, $\alpha(s)$ in $\mathbf{H}$, we can construct a $\mathbf{A}_1$-rotational surface by

$$\boxed{\mathbf{S}_\alpha : \qquad \phi_\alpha(s,t) = \mu_t(\alpha(s))}$$

★ The **converse also works**. Each $\mathbf{A}_1$-rotational surface is generated from a profile non-null curve in the Lorentzian plane $\mathbf{H}$. Therefore, the space of **$\mathbf{A}_1$-rotational surfaces can be identified**, up to congruences, with that of **non-null curves in the Lorentzian plane $\mathbf{H}$**.

$\boxed{\textbf{(2) Light-like Axis}}$

★ If $\vec{x}$ is null, then we can consider a basis, $\mathcal{B} = \{\vec{x}, \vec{y}, \vec{z}\}$ of $\mathbb{L}^3$ such that:

- (1) $\vec{y}$ is a **light-like vector** with $\langle \vec{x}, \vec{y} \rangle = -1$
- (2) $\vec{z}$ is a **unit space-like vector orthogonal to the Lorentzian plane** $\mathrm{Span}\{\vec{x}, \vec{y}\}$

Using this basis, it can be checked that the group $\mathbf{A}$ is identified with the following subgroup of $\mathbf{SO}^{\uparrow}(2,1)$

$$\mathbf{A}_2 = \left\{ \varsigma_t = \begin{pmatrix} 1 & \frac{1}{2}t^2 & t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \; : \; t \in \mathbb{R} \right\}$$

In this setting the orbit through the point $p = (a_1, a_2, a_3) = a_1\vec{x} + a_2\vec{y} + a_3\vec{z}$ is

- The **own point** $[\,p\,]_2 = \{p\}$ if $a_2 = a_3 = 0$
- The **straight line** $[\,p\,]_2 = \{t\vec{x} + a_3\vec{z} \; : \; t \in \mathbb{R}\}$ if $a_2 = 0$
- The **parabola**, in the plane $\boxed{y = a_2}$, given by $[\,p\,]_2 = \{(x, a_2, z) \; : \; x = \frac{1}{2a_2}(z - a_3)^2 + \frac{a_3}{a_2}(z - a_3) + a_1\}$ if $a_2 \neq 0$

Thus, $\mathbf{A}_2$ will be called the group of **parabolic rotations or parabolic motions**.

★ In this setting, the $\mathbf{A}_2$-rotational surfaces or **rotational surfaces with light-like axis are localized in the following fundamental regions** or half-spaces

$$\boxed{\mathbf{X}^+ \ : \ y > 0} \qquad \boxed{\mathbf{X}^- \ : \ y < 0}$$

★ To show it, cut these **half-spaces** with the plane $\boxed{z = 0}$ to obtain the following two Lorentzian **half-planes**

$$\boxed{\mathbf{H}^+ \ : \ y > 0, \quad z = 0} \qquad \boxed{\mathbf{H}^- \ : \ y < 0, \quad z = 0}$$

★ The **motion of these half-planes governed by the group $\mathbf{A}_2$ provide the original half-spaces**. More precisely, if we consider the space-like parabolas

$$\boxed{\mathcal{P}^+ \ : \ -2x + z^2 = 0, \quad y = 1} \qquad \boxed{\mathcal{P}^- \ : \ 2x + z^2 = 0, \quad y = -1}$$

then with the induced metric, we have the following **warped product** decompositions

$$\boxed{\mathbf{X}^+ = \mathbf{H}^+ \times_{f_+} \mathcal{P}^+} \quad f_+ : \mathbf{H}^+ \to \mathbb{R}, \quad \boxed{f_+(x, y, 0) = y}$$

$$\boxed{\mathbf{X}^- = \mathbf{H}^- \times_{f_-} \mathcal{P}^-} \quad f_- : \mathbf{H}^- \to \mathbb{R}, \quad \boxed{f_-(x, y, 0) = -y}$$

★ Now, choose **a non-null curve**, α in either $\mathbf{H}^+$ or $\mathbf{H}^-$ and **move it according to the group $\mathbf{A}_2$** to get the surface

$$\boxed{\Sigma_\alpha : \quad \Omega(s,t) = \varsigma_t(\alpha(s))}$$

it is a $\mathbf{A}_2$**-rotational surface which is Riemannian or Lorentzian** according to α is space-like or time-like, respectively. Obviously, the surface lies in a fundamental region just in that where the original curve yields.

★ The **converse** of this fact **also works**. Each $\mathbf{A}_2$-rotational surface contained in a fundamental half-space **is generated from a curve moving inside this fundamental region according to the group $\mathbf{A}_2$**.

★ To prove the statement, we only need to check that a $\mathbf{A}_2$-**rotational surface can not intersect the plane**

$$\boxed{\Pi : \quad y = 0}$$

Since the orbits contained in this plane are light-like, the statement is clear for Riemannian surfaces. The case of Lorentzian surfaces is treated through a technical argument which can be seen in the PhD. of Magdalena Caballero.

(3) Space-like Axis

★ If $\vec{x}$ is space-like, then we can consider an orthonormal basis, $\mathcal{B} = \{\vec{x}, \vec{y}, \vec{z}\}$ of $\mathbb{L}^3$ so $\{\vec{y}, \vec{z}\}$ determine a Lorentzian plane. In this setting, the group $\mathbf{A}$ is identified with the following subgroup of $\mathbf{SO}^{\uparrow}(2,1)$,

$$\mathbf{A}_3 = \{1\} \times \mathbf{SO}^{\uparrow}(1,1) = \left\{ \xi_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} \; : \; t \in \mathbb{R} \right\}$$

★ Given a point $p = a_1\vec{x} + a_2\vec{y} + a_3\vec{z} \in \mathbb{L}^3$, denote by P the Lorentzian plane in $\mathbb{L}^3$ passing through p and orthogonal to $\vec{x}$. It is clear that the orbit of p, $[\,p\,]_3$, is contained in P

- The **own point**: if $a_2 = a_3 = 0$, then $[\,p\,]_3 = \{p\}$
- A **half-straight line**: if $a_2^2 = a_3^2$, then $[\,p\,]_3$ is the open half straight line starting at $a_1\vec{x}$ and passing through p, i.e., $[\,p\,]_3 = \{(1-\lambda)\, a_1\vec{x} + \lambda\, p \; : \; 0 < \lambda\}$
- A **branch of hyperbola**: if $a_2^2 \neq a_3^2$, then $[\,p\,]_3$ is the branch of hyperbola in P centered at $a_1\vec{x}$ and passing through p, i.e., $[\,p\,]_3$ is the connected component of $\{a_1\vec{x} + y\vec{y} + z\vec{z} \in \mathbb{L}^3 \; : \; y^2 - z^2 = a_2^2 - a_3^2\}$ that contains p. Transformations in $\mathbf{A}_3$ will be usually called hyperbolic motions or hyperbolic rotations.

★ In this setting, to determine the $\mathbf{A}_3$-rotational surfaces or **rotational surfaces with space-like axis** we proceed as follows. First, we consider the following **two degenerate planes** containing the axis

$$\boxed{\Pi^+ : \quad y = z} \qquad \boxed{\Pi^- : \quad y = -z}$$

These two planes separate $\mathbb{L}^3$ in **four open regions**, which are **invariant under the $\mathbf{A}_3$-action**, that we will call the **fundamental regions**.

$$\boxed{\mathcal{R}^+ = \{x, y, z) \in \mathbb{L}^3 \,:\, z^2 - y^2 > 0 \quad z > 0\}}$$

$$\boxed{\mathcal{R}^- = \{x, y, z) \in \mathbb{L}^3 \,:\, z^2 - y^2 > 0 \quad z < 0\}}$$

$$\boxed{\mathcal{Q}^+ = \{x, y, z) \in \mathbb{L}^3 \,:\, z^2 - y^2 < 0 \quad y > 0\}}$$

$$\boxed{\mathcal{Q}^- = \{x, y, z) \in \mathbb{L}^3 \,:\, z^2 - y^2 < 0 \quad y < 0\}}$$

★ The **fundamental regions can be viewed as generated by suitable half-planes evolving according to the group $\mathbf{A}_3$**. In order to see it, consider the four half-planes

$$\boxed{R^+ = \mathcal{R}^+ \cap \{y = 0\} = \{x, 0, z) \in \mathbb{L}^3 \, : \, z > 0\}}$$

$$\boxed{R^- = \mathcal{R}^- \cap \{y = 0\} = \{x, 0, z) \in \mathbb{L}^3 \, : \, z < 0\}}$$

$$\boxed{Q^+ = \mathcal{Q}^+ \cap \{z = 0\} = \{x, y, 0) \in \mathbb{L}^3 \, : \, y > 0\}}$$

$$\boxed{Q^- = \mathcal{Q}^- \cap \{z = 0\} = \{x, y, 0) \in \mathbb{L}^3 \, : \, y < 0\}}$$

★ We also consider a branch of hyperbola in each fundamental region

$$\boxed{\mathcal{H}^+ = \{(0,y,z) \in \mathbb{L}^3 \,:\, z^2 - y^2 = 1 \quad z > 0\}} \subset \mathcal{R}^+$$

$$\boxed{\mathcal{H}^- = \{(0,y,z) \in \mathbb{L}^3 \,:\, z^2 - y^2 = 1 \quad z < 0\}} \subset \mathcal{R}^-$$

$$\boxed{\mathcal{J}^+ = \{(0,y,z) \in \mathbb{L}^3 \,:\, z^2 - y^2 = -1 \quad y > 0\}} \subset \mathcal{Q}^+$$

$$\boxed{\mathcal{J}^- = \{(0,y,z) \in \mathbb{L}^3 \,:\, z^2 - y^2 = -1 \quad y < 0\}} \subset \mathcal{Q}^-$$

It should be noticed that while $\mathcal{H}^+$ and $\mathcal{H}^-$ are **space-like**, $\mathcal{J}^+$ and $\mathcal{J}^-$ are **time-like**.

★ Finally, we consider the **positive functions**

$$\boxed{f_+ : R^+ \longrightarrow \mathbb{R}, \qquad f_+(x,0,z) = z}$$

$$\boxed{f_- : R^- \longrightarrow \mathbb{R}, \qquad f_-(x,0,z) = -z}$$

$$\boxed{h_+ : Q^+ \longrightarrow \mathbb{R}, \qquad h_+(x,y,0) = y}$$

$$\boxed{h_- : Q^- \longrightarrow \mathbb{R}, \qquad h_-(x,y,0) = -y}$$

★ The **conclusion** is that we have the following **warped product decomposition for the fundamental regions**, where metrics are always the induced from $\mathbb{L}^3$,

$$\boxed{\mathcal{R}^+ = R^+ \times_{f_+} (\mathcal{H}^+, dt^2)}$$

$$\boxed{\mathcal{R}^- = R^- \times_{f_-} (\mathcal{H}^-, dt^2)}$$

$$\boxed{\mathcal{Q}^+ = Q^+ \times_{h_+} (\mathcal{J}^+, -dt^2)}$$

$$\boxed{\mathcal{Q}^- = Q^- \times_{h_-} (\mathcal{J}^-, -dt^2)}$$

★ Now, it is **easy to construct $\mathbf{A}_3$-rotational surfaces living in one of the four fundamental regions**.

★ Thus, the **$\mathbf{A}_3$-rotational surfaces contained in the fundamental region $\mathcal{R}^+$ are generated from non-null curves in the half-plane R^+ evolving according to the $\mathbf{A}_3$-action**

$$\boxed{\alpha \subset R^+} \quad \Rightarrow \quad \boxed{\Sigma_\alpha : \quad \Omega(s,t) = \xi_t(\alpha(s)) \subset \mathcal{R}^+}$$

of course the **other three cases work similarly**.

$\boxed{\textbf{Some Remarks}}$

★ The half planes, with the induced metric, (R^+, g) and (R^-, g) are Lorentzian while (Q^+, g) and (Q^-, g) are Riemannian.

★ When one makes the obvious conformal changes dictated by the above warped product decompositions, it is noticed that

$$\boxed{\left(R^+, \frac{1}{f_+^2}\, g\right)} \quad \boxed{\left(R^-, \frac{1}{f_-^2}\, g\right)} \quad \textbf{de-Sitter planes, } \boxed{\mathbf{G} = 1}$$

$$\boxed{\left(Q^+, \frac{1}{h_+^2}\, g\right)} \quad \boxed{\left(Q^-, \frac{1}{h_-^2}\, g\right)} \quad \textbf{hyperbolic planes, } \boxed{\mathbf{G} = -1}$$

★ It should be also observed that the $\mathbf{A}_3$-rotational surfaces contained in the fundamental regions $\mathcal{R}^+$ and $\mathcal{R}^-$ are **either Riemannian or Lorentzian according the generatrix is a space-like or a time-like curve, respectively**.

★ **In contrast**, the $\mathbf{A}_3$-rotational surfaces contained in the fundamental regions $\mathcal{Q}^+$ and $\mathcal{Q}^-$ **are always Lorentzian**.

★ Each **Riemannian** $\mathbf{A}_3$-rotational surface is **contained in either** $\mathcal{R}^+$ or $\mathcal{R}^-$.

Are There Other Rotational Surfaces?

★ We have **learned a method to construct rotational surfaces**, in $\mathbb{L}^3$, **no matter the causal character of the axis**.

★ In this approach, **rotational surfaces appear generated from the evolution of non-null curves**, in certain half-planes, when **moving according a group of rotations**. Thus, surfaces keep **inside certain fundamental regions**.

★ In this way, **we obtain the whole moduli space of rotational surfaces with causal axis**.

★ If the **axis is space-like, we still obtain the whole moduli space of Riemannian rotational surfaces**.

★ However, the case of **Lorentzian rotational surfaces with space-like axis, needs some analysis**. We already know the existence of wide classes of these surfaces in four fundamental regions. Now, the problem is: **are there Lorentzian rotational surfaces, with space-like axis, leaving a fundamental regions and emerging in another one?**

★ In some sense, **we are asking for the existence of Lorentzian rotational surfaces, with space-like axis, which are obtained by gluing suitable pieces among the above obtained fundamental surfaces**.

The Answer is YES

★ The **class of nontrivial extended, or glued, Lorentzian rotational surfaces is non-empty**. One can find very **popular surfaces integrating this family**.

However, this has been **usually avoided in the literature, perhaps due to their difficulty and complexity, and on the other hand their subtleness and fineness**.

★ To get some idea on the gluing mechanism, let me analyze, from this point of view, a couple of well know surfaces.

(1) A Saddle Surface

★ In $\mathbb{L}^3$, we consider the following **piece of saddle surface**, just the piece where it is Lorentzian

$$\boxed{\mathbf{S} = \left\{ (x, y, z) \in \mathbb{L}^3 \,:\, x = y^2 - z^2 > -\frac{1}{4} \right\}}$$

★ This surface **admits a natural Monge parametrization as a graph**. Indeed, in the plane $x = 0$, define

$$\Omega : \mathbb{R} \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{S} \subset \mathbb{L}^3, \qquad \boxed{\Omega(y, z) = (y^2 - z^2, y, z)}$$

★ It is easy to see that this surface is **rotational with space-like axis**, the $\{x\}$-axis and so invariant under the group $\mathbf{A}_3$.

★ So, we have a **Lorentzian rotational surface with space-like axis**. However, it is **not a fundamental surface**, that means, it is **not contained in a unique fundamental region**.

★ Let me **analyze the pieces of saddle surface in each one fundamental region**

$$\boxed{\mathbf{S} \cap \mathcal{R}^+ = \Sigma_{\alpha^+}, \quad \alpha^+ : (0, 1/2) \to R^+, \quad \alpha^+(s) = (-s^2, 0, s)}$$

$$\boxed{\mathbf{S} \cap \mathcal{R}^- = \Sigma_{\alpha^-}, \quad \alpha^- : (-1/2, 0) \to R^-, \quad \alpha^-(s) = (-s^2, 0, s)}$$

$$\boxed{\mathbf{S} \cap \mathcal{Q}^+ = \Sigma_{\beta^+}, \quad \beta^+ : (0, +\infty) \to Q^+, \quad \beta^+(s) = (s^2, s, 0)}$$

$$\boxed{\mathbf{S} \cap \mathcal{Q}^- = \Sigma_{\beta^-}, \quad \beta^- : (-\infty, 0) \to Q^-, \quad \beta^-(s) = (s^2, s, 0)}$$

★ Certainly, these **four pieces are glued along the common boundaries** to obtain the saddle surface, **S**. Though, the **gluing mechanism is obvious in this case**, we will emphasize it as a motivation.

★ Let observe that actually, we have **two curves**

- A **time-like curve in the plane** $y = 0$

$$\boxed{\alpha : (-\delta, \delta) \to \{y = 0\}, \quad \alpha(s) = (f_\alpha(s), 0, s) = (-s^2, 0, s)}$$

- A **space-like curve in the plane** $z = 0$

$$\boxed{\beta : (-\delta, \delta) \to \{z = 0\}, \qquad \alpha(s) = (f_\beta(s), s, 0) = (s^2, s, 0)}$$

for a **suitable** $\delta \in (0, 1/2)$ to write **both curves as graphs of functions** f_α and f_β.

★ In addition, we have a **gluing smooth function F** : $\{(y, z) \in \mathbb{R}^2 \,:\, |z^2 - y^2| < \delta^2\} \longrightarrow \mathbb{R}$ defined by

$$\mathbf{F}(y, z) = \left\{ \begin{array}{lll} f_\alpha\left(\operatorname{sign}(z)\sqrt{z^2 - y^2}\right) = y^2 - z^2 & \text{if} & z^2 \geq y^2, \\ f_\beta\left(\operatorname{sign}(y)\sqrt{y^2 - z^2}\right) = y^2 - z^2 & \text{if} & y^2 \geq z^2. \end{array} \right\}$$

When we consider the **four pieces glued to shape the saddle surface, then that above considered Monge parametrization is obtained in terms of the gluing function F**. In fact, that is the Monge parametrization associated with the graph of the gluing function.

(2) A One-Sheet Hyperboloid

★ In $\mathbb{L}^3$, we consider the surface

$$\boxed{\mathbf{H} = \left\{(x, y, z) \in \mathbb{L}^3 \,:\, x^2 + y^2 - z^2 = 1\right\}}$$

it is a **Lorentzian rotational surface with space-like axis** the $\{x\}$-axis. Furthermore, it is not contained in any fundamental region. Indeed, the intersection of $\mathbf{H}$ with the fundamental regions **provides just six connected pieces or fundamental rotational surfaces**.

$$\mathbf{H} \cap \mathcal{R}^+ = \left\{ \begin{array}{lll} \Sigma_{\alpha_1^+} & \alpha_1^+(s) = (+\sqrt{1+s^2}, 0, s) & s \in (0, +\infty) \\ \Sigma_{\alpha_2^+} & \alpha_2^+(s) = (-\sqrt{1+s^2}, 0, s) & s \in (0, +\infty) \end{array} \right\}$$

$$\mathbf{H} \cap \mathcal{R}^- = \left\{ \begin{array}{lll} \Sigma_{\alpha_1^-} & \alpha_1^-(s) = (+\sqrt{1+s^2}, 0, s) & s \in (-\infty, 0) \\ \Sigma_{\alpha_2^-} & \alpha_2^-(s) = (-\sqrt{1+s^2}, 0, s) & s \in (-\infty, 0) \end{array} \right\}$$

$$\mathbf{H} \cap \mathcal{Q}^+ = \Sigma_{\beta^+} \quad \beta^+(t) = (t, +\sqrt{1-t^2}, 0) \quad t \in (-1, +1)$$

$$\mathbf{H} \cap \mathcal{Q}^- = \Sigma_{\beta^-} \quad \beta^-(t) = (t, -\sqrt{1-t^2}, 0) \quad t \in (-1, +1)$$

★ Denote by $p = (1, 0, 0)$ and $q = (-1, 0, 0)$ the **two points** in which **H intersects the axis**. Now, the **boundaries of the above six pieces** are just the **eight light-like orbits** with boundary either p or q. Thus, it is **necessary to glue twice** to obtain **H**.

★ First, we **work around p to glue four pieces**, $\Sigma_{\alpha_1^+}$, $\Sigma_{\alpha_1^-}$, Σ_{β^+} and Σ_{β^-}. A similar argument around the point q allows one to glue $\Sigma_{\alpha_2^+}$, $\Sigma_{\alpha_2^-}$, Σ_{β^+} and Σ_{β^-}. Notice that pieces Σ_{β^+} and Σ_{β^-} **must be glued twice**, first around p and then around q.

★ Choose δ satisfying $0 < \delta < 1$ and define:

- a time-like curve $\alpha_p : (-\delta, \delta) \rightarrow \{y = 0\}$, $\alpha_p(s) = (f_{\alpha_p}(s), 0, s) = (+\sqrt{1+s^2}, 0, s)$, and
- a space-like curve $\beta_p : (-\delta, \delta) \rightarrow \{z = 0\}$, $\beta_p(s) = (f_{\beta_p}(s), s, 0) = (+\sqrt{1-s^2}, s, 0)$,

which satisfy $\alpha_p(0) = \beta_p(0) = p$. Then, we define **the gluing smooth function** $\boxed{\mathbf{F_p} : \{(y,z) \in \mathbb{R}^2 \, : \, |z^2 - y^2| < \delta^2\} \longrightarrow \mathbb{R}}$ as

$$\mathbf{F}_p(y,z) =$$

$$\left\{ \begin{array}{llll} f_{\alpha_p}\left(\operatorname{sign}(z)\sqrt{z^2-y^2}\right) = +\sqrt{1-y^2+z^2} & \text{if} & z^2 \geq y^2, \\ f_{\beta_p}\left(\operatorname{sign}(y)\sqrt{y^2-z^2}\right) = +\sqrt{1-y^2+z^2} & \text{if} & y^2 \geq z^2. \end{array} \right\}$$

Now, in terms of this gluing function, we can define a parametrization of the one-sheet hyperboloid around p as follows

$$\Omega_p : \{(y,z) \in \mathbb{R}^2 \, : \, |z^2 - y^2| < \delta^2\} \longrightarrow \mathbf{H} \subset \mathbb{L}^3$$

$$\Omega_p(y,z) = (\mathbf{F}(y,z), y, z) = \left(+\sqrt{1-y^2+z^2}, y, z\right)$$

Finally, using the negative square root we obtain curves and a gluing function to paste the pieces around q.

(2) Rotational Surfaces Dissection

★ Let **S** be a **Lorentzian rotational surface**, with **space-like axis**. As above, we may assume that it is not a fundamental one and then to **submit it to a surgery process** to study the pieces in fundamental regions and the way they are glued.

★ Roughly speaking, and **up to a lot of technical details**, the result of this dissection can be summarized in the following points:

1. **There exists just one**, obviously space-like, **generatrix**, β, in the Euclidean plane $z = 0$ which could be either open (saddle surface) or closed (one sheet hyperboloid). This curve generates $\Sigma_\beta \subset$ providing connected pieces in $\mathcal{Q}^+$ and $\mathcal{Q}^-$.
2. **There is a class of time-like generatrices**, $\{\alpha_i : i \in \mathrm{I}\}$ in the Lorentzian plane $y = 0$. They generate $\{\Sigma_{\alpha_i} : i \in \mathrm{I}\}$ with pieces in $\mathcal{R}^+$ and $\mathcal{R}^-$.
3. **The rules to paste are:**
   - For each point $\beta(s) \in \langle \vec{x} \rangle$ there exists a curve α_i gluing appropriately with β in a neighborhood of $\beta(s)$ through a suitable gluing function.
   - For each time-like generatrix, α_i, there is a point, p, in the closure of β such that both generatrices glue nicely, through a suitable gluing function, in a neighborhood of p.

Gluing mechanism is well characterized and it provides an algorithm to construct infinitely many examples of Lorentzian rotational surfaces with spacelike axis.

Rotational Field Configurations

★ Once we have obtained the complete classification of rotational surfaces in $\mathbb{L}^3$, it seems natural to look for those **providing field configurations of the Lorentzian model**. In other words, **determinate the whole moduli space of Rotational Willmore Surfaces with prescribed first order boundary**.

★ For the sake of simplicity, we will obviate the boundary conditions which will be suitable and admissible.

★ In general and a priori, the main tools we have are **the extrinsic conformal invariance of the model** and **the principle of symmetric criticality**.

★★ These ingredients are more that **enough to solve the problem for rotational surfaces with time-like axis**. The main steps to do it are:

- First, we have the following **decomposition for the Lorentz metric**, g

  $$\mathbb{L}^3 - (\langle \vec{x} \rangle) = \mathbf{H} \times_f \mathbb{S}^1$$

  where $\mathbf{H}$ is the Lorentzian half-plane

  $$\mathbf{H} = \{\vec{v} \in \mathbb{L}^3 \,:\, \langle \vec{v}, \vec{z} \rangle = 0 \text{ and } \langle \vec{v}, \vec{y} \rangle > 0\}$$

  $$\boxed{f : \mathbf{H} \to \mathbb{R}, \quad f(\vec{v}) = \langle \vec{v}, \vec{y} \rangle}$$

- Make the **obvious conformal change** to see the following semi-Riemannian **product picture**

$$\left(\mathbb{L}^3 - (\langle \vec{x} \rangle), \frac{1}{f^2}\, g\right) = \left(\mathbf{H}, \frac{1}{f^2}\, g\right) \times \mathbb{S}^1$$

- Notice that the **first factor is nothing but an anti-de-Sitter plane** with curvature -1 and use that the group $\mathbf{A}_1$ is **compact to apply Palais**.

- Finally, observe that the **Willmore energy of a rotational surface**, with time-like axis, is the **elastic energy for**, clamped, **curves in the above anti-de-Sitter plane**.

- As a consequence, **rotational surfaces with time-like axis** are obtained by **moving**, via the group $\mathbf{A}_1$ of pure rotations, **free elasticae in an anti-de-Sitter plane**.

$$\{\mathbf{S}_\alpha :\equiv \mu_t(\alpha(s)), \quad \alpha \ \mathbf{elastica\, in\, AdS_2}\}$$

★★ **In contrast**, the cases where the axis is either light-like or space-like, are governed by **non-compact groups of rotations**. The above approach to reduce the symmetries could be used, however the corresponding **principle of symmetric criticality needs to be established**.

★ There exists an alternative method to solve the problem that consists in a **direct variational approach**. The crucial step to use this method is the **computation of the field equation** (Euler-Lagrange equations) for the **Willmore energy**.

★ Since one needs to make some conformal changes, the field equation is necessary in **a context more general that Lorentz-Minkowski's one**. The corresponding, long, computations **were made in the PhD of Magdalena Caballero**. There, she obtained the Euler-Lagrange equation providing Willmore surfaces in any semi-Riemannian three space. As a consequence one has the following big result

> **The field configurations in the two-dimensional $\mathbf{O}(2,1)$ Nonlinear Sigma Model which admit a one parameter group of motions are either**
>
> 1. **An open subset (with boundary) of a Lorentzian plane.**
> 2. **An open subset (with boundary) of a pseudosphere.**
> 3. **A surface generated, via rotations, by a clamped elastica according the following table.**

Symmetry Group and Axis	**Orbits**	**Character**	**Generating Curve**
$\mathbf{A}_1$ | Time-like	Circles	Riemannian	Space-like free elastica in anti de Sitter plane
$\mathbf{A}_1$ | Time-like	Circles	Lorentzian	Time-like free elastica in anti de Sitter plane
$\mathbf{A}_3$ | Space-like	Hyperbolas	Riemannian	Space-like free elastica in de Sitter plane
$\mathbf{A}_3$ | Space-like	Hyperbolas	Lorentzian	Time-like free elastica in de Sitter plane
$\mathbf{A}_3$ | Space-like	Hyperbolas	Lorentzian	Free elastica in hyperbolic plane
$\mathbf{A}_2$ | Light-like	Parabolas	Riemannian	Space-like free elastica in anti de Sitter plane
$\mathbf{A}_2$ | Light-like	Parabolas	Lorentzian	Time-like free elastica in anti de Sitter plane

$\boxed{\textbf{The Problem}}$

★ Study the dynamics of the model

$$\boxed{\mathcal{E} : \mathbf{I}_\Gamma(\mathbf{S}, \mathbb{R}^3) \to \mathbb{R}, \qquad \mathcal{E}(\phi) = \int_{\mathbf{S}} \|d\mathbf{N}_\phi\|^2 \, dA_\phi}$$

★ **The $\mathbf{NLSM}_2$ with boundary conditions** (Γ, N_o) **is an invariant under conformal transformations of the target Euclidean metric**. More precisely, **it is equivalent to a Willmore problem stated with the same boundary conditions**.

$\boxed{\textbf{PROBLEM: Solitons Foliated by Circles?}}$

Plateau versus Willmore

Or

Nambu-Goto versus Polyakov

★ The **Plateau problem deals with the critical points of the area action** acting on that class of immersions

$$\boxed{\mathcal{P} : \mathbf{I}^{\Gamma}(\mathbf{S}, \mathbb{R}^3) \to \mathbb{R}, \qquad \mathcal{P}(\phi) = \int_{\mathbf{S}} dA_\phi}$$

★ This is a very classical topic and it has been solved for different boundaries. In our context, to compare with the $\boxed{\mathbf{NLSM}_2}$, suppose that we wish to determine **minimal surfaces which admit a rotational symmetry**. **A unique solution, the catenoid**.

★ We already know that the **Plateau energy**, the area, was, up to a coupling constant, the first historically considered action to construct a bosonic string theory, the **Nambu-Goto bosonic string action**. So, when one considers the bosonic string theory governed by the **Polyakov action**, recall it corresponds with the Willmore action or equivalently the total Casorati curvature action or $\boxed{\mathbf{NLSM}_2}$, then we have just proved the **existence of a wide class of rotational string worldsheets**.

★ Rotational symmetric configurations are special cases of surfaces which are **foliated by circles**. In 1867, Hattendorf and Weber published a part of the **posthumous work of Riemann**. There, Riemann proposed and solved the following problem: **Find those minimal surfaces which are foliated by parallel circles**. Parallel circles means contained in parallel planes.

- B.Riemann, Abh.Königl.Ges.d.Wiss. Göttingen, Mathem.Cl. 13 (1867), 3-52 (*posthumous* paper published by K.Hattendorf and M.H.Weber).

★ Besides the catenoid, Riemann found a **one-parameter class of minimal surfaces which are foliated by parallel circles**,

$$\boxed{\{\mathbf{R}_\lambda \,:\, \lambda \in \mathbb{R}\},}$$

that nowadays are known as the **minimal surfaces of Riemann**.

★ A Result by Enneper.- In 1869, **A.Enneper** proved a result which combined with the above one of Riemann **allows one to obtain the moduli space of minimal surfaces which are foliated by circles**.

- A.Enneper, Z.Math.Phys. 14 (1869), 393-421.

★ **If a minimal surface is foliated by circles, then circles must be parallel, contained in parallel planes**.

What about Solitons foliated by Circles?

★ An Enneper result does not work in this model because $\boxed{\mathbf{NLSM}_2}$ **is defined in the conformal class of the Euclidean metric**, $[g]$. Therefore, we have a new degree of freedom. Roughly speaking, we can **travel inside the conformal class**, through conformal maps **preserving circles**. Certainly, we have a natural candidate: the **round sphere and the stereographic projection**.

The Clifford Parallelism

★ The round unit three sphere, $\mathbb{S}^3 \subset \mathbb{C}^2$ is **endowed with a parallelism structure**. It allows us to **decide when two lines, great circles, are parallel**.

★ Two great circles, $\mathbf{C}$ and $\mathbf{C}'$ are **Clifford parallel**, $\mathbf{C} \parallel \mathbf{C}'$, if the distance $d(\zeta, \mathbf{C}')$ is independent on the point $\zeta \in \mathbf{C}$.

★ Fix a great circle, say $\mathbf{C}$, then for any point $\zeta \in \mathbb{S}^3$ there exist **two great circles**, $\mathbf{C}'$, $\mathbf{C}''$, through that point which are **Clifford parallel with $\mathbf{C}$**.

★ Moreover, $\mathbf{C}' \neq \mathbf{C}''$ if $\zeta \in \mathbb{S}^3 - \{\mathbf{C} \cup \mathbf{C}^\perp\}$, where $\mathbf{C}^\perp$ is the great circle obtained when cutting the three sphere with the plane, $\mathbf{P}^\perp$, which is orthogonal to, $\mathbf{P}$, associated with $\mathbf{C}$. These great circles are called **first and second kind Clifford parallel** with $\mathbf{C}$.

★ Though the Clifford parallelism is not an equivalence relation, it can be decomposed in two equivalence relations. The idea is to obtain **each class of Clifford parallel circles as the space of orbits of a certain subgroup of isometries**.

★ Several **geometric constructions**, **similar** to those obtained for **Euclidean parallelism** in $\mathbb{R}^3$, can be **considered in the three sphere**.

★ For example, we can see those **surfaces in the three sphere** which **play the role of right circular cylinders**. Choose an axis, $\mathbf{C}$, and $\theta \in (0, \pi/2)$ to define

$$\boxed{\mathrm{T}_\theta = \{\zeta \in \mathbb{S}^3 \,:\, d(\zeta, \mathbf{C}) = \theta\}}$$

★ Now, for any $\zeta \in \mathrm{T}_\theta$ there exists a great circle, $\mathbf{C}'$, through that point which is **first kind Clifford parallel** to $\mathbf{C}$. In other words, **these tori can be viewed foliated with generatrices which are first kind Clifford parallel circles to the axis**, $\mathbf{C}$. A similar fact works for second kind Clifford parallel circles.

► We will restrict ourselves to first kind Clifford parallel circles though a similar study works for second kind.

★ This surface is obtained, in a suitable coordinate system, to be the following **rectangular torus**

$$\boxed{\mathrm{T}_\theta = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| = \cos\theta,\ |z_2| = \sin\theta\}}$$

So, **our immediate program is**

$\boxed{\textbf{Clifford Parallel Circles Foliated Solitons?}}$

$\boxed{\textbf{How We Look Clifford Parallelism in Euclidean Space?}}$

$\boxed{\textbf{How We Look these Solitons?}}$

★ To answer these questions, we first take advantage of the **high rigidity of the round three sphere**, $\mathbb{S}^3 \subset \mathbb{R}^4$. Thus, we can **choose a suitable coordinate system to see the first kind Clifford parallel great circles in an appropriate way**.

★ Certainly, we may choose the coordinate system in order to see the $\mathbf{G}^+$-action on $\mathbb{S}^3$ in a **familiar form**

$$\boxed{\mathbf{G}^+ \times \mathbb{S}^3 \to \mathbb{S}^3,} \qquad \boxed{(\varphi_t, \zeta) \mapsto \varphi_t(\zeta) = e^{it}\, \zeta}$$

★ Then, **the first kind Clifford parallel great circles appear as the fibres of the usual Hopf map**.

♣ *Once more the Hopf map* ♣

★ Now, **surfaces** in $\mathbb{S}^3$ which are **foliated by first kind Clifford parallel great circles** are nothing but

♣ Hopf Tubes ♣

Solving and Projecting

★ (1) Solving ★

- **Boundary Conditions.**

♠ The Prescribed Boundary A **pair** of first kind Clifford parallel great circles, or **Hopf fibres**,

$$\boxed{\Gamma = \{\gamma_1, \gamma_2\},}$$

♠ The Prescribed Gauss Map along the Boundary A $\mathbf{G}^+$-invariant unit vector field, N_o, along Γ with

$$\boxed{\langle \Gamma', N_o \rangle = 0,}$$

it is constructed as follows: choose points, $p_i = \gamma_i(0)$, and $x_i \in \mathrm{T}_{p_i}\mathbb{S}^3$ with $\langle \gamma_i'(0), x_i \rangle = 0$, then

$$\boxed{N_o\left(\gamma_i(t)\right) = N_o\left(\varphi_t(p_i)\right) = (d\varphi_t)_{p_i}(x_i).}$$

The boundary conditions determine a third unit vector field, ν, along Γ which is also $\mathbf{G}^+$-invariant

$$\boxed{\Gamma' \wedge \nu = N_o.}$$

- **Willmore Problem.**

♠ $\boxed{\text{The Source Surface}}$ Choose $\mathbf{S} = [a_1, a_2] \times \mathbb{S}^1$, a surface with boundary

$$\boxed{\partial\, \mathbf{S} = (\{a_1\} \times \mathbb{S}^1) \cup (\{a_2\} \times \mathbb{S}^1).}$$

♠ $\boxed{\text{The Elementary Fields}}$ Consider the space, $\mathbf{I}_\Gamma(\mathbf{S}, \mathbb{R}^3)$, of immersions, $\phi : \mathbf{S} \to \mathbb{R}^3$, that satisfy the following boundary conditions

1. $\phi(\{a_i\} \times \mathbb{S}^1) = \gamma_i \quad 1 \leq i \leq 2$

2. $d\phi_q(\mathrm{T}_q\mathbf{S}) \perp N_o(\phi(q)) \quad \forall q \in \mathbf{S}.$

♠ $\boxed{\text{The Willmore Action}}$

$$\boxed{\mathcal{W} : \mathbf{I}_\Gamma(\mathbf{S}, \mathbb{R}^3) \to \mathbb{R}}$$

$$\boxed{\mathcal{W}(\phi) = \int_{\mathbf{S}} (\mathbf{H}_\phi^2 + 1)\, dA_\phi + \int_{\partial\, \mathbf{S}} \kappa\, ds}$$

- **Willmore Hopf Tubes.**

♠ $\boxed{\text{Symmetric Points}}$ It is clear that the Willmore action is preserved by the $\mathbf{G}^+$-action, so we have all the ingredients to apply the principle of symmetric criticality.

$$\boxed{\textbf{Willmore} + \textbf{Symmetric} \;\Leftrightarrow\; \textbf{Symmetric} + \textbf{Willmore}}$$

So, we need, first, to **identify the space of symmetric points**. It is easy to see that this space corresponds with that of Hopf tubes with cross sections being clamped curves according the projection in the two sphere of the boundary conditions. More precisely, put

$$\boxed{\Pi(\gamma_i) = m_i} \qquad \boxed{d\Pi_{\gamma_i}(\nu) = v_i}$$

Now consider, in $\mathbb{S}^2(1/2)$ the space of curves, Λ, which are clamped according the above conditions

$$\boxed{\Lambda = \{\alpha : [s_1, s_2] \longrightarrow \mathbb{S}^2(1/2) \;:\; \alpha(s_i) = m_i,\; \alpha'(s_i) = v_i\}}$$

$$\boxed{\mathbf{I}_\Gamma^{\mathbf{G}^+}(\mathbf{S}, \mathbb{R}^3) = \{\Pi^{-1}(\alpha) \;:\; \alpha \in \Lambda\}}$$

♠ $\boxed{\text{Willmore Energy of Symmetric Points}}$ The the Willmore energy of a Hopf tube can be computed to be

$$\boxed{\mathcal{W}(\Pi^{-1}(\alpha)) = \frac{\pi}{2} \int_\alpha (\kappa_\alpha^2 + 4)\, ds}$$

We have reduced the search for **boundary Willmore Hopf tubes in** $\mathbb{S}^3$ to that **of clamped elastic curves in** $\mathbb{S}^2(1/2)$.

$$\boxed{\Pi^{-1}(\alpha) \in \mathbf{I}_\Gamma^{\mathbf{G}^+}(\mathbf{S}, \mathbb{R}^3) \qquad \textbf{Willmore}} \qquad \Leftrightarrow$$

$$\boxed{\alpha \;\; \text{critical point of} \qquad \mathcal{F} : \Lambda \to \mathbb{R} \;\; \text{defined by}}$$

$$\boxed{\mathcal{F}(\alpha) = \int_\alpha (\kappa_\alpha^2 + 4)\, ds}$$

♠ $\boxed{\text{Algorithm to Construct Willmore Hopf Tubes}}$ The following algorithm summarizes the approach that we have obtained to construct boundary Willmore Hopf tubes

1. For $\mu \geq \sqrt{2}$, choose the curve, $\alpha : \mathbb{I} \subset \mathbb{R} \to \mathbb{S}^2(1/2)$ with curvature function

$$\boxed{\kappa(s) = \sqrt{2(\mu^2 - 2)}\,\mathbf{cn}\left(\mu(s - s_o), \frac{\sqrt{\mu^2 - 2}}{\sqrt{2}\,\mu}\right)}$$

2. Choose $[s_1, s_2] \subset \mathbb{I}$ and put the clamped boundary conditions

$$\boxed{\alpha(s_i) = m_i} \qquad \boxed{\alpha'(s_i) = v_i}$$

3. Lift the curve to obtain its Hopf tube

$$\boxed{\mathbf{T}_{s_i}^{s_2}(\alpha) = \Pi^{-1}(\alpha([s_1, s_2]))}$$

4. Then $\mathbf{T}_{s_i}^{s_2}(\alpha)$ is a Willmore tube with bundary conditions $\Gamma = \{\gamma_1, \gamma_2\}$ and N_o defined by lifting the clamped boundary conditions

$$\boxed{\gamma_i = \Pi^{-1}(m_i)} \quad \boxed{N_o = \Gamma' \wedge \nu} \quad \boxed{\nu = \left(d\Pi_{\gamma_i}\right)^{-1}(v_i)}$$

♣ **All $\mathbf{G}^+$-invariant boundary Willmore surfaces are obtained according that algorithm**.

A similar method works to construct $\mathbf{G}^-$-invariant boundary Willmore surfaces.

Once the problem is solved in the three sphere, we need to project it to Euclidean space and so view the Euclidean picture of Clifford parallelism.

★ (2) Projecting ★

► First of all, let me give some historical information ◄

♠ Circles of Villarceau ♠

♦ **Antoine-Joseph Yvon Villarceau** (1813-1883) was a French astronomer, **mathematician**, and engineer. It seems that he was the first to notice the following fact.

♦ **There are, exactly, four circles passing through every point on an anchor ring, a revolution circular torus**.

♦ Two of them are obtained when cutting the torus by either a horizontal or a vertical plane. They correspond with the **parallel** and the **meridian** through the point and are the circles that most people can easily visualize.

♦ However, the other two, not so apparent, circles are the so **called Villarceau circles**. They appear by cutting the torus diagonally. Animations exhibiting these circles are amply extended all along the web. Thus, **through every point on a torus, there are four circles passing though it!** The two less obvious ones are called Villarceau circles.

♦ A simple way to show the Villarceau circles is using the **fundamental polygon representation** where a torus is represented as a rectangle whose opposite edges are glued together. After this identification the horizontal and the vertical edges provide the parallel and the meridian, respectively. In the same way, the **two diagonals give the Villarceau circles**.

♦ Therefore, on each anchor ring, one can find two families, $\mathcal{F}_1 = \{\Upsilon(t)\}$ and $\mathcal{F}_2 = \{\Xi(t)\}$, of these **exotic circles**. **Two circles of different families intersect in exactly two points while two circles of the same family not only do not intersect, but they are always linked**.

♠ Clifford versus Villarceau ♠

★ Without any loss of generality, choose a suitable stereographic projection. Say with pole $\zeta_o = (0, i) \in \mathbb{S}^3 \subset \mathbb{C}^2$.

$$\mathbf{E}_o : \mathbb{S}^3 - \{\zeta_o\} \longrightarrow \mathbb{R}^3,$$

Take $\mathbf{C}$ a great circle through the pole and pick the z-axis in $\mathbb{R}^3$ as $\mathbf{E}_o(\mathbf{C})$.

★ If we project down the set of great circles through the pole, $\zeta_o \in \mathbb{S}^3$, then we get the flow of straight lines **parallel, in the Euclidean sense**, with the z-axis.

★ The **natural idea of parallelism** in the three sphere, is defined among lines, geodesics or **great circles**, there, is the **Clifford parallelism**.

★ Now, we wish to describe the Clifford parallelism pictured in the Euclidean space. To start with, consider the rectangular torus generated by the great circles Clifford parallel with $\mathbf{C}$ at the same distance, say $\theta \in (0, \pi/2)$.

$$\boxed{\mathrm{T}_\theta = \{\zeta \in \mathbb{S}^3 \,:\, d(\zeta, \mathbf{C}) = \theta\}}$$

★ This surface is viewed as the following **rectangular torus**

$$\boxed{\mathrm{T}_\theta = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| = \cos\theta,\ |z_2| = \sin\theta\}}$$

★ Now, its **stereographic projection**, $\mathbf{T}_\theta = \mathbf{E}_o(\mathrm{T}_\theta)$ is an **anchor ring**, circular revolution torus, around the z-axis. Up to similarities, each anchor ring around the z-axis may be obtained in this way.

★ These surfaces, anchor rings around the z-axis, play the role of right circular cylinder around the z-axis when we **change the Euclidean parallelism picture by the conformal Clifford parallelism picture**.

★ **A pair of great circles passes through each point, ζ, in** T_θ **which are Clifford parallel to C (first and second kind Clifford parallel circles). Now, their stereographic images provide the Villarceau circles through** $\mathbf{E}_o(\zeta)$.

★ Consequently, Villarceau circles define, in $\mathbb{R}^3 - (\{z\} - \text{axis})$, a **pair of circle foliations**. These flows derive from **conformal Killing vector fields** which are, actually, conformal infinitesimal translations, which we call **first and second kind Villarceau flows**.

★ Now, we can add a new point to the above stated algorithm in order to get all the **solitons which are invariant under a Villarceau flow**.

★ **They are, stereographic images in the Euclidean space, of Hopf tubes in the three sphere with cross sections suitable clamped elasticae in the two sphere**.

★ **This family includes, on one hand, one-parameter classes of solitons foliated by circles which are not Euclidean parallel, but they are conformal Clifford or Villarceau parallel. On the other hand, a rational one parameter class of tori which are solitons for the free model**.

- M.Barros, M.Caballero and M.Ortega, JGP (2006)