Deformations of $2k$-Einstein structures

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Then,

- g is critical for $\mathcal{A} \Leftrightarrow \mathcal{T}_g = 0$, where

  $$\mathcal{T}_g = \mathrm{Ric}_g - \frac{1}{2}\mathcal{S}_g g$$

  is the *Einstein tensor* of g.
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Theorem

Let $T = T_{ij}$ be a twice covariant tensor on a Riemannian (or Lorentzian) manifold (X, g) such that:

1. $T_{ij} = T_{ji}$;
2. $\nabla_i T^i_j = 0$;
3. *Locally,* $T = \sum_{|\alpha|=2} A_\alpha(g, \partial g)\partial^\alpha g + B(g, \partial g)$.

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- In general, $\sigma^{(2k)}(\mathfrak{p})$ is the Gauss-Bonnet integrand of $\exp(\mathfrak{p})$ computed at $\exp(0)$.

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is a constant. In particular, $\mathcal{S}_g^{(2k)}$ is a constant as well.

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Definition

The *moduli space of $2k$-Einstein structures on X* is

$$\mathcal{E}^{(2k)}(X) = \frac{\mathrm{E}^{(2k)}(X)}{\mathbb{R}^+ \times \mathrm{D}(X)}.$$

The quotient map is represented by $g \mapsto [g]$.

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A $2k$-Einstein structure is *rigid* if it is isolated in $\mathcal{E}^{2k}(X)$.

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- $\mathcal{D}^p(X)$ differential p-forms on X;
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Theorem

Let (X, g_μ) be a compact non-flat space form not isometric to the round sphere and let f be a function on X that has the same sign as $\mathcal{S}_{g_\mu}^{(2k)} \approx \mu^k$ somewhere. Then f is the $2k$-Gauss-Bonnet curvature of some metric on X.