### Surfaces in exotic AdS spaces

Jorge H. S. Lira

Universidade Federal do Ceará Fortaleza - Brazil

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$$E_1 = \frac{1}{\tau} Z_{\varsigma_1}, \quad E_2 = Z_{\varsigma_2}, \quad E_3 = Z_{\varsigma_3}$$
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is  $g_{\tau}$ -orthonormal.

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$$R = 8 - 2\tau^2. \tag{7}$$



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This means that  $g_{\tau}$  is a stationary non-static metric in M. We describe that distribution in terms of the Hopf fibration  $\pi: SU_{1,1} \to \mathbb{H}^2$ .

When  $\tau=1$ , the Lie group  $SU_{1,1}$  acts isometrically in  $\mathfrak{su}_{1,1}=\mathbb{L}^3$  by its adjoint action

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Then

$$Z = \begin{pmatrix} z & w \\ \bar{w} & \bar{z} \end{pmatrix} \mapsto \sigma(Z) = \begin{pmatrix} z & -w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

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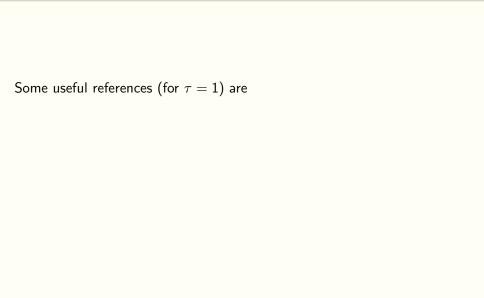
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$$\langle \nabla_{X_z} X_z, N \rangle = \frac{q}{2}, \quad \langle \nabla_{X_z} X_{\bar{z}}, N \rangle = -\frac{1}{2} H e^{2\omega}, \quad \langle \nabla_{X_{\bar{z}}} X_{\bar{z}}, N \rangle = \frac{\bar{q}}{2}, \quad (15)$$

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We define complex-valued functions  $Z^a(z,\bar{z}),\,a=1,2,3$ , so that

$$X_z = Z^1 E_1|_X + Z^2 E_2|_X + Z^3 E_3|_X$$

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(19)

(20)

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$$|3|^2 = \frac{1}{2}e^{\omega}$$
.

(21)

(22)

$$-|Z^1|^2+|Z^2|^2+|Z^3|^2=\frac{1}{2}e^{\omega}.$$

$$|^2=rac{1}{2}e^\omega.$$

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system of nine complex-scalar equations.

 $-(Z^1)^2 + (Z^2)^2 + (Z^3)^2 = 0.$ 

 $-|Z^{1}|^{2}+|Z^{2}|^{2}+|Z^{3}|^{2}=\frac{1}{2}e^{\omega}.$ 

In terms of  $Z^a$ , the Gauss-Weingarten equations become a first-order

 $X^{-1}X_7 = Z^1e_1 + Z^2e_2 + Z^3e_3 \in \mathfrak{su}_{1,1}$ 

(20)

(19)

(21)

(22)

$$\Upsilon = e_a \otimes (Z^a \mathrm{d} z + \bar{Z}^a \mathrm{d} \bar{z}),$$

(23)

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$$I = e^{2\omega} |\mathrm{d} z|^2. \tag{25}$$

$$X^{-1}\mathrm{d}X=\Upsilon$$

and it is locally given as an integral leaf of the distribution

$$(p,Z)\in\Sigma imes SU_{1,1}\mapsto \ker\Xi_{(p,Z)},$$

(27)

(26)

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(26)

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$$(p,Z) \in \Sigma \times SU_{1,1} \mapsto \ker \Xi_{(p,Z)},$$

where  $\Xi$  is a  $\mathfrak{su}_{1,1}$ -valued 1-form defined in  $\Sigma \times SU_{1,1}$  by

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 $\Xi = \Omega - \Upsilon$ .

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(32)

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where

$$U = -\frac{1}{2}(H - \frac{i}{\tau})(|\psi_2|^2 - |\psi_1|^2) - \frac{i}{2}(\tau - \frac{1}{\tau})(|\psi_2|^2 + |\psi_1|^2)$$

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(34)

(35)

(36)

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II 
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 satisfies the first order system

$$\partial_{z}\psi_{1} = \omega_{z}\psi_{1} + i\left(\tau - \frac{1}{\tau}\right)\psi_{1}^{2}\bar{\psi}_{2} - e^{-\omega}\frac{q}{2}\psi_{2},$$

$$\partial_{z}\bar{\psi}_{2} = \omega_{z}\bar{\psi}_{2} - i\left(\tau - \frac{1}{\tau}\right)\psi_{1}\bar{\psi}_{2}^{2} - e^{-\omega}\frac{q}{2}\bar{\psi}_{1},$$

$$\partial_z \bar{\psi}_2 = \omega_z \bar{\psi}_2 - i \left(\tau - \frac{1}{\tau}\right) \psi_1 \bar{\psi}_2^2 - e^{-\omega} \frac{q}{2} \bar{\psi}_1,$$

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then there exists an immersion  $X:\Sigma \to (M,g_\tau)$  with induced metric I, mean curvature H and Hopf differential  $q\mathrm{d}z^2$ .



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that is,

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$$K_{\text{int}} - K_{\text{ext}} = 4\tau \left(\tau - \frac{1}{\tau}\right)\alpha^2 + \tau^2 \tag{40}$$

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$$(\nabla_V^{\Sigma} K) W - (\nabla_W^{\Sigma} K) V = 4\alpha \tau (\tau - \frac{1}{\tau}) I(JV, W) J\beta$$
 (41)

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$$\tau'$$

$$lpha \mathcal{K} \mathcal{V} + 
abla_{\mathcal{V}} eta = -lpha au \mathcal{J} \mathcal{V}$$

(40)

(41)

(42)

$$\mathcal{K}_{\mathrm{int}} - \mathcal{K}_{\mathrm{ext}} = 4 au( au - rac{1}{ au})lpha^2 + au^2$$

$$(
abla_V^{\Sigma}K)W - (
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$$+\nabla_{\mathbf{V}}\beta = -\alpha\tau$$

$$\alpha KV + \nabla$$

$$\alpha KV + \nabla_V \beta = -\alpha \tau JV$$

$$d\alpha(V) + II(V, \beta) = \tau I(JV, \beta).$$

(43)

$$-\alpha \tau JV$$

(40)

$$\mathcal{K}_{\mathrm{int}} - \mathcal{K}_{\mathrm{ext}} = 4\tau(\tau - \frac{1}{\tau})\alpha^2 + \tau^2$$

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 $(\nabla_V^{\Sigma} K) W - (\nabla_W^{\Sigma} K) V = 4\alpha \tau (\tau - \frac{1}{2}) I(JV, W) J\beta$ 

as compatibility equations for 
$$I, II, \alpha$$
 and  $\beta$ .

(42)

(40)

(41)

(43)

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(40)

(41)

(42)

(43)

$$(\nabla_V^{\Sigma} K) W - (\nabla_W^{\Sigma} K) V = 4\alpha \tau (\tau - \frac{1}{\tau}) I(JV, W) J\beta$$

$$\alpha KV + \nabla_V \beta = -\alpha \tau JV$$

$$d\alpha(V)$$

$$d\alpha(V) + II(V,\beta) = \tau I(JV,\beta).$$

as compatibility equations for  $I, II, \alpha$  and  $\beta$ . Here, K is the Weingarten map  $I^{-1}II$ .

Let  $(\Sigma, I)$  be a simply-connected Riemannian surface,  $II \in \Gamma(T^*\Sigma \odot T^*\Sigma)$ ,  $\beta \in \Gamma(T\Sigma)$  and  $\alpha \in C^{\infty}(\Sigma, \mathbb{R})$  satisfying the compatibility equations

above. Then, there exists an isometric immersion 
$$X:\Sigma\to M$$
 such that 
$$I(V,M)=\langle Y,V,Y,M\rangle$$

$$I(V, W) = \langle X_* V, X_* W \rangle,$$

$$II(V, W) = \langle \nabla_{X_* V} X_* W, N \rangle,$$
(44)

where  $V,W\in\Gamma(T\Sigma)$  and  $p\in\Sigma\mapsto N|_{X(p))}\in T_{X(p)}M$  is a normal unit vector field along X given by

$$E_1|_{X(p)} = \alpha(p)N|_{X(p)} + \beta(p). \tag{46}$$

# Initial value problem

From now on, we assume that H = 0.

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Denoting by  $\mathbb D$  the unit disc in  $\mathbb C$ , the stereographic projection  $\pi:\mathbb H^2\to\mathbb D$  is defined by

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and denoting  $f=ar{\psi}_2^2$ , we may recover the results above in terms of f and g

The novelty is the following one

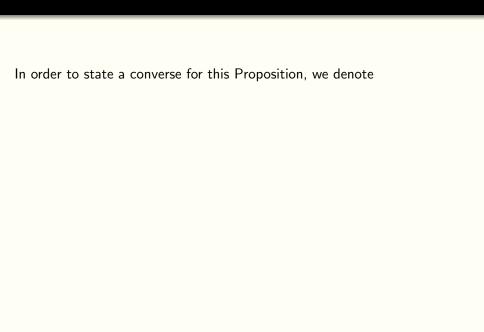
### Proposition

If the immersion  $X:\Sigma \to (SU_{1,1},g_{\tau})$  is maximal, then it holds that

If the immersion 
$$X:\Sigma o (SU_{1,1},g_ au)$$
 is maximal, then it holds that 
$$g_{z\bar z}+2\frac{2\varsigma^2-(1+|g|^2)}{(1+|g|^2)^2-4\varsigma^2|g|^2}\bar gg_zg_{\bar z}=0,$$

(50)

where 
$$\varsigma = 1/ au$$
.



In order to state a converse for this Proposition, we denote

$$\mathbb{D}_{ au} = \left\{ egin{array}{ll} \{w \in \mathbb{C} : |w| < au_0\}, & 0 < au \leq 1 \ \{w \in \mathbb{C} : |w| < 1\}, & 1 < au. \end{array} 
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(51)

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We then define the complete Riemannian metric in  $\mathbb{D}_{ au}$ 

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$$\mathrm{d} s_{\tau}^2 = \frac{|\mathrm{d} w|^2}{(1+|w|^2)^2 - 4c^2|w|^2}, \quad w \in \mathbb{D}_{\tau}.$$

(51)

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The Gauss map  $g: \Sigma \to (\mathbb{D}_{\tau}, ds_{\tau}^2)$  of a maximal immersion  $X: \Sigma \to (SU_{1,1}, g_{\tau})$  is a harmonic map.

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We observe that if g is harmonic, all maps  $g^{\vartheta}=e^{i\vartheta}g,\ \vartheta\in\mathbb{R},$  are also harmonic.

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#### Theorem

The quadratic differential

$$Q = (\psi_1 \partial_z \bar{\psi}_2 - \bar{\psi}_2 \partial_z \psi_1) dz^2$$
  
=  $(\frac{q}{2} - 2i(\tau - \frac{1}{\tau})f^2g^2) dz^2$  (53)

is holomorphic when H = 0.

#### Theorem

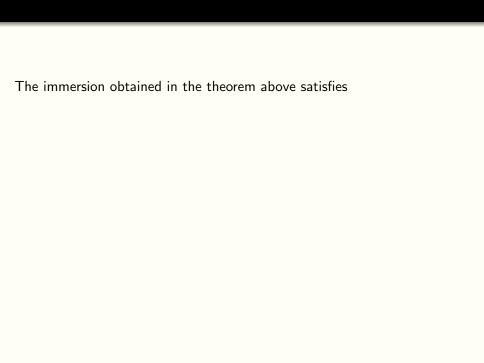
Let  $\Sigma$  be a simply connected Riemannian surface. Let  $g: \Sigma \to (\mathbb{D}_{\tau}, ds_{\tau}^2)$  be a harmonic never holomorphic map. Given  $z_0 \in \Sigma$  and  $Z_0 \in SU_{1,1}$ , there exists a unique conformal maximal immersion  $X: \Sigma \to SU_{1,1}$  whose Gauss map is g and such that  $X(z_0) = Z_0$ .

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The immersion obtained in the theorem above satisfies

$$-i\tau X^{-1}X_{z} = \frac{2g\bar{g}_{z}}{(1+|g|^{2})^{2} - 4\varsigma^{2}|g|^{2}} \hat{e}_{1}$$

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Moreover, there exists a family of maximal immersions  $X_{\vartheta}: \Sigma \to SU_{1,1}$ ,  $\vartheta \in \mathbb{R}$ , represented by the harmonic maps  $g^{\vartheta} = e^{i\vartheta}g$ , such that  $X_0 = X$ .

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#### Questions

- correspondence with CMC surfaces in some space with fibration over  $\mathbb{S}^2_1$ .
- more intrincate examples.

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$$\alpha_{\pm}(\lambda) = \alpha_{\pm \mathfrak{d}} + \lambda^{-1} \alpha'_{\pm \mathfrak{m}} + \lambda \alpha''_{\pm \mathfrak{m}}, \quad \lambda \in \mathbb{S}^{1}.$$
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with components  $\Phi=(\Phi_-,\Phi_+)$  we define a family of maximal immersions  $X_\lambda:\Sigma\to SU_{1,1}$  by

$$X_{\lambda} = \Phi^{+}(\lambda)\Phi^{-1}(\lambda), \quad \lambda \in \mathbb{S}^{1}.$$

An isometric immersion  $X: \Sigma \to SU_{1,1}$  with CMC H gives rise to a form  $A^0 \subset \Lambda^1(\Sigma, SU_{1,1})$  satisfying

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$$d\vartheta = -\frac{1}{2}[\vartheta,\vartheta]. \tag{57}$$

#### Flat Lorentz surfaces

#### Theorem (-, F. Vitório)

Let  $\Sigma$  be a simply-connected flat Lorentz surface. An isometric immersion  $X:\Sigma\to SL(2,\mathbb{R})$  whose mean curvature satisfies  $H\ge 1$  may be represented by

$$X = F_1 F_2^T,$$

where  $F_i: \Sigma \to SL(2,\mathbb{R}), i = 1,2$ , are given by

$$\mathrm{d}F_1 = F_1 \left( \begin{array}{cc} 0 & h \\ 1 & 0 \end{array} \right) \mathrm{d}u \quad \text{and} \quad \mathrm{d}F_2 = F_2 \left( \begin{array}{cc} 0 & 1 \\ g & 0 \end{array} \right) \mathrm{d}v$$

for any Lorentz-holomorphic function h and Lorentz-antiholomorphic function g defined in  $\Sigma$  in terms of a null coordinate system (u, v)

# Thank you!