# WILLMORE SURFACES AND ELASTIC CURVES

## *from $\mathbb{L}^3$ to 3-dimensional Generalized Robertson-Walker spacetimes and static spacetimes*


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This talk is based on

M. Barros, _ and M. Ortega, *Rotational Surfaces in* $\mathbb{L}^3$ *and Solutions of the Nonlinear Sigma Model*. Communications in Mathematical Physics (to appear).

_, Willmore surfaces in Generalized Robertson-Walker spacetimes and static spacetimes. (In progress)

# INTRODUCTION

# Willmore surfaces

**The Willmore functional**

$$\mathfrak{W}(\phi) = \int_S (H_\phi^2 + \bar{R}_\phi)\,dA_\phi + \int_{\phi(\partial S)} k^\phi\,ds,$$

$\phi : S \to \bar{M}$ $\longrightarrow$ non-degenerate immersion of a surface in a Lorentzian 3-manifold $(\bar{M}, \bar{g})$

$H_\phi$ $\longrightarrow$ mean curvature of $\phi$

$\bar{R}_\phi$ $\longrightarrow$ sectional curvature of $\phi(S)$ in $\bar{M}$

$k^\phi$ $\longrightarrow$ geodesic curvature of $\phi(\partial S)$ in $\phi(S)$

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# Elastic curves

Elastic energy:

$$\mathfrak{E}^{\lambda}(\alpha) = \int_{\alpha} (k^2 + \lambda) \qquad\qquad \lambda \geq 0$$

$(\bar{M}, \bar{g}) \longrightarrow$ Riemmanian or Lorentzian surface

$\alpha : I \to \bar{M} \longrightarrow$ non-degenerate immersed curve with curvature $k$

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$\lambda = 0 \longrightarrow$ free elastic curves

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# LINK

| Willmore surfaces | generated by | Elastic curves | |
|---|---|---|---|
| of revolution in $\mathbb{L}^3$ with spacelike axis | | anti de Sitter plane (free) | Barros,_ and Ortega |
| of revolution in $\mathbb{L}^3$ with null axis | | anti de Sitter plane (free) | Barros,_ and Ortega |
| $\mathbb{S}^1 \times \gamma$ in the warped product $(\mathbb{S}^1 \times M, \varepsilon dt^2 + f^2 g)$ | | $\gamma$ in $(M, g)$ | Barros |

## In all the previous results

Given:

$(\bar{M}, \bar{g}) \longrightarrow$ Lorentzian 3-manifold

$G \longrightarrow$ 1-parameter subgroup of isometries

They assure:

$G$-invariant Willmore surfaces in $(\bar{M}, \bar{g})$

are generated by

elastic curves in certain surface (either Riemannian or Lorentzian)

### Natural question

What must $(\bar{M}, \bar{g})$ and $G$ satisfy to obtain the previous thesis?

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# Technique

In examples 1 and 3

$G$ is COMPACT

The compactness is the key point in the proof of both results.

### Idea

Extend the technique used to prove 2 to get results for

$G$-invariant Willmore surfaces in Lorentzian 3-manifolds,

$G$ being a non necessarily compact 1-parameter subgroup of isometries

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# Rotational Willmore surfaces with null axis in $\mathbb{L}^3$

**THEOREM** BARROS, _ AND ORTEGA

$\phi : S \to \bar{M}$ is a Willmore surface if and only if

$$\int_S \bar{g}(\mathfrak{R}(\mathbb{H}_\phi) + \varepsilon N_\phi(\bar{R}^{\mathbf{V}})N_\phi, \mathbf{V}^\perp)dA = 0,$$

for any variational field $\mathbf{V}$ compatible with the boundary conditions.

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for any variational field **V** compatible with the boundary conditions.

$$\mathbb{H}_\phi \longrightarrow \text{mean curvature vector field}$$

$$\mathfrak{R} = \varepsilon(\triangle + \tilde{A}) + (\mathrm{Ric}(N_\phi, N_\phi) - 2(H_\phi^2 + \bar{R}_\phi))\mathbf{I}$$

is a kind of Schrödinger operator, being

$\triangle \longrightarrow$ Laplacian respect to the normal connection
$\tilde{A} \longrightarrow$ Simons' operator
$Ric \longrightarrow$ Ricci curvature

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for any variational field $\mathbf{V}$ compatible with the boundary conditions.

$\varepsilon \longrightarrow$ signature of $\phi$

$N_\phi \longrightarrow$ Gauss map along $\phi$

$\bar{R}^{\mathbf{V}}(m, v) \to$ sectional curvature of the level surface $v$, at the point $m$

# ROTATIONAL WILLMORE SURFACES IN $\mathbb{L}^3$ WITH NULL AXIS

Let $\phi : S \longrightarrow \mathbb{L}^3$ be a rotational surface with null axis.

$\phi(S)$ is contained in a semi-space of $\mathbb{L}^3$ conformal to

$$\mathcal{P} \times \mathrm{AdS}_2,$$

the product of an anti de Sitter plane and a spacelike parabola.

$\phi(S)$ is conformal to

$$\mathcal{P} \times \gamma,$$

where $\gamma$ is a non-degenerate curve in $\mathrm{AdS}_2$.

# Rotational Willmore surfaces in $^3$ with null axis

When $\bar{M} = \mathcal{P} \times \mathrm{AdS}_2$ and $S = \mathcal{P} \times \gamma$,

$$N_\phi(\mathrm{R}^{\mathbf{V}}) = 0,$$

so $\mathcal{P} \times \gamma$ is Willmore if and only if

$$\int_{\gamma \times \mathcal{P}} \bar{g}(\mathfrak{R}(\mathbb{H}_\phi), \mathbf{V}^\perp) dA = 0,$$

if and only if

$$\mathfrak{R}(\mathbb{H}_\phi) = 0$$

if and only if

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# Willmore surfaces in 3-dim Lorentzian product spaces

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$(M^1, ds^2) \longrightarrow$ 1-dimensional Riemannian manifold
$(M, g) \longrightarrow$ Riemannian or Lorentzian surface

$$(\bar{M}, \bar{g}) = (M^1 \times M, \, \bar{\varepsilon}\, ds^2 + g), \qquad \bar{\varepsilon} = \begin{cases} -1 & \text{if} \quad g \text{ Riemannian} \\ 1 & \text{if} \quad g \text{ Lorentzian} \end{cases}$$

$$S = M^1 \times \gamma, \qquad \gamma \text{ non-degenerate curve in M}$$

Is $M^1 \times \gamma$ Willmore?

$$N_\phi(\mathrm{R}^\mathrm{V}) = 0,$$

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## Willmore surfaces in a 3-dim Lorentzian product space

so $M^1 \times \gamma$ is Willmore if and only if

$$\int_{M^1 \times \gamma} \bar{g}(\Re(\mathbb{H}_\phi), \mathbf{V}^\perp) dA = 0,$$

if and only if

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if and only if

$$\gamma \text{ is a free elastic curve in } (M, g)$$

**Theorem**

$$M^1 \times \gamma \text{ is a Willmore surface in } (M^1 \times M, \bar{\varepsilon}\, ds^2 + g)$$

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# In a 3-dim warped product

Consider the warped product spacetimes

$$M^1 \times_f M = (M^1 \times M, \bar{\varepsilon} ds^2 + f^2 g) \quad \text{and} \quad M \times_h M^1 = (M^1 \times M, \bar{\varepsilon} h^2 ds^2 + g),$$

where $f : M^1 \longrightarrow \mathbb{R}^+$ and $h : M \longrightarrow \mathbb{R}^+$ are smooth

Since $\mathfrak{W}$ is invariant under conformal changes of the metric

**Corollary**

| | |
|---|---|
| $M^1 \times \gamma$ is Willmore in $M^1 \times_f M$ $\Updownarrow$ $\gamma$ is free elastic in $(M, g)$ | $M^1 \times \gamma$ is Willmore in $M \times_h M^1$ $\Updownarrow$ $\gamma$ is free elastic in $(M, \frac{1}{h^2} g)$ |

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## Corollary

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# In Generalized Robertson-Walker and standard static spacetimes

When $M^1$ is an interval and $(M, g)$ is Riemannian

**Corollary**

$I \times \gamma$ is Willmore in the Generalized Robertson-Walker spacetime $I \times_f M$

$$\Updownarrow$$

$\gamma$ is a free elastic curve in $(M, g)$

**Corollary**

$I \times \gamma$ is Willmore in the standard static spacetime $M \times_h I$

$$\Updownarrow$$

$\gamma$ is a free elastic curve in $(M, \frac{1}{h^2} g)$

# Willmore surfaces in Lorentzian 3-manifolds

# invariant under a 1-parameter subgroup of isometries

# Static and standard static vector fields

Let $(\bar{M}, \bar{g})$ be a Lorentzian 3-manifold.

A timelike Killing vector field $\xi$ in $(\bar{M}, \bar{g})$ is called

- static: if it is irrotational
- standard static: if there exists an isometry

$$\chi : (\bar{M}, \bar{g}) \longrightarrow (\mathbb{R} \times M, -f^2 dt^2 + g),$$

where $d\chi(\xi) = \partial_t$, $\xi(f \circ \chi) = 0$ and $(M, g)$ is a Riemannian surface.

Given $G$ $\longrightarrow$ 1-parameter subgroup of isometries
with timelike Killing vector field $\xi$

**COROLLARY**

If $\xi$ is standard static,

$G$-invariant Willmore surfaces in $(\bar{M}, \bar{g})$

are generated by

elastic curves in $\left(M, \dfrac{-1}{\bar{g}(\xi,\xi)}\bar{g}\right)$

$M$ being any maximal integral surface of the orthogonal distribution of $\xi$

Applying

**LEMMA** M.SÁNCHEZ

Let $\xi$ be a static vector field in $(\bar{M}, \bar{g})$ and let $(\tilde{M}, \tilde{g})$, $\Pi : \tilde{M} \to \bar{M}$, $\tilde{g} = \Pi^* \bar{g}$, its universal Lorentzian covering. If $\xi$ is complete, then $(\tilde{M}, \tilde{g})$ is standard static.

We get

**THEOREM** _

If $\xi$ is static,

$$G\text{-invariant Willmore surfaces are generated by elastic curves in } \left(M, \frac{-1}{g(\xi,\xi)} g\right)$$

$M$ being any maximal integral surface of the orthogonal distribution of $\xi$

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$M$ being any maximal integral surface of the orthogonal distribution of $\xi$

With similar techniques, the following result is obtained

Given $G$ $\longrightarrow$ 1-parameter subgroup of isometries with spacelike Killing vector field $\xi$.

### THEOREM

If $\xi$ has no zero and it is irrotational, then

$G$-invariant Willmore surfaces are generated by elastic curves in $\left(M, \dfrac{1}{g(\xi,\xi)}g\right)$

$M$ being any maximal integral surface of the orthogonal distribution of $\xi$

# The end