Height Estimates for r-Mean Curvature Spacelike Hypersurfaces in Product Spaces

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- where $\pi_{\mathbb{R}}$ and π_{M} denote the canonical projections of $-\mathbb{R} \times M$ onto each fator and $< \ , \ >_{M}$ stands for the Riemannian metric on M.
- In particular, when the Riemannian factor M is the Euclidean space $\mathbb{R}^{n}$ then $-\mathbb{R} \times \mathbb{R}^{n}$ is the Lorentz-Minkowski space $\mathbb{L}^{n+1}$.

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- Since $\partial_t = (\partial/\partial_t)_{(t,x)}, \ (t, x)\varepsilon - \mathbb{R} \times M$, is a unitary timelike vector field globally defined on the ambient, then there exists a unique unitary timelike normal field N globally defined on Σ which is in the same time-orientation as ∂_t, so that

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- We will refer to that normal field N as the future-pointing Gauss map of the hypersurface. Its opposite will be refered as the past-pointing Gauss map of Σ.

Higher order mean curvatures

- Let A be the shape operator of Σ with respect to either the future or the past-pointing Gauss map N. It is a self-adjoint linear operator on each tangent space $T_p\Sigma$ and its eigenvalues $k_1(p), \dots, k_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants given by

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  the mean curvature of Σ. By the choice of the sign $(-1)^k$, the mean curvature vector $\mathbf{H}$ is given by $\mathbf{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\mathbf{H}(p)$ is in the same time-orientation as N.

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$$L_k(f) = \operatorname{trace}(P_k \circ \nabla^2 f).$$

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- Then $L_k(f) = div(P_k(\nabla f)) \quad \Leftrightarrow \quad div(P_k) = 0.$

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Lemma 1

Let Σ be a spacelike hypersurface immersed into a Lorentzian product space. If there exists an elliptic point of Σ, with respect to an appropriate choice of the Gauss map N, and $H_{k+1} > 0$ on Σ, for $1 \leq k \leq n-1$, then for all $1 \leq j \leq k$ the operator L_j is elliptic or, equivalently, P_j is positive definite (for that appropriate choice of the Gauss map if j is odd)

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$$\binom{n}{k} H_j = \sigma_j(-k_1, \dots, -k_n) = (-1)^j S_j \,.$$

Height and support functions

- In this setting, we consider two particular functions naturally attached to a spacelike hypersurface Σ immersed into a Lorentzian product space $-\mathbb{R} \times \mathbb{R}^n$: the vertical height function $h = (\pi_{\mathbb{R}})_{|\Sigma}$ and the support function $\eta = \langle N, \partial_t \rangle$, where N denotes the Gauss map of Σ^n and ∂_t is the coordinate vector field induced on $-R \times M^n$.
- The following lemma corresponds to the analytical framework that we will use to obtain our main result.

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\begin{aligned}
L_r \eta &= \binom{n}{r+1} \langle \nabla H_{r+1}, \partial_t \rangle + \mathrm{tr}(A^2 \circ P_r)\eta \\
&= +\kappa_M \left((r+1)\binom{n}{r+1} H_r |\nabla h|^2 - \langle P_r \nabla h, \nabla h \rangle \right) \eta.
\end{aligned}
$$

Remark

The formulae collected in the above lemma are the Lorentzian versions of the ones obtained by X. Cheng and H. Rosenberg. We also note that L.J. Alías jointly with A. G. Colares obtained a generalization of these formulae in the context of the Generalized Robertson- Walker spacetimes. Moreover, A.L. Albujer and L.J. Alías obtained the corresponding formulae for the Laplacian of the height and support functions of a space-like surface immersed in a 3-dimensional Lorentzian product space.

Now, we are in the position to state and prove our main result.

Main theorem and proof

Theorem

Let Σ^n be a compact immersed spacelike hypersurface of a Lorentzian product space $-\mathbb{R} \times M^n$ whose Riemannian fiber M^n has nonnegative constant sectional curvature κ_M. Suppose that Σ^n has positive constant r-mean curvature H_r, for some $1 \leq r \leq n$, and that its boundary $\partial\Sigma^n$ is contained in the slice $\{0\} \times M^n$. Then, the vertical height of Σ^n satisfies the inequality

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$$|h| \leq \frac{C - 1}{H_r^{1/r}}$$

where $C = \max_{\partial\Sigma} |\eta|$. Moreover, in the case $r = 1$ one can replace the condition on the sectional curvature of M^n by that of the Ricci curvature of M^n being nonnegative.

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where c is a negative constant. We have that $\varphi_{|\partial\Sigma} \leq C$, where $C = \max_{\partial\Sigma} |\eta|$.

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Therefore,

$$\mathrm{tr}(A^2 \circ P_{r-1}) = \binom{n}{r}(nH\,H_r - (n-r)H_{r+1}) \geq r\binom{n}{r}H_r^{(r+1)/r}.$$

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we note that in the case $r = 1$ one can replace the condition on the sectional curvature of M^n by that of the Ricci curvature of M^n being nonnegative.

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for each $1 \leq r \leq n$ (if we choose the Gauss map N in the same time-orientation of e_1, for the case r odd).

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we conclude that our estimate for the vertical height function is sharp.

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If the support function η of Σ^n is bounded, then its end is not divergent.

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(Universidade Federal do Ceará, Brazil)

Joint work with H. F. de Lima
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V International Meeting on Lorentzian Geometry
Martina Franca, Italy

Thanks