V International Meeting on Lorentzian Geometry

Abstracts

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Contents

Mini Courses

Willmore actions in semi-Riemannian geometry with applications in physics

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Willmore's functionals appear naturally as Lagrangian governing field theories where the target space is a semi-Riemannian one, say (M, g). Actually, they are conformal field theories in the sense that just depend on the conformal class, $[g]$. The solutions of the corresponding field equations are usually known as Willmore submanifolds and they constitute the space of field configurations. To determine the moduli space of field configurations, even in the simplest cases, is a problem so important as difficult and, certainly, it is open anywhere. Consequently, the search of solutions satisfying certain prescribed conditions has high interest. These conditions can involve either the topology or the geometry of the source space.

In this mini-course, we will survey the Willmore program, including known results and open problems, with special attention to the cases where the target space is either Riemannian or Lorentzian. Furthermore, we will analyze the strong relationship between the Willmore variational principle and several topics in physics including *bosonic string theories, sigma models and theories of cell membranes and vesicles.*

Applications of variational methods to Lorentzian geometry

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In the last twenty years variational methods have been widely used to investigate the global properties of Lorentzian manifolds. In this mini-course we focus our attention to the Fermat principle of light rays in General Relativity and to the properties of the energy integral on a Lorentzian manifold, whose stationary points are geodesics. In particular we shall show how different geometries as Riemannian, Finslerian and Lorentzian interact in the study of these variational problems.

Main Talks

Cohomogeneity one Lorentzian manifolds

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A cohomogeneity one Lorentzian manifold is a Lorentzian n- dimensional manifold (M, g) with a connected isometry group G which has a codimension one orbit $P = Gp = G/H$. We describe the local structure of M in a G-invariant neighborhood U of a singular orbit $Q = G/K$ which belongs to the closure of the open submanifold which consists of codimension one orbits. In particular, we give a sufficient condition that U can be identified with the normal bundle $N(Q) = G \times_K N$ of the orbit and we describe the structure of such model G-manifolds $N(Q)$ which admit invariant Lorentzian metrics. Under the assumption that there exists a complete normal geodesic, we describe the structure of the orbit space M/G. We applies these results to a local description of 4-dimensional Lorentzian cohomogeneity one metrics.

Cauchy problem on a characteristic cone for the Einstein equations

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The Cauchy data on a characteristic cone for the Einstein equations in vacuo are only the values on the cone of the Lorentzian metric coefficients. These must satisfy constraints which turn out to be a hierarchy of ordinary differential equations along the generators. We show how to solve them and also obtain an existence theorem for the evolution problem, using a wave map gauge.

Periodic geodesics in compact Lorentzian manifolds

J. L. Flores
Universidad de Málaga, Spain

A well known result in differential geometry establishes the existence of a periodic geodesic in every compact Riemannian manifold. As far as we know, a Lorentzian analog of this result is still an open problem in its full generality. However, there are remarkable partial results in this direction. In this talk we are going to review these results and present a new one based on Lie groups theory, which, in particular, ensures the existence of at least two periodic timelike geodesics in any compact stationary Lorentzian manifold.

This talk is partially based on the following paper written in collaboration with M.A. Javaloyes and P. Piccione:
Periodic geodesics and geometry of compact Lorentzian manifolds with a Killing vector field, arXiv:0812.1163v2 [math.DG].

A genericity problem in Lorentzian Geometry

R. Giambò
Università di Camerino, Italy

Genericity issues are assuredly relevant in General Relativity - to establish stability of a given property with respect to perturbation of the data is crucial for experimental applications. The main purpose of the present talk is to discuss a problem justified by geometrical motivations as well (in particular Morse Theory) - genericity of the nondegeneracy property for light rays between source and observer in a Lorentzian manifold.

On the interplay between Lorentzian Causality and Finsler metrics of Randers type

M. A. Javaloyes
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We obtain some results in both, Lorentz and Finsler geometries, by using a correspondence between the conformal structure of standard stationary spacetimes on $M = \mathbb{R} \times S$ and Randers metrics on S. In particular:

(1) For stationary spacetimes: we give a simple characterization on when $\mathbb{R} \times S$ is causally continuous or globally hyperbolic (including in the latter case, when S is a Cauchy hypersurface), in terms of an associated Randers metric. Consequences for the computability of Cauchy developments are also derived.

(2) For Finsler geometry: Causality allows to determine that the natural sufficient condition for the convexity (i.e., geodesic connectedness by minimizing geodesics) of any Finsler manifold is the compactness of the symmetrized closed balls. Then, we show that for any Randers metric R with compact symmetrized closed balls, there exists another Randers metric $\widetilde{R}$ with the same pregeodesics and geodesically complete.

Moreover, results on the differentiability of Cauchy horizons yield consequences for the differentiability of the Randers distance to a subset, and viceversa

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http://arxiv.org/abs/0903.3501

On generalised Jacobi equations on Lorentzian manifolds

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The standard Jacobi equation (equation of geodesic deviation) describes the motion of a geodesic relative to a neighbouring geodesic in linear approximation. Several generalisations of this equation have been suggested, which go beyond the linear approximation to some extent. In particular, such generalised Jacobi equations and their relevance to general relativity have been discussed for timelike geodesics on Lorentzian manifolds. After reviewing these earlier results I will consider generalised Jacobi equations for lightlike geodesics on a Lorentzian manifold and discuss their physical relevance. The general results will be illustrated by applying them to Schwarzschild spacetime and to the spacetime of a plane gravitational wave.

Conformal transformations of spacetimes

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In General Relativity conformal transformations have been studied during many decades and many concepts in differential geometry and mathematical physics are conformally invariant. In the talk an overview on results concerning conformal symmetries of spacetimes from the point of view of differential geometry will be presented.

In particular results on conformal symmetries of Einstein manifolds and about essential conformal vector fields obtained in joint work with Wolfgang Kühnel (Stuttgart) will be discussed.

Contributed Talks

Some remarks on the null sectional curvature

A. L. Albujer*, S. Haesen
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In this talk, we first present a geometrical interpretation for the null sectional curvature of degenerate planes in a Lorentzian manifold. This interpretation is based on a generalization to the indefinite case of the squaroids of Levi-Civita. Further, we will prove that a three-dimensional, conformally flat Lorentzian manifold has isotropic and spatially constant null sectional curvature if and only if it is locally a Robertson-Walker manifold. In that way, we extend to the three-dimensional case a known result of Harris, Karcher and Koch.

References.

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Willmore surfaces and elastic curves, from $\mathbb{L}^3$ to 3-dimensional Generalized Robertson-Walker spacetimes and static spacetimes

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It is well known the existence of a closed link between Willmore surfaces and elastic curves. Many examples of this relation are:

- Willmore surfaces of revolution in $\mathbb{R}^3$ are obtained from clamped free elastic curves in the hyperbolic plane [2].
- Willmore surfaces of revolution in $\mathbb{L}^3$ with null axis are generated by clamped free elastic curves in the anti de Sitter plane [3].
- Willmore tori in $\mathbb{S}^5$ foliated by fibers of the standard Hopf map are generated by closed elastic curves in $\mathbb{CP}^2(4)$ [4].
- Given a semi-Riemannian manifold (M, g) and a nondegenerate curve immersed in (M, g), $\mathbb{S}^1 \times \gamma$ is a Willmore surface in $(\mathbb{S}^1 \times M, \varepsilon dt^2 + f^2 g)$ if and only if γ is a free elastic curve in (M, g) [1].

The Willmore surfaces appearing in all the previous examples, except for the second one, are invariant under the action of a compact group of transformations. The proof of those results is based on the Palais principle of symmetric criticality, for which the compactness is needed.

We focus on the technique used to obtain Willmore surfaces of revolution in $\mathbb{L}^3$ with null axis. We modify it to obtain a result for Willmore surfaces in 3-dimensional Lorentzian product spaces, from which we get two characterizations of Willmore surfaces invariant under a (not necessarily compact) group of transformations, in terms of elastic curves. The first one in Generalized Robertson-Walker spacetimes of dimension 3 (an extension in dimension 3 of the result given in the last item). The second one is a result for Willmore surfaces in 3-dimensional static spacetimes.

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[1] M. Barros, *Free elasticae and Willmore tori in warped product spaces*, Glasgow Math. J., 40 (1998), 265–272.

[2] M. Barros, *A geometric algorithm to construct new solitons in the* O(3) *nonlinear sigma model*, Physics Letters B, 553 (2003), 325–331.

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Curvature homogeneous Lorentzian three-manifolds

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A pseudo-Riemannian manifold (M, g) is said to be *curvature homogeneous up to order* k if, for any points $p, q \in M$, there exists a linear isometry $\phi : T_pM \to T_qM$ such that $\phi^*(\nabla^i R(q)) = \nabla^i R(p)$ for all $i \leq k$. When $k = 0$, (M, g) is simply called a *curvature homogeneous space*. A locally homogeneous space is curvature homogeneous of any order k. Conversely, curvature homogeneity up to order k implies local homogeneity when k is sufficiently high.

If $\dim M = 2$, then curvature homogeneity (up to order 0) implies local homogeneity, but when $\dim M \geq 3$, a curvature homogeneous space needs not to be locally homogeneous. Three-dimensional spaces are natural candidates for a deep investigation about curvature homogeneity, because their curvature tensor is completely determined by the Ricci tensor, and curvature homogeneity is equivalent to requiring that there exists, at least locally, a pseudo-orthonormal frame field with respect to which the Ricci components are constant.

Differences arising in dimension three between the Riemannian and the Lorentzian cases are essentially due to the different behavior of self-adjoint Ricci operator Q in these frameworks. At each point of a Riemannian manifold there exists an orthonormal basis diagonalizing Q, while for a Lorentzian manifold four different cases (*Segre types*) can occur, depending on the multiplicity of the Ricci eigenvalues and on the dimension of the corresponding eigenspaces. A natural question to ask is the following:

Do there exist nonhomogeneous curvature homogeneous Lorentzian three-manifolds for all Segre types of the Ricci operator?

Partial negative answers hold for: manifolds curvature homogeneous up to order one [1], semi-symmetric spaces [6], Einstein-like manifolds [3]. However, we proved that *three-dimensional nonhomogeneous curvature homogeneous Lorentzian metrics exist for all different Segre types of the Ricci operator* [2,4]. In some cases, the examples belong to the class of Lorentzian three-manifolds admitting a parallel degenerate line field [5]. In some other cases, a system of partial differential equations is introduced *ad hoc*.

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Riemannian extensions and curvature operators

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Riemannian extensions of torsion-free connections are a powerful technique for constructing many pseudo-Riemannian metrics with special curvature properties. In this communication we plan to emphasize this correspondence between affine and pseudo-Riemannian geometry by focusing on torsion-free connections with skew-symmetric Ricci tensor [3,4] and torsion-free connections with symmetric and degenerate Ricci tensor [1]. Both classes above are of special importance from the point of view of their curvature, since they correspond to Osserman and Ivanov-Petrova affine connections.

Also, attention will be paid to some generalizations of the Riemannian extensions and their relation with Einstein and Osserman metrics in dimension four [2].

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Height estimates for r-mean curvature spacelike hypersurfaces in product spaces

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It is obtained a height estimate concerning spacelike compact hypersurfaces $\sum^n$ in a $(n+1)-$dimensional Lorentzian product space $-R \times M^n$ with some positive constant r-mean curvature and whose boundary is contained into a slice $\{t\} \times M^n$. By considering hyperbolic caps in the Lorentz-Minkowski space L^{n+1}, it is shown that the estimate is sharp. Furthermore, this estimate is applied to the study of complete spacelike hypersurfaces immersed with some positive constant r-mean curvature into a Lorentzian product space yielding information about the ends that gives topological obstructions for the existence of such hypersurfaces.

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Deformations of $2k$-Einstein structures

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It is shown that the space of infinitesimal deformations of $2k$-Einstein structures is finite dimensional at compact non-flat space forms. Moreover, spherical space forms are shown to be rigid in the sense that they are isolated in the corresponding moduli space.

On related theorems about global hyperbolicity

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Various theorems are known which show that if a certain "metric" on a spacelike hypersurface of a spacetime is complete, this spacetime is globally hyperbolic. In that case, the hypersurface in question is a Cauchy surface. For example, by showing the completeness of the Riemannian metric on the spacelike slices, it has been established that warped product spacetimes and static spacetimes are globally hyperbolic. Moreover, recent theorems show the global hyperbolicity of conformally stationary spacetimes, based on the completeness of some Finslerian metric of Randers type – and of regularly sliced spacetimes as well, where a certain Riemannian metric is complete and the lapse function and shift vector satisfy certain boundedness conditions. In these cases the spacelike slices are not uniquely determined. Each of these theorems applies to a certain class of spacetimes, and we show the relations of these theorems on the intersection of these classes. In doing so, we also show that some of the theorems are in fact special cases of others. Finally, we present important examples.

On the reparametrization of homogeneous geodesics

Z. Dusek*, O. Kowalski, Z. Vlášek
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A geodesic in a homogeneous pseudo- Riemannian or affine manifold is homogeneous if it is an orbit of an one-parameter group of isometries, or, of affine diffeomorphisms, respectively. A homogeneous pseudo-Riemannian or affine manifold is a g.o. manifold if every geodesic is homogeneous.

On pseudo-Riemannian manifolds, the parameter of the group of isometries may be different from the affine parameter of the geodesic only for null homogeneous geodesics. Nevertheless, for all known examples of pseudo-Riemannian g.o. manifolds, these parameters are the same for all homogeneous geodesics. In the affine case, there are g.o. manifolds whose almost all geodesics must be reparametrized.

In the talk, the features of homogeneous geodesics in pseudo-Riemannian homogeneous manifolds and g.o. manifolds will be recalled, the new approach for the study of homogeneous geodesics in affine homogeneous manifold will be explained and the new phenomena which appear in the affine case will be illustrated.

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Einstein-like Walker manifolds and generalized locally symmetric spaces

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Locally symmetric spaces can be characterized by the fact that the Jacobi operator along each geodesic γ has constant eigenvalues ($\mathfrak{C}$) and parallel eigenspaces ($\mathfrak{P}$) along γ.

Two natural generalizations of locally symmetric spaces arise when considering either condition $\mathfrak{C}$ and $\mathfrak{P}$ separately [J. Berdnt and L. Vanhecke].

Our aim is to investigate $\mathfrak{C}$ and $\mathfrak{P}$ spaces in Lorentzian geometry by focusing on Walker metrics as a first step towards a complete description of such manifolds.

The causal boundary and its relations with the conformal boundary

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Penrose conformal boundary is the most common boundary in Mathematical Relativity. However, it has limitations, which has been put forward recently by some works about the holographic principle in string theory. As a consequence, the classical causal boundary have been redefined, and appears now as its optimal alternative.

Our aim in this talk is twofold. First, we are going to present some possible anomalies in the conformal boundary which are inherent to its approach. So, the causal boundary is selected as the physically and mathematically relevant boundary in Lorentzian Geometry. Second, we are going to show that, under certain regularity conditions, these anomalies are solved, and the conformal boundary becomes equal to the causal boundary. In particular, under these conditions the conformal boundary becomes a powerful tool to compute the causal one.

Lorentzian manifolds with Gray structure

W. Jelonek
Cracow Technical University, Poland

The aim of my talk is to describe Lorentzian manifolds (M, g) whose Ricci tensor ρ satises the Gray condition

$$\Sigma_{X,Y,Z}\nabla\rho(X,Y,Z) = \Sigma_{X,Y,Z}\frac{2X\tau}{n+2}g(Y,Z),$$

where $\tau = \mathrm{tr}_g$ is the scalar curvature of (M, g) and Σ means the cyclic sum. We construct Robertson-Walker manifolds with the Gray property. We also give many compact examples of Lorentzian manifolds satisfying Gray condition.

The space of Lorentzian flat tori in anti-de Sitter 3-space

M. A. León-Guzmán*, P. Mira, J. A. Pastor
Universidad de Murcia, Spain

We describe the space of isometric immersions from the Lorentz plane $\mathbb{L}^2$ into the anti-de Sitter 3-space $\mathbb{H}_1^3$, and solve some open problems of this context raised by M. Dajczer and K. Nomizu in 1981. We also obtain from the above result a description of the space of Lorentzian flat tori isometrically immersed in $\mathbb{H}_1^3$ in terms of pairs of closed curves with wave front singularities in the hyperbolic plane $\mathbb{H}^2$ satisfying some compatibility conditions.

Maximal surfaces in exotic anti-de Sitter spaces

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The three dimensional Anti de Sitter space AdS_3 may be endowed with a family of Lorentzian metrics arising from stretching or dilating the infinitesimal generator of the fibers of the Hopf fibration over the hyperbolic plane. In terms of the split-quaternionic Lie model for AdS_3, these are left-invariant metrics.

We then prove that a pair of spinors satisfying a Dirac type equation represent surfaces immersed in AdS_3 with prescribed mean curvature. Using this, we prove that the Gauss map of a maximal surface immersed in Anti de Sitter space is a harmonic map whose target is the unit disc. Conversely, we exhibit a representation of minimal surfaces in AdS_3 in terms of a given harmonic map. The examples we construct appear in associated families.

Finally, we present some results concerning $2+1$-gravity in these exotic metric in AdS_3 and flat timelike surfaces.

A representation formula for minimal surfaces in $\mathbb{R}^3$

F. Mercuri
Universidade Estadual de Campinas, Brazil

Foliations of globally hyperbolic manifolds

O. Müller
Universidad Nacional Autonoma de Mexico

In this talk I want to summarize different results of the last seven years on constructing metric foliations of globally hyperbolic manifolds, starting from the work of Antonio Bernal and Miguel Sánchez, passing to joint results with Miguel Sánchez about steep time functions and Nash type embedding theorems in Minkowski spaces to end up with some recent results in this area. I will also give a short overview over connections of this topic to some other areas of geometry and global analysis as holonomy theory (joint work with Helga Baum) and minimal surfaces.

Marginally trapped surfaces in Minkowski 4-space and 1-dimensional isometry groups

S. Haesen, M. Ortega*
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In Mathematics, marginally trapped surfaces in a Lorentzian manifold are those spacelike surfaces whose mean curvature vector is lightlike. In Physics, there exist similar definitions for spacelike surfaces with timelike mean curvature vector and satisfying some extra conditions. Some of these surfaces (MOTS) can be used to find regions of the Universe containing black holes.

We pay attention to marginally trapped surfaces in Minkowski 4-space which are boost, rotational or screw invariant, i. e., invariant by 1-dimensional isometry subgroups of O(4,1) whose axis is spacelike, timelike or lightlike. Depending on the chosen group, we fully describe them in terms of families of smooth functions. In the boost and rotational cases, we are able to connect two of them by an intermediate maximal surface. We also study classical geometric properties on them, like the Gaussian curvature or the harmonic mean curvature.

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Partially conformal vector fields

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We define the partially conformal vector fields and use them to give a characterization of Lorentzian manifolds which admit this kind of fields as doubly warped products with two 1-dimensional factors. In the particular case of Lorentzian space forms, partially conformal vector fields can be associated to foliations by rotation hypersurfaces. Adding the condition of the mean curvature of each leaf being constant, we prove that the foliation is by cylinders.

On Gödel-type metrics

M. Scherfner*, M. Gürses, M. Plaue, A. de Sousa, T. Schönfeld
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We will present new Gödel-type metrics with particular causal structure. Additionally we investigate the geodesic structure of these and other models – like A. Ori's time machine – by visualizing the behavior of geodesics and that of light cones. We concentrate on showing the formation of caustics and the tipping of light cones since these effects are related to closed timelike or null curves.

On infinitesimal orbit type theorems and non-normalizable orbits

D. Szeghy
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The principal orbit type theorem says, that if we have a compact Lie group G and a differentiable action of this group on a connected differentiable manifold M, then among the orbit types there is a unique one, the so called principal type, for which the orbits belonging to this type build an open, dense and connected set in M. This theorem doesn't hold in general if the Lie group is not compact. For example if we take the pseudo-orthogonal group $SO(2,1)$ and its action on the 3-dimensional Minkowski space, then the orbits in the interior of the lightcone and the orbits in the exterior of the lightcone will belong to different orbit types.

Recently J. Szenthe gave a kind of substitute of the principal orbit type theorem using the definition of infinitesimal orbit type, where the definition of the orbit type was modified the slightliest way, namely in the definition we do not take the isotropy subgroup G_x of a point x but its identity component G_x^0. In the paper of Szenthe the concept of the stable and unstable infinitesimal types are introduced, and the objective of his paper is to show that the union of orbits of stable infinitesimal types build an open and dense set. Szenthe gave criteria under which this will be true. Our point of view will be different. We will make assumptions under which there will be *a unique* infinitesimal type, the so called infinitesimal principal type, such that the orbits belonging to this type build an open and dense set. To state our result the following definitions are needed:

Definition 1. Let $\alpha : G \times M \to M$ be a differentiable action of a Lie group G on a differentiable manifold M. The orbit $G(x)$ is **normalizable** if there is a subspace $\widetilde{N_x}G(x) \subset T_xM$ for which the following holds:

- $\widetilde{N_x}G(x) \oplus T_xG(x) = T_xM$ is a decomposition;
- $\widetilde{N_x}G(x)$ is invariant under the action of G_x.
- Moreover the action α is called normalizable, if every orbit is normalizable.

Definition 2. Let $\alpha : G \times M \to M$ be a differentiable action of Lie group on a differentiable manifold M. Let $G(x)$ be a normalizable orbit and assume that there is a G-invariant neighbourhood $U \subset \widetilde{N}G(x)$ of the zero section and a G-equivariant locally diffeomorphic map $\varphi : U \to M$ for which $\varphi(0_x) = x$. Then $\widetilde{N}G(x)$ is called a local model.

Theorem 3. *Let $\alpha : G \times M \to M$ be a differentiable action of class C^2 of a Lie group G on the differentiable manifold M which is normalizable. Assume that for every orbit $G(x)$ the normal bundle $\widetilde{N}G(x)$ is a local model. Then in the canonical partial order of the infinitesimal orbit types there is a unique maximal infinitesimal type κ, such that the orbits of this type build an open and dense set in M.*

Corollary 4. *Let $\alpha : G \times M \to M$ be an isometric action of a Lie group G on a semi-Riemannian manifold (M, g) which is normalizable. Then there is a* *unique maximal infinitesimal type κ, such that the orbits of this type build an open and dense set in M.*

However the above corollary needs the assumption of normalizability. Therefore, we will also investigate the non-normalizable orbits in the Lorentzian case and prove theorem 6, but to state this result we need the definition:

DEFINITION 5. Let (M, g) be a Lorentz manifold and $\overline{X} : TM \to TM$ a Killing field, and $z \in M$ such that $\overline{X}(z) \neq 0_z$. If

$$\nabla_{\overline{X}(z)} \overline{X} = \mu \overline{X}(z),$$

holds for some $\mu \in \mathbb{R} \setminus \{0\}$ then the integral curve of $\overline{X}$ through z, which is a pregeodesic, is called a *homogeneous pregeodesic*, which is said to be *genuine* if $\mu \neq 0$.

THEOREM 6. *If (M, g) is a Lorentz manifold and $\alpha : G \times M \to M$ an isometric action of a Lie group G and the orbit $G(x)$ is non-normalizable, then it is a lightlike orbit, such that the lightlike curves of the orbit are lightlike homogeneous pregeodesics.*

The above result could be connected to some results of Z. Dusek and O. Kowalski on light-like homogeneous geodesics. We will investigate the neighbouring orbit of a lightlike one which helps us to prove the following result.

THEOREM 7. *If the Lorentz manifold (M, g) is geodetically complete and there is an isometric action of a Lie group such that the non-normalizable orbits have genuine homogeneous pregeodesics, then the normalizable orbits build a dense set in M.*

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Posters

Homogeneous geodesics of four-dimensional generalized symmetric pseudo-Riemannian spaces

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We study homogeneous geodesics on a relevant class of pseudo-Riemannian homogeneous spaces, namely, the generalized symmetric pseudo-Riemannian spaces ([9], [10]). We know that in dimension $n = 3, 4$, the classication of these spaces was made by J. Černý and O. Kowalski [1]. In dimension $n = 2$ all generalized symmetric pseudo-Riemannian spaces are symmetric, consequently all geodesics are homogeneous; in dimension $n = 3$, a generalized symmetric pseudo-Riemannian space may be identied with $\mathbb{R}^3$ endowed with a special metric, whose possible signatures are $(3, 0)$, $(0, 3), (2, 1), (1, 2)$. In dimension $n = 4$, a generalized symmetric pseudo-Riemannian space may be identied with $\mathbb{R}^4$ endowed with a special metric of four types: A), B), C), D). The metric of type A) has possible signatures $(4, 0)$, $(0, 4)$, $(2, 2)$; the metric of type B) has always signature $(2, 2)$; spaces of type C) are example of Lorentzian spaces, with possible signatures $(3, 1)$ and $(1, 3)$; the metric of type D) has always signature $(2, 2)$. A basic property of these spaces is that all of them are reductive homogeneous [1]. By using the Geodesic Lemma ([7],[8],[18]), this property allows us to obtain the set of all homogeneous geodesics of these spaces and then, to describe their behaviour. In particular, we prove that in any pseudo-Riemannian generalized symmetric space of dimension $n = 4$ there exist just four linearly independent homogeneous geodesics through each point.

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Global invariants of regular curves and null curves in the pseudo-Euclidean geometry

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Let $M(n,p)$ be the group of all motions of an n-dimensional pseudo-Euclidean space of index p. Definitions of an $M(n,p)$-equivalence of curves, a pseudo-Euclidean type of a regular curve and a null curve are introduced. Problems of the $M(n,p)$-equivalence of curves and null curves are reduced to that of paths. Generating systems of the differential fields of $M(n,p)$-invariant differential rational functions of paths are described. Global conditions of the $M(n,p)$-equivalence of curves and null curves are given in terms of the pseudo-Euclidean types of a curve and the generating invariants. All correlations between elements of the generating invariants are described. Similar results have obtained for the subgroup of $M(n,p)$ generated by unimodular pseudo-orthogonal transformations and translations of E_p^n. A similar invariant-theoretical approach was used in the theory of curves in the affine geometry [1, 2].

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On invariants of curves

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In this paper, the complete system of global differential invariants of curves in centro-equiaffine group are obtained. The types for centro-equiaffine curves and for every type all invariant parametrizations for such a curve are introduced. We found that the global and integral invariants are independent.

Uniqueness of Minkowski space

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The occurrence of a null line (ie an achronal null geodesic) that extends from $\mathscr{J}^-$ to $\mathscr{J}^+$ is an important feature of spacetime, closely related to the existence of eternal observer horizons. It has been proven recently that the above property characterizes de Sitter spacetime among all globally hyperbolic and asymptotically de Sitter solutions to the vacuum Einstein equations with $\Lambda > 0$. In this work we show that the same is true for Minkowski space. In other words, $\mathbb{M}^4$ is the only asymptotically flat and globally hyperbolic spacetime that satisfies $R_{ab} = 0$ and has a null line.

Lightlike submanifolds of a Lorentzian manifold with a semi-symmetric non-metric connection

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We study lightlike submanifolds of a Lorentzian manifold admitting semi- symmetric non-metric connection. We obtain a necessary and a sufficient condition for integrability of the screen distribution with semi-symmetric non-metric connection. Then we give the conditions under which the Ricci tensor of a lightlike submanifold with semi-symmetric non-metric connection is symmetric. Furthermore, we show that the Ricci tensor of a lightlike submanifold of Lorentzian space form is not parallel with respect to semi-symmetric non-metric connection.

Elastica on 2-dimensional de Sitter Space

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We derive differential equations for non-null elastic curves on 2-dimensional de Sitter space. Then, we solve these differential equations in terms of Jacobi's elliptic functions. Also, we give a relation between curvature and torsion of an elastic curve on a 2-dimensional de Sitter space.