Partially conformal vector fields

Oscar Palmas (Joint work with A. G. Colares)

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Abstract

We define the partially conformal vector fields and give a characterization of Lorentzian manifolds which admit this kind of fields as doubly warped products with two 1-dimensional factors.

These fields can be associated to foliations by $(n-1)$-umbilical hypersurfaces. As a particular case in Lorentzian space forms we have foliations by rotation hypersurfaces.

A natural question in this context is whether a given $(n-1)$-umbilical hypersurface is a leaf of such a foliation. We also give partial answers to this question.

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Previous works

As we may see, our general aim has a close relation with some characterization of totally umbilical hypersurfaces in spacetimes. In particular, we recall that

Alías-Romero-Sánchez and Montiel considered this problem in spacetimes satisfying the timelike convergence condition (Ricci curvature non negative on timelike [and null] directions) as well as those satisfying the weaker null convergence condition (Ricci curvature ≥ 0 along null directions) and CMC hypersurfaces.

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Previous works

They studied the class of ambient spacetimes $\bar{M}_1^{n+1}$ admitting a *closed conformal* timelike vector field K, which satisfies

$$\bar{\nabla}_u K = \phi u, \quad u \in TM,$$

for some real function ϕ.

Our way of generalization

As it is well-known, already in Riemannian space forms there are many hypersurfaces with constant mean curvature (or constant H_r) which are not totally umbilical; for example, rotational hypersurfaces in space forms.

In fact, Lorentzian space forms also admit plenty of rotational hypersurfaces, which are examples of $(n-1)$-umbilical hypersurfaces.

So our aim is to extend these results to this setting, namely, to a class of ambient spacetimes admitting a foliation by $(n-1)$-umbilical hypersurfaces.

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Our questions

In particular, we attacked the following questions:

- Given a spacetime $\bar{M}_1^{n+1}$, under which conditions it may be foliated by $(n-1)$-umbilical hypersurfaces?
- Given such a foliation, which conditions imply that a hypersurface with CMC (or constant H_r) is $(n-1)$-umbilical or, in particular, a leaf of the aforementioned foliation?

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An example

First we give an example of foliation of (open sets of) the de Sitter space. Recall that the scalar product in $\mathbb{R}_1^{n+2}$ is given by

$$\langle u, v\rangle = -u_0 v_0 + u_1 v_1 + \cdots + u_{n+1} v_{n+1}$$

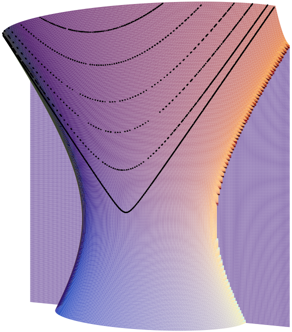
and that the de Sitter space is defined as

$$\mathbb{S}_1^{n+1} = \{\, u \in \mathbb{R}_1^{n+2} \mid |u|^2 = 1 \,\}$$

Hyperbolic cylinders

$$M = \{ p \in \mathbb{S}^{n+1}_1 \mid -x_0^2 + x_1^2 = -\sinh^2 r \},$$

where $r > 0$ and $n > 2$. Varying r we obtain a foliation, shown in the figure.



Closed partially conformal vector fields

Definition

Let $(\bar{M}_1^{n+1}, \bar{\nabla})$ be a spacetime and K be a timelike vector field defined on $\bar{M}$. We say that K is *closed, partially conformal* in $\bar{M}$ if and only if there is a vector field $W \in \mathfrak{X}(\bar{M})$ everywhere orthogonal to K such that

$$\bar{\nabla}_X K = \phi X \text{ for } X \in W^\perp \quad \text{and} \quad \bar{\nabla}_W K = \psi W$$

for some functions $\phi, \psi : \bar{M} \to \mathbb{R}$. W is called the vector field *associated* to K.

Example in $\mathbb{S}^{n+1}_1$

To obtain the foliation by hyperbolic cylinders, we take the unit vector field N normal to each of these hypersurfaces

$$N(p) = \frac{1}{\sinh\rho\cosh\rho}(x_0, \ldots, x_{n-1}, 0, 0) + (\tanh\rho)p$$

and prove that, if $e_0, \ldots, e_{n+1}$ is the canonical basis of $\mathbb{R}^{n+2}_1$, then

$$\nabla_{e_0+e_i} N = (\coth\rho)(e_0 + e_i), \quad i = 1, \ldots, n-1,$$

and

$$\nabla_{e_n - e_{n+1}} N = (\tanh\rho)(e_n - e_{n+1}).$$

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Basic properties

Take

$\bar{M}_1^{n+1}$: a spacetime possessing a closed partially conformal timelike vector field K,

and

$K^{\perp}$: the distribution defined by taking the orthogonal complement of K.

Theorem

$K^{\perp}$ is involutive and each leaf of the corresponding foliation $\mathcal{K}^{\perp}$ is a $(n-1)$-umbilical hypersurface with $n-1$ equal and constant principal curvatures.

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Relation with warped products

Note in the examples that these open subsets may be written as a warped product of the form $-I \times_f F \times J$, where I, J are intervals.

Theorem

Let $\bar{M}_1^{n+1}$ be a spacetime.

- *If $\bar{M} = -I \times_{f_1} F \times_{f_2} J$, then $K = f_1 \partial_0$ is a closed partially conformal timelike vector field defined on $\bar{M}$ (with $W = \frac{1}{f_2}\partial_{n+1}$).*
- *If $\bar{M}$ admits a closed partially conformal timelike vector field K and the vector field W is parallel, then $\bar{M}$ can be expressed locally as $-I \times_f F \times J$.*

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Is a given hypersurface a leaf?

Finally, we give conditions for a hypersurface with constant r-th mean curvature H_r to be a leaf of the foliation determined by a closed partially conformal vector field, thus being $(n-1)$-umbilical.

Our Theorem, first part

Theorem

Let $\bar{M}^{n+1} = -I \times_f F \times J$ be a spacetime with F compact. Suppose that the following strong null convergence condition holds:

$$K_F > \sup(ff'' - f'^2),$$

where K_F stands for the (not necessarily constant) sectional curvature of F.

Our Theorem, first part, continued

Let $M \subset \bar{M}$ be a spacelike hypersurface contained in a region $-[t_1, t_2] \times_f F \times J$ where f' does not vanish, and satisfying the following conditions:

1. M is everywhere transverse to K;
2. Denoting $\bar{M}_s = -I \times_f F \times \{s\}$ and $M_s = M \cap \bar{M}_s$, we have that M_s is compact and has constant r-th mean curvature H_r in $\bar{M}_s$ for $1 \leq r \leq n$.
3. If $\tilde{N}$ denotes a unit timelike vector field everywhere normal to M, there are constants a, b such that $\tilde{N} = aK + bW$.

Then M is $(n-1)$-umbilical in $\bar{M}$.

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Our Theorem, second part

Moreover, if

4. M has constant r-th mean curvature H_r, $1 \leq r \leq n$;
5. The leaves of the foliation determined by K have constant r-th mean curvature; and
6. There is a s_0 and a point p of M_{s_0} such that $\tilde{N}(p) = K(p)$ and locally, M lies above (or below) the leaf of $\mathcal{K}^{\perp}$ passing through p,

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Grazie!