Some remarks on the null sectional curvature

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(joint work with Stefan Haesen)

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- The null sectional curvature of π with respect to v is given by

$$K_v(p,\pi) = \frac{R(v,w,w,v)}{g(w,w)}.$$

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- The U-normalized null sectional curvature at $p \in M$ of the degenerate plane $\pi \subset T_pM$ is defined as

$$K^U(p,\pi) = K_v(p,\pi),$$

with $v \in (C_U M)_p = \{v \in T_pM | g(v,v) = 0 \text{ and } g(v,U_p) = -1\}$.

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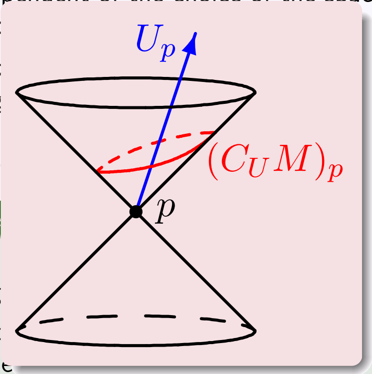

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- In the case of a time-oriented Lorentzian manifold, we can normalize $K_v(p, \pi)$ obtaining a well-defined function:

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- $K^U(p, \pi)$ is a purely Lorentzian concept.

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- We give a geometrical interpretation of the null sectional curvature in terms of the difference in length of two spacelike geodesics constructed from the degenerate plane.
- This interpretation is inspired in the one given by Levi-Civita (1917) for the sectional curvature in the Riemannian case.

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- Let (M^n, g) be a time-oriented Lorentzian manifold, $U \in TM$ a globally defined unitary timelike vector field and $\pi = \text{span}\{v, w\}$ a degenerate plane at $p \in M$ such that $v \in (C_U M)_p$ and $g(w, w) > 0$.

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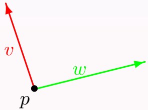

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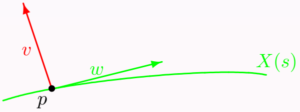

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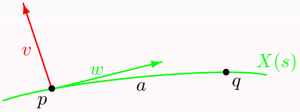

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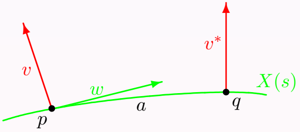

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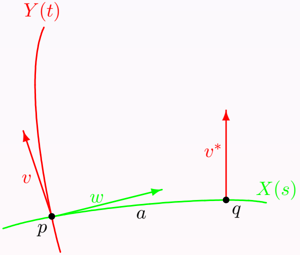

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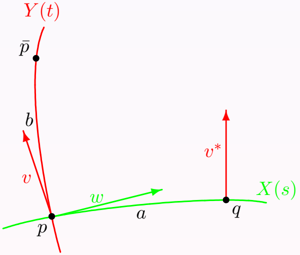

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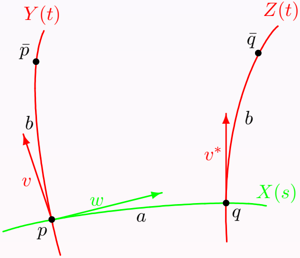

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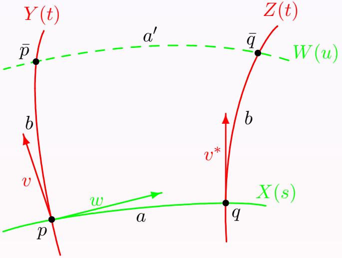

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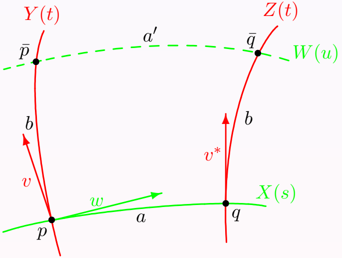

and we prove that

$$K^U(p,\pi) \simeq 6\,\frac{a' - a}{ab^2}.$$

Isotropic and spatially constant null sectional curvature

- The U-normalized null sectional curvature is said to be isotropic if it is only a point function, i.e., $K^U(p,\pi) = K^U(p)$ for all null planes $\pi \subset T_pM$.

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Proposition 1 (Harris, Koch-Sen)

Let $(M^n, g), n \geq 3$, be a time-orientable Lorentzian manifold and U a globally defined unitary timelike vector field. Then, the U-normalized null sectional curvature is isotropic if and only if the curvature tensor satisfies

i) $R(X,Y)Z = k(X \wedge_g Y)Z, \ \forall X, Y, Z \in U^\perp$,

ii) $R(X,U)U = \mu X$,

with $k, \mu : M \to \mathbb{R}$ and $K^U \equiv k + \mu$.

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- If the U-normalized null sectional curvature K^U is isotropic, it is said to be spatially constant if it is constant on the space orthogonal to the chosen timelike vector field U, i.e., $X[K^U] = 0$, for every $X \in U^\perp$.

Isotropic and spatially constant null sectional curvature

Theorem 1 (Harris, Karcher, Koch-Sen)

Let (M, g) be an $n(\geq 4)$-dimensional Lorentzian manifold and let U be a unitary timelike vector field on M. Suppose that the U-normalized null sectional curvature is non-zero, isotropic and spatially constant. Then, g is locally a Robertson-Walker metric.

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  - In fact, Palomo (2007) showed that the 3-dimensional sphere $\mathbb{S}^3$ endowed with a certain Lorentzian metric has non-zero constant null sectional curvature, whereas the distribution $U^{\perp}$ is not integrable and hence is not a Robertson-Walker space.

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   - In fact, Palomo (2007) showed that the 3-dimensional sphere $\mathbb{S}^3$ endowed with a certain Lorentzian metric has non-zero constant null sectional curvature, whereas the distribution $U^\perp$ is not integrable and hence is not a Robertson-Walker space.
- However, we can prove that the above theorem can be extended to the 3-dimensional case under some extra suitable assumptions: we will ask the space to be conformally flat.

Isotropic and spatially constant null sectional curvature

Lemma 1

A time-oriented, 3-dimensional Lorentzian manifold (M, g), with non-vanishing isotropic null sectional curvature is conformally flat if, and only if,

$$\nabla^{\perp}[k] = 0,$$

and

$$\nabla_{\xi} U = -\frac{U[k]}{2(k+\mu)}\xi^{\perp} + g(\xi, U)\nabla^{\perp}[\ln(k+\mu)],$$

with $\xi \in TM$, $\xi^{\perp}$ its projection on the 2-space perpendicular to U and $\nabla^{\perp}[f]$ the projection of the gradient of a function f on this 2-space.

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Proof. (M^3, g) is conformally flat if and only if the Schouten tensor

$$S = \mathrm{Ric} - \frac{\tau}{4} g$$

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- The proof follows easily by observing that by Proposition 1

$$\mathrm{Ric} = (k-\mu)g + (k+\mu)U^{\flat} \otimes U^{\flat},$$

being $U^{\flat}$ the one-form metrically equivalent with U.

Isotropic and spatially constant null sectional curvature

- As a first result we can prove the following,

Corollary 1

Every time-oriented, conformally flat, 3-dimensional Lorentzian manifold (M, g) with non-vanishing isotropic null sectional curvature can be locally and isometrically embedded as a quasi-umbilical hypersurface in a 4-dimensional pseudo-Euclidean space.

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Proof.

- The Gauss equation for a hypersurface in a 4-dimensional pseudo-Euclidean space can be written in general as

$$R(X, Y, Z, W) = \varepsilon\,(h(X, W)h(Y, Z) - h(X, Z)h(Y, W))$$

for any $X, Y, Z, W \in TM$, being h the second fundamental form of the hypersurface and ε the signature of the normal direction.

Proof of Corollary 1

- Using the expression of the Riemann curvature tensor given in Proposition 1, the Gauss equation is satisfied for

$$h(X,Y) = \alpha g(X,Y) + (\alpha+\beta) g(U,X) g(U,Y),$$

with $k = \varepsilon\alpha^2$ and $\mu = \varepsilon\alpha\beta$.

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- Finally, from the expression for the second fundamental form we conclude that M is quasi-umbilical.

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Let (M, g) be a 3-dimensional Lorentzian manifold. Then, (M, g) is conformally flat and has non-vanishing spatially constant, isotropic null sectional curvature if, and only if, (M, g) is locally a Robertson-Walker space.

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$$\nabla_\xi U = -\frac{U[k]}{2(k+\mu)}\xi^\perp + g(\xi, U)\nabla^\perp[\ln(k+\mu)],$$

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if follows easily that the vector field U is irrotational, geodesic and shear-free.

- Therefore, we can apply the Frobenius theorem, concluding that the distribution $U^\perp$ is integrable.

Proof of Theorem 2

- Moreover, the second fundamental form $\hat{h}$ of the spacelike 2-surfaces Σ of the distribution is given by

$$\hat{h} = -\frac{U[k]}{2(k+\mu)}\hat{g},$$

being $\hat{g}$ the induced metric on these surfaces.

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being $\hat{g}$ the induced metric on these surfaces.
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- Summing up, the integral curves of U are geodesics whose perpendicular spaces integrate to form 2-dimensional surfaces of constant curvature.
- Then, the metric of M can be locally be written as
$$ds^2 = -dt^2 + f^2(t)\left\{dr^2 + \Sigma^2(r,\kappa)d\theta^2\right\},$$
with $\Sigma(r,\kappa) = \sin r, r$ or $\sinh r$ if $\kappa = 1, 0$ or -1, respectively.

THANK YOU VERY MUCH FOR YOUR ATTENTION!