Surfaces in exotic AdS spaces

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Joint work in progress with Luis Alías (Universidad de Murcia, Spain) and J. Hinojosa (UFRPE, Brazil)

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In this sense, it is worth to mention some related contributions by I. Taimanov, B. Daniel, F. Mercuri, P. Piccione and collaborators.

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$$E_1 = \frac{1}{\tau} Z\varsigma_1, \quad E_2 = Z\varsigma_2, \quad E_3 = Z\varsigma_3 \tag{6}$$

is g_τ-orthonormal.

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This means that g_τ is a stationary non-static metric in M. We describe that distribution in terms of the Hopf fibration $\pi : SU_{1,1} \to \mathbb{H}^2$.

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$$\partial_z\psi_1 = \omega_z\psi_1 + i\big(\tau - \frac{1}{\tau}\big)\psi_1^2\bar{\psi}_2 - e^{-\omega}\,\frac{q}{2}\psi_2, \tag{35}$$

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Let J and ∇^{Σ} be the complex structure and Riemannian connection in Σ associated to I. We refer to the equations

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Theorem

Let (Σ, I) be a simply-connected Riemannian surface, $II \in \Gamma(T^*\Sigma \odot T^*\Sigma)$, $\beta \in \Gamma(T\Sigma)$ and $\alpha \in C^\infty(\Sigma, \mathbb{R})$ satisfying the compatibility equations above. Then, there exists an isometric immersion $X : \Sigma \to M$ such that

$$I(V, W) = \langle X_* V, X_* W \rangle, \tag{44}$$

$$II(V, W) = \langle \nabla_{X_* V} X_* W, N \rangle, \tag{45}$$

where $V, W \in \Gamma(T\Sigma)$ and $p \in \Sigma \mapsto N|_{X(p))} \in T_{X(p)}M$ is a normal unit vector field along X given by

$$E_1|_{X(p)} = \alpha(p) N|_{X(p)} + \beta(p). \tag{46}$$

Initial value problem

Maximal surfaces

From now on, we assume that $H = 0$.

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and denoting $f = \bar{\psi}_2^2$, we may recover the results above in terms of f and g

The novelty is the following one

Proposition

If the immersion $X : \Sigma \to (SU_{1,1}, g_\tau)$ is maximal, then it holds that

$$g_{z\bar{z}} + 2\frac{2\varsigma^2 - (1+|g|^2)}{(1+|g|^2)^2 - 4\varsigma^2|g|^2}\bar{g}g_z g_{\bar{z}} = 0, \tag{50}$$

where $\varsigma = 1/\tau$.

In order to state a converse for this Proposition, we denote

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$$\mathrm{d}s_\tau^2 = \frac{|\mathrm{d}w|^2}{(1+|w|^2)^2 - 4\varsigma^2|w|^2}, \quad w \in \mathbb{D}_\tau. \tag{52}$$

Theorem

The Gauss map $g : \Sigma \to (\mathbb{D}_\tau, ds^2_\tau)$ *of a maximal immersion* $X : \Sigma \to (SU_{1,1}, g_\tau)$ *is a harmonic map.*

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is holomorphic when $H = 0$.

The next result states that we may determine a maximal immersion of Σ into $SU_{1,1}$ starting with a harmonic map g with target in $\mathbb{D}_\tau$.

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Theorem

Let Σ be a simply connected Riemannian surface. Let $g : \Sigma \to (\mathbb{D}_\tau, ds^2_\tau)$ be a harmonic never holomorphic map. Given $z_0 \in \Sigma$ and $Z_0 \in SU_{1,1}$, there exists a unique conformal maximal immersion $X : \Sigma \to SU_{1,1}$ whose Gauss map is g and such that $X(z_0) = Z_0$.

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& + \frac{(1+g^2)\bar{g}_z}{(1+|g|^2)^2 - 4\varsigma^2|g|^2}\,\hat{e}_2 \\
& + \frac{(1-g^2)\bar{g}_z}{(1+|g|^2)^2 - 4\varsigma^2|g|^2}\,\hat{e}_3.
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\begin{aligned}
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& + \frac{(1+g^2)\bar{g}_z}{(1+|g|^2)^2-4\varsigma^2|g|^2}\,\hat{e}_2 \\
& + \frac{(1-g^2)\bar{g}_z}{(1+|g|^2)^2-4\varsigma^2|g|^2}\,\hat{e}_3.
\end{aligned}
\tag{54}
$$

Moreover, there exists a family of maximal immersions $X_\vartheta : \Sigma \to SU_{1,1}$, $\vartheta \in \mathbb{R}$, represented by the harmonic maps $g^\vartheta = e^{i\vartheta} g$, such that $X_0 = X$.

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with components $\Phi = (\Phi_-, \Phi_+)$ we define a family of maximal immersions $X_\lambda : \Sigma \to SU_{1,1}$ by

$$X_\lambda = \Phi^+(\lambda)\Phi_-^{-1}(\lambda), \quad \lambda \in \mathbb{S}^1.$$

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$$\mathrm{d}\vartheta = -\frac{1}{2}[\vartheta, \vartheta]. \tag{57}$$

Flat Lorentz surfaces

Theorem (–, F. Vitório)

Let Σ be a simply-connected flat Lorentz surface. An isometric immersion $X : \Sigma \to SL(2,\mathbb{R})$ whose mean curvature satisfies $H \geq 1$ may be represented by

$$X = F_1 F_2^T,$$

where $F_i : \Sigma \to SL(2,\mathbb{R}),\ i = 1, 2$, are given by

$$\mathrm{d}F_1 = F_1 \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix} \mathrm{d}u \quad \textit{and} \quad \mathrm{d}F_2 = F_2 \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} \mathrm{d}v$$

for any Lorentz-holomorphic function h and Lorentz-antiholomorphic function g defined in Σ in terms of a null coordinate system (u, v)

Thank you!