Marginally Trapped Surfaces in Minkowski 4-Space and 1-Dimensional Isometry Groups

Miguel Ortega

Universidad de Granada

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Summary

References

This talk is a summary of my works with Stefan Haesen (Simon Steven Institute for Geometry, The Nederlands)

S. Haesen, –, *Boost invariant marginally trapped surfaces in Minkowski space*, Class. Quant. Grav **24**(2007), 5441-5452

S. Haesen, –, *Screw Invariant Marginally Trapped Surfaces in Minkowski 4-space*, J. Math. Anal. Appl. **355**(2009), 639-648

S. Haesen, –, *Marginally trapped surfaces in Minkowski 4-space invariant under a rotation subgroup of the Lorentz group*, To appear in Gen. Rel. Grav (2009).

In Physics

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Given two future-pointing lightlike normal vectors $\mathbf{k}$ and $\mathbf{l}$ along S, with $\langle \mathbf{k}, \mathbf{l} \rangle = -1$

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A Marginally Outer Trapped Surface (MOTS)

is a surface whose mean curvature is always parallalel to either $\mathbf{k}$ or to $\mathbf{l}$. In particular, it might be zero.

In Physics

In certain spacetimes

If the mean curvature of a trapped surface points to the past, then the spacetime is expanding.

Viceversa, if the mean curvature of a trapped surface points to the future, then the the spacetime is contracting.

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In a spacetime, a region containing black holes is supposed to be surrounded by a 3-dim hypersurface which is foliated by MOTS. This hypersurface is called *generalized apparent horizon*.

Surfaces in Minkowski 4-Space

The (Lorentz-)Minkowski 4-space $\mathbb{L}^4$ is $\mathbb{R}^4$ endowed with the metric $\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. The time orientation is given by ∂_{x_1}.

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A maximal surface

is a surface whose mean curvature is everywhere zero.

We put $\overrightarrow{\mathrm{k}} = (1, 1, 0, 0)/\sqrt{2}$, $\overrightarrow{\mathrm{l}} = (1, -1, 0, 0)/\sqrt{2}$, $\overrightarrow{\mathbf{e}}_1 = (1, 0, 0, 0)$, $\overrightarrow{\mathbf{e}}_2 = (0, 1, 0, 0)$, $\overrightarrow{\mathbf{e}}_3 = (0, 0, 1, 0)$ and $\overrightarrow{\mathbf{e}}_4 = (0, 0, 0, 1)$.

We consider spacelike surfaces in $\mathbb{L}^4$ which are invariant under the following subgroups of direct, linear isometries of $\mathbb{L}^4$.

$$\textbf{Boost } \mathbf{G}_1 = \left\{ B_\theta = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

w.r.t. $\{\overrightarrow{\mathbf{e}}_1, \overrightarrow{\mathbf{e}}_2, \overrightarrow{\mathbf{e}}_3, \overrightarrow{\mathbf{e}}_4\}$.

$$\textbf{Rotational } \mathbf{G}_2 = \left\{ B_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

w.r.t. $\{\overrightarrow{\mathbf{e}}_1, \overrightarrow{\mathbf{e}}_2, \overrightarrow{\mathbf{e}}_3, \overrightarrow{\mathbf{e}}_4\}$,

$$\textbf{Screw } \mathbf{G}_3 = \left\{ B_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \theta^2 & 1 & \sqrt{2}\theta & 0 \\ \sqrt{2}\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

w.r.t. $\{\overrightarrow{k}, \overrightarrow{l}, \overrightarrow{\mathbf{e}}_3, \overrightarrow{\mathbf{e}}_4\}$.

A boost invariant spacelike surface S is contained in either

$$\mathcal{E}_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_1 > 0,\ x_1^2 > x_2^2\},\ \text{or in}$$

$$\mathcal{E}_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_1 < 0,\ x_1^2 > x_2^2\}.$$

We assume $S \subset \mathcal{E}_1$.

Given a point $p = a_k \overrightarrow{k} + a_l \overrightarrow{l} + a_3 \overrightarrow{\mathbf{e}}_3 + a_4 \overrightarrow{\mathbf{e}}_4 \in \mathbb{L}^4$, a screw invariant spacelike surface is contained in either

$$\mathcal{R}^+ = \{p \in \mathbb{L}^4 : a_k > 0\}, \quad \mathcal{R}^- = \{p \in \mathbb{L}^4 : a_k < 0\}.$$

We assume that a screw invariant spacelike surface $S \subset \mathcal{R}^+$.

We put

$$\begin{aligned}
\mathcal{P}_1 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_2 = 0, x_1 > 0\} \subset \mathcal{E}_1, \\
\mathcal{P}_2 &= \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_4 = 0, x_3 > 0\}, \\
\mathcal{P}_3 &= \{(a_k, a_l, a_3, a_4) \in \mathbb{L}^4 : a_3 = 0, a_k > 0\} \subset \mathcal{R}^+.
\end{aligned}$$

On S we introduce a (local) parametrization $X(s, \theta)$ on a dense open subset Σ_α of S as follows,

$$\Sigma_\alpha = \{X(s, \theta) = \alpha(s) \cdot B_\theta \; : \; s \in I, \theta \in \mathbb{R}\},$$

where α is a unit spacelike curve $\alpha : I \subset \mathbb{R} \to \mathcal{P}_i$, $(\langle \alpha', \alpha' \rangle = 1)$ according to $i = 1, 2, 3$ (boost, rotational or screw cases.)

- $\{\eta_1, \eta_2\}$ an orthonormal frame of $T^{\perp}\Sigma_\alpha$ in $\mathbb{L}^4$, η_1 future-pointing timelike, η_2 spacelike.
- A_i Weingarten endomorphism associated with η_i, $i = 1, 2$.
- The mean curvature vector is

$$\mathbf{H} = \frac{1}{2}\Big(- \mathrm{Tr}(A_1)\eta_1 + \mathrm{Tr}(A_2)\eta_2 \Big).$$

By the local theory of surfaces, we compute the coefficients of the first and the second fundamental forms...

$$E = \langle X_s, X_s \rangle, \quad F = \langle X_s, X_\theta \rangle, \quad G = \langle X_\theta, X_\theta \rangle,$$

$$i = 1, 2, \quad e_i = \langle X_{ss}, \eta_i \rangle, \quad f_i = \langle X_{s\theta}, \eta_i \rangle, \quad g_i = \langle X_{\theta\theta}, \eta_i \rangle.$$

$$h_i = \mathrm{tr}_{\langle , \rangle}(A_i) = \frac{e_i G - 2f_i F + g_i E}{EG - F^2} \;\Rightarrow\; \mathbf{H} = \frac{1}{2}\Big(- h_1\eta_1 + h_2\eta_2 \Big).$$

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Thus,

$$0 = \langle \mathbf{H}, \mathbf{H}\rangle \Longleftrightarrow h_1 = \pm h_2 \textit{ pointwise}.$$

Sign Choice Lemma

Given a smooth function $\rho : I \to \mathbb{R}$, there exist functions $\varepsilon : I \to \{-1, 1\} \subset \mathbb{R}$ such that $\varepsilon\rho$ is smooth.

The simplest choice consists of setting ε as a constant.

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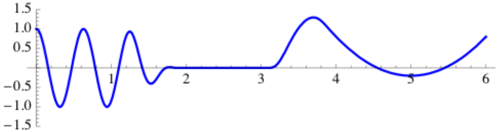

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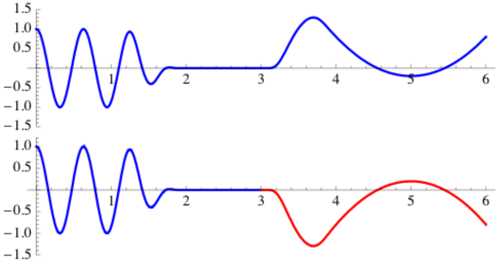

Note that ε is not always unique!

Remark

$\langle \mathbf{H}, \mathbf{H} \rangle = 0$ iff $h_1 = \varepsilon h_2$ for a suitable function ε.

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Main problem

To solve equation $h_1 = \varepsilon h_2$.

Boost case

We consider $\alpha : I \subset \mathbb{R} \to \mathcal{P}_1$, $\alpha = (\alpha_1, 0, \alpha_3, \alpha_4)$ with $\alpha_1 > 0$ and $\langle \alpha', \alpha' \rangle = 1$. The parametrization of Σ_α is

$$X(s, \theta) = \Big(\alpha_1(s)\cosh(\theta),\ \alpha_1(s)\sinh(\theta),\ \alpha_3(s),\ \alpha_4(s)\Big),$$

$s \in I$, $\theta \in \mathbb{R}$.

A globally defined orthonormal basis of the normal bundle of Σ_α is given by

$$\eta_1 = \frac{\Big(\cosh(\theta)(1+(\alpha_1')^2),\ \sinh(\theta)(1+(\alpha_1')^2),\ \alpha_1'\alpha_3',\ \alpha_1'\alpha_4'\Big)}{\sqrt{1+(\alpha_1')^2}},$$

$$\eta_2 = \frac{1}{\sqrt{1+(\alpha_1')^2}}\Big(0, 0, -\alpha_4',\ \alpha_3'\Big),$$

with η_1 future-pointing time-like and η_2 space-like.

$$h_1 = -\frac{1+(\alpha_1')^2+\alpha_1\alpha_1''}{\alpha_1\sqrt{1+(\alpha_1')^2}}, \quad h_2 = \frac{-\alpha_4'\alpha_3''+\alpha_3'\alpha_4''}{\sqrt{1+(\alpha_1')^2}}.$$

Given a smooth function $\alpha_1 > 0$, we define

$$\rho = -\frac{1 + (\alpha_1')^2 + \alpha_1 \alpha_1''}{\alpha_1}.$$

We choose a function ε as in Sign Choice Lemma, such that $\varepsilon\rho$ is smooth. We define

$$\text{Angle function: } \xi = \int \frac{\varepsilon\, \rho}{1 + (\alpha_1')^2} ds,$$

$$\alpha_3 = \int \sqrt{1 + (\alpha_1')^2} \cos(\xi) ds, \text{ and } \alpha_4 = \int \sqrt{1 + (\alpha_1')^2} \sin(\xi) ds.$$

Finally, we consider the curve

$$\alpha = (\alpha_1, 0, \alpha_3, \alpha_4) \left(\Rightarrow \langle \alpha', \alpha' \rangle = 1. \right)$$

Theorem 1

Any boost invariant, marginally trapped surface S (in $\mathcal{E}_1$) admits a dense open subset of the form Σ_α whose profile curve α is as above, with

$$\mathbf{H} = \frac{\rho}{2\sqrt{1+(\alpha_1')^2}}(-\varepsilon\eta_1 + \eta_2).$$

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$$\mathbf{H} = \frac{\rho}{2\sqrt{1+(\alpha_1')^2}}(-\varepsilon\eta_1 + \eta_2).$$

In addition, given any other unit spacelike curve $\beta = (\beta_1, 0, \beta_3, \beta_4)$ such that $\beta_1 = \alpha_1$, there exists an affine isometry F of $\mathbb{L}^4$ such that $F(\Sigma_\alpha) = \Sigma_\beta$.

Corollary 1

Let Σ_α be a boost invariant maximal surface (in $\mathcal{E}_1$). Then, a unit profile curve is given by

$$\alpha(s) = \left(f(s), 0, \cos(\xi_0)\sqrt{a_1}\,\arctan\left(\frac{s+a_2}{f(s)}\right), \right.$$

$$\left. \sin(\xi_0)\sqrt{a_1}\,\arctan\left(\frac{s+a_2}{f(s)}\right) \right),$$

where $f(s) = \sqrt{a_1 - (s+a_2)^2}$, and a_1, a_2, $\xi_0 \in \mathbb{R}$, $a_1 > 0$, being integration constants.

A Gluing Algorithm

Proposition 1

Take two unit curves, $\alpha^1, \alpha^2 :]p_i, q_i[\to \mathcal{P}_1$, with $-\infty \leqslant p_1 < q_1 < p_2 < q_2 \leqslant +\infty$, such that Σ_{α^i} are marginally trapped. Let $0 < d \leqslant \frac{1}{4} \min\{q_1 - p_1, p_2 - q_1, q_2 - p_2\}$.

Then, there exist a unit spacelike curve $\beta :]p_1, q_2[\to \mathcal{P}_1$, $\beta = (\beta_1, 0, \beta_3, \beta_4)$ which is the profile curve of a marginally trapped surface Σ_β such that

1. There exists a direct affine isometry $F_1 : \mathbb{L}^4 \to \mathbb{L}^4$ with $F_1 \circ {\alpha^1}_{|]p_1, q_1 - d[} = \beta_{|]p_1, q_1 - d[}$.
2. There exists a direct affine isometry $F_2 : \mathbb{L}^4 \to \mathbb{L}^4$ such that $F_2 \circ {\alpha^2}_{|]p_2 + d, q_2[} = \beta_{|]p_2 + d, q_2[}$.
3. The (intermediate) surface $\Sigma_{\beta_{|]q_1, p_2[}}$ is maximal.

New surfaces

This gluing method gives the possibility to construct a surface S satisfying the following conditions:

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1. S is boost invariant, marginally trapped, with (infinitely many countable) regions $\{S_n : n \in N \subset \mathbb{N}\}$ where its mean curvature vector $\mathbf{H} \neq 0$.
2. The mean curvature vector of each region S_n can be set either future or past-pointing, as *desired*.
3. Among two *adjacent* regions S_n and S_{n+1}, there is an open subset which is maximal, i.e. $\mathbf{H} = 0$.

Rotational Case

Recall

$$\mathcal{P}_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 : x_4 = 0, x_3 > 0\}$$

$$\textbf{Rotational} : \mathbf{G}_2 = \left\{ B_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

Given a rotational invariant surface S, we introduce a parametrization $X(t, \theta)$ with a smooth spacelike unit curve $\alpha : I \subset \mathbb{R} \to \mathcal{P}_2$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, 0)$, i. e.

$$\Sigma_\alpha = \{X(t, \theta) = \alpha(t) \cdot B_\theta : t \in I, \theta \in \mathbb{R}\} \subset S,$$

$$X(t, \theta) = \Big(\alpha_1(t), \alpha_2(t), \alpha_3(t)\cos(\theta), \alpha_3(t)\sin(\theta)\Big), \; t \in I, \theta \in \mathbb{R}.$$

A global orthonormal frame of the normal bundle of Σ_α is

$$\begin{aligned}\eta_1 &= \frac{1}{\sqrt{1+(\alpha_1')^2}}\Big(1+(\alpha_1')^2, \alpha_1'\alpha_2', \alpha_1'\alpha_3'\cos(\theta), \alpha_1'\alpha_3'\sin(\theta)\Big),\\ \eta_2 &= \frac{1}{\sqrt{1+(\alpha_1')^2}}\Big(0, -\alpha_3', \alpha_2'\cos(\theta), \alpha_2'\sin(\theta)\Big),\end{aligned}$$

with η_1 future-pointing timelike and η_2 spacelike.

$$h_1 = -\frac{\alpha_1'\,\alpha_3' + \alpha_3\,\alpha_1''}{2\alpha_3\sqrt{1+(\alpha_1')^2}} \quad \text{and} \quad h_2 = -\frac{\alpha_2' + \alpha_3(\alpha_2''\,\alpha_3' - \alpha_2'\,\alpha_3'')}{2\alpha_3\sqrt{1+(\alpha_1')^2}}.$$

We recall $0 = \langle \mathbf{H}, \mathbf{H}\rangle \iff h_1 = \varepsilon h_2$.

Surfaces of type A

Given a smooth function $\tau : I \subset (0, \infty) \to \mathbb{R}$, choose a function $\varepsilon : I \to \{1, -1\}$ such that $\varepsilon\tau$ is also smooth. Define the coordinate functions $\alpha_i : I \to \mathbb{R}$, $i = 1, 2, 3$, as follows

$$\alpha_1(t) = \int \varepsilon(t)\tau(t)dt, \ \ \alpha_2(t) = \int \tau(t)dt, \ \ \alpha_3(t) = t.$$

Then, the curve $\alpha = (\alpha_1, \alpha_2, \alpha_3, 0)$ defines a marginally trapped surface Σ_α whose mean curvature vector is

$$\mathbf{H} = \frac{\tau + t\tau'}{2t\sqrt{1+\tau^2}} \left(\varepsilon\eta_1 - \eta_2\right).$$

Surfaces of type B

Given a smooth positive function $\alpha_3 : I \subset \mathbb{R} \to \mathbb{R}$, and two constants $\varepsilon_1, \varepsilon_2 = \pm 1$, define

$$\xi(t) = \int \frac{dt}{\alpha_3(t)},$$

$$\alpha_1(t) = \varepsilon_1 \int \left\{ \sinh(\xi(t)) - \alpha_3'(t) \cosh(\xi(t)) \right\} dt,$$

$$\alpha_2(t) = \varepsilon_2 \int \left\{ \cosh(\xi(t)) - \alpha_3'(t) \sinh(\xi(t)) \right\} dt.$$

Then, the curve $\alpha = (\alpha_1, \alpha_2, \alpha_3, 0)$ defines a marginally trapped surface whose mean curvature vector is

$$\mathbf{H} = \frac{\cosh(\xi(t)) \left(1 - \alpha_3'(t)^2 - \alpha_3(t)\, \alpha_3''(t)\right)}{2\alpha_3(t) \sqrt{1 + \alpha_1'(t)^2}} \left(\varepsilon_1 \eta_1 - \varepsilon_2 \eta_2\right).$$

Theorem 2

Let Σ_α be a rotational invariant, marginally trapped surface in $\mathbb{L}^4$. Then, the the surface is locally congruent to a surface of type A or of type B.

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Let Σ_α be a rotational invariant, marginally trapped surface in $\mathbb{L}^4$. Then, the the surface is locally congruent to a surface of type A or of type B.
In addition, in Case B, given two unit curves $\alpha = (\alpha_1, \alpha_2, \alpha_3, 0)$ and $\beta = (\beta_1, \beta_2, \beta_3, 0)$, such that $\alpha_3 = \beta_3$, there exists an affine isometry F of $\mathbb{L}^4$ satisfying $F(\Sigma_\alpha) = \Sigma_\beta$.

Corollary 2

A rotational invariant, spacelike surface in $\mathbb{L}^4$ is maximal if, and only if, it is locally congruent to a surface Σ_α whose profile curve $\alpha = (\alpha_1, \alpha_2, \alpha_3, 0)$, is given by one of the following cases:

1. Given $a > 0$, $b, c \in \mathbb{R}$ and $\varepsilon_1, \varepsilon_2 = \pm 1$,

$$\alpha_1(t) = a\,\varepsilon_1 \ln(t) + b, \quad \alpha_2(t) = a\,\varepsilon_2 \ln(t) + c, \quad \alpha_3(t) = t.$$

2. Given $\varepsilon_1, \varepsilon_2 = \pm 1$, and $a, b \in \mathbb{R}$,

$$\begin{aligned}
\alpha_1(t) &= \frac{\varepsilon_1}{2}(a^2 + 1 - b) \ln\left|a + t + \sqrt{t^2 + 2at + b}\right|, \\
\alpha_2(t) &= \frac{\varepsilon_2}{2}(a^2 - 1 - b) \ln\left|a + t + \sqrt{t^2 + 2at + b}\right|, \\
\alpha_3(t) &= \sqrt{t^2 + 2at + b}.
\end{aligned}$$

A Gluing Algorithm

Proposition 2

Let Σ_{α^r}, $r = 1, 2$ be two surfaces of type A and/or B as in Theorem 2 with profile curves $\alpha^r : (a_r, b_r) \subset \mathbb{R} \longrightarrow \mathcal{P}_2$, $r = 1, 2$, with $b_1 < a_2$. Assume that there is $\omega \geqslant 0$ such that the associated functions ε^r are constant on the intervals $(b_1 - \omega, b_1)$ and $(a_2, a_2 + \omega)$, respectively (Type A: $\varepsilon^r = \varepsilon$; type B, $\varepsilon^r = \varepsilon_1$). Then, there exist two affine isometries $F_r : \mathbb{L}^4 \longrightarrow \mathbb{L}^4$, $r = 1, 2$, and a unit spacelike curve $\gamma : (a_1, b_2) \longrightarrow \mathcal{P}_2$, satisfying

1. the surface Σ_γ is rotational invariant and marginally trapped;
2. $F_1\left(\Sigma_{\alpha^1|_{(a_1, b_1 - \omega)}}\right) = \Sigma_{\gamma|_{(a_1, b_1 - \omega)}}$,
3. $F_2\left(\Sigma_{\alpha^2|_{(a_2 + \omega, b_2)}}\right) = \Sigma_{\gamma|_{(a_2 + \omega, b_2)}}$,
4. $\Sigma_{\gamma|_{(b_1, a_2)}}$ is maximal.

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2. Each region S_n can be of type either A or B.
3. The mean curvature vector of each region S_n can be set either future or past-pointing, as *desired*.
4. Among two *adjacent* regions S_n and S_{n+1}, there is an open subset which is maximal, i.e. $\mathbf{H} = 0$.

Screw Case

Recall

$$\textbf{Screw } \mathbf{G}_3 = \left\{ B_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \theta^2 & 1 & \sqrt{2}\theta & 0 \\ \sqrt{2}\theta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;:\; \theta \in \mathbb{R} \right\},$$

w.r.t. $\{\overrightarrow{k}, \overrightarrow{l}, \overrightarrow{\mathbf{e}}_3, \overrightarrow{\mathbf{e}}_4\}$.

$$\mathcal{P}_3 = \left\{ a_k \overrightarrow{k} + a_l \overrightarrow{l} + a_3 \overrightarrow{\mathbf{e}}_3 + a_4 \overrightarrow{\mathbf{e}}_4 \in \mathbb{L}^4 \;: a_3 = 0,\; a_k > 0 \right\}.$$

If S is a screw invariant spacelike surface in $\mathcal{R}^+ \subset \mathbb{L}^4$, we introduce a parametrization $X(t,\theta)$ on a dense open subset Σ_α of S as follows: $\Sigma_\alpha = \{X(t,\theta) = \alpha(t) \cdot B_\theta : t \in I, \theta \in \mathbb{R}\} \subset S$, where

$$\alpha : I \subset \mathbb{R} \to \mathcal{P}_3, \quad \langle \alpha', \alpha' \rangle = 1, \quad \alpha = \alpha_k \overrightarrow{k} + \alpha_l \overrightarrow{l} + \alpha_4 \overrightarrow{\mathbf{e}}_4,$$

$$X(t,\theta) = \alpha_k(t)\overrightarrow{k} + \left(\theta^2 \alpha_k(t) + \alpha_l(t)\right)\overrightarrow{l} + \sqrt{2}\theta\alpha_k(t)\overrightarrow{\mathbf{e}}_3 + \alpha_4(t)\overrightarrow{\mathbf{e}}_4.$$

Type I Surfaces

Given $\alpha_{k0}, \delta, t_0 \in \mathbb{R}$, with $\delta = \pm 1$, $\alpha_{k0} > 0$, define $\alpha_k, \alpha_l, \alpha_4 : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\alpha_k(t) = \alpha_{k0}$, $\alpha_4(t) = \delta t + t_0$ and $\alpha_l(t)$ is any smooth function. Then, the curve $\alpha = \alpha_k \overrightarrow{k} + \alpha_l \overrightarrow{l} + \alpha_4 \overrightarrow{e}_4$ is unit spacelike and defines a surface Σ_α with

$$\mathbf{H} = \frac{1 + \alpha_{k0}\, \alpha_l''(t)}{2\, \alpha_{k0}} \overrightarrow{l}.$$

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$$\mathbf{H} = \frac{1 + \alpha_{k0}\, \alpha_l''(t)}{2\, \alpha_{k0}} \overrightarrow{l}.$$

Type II Surfaces

Given $\rho : I \subset \mathbb{R} \rightarrow \mathbb{R}$ and $\varepsilon : I \subset \mathbb{R} \rightarrow \{\pm 1\}$, such that ρ and $\varepsilon\rho$ are smooth, and for constants α_{k0}, α_{l0}, α_{40}, α_{41}, $f_0 \in \mathbb{R}$, $\alpha_{k0} > 0$, define $f, \alpha_k, \alpha_4, \alpha_l : I \subset \mathbb{R} \rightarrow \mathbb{R}$ as follows,

$$
\begin{aligned}
f(t) &= \int_{t_0}^{t} \varepsilon(r)\rho(r)dr + f_0, \ \ \alpha_k(t) = \sqrt{\alpha_{k0} + \int_{t_0}^{t} \exp(f(s))ds}, \\
\alpha_4(t) &= \alpha_{40} + \alpha_{41}\alpha_k(t) \\
&\quad + \int_{t_0}^{t} \frac{\exp(f(s))}{\alpha_k(s)} \left(\int_{t_0}^{s} \frac{\alpha_k(w)\rho(w)}{\exp(f(w))} dw \right) ds, \\
\alpha_l(t) &= \alpha_{l0} - \int_{t_0}^{t} \frac{2\alpha_k(s)}{\exp(f(s))} ds \\
&\quad + \int_{t_0}^{t} \left\{ \frac{\alpha_k(s)}{4\exp(f(s))} \left(\alpha_{k0} + \int_{t_0}^{s} \frac{2\alpha_k(w)\rho(w)}{\exp(f(w))} dw \right)^2 \right\} ds.
\end{aligned}
$$

Then, the curve $\alpha = \alpha_k \overrightarrow{k} + \alpha_l \overrightarrow{l} + \alpha_4 \overrightarrow{\mathbf{e}}_4$ is unit spacelike and defines a surface Σ_α with

$$\mathbf{H} = \frac{\rho(t)}{2} (\varepsilon \eta_1 + \eta_2),$$

where

$$\begin{aligned}
\eta_1 &= \alpha_k'(t) \overrightarrow{k} + \left(\frac{1}{\alpha_k'(t)} + \theta^2 \alpha_k'(t) + \alpha_l'(t) \right) \overrightarrow{l} \\
&\quad + \sqrt{2}\theta \alpha_k'(t) \overrightarrow{\mathbf{e}}_3 + \alpha_4'(t) \overrightarrow{\mathbf{e}}_4, \\
\eta_2 &= \left(\alpha_{41} + 2 \int_{t_0}^{t} \frac{2\alpha_k(w)\rho(w)}{\exp(f(w))} dw \right) \overrightarrow{l} + \overrightarrow{\mathbf{e}}_4,
\end{aligned}$$

is an orthonormal frame of the normal bundle, with η_1 timelike and η_2 spacelike.

Theorem 3

Let S be a screw invariant, marginally trapped spacelike surface in $\mathcal{R}^+$. Then, S is locally congruent to a surface Σ_α of type I or type II.

Corollary 3

Let S be a maximal screw invariant spacelike surface in $\mathcal{R}^+$. Then, S is locally congruent to a surface Σ_α whose profile curve $\alpha = \alpha_k \overrightarrow{k} + \alpha_l \overrightarrow{l} + \alpha_4 \overrightarrow{\mathbf{e}}_4$ is one of the following two cases:

I. Given $\alpha_{k0}, \alpha_{l0}, \alpha_{l1}, \delta, t_0 \in \mathbb{R}$, with $\delta = \pm 1$, $\alpha_{k0} > 0$, define $\alpha_k, \alpha_l, \alpha_4 : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, $\alpha_k(t) = \alpha_{k0}$, $\alpha_4(t) = \delta t + t_0$ and $\alpha_l(t) = \alpha_{l0} + \alpha_{l1}\, t - t^2/(2\alpha_{k0})$.

II. Given $\alpha_{k0}, \alpha_{k1}, \alpha_{40}, \alpha_{41}, \alpha_{l0} \in \mathbb{R}$, with $\alpha_{k1} > 0$, define the functions $\alpha_k, \alpha_4, \alpha_l : I \subset (\frac{-\alpha_{k0}}{\alpha_{k1}}, +\infty) \subset \mathbb{R} \longrightarrow \mathbb{R}$ as follows,

$$\alpha_k(t) = \sqrt{\alpha_{k1}\, t + \alpha_{k0}}, \quad \alpha_4(t) = \alpha_{40} + \alpha_{41}\, \alpha_k(t),$$

$$\alpha_l(t) = \alpha_{l0} + \frac{(\alpha_{41})^2}{2}\, \alpha_k(t) - \frac{2}{3(\alpha_{k1})^2}\, \alpha_k(t)^3.$$

Type II surface

The coordinate function of the profile curve

$$\begin{aligned}\alpha_l(t) &= \alpha_{l0} - \int_{t_0}^{t} \frac{2\alpha_k(s)}{\exp(f(s))} ds \\ &+ \int_{t_0}^{t} \left\{ \frac{\alpha_k(s)}{4\exp(f(s))} \left(\alpha_{k0} + \int_{t_0}^{s} \frac{2\alpha_k(w)\rho(w)}{\exp(f(w))} dw \right)^2 \right\} ds\end{aligned}$$

makes impossible to glue two of them.

Gaussian curvature

In general,

$$\text{Gaussian curvature: } K = \frac{-e_1 g_1 + f_1^2 + e_2 g_2 - f_2^2}{EG - F^2}.$$

In each case, the Gaussian curvature can be easily computed:

$$\begin{aligned} \text{Boost:} \quad & \alpha = (\alpha_1, 0, \alpha_3, \alpha_4), & K = \frac{\alpha_1''}{\alpha_1} \\ \text{Rotational:} \quad & \alpha = (\alpha_1, \alpha_2, \alpha_3, 0), & K = -\frac{\alpha_3''}{\alpha_3} \\ \text{Screw:} \quad & \alpha = (\alpha_k, \alpha_l, 0, \alpha_4), & K = -\frac{\alpha_k''}{\alpha_k} \end{aligned}$$

- Given $\alpha_1 > 0$ and ε, $\Longrightarrow \exists$ boost invariant Σ_α
- Given $\alpha_3 > 0$ and ε_i, $i = 1, 2$, $\Longrightarrow \exists$ rotational invariant type B Σ_α
- Given $\rho > 0$ (or α_k) and ε, $\Longrightarrow \exists$ screw invariant type I or II Σ_α

Theorem 4

Let G be either boost, rotational or invariant group. Let $\kappa : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. Given $t_0 \in I$, there exist $\epsilon > 0$ and a unit space-like curve $\alpha : (t_0 - \epsilon, t_0 + \epsilon) \to \mathcal{P}_i$ such that α is a unit profile curve of the marginally trapped G-invariant surface Σ_α whose Gaussian curvature satisfies $K(t, \theta) = \kappa(t)$ for any $(t, \theta) \in (t_0 - \epsilon, t_0 + \epsilon) \times \mathbb{R}$.

Thank you very much for your attention