On Related Theorems About Global Hyperbolicity

Alexander Dirmeier

Technische Universität Berlin

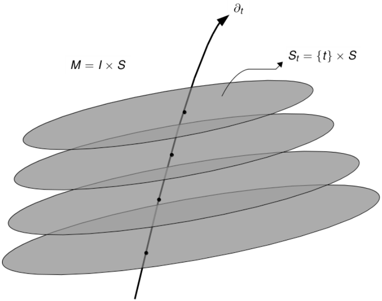
2009-07-10

Outline

Sliced Spacetimes

- Metric
  $g = -N^2(t,x)dt^2 + 2\beta(t,x)dt + h(t,x)$
- Lapse
  $N : M \to \mathbb{R},\ (t,x) \mapsto N(t,x) > 0$
- Shift
  $\beta = \beta_i dx^i \in \Lambda^1 S_t$
- Time Function
  $t : I \times S \to \mathbb{R},\ (t,x) \mapsto t \in I$

$\Rightarrow$ stably causal



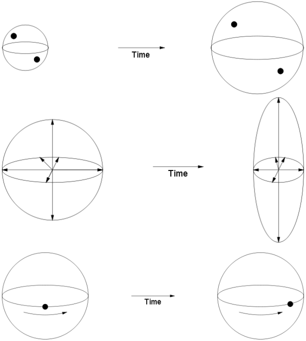
$(t,x) \in M = I \times S$, $I \subset \mathbb{R}$ and $\dim(S) = 3$

Kinematical Invariants

Let $V \parallel \partial_t$ and $g(V,V) = -1$, $u := g(V,\cdot) \Rightarrow$ Acceleration $\dot{u} = \nabla_V u$

$$\nabla u = \frac{\Theta}{3} p + \sigma + \omega - \dot{u} \otimes u$$

- Projection on $V^\perp$: $p = g + u \otimes u$
- Expansion $\Theta = \mathrm{div}(V)$
- Shear $\sigma = \mathrm{sym}(\nabla u) + \dot{u} \vee u - \frac{\Theta}{3} p$
- Rotation $\omega = du + \dot{u} \wedge u$
- Red-shift one-form $\rho = \dot{u} - \frac{\Theta}{3} u$



Finsler Geometry

Following [Bao2000], let $F : TS \to [0, \infty)$ be a Finslerian metric on S.

- Forward (Backward) completeness: Cauchy sequence $\{x_i\} \subset S$ with $d(x_i, x_j)\,(d(x_j, x_i)) < \epsilon$ for a $\nu(\epsilon) < i \leq j$ converges
- Randers metric: Riemannian metric $\tilde{g}$ and one-form b on S imply $R : TS \to [0, \infty)$ with $R = \sqrt{\tilde{g}} + b$ being Finslerian if $\|b\|_x^{\tilde{g}} < 1$
- [Caponio,Javaloyes,Masiello2007]:

$\tilde{g}$ complete and $\|b\|^{\tilde{g}} := sup_{x \in S}\|b\|_x^{\tilde{g}} < 1$ $\Rightarrow$ $R = \sqrt{\tilde{g}} + b$ forward and backward complete

Regularly Sliced Spacetimes

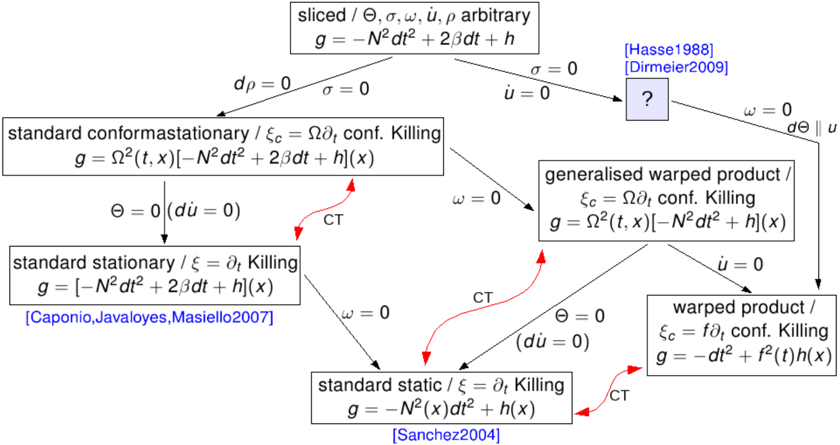
- A sliced spacetime $(M, -N^2 dt^2 + 2\beta dt + h)$ is regularly sliced if
  - (i) bounded lapse: $0 < N_m \leq N \leq N_M$ with $0 < N_m \leq N_M$ constant
  - (ii) $h(t)$ is a complete Riemannian metric on every S_t
  - (iii) $h(t)$ is bounded from below by a $\gamma := h(t_0)$, such that $A\gamma_{ij}v^i v^j \leq h_{ij}v^i v^j$ with $0 < A = \mathrm{const}$
  - (iv) shift is h-norm bounded: $\sqrt{h^{ij}\beta_i\beta_j} = \|\beta\|^h_{(t,x)} < B$
- [Choquet-Bruhat,Cotsakis2002]:

(M, g) regularly sliced $\Rightarrow$ (M, g) globally hyperbolic with Cauchy surfaces S_t

Subclasses of Sliced Spacetimes / General Remarks

- *Goal:* metric and kinematic characterisation of subclasses
- $\rightarrow$ Kinematic characterisation allows for non-trivial generalisations of known subclasses
- Self-contained theorems on global hyperbolicity for subclasses exist
- $\rightarrow$ Identify interdependence and special cases of theorems on global hyperbolicity for subclasses
- ($\rightarrow$ Relations of causality and kinematical invariants)

Subclasses


sliced / $\Theta, \sigma, \omega, \dot{u}, \rho$ arbitrary
$g = -N^2 dt^2 + 2\beta dt + h$
$d\rho = 0$
$\sigma = 0$
$\sigma = 0$
$\dot{u} = 0$
[Hasse1988]
[Dirmeier2009]
?
$\omega = 0$
$d\Theta \parallel u$
standard conformastationary / $\xi_c = \Omega\partial_t$ conf. Killing
$g = \Omega^2(t,x)[-N^2 dt^2 + 2\beta dt + h](x)$
$\Theta = 0\,(d\dot{u} = 0)$
CT
$\omega = 0$
generalised warped product /
$\xi_c = \Omega\partial_t$ conf. Killing
$g = \Omega^2(t,x)[-N^2 dt^2 + h](x)$
standard stationary / $\xi = \partial_t$ Killing
$g = [-N^2 dt^2 + 2\beta dt + h](x)$
[Caponio,Javaloyes,Masiello2007]
$\omega = 0$
CT
$\Theta = 0$
$(d\dot{u} = 0)$
$\dot{u} = 0$
warped product /
$\xi_c = f\partial_t$ conf. Killing
$g = -dt^2 + f^2(t)h(x)$
standard static / $\xi = \partial_t$ Killing
$g = -N^2(x)dt^2 + h(x)$
CT
[Sanchez2004]

Globally Hyperbolic Standard Stationary Spacetimes

- Let $(M = I \times S, g = -N^2(x)dt^2 + \beta(x)dt + h(x))$ be standard stationary and the Randers metric on S (Fermat metric) be

$$R = \sqrt{\frac{1}{N^2}h + \frac{1}{N^4}\beta^2} + \frac{1}{N^2}\beta.$$

- [Caponio,Javaloyes,Masiello2007]

R is forward and backward complete $\Leftrightarrow$ S is Cauchy surface for (M, g)

- Global hyperbolicity of standard conformastationary $g' = \Omega^2 g$ by conformal transformations

Globally Hyperbolic Standard Static Spacetimes

- Standard stationary: $\beta \to 0$ implies standard static with $R = \frac{1}{N}\sqrt{h}$ as special case
- Standard static global hyperbolicity follows as special case (e.g. [Sanchez2004]):

$\frac{1}{N^2} h$ complete	$\Leftrightarrow$	$(M, -N^2(x)dt^2 + h(x))$ globally hyperbolic

- Then also (e.g. [Beem1996]) warped product $(I \times S, -dt^2 + f^2(t)h)$ is globally hyperbolic iff h is complete

Relatedness of the Subclasses I

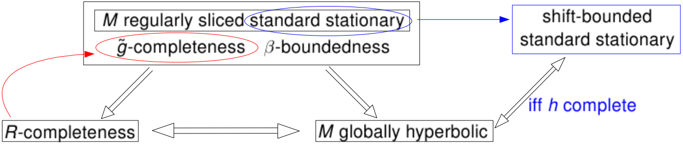
- $R = \sqrt{\tilde{g}} + \frac{1}{N^2}\beta$ with $\tilde{g} = \frac{1}{N^2}h + \frac{1}{N^4}\beta^2$
- Let h be a Riemannian metric, $N > 0$ *bounded* lapse function and β h-bounded shift: h complete $\Leftrightarrow$ $\tilde{g}$ complete complete
- (M, g) regularly sliced standard stationary
  $\Rightarrow$ $\tilde{g}$ complete and $\|\frac{1}{N^2}\beta\|^{\tilde{g}}$ uniformly bounded by 1
  $\Rightarrow$ R forward and backward complete
- [Caponio,Javaloyes,Sanchez2009]

R forward and backward complete $\Rightarrow$ $\tilde{g}$ complete

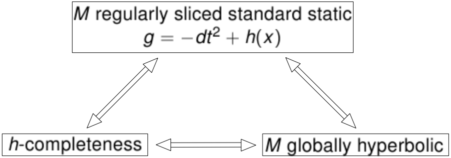
Relatedness of the Subclasses II

- shift-bounded standard stationary (M, g) with metric $g = -dt^2 + 2\beta(x)dt + h(x)$ and $\|\beta\|_x^h < B$

- stationary:



- static:



Hubble-Isotropic Spacetimes

- A sliced spacetime $(I \times S, g)$ is called Hubble-isotropic (H-isotropic) if in comoving coordinates

$$g = -dt^2 + 2b(x)dt + s^2(t,x)\gamma(x) - b(x) \otimes b(x)$$

or $\sigma = 0$ and $\dot{u} = 0$ for $V = \partial_t$.

- Thereby s is a function on M, γ is a Riemannian metric and b is a one-form on S and $h(t,x) = s^2\gamma - b^2$ on S_t with $s^2(t,x) > (\|b\|_x^\gamma)^2$
- Hubble function (e.g. [HassePerlick1999])

$$H : \mathcal{C}M \to \mathbb{R}; \quad H = g(\nabla_{K_x} V, K_x) = \tilde{H}(x) \circ \pi$$

with $\mathcal{C}M = \{K \in TM \mid g(K,K) = 0\}$ is isotropic.

Global Hyperbolicity of H-isotropic Spacetimes

- [Dirmeier2009] Let $s^2\gamma$ be complete and

$$\inf_{(t,x)\in M} s^2(t,x) > \sup_{x\in S}(\|b\|_x^\gamma)^2,$$

then the H-isotropic spacetime is globally hyperbolic.

- The h-boundedness of the shift b is matched if

$$\sup_{(x,t)\in S} \sup_{v\in T_x S\setminus 0} \sqrt{\frac{(b(x)[v])^2}{s^2(t,x)\gamma(x)[v,v]-(b(x)[v])^2}} < B.$$

- This is fulfilled if $\inf s^2 > \sup(\|b\|^\gamma)^2$.
- Then also h is complete and bounded from below
$\Rightarrow$ the spacetime is regularly sliced.

Example for a Globally Hyperbolic H-isotropic Spacetime

- [Gürses,Plaue,Scherfner,Schönfeld,Sousa2009]: $M = \mathbb{R} \times S$ and
$$g = -dt^2 + 2e^x dydt + s^2(t)dx^2 + \frac{1}{2}e^{2x}(s^2(t) - 2)dy^2 + s^2(t)dz^2,$$
with $S = \mathbb{R} \times \mathbb{H}^2$ homogeneous Bianchi III surface.
- Globally hyperbolic for $s^2 > C > 2 = (\|e^x dy\|^\gamma)^2$.
- But g stays Lorentzian for $s^2 \in (0, 2] \Rightarrow$ horizon and CTCs.

References «Thank you for your attention!»

- [Hasse1988] *Geometrische Auswertungsmethoden fuer kosmologische Beobachtungen*, Ph.D. thesis, TU Berlin (in German)
- [Beem1996] *Global Lorentzian Geometry*, Marcel Dekker, Inc., New-York
- [HassePerlick1999] *On spacetime models with an isotropic Hubble law*, Class. Quant. Grav. 16, 2559-2576
- [Bao2000] *An Introduction to Riemann-Finsler Geometry*, Springer, New-York
- [Choquet-Bruhat,Cotsakis2002] *Global Hyperbolicity and Completeness*, J. Geom. Phys. 43(4), 345-50
- [Sanchez2004] *On the Geometry of Static Spacetimes*, arXiv: math/0406332
- [Caponio,Javaloyes,Masiello2007] *On the Energy Functional on Finsler Manifolds and Application to Stationary Spacetimes*, arXiv: math/0702323v3
- [Caponio,Javaloyes,Sanchez2009] *On the Interplay Between Lorentzian Causality and Finsler Metrics of Randers Type*, arXiv: 0909.3501v1
- [Gürses,Plaue,Scherfner,Schönfeld,Sousa2009] *On spacetimes with given kinematical invariants: Construction and examples*, to appear in Gödel-type Spacetimes: History and New Developments
- [Dirmeier2009] *On Classes of Globally Hyperbolic Spacetimes with Rotation and Expansion*, (preprint)