WILLMORE ACTIONS IN SEMI RIEMANNIAN GEOMETRY WITH APPLICATIONS IN PHYSICS

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As a Presentation

★★ Roughly speaking, the reasons and motivations that stimulated me to prepare the material of this mini-course could be inserted in a philosophical context that may be described as follows:

1. Most of experimental sciences study nature. In particular, Physics studies physical objects which provide the physical systems. These are modelled inside the spaces by means of geometrical structures, so we **need Geometry. More precisely, extrinsic Geometry**.

2. However, this is not enough. Physical systems evolve, move and change, in the space due to forces and interactions producing physical phenomena. **To model the motion, we need Calculus**.

3. Consequently, to obtain models describing physical phenomena, we need Differential Geometry and more precisely **extrinsic Differential Geometry**. In other words, Geometry of **Curves, Surfaces and Submanifolds**.

4. The requirements of Poincaré and reparametrization invariance give special relevance to the extrinsic geometric invariants. These are encoded in two main structures. On one hand, the **second fundamental form: mean curvature and length of the second fundamental form**. On the other hand, **the normal connection and the corresponding normal curvature**.

For example

- The second fundamental form of any curve, viewed as a one dimensional submanifold, in a space no matter dimension, is nothing but its curvature function, the first curvature function, also called its proper acceleration. Notice that the intrinsic geometry of a curve is trivial.
- The normal curvature of a curve in a three dimensional background is encoded in its torsion or second curvature function.
- The extrinsic geometry of a surface in a three dimensional space is mainly encoded in its mean curvature function.
- More generally, the extrinsic geometry of a hypersurface is encoded exclusively in its second fundamental form, or shape operator, and mainly in its mean curvature, because its normal connection is trivial.

My Phylosophy

In this framework, the philosophy around which this mini-course was constructed can be explained as follows. Many physical phenomena can be modelled in the context of the theory of submanifolds. Essentially, in a physical terminology, they are field theories. In a geometrical context, they are geometrical variational problems. The elementary fields or dynamical variables in both contexts are mappings, immersions or submanifolds, from a source space into a target one. In both terminologies, the densities of the Lagrangians, governing either the field theory or the variational approach, involve the extrinsic geometry of submanifolds with a fixed topology, because we admit that fluctuations do not change the topology of the evolved elements. So the beauty of this approach and its aesthetically attractive point is assume that the quantum states, quantum numbers, are encoded in the geometry of the extended structures, submanifolds.

On the Content

★ I understood the Scientific Committee order in the following sense.

- The mini-course should be developed in two sections of one hour and a half each.
- The course should contains some basic materials.
- It should be related with the Lorentzian Geometry

In this sense, the mini-course is divided in two parts. With the main purpose of exhibiting a proposal on how to explore the **Lorentzian non linear sigma model of dimension two**. On the way a field almost virgin, at least from the point of view of this approach, and many applications to different physical phenomena.

★ PART I: Willmore has been and still is Highly Fashionable ★

1. **The Classical Willmore Program**
   - The Strategy of Blaschke
   - An early theorem by Willmore
2. **The Willmore Conjecture**
   - Motivation: The Clifford Torus, a Test Surface
   - Some Partial Answers
3. **A Break to Connect with Physics**
   - Dirac versus Hopf
   - The sine-Gordon Equation
   - Kirchhoff Elastic Rods
   - Magnetic Fields
   - Vortex Filaments
   - The Schrödinger Equation
4. **Willmore, several Times in Physics**
   - Elastic Membranes and Plates
   - Bosonic String Theories
   - Geometric Approach to Obtain Willmore Surfaces in Kaluza-Klein Conformal Classes
   - Coming Soon: Sigma Models

★ PART II: Sigma Models ★

1. **Functionals of Curvatures**
   - The Plateau problem
   - The Willmore problem with boundary
   - The Total Casorati Curvature
   - Casorati versus Willmore
2. **What a Sigma Model is?**
   - Sigma Models as Field Theories
   - The Geometric Approach to $\mathbf{NLSM}_2$
   - The Conformal Invariance of $\mathbf{NLSM}_2$
3. **Solitons Foliated by Circles**
   - Rotational Solitons and Clamped Elastic Curves
   - An Algorithm to Obtain the Whole Space of Rotational Solitons
   - Plateau versus Willmore or Nambu-Goto versus Polyakov
   - The Clifford Parallelism in the Three Sphere
   - Solitons Foliated by Clifford Parallel Circles
   - The Villarceau Parallelism or Clifford versus Villarceau
4. **The Lorentzian World**
   - The sigma model with symmetry $\mathbf{O}(2,1)$
   - Geometrical Approach and Conformal Invariance
   - The Whole Space of Rotational Surfaces in $\mathbb{L}^3$
   - The Moduli Space of Rotational Field Configurations

PART I:

WILLMORE HAS BEEN AND STILL IS HIGHLY FASHIONABLE

♠♠ For orientable surfaces, $\mathbf{S}$, in the Euclidean three space, $\mathbb{R}^3$, one has the Gauss map

$$\boxed{\mathbf{N} : \mathbf{S} \longrightarrow \mathbb{S}^2(1),}$$

which encodes an important part of both the geometry of $\mathbf{S}$ and the way how $\mathbf{S}$ is viewed in $\mathbb{R}^3$.

♠♠ In particular, one obtains the shape operator

$$\boxed{d\mathbf{N} : T\mathbf{S} \longrightarrow T\mathbf{S},}$$

that provides the main two geometrical invariants for surfaces in the Euclidean space:

♣♡♣ The **Gaussian curvature**, **G**, and the **mean curvature**, **H**

$$\boxed{\mathbf{G} = \det(d\mathbf{N}),} \qquad \boxed{\mathbf{H} = \mathrm{trace}(d\mathbf{N}).}$$

►► These two invariants behave quite different. Despite the definition we gave for Gaussian curvature, it has an intrinsic nature. According to the popular theorem *EGREGIUM*, the Gaussian curvature is an intrinsic property of surfaces, it only depends on the metric, no matter of how the surface is immersed in the space.

►► In addition to this deep result, for compact surfaces, the total Gaussian curvature only depends on the topology of the surface. This is the celebrated theorem of Gauss-Bonnet

$$\boxed{\int_{\mathbf{S}} \mathbf{G}\, dA = 2\pi\chi(\mathbf{S}),}$$

where $\chi(\mathbf{S})$ is the Euler characteristic of the surface and dA its area element.

★ ► In contrast, the mean curvature, $\mathbf{H}$ of a surface is extrinsic. It strongly depends on the way the surface is viewed in the space, being an essential ingredient to understand the extrinsic geometry of surfaces. The study of mean curvature has provided important big theories as those of minimal ($\mathbf{H} = 0$) and constant mean curvature surfaces. These theories have important applications not only in Mathematics but also in Physics. In particular, they are of practical interest for fluid mechanics to study capillary surfaces or surfaces that represent the interfaces between two different fluids.

★ ► Minimal surfaces appear as extremals of a variational problem, that associated with the area functional for surfaces that have certain boundary

$$\boxed{\mathcal{M}(\mathbf{S}) = \int_{\mathbf{S}} dA_{\mathbf{S}}, \qquad \partial\mathbf{S} = \Gamma.}$$

★ ► Constant mean curvature surfaces ($\mathbf{H} = r$) also appear as extremals of a variational problem, that associated with the area subject to the constraint that the enclosed volume is preserved

$$\boxed{\mathcal{CMC}(\mathbf{S}) = \int_{\mathbf{S}} dA_{\mathbf{S}} - r\,\mathrm{Vol}(\mathbf{Q}), \qquad \partial\mathbf{Q} = \mathbf{S}.}$$

★ ► The idea of integrating the squared mean curvature for compact surfaces is very old. As early as 1812, S.D.Poisson (*Mémoire sur les surfaces élastiques. Cl. Sci. Mathém. Phys. Inst. de France, 2nd printing, 167-225, 1812*) wrote the free energy for a solid membrane as

$$\boxed{\mathcal{P}(\mathbf{S}) = \frac{k_c}{2} \int_{\mathbf{S}} (2\mathbf{H})^2 \, dA}$$

where k_c is a constant, the bending modulus.

★ ► However, a systematic study of this functional was proposed by **T.J.Willmore** in Oberwolfach (the 1960's). From then, the action is popularly known as **the Willmore functional**.

★ ► The associated variational problem, **Willmore variational problem**, has been widely studied and considered for many people along more than 45 years

$$\boxed{\mathcal{W}(\mathbf{S}) = \int_{\mathbf{S}} \mathbf{H}^2 \, dA}$$

⊛ ►► I am sure that everybody here heard about Willmore (surfaces, variational problem, conjecture...) before attending this mini-course.

⊛ ►► Perhaps, a powerful reason to this popularity is the above mentioned **Willmore Conjecture**. Though its own interest comes from its invariance under conformal changes in the Euclidean metric.

The strategy of Blaschke and How we define

Willmore in non flat spaces

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a **conformal transformation**. Denote by g the Euclidean and define the metric, $\bar{g}$, by $g = \phi^*(\bar{g})$. Now, ϕ can decomposed into a product of similarity transformations (homotheties and motions) and inversions. Obviously, a similarity preserves both curvature of the space, which is still flat, and the Willmore action. However an inversion does not preserve the flatness of the metric, so we wish to see how Willmore behaves under an inversion.

$$\boxed{\phi(p) = \bar{p} = c^2 \frac{p}{r^2},} \qquad r^2 = \langle p, p \rangle$$

where we have chosen the origin as the center of the inversion and called c its radius.

A direct computation allows one to relate the area elements of $\mathbf{S}$ and $\bar{\mathbf{S}}$, respectively as

$$\boxed{d\bar{A} = \frac{c^4}{r^4}\, dA} \tag{1}$$

On the other hand, the unit normal vector fields of both surfaces are related by

$$\boxed{\bar{N} = \frac{2}{r^2}\, \langle p, N \rangle p - N}$$

which allows one to relate the principal curvatures of both surfaces

$$\boxed{\bar{\kappa}_i = -\frac{r^2}{c^2}\, \kappa_i - \frac{2r^2}{c^2} \langle p, N \rangle}$$

so, we get the following relation

$$\boxed{(\bar{\kappa}_1 + \bar{\kappa}_2)^2 - 4\bar{\kappa}_1\bar{\kappa}_2 = \frac{r^4}{c^4}[(\kappa_1 + \kappa_2)^2 - 4\kappa_1\kappa_2]}$$

The product of both principal curvatures is the Gaussian curvature only for flat metrics, like g, otherwise, we need to consider the sectional curvature function $\bar{\mathbf{R}}$ of $\bar{g}$ over the tangent plane of the surface

$$\boxed{\bar{\kappa}_1 \bar{\kappa}_2 = \bar{\mathbf{G}} - \bar{\mathbf{R}}} \qquad \boxed{\kappa_1 \kappa_2 = \mathbf{G}}$$

Thus, we have

$$\boxed{\bar{\mathbf{H}}^2 + \bar{\mathbf{R}} - \bar{\mathbf{G}} = \frac{r^4}{c^{\,4}}(\mathbf{H}^2 - \mathbf{G})}$$

and then, using Gauss-Bonnet, we get

$$\boxed{\int_{\bar{\mathbf{S}}} (\bar{\mathbf{H}}^2 + \bar{\mathbf{R}})\, d\bar{A} - 2\pi\chi(\bar{\mathbf{S}}) = \int_{\mathbf{S}} \mathbf{H}^2\, dA - 2\pi\chi(\mathbf{S})} \tag{2}$$

however, $\mathbf{S}$ and $\bar{\mathbf{S}}$ are the same topological surface, viewed with different metrics. Then, combine (1) with (2) to have

$$\boxed{\bar{\mathcal{W}}(\bar{\mathbf{S}}) = \int_{\bar{\mathbf{S}}} (\bar{\mathbf{H}}^2 + \bar{\mathbf{R}})\, d\bar{A} = \int_{\mathbf{S}} \mathbf{H}^2\, dA = \mathcal{W}(\mathbf{S})}$$

this provides the way to define the Willmore functional for any metric even if this is not flat. Furthermore, this functional is invariant under extrinsic conformal changes.

An Early Result by Willmore

♡♡ ►► **Round spheres are, globally, the least curved compact surfaces in Euclidean space**.

♡♡ ►► To be precise,

$$\boxed{\mathbf{S} \subset \mathbb{R}^3, \quad \text{compact} \quad \Rightarrow \quad \mathcal{W}(\mathbf{S}) = \int_{\mathbf{S}} \mathbf{H}^2 \, dA \geq 4\pi}$$

Equality holds if and only if $\boxed{\mathbf{S} = \mathbb{S}^2(r)}$, **a round sphere no matter the radius**.

How We Prove That ?

♡♡ ►► Nowadays, we can give a beautiful prove of the **Willmore** theorem which is based on a powerful inequality due to **Chern and Lashof**.

$$\boxed{\mathbf{S} \subset \mathbb{R}^3, \quad \text{compact} \quad \Rightarrow \quad \int_{\mathbf{S}} |\mathbf{G}|\, dA \geq 2\pi(4 - \chi(\mathbf{S}))}$$

▲▲ Proof of Willmore assuming true Chern-Lashof.-

Combine Chern-Lashof with Gauss-Bonnet in the following way

$$\int_{\mathbf{S}^+} \mathbf{G}\, dA + \int_{\mathbf{S}^-} \mathbf{G}\, dA = 2\pi\chi(\mathbf{S})$$

$$\int_{\mathbf{S}^+} \mathbf{G}\, dA - \int_{\mathbf{S}^-} \mathbf{G}\, dA \geq 2\pi(4 - \chi(\mathbf{S}))$$

where $\mathbf{S}^+ = \{p \in \mathbf{S} \,:\, \mathbf{G}(p) > 0\}$ and $\mathbf{S}^- = \mathbf{S} - \mathbf{S}^+$

To obtain

$$\boxed{\int_{\mathbf{S}^+} \mathbf{G}\, dA \geq 4\pi}$$

►► On the other hand, we know that

$$\mathbf{H}^2 - \mathbf{G} \geq 0, \quad \text{equality holding at umbilical points}$$

Then, we get

$$\boxed{\int_{\mathbf{S}} \mathbf{H}^2\, dA \geq \int_{\mathbf{S}^+} \mathbf{H}^2\, dA \geq \int_{\mathbf{S}^+} \mathbf{G}\, dA \geq 4\pi}$$

Equality ?

►► Certainly, for any round sphere one obtains equality.

$\boxed{\textbf{Converse ?}}$

►► Suppose we have the equality for a certain surface, $\mathbf{S}$, then

- The open $\mathbf{S}^+$ is made up of umbilical points and so it must be contained in a certain sphere, $\mathbf{S}^+ \subset \mathbb{S}^2(r)$

$$\boxed{\mathbf{H} = \frac{1}{r} \qquad \text{in} \qquad \mathbf{S}^+}$$

- Its complementary, $\mathbf{S} - \mathbf{S}^+$ is made up of minimal points

$$\boxed{\mathbf{H} = 0 \qquad \text{in} \qquad \mathbf{S}^-}$$

However, there are no compact minimal surfaces in $\mathbb{R}^3$

$$\boxed{\Rightarrow\Rightarrow \qquad \mathbf{S} = \mathbb{S}^2(r)}$$

▲▲ $\boxed{\textbf{Proof of Chern-Lashof.-}}$

⊛ ► As $\mathbf{S} \subset \mathbb{R}^3$ is compact, then it is orientable and so there exists, globally defined the Gauss map

$$\mathbf{N} : \mathbf{S} \to \mathbb{S}^2(1)$$

⊛ ► Now, $\int_{\mathbf{S}} |\mathbf{G}| \, dA$ measures the *average number of times* that $\mathbb{S}^2(1)$ is covered by the Gauss map.

Using Morse Theory

⊛ ► Given $\xi \in \mathbb{S}^2(1)$, define the corresponding height function

$$\boxed{h_\xi : \mathbf{S} \to \mathbb{R} \qquad h_\xi(p) = \langle p, \xi \rangle}$$

⊛ ► The critical points of these functions are nicely characterized by

$$\boxed{p \quad \text{critical point of} \quad h_\xi \quad \Leftrightarrow \quad \xi \perp T_p\mathbf{S}}$$

⊛ ► Functions all whose critical points are non-degenerate are called **Morse functions**. Then as an application of the **Sard theorem** one can see that

$$\boxed{\mathbf{A} = \{\xi \in \mathbb{S}^2(1) \ : \ h_\xi \, \text{is Morse}\} \quad \text{open}\, \textbf{dense} \,\text{in} \quad \mathbb{S}^2}$$

⊛ ► **Morse inequalities** Given a Morse function on a compact differentiable manifold, $f : \mathbf{M} \to \mathbb{R}$ denote by $c_i(f)$ the number of critical points with index i and name by b_i the i-th **Betti** number of $\mathbf{M}$ then

$$\boxed{b_i \leq c_i(f)}$$

⊛ ► Notice that if p is a critical point of h_ξ then it is a critical point of $h_{-\xi}$ too. However, $\mathbf{N}(p)$ is ξ or $-\xi$, consequently

$$\boxed{\int_{\mathbf{S}} |\mathbf{G}|\, dA = \frac{1}{2} \int_{\xi \in \mathbf{A}} c\,(h_\xi)\, d\sigma^2}$$

where $c\,(h_\xi)$ is the number of critical points of h_ξ, $d\sigma^2$ is the area density on the unit two-sphere and of course, we used the Sard theorem.

⊛ ► Now use the Morse inequalities

$$\boxed{\int_{\mathbf{S}} |\mathbf{G}|\, dA \geq \frac{1}{2}(b_0 + b_1 + b_2) \int_{\xi \in \mathbf{N}} d\sigma^2 = \frac{1}{2}(b_0 + b_1 + b_2) \int_{\xi \in \mathbb{S}^2(1)} d\sigma^2 =}$$

$$\boxed{2\pi(b_0 + b_1 + b_2) = 2\pi(2 + b_1) = 2\pi(4 - \chi(\mathbf{S}))}$$

⊛ ► This cloncludes the proof of the Chern-Lashof inequality and so that of the Willmore result.

As a First Conclusion: Summary

Willmore Outside Euclidean Three Space

♦ ►► The Willmore functional can be defined for, compact, surfaces, **S**, immersed in a Riemannian space $(M^n; g = \langle , \rangle)$. However, in this case, Willmore density must be modified by adding the sectional curvature function **R**, measured in the ambient space, restricted to the tangent plane of surface, i.e.

$$\boxed{\mathcal{W}(\mathbf{S}) = \int_{\mathbf{S}} \left(\mathbf{H}^2 + \mathbf{R}\right)\, dA}$$

♦ ►► **Conformal Invariance.-** Part of the beauty of this action comes from its **invariance** under **conformal changes** in the surrounding metric $g = \langle , \rangle$. Therefore, actually the **Willmore program** is stated in the conformal class, $[g]$, of the original metric.

THE WILLMORE CONJECTURE

★ ►► The **Euler-Lagrange** equations, which provide the critical points also known as **Willmore surfaces**, for Willmore functional in spaces with constant curvature were computed by **J.L.Weiner**. In particular, in $\mathbb{R}^3$ or in the unit three sphere, $\mathbb{S}^3$, the Willmore surfaces are the solutions of

$$\boxed{\Delta \mathbf{H} = 2(\mathbf{H}^2 - \mathbf{G})\mathbf{H} \qquad \textbf{impossible to be solved}}$$

★ ►► The last equation shows that **minimal surfaces, $\mathbf{H} = 0$, are Willmore surfaces**. This statement has no sense in Euclidean space because the non existence of compact minimal surfaces. However, in the unit three sphere the class of compact minimal surfaces is very big.

★ ►► Using a stereographic projection, which is a conformal map, we can project compact minimal surfaces from $\mathbb{S}^3$ to obtain a wide class of Willmore surfaces in the Euclidean space.

$$\boxed{\mathbf{E}_o : \mathbb{S}^3 - \{z_o\} \to \mathbb{R}^3, \qquad \mathbf{E}_o(\mathbf{S}) = \text{Willmore},}$$

if **S** is minimal in the three sphere.

★★★ **A natural problem.-** From the early result by Willmore, it seems natural to look for minima of Willmore energy in the class of surfaces with a prescribed topology.

★★★ More generally, study Willmore surfaces with prescribed positive genus.

★★★ As you already know, the more popular and star case correspond to genus one surfaces.

★★★ **Popular problem.-** Searching for genus one (tori) which are solutions of the equation

$$\boxed{\Delta\mathbf{H} = 2(\mathbf{H}^2 - \mathbf{G})\mathbf{H},}$$

and obtain the critical values of the Willmore energy.

★★★ In other words, look for Willmore tori and corresponding critical values of the Willmore energy.

The Clifford Torus, A Test Surface

★★★ Let us check the Willmore measure of a circular revolution torus. Certainly, one can try to directly compute the Willmore energy of an arbitrary circular revolution torus, $\mathbf{T}(R, r)$, $R > r$, and then to get consequences. However, we propose other, perhaps more elegant, way to analyze this problem.

♠ **(1)** Start with a two-parameter class of isometric embeddings. For any pair, a, b, of nonzero real numbers, define

$$\boxed{\Phi_{ab} : \mathbb{R}^2 \to \mathbb{C}^2 \qquad \Phi_{ab}(s,t) = \left(a\, e^{\frac{is}{a}}, b\, e^{\frac{it}{b}}\right).}$$

► Choose $a^2 + b^2 = 1$ to obtain spherical embeddings in the unit three sphere, say $a = \cos\theta$ and $b = \sin\theta$, with $\theta \in \left(0, \frac{\pi}{2}\right)$

$$\Phi_{ab}(\mathbb{R}^2) \subset \mathbb{S}^3(1)$$

► It is obvious that each embedding can be induced to a flat torus

$$\boxed{\Phi_{ab} : \mathbf{T}_{ab} \to \mathbb{S}^3(1) \subset \mathbb{C}^2,}$$

$$\boxed{\mathbf{T}_{ab} = \mathbb{R}^2/\Gamma_{ab}, \qquad \Gamma_{ab} = \mathrm{Span}\{(2\pi a, 0); (0, 2\pi b)\}.}$$

► In other words, these flat tori are the Riemannian product of two circles with radii $a = \cos\theta$ and $b = \sin\theta$, respectively

$$\boxed{\mathbf{T}_{ab} = \mathbb{S}^1(\cos\theta) \times \mathbb{S}^1(\sin\theta) = \mathbf{T}_\theta.}$$

♠ **(2)** Notice that the Willmore density for surfaces in the unit three sphere, $\phi : \mathbf{S} \to \mathbb{S}^3$, is $\mathbf{H}^2 + 1$ which is nothing but the squared mean curvature when regarded in $\mathbb{C}^2$, which according to the Laplace equation is $\frac{1}{4}|\Delta\phi|^2$, where Δ is the Euclidean Laplacian. Therefore, since those tori are flat,

$$\mathcal{W}(\mathbf{T}_\theta) = \frac{1}{4}\int_0^{2\pi\cos\theta}\int_0^{2\pi\sin\theta}\left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right) ds\, dt$$

$$= \frac{1}{\cos\theta\sin\theta}\,\pi^2 = \frac{2}{\sin 2\,\theta}\pi^2, \qquad a = \cos\theta, \quad b = \sin\theta.$$

► Certainly, this function present a minimum in $\theta = \frac{\pi}{4}$ which corresponds with a squared lattice, $a = b = \sqrt{2}/2$. In this case the torus is called **Clifford** torus, and so we have

$$\boxed{\mathcal{W}(\mathbf{T}_\theta) \geq \mathcal{W}(\mathbf{T}_{\text{Clifford}}) = 2\,\pi^2,}$$

and the equality holds if and only if $\mathbf{T}_\theta = \mathbf{T}_{\text{Clifford}}$

♠ (3) $\boxed{\textbf{How we look this setting in Euclidean space?}}$

► Choose a suitable stereographic projection. Say with pole $z_o = (0, i) \in \mathbb{S}^3 \subset \mathbb{C}^2$.

$$\mathbf{E}_o : \mathbb{S}^3 - \{z_o\} \longrightarrow \mathbb{R}^3,$$

Take $\mathbf{C}$ a great circle through the pole and pick the z-axis in $\mathbb{R}^3$ as $\mathbf{E}_o(\mathbf{C})$.

► In this framework, it is not hard to see that

$$\boxed{\mathbf{E}_o(\mathbf{T}_\theta) = \mathbf{T}(\csc\theta, \tan\theta), \qquad \theta \in (0, \frac{\pi}{2}),}$$

a revolution torus obtained by rotating around the z-axis the circle, in the plane $y = 0$, with center $(\csc\theta, 0, 0)$ and radius $r = \tan\theta$.

► Notice that a circular revolution torus, $\mathbf{T}(R, r)$ is the stereographic image of a constant mean curvature, flat torus in the three sphere (i.e. a Riemannian product of two plane circles) if and only the radius R and r are the hypotenuse and the catet of an appropriate right triangle, $R = \csc\theta$ and $r = \tan\theta$. Write

$$\boxed{\mathcal{C} = \{\mathbf{E}_o(\mathbf{T}_\theta) \ : \ \theta \in (0, \frac{\pi}{2})\}.}$$

♠ (4) What happens with Willmore energy of circular revolution tori outside of $\mathcal{C}$?

► Suppose that $\mathbf{T}(R,r)$ is a circular revolution torus which is not in the class $\mathcal{C}$. Then, the profile circle

$$\{(x,o,z)\,:\,(x-R)^2+z^2=r^2\}$$

can be parametrized, in $[0,2\pi]$, as follows

$$\gamma(t)=\left(\frac{\rho^2}{R-r\sin t};0;\frac{r\,\rho\,\cos t}{R-r\sin t}\right),\qquad \rho=\sqrt{R^2-r^2}.$$

In $\mathbf{T}(R,r)=\mathbf{R}_\varphi(\gamma(t))$, with $(\varphi,t)\in[0,2\pi]^2$, we change parameters

$$\varphi=\frac{r}{\rho^2}\,u\qquad t=\frac{R}{r\,\rho}\,v,$$

to obtain a new parametrization $\mathbf{T}(R,r)=X(u,v)$ with

$$u\in\left[0,\frac{2\pi\rho^2}{R}\right],\qquad v\in\left[0,\frac{2\pi r\rho}{R}\right].$$

▶ Using this new parametrization, it is not difficult to see that the Willmore measure is

$$\boxed{(\mathbf{H}^2 - \mathbf{G})\, dA = \frac{R^4}{4r^2\rho^4}\, du\, dv,}$$

consequently, the Willmore energy is computed to be

$$\boxed{\mathcal{W}(\mathbf{T}(R,r)) = \frac{R^2}{r\sqrt{R^2 - r^2}}\, \pi^2.}$$

▶ However, if we choose $\theta \in (0, \pi/2)$ such that $\boxed{\sin\theta = r/R}$, then we observe that

$$\boxed{\mathcal{W}((\mathbf{T}(R,r)) = \mathcal{W}(\mathbf{E}_o(\mathbf{T}_\theta)),}$$

and consequently

$$\mathcal{W}((\mathbf{T}(R,r)) = \mathcal{W}(\mathbf{E}_o(\mathbf{T}_\theta)) \geq \mathcal{W}(\mathbf{E}_o(\mathbf{T}_{\text{Clifford}})) = 2\pi^2.$$

$$\boxed{\mathcal{W}((\mathbf{T}(R,r)) \geq 2\pi^2,}$$

and the equality holds if and only if $\boxed{R = \sqrt{2}\, r}$.

Several Conjectures On The Clifford Torus

⊛ ►► The Clifford torus is a real, main **star** in many different contexts. Several conjectures have been formulated, and still are open, on this surface. I wish to mention the more important

★ The Clifford torus is minimal in the three sphere.

► There are several ways to check this claim: one can use the Hopf map to exhibit the Clifford torus as a Hopf tube on a geodesic in the two sphere. However, since this technique will be considered several times along this minicourse, we use a result due to Takahashi to this purpose.

► Minimal submanifolds in any sphere are nicely characterized from the fact that they are constructed in the Euclidean space using eigenfunctions of the induced metric Laplacian associated with the same eigenvalue.

$$\boxed{\psi : M^n \to \mathbb{R}^m, \qquad \Delta\psi = \lambda\psi, \qquad \Leftrightarrow}$$

it is minimal in a sphere with center the origin and radius r with

$$\boxed{\lambda = \frac{n}{r^2}.}$$

▶ For the Clifford torus,

$$\boxed{\Phi = \Phi_{aa} : \mathbf{T}_{\text{Clifford}} \to \mathbb{S}^3 \subset \mathbb{C}^2, \qquad a = \frac{\sqrt{2}}{2},}$$

$$\boxed{\Delta\Phi = 2\,\Phi \qquad \Rightarrow \qquad \text{minimal} \quad \text{in} \quad \mathbb{S}^3,}$$

▶ Notice also that $\lambda = 2$ is the first eigenvalue, λ_1, of the Clifford torus.

♡♡ ▶ **The Yau Conjecture:**

Every minimal embedding of a compact surface in the three sphere is constructed with eigenfunctions associated with the first Laplacian eigenvalue, like Clifford torus

▶ The only known torus minimally embedded in $\mathbb{S}^3$ is the Clifford torus.

♡♡ ▶ **The Lawson Conjecture:**

The Clifford torus is the only minimal embedding of a genus one surface in the three sphere

▶ It is known that the only torus minimally embedded (immersed) into $\mathbb{S}^3$ with eigenfunctions associated with its first eigenvalue is the Clifford torus.

▶ This result provides a connection between the Yau and the Lawson conjectures in the following sense

★ $\boxed{\text{Yau conjecture true}} \Rightarrow \boxed{\text{Lawson conjecture true}}$

▶ $\boxed{\text{However both conjectures are still open.}}$

♡♡ ▶ **The Willmore Conjecture:**

▶ A round sphere is a minimum of the Willmore functional on the whole class of compact surfaces, no matter the topology.

▶ The Willmore value in the Clifford torus is

$$\boxed{\mathcal{W}(\mathbf{T}_{\text{Clifford}}) = 2\pi^2,}$$

▶ Moreover, we already know that

$$\boxed{\mathcal{W}(\mathbf{T}(R, r)) \geq 2\pi^2,}$$

for any circular revolution torus, $\mathbf{T}(R, r)$, and the equality holds if and only if that revolution torus is the stereographic projection of the Clifford torus.

▶ Under a suitable stereographic projection, the image of the Clifford torus can be viewed as a revolution torus with radii in the ratio $\sqrt{2}$. However, it can be conformally deformed, for example changing the projection pole, and so viewed as, for example, a cyclide of Dupin.

► **Let $\mathbf{S}$ be a genus one surface and $\mathbf{I}(\mathbf{S},\mathbb{R}^3)$ the space of immersions, $\psi : \mathbf{S} \to \mathbb{R}^3$, the Willmore functional can be viewed as defined**

$$\boxed{\mathcal{W} : \mathbf{I}(\mathbf{S},\mathbb{R}^3) \to \mathbb{R}, \qquad \mathcal{W}(\psi) = \int_{\mathbf{S}} \mathbf{H}_\psi^2 \, dA_\psi.}$$

► Now, the Willmore conjecture is established as follows

$$\boxed{\mathcal{W}(\psi) \geq 2\pi^2,} \qquad \boxed{\text{equality} \quad \Leftrightarrow \quad \textbf{Clifford}.}$$

This Conjecture Is Still Unsolved

However, we know some partial positive answers

$\boxed{\textbf{Some Partial Answers}}$

1. ★ $\boxed{\textbf{Tubes.}}$ The conjecture is true for circular revolution tori. More generally, for tubes constructed over closed curves in $\mathbb{R}^3$ (K.Shiohama and R.Takagi, JDG, 1970).

2. ★ $\boxed{\textbf{Revolution Surfaces.}}$ The conjecture is true for revolution tori (J.Langer and D.A.Singer, JDG, 1984 and Bull. London Math.Soc, 1983).

► To prove this result, we proceed as follows. Let γ be a closed curve in the half plane

$$\boxed{\mathbf{P} = \{(x, 0, z) \in \mathbb{R}^3 \,:\, x > 0\},}$$

and denote by $\mathbf{T}_\gamma$ the revolution torus obtained when rotate γ around the z-axis.

► If g_o is the Euclidean metric put

$$\boxed{f : \mathbf{P} \to \mathbb{R}, \qquad f(x, 0, z) = x,}$$

then

$$\boxed{\left(\mathbb{R}^3 - \{z - \text{axis}\}, g_o\right) = (\mathbf{P}, g_o) \times_f (\mathbb{S}^1, dt^2).}$$

▶ We make an obvious conformal change to get

$$\boxed{\left(\mathbb{R}^3 - \{z - \text{axis}\}, \frac{1}{f^2}\, g_o\right) = \left(\mathbf{P}, \frac{1}{f^2}\, g_o\right) \times (\mathbb{S}^1, dt^2).}$$

▶ To compute the Willmore energy of $\mathbf{T}_\gamma$, we may use the new metric because the conformal invariance. Since the new metric is a Riemannian product and the tangent plane of $\mathbf{T}_\gamma$is anytime a mixed section, then $\mathbf{R} = 0$ and

$$\boxed{\mathcal{W}(\mathbf{T}_\gamma) = \int_0^{2\pi} \int_\gamma \frac{1}{4}\, \kappa^2\, ds\, dt = \frac{\pi}{2} \int_\gamma \kappa^2\, ds,}$$

where κ is the curvature of γ in the hyperbolic half plane, with curvature -1, $\left(\mathbf{P}, \frac{1}{f^2}\, g_o\right)$.

▶ Now, a result by Langer-Singer guarantees that for closed curves in the hyperbolic plane one has

$$\boxed{\int_\gamma \kappa^2\, ds \geq 4\pi,}$$

and the equality holds if and only if γ is the geodesic circle with radius $\sinh^{-1}(1)$. This proves the conjecture.

3. ★ $\boxed{\textbf{The Tangency Map of Spherical Surfaces.}}$

▶ Consider a compact surface in the three sphere

$$\boxed{\phi : \mathbf{S} \longrightarrow \mathbb{S}^3 \subset \mathbb{C}^2.}$$

Then its Gauss map, in $\mathbb{C}^2$, takes values in the $\boxed{\text{Grassmannian}}$ of oriented two planes of $\mathbb{R}^4 \equiv \mathbb{C}^2$

$$\boxed{\mathbf{N}_\phi : \mathbf{S} \longrightarrow \mathbf{G}_{2,4} \equiv \mathbb{S}^2 \times \mathbb{S}^2, \qquad \mathbf{N}_\phi = \phi \wedge \eta,}$$

where η is the unit normal vector field of the surface in the sphere.

▶ Certainly, $\mathbf{G}_{2,4}$ is identified, via intersection with $\mathbb{S}^3$, to the space of oriented great circles in $\mathbb{S}^3$. Each great circle determines a foliation, by tori, of $\mathbb{S}^3 \subset \mathbb{C}^2$, as follows. Denote by $\mathbf{P}$ the plane, trough the origin, that contains the given great circle, so we have $\mathbb{C}^2 = \mathbf{P} \oplus \mathbf{P}^\perp$. Using this framework, we have

$$\boxed{\mathbf{T}_r = \{(z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1| = r, \ |z_2| = \sqrt{1 - r^2}\},}$$

where $0 \leq r \leq 1$. Of course $\mathbf{T}_1$ is the original great circle and $\mathbf{T}_0$ is the great circle in $\mathbf{P}^\perp$.

▶ In the above setting, given any immersed surface, $\phi : \mathbf{S} \longrightarrow \mathbb{S}^3 \subset \mathbb{C}^2$, we define its $\boxed{\text{tangential map}}$ as follows

$$\boxed{\Upsilon_\phi : \mathbf{G}_{2,4} \longrightarrow \mathbb{N} \cup \{0, \infty\},}$$

then, the image under Υ_ϕ of a great circle is the number of points of tangency between its associated foliation of tori, $\{\mathbf{T}_r : r \in [0,1]\}$ and the surface itself.

▶ In 2000, P.Topping (Calc.Var.) showed the following result: **If S has genus one, it is a torus, then**

$$\boxed{\mathcal{W}(\phi) \geq \frac{\pi^2}{4} \int_{\mathbf{G}_{2,4}} \Upsilon_\phi \, d\sigma^2.}$$

▶ In particular, **the Willmore conjecture holds for any torus for which the average value of its tangency map is at least** 8.

▶ One can follow similar arguments to those used in the proof of the Chern-Lashof inequality, including Morse theory and the Sard theorem, to obtain a pointwise lower estimate for tangency number of an immersion in terms of the topology of the surface. In fact

$$\boxed{\int_{\mathbf{G}_{2,4}} \Upsilon_\phi \geq 2(1+g), \qquad g = \text{genus.}}$$

► Certainly, the above estimate is not enough to prove the Willmore conjecture. However, **if one assumes that the torus is embedded in $\mathbb{S}^3$ and it is invariant under the antipodal map of the sphere, then one has the following stronger estimate**

$$\boxed{\int_{\mathbf{G}_{2,4}} \Upsilon_\phi \geq 8.}$$

This is an alternative proof of a result by A.Ros (Math.Res.Lett., 1999).

4. ★ Hopf Tori.

► As an illustration of the above result, we exhibit a class of tori in the three sphere which are invariant under the antipodal map and consequently the Willmore conjecture is true in this class.

► In $\mathbb{S}^3 = \{\zeta = (z_1, z_2) \in \mathbb{C}^2 \,:\, |\zeta|^2 = |z_1|^2 + |z_2|^2 = 1\}$, we consider the usual action,

$$\boxed{\mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow \mathbb{S}^3, \qquad (\mathrm{e}^{\mathrm{i}t}, \zeta) \mapsto \mathrm{e}^{\mathrm{i}t}\zeta = (\mathrm{e}^{\mathrm{i}t} z_1, \mathrm{e}^{\mathrm{i}t} z_2).}$$

► The orbits under this action are great circles and they generate a $\boxed{\text{Hopf flow}}$ (Killing vector field with constant length)

$$\boxed{V(\zeta) = i\,\zeta, \qquad \text{infinitesimal translation.}}$$

► The space of orbits is identified with a two-sphere, which endowed with the metric of constant curvature 4, becomes the natural projection to be a Riemannian submersion.

$$\boxed{\Pi : \mathbb{S}^3 \to \mathbb{S}^2(1/2), \qquad \text{Riemannian} \quad \text{submersion,}}$$

$$\boxed{\Pi(z_1, z_2) = (z_1\bar{z}_2, \frac{1}{2}(|z_1|^2 - |z_2|^2)).}$$

where we regard two-sphere in $\mathbb{C} \times \mathbb{R}$.

► We can use all the paraphernalia associated with the Riemannian submersions. In particular, we can talk about the horizontal lifts, in $\mathbb{S}^3$, of a given curve, $\delta(s)$, in $\mathbb{S}^2(1/2)$. They are curves, $\bar{\delta}(s)$ which satifies

$$\Pi \circ \bar{\delta} = \delta \quad \boxed{d\Pi_{\bar{\delta}(s)}(\bar{\delta}'(s)) = \delta'(s),} \qquad \boxed{\langle \bar{\delta}'(s), V(\bar{\delta}(s)\rangle = 0}$$

► If the original curve is arclength parametrized, then all the horizontal liftings are arclengeth parametrized. Furthermore, one can obtain all the horizontal lifts of a given curve starting from a fixed one and using the Hopf action. All of them are collected in the so called complete lift

$$\boxed{\mathbf{T}_\delta = \Pi^{-1}(\delta) = \{e^{it}\,\bar{\delta}(s) \;:\; (s,t) \in \mathbb{R}^2\}}$$

► The complete lift, $\mathbf{T}_\delta$, of a curve, δ, is a flat surface which is named the Hopf tube with cross section that curve. It is embedded, in $\mathbb{S}^3$, if the curve is simple, free of self-intersections, in the two sphere. Furthermore, if the cross section is closed, then its Hopf tube becomes a torus, Hopf torus.

► Most of the geometry of a Hopf tube is reflected in the geometry of its cross section. First notice that a Hopf tube can be parametrized, via its covering Riemannian map

$$\boxed{X : \mathbb{R}^2 \to \mathbf{T}_\delta, \qquad X(s,t) = e^{it}\,\bar{\delta}(s),}$$

so the coordinate curves are, respectively, horizontal lifts ($t =$ constant) and orbits ($s =$ constant).

► In particular, when δ is a closed curve in $\mathbb{S}^2(1/2)$, then the isometry type of the flat torus $\mathbf{T}_\delta$ can be computed in terms of the invariants associated with its cross section, δ. To explain how we do it, we start by regarding the Hopf map as a circle bundle on the two sphere which is endowed with a natural gauge potential or principal connection.

► In fact, it is defined as the dual one form of the Hopf vector field

$$\boxed{\omega = V^\sharp,}$$

▶ In other words, it is associated with the following $\boxed{\text{horizontal}}$ distribution

$$\boxed{\mathcal{H}_z = \operatorname{Ker}\omega_z =< iz >^{\perp} .}$$

▶ The $\boxed{\text{curvature}}$ or $\boxed{\text{strength}}$ of this gauge potential is computed to be

$$\boxed{\Omega = \Pi^*(\Theta), \qquad \text{with} \qquad \Theta = 2\, d\sigma^2,}$$

and $d\,\sigma^2$ standing for the element of area in $\mathbb{S}^2(1/2)$.

▶ Certainly, ω has non trivial $\boxed{\text{holonomy}}$, which makes that the horizontal liftings of closed curves do not close. Thus, assume δ is closed with length L and enclosing an oriented area $A \in [-\pi, \pi]$ in $\mathbb{S}^2(1/2)$. Then, there exists $h(\delta) \in [-\pi, \pi)$ such that

$$\boxed{\bar{\beta}(L) = e^{i\delta}\bar{\beta}(0)},$$

and it does not depend on the horizontal lift one chooses.

▶ The $\boxed{\text{holonomy index}}$, $h(\delta)$ of a closed curve can be computed as follows. Consider a two chain, say c, such that $\partial c = \delta$ then

$$\boxed{h(\delta) = \int_c \Theta = 2\, A.}$$

► Certainly, the group of desk transformation or the lattice, in Euclidean plane, corresponding to $\mathbf{T}_\delta$ is generated by $(L, h(\delta))$ and $(0, 2\pi)$. As a consequence, we have: **let δ be a closed curve in $\mathbb{S}^2(1/2)$ with length L and enclosed area A, then its Hopf torus is**

$$\boxed{\mathbf{T}_\delta = \frac{\mathbb{R}^2}{\Gamma},}$$

where Γ is the lattice in the Euclidean plane generated by $(L, 2\,A)$ **and** $(0, 2\,\pi)$.

► Therefore, **the intrinsic geometry of a Hopf torus is completely encoded in the geometry of its cross section.**

► The cross section also encodes the extrinsic geometry of its Hopf tube. We can compute the shape operator of a Hopf tube in $\mathbb{S}^3$, which in the above framework is given by the following matrix

$$\begin{pmatrix} \kappa & 1 \\ 1 & 0 \end{pmatrix}$$

where κ is the curvature function, actually it is the lift, of the cross section in the two-sphere.

► Therefore, the mean curvature of a Hopf tube is given by

$$\boxed{\mathbf{H}^2 = \frac{1}{4}\,\kappa^2.}$$

In particular a Hopf torus is minimal if and only of its cross section is a great circle of the two-sphere. Since flat and minimal torus implies Clifford, then the Clifford torus, $\mathbf{T}_{\text{Clifford}}$, belongs to class of Hopf tori.

► Notice that Hopf tori are invariant under the antipodal map of $\mathbb{S}^3$, consequently we have

$$\boxed{\mathcal{W}(\mathbf{T}_\delta) \geq \mathcal{W}(\mathbf{T}_{\text{Clifford}}) = 2\pi^2,}$$

$$\boxed{\text{Equality} \quad \Leftrightarrow \quad \mathbf{T}_\delta = \mathbf{T}_{\text{Clifford}}.}$$

5. ★ Other partial solutions to the Willmore conjecture use spectral geometry. Perhaps, the deeper in that direction was given by P.Li and S.T.Yau (Invent.Math., 1982). **For an isometric immersion of a compact surface**

$$\boxed{\phi : \mathbf{S} \longrightarrow \mathbb{S}^3 \subset \mathbb{C}^2,}$$

we have

$$\boxed{\mathcal{W}(\phi) \geq \lambda_1 \,\mathrm{Area}(\mathbf{S}),}$$

where λ_1 **is the first eigenvalue of the Laplacian of S, of course, endowed with the induced metric**.

► It is well known that every Riemannian torus is conformally equivalent to a flat one, say

$$\boxed{\mathbf{T} = \mathbb{R}^2/\Gamma, \qquad \Gamma = \text{lattice in } \mathbb{R}^2.}$$

► Now, up to dilatations (homotheties) the generator of the lattice, Γ, can be chosen to be

$$\boxed{\varepsilon_1 = (1,0), \quad \varepsilon_2 = (x,y),\ 0 \le x \le \frac{1}{2},\ x^2 + y^2 \ge 1,\ y > 0.}$$

The smaller point in the dual lattice is $\upsilon = (0, 1/y)$ and the area of this flat torus is y, consequently

$$\boxed{\lambda_1 = 4\,\pi^2\,|\upsilon|^2 = \frac{4\,\pi^2}{y^2},}$$

and so

$$\boxed{\mathcal{W}(\mathbf{T}) \ge \frac{2\,\pi^2}{y},}$$

which proves the Willmore conjecture for those conformal structures corresponding to $y \le 1$.

The Willmore conjecture has been proved for other conformal structures, however, as long as I know, they use similar arguments.

$\boxed{\text{A Break to Connect with Physics}}$

★ **The research worker, in his effort to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence.**

★ $\boxed{\text{Paul Dirac, 1939.}}$

Dirac-Hopf

♠ It has been (and still is) of great interest to construct **mathematical models** to better understand nature and, in particular, **physical phenomena**.

♠ In 1931, **P.Dirac** published a paper (Quantised Singularities in the Electromagnetic Field. Proc. Roy. Soc. (London) A 133, 60). In this paper, Dirac assumes the existence of a **magnetic charge** (magnetic monopole) and then he proves non only the quantization of the electric charge but also the quantization of the magnetic charge itself. This is the popular Dirac **quantization principle**.

♠ In 1931, **H.Hopf** constructed his popular, nowadays known as, **Hopf map**; to show that the third homotopy group of a two-sphere is not trivial.

♠ Modern interest in monopoles stems from **particle theories, unified theories and superstrings theories** which predict their existence, though detection is an open problem in experimental physics.

♠ Nowadays, we already know that a Dirac magnetic monopole is nothing but a circle bundle on a two-sphere. Actually they are the same thing. In this identification, the Dirac quantization principle correspond with the moduli space of circle bundles on a two-sphere.

$\boxed{\text{A Kind of Universality}}$

♣ It is common that a mathematical model has interest, and so it applies, to different and apparently unrelated physical phenomena. In particular, this is very usual in the soliton equation theory which has had (and still has) an enormous impact non only in applied mathematics but also in geometry and in a wide variety of non linear phenomena in physics.

♣ In my opinion, this kind of universality may be strongly related to the fact that these equations have an underlying geometric meaning. Let me recall the following example.

♣ Let **S** be a surface in $\mathbb{R}^3$. The induced metric in a **net of Chebyshev** is given by

$$\boxed{(g_{ij}) = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}}$$

where the function $\omega(u, v)$ measures the angle that make both classes of coordinate curve of that net.

♣ Apply Codazzi-Mainardi to get the Gauss curvature of **S** in this framework satisfying

$$\boxed{\omega_{uv} = \frac{\partial^2 \omega}{\partial u \partial v} = -\mathbf{G}\, \sin\omega.}$$

♣ In particular, if the surface has **constant negative Gaussian curvature**, say -1, then the above equation turns out to be

$$\boxed{\omega_{uv} = \sin\omega.}$$

This is the popular **sine-Gordon equation** which appeared for the first time in 1880 (J.N.Hazzidakis, J. Reine Angew.Math. 88, 68-73) just when the author studied surfaces in the space with constant negative curvature.

♣ In these surfaces, **the asymptotic lines form a Chebyshev net** so in this framework, we can compute the second fundamental form to be

$$\boxed{(h_{ij}) = \begin{pmatrix} 0 & \sin\omega \\ \sin\omega & 0 \end{pmatrix}}$$

♣ The number of functions, locally, specifying a surface with constant negative curvature is reduced to the minimum possible, namely one, that measures the angle between the asymptotic lines. Moreover, this function must be a solution of the sine-Gordon equation.

♣ The converse also works. Let ω be a solution of the sine-Gordon equation, then we define

$$\boxed{(g_{ij}) = \begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}} \qquad \boxed{(h_{ij}) = \begin{pmatrix} 0 & \sin\omega \\ \sin\omega & 0 \end{pmatrix}}$$

One can check that these matrices satisfy the compatibility conditions to be, locally, the first and the second fundamental forms of a unique, up to motions, surface with constant negative curvature, $\mathbf{G} = -1$.

♣ **Surfaces of constant negative curvature are locally in a one-to-one correspondence with the solutions of the sine-Gordon equation.**

$$\boxed{\{\mathbf{G} = -1,\ \text{locally}\}} \qquad \Leftrightarrow \qquad \boxed{\{\omega\ :\ \omega_{uv} = \sin\omega\}.}$$

♣ The equation, as well as several solution techniques, were known in the 19th century, but the equation grew greatly in importance later. The sine-Gordon equation also appears in a number of other physical applications including the propagation of fluxons in Josephson junctions (a junction between two superconductors), the motion of rigid pendula attached to a stretched wire, and dislocations in crystals.

I can't resist quoting another example

(1) Kirchhoff Elastic Rods. At the end of XIX century, Kirchhoff formulated his theory on the elastic rod (A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity* Cambridge University Press, 1927). It is a variational principle where rod configurations are represented by adapted framed curves: a curve, the **centerline of the rod**, and an **adapted material frame** along the curve.

However, more that one century later, the model was showed to be equivalent to that corresponding with **generalized elastic curves** (J.Langer and D.A.Singer, SIAM Review 1996, T.A.Ivey and D.A.Singer, Proc. London Mathematical Society,1999). That is, a variational problem which is governed by a Lagrangian, acting on a suitable space of curves, with density being linear in the length, torsion and squared curvature of curves, respectively

$$\boxed{\mathcal{F}(\gamma) = \int_\gamma \left(a\,\kappa^2 + b\,\tau + c\right)\, ds.}$$

where a, b, c are three constants.

(2) Killing Magnetic Fields. A magnetic field on a Riemannian space is nothing but a closed two-form. In dimension three, it coincides with a **divergence free vector field**. In this framework, an electric test charge, submitted to a magnetic field, V, evolves along curves that are solutions of the **Lorentz force equation**.

$$\boxed{\nabla_{\gamma'}\gamma' = V \wedge \gamma'.}$$

If the magnetic field admits a **vector potential**, $V = \operatorname{Curl} X$, then the **Lorentz equation** can be regarded as the **Euler-Lagrange** one associated with a variational problem stated in terms of the vector potential. However, the existence of a vector potential is **only local** (it is global only for **special topologies** of the background space). Even, if it exists, the **existence is highly not unique**.

The **Lorentz equation of a Gauss magnetic field** in dimension two, **no matter the topology**, is the **field equation of a Lagrangian with density being linear in the curvature of curves**. Notice that while the former equation is of **order two**, the latter is of **order four**. In some sense this is the **prize that we need to pay** in order to avoid the already commented troubles.

★ In **dimension three and constant curvature, no matter the topology**, the **Lorentz equation** associated with a **Killing magnetic field** can be regarded as the **field equation** associated with a **Lagrangian of the type**

$$\boxed{\mathcal{F}(\gamma) = \int_\gamma \big(a\,\kappa^2 + b\,\tau + c\big)\; ds.}$$

More precisely, **a curve is a magnetic trajectory of a charge in a Killing magnetic field if and only if it is the centerline of a Kirchhoff elastic rod**.

Therefore, **the same variational model can be used to describe two apparently unrelated phenomena in physics**. Furthermore, this equivalence **allows one to solve intrinsic, stated problems in both theories**. On one hand, magnetic trajectories are extremals of an action which does not depend on the existence of a potential, on the other hand equations describing Kirchhoff elastic rods reduce order throughout two first integrals.

However, The Surprises Continues

(3) Vortex Filaments. Among soliton equations, the **filament flow** or **binormal flow**

$$\boxed{\gamma_t = \gamma_s \wedge \gamma_{ss},}$$

is particularly simple in form and easy to interpret geometrically. It describes a curve, $\gamma(s,t)$, evolving in dimension three, and aroses as a model to describe thin vortex tubes in ideal three-dimensional fluids. This is also known as the **localized induction equation, LIE** or the **Betchov-Da Rios equation**.

★ This flow was found for the first time by **Luigi Sante Da Rios**, a student in the **University of Padua** of **Tullio Levi Civita**, in the work for his **Laurea in Mathematics** published in 1906 (Rend.Circ.Mat. Palermo). Then, it was rediscovered for **R.Betchov** in 1965 (J. Fluid Mech.) and then for many others (see Renzzo L. Ricca, *Rediscovery of Da Rios equations*, Nature 352, 1991, 561-562).

★ Our interest here in this flow comes from the following fact: **every Killing magnetic trajectory is an initial condition for a soliton solution of the Betchov-Da Rios equation. More precisely, the evolution under the binormal flow of a Killing magnetic trajectory is described by a screw motion together with slippage along the curve**. The converse of this statement is also true.

(4) Schrödinger Equation. The **Hasimoto transformation** (J. Fluid Mech. 1972) identifies the space of connected smooth curves, free of inflections, up to motions, with that of complex valued smooth functions free of zeroes. It associates to each curve its **wave function** as follows

$$\boxed{\mathrm{H}(\gamma(s)) = \psi(s), \qquad \psi(s) = \kappa(s)\, \exp\left(i \int_0^s \tau(u)\, du\right).}$$

★ Suppose that the curve $\gamma(s)$ evolves in the space as a function of time t to produce a space curve $\gamma(s,t)$ which, via Hasimoto transformation, has a wave function, $\psi(s,t)$. Now, if the curve evolves according to the vortex filament (Betchov-Da Rios) equation, then the wave function is a solution of the cubic Schorödinger equation, actually both statements are equivalent

$$\boxed{\gamma_t = \gamma_s \wedge \gamma_{ss}} \quad \Leftrightarrow \quad \boxed{\psi_t = i\left(\psi_{ss} + \left(\frac{1}{2}|\psi|^2 - A\right)\psi\right)}$$

WILLMORE SEVERAL TIMES IN PHYSICS

►► It is my plan here to exhibit the above mentioned universality for Willmore program. So, I will describe several non linear physical phenomena where the Willmore functional naturally appears. This is a consequence of the strong underlying geometrical meaning of such models, and the equations governing them.

ELASTIC SURFACES AND MEMBRANES

◊◊◊

Origin of the elasticity theory.- As far as I know, the origin of the **theory of elastic membranes** comes from the early nineteenth century. At that time, 1808, the german physicist, and amateur violinist, **M.Chladni** went to Paris to present his intriguing experiment on the vibration of elastic plates. He sprinkled sand on an elastic plate. Then, he played the edge of the plate with his violin bow and a variety of interesting shapes were observed to be formed by the sand.

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The French Academy Prize.- In 1809, the French Academy sponsored a competition with the challenge of formulating a mathematical theory of elastic surfaces in keeping with the empirical evidence observed by Chladni. We have already mentioned that in 1812, S.D.Poisson wrote the free energy for a solid elastic membrane as

$$\boxed{\mathcal{P}(\mathbf{S}) = \frac{k_c}{2} \int_{\mathbf{S}} (2\mathbf{H})^2 \, dA,}$$

Poisson would be the man expected to take the hint. However, only one enter was in, and it was not to Poisson. In fact, along three years, Sophie Germain submitted tree separate papers to get the prize. The only entrant in the competition all three years. She failed to win anything in her first two submissions, but on her third attempt, submitted around May 1816, the judges deemed her paper entitled Memoir on the Vibrations of Elastic Plates worthy of a prize, despite pointing out some remaining mathematical deficiencies. Germain refused to attend the award ceremony. She apparently felt the judges did not fully appreciate her work, and that the scientific community as a whole did not show her the respect she believed she had clearly earned. Germain had a point. Poisson, her chief rival on the subject of elasticity, was one of the judges and technically a colleague, yet he pointedly avoided having serious discussions with her, and snubbed her in public. Nonetheless, Germain became the first woman to attend lectures at the French Academy of Sciences who was not the wife of a member the highest honor that body had ever conferred on a woman.

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It is a constant along the history, when one tries to provide a physical model at least variationally, the problem of searching Lagrangian densities. The theory of elastic surfaces is not an exception. The search for Lagrangian to do the role of elastic energy has a long history. **Sophie Germain** proposed the following idea to construct elastic Lagrangians: **the elastic energy of a membrane is the Lagrangian associated with a density which is an even, symmetric function of the principal curvatures of the membrane**.

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In the last half century, physicists turned their interest to cell membranes and bi-layers (lipid, proteins and carbohydrates). Therefore, in the 1970's **Canham and Helfrich**, according to the Germain philosophy, proposed a quadratic, in the principal curvatures, elastic energy density, namely

$$\mathcal{E}(\mathbf{S}) = \int_{\mathbf{S}} \left(a + b\,\mathbf{H}^2 + c\,\mathbf{G}\right)\, dA$$

and elastic energy for bi-layers

$$\mathcal{E}(\mathbf{S}) = \int_{\mathbf{S}} \left(a + b(\mathbf{H} + b_o)^2 + c\,\mathbf{G}\right)\, dA$$

- P.B.Canham, Theor.Biol. 26 (1970), 61.
- W.Helfrich, Z.Naturforsch 28c (1973) 693.

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The critical points of the elastic energy action are called equilibrium elastic surfaces.

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Stable equilibrium **spherical shapes** are known for a long time.

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Besides these spherical solutions, nowadays are known other analytic solutions: **revolution tori with radii in the rate** $\sqrt{2}$ and **biconcave disks as red blood cells**.

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There are also a lot of known numerical solutions of the corresponding field equations.

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Notice that if we consider that fluctuations of elastic surfaces do not change the topology, then the total Gaussian curvature is constant from Gauss-Bonnet and so the Canham-Helfrich elastic energy is

$$\mathcal{E}(\mathbf{S}) = \int_{\mathbf{S}} \left(a + b\,\mathbf{H}^2\right)\, dA$$

$\diamondsuit\diamondsuit\diamondsuit$

Moreover, we may assume that $b \neq 0$, otherwise we are talking about minimal surfaces which is not real. Therefore, the Canham-Helfrich elastic energy looks like a kind of Willmore functional.

BOSONIC STRING THEORIES

★ In contrast with the theories of membranes and vesicles, when one studies a theory to describe a **bosonic string**, the surfaces, **worldsheets**, which are generated from the evolution of curves, **strings**, and more generally the submanifolds that play the role of **branes**, **worldvolumes**, could self-intersect, so they are immersed and not necessarily embedded.

★ Roughly speaking, a bosonic string theory on a gravitational space, say (M, g), is defined when one has **curves, strings**, that **dynamically evolve generating surfaces, worldsheets**, which are s**olutions of the field equations associated with some string action**. This idea, obviously, **extend to higher dimensions** and then one has the notion of **brane generating a worldvolume**. Consequently, as usual, the **first important problem one has to define a consistent bosonic string theory is the choice of a suitable string action**.

★ Certainly, the above suitable meaning, should include, at least, that the action is invariant under motions in the space. Now, the simplest invariant for worldsheets, surfaces, is the area. In fact, historically, the first bosonic string theory was constructed by **Yoichiro Nambu and Tetsuo Goto** and it correspond with a string action that, up to a coupling constant, measures the are of the worldsheets.

$$\boxed{\mathcal{NG}(\mathbf{S}) = \mu \int_{\mathbf{S}} dA}$$

★ The **Nambu-Goto string theory** presents serious problems. The main is the great difficulties to be **consistently quantized**. Moreover, **it can not describe compact worldsheets generated by closed strings because compactness and minimality are not good friends**.

★ To **overcome these troubles**, it is necessary to consider another, perhaps more subtle, string action. There are several **good reasons to invite** the **extrinsic geometry of worldsheets**, in (M, g), to get involved in the string action. From a geometrical point of view, it seems clear that the **worldsheets are extended objects** obtained when strings evolve in (M, g). Now, the nature of strings is exclusively extrinsic in (M, g) so string configurations must involve the extrinsic geometry of the extended objets.

★ In the eighties of the last century, several physicists, (S.Deser and B.Zumino and independently L.Brink, P. Di Vecchia and P.S.Howe), proposed a **string action a la Willmore**. However, nowadays that action is known as the **Polyakov action**, after he (and independently Kleinert) made use of it to obtain the **quantum chromo-dynamics**, QCD (A.Polyakov (Nucl.Phys. B 268 (1986) 406-412) and H.Kleinert (Phys.Lett. B 174 (1986) 335-338). Therefore, **they added to the Nambu-Goto action a multiple of the Willmore functional**

$$\boxed{\mathcal{PK}(\mathbf{S}) = \mu \int_{\mathbf{S}} dA + \nu \int_{\mathbf{S}} \mathbf{H}^2 \, dA,}$$

$$\boxed{\mathcal{PK}(\mathbf{S}) = \mu \int_{\mathbf{S}} dA + \frac{\nu}{4} \int_{\mathbf{S}} |\Delta\psi|^2 \, dA,}$$

★ In this framework a string configuration is a curve that evolves in the background generating a Willmore, perhaps constraint, worldsheet.

★ Now, the natural problem is

How We Obtain Willmore Surfaces?

How We Obtain Willmore Tori?

WILLMORE SURFACES

★ We already know that **elastic membranes** and **string configurations** are strongly related with **Willmore actions**. In this first approach, we will restrict ourselves to the case of **compact boundary free configurations**. However, as we will see later, boundary configurations and corresponding boundary value problems can be considered.

★ In this context and, may be, under the influence of the **Willmore conjecture** there was a *golden fever* to get Willmore surfaces and specially **Willmore tori**.

★ This problem, in spaces with constant curvature, is equivalent to that of looking for genus one compact solutions of the following equation

$$\boxed{\Delta \mathbf{H} = 2(\mathbf{H}^2 - \mathbf{G})\mathbf{H},}$$

★ Certainly, this equation is impossible to be completely solved. However, **one can look for solutions with some prescribed degree of symmetry**. The approach, we are going to explain, uses, as an important ingredient, the **principle of symmetric criticality**.

Principle of Symmetric Criticality

A version due to R.S.Palais

♦ In many areas, including Mathematics and Physics, it has proved extremely useful to look for symmetries and to exploit them, if they exist, in problem solving. The success of this procedure is based in the **principle of symmetric criticality**. This has been used in many applications of the calculus of variations, without being particularly noticed.

♦ An early, typical example of the **implicit use** of this principle can be found in the **Weyl derivation** of the **Schwarzshild solution** of the **Einstein field equations**.

- H.Weyl, *Space-Time-Matter*, Dover, 1951.

♦ Let me expose a **simplified version** of that situation. Consider a classical field theory, say on $\mathbb{R}^3$, which is governed by a $\mathbf{SO}(3)$ invariant Lagrangian, the **total scalar curvature**, acting on a space of dynamical variables, the space of **metrics**

$$\boxed{\mathcal{E} : \mathcal{M} \to \mathbb{R}.}$$

♦ Suppose, we wish to obtain **critical points** of the **above action which are themselves SO**(3) **invariant**. Then, **compute the variation of** $\mathcal{E}$ **along the points, metrics, which are SO**(3) **invariant, equal to zero and solve this equation**.

♦ This is the procedure used in many places and in particular by H.Weyl to get the Schwarzschild solution of the Einstein field equations. The mistake comes because it was used as a **necessary and sufficient condition for a SO**(3) **invariant field configuration**. Obviously, this represents a necessary condition for a field configuration, metric, to be a critical point of $\mathcal{E}$. However, **sufficient condition needs proof**. It constitutes **the essence of the principle**.

♦ In 1975, **S.Coleman** made reference to this principle. It was not until 1979 that **R.S.Palais** gave a precise formulation of it and a proof of its validity in fairly general circumstances.

- S.Coleman, *Classical Lumps and their Quantum descendents*, International School of Subnuclear Physics, Erice 1975.
- R.S.Palais, Comm.Math.Phys. 1979.

♦ **We discuss here a simplified version of the Palais one, which will be enough to our purposes**.

The Setting we need

⧫ **Suggestive Formulation: Any critical symmetric point is a symmetric critical point?**

⧫ Let start with the following ingredients

1. A **smooth manifold**, $\mathcal{N}$, on which a **group**, $\mathbf{G}$, acts by **diffeomorphisms**.

2. A **functional**, $\mathcal{W} : \mathcal{N} \to \mathbb{R}$, which is **G-invariant**.

$$\mathcal{W}(a.\varphi) = \mathcal{W}(\varphi), \quad \forall a \in \mathbf{G}$$

3. In this setting, one can define the set of **symmetric points**

$$\boxed{\mathcal{N}_{\mathbf{G}} = \{\varphi \in \mathcal{N} / a.\varphi = \varphi, \forall a \in \mathbf{G}\}.}$$

♦ Now, we have the followings sets of critical points

- The **set of critical points**, Σ, of $\mathcal{W} : \mathcal{N} \to \mathbb{R}$.

- The **set of critical points**, $\Sigma_{\mathbf{G}}$, of the **restriction** of $\mathcal{W}$ to the set of symmetric points, $\mathcal{N}_{\mathbf{G}}$.

♦ Certainly, in order to talk about $\Sigma_{\mathbf{G}}$, it is necessary that $\mathcal{N}_G$ is endowed with a structure of differentiable manifold, usually a submanifold of $\mathcal{N}$. Therefore, the first part of the Palais procedure is to **look for conditions to ensure that $\mathcal{N}_G$ is a submanifold of $\mathcal{N}$.**

♦ Once one has the above, put

$$\boxed{\mathbf{T}_\varphi \mathcal{N} = \mathbf{T}_\varphi \mathcal{N}_{\mathbf{G}} \oplus \mathbf{T}_\varphi \mathcal{N}_{\mathbf{G}}^{\perp}}$$

♦ Obviously, one has

$$\boxed{\Sigma \cap \mathcal{N}_{\mathbf{G}} \subset \Sigma_{\mathbf{G}}},$$

♦ So in order to ensure the converse, it is necessary to show

$$\boxed{D_\varphi \mathcal{W}_{/\mathbf{T}_\varphi \mathcal{N}_{\mathbf{G}}} = 0} \qquad \Rightarrow \qquad \boxed{D_\varphi \mathcal{W}_{/\mathbf{T}_\varphi \mathcal{N}_{\mathbf{G}}^{\perp}} = 0}$$

♦ The **Palais procedure codifies several sufficient conditions to ensure both statements**. Now, the **principle is established as follows**

$$\boxed{\Sigma \cap \mathcal{N}_G = \Sigma_G.}$$

♦ For example if the group $\mathbf{G}$ is **compact**, then the Palais principle holds and this will be enough to our purposes.

$\boxed{\boxed{\textbf{A Counter-example for sceptics}}}$

On $\mathcal{N} = \mathbb{R}^2$, we consider the action of the additive group $\mathbf{G} = \mathbb{R}$ defined by

$$\boxed{\mathbf{G} \times \mathcal{N} \to \mathcal{N}, \qquad (t, (x, y)) = \phi_t(x, y) = (x + y^k t, y).}$$

where $k \in \mathbb{N}$. The set of symmetric points is

$$\boxed{\mathcal{N}_{\mathbf{G}} = \{p = (x, 0) \ : \ x \in \mathbb{R}\}}$$

For an arbitrary function, $f : \mathbb{R} \to \mathbb{R}$, define the functional

$$\boxed{\mathcal{W} : \mathcal{N} \to \mathbb{R}, \qquad \mathcal{W}(x, y) = f(y).}$$

$$\mathcal{W}(x, y) = f(y), \quad f \text{ arbitrary}, \quad (\mathcal{W}/\mathcal{N}_G)(x, 0) = f(0).$$

Then, (x, y) is a critical point of $\mathcal{W}$ if and only if $f'(y) = 0$ and so

$$\boxed{\Sigma = \{(x, y) \ : \ f'(y) = 0\}}$$

$$\boxed{\Sigma \cap \mathcal{N}_{\mathbf{G}} = \{(x, 0) \ : \ f'(0) = 0\}}$$

On the other hand, the restriction of $\mathcal{W}$ to $\mathcal{N}_{\mathbf{G}}$ is a constant

$$\boxed{\mathcal{W} : \mathcal{N}_{\mathbf{G}} :\longrightarrow \mathbb{R}, \qquad \mathcal{W}(x,0) = f(0),}$$

$$\boxed{\Sigma_{\mathbf{G}} = \mathcal{N}_{\mathbf{G}}.}$$

Consequently, if we choose a function, f, with $f'(0) \neq 0$, then

$$\boxed{\Sigma_{\mathbf{G}} = \mathcal{N}_{\mathbf{G}}} \quad \text{the } \{x\} - \text{axis},$$

while

$$\boxed{\Sigma \cap \mathcal{N}_{\mathbf{G}} = \emptyset.}$$

SOME APPLICATIONS

(1) Willmore Revolution Tori

The class of revolution tori that are Willmore can be obtained using an approach that mainly involves the following ingedients

- **The principle of Palais.**
- **A suitable conformal change.**
- **Some geometric spices.**

Preparing the Space

A conformal Change

Recall that the space $\mathbf{M} = \mathbb{R}^3 - (\{z\} - axis)$ endowed with the Euclidean metric, $\overline{g}$, is the warped product of the Euclidean half plane $(\mathbf{P}, g)$, with

$$\boxed{\mathbf{P} = \{(x, 0, z) \in \mathbb{R}^3 \;:\; x > 0\}}$$

and the unit circle

$$\boxed{\mathbf{M} = \mathbf{P} \times_f \mathbb{S}^1, \quad \overline{g} = g + f^2 dt^2,}$$

and the warping function

$$f : \mathbf{P} \to \mathbb{R}, \quad f(x, 0, z) = x.$$

Make the following obvious conformal change

$$\boxed{\tilde{g} = \frac{1}{f^2}\,\overline{g} = \frac{1}{f^2}\,g + dt^2, \quad \text{Riemannian product.}}$$

Now, notice that

$$\boxed{\left(\mathbf{P}, \frac{1}{f^2}\,g\right) = \text{hyperbolic plane.}}$$

Therefore, the Euclidean metric in M is conformal to a Riemannian product of a hyperbolic plane and a circle.

$\boxed{\textbf{Taking out the Palais ingredients}}$

- For a torus, $\mathbf{T}$ let $\mathcal{N} = \mathbf{I}(\mathbf{T}, \mathbf{M})$ be the **space of immersions** in Euclidean space.

- The group $\mathbf{G} = \mathbb{S}^1$ acts on $\mathbf{M}$ through **rotations around the $\{z\}$-axis** and so on $\mathcal{N}$ in an obvious way.

- $\overline{\mathcal{W}} : \mathcal{N} \to \mathbb{R}$ is the **Willmore action for Euclidean metric**.

- $\mathcal{N}_{\mathbf{G}}$ is the **space of revolution tori with axis $\{z\}$**. Certainly, it is **identified with the space, C, of closed curves in the half plane P**.

- **Apply Palais: A revolution torus is Willmore if and only if it is Willmore among revolution tori**.

Computing Willmore Energy for Revolution Tori

▶ The computation of Willmore energy for a revolution torus is not a good idea.

▶ However, since it is invariant under extrinsic conformal changes, one can uses to that purpose the conformal metric $\tilde{g}$ which is more convenient.

▶ Let γ be a closed curve in the half plane $\mathbf{P}$ and denote by $\mathbf{T}_\gamma$ the revolution torus obtained when rotating the curve around the $\{z\}$-axis, then

$$\boxed{\overline{\mathcal{W}}(\mathbf{T}_\gamma) = \tilde{\mathcal{W}}(\mathbf{T}_\gamma) = \int_{\mathbf{T}} \left(\tilde{\mathbf{H}}^2 + \tilde{\mathbf{R}}\right) dA}$$

Seasoning with Spices from Elementary Geometry

- $(\mathbf{M}, \tilde{g})$ is a Riemannian product $\Rightarrow$ $\boxed{\tilde{\mathbf{R}} = 0}$.
- Parallels are geodesics in the metric $\tilde{g} \Rightarrow \boxed{\tilde{\mathbf{H}} = \frac{1}{2}\kappa}$, where κ is the curvature of γ in the hyperbolic plane $\left(\mathbf{P}, \frac{1}{f^2}\, g\right)$.

▶ Consequently, we have

$$\boxed{\overline{\mathcal{W}}(\mathbf{T}_\gamma) = \tilde{\mathcal{W}}(\mathbf{T}_\gamma) = \frac{\pi}{2} \int_\gamma \kappa^2(s)\, ds}$$

CONCLUSION.- $\mathbf{T}_\gamma$ is Willmore if and only if γ is a critical point of the following functional acting on closed curves in the hyperbolic plane

$$\boxed{\mathcal{E} : \mathbf{C} \to \mathbb{R}, \quad \mathcal{E}(\gamma) = \int_\gamma \kappa^2(s)\, ds}$$

▶ Those curves correspond with the classical idea of **elastica**, in the sense of the **Daniel Bernoulli** model.

$\boxed{\mathbf{T}_\gamma \text{ is Willmore}} \quad \Leftrightarrow \quad \boxed{\gamma \text{ is elastica in hyperbolic plane}}$.

(2) Willmore Hopf Tori and More

★ The search for **Willmore Hopf tori** is a special case of a technique working to obtain Willmore tori, with a rotational symmetry, in a wide family of conformal classes which are related with the **Kaluza-Klein metrics on circle bundles**.

Kaluza-Klein Metrics.

★ Start with a principal fiber bundle,

$$\boxed{\mathbf{p} : \mathbf{P} \to \mathbf{M}, \quad \text{with group } \mathbf{G}}$$

★ Now consider the following ingredients

- A metric, g on the basis, $\mathbf{M}$,
- A connection, gauge potential, $\omega : \mathbf{TP} \to g$,
- An invariant metric, $d\sigma^2$, on the group, $\mathbf{G}$,
- A positive function, f, on the basis.

★ With these ingredients, we construct a metric on $\mathbf{P}$ as

$$\boxed{\bar{g} = \mathbf{p}^*(g) + (f \circ \mathbf{p})^2\, \omega^*(d\sigma^2),}$$

which is called a **generalized Kaluza-Klein metric**. In particular, if f is constant, then one has a **Kaluza-Klein metric** which, in other settings, is also called a **bundle-like metric**. These metrics were used to provide a natural model to unify **gauge fields** and **gravitation**.

Some Properties of these Metrics.

1. $\mathbf{p} : (\mathbf{P}, \bar{g}) \rightarrow (\mathbf{M}, g)$ is a **Riemannian submersion**. Furthermore, the leaves are totally geodesics if and only if f is constant, Kaluza-Klein or bundle-like.

2. The action of $\mathbf{G}$ on $\mathbf{P}$ **is made through isometries** of $(\mathbf{P}, \bar{g})$.

3. The above fact and the principle of symmetric criticality, when it works for example if $\mathbf{G}$ is compact, motivate the origin in physics of these metrics. Therefore, assume that $d\sigma^2$ is fixed and choose $f = 1$. Then, the **Kaluza-Klein metrics can be considered as functions depending on both g and ω**

$$\boxed{\bar{g}(g, \omega) = \mathbf{p}^*(g) + \omega^*(d\sigma^2).}$$

Suppose that $\bar{g}$ is a solution of the **Einstein field equations**, i.e. a **critical points of the total scalar curvature**. Then, we can take particular variations of this metric:

- Change $\bar{g}$ through variations of g and obtain that g satisfies the Einstein field equations in the basis.

- Change $\bar{g}$ through variations of ω and obtain the Yang-Mills equations for ω.

Thus, the Einstein field equations and the Yang-Mills equations arise simultaneously from a single variational principle that derives from the total scalar curvature of the Kaluza-Klein metrics.

4. The special case where $\mathbf{G} = \mathbb{S}^1$, $(d\sigma^2 = dt^2)$, was the first considered by **T.Kaluza** (1921) and **O.Klein**(1926) to unify the **gravity** and **electromagnetism**. In this case, Einstein field equations and Maxwell equations appear from Einstein field equations of Kaluza-Klein metrics on **circle fiber bundles**.

5. Notice that given a generalized Kaluza-Klein metric, we can find a Kaluza-Klein one which is **conformal** to the former.

$$\tilde{g} = \frac{1}{(f \circ \mathbf{p})^2}\, \bar{g} = \mathbf{p}^* \left(\frac{1}{f^2}\, g \right) + \omega^*(dt^2)$$

In particular the Willmore programs in $(\mathbf{P}, \tilde{g})$ **and** $(\mathbf{P}, \bar{g})$ **coincide**.

Willmore Tori In Conformal Kaluza-Klein Classes

$$\boxed{\mathbf{p} : \mathbf{P} \to \mathbf{M}, \qquad \mathbf{G} = \mathbb{S}^1}$$

★ Let γ be an immersed curve in $\mathbf{M}$, then $\mathbf{N}_\gamma = \mathbf{p}^{-1}(\gamma)$ is a surface immersed in $\mathbf{P}$, which is invariant under the $\mathbb{S}^1$-action. Certainly, $\mathbf{N}_\gamma$ is embedded if γ is simple.Moreover, all the surfaces in $\mathbf{P}$ that are $\mathbb{S}^1$-invariant are obtained in this way, they are complete lifts of curves in $\mathbf{M}$.

★ The surfaces $\mathbf{N}_\gamma = \mathbf{p}^{-1}(\gamma)$ are flat. They can be parametrized through fibers and horizontal lifts of the cross section γ.

$$\boxed{\Phi(s,t) = e^{it}\,\bar{\gamma}(s).}$$

★ If γ is closed then, $\mathbf{N}_\gamma$ is a torus. However, the horizontal lifts of γ are not closed, in general, because ω could have non trivial holonomy.

★ The mean curvature, $\mathbf{H}$, of $\mathbf{N}_\gamma$ in $(\mathbf{P}, \tilde{g})$ and the curvature, κ, of γ in $(\mathbf{M}, \frac{1}{f^2}\, g)$ are nicely related by

$$\boxed{\mathbf{H}^2 = \frac{1}{4}(\kappa^2 \circ \mathbf{p})}$$

★ Now, the natural problem that appears in the setting provided by the Kaluza-Klein conformal classes is

How choose a closed curve, γ, in $(\mathbf{M}, \frac{1}{f^2}\, g)$ in order to $\mathbf{N}_\gamma$ is a Willmore torus in $(\mathbf{P}, \tilde{g})$?

★ To answer this problem, we apply Palais. Then we only need to compute the additional term $\mathbf{R}$, associated with the metric $\tilde{g}$, that appear in the Willmore density. It is computed as measuring the divergence of the Ricci curvature that provides the holonomy. In other words

$$\boxed{\mathbf{R} = \frac{1}{2}\left(\mathbf{r}(\gamma', \gamma') \circ \mathbf{p} - \tilde{\mathbf{r}}(\bar{\gamma'}, \bar{\gamma'})\right),}$$

where $\mathbf{r}$ and $\tilde{\mathbf{r}}$ stand for corresponding Ricci curvatures.

★ In the unit vector bundle, $\mathbf{UM}$ of $(\mathbf{M}, \frac{1}{f^2}\, g)$ define a potential as

$$\boxed{\Psi : \mathbf{UM} \to \mathbb{R}, \quad \Psi(v) \circ \mathbf{p} = 2\left(\mathbf{r}(v, v) \circ \mathbf{p} - \tilde{\mathbf{r}}(\bar{v}, \bar{v})\right).}$$

★ Then, the restriction of the Willmore energy to the class of $\mathbb{S}^1$-invariant tori is

$$\boxed{\mathcal{W}(\mathbf{N}_\gamma) = \frac{1}{4}\int_{\mathbf{N}_\gamma} \left(\left(\kappa^2 + \Psi(\gamma')\right) \circ \mathbf{p}\right)\, dA}$$

integrate along the fibers to get

$$\boxed{\mathcal{W}(\mathbf{N}_\gamma) = \frac{\pi}{2}\int_\gamma \left(\kappa^2 + \Psi(\gamma')\right)\, ds}$$

ANSWER: $\mathbf{N}_\gamma = p^{-1}(\gamma)$ is Willmore in $(\mathbf{P}, [\bar{g}])$ if and only if γ is a critical point in $(\mathbf{M}, \frac{1}{f^2}\, g)$ of the action $\mathcal{E} : \Gamma \to R$ acting on closed curves and defined by

$$\boxed{\mathcal{E}(\gamma) = \int_\gamma \left(\kappa^2 + \Psi(\gamma')\right) ds.}$$

Some Examples

(1) Revolution Tori. It should be noticed that the revolution tori, in Euclidean space, also appear in this context. In fact, choose

- $\mathbf{P} = \mathbb{R}^3 - (\{z\} - axis)$
- $\mathbf{M} = \{(x, 0, z) \in R^3 \, : \, x > 0\}.$

Since $\mathbf{P} = \mathbf{M} \times \mathbb{S}^1$, it can be seen as a trivial $\mathbf{S}^1$ bundle. Also recall that

$$\boxed{\bar{g} = g + f^2 dt^2,} \qquad \textbf{generalized Kaluza-Klein}$$

$$\boxed{\tilde{g} = \frac{1}{f^2}\,\bar{g} = \frac{1}{f^2}\,g + dt^2,} \qquad \textbf{Kaluza-Klein}$$

Now, notice that the potential is zero, $\Psi = 0$, to get the already well known result.

$\boxed{\textbf{(2) Hopf Tori.}}$ Let consider the well known Hopf map regarded as a circle principal bundle

- $\mathbf{P} = \mathbb{S}^3$ endowed with the metric, $\bar{g}$, with curvature 1.
- $\mathbf{M} = \mathbb{S}^2$ endowed with the metric, g, with curvature 4.
- $\mathbf{p} : \mathbb{S}^3 \to \mathbb{S}^2$ the usual Hopf map.

If ω is the gauge potential, $\omega = V^\sharp$, where V is a Hopf vector field, then

$$\boxed{\bar{g} = \mathbf{p}^*(g) + \omega^*(dt^2),} \quad \textbf{Kaluza-Klein}$$

One can compute the potential to be constant, $\Psi(v) = 4$ for all $v \in \mathbf{U}\mathbb{S}^2$ and then obtain the following result by **U.Pinkall** (Inventiones Mathematicae, 81 (1985), 374-379)

A Hopf torus, $\mathbf{T}_\gamma$ is Willmore if and only its cross section is a 4-constraint elastica in the two sphere. That is, a critical point of the following action acting on closed curves in the two sphere

$$\boxed{\mathcal{E}(\gamma) = \int_\gamma (\kappa^2 + 4)\, ds}$$

(3) Willmore tori in conformal Berger's spheres.

♠ Let consider the so-called **canonical variation of the Hopf map**,

$$\boxed{\{\mathbf{p} : (\mathbb{S}^3, \bar{g}_r) \rightarrow (\mathbb{S}^2, g) \,/\, r > 0\}}$$

$$\boxed{\bar{g}_r = \mathbf{p}^*(g) + r^2 dt^2,}$$

it provides a one-parameter family of Kaluza-Klein metrics on the three sphere.

♠ It is clear that $\bar{g}_1 = \bar{g}$, the unit round sphere, and it is the unique Einstein metric in this family. However, all these metrics have **constant scalar curvature**. Notice that, in dimension three, Einstein implies constant curvature.

♠ Geometrically the three spheres $(\mathbb{S}^3, \bar{g}_r)$ are the so called **Berger's spheres**, which can be regarded as geodesic spheres in the complex projective plane, $\mathbb{CP}^2$, endowed with the **Fubini-Study metric**. Moreover in the conformal classes $\{[\bar{g}_r]\}$ with $r \neq 1$, one can not find any metric with constant curvature.

♠ One can compute the potential to be constant, $\Psi = 4r$, and then we get the following result (**M.Barros**, Math.Proc.Camb. Phil.Soc., 1997)

$\mathbf{p}^{-1}(\gamma)$ is Willmore in the conformal Berger sphere, $(\mathbb{S}^3, [\bar{g}_r])$ if and only if the curve, γ, is an elastica in the round two sphere with Lagrange multiplier $4r$.

$$\boxed{\mathcal{E}(\gamma) = \int_\gamma (\kappa^2 + 4r)\, ds.}$$

♠ One can also see that there exist Willmore tori in $(\mathbb{S}^3, [\bar{g}_r])$, with $0 < r < 1$, and Willmore energy

$$\boxed{\mathcal{W}(p^{-1}(\gamma)) < 2\pi^2,}$$

♠ which contrasts with the Willmore conjecture in $[\bar{g}_1]$.

(4) More Applications

★ This method has been widely exploited along the literature to obtain examples of Willmore tori and Willmore-Chen submanifolds, its natural higher dimensional extension, in many different conformal classes of either Riemannian or Lorentzian metrics. Let me mention a few of them

- The **anti-de-Sitter conformal class and the conformal classes of its canonical variations**.
- The **Kaluza-Klein conformal classes in circle fiber bundles on a surface of revolution**.
- The **Kaluza-Klein conformal classes in circle fiber bundles on a lens space**.
- The **odd dimensional conformal round spheres and those corresponding to its canonical variations**.
- The **conformal Stiefel spaces and and their conformal classes in the canonical variations**.

★ The Stiefel space is the unit tangent bundle of a round sphere, $\mathbf{U}\mathbb{S}^n$. This space admits a unique $\mathbb{S}^1$-invariant Einstein metric, the so called Kobayashi's metric (which by the way is Kaluza-Klein), which provides a Riemannian submersion to the complex quadric endowed with its usual metric as a complex hypersurface of $\mathbb{CP}^n$.