The space of Lorenzian flat tori in anti-de Sitter 3-space

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Objective: To describe all the isometric immersions from $\mathbb{L}^2$ to $\mathbb{H}^3_1$

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  Hopf cylinders and Hopf torus.
- Classic result [Dajczer, Nomizu (1981)]
  Construction of i.i. $\mathbb{L}^2 \longrightarrow \mathbb{H}^3_1$ as the product of two adequate curves in $\mathbb{H}^3_1$ (one of them time-like and the other space-like).

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  Open problems.

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- Are there other examples of i.i. $\mathbb{L}^2 \to \mathbb{H}^3_1$?

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  Open problems.

- Are there other examples of i.i. $\mathbb{L}^2 \to \mathbb{H}^3_1$?
- Which of them are tori?
- Solve Dajczer-Nomizu problems.

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- Bianchi (1896) - Spivak(1975)

  General method to construct i.i. $\mathbb{R}^2 \longrightarrow \mathbb{S}^3$

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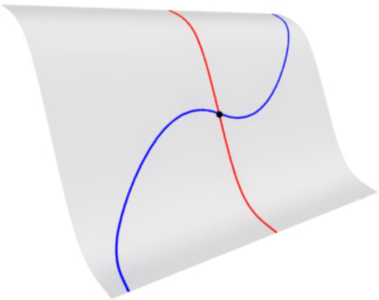
  Two asymptotic curves through each point.



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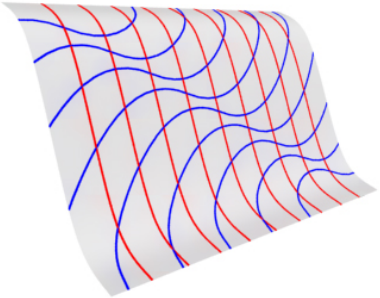
  Two families of asymptotic curves that differ by a translation in $\mathbb{S}^3$. One has torsion $\tau = 1$, the other $\tau = -1$.



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Strategy: **Try to extend Kitagawa's ideas to the Lorentzian setting**

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- Asymptotic curves do not have, in general, constant causal character $\rightarrow$ We do not have Tchebysheff parameters.
- We cannot restrict ourselves to regular curves in $\mathbb{H}^2$, we need to consider curves with wavefront singularities.

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A pseudo-quaternionic model

- $\mathbb{R}^4_2 \rightsquigarrow \mathbb{R}^4$ with the metric $\langle\ ,\ \rangle = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2$,

  $\mathbb{H}^3_1 = \{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\}$.

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  $\mathbb{H}^3_1 = \{x \in \mathbb{R}^4_2 : \langle x, x\rangle = -1\}$.
- We identify $\mathbb{R}^4_2 \equiv \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$

Product rules:
$$\begin{array}{rclrclrcl} i^2 &=& -1, & ji &=& -k, & ki &=& j, \\ ij &=& k, & j^2 &=& 1, & kj &=& i, \\ ik &=& -j, & jk &=& -i, & k^2 &=& 1. \end{array}$$

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- Conjugation: $z = a + bi + cj + dk \rightsquigarrow \bar{z} = a - bi - cj - dk$

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- Conjugation: $z = a + bi + cj + dk \rightsquigarrow \bar{z} = a - bi - cj - dk$
- $z \in \mathbb{H}^3_1 \Leftrightarrow z^{-1} = \bar{z}$
- The Lie group structure induced on $\mathbb{H}^3_1$ by this product is the one for which $\langle\ ,\ \rangle$ is bi-invariant.
  i.e. $\langle x, y\rangle = \langle zx, zy\rangle = \langle xz, yz\rangle \ \forall\, x, y, z \in \mathbb{H}^3_1$

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The characteristic parameters

$f(x, y) : \mathbb{L}^2 \longrightarrow \mathbb{H}^3_1$ isometric immersion

$N(x, y)$ unit normal, chosen so that $\{f, f_x, f_y, N\}$ $+$oriented in $\mathbb{R}^4_2$

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$$f(x,y) : \mathbb{L}^2 \longrightarrow \mathbb{H}^3_1$$

$I = -dx^2 + dy^2$

$II = \phi_{xx} dx^2 + 2\phi_{xy} dxdy + \phi_{yy} dy^2$

with $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1.$

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In our case the asymptotic curves have in general varying causal character. We do not have Tchebysheff parameters.

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This correspondence is not *geometric*, it depends on the choice of the specific coordinates (x,y)

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with $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1$.

$$\tilde{f}(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{S}^3$$

$\tilde{I} = dx^2 + dy^2$

$II = \phi_{xx} dx^2 + 2\phi_{xy} dxdy + \phi_{yy} dy^2$

with $\phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1$.

This correspondence is not *geometric*, it depends on the choice of the specific coordinates (x,y)

$\tilde{f}$ complete flat surface in $\mathbb{S}^3$ $\rightarrow$ globally defined Tchebysheff coordinates (u,v)

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We can express the Lorentzian metric I in terms of the coordinates (u, v)

$$
\begin{aligned}
I &= -\cos(2\omega_1)du^2 - 2\cos(\omega_1 - \omega_2)dudv - \cos(2\omega_2)dv^2 \\
II &= 2\sin(\omega_1 + \omega_2)dudv
\end{aligned}
$$

$$\omega_1(u), \omega_2(v) \in C^\infty(\mathbb{R}), \qquad 0 < \omega_1(u) + \omega_2(v) < \pi$$

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$$\omega_1(u), \omega_2(v) \in C^\infty(\mathbb{R}), \qquad 0 < \omega_1(u) + \omega_2(v) < \pi$$

$(u, v) \quad \longrightarrow \quad$ **Characteristic parameters**

The characteristic parameters

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$(u, v) \quad \rightarrow \quad$ **Characteristic parameters**

Formula for the change of coordinates $(u, v) \rightarrow (x, y)$

$$
\begin{cases}
x(u, v) = \int \cos\omega_1 du + \int \cos\omega_2 dv \\
y(u, v) = \int \sin\omega_1 du - \int \sin\omega_2 dv
\end{cases}
\qquad [**]
$$

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Proposition

$$\begin{array}{c} \omega_1(u), \omega_2(v) \in C^\infty(\mathbb{R}) \\ 0 < \omega_1(u) + \omega_2(v) < \pi \end{array} \longleftrightarrow \begin{array}{c} f \text{ flat Lorentzian surface in } \mathbb{H}^3_1 \\ I \text{ and } II \text{ given by } [*] \end{array}$$

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Completeness (i.e. $f : \mathbb{L}^2 \to \mathbb{H}^3_1$) $\Longleftrightarrow$ $(x(u,v), y(u,v))$ given by $[**]$ is a global diffeomorfism

The characteristic parameters

Proposition

$$\begin{array}{c} \omega_1(u), \omega_2(v) \in C^\infty(\mathbb{R}) \\ 0 < \omega_1(u) + \omega_2(v) < \pi \end{array} \longleftrightarrow \begin{array}{c} f \text{ flat Lorentzian surface in } \mathbb{H}^3_1 \\ I \text{ and } II \text{ given by } [*] \end{array}$$

$$\begin{array}{c} \text{Completeness} \\ (\text{i.e. } f : \mathbb{L}^2 \to \mathbb{H}^3_1) \end{array} \Longleftrightarrow \begin{array}{l} (x(u,v), y(u,v)) \text{ given by } [**] \\ \text{is a global diffeomorfism} \end{array}$$

Sufficient condition:

$$0 < c_1 \leq \omega_1(u) + \omega_2(v) \leq c_2 < \pi$$

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A flat surface as a product of its asymptotic curves

Theorem 1

Let $f(u,v) : \mathbb{L}^2 \to \mathbb{H}^3_1$ be an isometric immersion, (u,v) its characteristic parameters. We assume w.l.o.g. that $f(0,0) = 1$ and $N(0,0) = j$. Then, for $a_1(u) := f(u,0)$ and $a_2(v) := f(0,v)$,

$$f(u,v) = a_1(u)a_2(v),$$

And these curves verify $\langle a_1'(u), a_1(u)j \rangle = 0 = \langle a_2'(v), ja_2(v) \rangle$.

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For a fixed v_0 we define $\Gamma_1(u) = f(u,v_0)$, $\Gamma_2(u) = a_1(u)a_2(v_0)$

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We construct frames $\{\vec{t_1}, \vec{n_1}, \vec{b_1}\}$ along Γ_1, $\{\vec{t_2}, \vec{n_2}, \vec{b_2}\}$ along Γ_2

Lorenzian flat tori in $\mathbb{H}^3_1$
María Amelia León-Guzmán
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Let $f(u,v):\mathbb{L}^2\to\mathbb{H}^3_1$ be an isometric immersion, (u,v) its characteristic parameters. We assume w.l.o.g. that $f(0,0)=1$ and $N(0,0)=j$. Then, for $a_1(u):=f(u,0)$ and $a_2(v):=f(0,v)$,

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- they coincide at $u=0$
- they verify the same system of differential equation

A flat surface as a product of its asymptotic curves

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- they coincide at $u = 0$
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Γ_1 and Γ_2 do not have constant causal character. Thus, these frames are not the Frenet frame of each curve.

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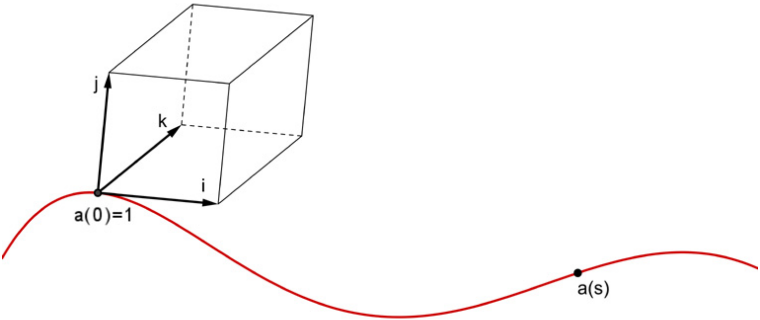
  What does $\bar{a}a'$ represent?



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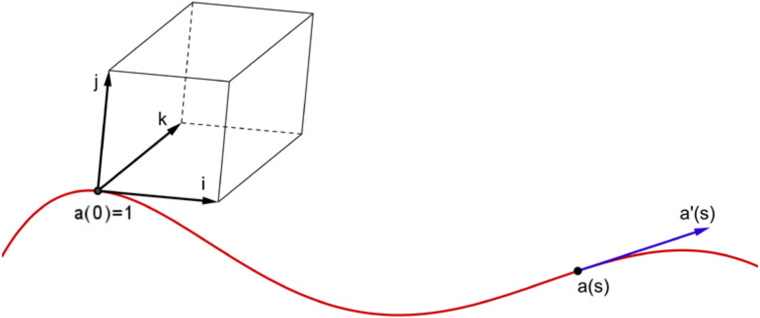

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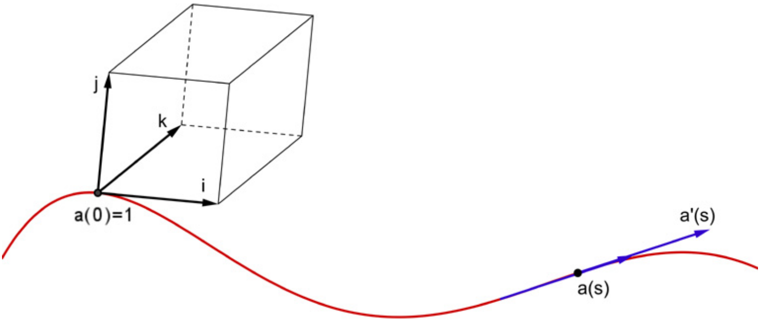

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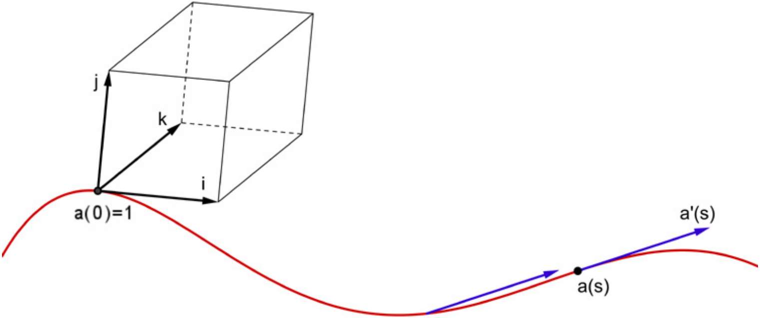

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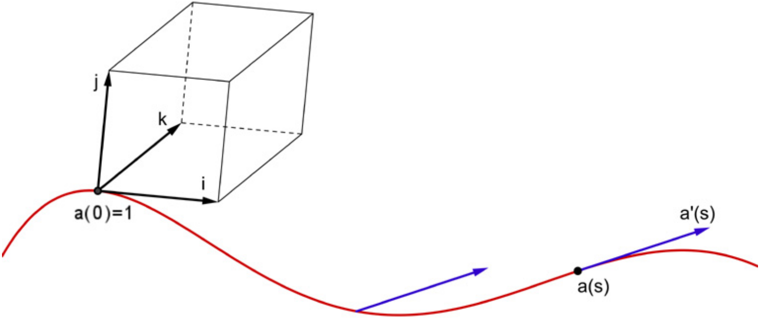

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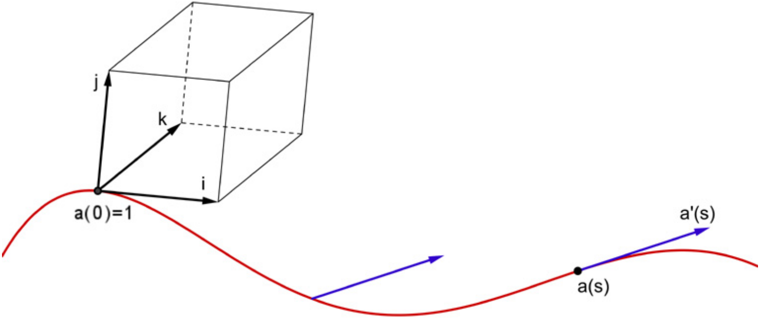

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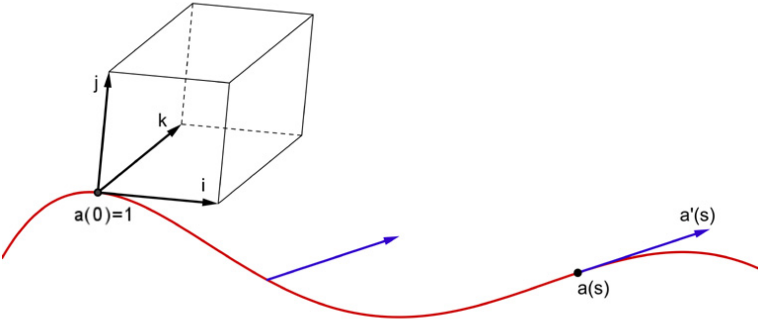

Constructing flat surfaces by multiplying curves

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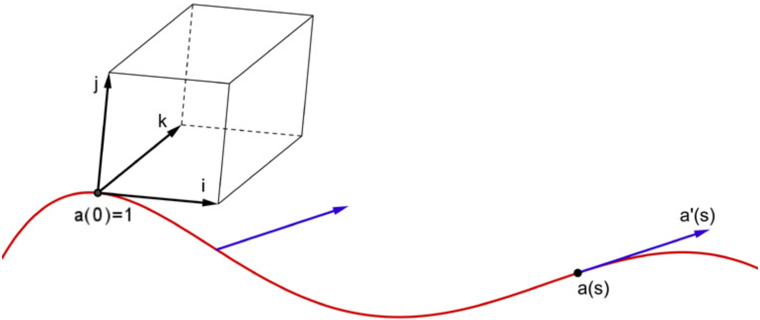

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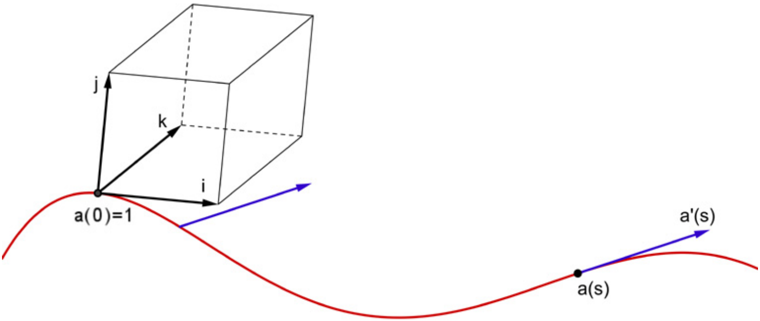

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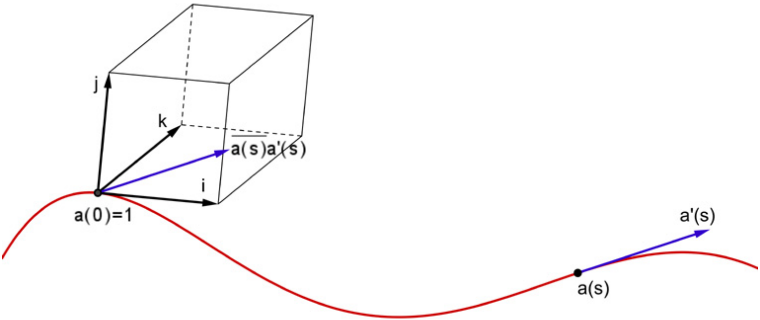

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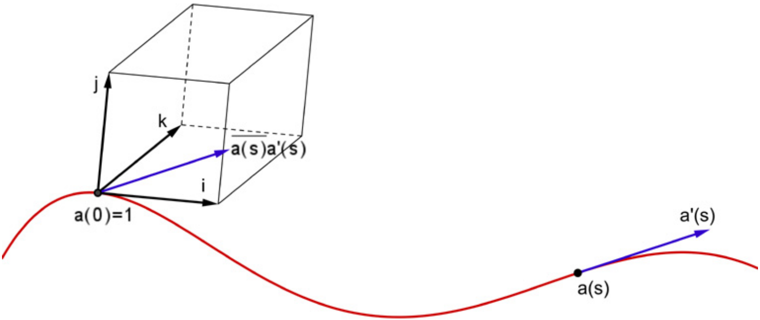

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- In this situation, s is the **asymptotic parameter** of a if $\lambda^2 + \mu^2 \equiv 1$,

Constructing flat surfaces by multiplying curves

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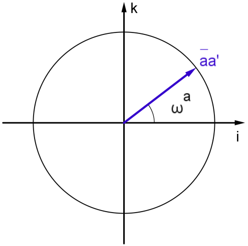
- In this situation, s is the **asymptotic parameter** of a if $\lambda^2 + \mu^2 \equiv 1$, i. e. if $\bar{a}a' = \cos(\omega^a) i + \sin(\omega^a) k$



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Constructing flat surfaces by multiplying curves

Theorem 2

$a_1(u), a_2(v)$ regular curves, with $a_1(0) = 1 = a_2(0)$, satisfying:

i) $\langle a_1', a_1 j\rangle = 0, \quad \langle a_2', a_2 j\rangle = 0$

ii) u, v are the asymptotic parameters of a_1 and a_2, resp.

iii) $\omega_1 = \omega^{a_1}$ and $\omega_2 = \pi - \omega^{a_2}$ verify $0 < \omega_1(u) + \omega_2(v) < \pi \ \forall u, v$

iv) The map $(x(u,v), y(u,v))$ in [**] is a global diffeomorphism.

Then, $f(u,v) = a_1(u)\overline{a_2(v)}$ describes an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, and (u,v) are its global characteristic parameters.

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ii) u, v are the asymptotic parameters of a_1 and a_2, resp.

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iv) The map $(x(u,v), y(u,v))$ in [∗∗] is a global diffeomorphism.

Then, $f(u,v) = a_1(u)\overline{a_2(v)}$ describes an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, and (u,v) are its global characteristic parameters.

ii) $\rightarrow$ adequate parameters

iii) $\rightarrow$ regularity

iv) $\rightarrow$ completeness

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Curves with wavefront singularities in $\mathbb{H}^2$

Wavefront in $\mathbb{H}^2$ (front)

curve $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2$

globally defined unitary normal ν

$\langle \gamma, \nu \rangle = 0, \quad \langle \gamma', \nu \rangle = 0$

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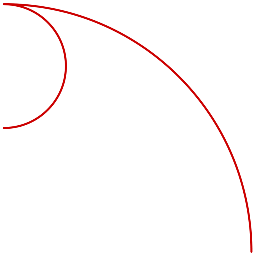
Curves with wavefront singularities in $\mathbb{H}^2$

Wavefront in $\mathbb{H}^2$ (front)

curve $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2$

globally defined unitary normal ν

$\langle \gamma, \nu \rangle = 0, \quad \langle \gamma', \nu \rangle = 0$



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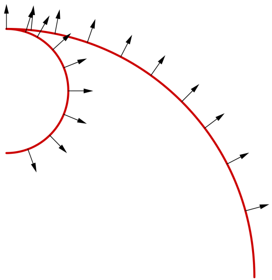

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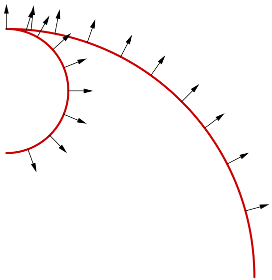

For a front we can define the metric

$$\langle \gamma', \gamma' \rangle_S = \langle \gamma', \gamma' \rangle + \langle \nu, \nu \rangle$$

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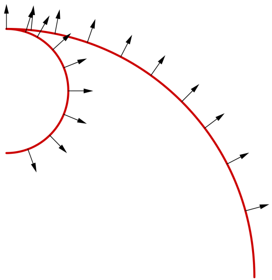

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$$\begin{array}{rccc} h: & \mathbb{H}^3_1 & \longrightarrow & \mathbb{H}^2 \\ & z & \longmapsto & z\, i\, \bar{z} \end{array}$$

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Geodesic curvature of a front

$$k_g(s) = \begin{cases} \dfrac{\langle \gamma''(s), \nu(s)\rangle}{||\gamma'(s)||^2} & \text{if} \quad \gamma'(s) \neq 0 \\ \infty & \text{if} \quad \gamma'(s) = 0 \end{cases}$$

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Flat surfaces in terms of pairs of fronts

Flat surfaces in terms of pairs of fronts

Theorem 3 (Classification of complete examples)

$\gamma_1(u), \gamma_2(v)$ fronts in $\mathbb{H}^2$ with $\gamma_i(0) = i$ and $\nu_i(0) = k$. If they verify:

i) $u/2, v/2$ arc-length parameters of γ_1, γ_2 with respect to $\langle\ ,\ \rangle_{\mathcal{S}}$

ii) $k_1(u) \neq k_2(v) \quad \forall\, u, v$

iii) For $\omega_1(u) := \cot^{-1}(k_1(u))$ and $\omega_2(v) := \pi - \cot^{-1}(k_2(v))$, the map $(x(u,v), y(u,v))$ in [$**$] is a global diffeomorphism.

Then, $f(u,v) = a_1(u)\overline{a_2(v)}$ is an isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$, and (u,v) are its characteristic parameters.

Conversely, every isometric immersion of $\mathbb{L}^2$ into $\mathbb{H}^3_1$ can be recovered by this process.

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ii) $\rightarrow$ regularity

iii) $\rightarrow$ completeness

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No differential equation

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Theorem 4 (Classification of flat tori)

Flat torus $\iff$ γ_1, γ_2 are closed fronts in $\mathbb{H}^2$ with $k_1 \neq k_2$

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Method to construct flat surfaces in $\mathbb{H}^3_1$ as a product of two regular curves b_1, b_2 , such that:

- b_1 time-like with $\tau = 1$
- b_2 space-like with $\tau = -1$

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If we think of this surface in terms of pairs of fronts $\gamma_1, \gamma_2 \in \mathbb{H}^2$

Regularity $\Longleftrightarrow k_1 \neq k_2$

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If we think of this surface in terms of pairs of fronts $\gamma_1, \gamma_2 \in \mathbb{H}^2$

Regularity $\iff k_1 \neq k_2$

b_1 time-like
b_2 space-like $\longrightarrow$ $|k_1| > 1$
$|k_2| < 1$ $\longrightarrow$ **Yes**

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Q2 It is also complete?

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In general, the answer is **No**

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As in the riemmanian case, we can find some **sufficient conditions** that ensure the **completeness** of the flat surface.

$$0 < c_1 \leq \omega_1(u) + \omega_2(v) \leq c_2 < \pi$$

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Q4 These two curves can be chosen so that one of them is everywhere timelike and the other is everywhere spacelike?

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Yes, Theorem 1

Q4 These two curves can be chosen so that one of them is everywhere timelike and the other is everywhere spacelike?

No, consider a flat surface generated from fronts γ_1, γ_2 both with $|k_i| > 1$ (or both with $|k_i| < 1$)

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$$\begin{cases} x(u,v) = \int \cos\omega_1 du + \int \cos\omega_2 dv \\ y(u,v) = \int \sin\omega_1 du - \int \sin\omega_2 dv \end{cases} \qquad [**]$$

A sufficient condition for $(x(u,v), y(u,v))$ to be a global diffeomorfism is that:

$$0 < c_1 \leq \omega_1(u) + \omega_2(v) \leq c_2 < \pi \qquad \forall (u,v) \in \mathbb{R}^2$$