A Genericity problem in Lorentzian Geometry

Roberto Giambò

Dipartimento di Matematica e Informatica
Università di Camerino (Italy)
roberto.giambo@unicam.it
http://www.cs.unicam.it/giambo

GeLoBa2009

Martina Franca, July 9, 2009

Outline

Outline

Why genericity?

S W Hawking, *Stable and Generic properties in General Relativity*, Gen Rel Grav 1 (1971)

...

The accuracy of the observation is always limited by practical difficulties and by the uncertainty principle. Thus the only properties of spacetime that are physically significant are those that are stable in some appropriate topology.

...

A couple of examples...

Global stability of exact solutions

- Minkowski solution

  D Christodoulou and S Klainerman, *The Global nonlinear stability of the Minkowski space*, Princeton Math Series **41** (1993), S Klainerman and F Nicolò, Class Quantum Grav **20** (2003), H Lindblad and I Rodnianski, Commun Math Phys **256** (2005)

- flat Bianchi type III model

  Y Choquet-Bruhat and V Moncrief, Ann H Poincaré **2** (2001), C-B, in *The Einstein Equations and the large scale behavior of gravitational fields*, Birkhäuser (2004)

Gravitational collapse and Cosmic Censorship

Stability wrt metric perturbations of scalar field solutions collapsing to a naked singularity (es. D Christodoulou, Ann Math, 149 (1999), but BV data are used!)

A couple of examples...

Global stability of exact solutions

- Minkowski solution

  D Christodoulou and S Klainerman, *The Global nonlinear stability of the Minkowski space*, Princeton Math Series **41** (1993), S Klainerman and F Nicolò, Class Quantum Grav **20** (2003), H Lindblad and I Rodnianski, Commun Math Phys **256** (2005)

- flat Bianchi type III model

  Y Choquet-Bruhat and V Moncrief, Ann H Poincaré **2** (2001), C-B, in *The Einstein Equations and the large scale behavior of gravitational fields*, Birkhäuser (2004)

Gravitational collapse and Cosmic Censorship

Stability wrt metric perturbations of scalar field solutions collapsing to a naked singularity (es. D Christodoulou, Ann Math, 149 (1999), but BV data are used!)

A couple of examples...

Global stability of exact solutions

- Minkowski solution

  D Christodoulou and S Klainerman, *The Global nonlinear stability of the Minkowski space*, Princeton Math Series **41** (1993), S Klainerman and F Nicolò, Class Quantum Grav **20** (2003), H Lindblad and I Rodnianski, Commun Math Phys **256** (2005)

- flat Bianchi type III model

  Y Choquet-Bruhat and V Moncrief, Ann H Poincaré **2** (2001), C-B, in *The Einstein Equations and the large scale behavior of gravitational fields*, Birkhäuser (2004)

Gravitational collapse and Cosmic Censorship

Stability wrt metric perturbations of scalar field solutions collapsing to a naked singularity (es. D Christodoulou, Ann Math, 149 (1999), but BV data are used!)

A couple of examples...

Global stability of exact solutions

- Minkowski solution

  D Christodoulou and S Klainerman, *The Global nonlinear stability of the Minkowski space*, Princeton Math Series **41** (1993), S Klainerman and F Nicolò, Class Quantum Grav **20** (2003), H Lindblad and I Rodnianski, Commun Math Phys **256** (2005)

- flat Bianchi type III model

  Y Choquet-Bruhat and V Moncrief, Ann H Poincaré **2** (2001), C-B, in *The Einstein Equations and the large scale behavior of gravitational fields*, Birkhäuser (2004)

Gravitational collapse and Cosmic Censorship

Stability wrt metric perturbations of scalar field solutions collapsing to a naked singularity (es. D Christodoulou, Ann Math, 149 (1999), but BV data are used!)

A couple of examples...

Global stability of exact solutions

- Minkowski solution

  D Christodoulou and S Klainerman, *The Global nonlinear stability of the Minkowski space*, Princeton Math Series **41** (1993), S Klainerman and F Nicolò, Class Quantum Grav **20** (2003), H Lindblad and I Rodnianski, Commun Math Phys **256** (2005)

- flat Bianchi type III model

  Y Choquet-Bruhat and V Moncrief, Ann H Poincaré **2** (2001), C-B, in *The Einstein Equations and the large scale behavior of gravitational fields*, Birkhäuser (2004)

Gravitational collapse and Cosmic Censorship

Stability wrt metric perturbations of scalar field solutions collapsing to a naked singularity (es. D Christodoulou, Ann Math, 149 (1999), but BV data are used!)

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann 197 (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR 46 (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

- counterexample (K Meyer and J Palmore, *Global Analysis* (1970) AMS)
- flow may possibly arrange differently on energy levels

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

- counterexample (K Meyer and J Palmore, *Global Analysis* (1970) AMS)
- flow may possibly arrange differently on energy levels

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

- counterexample (K Meyer and J Palmore, *Global Analysis* (1970) AMS)
- flow may possibly arrange differently on energy levels

Generic properties of Lorentzian geodesic flow

One step behind: Bumpy Theorem in Riemannian geometry

the set of the metric in a fixed compact manifold M st every closed geo is nondegenerate is generic (genericity = G_δ dense)

- first stated by R Abraham (*Global Analysis* (1970) AMS)
- proof: some ideas by W Klingenberg (Math Ann **197** (1972)), complete argument by D V Anosov (Izv Akad Nauk SSSR **46** (1982))

Application: Morse theory for closed geodesics on compact manifolds

- R Bott (nondegenerate case) Comm. Pure Appl. Math. **9** (1956)
- D Gromoll and W Meyer (possibly degenerate case) J. Diff. Geom. **3** (1969)

Bumpy theorem for general Hamiltonian flows?

- counterexample (K Meyer and J Palmore, *Global Analysis* (1970) AMS)
- flow may possibly arrange differently on energy levels

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology

$\Rightarrow$ Morse theory of light rays between p and Ω unchanged in a set of Lorentzian manifolds st nondegeneracy generically holds

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology

$\Rightarrow$ Morse theory of light rays between p and U **unchanged** in a set of Lorentzian manifolds st nondegeneracy generically holds

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology

$\Rightarrow$ Morse theory of light rays between p and U **unchanged** in a set of Lorentzian manifolds st nondegeneracy generically holds

Generic properties of Lorentzian geodesic flow

Lorentzian case: a HUGE literature...

Morse theory for light rays and timelike geodesics

- K Uhlenbeck (Topology **14** (1975))
- V Perlick (J Math Phys **36** (1995), *Current topics in mathematical cosmology*, World Sci (1998)), w/ W Hasse (J Math Phys **47** (2006)) . . .
- D Fortunato, F Giannoni, A Masiello (J Geom Phys **15** (1995)), FG, AM, P Piccione (J Geom Phys **35** (2000), J Math Phys **43** (2002)). . .
- . . .

Key requisite

Nondegeneracy of critical points

A Abbondandolo and P Majer, Asian J Math 12 (2008)

C^0-stability of Morse complex homology
$\Rightarrow$ Morse theory of light rays between p and U **unchanged** in a set of Lorentzian manifolds st nondegeneracy generically holds

Geometric motivations

Problem

Genericity of non conjugacy along light rays joining an event p and an observer U

- keeping g fixed, wrt (p, U) (using exp $\longrightarrow$ OK)
- keeping (p, U) fixed, wrt g?

Obstructions

- regularity of curves space
- light curves space varies with g

Assumption

Existence of a splitting structure for (M, g)

Geometric motivations

Problem

Genericity of non conjugacy along light rays joining an event p and an observer U

- keeping g fixed, wrt (p, U) (using $\exp \to$ OK)
- keeping (p, U) fixed, wrt g?

Obstructions

- regularity of curves space
- light curves space varies with g

Assumption

Existence of a splitting structure for (M, g)

Geometric motivations

Problem

Genericity of non conjugacy along light rays joining an event p and an observer U

- keeping g fixed, wrt (p, U) (using exp $\to$ OK)
- keeping (p, U) fixed, wrt g?

Obstructions

- regularity of curves space
- light curves space *varies* with g

Assumption

Existence of a splitting structure for (M, g)

Geometric motivations

Problem

Genericity of non conjugacy along light rays joining an event p and an observer U

- keeping g fixed, wrt (p, U) (using exp $\longrightarrow$ OK)
- keeping (p, U) fixed, wrt g?

Obstructions

- regularity of curves space
- light curves space *varies* with g

Assumption

Existence of a splitting structure for (M, g)

Geometric motivations

Problem

Genericity of non conjugacy along light rays joining an event p and an observer U

- keeping g fixed, wrt (p, U) (using exp $\to$ OK)
- keeping (p, U) fixed, wrt g?

Obstructions

- regularity of curves space
- light curves space *varies* with g

Assumption

Existence of a splitting structure for (M, g)

Geometric motivations

Problem

Genericity of non conjugacy along light rays joining an event p and an observer U

- keeping g fixed, wrt (p, U) (using exp $\rightarrow$ OK)
- keeping (p, U) fixed, wrt g?

Obstructions

- regularity of curves space
- light curves space *varies* with g

Assumption

Existence of a splitting structure for (M, g)

The main result

RG, F Giannoni, P Piccione, Commun Math Phys 287 (2009)

M_0 diff manifold, $p_0 \neq p_1 \in M_0$
The set of all *standard stationary* metrics defined in $M_0 \times \mathbb{R}$ with only nondegenerate light rays from $(p_0, 0)$ to $U = \{p_1\} \times \mathbb{R}$ is generic in the C^∞ topology

Return

Plan of the proof

- C^k genericity $\forall k$
- extension to C^∞ genericity (RG, MA Javaloyes (2009) preprint)

The main result

RG, F Giannoni, P Piccione, Commun Math Phys 287 (2009)

M_0 diff manifold, $p_0 \neq p_1 \in M_0$
The set of all *standard stationary* metrics defined in $M_0 \times \mathbb{R}$ with only nondegenerate light rays from $(p_0, 0)$ to $U = \{p_1\} \times \mathbb{R}$ is generic in the C^∞ topology

Return

Plan of the proof

- C^k genericity $\forall k$
- extension to C^∞ genericity (RG, MA Javaloyes (2009) preprint)

The main result

RG, F Giannoni, P Piccione, Commun Math Phys 287 (2009)

M_0 diff manifold, $p_0 \neq p_1 \in M_0$
The set of all *standard stationary* metrics defined in $M_0 \times \mathbb{R}$ with only nondegenerate light rays from $(p_0, 0)$ to $U = \{p_1\} \times \mathbb{R}$ is generic in the C^∞ topology

Return

Plan of the proof

- C^k genericity $\forall k$
- extension to C^∞ genericity (RG, MA Javaloyes (2009) preprint)

The main result

RG, F Giannoni, P Piccione, Commun Math Phys 287 (2009)

M_0 diff manifold, $p_0 \neq p_1 \in M_0$
The set of all *standard stationary* metrics defined in $M_0 \times \mathbb{R}$ with only nondegenerate light rays from $(p_0, 0)$ to $U = \{p_1\} \times \mathbb{R}$ is generic in the C^∞ topology

Return

Plan of the proof

- C^k genericity $\forall k$
- extension to C^∞ genericity (RG, MA Javaloyes (2009) preprint)

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin\frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x > 0, \quad a, b < 1$$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin \frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x > 0, \quad a, b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n, b_n) \to (a,b)$ st x_n simple zero of $f(x, a_n, b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin \frac{1}{x-u(a,b)}, \;\; u(a,b) = \frac{b}{a+b}, \;\; x > 0, \;\; a,b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n,b_n) \to (a,b)$ st x_n simple zero of $f(x,a_n,b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin \frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x > 0, \quad a, b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n, b_n) \to (a,b)$ st x_0 simple zero of $f(x, a_n, b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin\frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x>0, \quad a,b<1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n,b_n) \to (a,b)$ st x_0 simple zero of $f(x,a_n,b_n)$

Anosov criticism to Klingenberg proof

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin\frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x > 0, \quad a, b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n, b_n) \to (a,b)$ st x_0 simple zero of $f(x, a_n, b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin \frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x > 0, \quad a,b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n, b_n) \to (a,b)$ st x_0 simple zero of $f(x, a_n, b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin \frac{1}{x-u(a,b)}, \;\; u(a,b) = \frac{b}{a+b}, \;\; x > 0, \;\; a, b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n, b_n) \to (a,b)$ st x_0 simple zero of $f(x, a_n, b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

Techniques for genericity results

Riemannian bumpy theorem: Klingenberg argument

- local perturbation argument to adjust the flow near a closed geo
- if g has a degenerate closed geo $\gamma(t) \Rightarrow \exists g_n \to g$ such that $\gamma(t)$ nondegenerate g_n-closed geo

Issue (just to fix ideas...)

$$f(x,a,b) = e^{-\frac{1}{a+b}} e^{-\frac{1}{(x-u(a,b))^2}} \sin \frac{1}{x-u(a,b)}, \quad u(a,b) = \frac{b}{a+b}, \quad x > 0, \quad a,b < 1$$

- $\forall (a,b)\ \exists x_0 = u(a,b)$ non simple zero of $x \to f(x,a,b)$
- but $\exists (a_n, b_n) \to (a,b)$ st x_0 simple zero of $f(x, a_n, b_n)$

Anosov criticism to Klingenberg proof

- local perturbation arguments suffer from uniformity issues
- way out: Sard's theorem (or transversality theorem) appears as an essential resource

An abstract genericity result

B White, Indiana Univ Math J (1991), L Biliotti, MA Javaloyes, P Piccione, Indiana Univ Math J (2009)

B Banach, $\mathcal{H}$ Hilbert; B, $\mathcal{H}$ separable, $A \subset B \times \mathcal{H}$ open
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $\quad F_b : x \to F(b, x)$

$$\mathfrak{C} = \big\{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\big\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}\big(\mathrm{d}^2 F_{b_0}(x_0)\big)$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b\, \partial x}(b_0, x_0)\big[(\beta, \xi)\big] \neq 0.$$

$\Longrightarrow \{b \in \Pi(A) \ : \ F_b \text{ Morse}\}$ generic in $\Pi(A)$ $(\Pi : B \times \mathcal{H} \to B)$

Return

An abstract genericity result

B White, Indiana Univ Math J (1991), L Biliotti, MA Javaloyes, P Piccione, Indiana Univ Math J (2009)

B Banach, $\mathcal{H}$ Hilbert; B, $\mathcal{H}$ separable, $A \subset B \times \mathcal{H}$ open
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $\quad F_b : x \to F(b, x)$

$$\mathfrak{C} = \{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}(\mathrm{d}^2 F_{b_0}(x_0))$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b \, \partial x}(b_0, x_0)[(\beta, \xi)] \neq 0.$$

$\Longrightarrow \{ b \in \Pi(A) \ : \ F_b \text{ Morse } \}$ generic in $\Pi(A)$ ($\Pi : B \times \mathcal{H} \to B$)

Return

An abstract genericity result

B White, Indiana Univ Math J (1991), L Biliotti, MA Javaloyes, P Piccione, Indiana Univ Math J (2009)

B Banach, $\mathcal{H}$ Hilbert; B, $\mathcal{H}$ separable, $A \subset B \times \mathcal{H}$ open
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $F_b : x \to F(b, x)$

$$\mathfrak{C} = \{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}\big(\mathrm{d}^2 F_{b_0}(x_0)\big)$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b\, \partial x}(b_0, x_0)\big[(\beta, \xi)\big] \neq 0.$$

$\Longrightarrow \{b \in \Pi(A) \;:\; F_b \text{ Morse}\}$ generic in $\Pi(A)$ $(\Pi : B \times \mathcal{H} \to B)$

Return

An abstract genericity result

B White, Indiana Univ Math J (1991), L Biliotti, MA Javaloyes, P Piccione, Indiana Univ Math J (2009)

B Banach, $\mathcal{H}$ Hilbert; B, $\mathcal{H}$ separable, $A \subset B \times \mathcal{H}$ open
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $\quad F_b : x \to F(b,x)$

$$\mathfrak{C} = \big\{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\big\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}\big(\mathrm{d}^2 F_{b_0}(x_0)\big)$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b\, \partial x}(b_0, x_0)\big[(\beta, \xi)\big] \neq 0.$$

$\Longrightarrow \{ b \in \Pi(A) \; : \; F_b \text{ Morse } \}$ generic in $\Pi(A)$ $(\Pi : B \times \mathcal{H} \to B)$

Return

An abstract genericity result

B White, Indiana Univ Math J (1991), L Biliotti, MA Javaloyes, P Piccione, Indiana Univ Math J (2009)

B Banach, $\mathcal{H}$ Hilbert; B, $\mathcal{H}$ separable, $A \subset B \times \mathcal{H}$ open
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $\quad F_b : x \to F(b, x)$

$$\mathfrak{C} = \big\{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\big\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}\big(\mathrm{d}^2 F_{b_0}(x_0)\big)$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b \, \partial x}(b_0, x_0)\big[(\beta, \xi)\big] \neq 0.$$

$\Longrightarrow \{b \in \Pi(A) \; : \; F_b \text{ Morse }\}$ generic in $\Pi(A)$ ($\Pi : B \times \mathcal{H} \to B$) Return

An abstract gener

Condition (b)

$\Rightarrow \frac{\partial F}{\partial x} : B \times \mathcal{H} \to T^*\mathcal{H}$ transversal to the 0-section of $T^*\mathcal{H} \Rightarrow \mathfrak{C} \subseteq B \times \mathcal{H}$ submanifold

B White, Indiana Univ
Piccione, Indiana Univ

B Banach, $\mathcal{H}$ Hilbert; B, $\mathcal{H}$ separable, $A \subset B \times \mathcal{H}$ open
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $\quad F_b : x \to F(b, x)$

$$\mathfrak{C} = \big\{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\big\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}\big(\mathrm{d}^2 F_{b_0}(x_0)\big)$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b\, \partial x}(b_0, x_0)\big[(\beta, \xi)\big] \neq 0.$$

$\Longrightarrow \{ b \in \Pi(A) \ : \ F_b \text{ Morse } \}$ generic in $\Pi(A)$ ($\Pi : B \times \mathcal{H} \to B$) Return

An abstract gener

Condition (a)

$\Rightarrow \Pi|_{\mathfrak{C}} : \mathfrak{C} \to B$ is Fredholm (w/ index 0)
$\{b \in \Pi(A) : F_b$ not Morse $\} = \{$crit vals of $\Pi|_{\mathfrak{C}}\}$
$\Rightarrow$ result follows from Sard-Smale

B White, Indiana Univ
Piccione, Indiana Uni

B Banach, $\mathcal{H}$ Hilbert; B
$F : A \to \mathbb{R}$, $F \in \mathcal{C}^k (k \geq 2)$; $F_b : x \to F(b, x)$

$$\mathfrak{C} = \{(b_0, x_0) \in A : x_0 \text{ critical point of } F_{b_0}\}.$$

Assume $\forall (b_0, x_0) \in \mathfrak{C}$:

(a) the Hessian $\mathrm{d}^2 F_{b_0}(x_0) = \frac{\partial^2 F}{\partial x^2}(b_0, x_0)$ is Fredholm;

(b) $\forall \xi \in \mathrm{Ker}\big(\mathrm{d}^2 F_{b_0}(x_0)\big)$ with $\xi \neq 0$ $\exists \beta \in T_{b_0} B$ such that

$$\frac{\partial^2 F}{\partial b\, \partial x}(b_0, x_0)\big[(\beta, \xi)\big] \neq 0.$$

$\Longrightarrow \{b \in \Pi(A) : F_b \text{ Morse } \}$ generic in $\Pi(A)$ $(\Pi : B \times \mathcal{H} \to B)$ Return

Outline

1. Introduction
   - Genericity and stability in GR
   - Genericity of lightlike nondegeneracy

2. **Variational setup**
   - Fermat principles

3. Proof of the main result
   - C^k genericity of lightlike nondegeneracy
   - C^∞ genericity of lightlike nondegeneracy

Fermat principles

$M = M_0 \times \mathbb{R}$, $\mathfrak{g} \in \mathrm{Riem}(M_0)$, $\delta \in \mathfrak{X}(M_0)$
$$g_{(x,s)}\big((v,r),(\bar{v},\bar{r})\big) = \mathfrak{g}_x(v,\bar{v}) + \mathfrak{g}_x\big(\delta(x),v\big)\bar{r} + \mathfrak{g}_x\big(\delta(x),\bar{v}\big)r - \beta(x)r\bar{r}$$

E Minguzzi and M Sánchez `arXiv:gr-qc/0609119`

$\beta(x) = 1$

Randers metric on M_0

$$\mathfrak{h}(v,w) = \mathfrak{g}(v,w) + \mathfrak{g}(\delta_p,v)\mathfrak{g}(\delta_p,w), \quad \omega_p(v) = \mathfrak{g}(\delta_p,v)$$

A Finsler metric f on M_0 is induced

$$f_{(\mathfrak{h},\omega)}(v) = \sqrt{\mathfrak{h}(v,v)} + \omega(v), \quad v \in TM_0$$

1st order Fermat principle

$\exists$ bijection $\gamma = (x,t) \leftrightarrow x$ from future pointing light geo in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ to Randers geo from p_0 to p_1.
Moreover $c_\gamma \equiv \mathfrak{g}(\dot{x},\delta) - \dot{t} = -\sqrt{\mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2} \equiv c_x$.

$$M = M_0 \times \mathbb{R},\ \mathfrak{g} \in \mathrm{Riem}(M_0),\ \delta \in \mathfrak{X}(M_0)$$
$$g_{(x,s)}\big((v,r),(\bar v,\bar r)\big) = \mathfrak{g}_x(v,\bar v) + \mathfrak{g}_x\big(\delta(x),v\big)\bar r + \mathfrak{g}_x\big(\delta(x),\bar v\big)r - \beta(x) r\bar r$$

E Minguzzi and M Sánchez `arXiv:gr-qc/0609119`

$$\beta(x) = 1$$

Randers metric on M_0

$$\mathfrak{h}(v,w) = \mathfrak{g}(v,w) + \mathfrak{g}(\delta_p,v)\mathfrak{g}(\delta_p,w), \quad \omega_p(v) = \mathfrak{g}(\delta_p,v)$$

A Finsler metric f on M_0 is induced

$$f_{(\mathfrak{h},\omega)}(v) = \sqrt{\mathfrak{h}(v,v)} + \omega(v), \quad v \in TM_0$$

1st order Fermat principle

$\exists$ bijection $\gamma = (x,t) \leftrightarrow x$ from future pointing light geo in $M_0 \times \mathbb{R}$ between $(p_0,0)$ and $\{p_1\} \times \mathbb{R}$ to Randers geo from p_0 to p_1.
Moreover $c_\gamma \equiv \mathfrak{g}(\dot x,\delta) - \dot t = -\sqrt{\mathfrak{g}(\dot x,\dot x) + \mathfrak{g}(\dot x,\delta)^2} \equiv c_x$.

$M = M_0 \times \mathbb{R}$, $\mathfrak{g} \in \mathrm{Riem}(M_0)$, $\delta \in \mathfrak{X}(M_0)$
$g_{(x,s)}\big((v,r),(\bar{v},\bar{r})\big) = \mathfrak{g}_x(v,\bar{v}) + \mathfrak{g}_x\big(\delta(x),v\big)\bar{r} + \mathfrak{g}_x\big(\delta(x),\bar{v}\big)r - \beta(x)r\bar{r}$

E Minguzzi and M Sánchez `arXiv:gr-qc/0609119`

$\beta(x) = 1$

Randers metric on M_0

$\mathfrak{h}(v,w) = \mathfrak{g}(v,w) + \mathfrak{g}(\delta_p,v)\mathfrak{g}(\delta_p,w), \quad \omega_p(v) = \mathfrak{g}(\delta_p,v)$

A Finsler metric f on M_0 is induced

$f_{(\mathfrak{h},\omega)}(v) = \sqrt{\mathfrak{h}(v,v)} + \omega(v), \quad v \in TM_0$

1 st order Fermat principle

$\exists$ bijection $\gamma = (x,t) \leftrightarrow x$ from future pointing light geo in $M_0 \times \mathbb{R}$ between $(p_0,0)$ and $\{p_1\} \times \mathbb{R}$ to Randers geo from p_0 to p_1.
Moreover $c_\gamma \equiv \mathfrak{g}(\dot{x},\delta) - \dot{t} = -\sqrt{\mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2} \equiv c_x$.

Fermat principles

$M = M_0 \times \mathbb{R}$, $\mathfrak{g} \in \mathrm{Riem}(M_0)$, $\delta \in \mathfrak{X}(M_0)$
$g_{(x,s)}((v,r),(\bar{v},\bar{r})) = \mathfrak{g}_x(v,\bar{v}) + \mathfrak{g}_x(\delta(x),v)\bar{r} + \mathfrak{g}_x(\delta(x),\bar{v})r - \beta(x)r\bar{r}$

E Minguzzi and M Sánchez `arXiv:gr-qc/0609119`

$\beta(x) = 1$

Randers metric on M_0

$\mathfrak{h}(v,w) = \mathfrak{g}(v,w) + \mathfrak{g}(\delta_p,v)\mathfrak{g}(\delta_p,w), \quad \omega_p(v) = \mathfrak{g}(\delta_p,v)$

A Finsler metric f on M_0 is induced

$f_{(\mathfrak{h},\omega)}(v) = \sqrt{\mathfrak{h}(v,v)} + \omega(v), \quad v \in TM_0$

1^{st} order Fermat principle

$\exists$ bijection $\gamma = (x,t) \leftrightarrow x$ from future pointing light geo in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ to Randers geo from p_0 to p_1.
Moreover $c_\gamma \equiv \mathfrak{g}(\dot{x},\delta) - \dot{t} = -\sqrt{\mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2} \equiv c_x$.

$$M = M_0 \times \mathbb{R},\ \mathfrak{g} \in \mathrm{Riem}(M_0),\ \delta \in \mathfrak{X}(M_0)$$
$$g_{(x,s)}\big((v,r),(\bar v,\bar r)\big) = \mathfrak{g}_x(v,\bar v) + \mathfrak{g}_x\big(\delta(x),v\big)\bar r + \mathfrak{g}_x\big(\delta(x),\bar v\big)r - \beta(x)r\bar r$$

E Minguzzi and M Sánchez `arXiv:gr-qc/0609119`

$\beta(x) = 1$

Randers metric on M_0

$$\mathfrak{h}(v,w) = \mathfrak{g}(v,w) + \mathfrak{g}(\delta_p,v)\mathfrak{g}(\delta_p,w), \quad \omega_p(v) = \mathfrak{g}(\delta_p,v)$$

A Finsler metric f on M_0 is induced

$$f_{(\mathfrak{h},\omega)}(v) = \sqrt{\mathfrak{h}(v,v)} + \omega(v), \quad v \in TM_0$$

1^{st} order Fermat principle

$\exists$ bijection $\gamma = (x,t) \leftrightarrow x$ from future pointing light geo in $M_0 \times \mathbb{R}$ between $(p_0,0)$ and $\{p_1\} \times \mathbb{R}$ to Randers geo from p_0 to p_1.

Moreover $c_\gamma \equiv \mathfrak{g}(\dot x,\delta) - \dot t = -\sqrt{\mathfrak{g}(\dot x,\dot x) + \mathfrak{g}(\dot x,\delta)^2} \equiv c_x$.

Reduction to the base space

The functional

$\Omega_{p_0,p_1}(M_0) = H^1$ curves $x : [0,1] \longrightarrow M_0$ from p_0 to p_1

$$F(x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$$

- F smooth
- critical points = Finsler geodesics w/ $\sqrt{\mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2} \equiv -c_x$ (constant)

Euler–Lagrange equation

$$\frac{\mathrm{D}}{\mathrm{d}s}\dot{x} + \frac{\mathrm{D}}{\mathrm{d}s}\left(\mathfrak{g}(\dot{x},\delta)\,\delta\right) - \mathfrak{g}(\dot{x},\delta)(\nabla\delta)^{\star}\dot{x} + c_x\left[(\nabla\delta)^{\star}\dot{x} - (\nabla\delta)\dot{x}\right] = 0$$

Reduction to the base space

The functional

$\Omega_{p_0,p_1}(M_0) = H^1$ curves $x : [0,1] \to M_0$ from p_0 to p_1

$$F(x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2 \,\mathrm{d}s \right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$$

- F smooth
- critical points = Finsler geodesics w/ $\sqrt{\mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2} \equiv -c_x$ (constant)

Euler–Lagrange equation

$$\frac{\mathrm{D}}{\mathrm{d}s}\dot{x} + \frac{\mathrm{D}}{\mathrm{d}s}\big(\mathfrak{g}(\dot{x},\delta)\,\delta\big) - \mathfrak{g}(\dot{x},\delta)(\nabla\delta)^*\dot{x} + c_x\big[(\nabla\delta)^*\dot{x} - (\nabla\delta)\dot{x}\big] = 0$$

Reduction to the base space

The functional

$\Omega_{p_0,p_1}(M_0) = H^1$ curves $x : [0,1] \to M_0$ from p_0 to p_1

$$F(x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$$

- F smooth
- critical points = Finsler geodesics w/ $\sqrt{\mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2} \equiv -c_x$ (constant)

Euler–Lagrange equation

$$\frac{\mathrm{D}}{\mathrm{d}s}\dot{x} + \frac{\mathrm{D}}{\mathrm{d}s}\big(\mathfrak{g}(\dot{x},\delta)\,\delta\big) - \mathfrak{g}(\dot{x},\delta)(\nabla\delta)^*\dot{x} + c_x\left[(\nabla\delta)^*\dot{x} - (\nabla\delta)\dot{x}\right] = 0$$

Reduction to the base space

The functional

$\Omega_{p_0,p_1}(M_0) = H^1$ curves $x : [0,1] \longrightarrow M_0$ from p_0 to p_1

$$F(x) = \left(\int_0^1 \mathfrak{g}(\dot x, \dot x) + \mathfrak{g}(\dot x, \delta)^2 \, \mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot x, \delta) \, \mathrm{d}s$$

- F smooth
- critical points = Finsler geodesics w/ $\sqrt{\mathfrak{g}(\dot x, \dot x) + \mathfrak{g}(\dot x, \delta)^2} \equiv -c_x$ (constant)

Euler–Lagrange equation

$$\frac{\mathrm{D}}{\mathrm{d}s}\dot x + \frac{\mathrm{D}}{\mathrm{d}s}\big(\mathfrak{g}(\dot x, \delta)\,\delta\big) - \mathfrak{g}(\dot x, \delta)(\nabla\delta)^\star \dot x + c_x\big[(\nabla\delta)^\star \dot x - (\nabla\delta)\dot x\big] = 0$$

2^{nd} order Fermat principle

Let $x \in \Omega_{p_0,p_1}(M_0)$ cp of F. Recall the bijection

$$x \leftrightarrow \gamma = (x,t) : [0,1] \to M = M_0 \times \mathbb{R}$$

A Masiello, Pitman Research Notes in Mathematics 309 (1994)

- $\gamma(1)$ conjugate to $\gamma(0)$ along $\gamma \Leftrightarrow x$ degenerate cp of F
- $\mu(\gamma(1)) = \dim \operatorname{Ker} \mathrm{d}^2 F(x)$

$\mathrm{d}^2 F(x)$ is Fredholm

$\mathrm{d}^2 F(x) = -\frac{1}{c_x} \int_0^1 \mathfrak{g}\left(\frac{\mathrm{D}}{\mathrm{d}s}\xi, \frac{\mathrm{D}}{\mathrm{d}s}\eta\right) \mathrm{d}s + D[\xi,\eta]$, D compact

2^{nd} order Fermat principle

Let $x \in \Omega_{p_0,p_1}(M_0)$ cp of F. Recall the bijection

$$x \leftrightarrow \gamma = (x,t) : [0,1] \to M = M_0 \times \mathbb{R}$$

A Masiello, Pitman Research Notes in Mathematics 309 (1994)

- $\gamma(1)$ conjugate to $\gamma(0)$ along $\gamma \Leftrightarrow x$ degenerate cp of F
- $\mu(\gamma(1)) = \dim \operatorname{Ker} \mathrm{d}^2 F(x)$

$\mathrm{d}^2 F(x)$ is Fredholm

$\mathrm{d}^2 F(x) = -\frac{1}{c_x}\int_0^1 \mathfrak{g}\left(\frac{\mathrm{D}}{\mathrm{d}s}\xi, \frac{\mathrm{D}}{\mathrm{d}s}\eta\right) \mathrm{d}s + D[\xi,\eta]$, D compact

2^{nd} order Fermat principle

Let $x \in \Omega_{p_0,p_1}(M_0)$ cp of F. Recall the bijection

$$x \leftrightarrow \gamma = (x,t) : [0,1] \to M = M_0 \times \mathbb{R}$$

A Masiello, Pitman Research Notes in Mathematics 309 (1994)

- $\gamma(1)$ conjugate to $\gamma(0)$ along $\gamma \Leftrightarrow x$ degenerate cp of F
- $\mu(\gamma(1)) = \dim \operatorname{Ker} \mathrm{d}^2 F(x)$

$\mathrm{d}^2 F(x)$ is Fredholm

$\mathrm{d}^2 F(x) = -\frac{1}{c_x} \int_0^1 \mathfrak{g}\left(\frac{\mathrm{D}}{\mathrm{d}s}\xi, \frac{\mathrm{D}}{\mathrm{d}s}\eta\right) \mathrm{d}s + D[\xi,\eta]$, D compact

2^{nd} order Fermat principle

Let $x \in \Omega_{p_0,p_1}(M_0)$ cp of F. Recall the bijection

$$x \leftrightarrow \gamma = (x,t) : [0,1] \to M = M_0 \times \mathbb{R}$$

A Masiello, Pitman Research Notes in Mathematics 309 (1994)

- $\gamma(1)$ conjugate to $\gamma(0)$ along $\gamma \Leftrightarrow x$ degenerate cp of F
- $\mu(\gamma(1)) = \dim \mathrm{Ker}\, \mathrm{d}^2 F(x)$

$\mathrm{d}^2 F(x)$ is Fredholm

$\mathrm{d}^2 F(x) = -\frac{1}{c_x} \int_0^1 \mathfrak{g}\left(\frac{\mathrm{D}}{\mathrm{d}s}\xi, \frac{\mathrm{D}}{\mathrm{d}s}\eta\right) \mathrm{d}s + D[\xi,\eta]$, D compact

Outline

1. Introduction
   - Genericity and stability in GR
   - Genericity of lightlike nondegeneracy
2. Variational setup
   - Fermat principles
3. **Proof of the main result**
   - C^k genericity of lightlike nondegeneracy
   - C^∞ genericity of lightlike nondegeneracy

Begin the begin...

▸ Main Theorem

Applying the abstract genericity result ▸ recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \to \mathbb{R}$

$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot x,\dot x) + \mathfrak{g}(\dot x,\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot x,\delta)\,\mathrm{d}s$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g}, \delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi, \tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

$\Longrightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)[h,\xi] \neq 0$

Begin the begin...

Main Theorem

Applying the abstract genericity result recall it

$$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \to \mathbb{R}$$

$$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed

Begin the begin...

Main Theorem

Applying the abstract genericity result recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \to \mathbb{R}$

$$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi,\tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

   $\Rightarrow \exists h = (\mathfrak{h},\zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)[h,\xi] \neq 0$

Begin the begin...

Main Theorem

Applying the abstract genericity result recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \longrightarrow \mathbb{R}$

$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi,\tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

   $\Rightarrow \exists h = (\mathfrak{h},\zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)[h,\xi] \neq 0$

Begin the begin...

Main Theorem

Applying the abstract genericity result recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \longrightarrow \mathbb{R}$

$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi,\tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

$\Rightarrow \exists h = (\mathfrak{h},\zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)\big[h,\xi\big] \neq 0$

Begin the begin...

▸ Main Theorem

Applying the abstract genericity result ▸ recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \longrightarrow \mathbb{R}$

$F((\mathfrak{g}, \delta), x) = \left(\int_0^1 \mathfrak{g}(\dot{x}, \dot{x}) + \mathfrak{g}(\dot{x}, \delta)^2 \, \mathrm{d}s \right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x}, \delta) \, \mathrm{d}s$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g}, \delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$
   - $V = (\xi, \tau)$ nontrivial Jacobi along $\gamma = (x, t)$ st $V(0) = V(1) = 0$

$\Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g \, \partial x}(g, x)\left[h, \xi\right] \neq 0$

Begin the begin...

Main Theorem

Applying the abstract genericity result

recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \to \mathbb{R}$

$$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi,\tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

   $\Rightarrow \exists h = (\mathfrak{h},\zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)\big[h,\xi\big] \neq 0$

Begin the begin...

Main Theorem

Applying the abstract genericity result

recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \longrightarrow \mathbb{R}$

$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi,\tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

$\Rightarrow \exists h = (\mathfrak{h},\zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)\left[h,\xi\right] \neq 0$

Begin the begin...

Main Theorem

Applying the abstract genericity result

recall it

$F : \mathfrak{A} \times \Omega_{p_0,p_1}(M_0) \longrightarrow \mathbb{R}$

$F((\mathfrak{g},\delta),x) = \left(\int_0^1 \mathfrak{g}(\dot{x},\dot{x}) + \mathfrak{g}(\dot{x},\delta)^2\,\mathrm{d}s\right)^{\frac{1}{2}} + \int_0^1 \mathfrak{g}(\dot{x},\delta)\,\mathrm{d}s$

Ingredients:

1. find a suitable set of admissible metrics $g = (\mathfrak{g},\delta) \in \mathfrak{A}$
2. prove that fixed
   - $g \in \mathfrak{A}$
   - $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$
   - $V = (\xi,\tau)$ nontrivial Jacobi along $\gamma = (x,t)$ st $V(0) = V(1) = 0$

$\Rightarrow \exists h = (\mathfrak{h},\zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)\left[h,\xi\right] \neq 0$

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

◂ Return

1. $\|h\|_k = \max_{j=0,\dots,k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j h_x \| \right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots,k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j V_x \| \right] < +\infty$
3. $\inf_{x \in M_0} \min_{\substack{v \in T_x M_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v,v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\mathfrak{g}(h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in $\mathcal{C}_0^\infty$
- $\|\cdot\|$ convergence $\to C^k$ convergence on compact sets

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

Return

1. $\|h\|_k = \max_{j=0,\dots,k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j h_x \| \right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots,k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j V_x \| \right] < +\infty$
3. $\inf_{x \in M_0} \min_{\substack{v \in T_x M_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v, v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\forall (h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in C_c^∞
- $\|\cdot\|_k$ convergence $\Rightarrow$ C^k convergence on compact sets

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

Return

1. $\|h\|_k = \max_{j=0,\dots k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j h_x \|\right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j V_x \|\right] < +\infty$
3. $\inf_{x \in M_0} \min_{\substack{v \in T_x M_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v, v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\forall (h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in C_0^∞
- $\| \cdot \|$ convergence $\Rightarrow C^k$ convergence on compact sets

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

Return

1. $\|h\|_k = \max_{j=0,\dots k}\left[\sup_{x\in M_0} \|\left(\nabla^0\right)^j h_x\|\right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots k}\left[\sup_{x\in M_0} \|\left(\nabla^0\right)^j V_x\|\right] < +\infty$
3. $\inf_{x\in M_0} \min_{\substack{v\in T_x M_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v, v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\forall (h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in C_0^∞
- $\|\cdot\|$ convergence $\Rightarrow C^k$ convergence on compact sets

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

Return

1. $\|h\|_k = \max_{j=0,\dots,k}\left[\sup_{x\in M_0} \|\left(\nabla^0\right)^j h_x\|\right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots,k}\left[\sup_{x\in M_0} \|\left(\nabla^0\right)^j V_x\|\right] < +\infty$
3. $\inf_{x\in M_0} \min_{\substack{v\in T_xM_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v, v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\forall (h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in $\mathcal{C}^\infty_0$
- $\|\cdot\|$ convergence $\Rightarrow \mathcal{C}^k$ convergence on compact sets

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

Return

1. $\|h\|_k = \max_{j=0,\dots k}\left[\sup_{x\in M_0} \|\left(\nabla^0\right)^j h_x\|\right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots k}\left[\sup_{x\in M_0} \|\left(\nabla^0\right)^j V_x\|\right] < +\infty$
3. $\inf_{x\in M_0} \min_{\substack{v\in T_x M_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v, v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\forall (h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in $\mathcal{C}^\infty_0$
- $\|\cdot\|$ convergence $\Rightarrow \mathcal{C}^k$ convergence on compact sets

Find admissible metrics

Let $\mathfrak{g}_0 \in \mathrm{Riem}(M_0)$ ($\rightsquigarrow \nabla^0$) and $k \geq 2$ fixed

Assumptions on $(h, V) \in \mathfrak{A}$

Return

1. $\|h\|_k = \max_{j=0,\dots,k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j h_x \|\right] < +\infty$
2. $\|V\|_k = \max_{j=0,\dots,k} \left[\sup_{x \in M_0} \| \left(\nabla^0\right)^j V_x \|\right] < +\infty$
3. $\inf_{x \in M_0} \min_{\substack{v \in T_x M_0 \\ (\mathfrak{g}_0)_x(v,v)=1}} h_x(v, v) > 0$

Remarks on $\mathfrak{A}$

- it is an open subset of a Banach space
- $\forall (h, V)$ defines a stationary Lorentzian metric on $M_0 \times \mathbb{R}$
- it contains tensors in $\mathcal{C}^\infty_0$
- $\| \cdot \|$ convergence $\Rightarrow \mathcal{C}^k$ convergence on compact sets

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\left[h, \xi\right] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a, b[\subset [0, 1]$, $U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0,1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a,b[\subset [0,1], U \subset M_0$ w/ $x(]a,b[)$ embedded in U, $\xi_s \nparallel \dot{x}(s)$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g,x)\big[h,\xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0,1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a,b[\subset [0,1]$, $U \subset M_0$ w/ $x(]a,b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a, b[\subset [0, 1], U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$
- observe that $W = 2[c_x - \mathfrak{g}(\dot{x}, \delta)]\delta - \dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{T}^2_0(x(]a, b[)$ w/ $\int_a^b K_t(\dot{x}, W)\, ds \neq 0$
- $\exists \mathfrak{h} \in \mathfrak{T}^2_0(U)$ w/ $\mathfrak{h}_x = 0, \nabla_\xi \mathfrak{h} = K$ on $x(]a, b[)$.
- observe that $\zeta = \nabla_\xi \zeta = 0$ and $\mathfrak{h} = 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)[h, \xi] = \frac{1}{2c_x} \int_0^1 \nabla_\xi \mathfrak{h}(\dot{x}, W)\, ds$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{$self-$\cap$ of $x\} < +\infty$ and $\sharp\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a, b[\subset [0, 1], U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$
- observe that $W = 2[c_x - \mathfrak{g}(\dot{x}, \delta)]\delta - \dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{T}^2_0(x(]a, b[)$ w/ $\int_a^b K_t(\dot{x}, W)\,\mathrm{d}s \neq 0$
- $\exists \mathfrak{h} \in \mathfrak{T}^2_0(U)$ w/ $\mathfrak{h}_x = 0, \nabla_\xi \mathfrak{h} = K$ on $x(]a, b[)$.
- observe that $\zeta = \nabla_\xi \zeta \equiv 0$ and $\mathfrak{h} \equiv 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \nabla_\xi \mathfrak{h}(\dot{x}, W)\,\mathrm{d}s$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\,\partial x}(g, x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a, b[\subset [0, 1]$, $U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \nparallel \dot{x}(s)$
- observe that $W = 2[c_x - \mathfrak{g}(\dot{x}, \delta)]\delta - \dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{T}^2_0(x(]a, b[)$ w/ $\int_a^b K_t(\dot{x}, W)\,\mathrm{d}s \neq 0$
- $\exists \mathfrak{h} \in \mathfrak{T}^2_0(U)$ w/ $\mathfrak{h}_x = 0, \nabla_\xi \mathfrak{h} = K$ on $x(]a, b[)$.
- observe that $\zeta = \nabla_\xi \zeta = 0$ and $\mathfrak{h} = 0$ implies $\frac{\partial^2 F}{\partial g\,\partial x}(g, x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \nabla_\xi \mathfrak{h}(\dot{x}, W)\,\mathrm{d}s$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0,1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a,b[\subset [0,1], U \subset M_0$ w/ $x(]a,b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$
- observe that $W = 2[c_x - \mathfrak{g}(\dot{x}, \delta)]\delta - \dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{T}^2_0(x(]a,b[)$ w/ $\int_a^b K_t(\dot{x}, W)\,\mathrm{d}s \neq 0$
- $\exists \mathfrak{h} \in \mathfrak{T}^2_0(U)$ w/ $\mathfrak{h}_x = 0$, $\nabla_\xi \mathfrak{h} = K$ on $x(]a,b[)$.
- observe that $\zeta = \nabla_\xi \zeta \equiv 0$ and $\mathfrak{h} \equiv 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \nabla_\xi \mathfrak{h}(\dot{x}, W)\,\mathrm{d}s$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\big\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\big\} < +\infty$
- $\exists]a, b[\subset [0, 1], U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \nparallel \dot{x}(s)$

similarly...

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{$self-$\cap$ of $x\} < +\infty$ and $\sharp\{s \in [0,1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a,b[\subset [0,1]$, $U \subset M_0$ w/ $x(]a,b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$
- observe that $Z = [c_x - \mathfrak{g}(\dot{x}, \delta)]\dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{X}(x(]a,b[)$ w/ $\int_a^b \mathfrak{g}(Z, K)\, ds \neq 0$
- $\exists \zeta \in \mathfrak{X}(U)$ w/ $\zeta_x = 0, \nabla_\xi \zeta = K$ on $x(]a,b[)$.
- observe that $\zeta = 0$ and $\mathfrak{h} = \nabla_\xi \mathfrak{h} = 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \mathfrak{g}(Z, \nabla_\xi \zeta)\, ds$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0,1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a,b[\subset [0,1]$, $U \subset M_0$ w/ $x(]a,b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$
- observe that $Z = [c_x - \mathfrak{g}(\dot{x}, \delta)]\dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{X}(x(]a,b[)$ w/ $\int_a^b \mathfrak{g}(Z, K)\, \mathrm{d}s \neq 0$
- $\exists \zeta \in \mathfrak{X}(U)$ w/ $\zeta_x = 0, \nabla_\xi \zeta = K$ on $x(]a,b[)$.
- observe that $\zeta = 0$ and $\mathfrak{h} = \nabla_\xi \mathfrak{h} \equiv 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \mathfrak{g}(Z, \nabla_\xi \zeta)\, \mathrm{d}s$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a, b[\subset [0, 1]$, $U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \not\parallel \dot{x}(s)$
- observe that $Z = [c_x - \mathfrak{g}(\dot{x}, \delta)]\dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{X}(x(]a, b[)$ w/ $\int_a^b \mathfrak{g}(Z, K)\, \mathrm{d}s \neq 0$
- $\exists \zeta \in \mathfrak{X}(U)$ w/ $\zeta_x = 0, \nabla_\xi \zeta = K$ on $x(]a, b[)$.
- observe that $\zeta = 0$ and $\mathfrak{h} = \nabla_\xi \mathfrak{h} \equiv 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \mathfrak{g}(Z, \nabla_\xi \zeta)\, \mathrm{d}s$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}, x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g, x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- $\sharp\{\text{self-}\cap \text{ of } x\} < +\infty$ and $\sharp\{s \in [0, 1] : \xi_s \parallel \dot{x}(s)\} < +\infty$
- $\exists]a, b[\subset [0, 1], U \subset M_0$ w/ $x(]a, b[)$ embedded in U, $\xi_s \nparallel \dot{x}(s)$
- observe that $Z = [c_x - \mathfrak{g}(\dot{x}, \delta)]\dot{x} \neq 0 \forall s$
- $\exists K \in \mathfrak{X}(x(]a, b[)$ w/ $\int_a^b \mathfrak{g}(Z, K)\, \mathrm{d}s \neq 0$
- $\exists \zeta \in \mathfrak{X}(U)$ w/ $\zeta_x = 0, \nabla_\xi \zeta = K$ on $x(]a, b[)$.
- observe that $\zeta = 0$ and $\mathfrak{h} = \nabla_\xi \mathfrak{h} \equiv 0$ implies $\frac{\partial^2 F}{\partial g\, \partial x}(g, x)\big[h, \xi\big] = \frac{1}{2c_x} \int_0^1 \mathfrak{g}(Z, \nabla_\xi \zeta)\, \mathrm{d}s$

The transversality condition

Problem

$\forall g = (\mathfrak{g}, \delta) \in \mathfrak{A}$, $x \in \Omega_{p_0,p_1}(M_0)$ cp of $x \mapsto F(g,x)$, $V = (\xi, \tau)$ Jacobi w/ $V_0 = V_1 = 0 \Rightarrow \exists h = (\mathfrak{h}, \zeta) \in T_g\mathfrak{A}$ st $\frac{\partial^2 F}{\partial g\, \partial x}(g,x)\big[h, \xi\big] \neq 0$

A sketch of the proof when x not periodic

- Note that in this way a **stronger** result is proven
  1. $\forall \mathfrak{g}$, the set of δ such that g has only nondegenerate light rays from $(p_0, 0)$ to $\{p_1\} \times \mathbb{R}$ is generic in $\{\delta : (\mathfrak{g}, \delta) \in \mathfrak{A}\}$
  2. $\forall \delta$, the set of $\mathfrak{g}$ such that g has only nondegenerate light rays from $(p_0, 0)$ to $\{p_1\} \times \mathbb{R}$ is generic in $\{\mathfrak{g} : (\mathfrak{g}, \delta) \in \mathfrak{A}\}$

Extension to C^∞ genericity

Obstruction

▸ C^k admissible tensors

C^∞–topology makes the space of admissible metrics Frechet

Solution

Use an idea from A Floer, H Hofer and D Salamon (1995), Duke Math J **80** 251, used in L Biliotti, MA Javaloyes and P Piccione, Indiana Univ. Math. J (2009) for the fixed-point case.

To begin

- From now on denote the set of admissible tensor by $\mathfrak{A}_k$ (to stress dependence on $k \in \mathbb{N}$).
- fixed $p_0, p_1 \in M_0$ denote by $\mathfrak{A}_{k,}$[illegible] the set of $(g, \delta) \in \mathfrak{A}_k$ such that [illegible] light ray in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ is nondeg[illegible]
- what we proved so far: $\mathfrak{A}_{k,}$[illegible] $\subset \mathfrak{A}_k$ generic $\forall k \geq 2$

Extension to C^∞ genericity

Obstruction

▸ C^k admissible tensors

C^∞–topology makes the space of admissible metrics Frechet

Solution

Use an idea from A Floer, H Hofer and D Salamon (1995), Duke Math J **80** 251, used in L Biliotti, MA Javaloyes and P Piccione, Indiana Univ. Math. J (2009) for the fixed-point case.

To begin

- From now on denote the set of admissible tensor by $\mathfrak{A}_k$ (to stress dependence on $k \in \mathbb{N}$).
- fixed $p_0, p_1 \in M_0$ denote by $\mathfrak{A}_{k,*}$ the set of $(g,\delta) \in \mathfrak{A}_k$ such that $\forall$ light ray in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ is nondeg.
- what we proved so far: $\mathfrak{A}_{k,*} \subset \mathfrak{A}_k$ generic $\forall k \geq 2$

Extension to C^∞ genericity

Obstruction C^k admissible tensors

C^∞–topology makes the space of admissible metrics Frechet

Solution

Use an idea from A Floer, H Hofer and D Salamon (1995), Duke Math J **80** 251, used in L Biliotti, MA Javaloyes and P Piccione, Indiana Univ. Math. J (2009) for the fixed-point case.

To begin

- From now on denote the set of admissible tensor by $\mathfrak{A}_k$ (to stress dependence on $k \in \mathbb{N}$).
- fixed $p_0, p_1 \in M_0$ denote by $\mathfrak{A}_{k,\star}$ the set of $(\mathfrak{g}, \delta) \in \mathfrak{A}_k$ such that $\forall$ light ray in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ is nondeg.
- what we proved so far: $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

Extension to C^∞ genericity

Obstruction

C^k admissible tensors

C^∞–topology makes the space of admissible metrics Frechet

Solution

Use an idea from A Floer, H Hofer and D Salamon (1995), Duke Math J **80** 251, used in L Biliotti, MA Javaloyes and P Piccione, Indiana Univ. Math. J (2009) for the fixed-point case.

To begin

- From now on denote the set of admissible tensor by $\mathfrak{A}_k$ (to stress dependence on $k \in \mathbb{N}$).
- fixed $p_0, p_1 \in M_0$ denote by $\mathfrak{A}_{k,\star}$ the set of $(\mathfrak{g}, \delta) \in \mathfrak{A}_k$ such that $\forall$ light ray in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ is nondeg.
- what we proved so far: $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

Extension to C^∞ genericity

Obstruction

C^k admissible tensors

C^∞–topology makes the space of admissible metrics Frechet

Solution

Use an idea from A Floer, H Hofer and D Salamon (1995), Duke Math J **80** 251, used in L Biliotti, MA Javaloyes and P Piccione, Indiana Univ. Math. J (2009) for the fixed-point case.

To begin

- From now on denote the set of admissible tensor by $\mathfrak{A}_k$ (to stress dependence on $k \in \mathbb{N}$).
- fixed $p_0, p_1 \in M_0$ denote by $\mathfrak{A}_{k,\star}$ the set of $(\mathfrak{g}, \delta) \in \mathfrak{A}_k$ such that $\forall$ light ray in $M_0 \times \mathbb{R}$ between $(p_0, 0)$ and $\{p_1\} \times \mathbb{R}$ is nondeg.
- what we proved so far: $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k, \; \mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|x\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$
  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k \; \forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k$, $\mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$
  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$ $\forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k$, $\mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$

  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$ $\forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k, \ \mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$
  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k \ \forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k$, $\mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$
  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$ $\forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k$, $\mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$
  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$ $\forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k, \ \mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$
  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$ $\forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Extension to C^∞ genericity

Problem: start from $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_k$ generic $\forall k \geq 2$

- $\mathfrak{A}_\infty = \cap_k \mathfrak{A}_k$, $\mathfrak{A}_{\infty,\star} = \cap_k \mathfrak{A}_{k,\star}$
- claim: $\mathfrak{A}_{\infty,\star} \subseteq \mathfrak{A}_\infty$ generic

Sketch of the proof

- define $\mathfrak{A}_{k,\star,N}$ such that every light ray between p and γ st $\|\dot{x}\|_\infty \leq N$ is nondegenerate and let $\mathfrak{A}_{\infty,\star,N} = \cap_k \mathfrak{A}_{k,\star,N}$
- $\mathfrak{A}_{k,\star,N} \subseteq \mathfrak{A}_k$ is open $\forall N$ and $\forall 2 \leq k \leq +\infty$
- observe $\mathfrak{A}_{k,\star} \subseteq \mathfrak{A}_{k,\star,N} \Rightarrow \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$
- observe $\mathfrak{A}_\infty$ dense in $\mathfrak{A}_k$

  $\Rightarrow \mathfrak{A}_{\infty,\star,N} = \mathfrak{A}_\infty \cap \mathfrak{A}_{k,\star,N}$ dense in $\mathfrak{A}_k$ $\forall k \Rightarrow \mathfrak{A}_{\infty,\star,N}$ dense in $\mathfrak{A}_\infty$
- observe $\mathfrak{A}_{\infty,\star} = \cap_N \mathfrak{A}_{\infty,\star,N}$

Summary

Conclusions

- motivations for studying genericity features in GR are observational and geometrical as well
- genericity problems in Lorentzian geometry can be tackled using global abstract results
- these techniques also successfully extend to general splitting spacetimes (work in progress)
- open problem: genericity of light rays nondegeneracy for wider classes of spacetimes

Summary

Conclusions

- motivations for studying genericity features in GR are observational and geometrical as well
- genericity problems in Lorentzian geometry can be tackled using global abstract results
- these techniques also successfully extend to general splitting spacetimes (work in progress)
- open problem: genericity of light rays nondegeneracy for wider classes of spacetimes

Summary

Conclusions

- motivations for studying genericity features in GR are observational and geometrical as well
- genericity problems in Lorentzian geometry can be tackled using global abstract results
- these techniques also successfully extend to general splitting spacetimes (work in progress)
- open problem: genericity of light rays nondegeneracy for wider classes of spacetimes

Summary

Conclusions

- motivations for studying genericity features in GR are observational and geometrical as well
- genericity problems in Lorentzian geometry can be tackled using global abstract results
- these techniques also successfully extend to general splitting spacetimes (work in progress)
- open problem: genericity of light rays nondegeneracy for wider classes of spacetimes