Let us denote by S_t for $t = 0, 1, \ldots$ the set of users that decide to subscribe to the service during time slot t in the temporary promotion model. Let us define for each subset $A \subseteq V$ the function f_θ as

$$f_\theta(A) = \{\ v \in V \ \mid \ \text{at least a fraction } q \geq \theta \text{ of neighbors of } v \text{ are in } A \quad \}\ .$$

1. ($\rightarrowtail$)Show that for any $t \geq 0$, $S_t = f_\theta^{(t)}(S_0)$ where $f_\theta^{(t)}$ denotes the function f_θ applied t times.

Let us denote respectively by S'_t and S''_t for $t = 0, 1, \ldots$ the set of users that decide to subscribe to the service during time slot t in the targeted permanent promotion model and the general permanent promotion model.

2. ($\curvearrowright$) Show that $S'_t = S''_t = g_\theta^{(t)}(S_0)$ where for any subset $A \subseteq V$, $g_\theta(A) = f_\theta(A) \cup A$. What can you deduce w.r.t. the efficiency of the general permanent promotion strategy?

We consider an infinite graph $G = (V, E)$ where all nodes have finite degree. For a given threshold θ, we say that a set S_0 is an infectious set for the temporary promotion model if, starting from S_0 the sequence S_t of nodes that subscribe the service eventually reach all nodes. Formally, S_0 is infectious if

$$\forall v \in V, \exists k \geq 0 \ \text{ such that } \ \forall t \geq k, v \in S_t\ .$$

Note that this definition is shown here for the temporary promotion model (*i.e.,* using the sequence $(S_t)_{t\geq 0}$) and that the same definition could be used with S'_t and S''_t to define infectious set in the two other models.

3. ($\curvearrowright$) Give an example of a graph $G = (V, E)$ and a set S_0 that is infectious for the general permanent promotion model but not for the temporary promotion model.

Let S_0 be a finite subset that is infectious for the general permanent promotion model. We define S^+ as the subset that contains all nodes in S_0 as well as all neighbors of nodes in S_0. Since S_0 is infectious for the general permanent promotion model, there exists t_0 such that $S^+ \subseteq S'_{t_0}$, where S'_t is the sequence of subscribing nodes starting from S_0.

4. ($\curvearrowright$) Show that the subset $T = S'_{t_0}$ is infectious for the temporary promotion model (*i.e.,* that the sequence $(S_t)_{t\geq 0}$ starting from T eventually contains the whole set).

Exercise 3: Connection between two general models of influence (6 pt)

Motivation In this exercice, we prove that the two general models of influence with random thresholds are, under a natural condition, equivalent.

As a quick reminder from the lecture, one can define more general model of influence in one of the two following manner:

- Define for any node $u \in V$ a function g_u taking value in $[0; 1]$ and which is defined on all subset of neighbors of u (*i.e.,* for any $S \subseteq N(U)$ we define a value $g_u(S) \in [0; 1]$).

  Node's behavior is then characterized as follows. First, we assume that a set S_0 of nodes initially adopt the service, and that for any node $v \in V$ there exists a threshold θ_v which chosen once for all in $[0; 1]$ according to a uniform distribution.

  Then, for any time slot t, if during this time slot t, the set of neighbors of v which adopt the service is S, v will adopt the service if and only if we have $\theta_v \leq g_v(S)$.