Preface

It is gratifying to note that education as a whole and school education in particular witness marked changes in the state of Tamil Nadu resulting in the implementation of uniform curriculum for all streams in the school education system. This is a golden opportunity given by the Government of Tamil Nadu which must be utilized for the over all improvement of education in Tamil Nadu.

Mathematics, the queen of all sciences, remains and will remain as a subject with great charm having an intrinsic value and beauty of its own. It plays an indispensable role in sciences, engineering and other subjects as well. So, mathematical knowledge is essential for the growth of science and technology, and for any individual to shine well in the field of one's choice. In addition, a rigorous mathematical training gives one not only the knowledge of mathematics but also a disciplined thought process, an ability to analyze complicated problems.

Thiruvalluvar, the prophetic Tamil poet, had as far back as at least two thousand years ago, underlined the importance and the value of mathematical education by saying,

எண்ணென்ப ஏனை எழுத்தென்ப இவ்விரண்டும்
கண்ணென்ப வாழும் உயிர்க்கு. – குறள் (392)

The two that are known as numbers and letters
They say are the eyes of people on the earth.

– Kural (392)

We need the power and prowess of mathematics to face and solve the ever increasing complex problems that we encounter in our life. Furthermore, mathematics is a supremely creative force and not just a problem solving tool. The learners will realize this fact to their immense satisfaction and advantage as they learn more and more of mathematics.

Besides, a good mathematical training is very much essential to create a good work force for posterity. The rudiments of mathematics attained at the school level form the basis of higher studies in the field of mathematics and other sciences. Besides learning the basics of mathematics, it is also important to learn how to apply them in solving problems.

Deeper understanding of basic principles and problem solving are the two important components of learning mathematics. This book is a step in this direction. It is intended to help the students to grasp the fundamentals of mathematics and apply them in problem solving. It also fosters an informed awareness of how mathematics develops and works in different situations. With this end in view, the chapters are arranged in their natural and logical order with a good number of worked out examples. Each section of a chapter is designed in such a way as to provide the students the much needed practice which promotes a thorough understanding of the concepts. We suggest that before going into the problems, the teachers and the students get themselves acquainted with the underlying mathematical ideas and their connections which precede the set of problems given in the exercises.

However, be it remembered that mathematics is more than the science of numbers. The teacher in the classroom is the most important person whose help and guidance are indispensable in learning mathematics. During the stage of transition from basic mathematics to higher mathematics, the teachers have a more significant role to play. In this context we hope that this book serves the purpose and acts as a catalyst. To reap the maximum benefit out of this, the teacher should necessarily strive for a two-way communication. This endeavour will undoubtedly pave the way for learner-centered activities in the class rooms. Moreover, this text book is aimed at giving the students a space to explore mathematics and develop skills in all directions. As we have mentioned already, there are two parts in learning mathematics. One is learning the basics and the other is applying the basics in problem solving. Going through the examples in the text does help in understanding the methods; but learning basics, solving exercise problems on one's own and then trying to create new related problems alone will help consolidate one's mathematical knowledge.

We learn Mathematics by doing Mathematics.

We would be grateful for suggestions and comments from experts, teachers and students for the improvement of this book.

Murthy R
Chairperson

SYLLABUS

Topic	Content	Expected Learning Outcomes	Transactional Teaching Strategy	No. of Periods
I. Sets and Functions	i. Introduction ii. Properties of operations on sets iii. De Morgan's laws-verification using example Venn diagram iv. Formula for v. Functions	• To revise the basic concepts on Set operations • To understand the properties of operations of sets - commutative, associative, and distributive restricted to three sets. • To understand the laws of complementation of sets. • To understand De Morgan's laws and demonstrating them by Venn diagram as well. • To solve word problems using the formula as well as Venn diagram. • To understand the definition , types and representation of functions. • To understand the types of functions with simple examples.	Use Venn diagrams for all illustrations Give examples of functions from economics, medicine, science etc.	26
II. Sequences and Series of Real Numbers	i. Introduction ii. Sequences iii. Arithmetic Progression (A.P) iv. Geometric Progression (G.P) v. Series	• To understand to identify an Arithmetic Progression and a Geometric Progression. • Able to apply to find the nth term of an Arithmetic Progression and a Geometric Progression. • To determine the sum of n terms of an Arithmetic Progression and a Geometric Progression. • To determine the sum of some finite series.	Use pattern approach Use dot pattern as teaching aid Use patterns to derive formulae Examples to be given from real life situations	27
III. Algebra	i. Solving linear equations ii. Polynomials iii. Synthetic division iv. Greatest common divisor (GCD) and Least common multiple (LCM) v. Rational expressions vi. Square root vii. Quadratic Equations	• To understand the idea about pair of linear equations in two unknowns. Solving a pair of linear equations in two variables by elimination method and cross multiplication method. • To understand the relationship between zeros and coefficients of a polynomial with particular reference to quadratic polynomials.	Illustrative examples – Use charts as teaching aids Recall GCD and LCM of numbers initially	

III. Algebra		• To determine the remainder and the quotient of the given polynomial using Synthetic Division Method. • To determine the factors of the given polynomial using Synthetic Division Method. • Able to understand the difference between GCD and LCM, of rational expression. • Able to simplify rational expressions (Simple Problems), • To understand square roots. • To understand the standard form of a quadratic equation . • To solve quadratic equations (only real root) - by factorization, by completing the square and by using quadratic formula. • Able to solve word problems based on quadratic equations. • Able to correlate relationship between discriminant and nature of roots. • Able to Form quadratic equation when the roots are given.	Compare with operations on fractions Compare with the square root operation on numerals. Help students visualize the nature of roots algebraically and graphically.	40
IV. Matrices	i. Introduction ii. Types of matrices iii. Addition and subtraction iv. Multiplication v. Matrix equation	• Able to identify the order and formation of matrices • Able to recognize the types of matrices • Able to add and subtract the given matrices. • To multiply a matrix by a scalar, and the transpose of a matrix. • To multiply the given matrices (2x2; 2x3; 3x2 Matrices). • Using matrix method solve the equations of two variables.	Using of rectangular array of numbers. Using real life situations. Arithmetic operations to be used	16

V. Coordinate Geometry	i. Introduction ii. Revision :Distance between two points iii. Section formula, Mid point formula, Centroid formula iv. Area of a triangle and quadrilateral v. Straight line	• To recall the distance between two points, and locate the mid point of two given points. • To determine the point of division using section formula (internal). • To calculate the area of a triangle. • To determine the slope of a line when two points are given, equation is given. • To find an equation of line with the given information. • Able to find equation of a line in: slope-intercept form, point -slope form, two -point form, intercept form. • To find the equation of a straight line passing through a point which is (i) parallel (ii) perpendicular to a given straight line.	Simple geometrical result related to triangle and quadrilaterals to be verified as applications. the form $y = mx + c$ to be taken as the starting point	25
VI. Geometry	i. Basic proportionality theorem (with proof) ii. Converse of Basic proportionality theorem (with proof) iii. Angle bisector theorem (with proof - internal case only) iv. Converse of Angle bisector theorem (with proof - internal case only) v. Similar triangles (theorems without proof)	• To understand the theorems and apply them to solve simple problems only.	Paper folding symmetry and transformation techniques to be adopted. Formal proof to be given Drawing of figures Step by step logical proof with diagrams to be explained and discussed	20
VII. Trigonometry	i. Introduction ii. Identities iii. Heights and distances	• Able to identify the Trigonometric identities and apply them in simple problems. • To understand trigonometric ratios and applies them to calculate heights and distances. (not more than two right triangles)	By using Algebraic formulae Using trigonometric identities. The approximate nature of values to be explained	21

VIII. Mensuration	i. Introduction ii. Surface Area and Volume of Cylinder, Cone, Sphere, Hemisphere, Frustum iii. Surface area and volume of combined figures iv. Invariant volume	• To determine volume and surface area of cylinder, cone, sphere, hemisphere, frustum • Volume and surface area of combined figures (only two). • Some problems restricted to constant Volume.	Use 3D models to create combined shapes Use models and pictures ad teaching aids. Choose examples from real life situations.	24
IX. Practical Geometry	i. Introduction ii. Construction of tangents to circles iii. Construction of Triangles iv. Construction of cyclic quadrilateral	• Able to construct tangents to circles. • Able to construct triangles, given its base, vertical angle at the opposite vertex and (a) median (b) altitude (c) bisector. • Able to construct a cyclic quadrilateral	To introduce algebraic verification of length of tangent segments. Recall related properties of angles in a circle before construction. Recall relevant theorems in theoretical geometry	15
X. Graphs	i. Introduction ii. Quadratic graphs iii. Some special graphs	• Able to solve quadratic equations through graphs • To solve graphically the equations . • Able to apply graphs to solve word problems	Interpreting skills also to be taken care of graphs of quadratics to precede algebraic treatment. Real life situations to be introduced.	10
XI. Statistics	i. Recall Measures of central tendency ii. Measures of dispersion iii. Coefficient of variation	• To recall Mean for grouped and ungrouped data situation to be avoided). • To understand the concept of Dispersion and able to find Range, Standard Deviation and Variance. • Able to calculate the coefficient of variation.	Use real life situations like performance in examination, sports, etc.	16
XII. Probability	i. Introduction ii. Probability-theoretical approach iii. Addition Theorem on Probability	• To understand Random experiments, Sample space and Events – Mutually Exclusive, Complementary, certain and impossible events. • To understand addition Theorem on probability and apply it in solving some simple problems.	Diagrams and investigations on coin tossing, die throwing and picking up the cards from a deck of cards are to be used.	15

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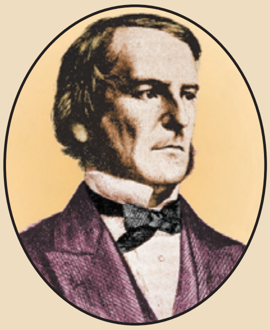
1

SETS AND FUNCTIONS

A set is Many that allows itself to be thought of as a One
- Georg Cantor

- Introduction
- Sets
- Properties of set operations
- De Morgan's Laws
- Functions



GEORGE BOOLE

(1815-1864)
England

Boole believed that there was a close analogy between symbols that represent logical interactions and algebraic symbols.

He used mathematical symbols to express logical relations. Although computers did not exist in his day, Boole would be pleased to know that his Boolean algebra is the basis of all computer arithmetic.

As the inventor of Boolean logic-the basis of modern digital computer logic - Boole is regarded in hindsight as a founder of the field of computer science.

1.1 Introduction

The concept of set is one of the fundamental concepts in mathematics. The notation and terminology of set theory is useful in every part of mathematics. So, we may say that set theory is the language of mathematics. This subject, which originated from the works of George Boole (1815-1864) and Georg Cantor (1845-1918) in the later part of 19th century, has had a profound influence on the development of all branches of mathematics in the 20th century. It has helped in unifying many disconnected ideas and thus facilitated the advancement of mathematics.

In class IX, we have learnt the concept of set, some operations like union, intersection and difference of two sets. Here, we shall learn some more concepts relating to sets and another important concept in mathematics namely, function. First let us recall basic definitions with some examples. We denote all positive integers (natural numbers) by $\mathbb{N}$ and all real numbers by $\mathbb{R}$.

1.2 Sets

Definition

A set is a collection of well-defined objects. The objects in a set are called elements or members of that set.

Here, "well-defined" means that the criteria for deciding if an object belongs to the set or not, should be defined without confusion.

For example, the collection of all "tall people" in Chennai does not form a set, because here, the deciding criteria "tall people" is not clearly defined. Hence this collection does not define a set.

Notation

We generally use capital letters like A, B, X, etc. to denote a set. We shall use small letters like x, y, etc. to denote elements of a set. We write $x \in Y$ to mean x is an element of the set Y. We write $t \notin Y$ to mean t is not an element of the set Y.

Examples

(i) The set of all high school students in Tamil Nadu.

(ii) The set of all students either in high school or in college in Tamil Nadu.

(iii) The set of all positive even integers.

(iv) The set of all integers whose square is negative.

(v) The set of all people who landed on the moon.

Let A, B, C, D and E denote the sets defined in (i), (ii), (iii), (iv), and (v) respectively. Note that square of any integer is an integer that is either zero or positive and so there is no integer whose square is negative. Thus, the set D does not contain any element. Any such set is called an empty set. We denote the empty set by ϕ.

Definition

(i) A set is said to be a finite set if it contains only a finite number of elements in it.

(ii) A set which is not finite is called an infinite set.

Observe that the set A given above is a finite set, whereas the set C is an infinite set. Note that empty set contains no elements in it. That is, the number of elements in an empty set is zero. Thus, empty set is also a finite set.

Definition

(i) If a set X is finite, then we define the cardinality of X to be the number of elements in X. Cardinality of a set X is denoted by $n(X)$.

(ii) If a set X is infinite, then we denote the cardinality of X by a symbol ∞.

Now looking at the sets A, B in the above examples, we see that every element of A is also an element of B. In such cases we say A is a subset of B.

Let us recall some of the definitions that we have learnt in class IX.

Subset Let X and Y be two sets. We say X is a subset of Y if every element of X is also an element of Y. That is, X is a subset of Y if $z \in X$ implies $z \in Y$. It is clear that every set is a subset of itself.

If X is a subset of Y, then we denote this by $X \subseteq Y$.

Set Equality Two sets X and Y are said to be equal if both contain exactly same elements. In such a case, we write $X = Y$. That is, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Equivalent Sets Two finite sets X and Y are said to be equivalent if $n(X) = n(Y)$.

For example, let $P = \{x \mid x^2 - x - 6 = 0\}$ and $Q = \{3, -2\}$. It is easy to see that both P, Q contain same elements and so $P = Q$. If $F = \{3, 2\}$, then F, Q are equivalent sets but $Q \neq F$. Using the concept of function, one can define the equivalent of two infinite sets

Power Set Given a set A, let $P(A)$ denote the collection of all subsets of A. The set $P(A)$ is called the power set of A.

If $n(A) = m$, then the number of elements in $P(A)$ is given by $n(P(A)) = 2^m$.

For example, if $A = \{a,b,c\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}. \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and hence $n(P(A)) = 8$.

Now, given two sets, how can we create new sets using the given sets?

One possibility is to put all the elements together from both sets and create a new set. Another possibility is to create a set containing only common elements from both sets. Also, we may create a set having elements from one set that are not in the other set. Following definitions give a precise way of formalizing these ideas. We include Venn diagram next to each definition to illustrate it.

1.3 Operations on sets

Let X and Y be two sets. We define the following new sets:

(i) **Union** $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$

(read as "X union Y")

Note that $X \cup Y$ contains all the elements of X and all the elements of Y and the Fig. 1.1 illustrates this.

It is clear that $X \subseteq X \cup Y$ and also $Y \subseteq X \cup Y$.

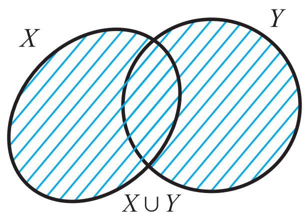


Fig. 1.1

(ii) **Intersection** $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$

(read as "X intersection Y")

Note that $X \cap Y$ contains only those elements which belong to both X and Y and the Fig. 1.2 illustrates this.

It is trivial that $X \cap Y \subseteq X$ and also $X \cap Y \subseteq Y$.

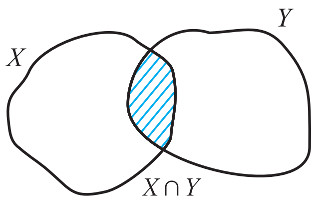


Fig. 1.2

(iii) **Set difference** $X \setminus Y = \{z \mid z \in X \text{ but } z \notin Y\}$

(read as "X difference Y")

Note that $X \setminus Y$ contains only elements of X that are not in Y and the Fig. 1.3 illustrates this. Also, some authors use $A - B$ for $A \setminus B$. We shall use the notation $A \setminus B$ which is widely used in mathematics for set difference.

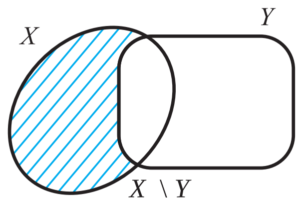


Fig. 1.3

(iv) **Symmetric Difference** $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$

(read as "X symmetric difference Y"). Note that $X \triangle Y$ contains all elements in $X \cup Y$ that are not in $X \cap Y$.

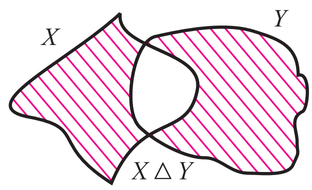


Fig. 1.4

(v) **Complement** If $X \subseteq U$, where U is a universal set, then $U \setminus X$ is called the complement of X with respect to U. If underlying universal set is fixed, then we denote $U \setminus X$ by X' and is called complement of X. The difference set $A \setminus B$ can also be viewed as the complement of B with respect to A.

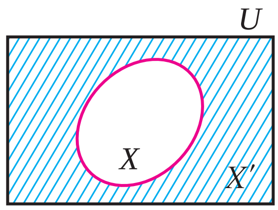


Fig. 1.5

(vi) **Disjoint sets** Two sets X and Y are said to be disjoint if they do not have any common element. That is, X and Y are disjoint if $X \cap Y = \phi$.
It is clear that $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint finite sets.

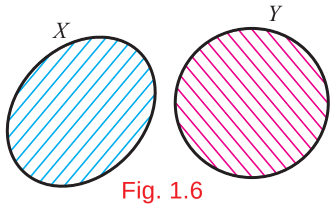


Fig. 1.6

Remarks

Usually circles are used to denote sets in Venn diagrams. However any closed curve may also be used to represent a set in a Venn diagram. While writing the elements of a set, we do not allow repetitions of elements in that set.

Now, we shall see some examples.

Let $A = \{x \mid x \text{ is a positive integer less than } 12\}$, $B = \{1, 2, 4, 6, 7, 8, 12, 15\}$ and $C = \{-2, -1, 0, 1, 3, 5, 7\}$. Now let us find the following:

(i) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$= \{x \mid x \text{ is a positive integer less than } 12, \text{ or } x = 12, \text{ or } 15\}$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15\}$.

(ii) $C \cap B = \{y \mid y \in C \text{ and } y \in B\} = \{1, 7\}$.

(iii) $A \setminus C = \{x \mid x \in A \text{ but } x \notin C\} = \{2, 4, 6, 8, 9, 10, 11\}$.

(iv) $A \triangle C = (A \setminus C) \cup (C \setminus A)$
$= \{2, 4, 6, 8, 9, 10, 11\} \cup \{-2, -1, 0\} = \{-2, -1, 0, 2, 4, 6, 8, 9, 10, 11\}$.

(v) Let $U = \{x \mid x \text{ is an integer}\}$ be the universal set.
Note that 0 is neither positive nor negative. Therefore, $0 \notin A$.

Now, $A' = U \setminus A = \{x : x \text{ is an integer but it should not be in } A\}$
$= \{x \mid x \text{ is either zero or a negative integer or positive integer greater than or equal to } 12\}$
$= \{\cdots, -4, -3, -2, -1, 0\} \cup \{12, 13, 14, 15, \cdots\}$
$= \{\cdots, -4, -3, -2, -1, 0, 12, 13, 14, 15, \cdots\}$.

Let us list out some useful results.

Let U be an universal set and A, B are subsets of U. Then the following hold:

(i) $A \setminus B = A \cap B'$ (ii) $B \setminus A = B \cap A'$

(iii) $A \setminus B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A \setminus B) \cup B = A \cup B$

(v) $(A \setminus B) \cap B = \phi$ (vi) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

Let us state some properties of set operations.

1.4 Properties of set operations

For any three sets A, B and C, the following hold.

(i) Commutative property

(a) $A \cup B = B \cup A$ (set union is commutative)

(b) $A \cap B = B \cap A$ (set intersection is commutative)

(ii) Associative property

(a) $A \cup (B \cup C) = (A \cup B) \cup C$ (set union is associative)

(b) $A \cap (B \cap C) = (A \cap B) \cap C$ (set intersection is associative)

(iii) Distributive property

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (intersection distributes over union)

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (union distributes over intersection)

Mostly we shall verify these properties with the given sets. Instead of verifying the above properties with examples, it is always better to give a mathematical proof. But this is beyond the scope of this book. However, to understand and appreciate a rigorous mathematical proof, let us take one property and give the proof.

(i) Commutative property of union

In this part we want to prove that for any two sets A and B, the sets $A \cup B$ and $B \cup A$ are equal. Our definition of equality of sets says that two sets are equal only if they contain same elements.

First we shall show that every element of $A \cup B$, is also an element of $B \cup A$.
Let $z \in A \cup B$ be an arbitrary element. Then by the definition of union of A and B we have $z \in A$ or $z \in B$. That is,

$$
\begin{aligned}
\text{for every } z \in A \cup B &\implies z \in A \text{ or } z \in B \\
&\implies z \in B \text{ or } z \in A \\
&\implies z \in B \cup A \text{ by the definition of } B \cup A. \qquad (1)
\end{aligned}
$$

Since (1) is true for every $z \in A \cup B$, the above work shows that every element of $A \cup B$ is also is an element of $B \cup A$. Hence, by the definition of a subset, we have $(A \cup B) \subseteq (B \cup A)$.

Next, we consider an arbitrary $y \in B \cup A$ and show that this y is also an element of $A \cup B$.

$$
\begin{aligned}
\text{Now, for every } y \in B \cup A &\implies y \in B \text{ or } y \in A \\
&\implies y \in A \text{ or } y \in B \\
&\implies y \in A \cup B \text{ by the definition of } A \cup B. \qquad (2)
\end{aligned}
$$

Since (2) is true for every $y \in B \cup A$, the above work shows that every element of $B \cup A$ is also an element of $A \cup B$. Hence, by the definition of a subset, we have $(B \cup A) \subseteq (A \cup B)$.

So, we have shown that $(A \cup B) \subseteq (B \cup A)$ and $(B \cup A) \subseteq (A \cup B)$. This can happen only when $(A \cup B) = (B \cup A)$. One could follow above steps to prove other properties listed above by exactly the same method.

About proofs in Mathematics

In mathematics, a statement is called a **true statement** if it is always true. If a statement is not true even in one instance, then the statement is said to be a **false statement**. For example, let us consider a few statements:

(i) Any positive odd integer is a prime number (ii) Sum of all angles in a triangle is 180°

(iii) Every prime number is an odd integer (iv) For any two sets A and B, $A \setminus B = B \setminus A$

Now, the statement (i) is false, though very many odd positive integers are prime, because integers like $9, 15, 21, 45$ etc. are positive and odd but not prime.

The statement (ii) is a true statement because no matter which triangle you consider, the sum of its angles equals 180°.

The statement (iii) is false, because 2 is a prime number but it is an even integer. In fact, the statement (iii) is true for every prime number except for 2. So, if we want to prove a statement we have to prove that it is true for all instances. If we want to disprove a statement it is enough to give an example of one instance, where it is false.

The statement (iv) is false. Let us analyze this statement. Basically, when we form $A \setminus B$ we are removing all elements of B from A. Similarly, for $B \setminus A$. So it is highly possible that the above statement is false. Indeed, let us consider a case where $A = \{2, 5, 8\}$ and $B = \{5, 7, -1\}$. In this case, $A \setminus B = \{2, 8\}$ and $B \setminus A = \{7, -1\}$ and we have $A \setminus B \neq B \setminus A$. Hence the statement given in (iv) is false.

Example 1.1

For the given sets $A = \{-10, 0, 1, 9, 2, 4, 5\}$ and $B = \{-1, -2, 5, 6, 2, 3, 4\}$, verify that (i) set union is commutative. Also verify it by using Venn diagram.

(ii) set intersection is commutative. Also verify it by using Venn diagram.

Solution

(i) Now, $A \cup B = \{-10, 0, 1, 9, 2, 4, 5\} \cup \{-1, -2, 5, 6, 2, 3, 4\}$

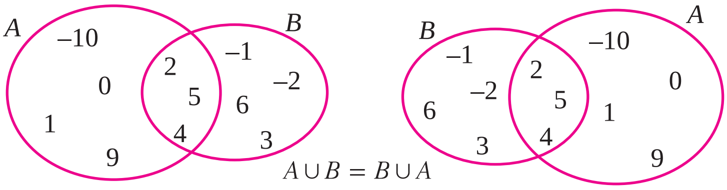
$$= \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\} \quad (1)$$

Also, $B \cup A = \{-1, -2, 5, 6, 2, 3, 4\} \cup \{-10, 0, 1, 9, 2, 4, 5\}$

$$= \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\} \quad (2)$$

Thus, from (1) and (2) we have verified that $A \cup B = B \cup A$.

By Venn diagram, we have



$A \cup B = B \cup A$

Fig. 1.7

Hence, it is verified that set union is commutative.

(ii) Let us verify that intersection is commutative.

Now, $A \cap B = \{-10, 0, 1, 9, 2, 4, 5\} \cap \{-1, -2, 5, 6, 2, 3, 4\}$

$= \{2, 4, 5\}$. (1)

Also, $B \cap A = \{-1, -2, 5, 6, 2, 3, 4\} \cap \{-10, 0, 1, 9, 2, 4, 5\}$

$= \{2, 4, 5\}$. (2)

From (1) and (2), we have $A \cap B = B \cap A$ for the given sets A and B.

By Venn diagram, we have

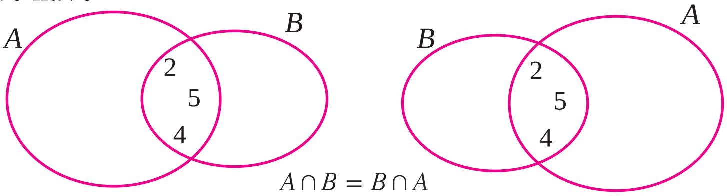


Fig. 1.8

Hence, it is verified.

Example 1.2

Given, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$, show that

(i) $A \cup (B \cup C) = (A \cup B) \cup C$. (ii) Verify (i) using Venn diagram.

Solution

(i) Now, $B \cup C = \{3, 4, 5, 6\} \cup \{5, 6, 7, 8\} = \{3, 4, 5, 6, 7, 8\}$

$\therefore\ A \cup (B \cup C) = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (1)

Now, $A \cup B = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$

$\therefore\ (A \cup B) \cup C = \{1, 2, 3, 4, 5, 6\} \cup \{5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (2)

From (1) and (2), we have $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) Using Venn diagram, we have

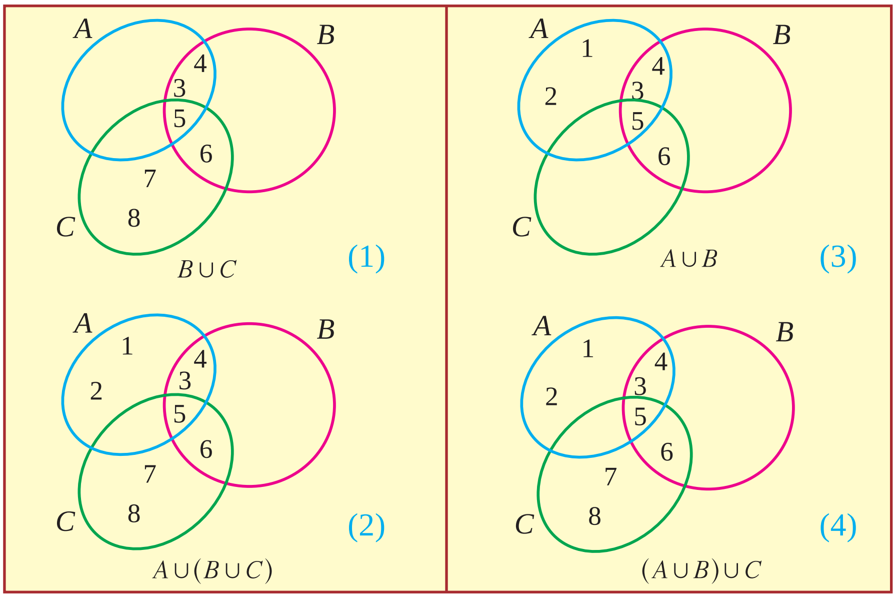


Fig. 1.9

Thus, from (2) and (4), we have verified that the set union is associative.

Example 1.3

Let $A = \{a,b,c,d\}$, $B = \{a,c,e\}$ and $C = \{a,e\}$.

(i) Show that $A \cap (B \cap C) = (A \cap B) \cap C$. (ii) Verify (i) using Venn diagram.

Solution

(i) We are given $A = \{a,b,c,d\}$, $B = \{a,c,e\}$ and $C = \{a,e\}$.

We need to show $A \cap (B \cap C) = (A \cap B) \cap C$. So, we first consider $A \cap (B \cap C)$.

Now, $B \cap C = \{a,c,e\} \cap \{a,e\} = \{a,e\}$; thus,

$$A \cap (B \cap C) = \{a,b,c,d\} \cap \{a,e\} = \{a\}. \quad (1)$$

Next, we shall find $A \cap B = \{a,b,c,d.\} \cap \{a,c,e\} = \{a,c\}$. Hence

$$(A \cap B) \cap C = \{a,c\} \cap \{a,e\} = \{a\} \quad (2)$$

Now (1) and (2) give the desired result.

(ii) Using Venn diagram, we have

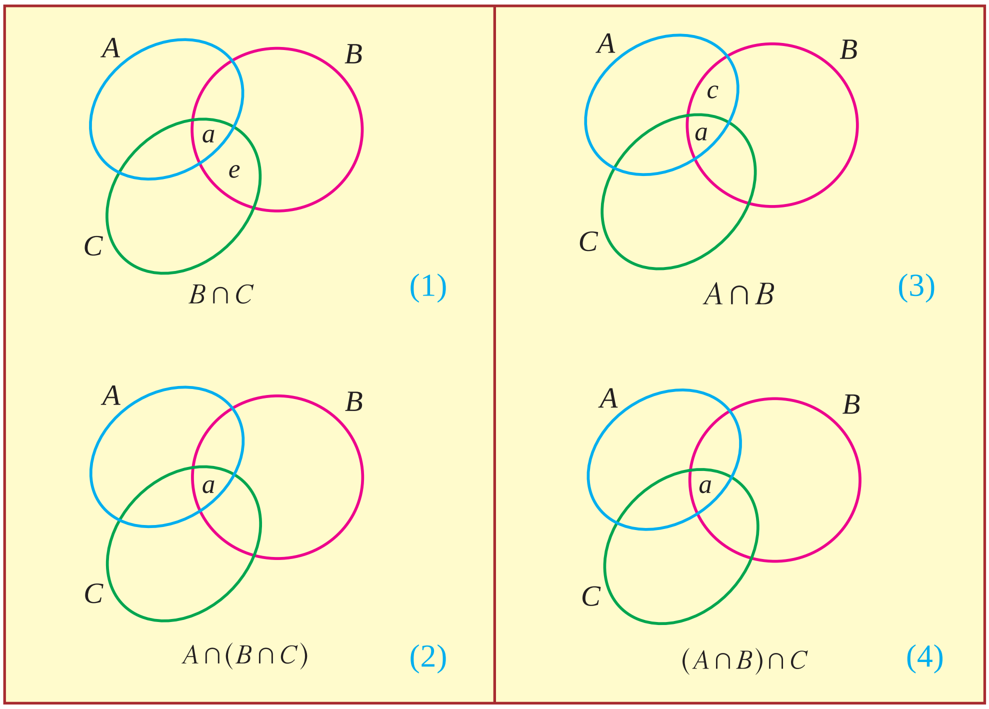


Fig. 1.10

Thus, from (2) and (4) , it is verified that $A \cap (B \cap C) = (A \cap B) \cap C$

Example 1.4

Given $A = \{a,b,c,d,e\}$, $B = \{a,e,i,o,u\}$ and $C = \{c,d,e,u\}$.

(i) Show that $A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$. (ii) Verify (i) using Venn diagram.

Solution

(i) First let us find $A\setminus(B\setminus C)$. To do so, consider

$$(B\setminus C) = \{a,e,i,o,u\}\setminus\{c,d,e,u\} = \{a,i,o\}.$$

Thus, $A\setminus(B\setminus C) = \{a,b,c,d,e\}\setminus\{a,i,o\} = \{b,c,d,e\}$. (1)

Next, we find $(A\setminus B)\setminus C$.

$$A\setminus B = \{a,b,c,d,e\}\setminus\{a,e,i,o,u\} = \{b,c,d\}.$$

Hence, $(A\setminus B)\setminus C = \{b,c,d\}\setminus\{c,d,e,u\} = \{b\}$. (2)

From (1) and (2) we see that $A\setminus(B\setminus C) \neq (A\setminus B)\setminus C$.

Thus, the set difference is not associative.

(ii) Using Venn diagram, we have

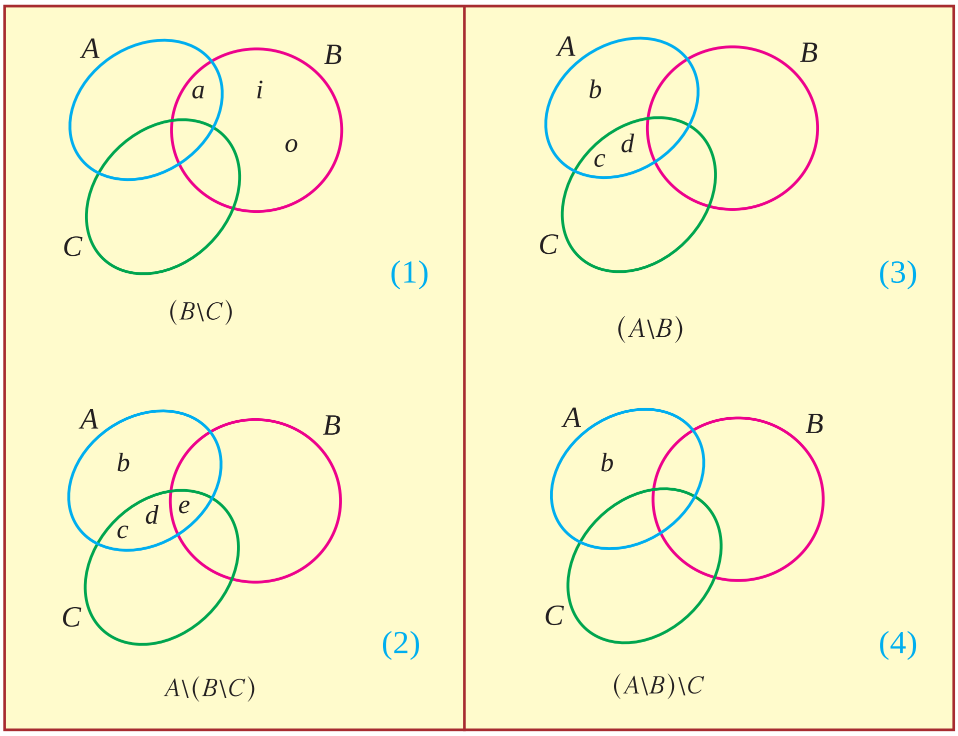


Fig. 1.11

From (2) and (4), it is verified that $A\setminus(B\setminus C) \neq (A\setminus B)\setminus C$.

Remarks

The set difference is not associative. However, if the sets A, B and C are mutually disjoint, then $A\setminus(B\setminus C) = (A\setminus B)\setminus C$. This is very easy to prove; so let us prove it. Since B and C are disjoint we have $B\setminus C = B$. Since A, B are disjoint we have $A\setminus B = A$. Thus, we have $A\setminus(B\setminus C) = A$. Again, $A\setminus B = A$ and A, C are disjoint and so we have $A\setminus C = A$. Hence, $(A\setminus B)\setminus C = A$. So we have $A\setminus(B\setminus C) = (A\setminus B)\setminus C$ as desired. Thus, for sets which are mutually disjoint, the set difference is associative.

Example 1.5

Let $A = \{0,1,2,3,4\}$, $B = \{1,-2,3,4,5,6\}$ and $C = \{2,4,6,7\}$.

(i) Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (ii) Verify using Venn diagram.

Solution

(i) First, we find $A \cup (B \cap C)$.

Consider $B \cap C = \{1,-2,3,4,5,6\} \cap \{2,4,6,7\} = \{4,6\}$;

$$A \cup (B \cap C) = \{0,1,2,3,4\} \cup \{4,6\} = \{0,1,2,3,4,6\}. \qquad (1)$$

Next, consider
$$A \cup B = \{0,1,2,3,4\} \cup \{1,-2,3,4,5,6\}$$
$$= \{-2,0,1,2,3,4,5,6\},$$
$$A \cup C = \{0,1,2,3,4\} \cup \{2,4,6,7\} = \{0,1,2,3,4,6,7\}.$$

Thus, $(A \cup B) \cap (A \cup C) = \{-2,0,1,2,3,4,5,6\} \cap \{0,1,2,3,4,6,7\}$
$$= \{0,1,2,3,4,6\}. \qquad (2)$$

From (1) and (2), we get $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) Using Venn diagram, we have

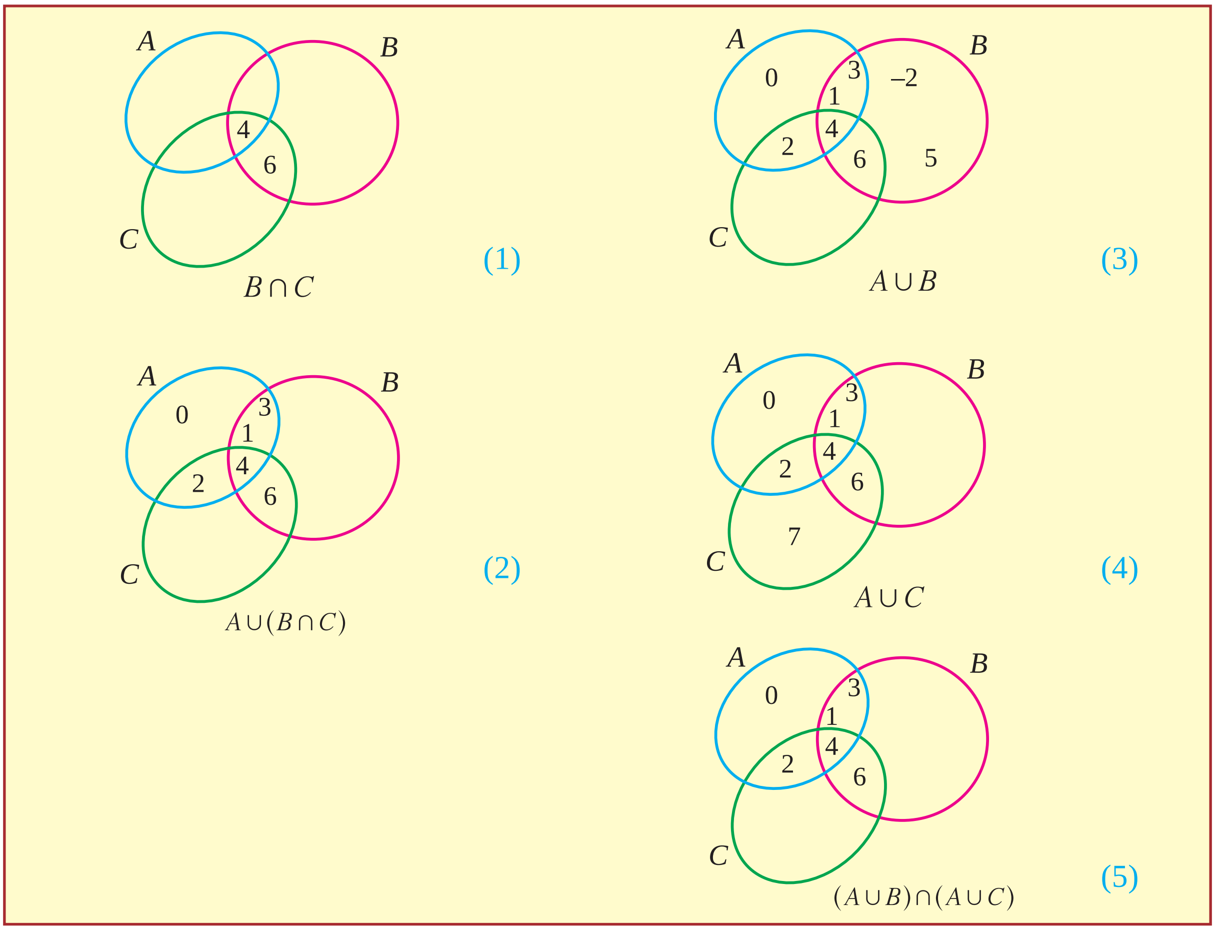


Fig. 1.12

From (2) and (5) it is verified that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example 1.6

For $A = \{x \mid -3 \le x < 4, x \in \mathbb{R}\}$, $B = \{x \mid x < 5, x \in \mathbb{N}\}$ and

$C = \{-5, -3, -1, 0, 1, 3\}$, Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution First note that the set A contains all the real numbers (not just integers) that are greater than or equal to -3 and less than 4.

On the other hand the set B contains all the positive integers that are less than 5. So,

$A = \{x \mid -3 \le x < 4, x \in \mathbb{R}\}$; that is, A consists of all real numbers from – 3 upto 4 but 4 is not included.

Also, $B = \{x \mid x < 5, x \in \mathbb{N}\} = \{1, 2, 3, 4\}$. Now, we find

$$B \cup C = \{1, 2, 3, 4\} \cup \{-5, -3, -1, 0, 1, 3\}$$

$$= \{1, 2, 3, 4, -5, -3, -1, 0\}; \text{ thus}$$

$$A \cap (B \cup C) = A \cap \{1, 2, 3, 4, -5, -3, -1, 0\}$$

$$= \{-3, -1, 0, 1, 2, 3\}. \qquad (1)$$

Next, to find $(A \cap B) \cup (A \cap C)$, we consider

$$A \cap B = \{x \mid -3 \le x < 4, x \in \mathbb{R}\} \cap \{1, 2, 3, 4\} = \{1, 2, 3\};$$

and

$$A \cap C = \{x \mid -3 \le x < 4, x \in \mathbb{R}\} \cap \{-5, -3, -1, 0, 1, 3\}$$

$$= \{-3, -1, 0, 1, 3\}.$$

Hence,

$$(A \cap B) \cup (A \cap C) = \{1, 2, 3,\} \cup \{-3, -1, 0, 1, 3\}$$

$$= \{-3, -1, 0, 1, 2, 3\}. \qquad (2)$$

Now (1) and (2) imply $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.1

1. If $A \subset B$, then show that $A \cup B = B$ (use Venn diagram).
2. If $A \subset B$, then find $A \cap B$ and $A \setminus B$ (use Venn diagram).
3. Let $P = \{a, b, c\}$, $Q = \{g, h, x, y\}$ and $R = \{a, e, f, s\}$. Find the following:

   (i) $P \setminus R$ (ii) $Q \cap R$ (iii) $R \setminus (P \cap Q)$.
4. If $A = \{4, 6, 7, 8, 9\}$, $B = \{2, 4, 6\}$ and $C = \{1, 2, 3, 4, 5, 6\}$, then find

   (i) $A \cup (B \cap C)$ (ii) $A \cap (B \cup C)$ (iii) $A \setminus (C \setminus B)$
5. Given $A = \{a, x, y, r, s\}$, $B = \{1, 3, 5, 7, -10\}$, verify the commutative property of set union.

6. Verify the commutative property of set intersection for $A = \{l, m, n, o, 2, 3, 4, 7\}$ and $B = \{2, 5, 3, -2, m, n, o, p\}$.

7. For $A = \{x \mid x$ is a prime factor of $42\}$, $B = \{x \mid 5 < x \leq 12, x \in \mathbb{N}\}$ and $C = \{1, 4, 5, 6\}$, verify $A \cup (B \cup C) = (A \cup B) \cup C$.

8. Given $P = \{a, b, c, d, e\}, Q = \{a, e, i, o, u\}$ and $R = \{a, c, e, g\}$. Verify the associative property of set intersection.

9. For $A = \{5, 10, 15, 20\}$; $B = \{6, 10, 12, 18, 24\}$ and $C = \{7, 10, 12, 14, 21, 28\}$, verify whether $A \setminus (B \setminus C) = (A \setminus B) \setminus C$. Justify your answer.

10. Let $A = \{-5, -3, -2, -1\}, B = \{-2, -1, 0\}$, and $C = \{-6, -4, -2\}$. Find $A \setminus (B \setminus C)$ and $(A \setminus B) \setminus C$. What can we conclude about set difference operation?

11. For $A = \{-3, -1, 0, 4, 6, 8, 10\}, B = \{-1, -2, 3, 4, 5, 6\}$ and $C = \{-1, 2, 3, 4, 5, 7\}$, show that (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (iii) Verify (i) using Venn diagram (iv) Verify (ii) using Venn diagram.

1.5 De Morgan's laws

De Morgan's father (a British national) was in the service of East India Company, India. Augustus De Morgan (1806-1871) was born in Madurai, Tamilnadu, India. His family moved to England when he was seven months old. He had his education at Trinity college, Cambridge, England. De Morgan's laws relate the three basic set operations Union, Intersection and Complementation.

De Morgan's laws for set difference

For any three sets A, B and C, we have

(i) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ (ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

De Morgan's laws for complementation

Let U be the universal set containing sets A and B. Then

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

Observe that proof of the laws for complementation follows from that of the set difference because for any set D, we have $D' = U \setminus D$. Again we shall not attempt to prove these; but we shall learn how to apply these laws in problem solving.

Example 1.7

Use Venn diagrams to verify $(A \cap B)' = A' \cup B'$.

Solution

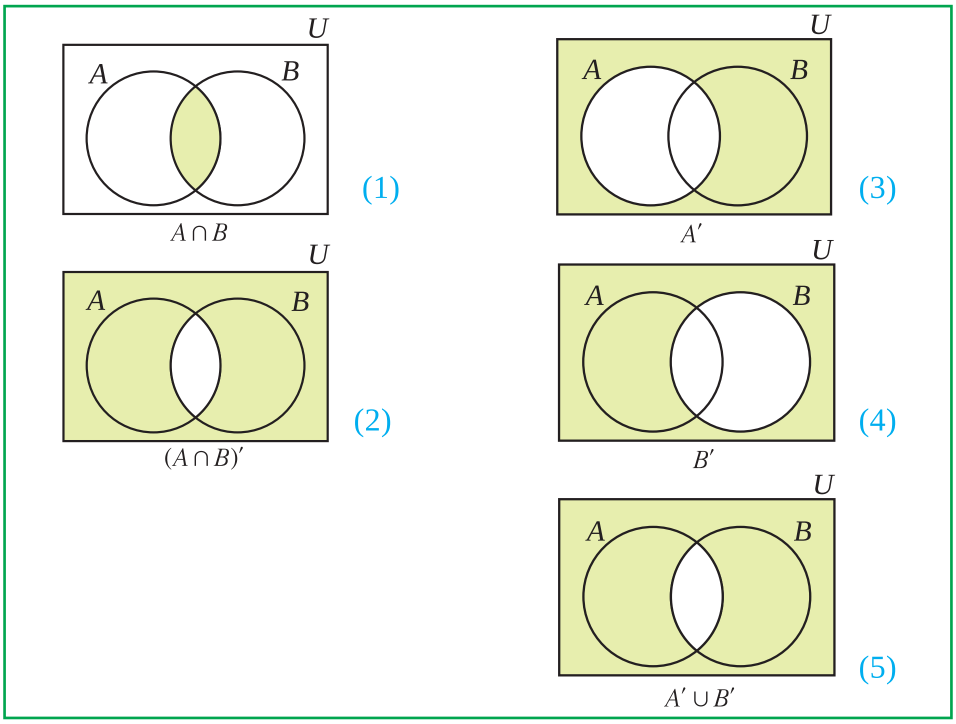


Fig. 1.13

From (2) and (5) it follows that $(A \cap B)' = A' \cup B'$.

Example 1.8

Use Venn diagrams to verify De Morgan's law for set difference

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Solution

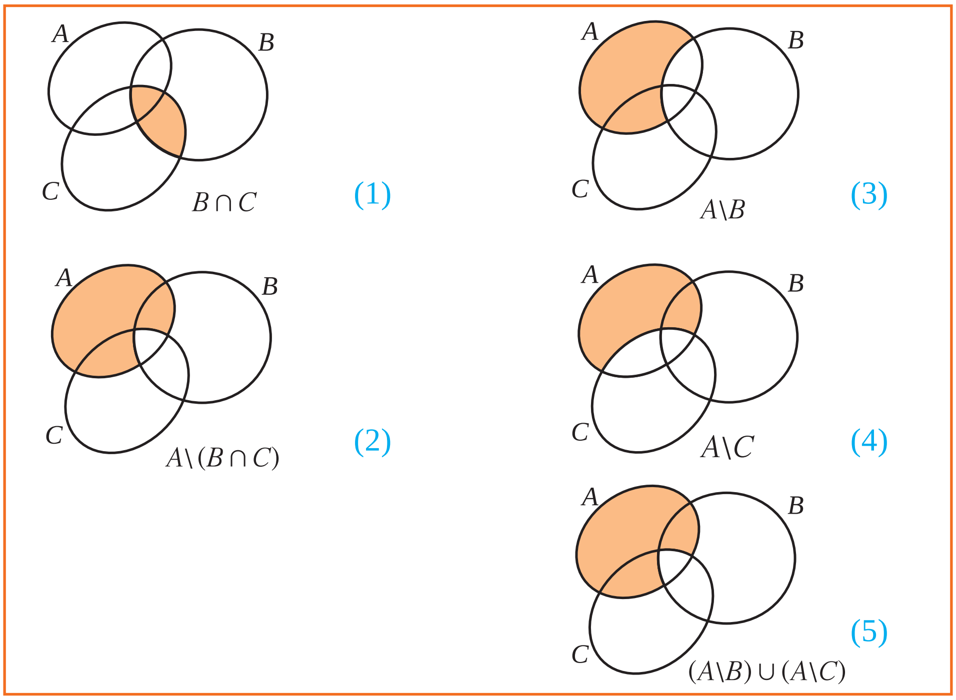


Fig. 1.14

From (2) and (5) we have $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Example 1.9

Let $U = \{-2, -1, 0, 1, 2, 3, \cdots, 10\}$, $A = \{-2, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 8, 9\}$.

Verify De Morgan's laws of complementation.

Solution First we shall verify $(A \cup B)' = A' \cap B'$. To do this we consider

$$A \cup B = \{-2, 2, 3, 4, 5\} \cup \{1, 3, 5, 8, 9\} = \{-2, 1, 2, 3, 4, 5, 8, 9\};$$

which implies

$$(A \cup B)' = U \setminus \{-2, 1, 2, 3, 4, 5, 8, 9\} = \{-1, 0, 6, 7, 10\}. \quad (1)$$

Next, we find
$$A' = U \setminus A = \{-1, 0, 1, 6, 7, 8, 9, 10\}$$
$$B' = U \setminus B = \{-2, -1, 0, 2, 4, 6, 7, 10\}.$$

Thus, we have
$$A' \cap B' = \{-1, 0, 1, 6, 7, 8, 9, 10\} \cap \{-2, -1, 0, 2, 4, 6, 7, 10\}$$
$$= \{-1, 0, 6, 7, 10\}. \quad (2)$$

From (1) and (2) it follows that $(A \cup B)' = A' \cap B'$.

Similarly, one can verify $(A \cap B)' = A' \cup B'$ for the given sets above. We leave the details as an exercise.

Example 1.10

Let $A = \{a, b, c, d, e, f, g, x, y, z\}$, $B = \{1, 2, c, d, e\}$ and $C = \{d, e, f, g, 2, y\}$.

Verify $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Solution First, we find
$$B \cup C = \{1, 2, c, d, e\} \cup \{d, e, f, g, 2, y\}$$
$$= \{1, 2, c, d, e, f, g, y\}.$$

Then
$$A \setminus (B \cup C) = \{a, b, c, d, e, f, g, x, y, z\} \setminus \{1, 2, c, d, e, f, g, y\}$$
$$= \{a, b, x, z\}. \quad (1)$$

Next, we have $A \setminus B = \{a, b, f, g, x, y, z\}$ and $A \setminus C = \{a, b, c, x, z\}$

and so
$$(A \setminus B) \cap (A \setminus C) = \{a, b, x, z\}. \quad (2)$$

Hence from (1) and (2) it follows that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

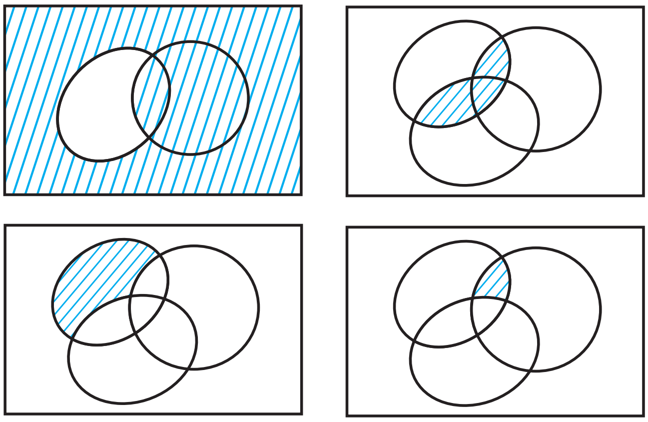
Exercise 1.2

1. Represent the following using Venn diagrams

   (i) $U = \{5, 6, 7, 8, ...13\}$, $A = \{5, 8, 10, 11\}$, and $B = \{5, 6, 7, 9, 10\}$

   (ii) $U = \{a, b, c, d, e, f, g, h\}$, $M = \{b, d, f, g\}$, and $N = \{a, b, d, e, g\}$

2. Write a description of each shaded area. Use symbols U, $A, B, C, \cup, \cap,\ '$ and $\setminus$ as necessary.



3. Draw Venn diagram of three sets A, B and C illustrating the following:

(i) $A \cap B \cap C$ (ii) A and B are disjoint but both are subsets of C

(iii) $A \cap (B \setminus C)$ (iv) $(B \cup C) \setminus A$ (v) $A \cup (B \cap C)$

(vi) $C \cap (B \setminus A)$ (vii) $C \cap (B \cup A)$

4. Use Venn diagram to verify $(A \cap B) \cup (A \setminus B) = A$.

5. Let $U = \{4, 8, 12, 16, 20, 24, 28\}$, $A = \{8, 16, 24\}$ and $B = \{4, 16, 20, 28\}$. Find $(A \cup B)'$ and $(A \cap B)'$.

6. Given that $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, f, g\}$, and $B = \{a, b, c\}$, verify De Morgan's laws of complementation.

7. Verify De Morgan's laws for set difference using the sets given below:

$$A = \{1, 3, 5, 7, 9, 11, 13, 15\},\ B = \{1, 2, 5, 7\} \text{ and } C = \{3, 9, 10, 12, 13\}.$$

8. Let $A = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$, $B = \{1, 5, 10, 15, 20, 30\}$ and $C = \{7, 8, 15, 20, 35, 45, 48\}$. Verify $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

9. Using Venn diagram, verify whether the following are true:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iii) $(A \cup B)' = A' \cap B'$ (iv) $(A \cap B)' = A' \cup B'$

(v) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ (vi) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

1.6 Cardinality of sets

In class IX, we have learnt to solve problems involving two sets, using the formula $n(A\cup B) = n(A) + n(B) - n(A\cap B)$. This formula helps us in calculating the cardinality of the set $A\cup B$ when the cardinalities of A, B and $A\cap B$ are known. Suppose we have three sets A, B and C and we want to find the cardinality of $A\cup B\cup C$, what will be the corresponding formula? The formula in this case is given by

$$n(A\cup B\cup C)= n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C).$$

Following example illustrates the usage of the above formula.

Example 1.11

In a group of students, 65 play foot ball, 45 play hockey, 42 play cricket, 20 play foot ball and hockey, 25 play foot ball and cricket, 15 play hockey and cricket and 8 play all the three games. Find the number of students in the group.

Solution Let F, H and C represent the set of students who play foot ball, hockey and cricket respectively. Then $n(F) = 65$, $n(H) = 45$, and $n(C) = 42$.

Also, $n(F\cap H) = 20$, $n(F\cap C) = 25$, $n(H\cap C) = 15$ and $n(F\cap H\cap C) = 8$.

We want to find the number of students in the whole group; that is $n(F\cup H\cup C)$.

By the formula, we have

$$\begin{aligned} n(F\cup H\cup C) &= n(F)+n(H)+n(C)-n(F\cap H) \\ &\quad -n(H\cap C)-n(F\cap C)+n(F\cap H\cap C) \\ &= 65+45+42-20-25-15+8 = 100. \end{aligned}$$

Hence, the number of students in the group = 100.

Alternate method

The same problem can also be solved using Venn diagram. Nowadays, it is possible to solve some of the problems that we come across in daily life using Venn diagrams and logic. The Venn diagram will have three intersecting sets, each representing a game. Look at the diagram and try to find the number of players in the group by working carefully through the statements and fill in as you go along.

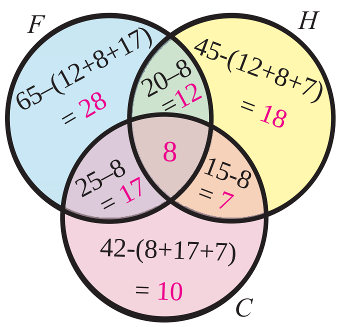


Fig. 1.15

Number of students in the group

$= 28 + 12 + 18 + 7 + 10 + 17 + 8 = 100.$

Example 1.12

In a survey of university students, 64 had taken mathematics course, 94 had taken computer science course, 58 had taken physics course, 28 had taken mathematics and physics, 26 had taken mathematics and computer science, 22 had taken computer science and physics course, and 14 had taken all the three courses. Find the number of students who were surveyed. Find how many had taken one course only.

Solution Let us represent the given data in a Venn diagram.

Let M, C, P represent sets of students who had taken mathematics, computer science and physics respectively. The given details are filled in the Venn diagram

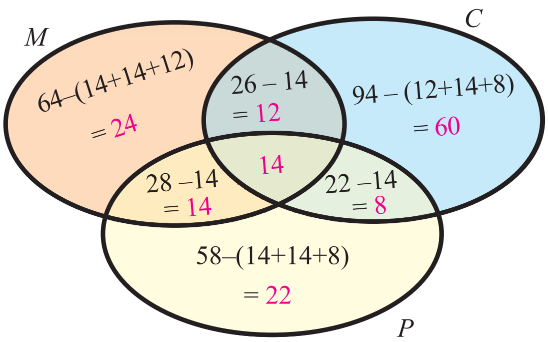


Fig. 1.16

$n(M \cap C \cap P') = 26 - 14 = 12$

$n(M \cap P \cap C') = 28 - 14 = 14$

$n(C \cap P \cap M') = 22 - 14 = 8$

Number of students surveyed

$= 24 + 12 + 60 + 8 + 22 + 14 + 14 = 154$

The number of students who had taken only mathematics = 64–(14+14+12) = 24

The number of students who had taken only computer science = 94 – (12+14+8) = 60

The number of students who had taken only physics = 58 – (14+14+8) = 22

Example 1.13

A radio station surveyed 190 students to determine the types of music they liked. The survey revealed that 114 liked rock music, 50 liked folk music, and 41 liked classical music, 14 liked rock music and folk music, 15 liked rock music and classical music, 11 liked classical music and folk music. 5 liked all the three types of music.

Find (i) how many did not like any of the 3 types?

(ii) how many liked any two types only?

(iii) how many liked folk music but not rock music?

Solution Let R, F and C represent the sets of students who liked rock music, folk music and classical music respectively. Let us fill in the given details in the Venn diagram. Thus, we have

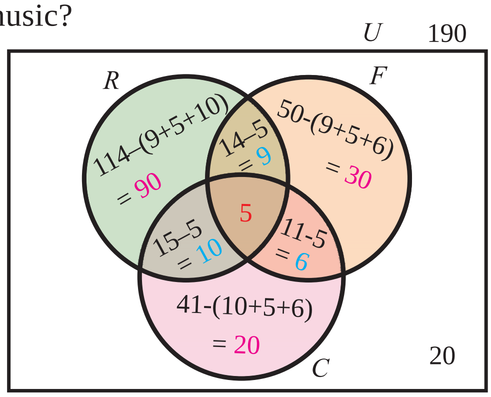


Fig. 1.17

$n(R \cap F \cap C') = 14 - 5 = 9$

$n(R \cap C \cap F') = 15 - 5 = 10$

$n(F \cap C \cap R') = 11 - 5 = 6.$

From the Venn diagram, the number of students who liked any one of the three types of music equals 90 + 9 + 30 + 6 + 20 + 10 + 5 = 170.

Number of students surveyed = 190.

Number of students who did not like any of the three types = $190 - 170 = 20$.

Number of students who liked any two types only = 9 + 6 + 10 = 25.

Number of students who liked folk music but not rock music = 30 + 6 = 36.

Exercise 1.3

1. If A and B are two sets and U is the universal set such that $n(U) = 700$, $n(A) = 200, n(B) = 300$ and $n(A \cap B) = 100$, find $n(A' \cap B')$.

2. Given $n(A) = 285, n(B) = 195, n(U) = 500, n(A \cup B) = 410$, find $n(A' \cup B')$.

3. For any three sets A, B and C if $n(A) = 17$ $n(B) = 17$, $n(C) = 17$, $n(A \cap B) = 7$ $n(B \cap C) = 6$, $n(A \cap C) = 5$ and $n(A \cap B \cap C) = 2$, find $n(A \cup B \cup C)$.

4. Verify $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) -$
$n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$ for the sets given below:
   (i) $A = \{4,5,6\}, B = \{5,6,7,8\}$ and $C = \{6,7,8,9\}$
   (ii) $A = \{a,b,c,d,e\}, B = \{x,y,z\}$ and $C = \{a,e,x\}$.

5. In a college, 60 students enrolled in chemistry, 40 in physics, 30 in biology, 15 in chemistry and physics, 10 in physics and biology, 5 in biology and chemistry. No one enrolled in all the three. Find how many are enrolled in at least one of the subjects.

6. In a town 85% of the people speak English, 40% speak Tamil and 20% speak Hindi. Also, 42% speak English and Tamil, 23% speak Tamil and Hindi and 10% speak English and Hindi, find the percentage of people who can speak all the three languages.

7. An advertising agency finds that, of its 170 clients, 115 use Television, 110 use Radio and 130 use Magazines. Also, 85 use Television and Magazines, 75 use Television and Radio, 95 use Radio and Magazines, 70 use all the three. Draw Venn diagram to represent these data. Find
   (i) how many use only Radio? (ii) how many use only Television?
   (iii) how many use Television and magazine but not radio?

8. In a school of 4000 students, 2000 know French, 3000 know Tamil and 500 know Hindi, 1500 know French and Tamil, 300 know French and Hindi, 200 know Tamil and Hindi and 50 know all the three languages.
   (i) How many do not know any of the three languages?
   (ii) How many know at least one language?
   (iii) How many know only two languages?

9. In a village of 120 families, 93 families use firewood for cooking, 63 families use kerosene, 45 families use cooking gas, 45 families use firewood and kerosene, 24 families use kerosene and cooking gas, 27 families use cooking gas and firewood. Find how many use firewood, kerosene and cooking gas.

1.7 Relations

In the previous section, we have seen the concept of Set. We have also seen how to create new sets from the given sets by taking union, intersection and complementation. Here we shall see yet another way of creating a new set from the given two sets *A* and *B*. This new set is important in defining other important concepts of mathematics "relation, function".

Given two non empty sets *A* and *B*, we can form a new set $A \times B$, read as '*A* cross *B*', called the cartesian product of A with B. It is defined as

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Similarly, the set B cross A is defined as

$$B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}.$$

Note

(i) The order in the pair (a, b) is important. That is, $(a, b) \neq (b, a)$ if $a \neq b$.

(ii) It is possible that the sets A and B are equal in the cartesian product $A \times B$.

Let us look at an example.

Suppose that a cell phone store sells three different types of cell phones and we call them C_1, C_2, C_3. Let us also suppose that the price of C_1 is ₹ 1200, price of C_2 is ₹ 2500 and price of C_3 is ₹ 2500.

We take $A = \{ C_1, C_2, C_3 \}$ and $B = \{ 1200, 2500 \}$.

In this case, $A \times B = \{(C_1, 1200), (C_1, 2500), (C_2, 1200), (C_2, 2500), (C_3, 1200), (C_3, 2500)\}$

but $B \times A = \{(1200, C_1), (2500, C_1), (1200, C_2), (2500, C_2,),(1200, C_3), (2500, C_3)$.

It is easy to see that $A \times B \neq B \times A$ if $A \neq B$.

Let us consider a subset $F = \{(C_1, 1200), (C_2, 2500), (C_3, 2500)\}$ of $A \times B$.

Every first component in the above ordered pairs is associated with a unique element. That is no element in the first place is paired with more than one element in the second place.

For every element in F, basically the second component indicates the price of the first component. Next, consider a subset $E = \{(1200, C_1), (2500, C_2), (2500, C_3)\}$ of $B \times A$

Here, the first component 2500 is associated with two different elements C_2 and C_3.

Definition

Let A and B be any two non empty sets. A relation R from A to B is a non-empty subset of $A \times B$. That is, $R \subseteq A \times B$.

Domain of $R = \{x \in A \mid (x,y) \in R$ for some $y \in B\}$

Range of $R = \{y \in B \mid (x,y) \in R$ for some $x \in A\}$.

1.8 Functions

Peter Dirichlet

(1805-1859)
Germany

Dirichlet made major contributions in the fields of number theory, analysis and mechanics.

In 1837 he introduced the modern concept of a function with notation $y = f(x)$. He also formulated the well known Pigeonhole principle.

Let A and B be any two non empty sets. A function from A to B is a relation

$f \subseteq A \times B$ such that the following hold:

(i) Domain of f is A.

(ii) For each $x \in A$, there is only one $y \in B$ such that $(x, y) \in f$.

Note that a function from A to B is a special kind of relation that satisfies (i) and (ii). A function is also called as a mapping or a transformation.

A function from A to B is denoted by $f: A \to B$, and if $(x, y) \in f$, then we write $y = f(x)$.

We can reformulate the definition of a function without using the idea of relation as follows: In fact, most of the time this formulation is used as a working definition of a function,

Definition

Let A and B be any two non empty sets. A function f from A to B is a rule of correspondence that assigns each element $x \in A$ to a unique element $y \in B$. We denote $y = f(x)$ to mean y is a function of x.

The set A is called the **domain** of the function and set B is called the **co-domain** of the function. Also, y is called **the image** of x under f and x is called **a preimage** of y. The set of all images of elements of A under f is called the **range** of f. Note that the range of a function is a subset of its co-domain.

This modern definition of a function, given above, was given by Nikolai Labachevsky and **Peter Dirichlet** independently around 1837. Prior to this, there was no clear definition of a function.

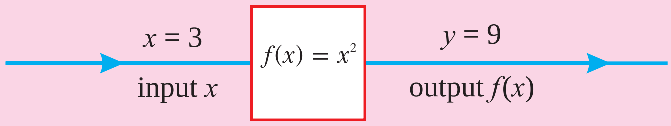
In the example we considered in section 1.7, prior to the above definitions, the set

$F = \{(C_1, 1200), (C_2, 2500), (C_3, 2500)\}$ represents a function; because $F \subseteq A \times B$ is a relation satisfying conditions (i) and (ii) given above.

But $E = \{(1200, C_1), (2500, C_2), (2500, C_3)\}$ does not represent a function, because condition (ii) given above is not satisfied as $(2500, C_2), (2500, C_3) \in E$.

Remarks

(i) A function f may be thought of as a machine which yields a unique output y for every input value of x.



(ii) In defining a function we need a domain, co-domain and a rule that assigns each element of the domain to a unique element in the co-domain.

Example 1.14

Let $A = \{1, 2, 3, 4\}$ and $B = \{-1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12\}$.

Let $R = \{(1, 3), (2, 6), (3, 10), (4, 9)\} \subseteq A \times B$ be a relation. Show that R is a function and find its domain, co-domain and the range of R.

Solution The domain of $R = \{1, 2, 3, 4\} = A$.

Also, for each $x \in A$ there is only one $y \in B$ such that $y = R(x)$.

So, given R is a function. The co-domain is obviously B. Since

$R(1) = 3, R(2) = 6, R(3) = 10$ and $R(4) = 9$, the range of R is given by $\{3, 6, 10, 9\}$.

Example 1.15

Does each of the following arrow diagrams represent a function? Explain.

(i)

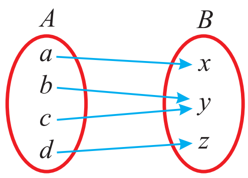


Fig. 1.18

(ii)

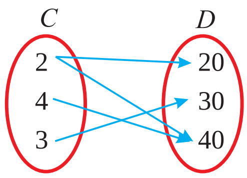


Fig. 1.19

Solution In arrow diagram (i), every element in A has a unique image. Hence it is a function. In arrow diagram (ii), the element 2 in C has two images namely 20 and 40. Hence, it is not a function.

Example 1.16

Let $X = \{1, 2, 3, 4\}$. Examine whether each of the relations given below is a function from X to X or not. Explain.

(i) $f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}$

(ii) $g = \{(3, 1), (4, 2), (2, 1)\}$ (iii) $h = \{(2, 1), (3, 4), (1, 4), (4, 3)\}$

Solution

(i) Now, f = { (2, 3), (1, 4), (2, 1), (3, 2), (4, 4) }

f is not a function because 2 is associated with two different elements 3 and 1.

(ii) The relation g = { (3, 1), (4, 2), (2, 1)} is not a function because the element 1 does not have a image. That is, domain of $g = \{2, 3, 4\} \neq X$.

(iii) Next, we consider h = { (2, 1), (3, 4), (1, 4), (4, 3) }.

Each element in X is associated with a unique element in X.

Thus, h is a function.

Example 1.17

Which of the following relations are functions from A = { 1, 4, 9, 16 } to B = { –1, 2, –3, –4, 5, 6 }? In case of a function, write down its range.

(i) f_1 = { (1, –1), (4, 2), (9, –3), (16, –4) }

(ii) f_2 = { (1, –4), (1, –1), (9, –3), (16, 2) }

(iii) f_3 = { (4, 2), (1, 2), (9, 2), (16, 2) }

(iv) f_4 = { (1, 2), (4, 5), (9, –4), (16, 5) }

Solution (i) We have f_1 = { (1, –1), (4, 2), (9, – 3), (16,– 4) }.

Each element in A is associated with a unique element in B.

Thus, f_1 is a function.

Range of f_1 is $\{-1, 2, -3, -4\}$.

(ii) Here, we have f_2 = { (1, – 4), (1, –1), (9, – 3), (16, 2) }.

f_2 is not a function because 1 is associated with two different image elements -4 and -1. Also, note that f_2 is not a function since 4 has no image.

(iii) Consider f_3 = { (4, 2), (1, 2), (9, 2), (16, 2) }.

Each element in A is associated with a unique element in B.

Thus, f_3 is a function.

Range of f_3 = { 2 }.

(iv) We have f_4 = { (1, 2), (4, 5), (9, – 4), (16, 5) }.

Each element in A is associated with a unique element in B.

Hence, f_4 is a function.

Range of f_4 = { 2, 5, – 4}.

Example 1.18

Let $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, where $x \in \mathbb{R}$. Does the relation $\{ (x,y) \mid y = |x|, x \in \mathbb{R} \}$ define a function? Find its range.

Solution For every value of x, there exists a unique value $y = |x|$. Therefore, the given relation defines a function.

The domain of the function is the set $\mathbb{R}$ of all real numbers.

Since $|x|$ is always either zero or positive for every real number x, and every positive real number can be obtained as an image under this function, the range will be the set of non-negative real numbers (either positive or zero).

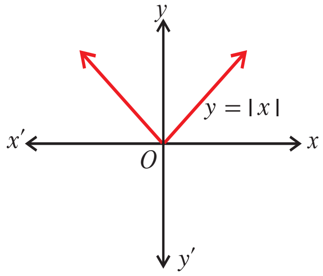


Fig. 1.20

Remarks

The function $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, where $x \in \mathbb{R}$, is known as **modulus** or **absolute value** function.

Thus, for example, $|-8| = -(-8) = 8$ and also $|8| = 8$.

1.8.1 Representation of functions

A function may be represented by

(i) a set of ordered pairs, (ii) a table, (iii) an arrow diagram, (iv) a graph

Let $f : A \to B$ be a function.

(i) The set $f = \{(x, y) : y = f(x), x \in A\}$ of all ordered pairs represents the function.

(ii) The values of x and the values of their respective images under f can be given in the form of a table.

(iii) An arrow diagram indicates the elements of the domain of f and their respective images by means of arrows.

(iv) The ordered pairs in the collection $f = \{(x, y) : y = f(x), x \in A\}$ are plotted as points in the x-y plane. The graph of f is the totality of all such points.

Let us illustrate the representation of functions in different forms through some examples.

For many functions we can obtain its graph. But not every graph will represent a function. Following test helps us in determining if the given graph is a function or not.

1.8.2 Vertical line test

A graph represents a function only if every vertical line intersects the graph in at most one point.

Note

It is possible that some vertical lines may not intersect the graph, which is alright. If there is even one vertical line that meets the graph in more than one point, then that graph cannot represent a function, because in this case, we shall have at least two y-values for the same x-value. For example, the graph of $y^2 = x$ is not a function.

Example 1.19

Use the vertical line test to determine which of the following graphs represent a function.

(i)

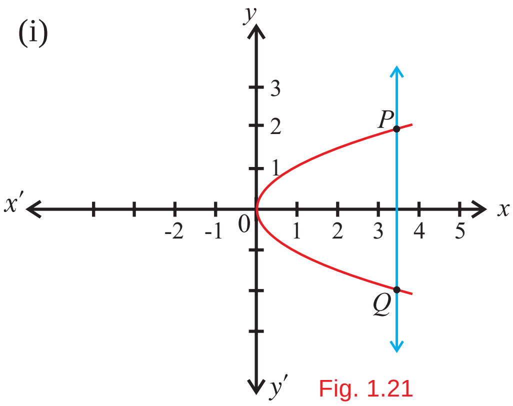


Fig. 1.21

(ii)

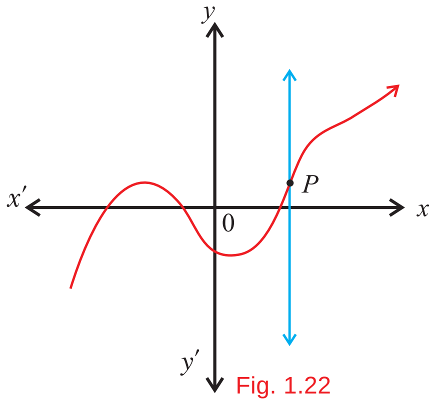


Fig. 1.22

(iii)

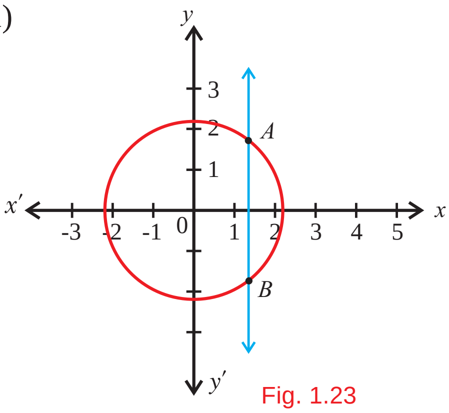


Fig. 1.23

(iv)

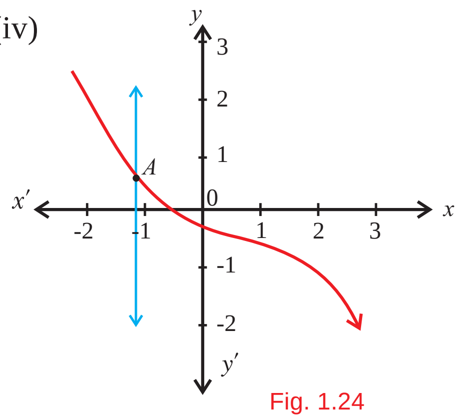


Fig. 1.24

Solution

(i) The given graph does not represent a function as a vertical line cuts the graph at two points P and Q.

(ii) The given graph represents a function as any vertical line will intersect the graph at most one point P.

(iii) The given graph does not represent a function as a vertical line cuts the graph at two points A and B.

(iv) The given graph represents a function as the graph satisfies the vertical line test.

Example 1.20

Let $A = \{ 0, 1, 2, 3 \}$ and $B = \{ 1, 3, 5, 7, 9 \}$ be two sets. Let $f : A \to B$ be a function given by $f(x) = 2x + 1$. Represent this function as (i) a set of ordered pairs (ii) a table (iii) an arrow diagram and (iv) a graph.

Solution $A = \{ 0, 1, 2, 3 \}$, $B = \{ 1, 3, 5, 7, 9 \}$, $f(x) = 2x + 1$

$f(0) = 2(0) + 1 = 1$, $f(1) = 2(1)+1 = 3$, $f(2) = 2(2) + 1 = 5$, $f(3) = 2(3) + 1 = 7$

(i) Arrow Diagram

Let us represent f by an arrow diagram.

We draw two closed curves to represent the sets A and B.

Then each element of A and its unique image element in B are related with an arrow.

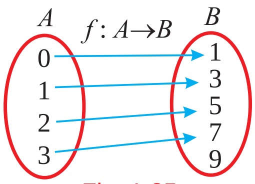


Fig. 1.25

(ii) Table form

Let us represent f using a table as shown below.

x	0	1	2	3
$f(x)$	1	3	5	7

(iii) Set of ordered pairs

The given function f can be represented as a set of ordered pairs as

$$f = \{ (0, 1), (1, 3), (2, 5), (3, 7) \}$$

(iv) Graph

We are given that

$f = \{(x, f(x)) \mid x \in A\} = \{(0,1), (1,3), (2,5), (3,7)\}$.

Now, the points (0, 1), (1, 3), (2, 5) and (3, 7) are plotted on the plane as shown below.

The totality of all points represent the graph of the function.

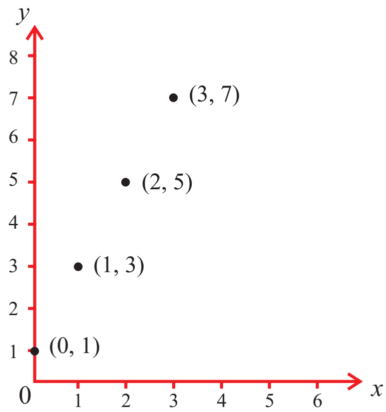


Fig. 1.26

1.8.3 Types of functions

Based on some properties of a function, we divide functions into certain types.

(i) One-One function

Let $f: A \to B$ be a function. The function f is called an **one-one** function if it takes different elements of A into different elements of B. That is, we say f is one-one if $u \neq v$ in A always imply $f(u) \neq f(v)$. In other words f is one-one if no element in B is associated with more than one element in A.

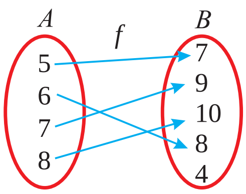


Fig. 1.27

A one-one function is also called an injective function. The above figure represents a one-one function.

(ii) Onto function

A function $f: A \to B$ is said to be an onto function if every element in B has a pre-image in A. That is, a function f is onto if for each $b \in B$, there is atleast one element $a \in A$, such that $f(a) = b$. This is same as saying that B is the range of f. An onto function is also called a surjective function. In the above figure, f is an onto function.

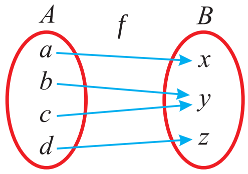


Fig. 1.28

(iii) One-One and onto function

A function $f: A \to B$ is called a one-one and onto or a bijective function if f is both a one-one and an onto function. Thus $f: A \to B$ is one-one and onto if f maps distinct elements of A into distinct images in B and every element in B is an image of some element in A.

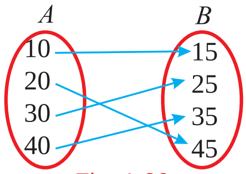


Fig. 1.29

Note

(i) A function $f: A \to B$ is onto if and only if $B =$ range of f.

(ii) $f: A \to B$ is one-one and onto, if and only if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ in A and every element in B has exactly one pre-image in A.

(iii) If $f: A \to B$ is a bijective function and if A and B are finite sets, then the cardinalities of A and B are same. In Fig.1.29, the function f is one - one and onto.

(iv) If $f: A \to B$ is a bijective function, then A and B are equivalent sets

(v) A one-one and onto function is also called a one-one correspondence.

(iv) Constant function

A function $f: A \to B$ is said to be a constant function if every element of A has the same image in B.

Range of a constant function is a singleton set.

Let $A = \{x, y, u, v, 1\}$, $B = \{3, 5, 7, 8, 10, 15\}$ $f: A \to B$ is defined by $f(x) = 5$ for every $x \in A$. The given figure represents constant function.

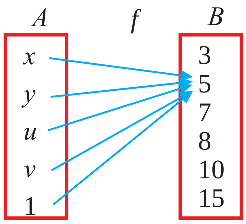


Fig. 1.30

(v) Identity function

Let A be a non-empty set. A function $f: A \to A$ is called an identity function of A if $f(a) = a$ for all $a \in A$. That is, an identity function maps each element of A into itself.

For example, let $A = \mathbb{R}$. The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$ is the identity function on $\mathbb{R}$. Fig.1.31 represents the graph of the identity function on $\mathbb{R}$.

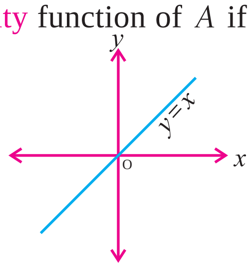


Fig. 1.31

Example 1.21

Let $A = \{1, 2, 3, 4, 5\}$, $B = \mathbb{N}$ and $f : A \to B$ be defined by $f(x) = x^2$. Find the range of f. Identify the type of function.

Solution Now, $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 2, 3, 4, \cdots\}$

Given $f : A \to B$ and $f(x) = x^2$

$\therefore$ $f(1) = 1^2 = 1$; $f(2) = 4$; $f(3) = 9$; $f(4) = 16$; $f(5) = 25$.

Range of $f = \{1, 4, 9, 16, 25\}$

Since distinct elements are mapped into distinct images, it is a one-one function. However, the function is not onto, since $3 \in B$ but there is no $x \in A$ such that

$f(x) = x^2 = 3$.

Remarks

However, a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x^2$ is not one-one because, if $u = 1$ and $v = -1$ then $u \neq v$ but $g(u) = g(1) = 1 = g(-1) = g(v)$. So, just formula alone does not make a function one-one or onto. We need to consider the rule, its domain and co-domain in deciding one-to-one and onto.

Example 1.22

A function $f : [1, 6) \longrightarrow \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 1 + x & 1 \le x < 2 \\ 2x - 1 & 2 \le x < 4 \\ 3x^2 - 10 & 4 \le x < 6 \end{cases} \qquad (\text{ Here, } [1, 6) = \{x \in \mathbb{R} : 1 \le x < 6\})$$

Find the value of (i) $f(5)$ (ii) $f(3)$ (iii) $f(1)$

(iv) $f(2) - f(4)$ (v) $2f(5) - 3f(1)$

Solution

(i) Let us find $f(5)$. Since 5 lies between 4 and 6, we have to use $f(x) = 3x^2 - 10$.

Thus, $f(5) = 3(5^2) - 10 = 65$.

(ii) To find $f(3)$, note that 3 lies between 2 and 4.

So, we use $f(x) = 2x - 1$ to calculate $f(3)$.

Thus, $f(3) = 2(3) - 1 = 5$.

(iii) Let us find $f(1)$.

Now, 1 is in the interval $1 \le x < 2$

Thus, we have to use $f(x) = 1 + x$ to obtain $f(1) = 1 + 1 = 2$.

(iv) $f(2) - f(4)$

Now, 2 is in the interval $2 \le x < 4$ and so, we use $f(x) = 2x - 1$.

Thus, $f(2) = 2(2) - 1 = 3$.

Also, 4 is in the interval $4 \le x < 6$. Thus, we use $f(x) = 3x^2 - 10$.

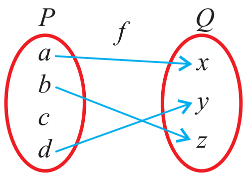
Therefore, $f(4) = 3(4^2) - 1 = 3(16) - 1 = 48 - 1 = 47$.

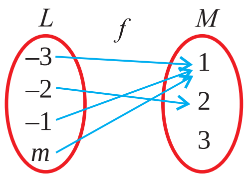
Hence $f(2) - f(4) = 3 - 47 = -44$.

(v) To calculate $2f(5) - 3f(1)$, we shall make use of the values that we have already calculated in (i) and (iii). Thus, $2f(5) - 3f(1) = 2(65) - 3(2) = 130 - 6 = 124$.

Exercise 1.4

1. State whether each of the following arrow diagrams define a function or not. Justify your answer.

(i) 

(ii) 

2. For the given function $F = \{ (1, 3), (2, 5), (4, 7), (5, 9), (3, 1) \}$, write the domain and range.

3. Let $A = \{ 10, 11, 12, 13, 14 \}$; $B = \{ 0, 1, 2, 3, 5 \}$ and $f_i : A \to B$, $i = 1,2,3$. State the type of function for the following (give reason):

$f_1 = \{ (10, 1), (11, 2), (12, 3), (13, 5), (14, 3) \}$

$f_2 = \{ (10, 1), (11, 1), (12, 1), (13, 1), (14, 1) \}$

$f_3 = \{ (10, 0), (11, 1), (12, 2), (13, 3), (14, 5) \}$

4. If $X = \{ 1, 2, 3, 4, 5 \}$, $Y = \{ 1, 3, 5, 7, 9 \}$ determine which of the following relations from A to B are functions? Give reason for your answer. If it is a function, state its type.

(i) $R_1 = \{ (x, y) | y = x + 2, x \in X, y \in Y \}$

(ii) $R_2 = \{ (1, 1), (2, 1), (3, 3), (4, 3), (5, 5) \}$

(iii) $R_3 = \{ (1, 1), (1, 3), (3, 5), (3, 7), (5, 7) \}$

(iv) $R_4 = \{ (1, 3), (2, 5), (4, 7), (5, 9), (3, 1) \}$

5. If $R = \{(a, -2), (-5, b), (8, c), (d, -1)\}$ represents the identity function, find the values of a, b, c and d.

6. $A = \{ -2, -1, 1, 2 \}$ and $f = \left\{\left(x, \frac{1}{x}\right) : x \in A\right\}$. Write down the range of f. Is f a function from A to A ?

7. Let $f = \{ (2, 7), (3, 4), (7, 9), (-1, 6), (0, 2), (5, 3) \}$ be a function from

$A = \{ -1, 0, 2, 3, 5, 7 \}$ to $B = \{ 2, 3, 4, 6, 7, 9 \}$. Is this (i) an one-one function (ii) an onto function (iii) both one-one and onto function?

8. Write the pre-images of 2 and 3 in the function
$f = \{ (12, 2), (13, 3), (15, 3), (14, 2), (17, 17) \}$.

9. The following table represents a function from $A = \{ 5, 6, 8, 10 \}$ to

$B = \{ 19, 15, 9, 11 \}$ where $f(x) = 2x - 1$. For what values of a and b it represents a one-one function ?

x	5	6	8	10
$f(x)$	a	11	b	19

10. Let $A = \{ 5, 6, 7, 8 \}$; $B = \{ -11, 4, 7, -10, -7, -9, -13 \}$ and

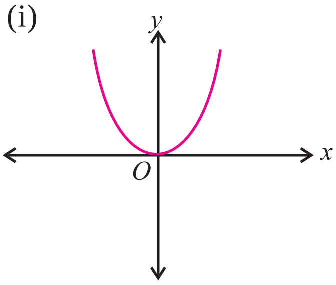
$f = \{(x,y) : y = 3 - 2x,\ x \in A,\ y \in B \}$

(i) Write down the elements of f. (ii) What is the co-domain?

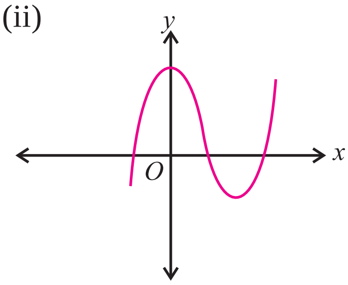
(iii) What is the range ? (iv) Identify the type of function.

11. State whether the following graphs represent a function. Give reason for your answer.

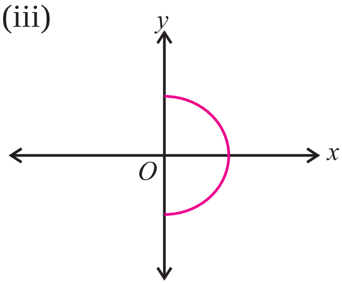
(i)



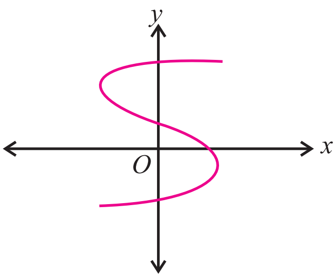
(ii)



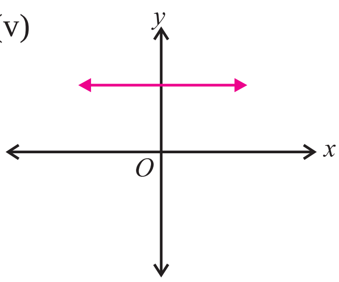
(iii)



(iv)



(v)



12. Represent the function $f = \{ (-1, 2), (-3, 1), (-5, 6), (-4, 3) \}$ as

(i) a table (ii) an arrow diagram

13. Let $A = \{ 6, 9, 15, 18, 21 \}$; $B = \{ 1, 2, 4, 5, 6 \}$ and $f : A \to B$ be defined by $f(x) = \frac{x-3}{3}$. Represent f by

(i) an arrow diagram (ii) a set of ordered pairs

(iii) a table (iv) a graph .

14. Let $A = \{4, 6, 8, 10\}$ and $B = \{3, 4, 5, 6, 7\}$. If $f : A \to B$ is defined by $f(x) = \frac{1}{2}x + 1$ then represent f by (i) an arrow diagram (ii) a set of ordered pairs and (iii) a table.

15. A function $f : [-3,7) \Longrightarrow \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 4x^2 - 1; & -3 \le x < 2 \\ 3x - 2; & 2 \le x \le 4. \\ 2x - 3; & 4 < x \le 6 \end{cases}$$

Find (i) $f(5)+f(6)$ (ii) $f(1)-f(-3)$

(iii) $f(-2)-f(4)$ (iv) $\dfrac{f(3)+f(-1)}{2f(6)-f(1)}$.

16. A function $f : [-7,6) \Longrightarrow \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} x^2 + 2x + 1; & -7 \le x < -5 \\ x + 5; & -5 \le x \le 2 \\ x - 1; & 2 < x < 6 . \end{cases}$$

Find (i) $2f(-4) + 3f(2)$ (ii) $f(-7) - f(-3)$ (iii) $\dfrac{4f(-3) + 2f(4)}{f(-6) - 3f(1)}$.

Exercise 1.5

Choose the correct answer

1. For two sets A and B, $A \cup B = A$ if and only if

(A) $B \subseteq A$ (B) $A \subseteq B$ (C) $A \neq B$ (D) $A \cap B = \phi$

2. If $A \subset B$, then $A \cap B$ is

(A) B (B) $A \backslash B$ (C) A (D) $B \backslash A$

3. For any two sets P and Q, $P \cap Q$ is

(A) $\{x : x \in P \text{ or } x \in Q\}$ (B) $\{x : x \in P \text{ and } x \notin Q\}$

(C) $\{x : x \in P \text{ and } x \in Q\}$ (D) $\{x : x \notin P \text{ and } x \in Q\}$

4. If $A = \{ p, q, r, s \}$, $B = \{ r, s, t, u \}$, then $A \backslash B$ is

(A) $\{ p, q \}$ (B) $\{ t, u \}$ (C) $\{ r, s \}$ (D) $\{p, q, r, s \}$

5. If $n[p(A)] = 64$, then $n(A)$ is

(A) 6 (B) 8 (C) 4 (D) 5

6. For any three sets A, B and C, $A \cap (B \cup C)$ is

(A) $(A \cup B) \cup (B \cap C)$ (B) $(A \cap B) \cup (A \cap C)$

(C) $A \cup (B \cap C)$ (D) $(A \cup B) \cap (B \cup C)$

7. For any two sets A and B, $\{(A \backslash B) \cup (B \backslash A)\} \cap (A \cap B)$ is

(A) ϕ (B) $A \cup B$ (C) $A \cap B$ (D) $A' \cap B'$

8. Which one of the following is not true ?

(A) $A \backslash B = A \cap B'$ (B) $A \backslash B = A \cap B$

(C) $A \backslash B = (A \cup B) \cap B'$ (D) $A \backslash B = (A \cup B) \backslash B$

9. For any three sets A, B and C, $B \backslash (A \cup C)$ is

(A) $(A \backslash B) \cap (A \backslash C)$ (B) $(B \backslash A) \cap (B \backslash C)$

(C) $(B \backslash A) \cap (A \backslash C)$ (D) $(A \backslash B) \cap (B \backslash C)$

10. If $n(A) = 20$, $n(B) = 30$ and $n(A \cup B) = 40$, then $n(A \cap B)$ is equal to

(A) 50 (B) 10 (C) 40 (D) 70.

11. If $\{ (x, 2), (4, y) \}$ represents an identity function, then (x, y) is

(A) (2, 4) (B) (4, 2) (C) (2, 2) (D) (4, 4)

12. If $\{ (7, 11), (5, a) \}$ represents a constant function, then the value of 'a' is

(A) 7 (B) 11 (C) 5 (D) 9

13. Given $f(x) = (-1)^x$ is a function from $\mathbb{N}$ to $\mathbb{Z}$. Then the range of f is

(A) $\{ 1\}$ (B) $\mathbb{N}$ (C) $\{ 1, -1 \}$ (D) $\mathbb{Z}$

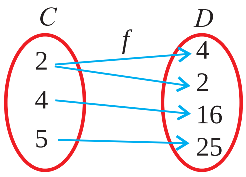
14. If $f = \{ (6, 3), (8, 9), (5, 3), (-1, 6) \}$, then the pre-images of 3 are

(A) 5 and –1 (B) 6 and 8 (C) 8 and –1 (D) 6 and 5.

15. Let $A = \{ 1, 3, 4, 7, 11 \}$, $B = \{-1, 1, 2, 5, 7, 9 \}$ and $f : A \to B$ be given by

$f = \{ (1, -1), (3, 2), (4, 1), (7, 5), (11, 9) \}$. Then f is

(A) one-one (B) onto (C) bijective (D) not a function

16. 

The given diagram represents

(A) an onto function (B) a constant function

(C) an one-one function (D) not a function

17. If $A = \{5, 6, 7\}$, $B = \{1, 2, 3, 4, 5\}$ and $f : A \to B$ is defined by $f(x) = x - 2$, then the range of f is

(A) $\{1, 4, 5\}$ (B) $\{1, 2, 3, 4, 5\}$ (C) $\{2, 3, 4\}$ (D) $\{3, 4, 5\}$

18. If $f(x) = x^2 + 5$, then $f(-4) =$

(A) 26 (B) 21 (C) 20 (D) –20

19. If the range of a function is a singleton set, then it is

(A) a constant function (B) an identity function

(C) a bijective function (D) an one-one function

20. If $f : A \to B$ is a bijective function and if $n(A) = 5$, then $n(B)$ is equal to

(A) 10 (B) 4 (C) 5 (D) 25

Points to Remember

SETS

- A set is a collection of well defined objects.
  - Set union is commutative and associative.
  - Set intersection is commutative and associative.
  - Set difference is not commutative.
  - Set difference is associative only when the sets are mutually disjoint.
- Distributive Laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's Laws for set difference
  - $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$
  - $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$
- De Morgan's Laws for complementation.
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$
- Formulae for the cardinality of union of sets
  - $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
  - $n(A \cup B \cup C)$
    $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

FUNCTIONS

- The cartesian product of A with B is defined as

  $A \times B = \{(a,b) \mid a \in A$ and $b \in B\}$.

- A relation R from A to B is a non-empty subset of $A \times B$. That is, $R \subseteq A \times B$.
- A function $f: X \to Y$ is defined if the following condition hold:

  Every $x \in X$ is associated with only one $y \in Y$.

- Every function can be represented by a graph. However, the converse is not true in general.
- If every vertical line intersects a graph in at most one point, then the graph represents a function.
- A function can be described by

  ➢ a set of ordered pairs ➢ an arrow diagram ➢ a table and ➢ a graph.

- The modulus or absolute value function $y = |x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Some types of functions:
  - One-One function (injective function) (distinct elements have distinct images)
  - Onto function (surjective function) (the range and the co-domain are equal)
  - Bijective function (both one-one and onto)
  - Constant function (range is a singleton set)
  - Identity function (which leaves each input as it is)

Do you Know?

The Millennium Prize problems are seven problems in Mathematics that were stated by the Clay Mathematics Institute in USA in 2000. As of August 2010, six of the problems remain unsolved. A correct solution to any of the problems results in a US \$1000,000 being awarded by the institute. Only Poincare conjecture has been solved by a Russian Mathematician Girigori Perelman in 2010. However, he declined the Millinnium Prize award. (Here, the word conjecture means a mathematical problem is to be proved or disproved)

2

- Introduction
- Sequences
- Arithmetic Progression (A.P.)
- Geometric Progression (G.P.)
- Series

Leonardo Pisano (Fibonacci)

(1170-1250)

Italy

Fibonacci played an important role in reviving ancient mathematics. His name is known to modern mathematicians mainly because of a number sequence named after him, known as the 'Fibonacci numbers', which he did not discover but used as an example.

SEQUENCES AND SERIES OF REAL NUMBERS

Mathematics is the Queen of Sciences, and arithmetic is the Queen of Mathematics - C.F.Gauss

2.1 Introduction

In this chapter, we shall learn about sequences and series of real numbers. Sequences are fundamental mathematical objects with a long history in mathematics. They are tools for the development of other concepts as well as tools for mathematization of real life situations.

Let us recall that the letters $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and real numbers respectively.

Let us consider the following real-life situations.

(i) A team of ISRO scientists observes and records the height of a satellite from the sea level at regular intervals over a period of time.

(ii) The Railway Ministry wants to find out the number of people using Central railway station in Chennai on a daily basis and so it records the number of people entering the Central Railway station daily for 180 days.

(iii) A curious 9th standard student is interested in finding out all the digits that appear in the decimal part of the irrational number $\sqrt{5} = 2.236067978\cdots$ and writes down as

$2, 3, 6, 0, 6, 7, 9, 7, 8, \cdots$.

(iv) A student interested in finding all positive fractions with numerator 1, writes $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$.

(v) A mathematics teacher writes down the marks of her class according to alphabetical order of the students' names as $75, 95, 67, 35, 58, 47, 100, 89, 85, 60.$.

(vi) The same teacher writes down the same data in an ascending order as

35, 47, 58, 60, 67, 75, 85, 89, 95, 100..

In each of the above examples, some sets of real numbers have been listed in a specific order.

Note that in (iii) and (iv) the arrangements have infinite number of terms. In (i), (ii), (v) and (vi) there are only finite number of terms; but in (v) and (vi) the same set of numbers are written in different order.

2.2 Sequences

Definition

A sequence of real numbers is an arrangement or a list of real numbers in a specific order.

(i) If a sequence has only finite number of terms, then it is called a finite sequence.

(ii) If a sequence has infinitely many terms, then it is called an infinite sequence.

We denote a finite sequence as $S: a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ and an infinite sequence as $S: a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ where a_k denotes the k^{th} term of the sequence. For example, a_1 denotes the first term and a_7 denotes the seventh term in the sequence.

Note that in the above examples, (i), (ii), (v) and (vi) are finite sequences, whereas (iii) and (iv) are infinite sequences

Observe that, when we say that a collection of numbers is listed in a sequence, we mean that the sequence has an identified first member, second member, third member and so on. We have already seen some examples of sequences. Let us consider some more examples below.

(i) 2, 4, 6, 8, $\cdots$, 2010. (finite number of terms)

(ii) 1, -1, 1, -1, 1, -1, 1, $\cdots$. (terms just keep oscillating between 1 and -1)

(iii) π, π, π, π, π. (terms are same; such sequences are constant sequences)

(iv) 2, 3, 5, 7, 11, 13, 17, 19, 23, $\cdots$. (list of all prime numbers)

(v) 0.3, 0.33, 0.333, 0.3333, 0.33333, $\cdots$.

(vi) $S = \{a_n\}_1^{\infty}$ where $a_n = 1$ or 0 according to the outcome head or tail in the n^{th} toss of a coin.

From the above examples, (i) and (iii) are finite sequences and the other sequences are infinite sequences. One can easily see that some of them, i.e., (i) to (v) have a definite pattern or rule in the listing and hence we can find out any term in a particular position in the sequence. But in (vi), we cannot predict what a particular term is, however, we know it must be either 1 or 0. Here, we have used the word "pattern" to mean that the n^{th} term of a sequence is found based on the knowledge of its preceding elements in the sequence. In general, sequences can be viewed as functions.

2.2.1 Sequences viewed as functions

A finite real sequence $a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ can be viewed as a function $f : \{1, 2, 3, 4, \cdots, n\} \to \mathbb{R}$ defined by $f(k) = a_k$, $k = 1, 2, 3, \cdots, n$.

An infinite real sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ can be viewed as a function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(k) = a_k$, $\forall\, k \in \mathbb{N}$.

The symbol $\forall$ means "for all". If the general term a_k of a sequence $\{a_k\}_1^{\infty}$ is given, we can construct the whole sequence. Thus, a sequence is a function whose domain is the set{ 1, 2, 3, $\cdots$, }of natural numbers, or some subset of the natural numbers and whose range is a subset of real numbers.

Remarks

A function is not necessarily a sequence. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2x + 1, \forall\, x \in \mathbb{R}$ is not a sequence since the required listing is not possible. Also, note that the domain of f is not $\mathbb{N}$ or a subset $\{1, 2, \cdots, n\}$ of $\mathbb{N}$.

Example 2.1

Write the first three terms in a sequence whose n^{th} term is given by

$$c_n = \frac{n(n+1)(2n+1)}{6}, \ \forall\, n \in \mathbb{N}$$

Solution Here, $$c_n = \frac{n(n+1)(2n+1)}{6}, \ \forall\, n \in \mathbb{N}$$

For $n = 1$, $$c_1 = \frac{1(1+1)(2(1)+1)}{6} = 1.$$

For $n = 2$, $$c_2 = \frac{2(2+1)(4+1)}{6} = \frac{2(3)(5)}{6} = 5.$$

Finally $n = 3$, $$c_3 = \frac{3(3+1)(7)}{6} = \frac{(3)(4)(7)}{6} = 14.$$

Hence, the first three terms of the sequence are 1, 5, and 14.

In the above example, we were given a formula for the general term and were able to find any particular term directly. In the following example, we shall see another way of generating a sequence.

Example 2.2

Write the first five terms of each of the following sequences.

(i) $a_1 = -1$, $\quad a_n = \dfrac{a_{n-1}}{n+2}$, $\quad n > 1$ and $\forall\, n \in \mathbb{N}$

(ii) $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $\quad n = 3, 4, \cdots$.

Solution

(i) Given $a_1 = -1$ and $a_n = \dfrac{a_{n-1}}{n+2}, \quad n > 1$

$$a_2 = \frac{a_1}{2+2} = -\frac{1}{4}$$

$$a_3 = \frac{a_2}{3+2} = \frac{-\frac{1}{4}}{5} = -\frac{1}{20}$$

$$a_4 = \frac{a_3}{4+2} = \frac{-\frac{1}{20}}{6} = -\frac{1}{120}$$

$$a_5 = \frac{a_4}{5+2} = \frac{-\frac{1}{120}}{7} = -\frac{1}{840}$$

$\therefore$ The required terms of the sequence are $-1, -\frac{1}{4}, -\frac{1}{20}, -\frac{1}{120}$ and $-\frac{1}{840}$.

(ii) Given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n = 3, 4, 5, \cdots$.

Now, $F_1 = 1, F_2 = 1$

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$\therefore$ The first five terms of the sequence are 1, 1, 2, 3, 5.

Remarks

The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence. Its terms are listed as 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$. The Fibonacci sequence occurs in nature, like the arrangement of seeds in a sunflower. The number of spirals in the opposite directions of the seeds in a sunflower are consecutive numbers of the Fibonacci sequence.

Exercise 2.1

1. Write the first three terms of the following sequences whose n^{th} terms are given by

   (i) $a_n = \dfrac{n(n-2)}{3}$ (ii) $c_n = (-1)^n 3^{n+2}$ (iii) $z_n = \dfrac{(-1)^n n(n+2)}{4}$

2. Find the indicated terms in each of the sequences whose n^{th} terms are given by

   (i) $a_n = \dfrac{n+2}{2n+3}$; a_7, a_9 (ii) $a_n = (-1)^n 2^{n+3}(n+1)$; a_5, a_8

   (iii) $a_n = 2n^2 - 3n + 1$; a_5, a_7. (iv) $a_n = (-1)^n(1 - n + n^2)$; a_5, a_8

3. Find the 18th and 25th terms of the sequence defined by

$$a_n = \begin{cases} n(n+3), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ \dfrac{2n}{n^2+1}, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

4. Find the 13th and 16th terms of the sequence defined by

$$b_n = \begin{cases} n^2, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ n(n+2), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

5. Find the first five terms of the sequence given by

$$a_1 = 2, a_2 = 3 + a_1 \text{ and } a_n = 2a_{n-1} + 5 \quad \text{for } n > 2.$$

6. Find the first six terms of the sequence given by

$$a_1 = a_2 = a_3 = 1 \text{ and } a_n = a_{n-1} + a_{n-2} \quad \text{for } n > 3.$$

2.3 Arithmetic sequence or Arithmetic Progression (A.P.)

In this section we shall see some special types of sequences.

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference. An arithmetic sequence is also called an Arithmetic Progression (A.P.).

Examples

(i) $2, 5, 8, 11, 14, \cdots$ is an A.P. because $a_1 = 2$ and the common difference $d = 3$.

(ii) $-4, -4, -4, -4, \cdots$ is an A.P. because $a_1 = -4$ and $d = 0$.

(iii) $2, 1.5, 1, 0.5, 0, -0.5, -1.0, -1.5, \cdots$ is an A.P. because $a_1 = 2$ and $d = -0.5$.

The general form of an A.P.

Let us understand the general form of an A.P. Suppose that a is the first term and d is the common difference of an arithmetic sequence $\{a_k\}_{k=1}^{\infty}$. Then, we have

$$a_1 = a \text{ and } a_{n+1} = a_n + d, \ \forall n \in \mathbb{N}.$$

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 + d = a + d = a + (2-1)d$$
$$a_3 = a_2 + d = (a+d) + d = a + 2d = a + (3-1)d$$
$$a_4 = a_3 + d = (a+2d) + d = a + 3d = a + (4-1)d$$

Following the pattern, we see that the n^{th} term a_n as

$$a_n = a_{n-1} + d = [a + (n-2)d] + d = a + (n-1)d.$$

Thus, we have $a_n = a + (n-1)d$ for every $n \in \mathbb{N}$.

So, a typical arithmetic sequence or A.P. looks like

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d,\ a+nd,\cdots$$

Also, the formula for the general term of an Arithmetic sequence is of the form

$$t_n = a + (n-1)d \quad \text{for every } n \in \mathbb{N}.$$

Note

(i) Remember a sequence may also be a finite sequence. So, if an A.P. has only n terms, then the last term l is given by $l = a + (n-1)d$

(ii) $l = a + (n-1)d$ can also be rewritten as $n = \left(\frac{l-a}{d}\right) + 1$. This helps us to find the number of terms when the first, the last term and the common difference are given.

(iii) Three consecutive terms of an A.P. may be taken as $m-d,\ m,\ m+d$

(iv) Four consecutive terms of an A.P. may be taken as $m-3d,\ m-d,\ m+d,\ m+3d$ with common difference $2d$.

(v) An A.P. remains an A.P. if each of its terms is added or subtracted by a same constant.

(vi) An A.P. remains an A.P. if each of its terms is multiplied or divided by a non-zero constant.

Example 2.3

Which of the following sequences are in an A.P.?

(i) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \cdots$. (ii) $3m-1, 3m-3, 3m-5, \cdots$.

Solution

(i) Let t_n, $n \in \mathbb{N}$ be the n^{th} term of the given sequence.

$\therefore$ $$t_1 = \frac{2}{3},\ t_2 = \frac{4}{5},\ t_3 = \frac{6}{7}$$

So $$t_2 - t_1 = \frac{4}{5} - \frac{2}{3} = \frac{12-10}{15} = \frac{2}{15}$$

$$t_3 - t_2 = \frac{6}{7} - \frac{4}{5} = \frac{30-28}{35} = \frac{2}{35}$$

Since $t_2 - t_1 \neq t_3 - t_2$, the given sequence is not an A.P.

(ii) Given $3m-1, 3m-3, 3m-5, \cdots$.

Here $$t_1 = 3m-1,\ t_2 = 3m-3,\ t_3 = 3m-5, \cdots.$$

$\therefore$ $$t_2 - t_1 = (3m-3) - (3m-1) = -2$$

Also, $$t_3 - t_2 = (3m-5) - (3m-3) = -2$$

Hence, the given sequence is an A.P. with first term $3m-1$ and the common difference -2.

Example 2.4

Find the first term and common difference of the A.P.

(i) $5, 2, -1, -4, \cdots$. (ii) $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \cdots, \frac{17}{6}$

Solution

(i) First term $a = 5$, and the common difference $d = 2 - 5 = -3$.

(ii) $a = \frac{1}{2}$ and the common difference $d = \frac{5}{6} - \frac{1}{2} = \frac{5-3}{6} = \frac{1}{3}$.

Example 2.5

Find the smallest positive integer n such that t_n of the arithmetic sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, \cdots$ is negative.?

Solution Here we have $a = 20$, $d = 19\frac{1}{4} - 20 = -\frac{3}{4}$.

We want to find the first positive integer n such that $t_n < 0$.

This is same as solving $a + (n-1)d < 0$ for smallest $n \in \mathbb{N}$.

That is solving $20 + (n-1)\left(-\frac{3}{4}\right) < 0$ for smallest $n \in \mathbb{N}$.

Now, $(n-1)\left(-\frac{3}{4}\right) < -20$

$\Longrightarrow (n-1) \times \frac{3}{4} > 20$ (The inequality is reversed on multiplying both sides by -1)

$\therefore$ $n - 1 > 20 \times \frac{4}{3} = \frac{80}{3} = 26\frac{2}{3}$.

This implies $n > 26\frac{2}{3} + 1$. That is, $n > 27\frac{2}{3} = 27.66$

Thus, the smallest positive integer $n \in \mathbb{N}$ satisfying the inequality is $n = 28$.

Hence, the 28^{th} term, t_{28} is the first negative term of the A.P.

Example 2.6

In a flower garden, there are 23 rose plants in the first row, 21 in the second row, 19 in the third row and so on. There are 5 rose plants in the last row. How many rows are there in the flower garden?

Solution Let n be the number of rows in the flower garden .

The number of rose plants in the $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}, \cdots, n^{th}$ rows are $23, 21, 19, \cdots, 5$ respectively.

Now, $t_k - t_{k-1} = -2$ for $k = 2, \cdots, n$.

Thus, the sequence $23, 21, 19, \cdots, 5$ is in an A.P.

We have $a = 23, \quad d = -2$, and $l = 5$.

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{5-23}{-2} + 1 = 10.$$

So, there are 10 rows in the flower garden.

Example 2.7

If a person joins his work in 2010 with an annual salary of ₹30,000 and receives an annual increment of ₹600 every year, in which year, will his annual salary be ₹39,000?

Solution Suppose that the person's annual salary reaches ₹39,000 in the n^{th} year.

Annual salary of the person in 2010, 2011, 2012, $\cdots$, $[2010 + (n-1)]$ will be

₹30,000, ₹30,600, ₹31,200, $\cdots$, ₹39000 respectively.

First note that the sequence of salaries form an A.P.

To find the required number of terms, let us divide each term of the sequence by a fixed constant 100. Now, we get the new sequence 300, 306, 312, $\cdots$, 390.

Here $a = 300$, $d = 6$, $l = 390$.

So, $$n = \frac{l-a}{d} + 1$$

$$= \frac{390-300}{6} + 1 = \frac{90}{6} + 1 = 16$$

Thus, 16$^{\text{th}}$ annual salary of the person will be ₹39,000.

$\therefore$ His annual salary will reach ₹39,000 in the year 2025.

Example 2.8

Three numbers are in the ratio $2:5:7$. If 7 is subtracted from the second, the resulting numbers form an arithmetic sequence. Determine the numbers.

Solution Let the numbers be $2x, 5x$ and $7x$ for some unknown $x, (x \neq 0)$

By the given information, we have that $2x, 5x-7, 7x$ are in A.P.

$\therefore \ (5x-7) - 2x = 7x - (5x-7) \implies 3x - 7 = 2x + 7$ and so $x = 14$.

Thus, the required numbers are 28, 70, 98.

Exercise 2.2

1. The first term of an A.P. is 6 and the common difference is 5. Find the A.P. and its general term.
2. Find the common difference and 15$^{\text{th}}$ term of the A.P. 125, 120, 115, 110, $\cdots$.
3. Which term of the arithmetic sequence $24, 23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4}, \cdots$ is 3?

4. Find the 12$^{\text{th}}$ term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \cdots$.

5. Find the 17$^{\text{th}}$ term of the A.P. $4, 9, 14, \cdots$.

6. How many terms are there in the following Arithmetic Progressions?

   (i) $-1, \ -\frac{5}{6}, \ -\frac{2}{3}, \cdots, \frac{10}{3}$. (ii) $7, 13, 19, \cdots, 205$.

7. If 9$^{\text{th}}$ term of an A.P. is zero, prove that its 29$^{\text{th}}$ term is double (twice) the 19$^{\text{th}}$ term.

8. The 10$^{\text{th}}$ and 18$^{\text{th}}$ terms of an A.P. are 41 and 73 respectively. Find the 27$^{\text{th}}$ term.

9. Find n so that the n^{th} terms of the following two A.P.'s are the same.

   $1, 7, 13, 19, \cdots$ and $100, 95, 90, \cdots$.

10. How many two digit numbers are divisible by 13?

11. A TV manufacturer has produced 1000 TVs in the seventh year and 1450 TVs in the tenth year. Assuming that the production increases uniformly by a fixed number every year, find the number of TVs produced in the first year and in the 15$^{\text{th}}$ year.

12. A man has saved ₹640 during the first month, ₹720 in the second month and ₹800 in the third month. If he continues his savings in this sequence, what will be his savings in the 25$^{\text{th}}$ month?

13. The sum of three consecutive terms in an A.P. is 6 and their product is –120. Find the three numbers.

14. Find the three consecutive terms in an A. P. whose sum is 18 and the sum of their squares is 140.

15. If m times the m^{th} term of an A.P. is equal to n times its n^{th} term, then show that the $(m+n)^{\text{th}}$ term of the A.P. is zero.

16. A person has deposited ₹25,000 in an investment which yields 14% simple interest annually. Do these amounts (principal + interest) form an A.P.? If so, determine the amount of investment after 20 years.

17. If a, b, c are in A.P. then prove that $(a-c)^2 = 4(b^2 - ac)$.

18. If a, b, c are in A.P. then prove that $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A.P.

19. If a^2, b^2, c^2 are in A.P. then show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A.P.

20. If $a^x = b^y = c^z, x \neq 0, y \neq 0, z \neq 0$ and $b^2 = ac$, then show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

2.4 Geometric Sequence or Geometric Progression (G.P.)

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, $n \in \mathbb{N}$, where r is a non-zero constant. Here, a_1 is the first term and the constant r is called the common ratio. A geometric sequence is also called a Geometric Progression (G.P.).

Let us consider some examples of geometric sequences.

(i) $3, 6, 12, 24, \cdots$.

A sequence $\{a_n\}_1^\infty$ is a geometric sequence if $\dfrac{a_{n+1}}{a_n} = r \neq 0$, $n \in \mathbb{N}$.

Now, $\dfrac{6}{3} = \dfrac{12}{6} = \dfrac{24}{12} = 2 \neq 0$. So the given sequence is a geometric sequence.

(ii) $\dfrac{1}{9}, -\dfrac{1}{27}, \dfrac{1}{81}, -\dfrac{1}{243}, \cdots$.

Here, we have $\dfrac{-\frac{1}{27}}{\frac{1}{9}} = \dfrac{\frac{1}{81}}{-\frac{1}{27}} = \dfrac{-\frac{1}{243}}{\frac{1}{81}} = -\dfrac{1}{3} \neq 0$.

Thus, the given sequence is a geometric sequence.

The general form of a G.P.

Let us derive the general form of a G.P. Suppose that a is the first term and r is the common ratio of a geometric sequence $\{a_k\}_{k=1}^\infty$. Then, we have

$$a_1 = a \quad \text{and} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for } n \in \mathbb{N}.$$

Thus, $\quad a_{n+1} = r\, a_n \quad$ for $n \in \mathbb{N}$.

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 r = ar = ar^{2-1}$$

$$a_3 = a_2 r = (ar)r = ar^2 = ar^{3-1}$$

$$a_4 = a_3 r = (ar^2)r = ar^3 = ar^{4-1}$$

Following the pattern, we have

$$a_n = a_{n-1} r = (ar^{n-2})r = ar^{n-1}.$$

Thus, $\quad a_n = ar^{n-1}$ for every $n \in \mathbb{N}$, gives n^{th} term of the G.P.

So, a typical geometric sequence or G.P. looks like

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}, ar^n, \cdots.$$

Thus , the formula for the general term of a geometric sequence is

$$t_n = ar^{n-1}, \; n = 1, 2, 3, \cdots.$$

Suppose we are given the first few terms of a sequence, how can we determine if the given sequence is a geometric sequence or not?

If $\frac{t_{n+1}}{t_n} = r, \forall\, n \in \mathbb{N}$, where r is a non-zero constant, then $\{t_n\}_1^\infty$ is in G.P.

Note

(i) If the ratio of any term other than the first term to its preceding term of a sequence is a non-zero constant, then it is a geometric sequence.

(ii) A geometric sequence remains a geometric sequence if each term is multiplied or divided by a non zero constant.

(iii) Three consecutive terms in a G.P may be taken as $\frac{a}{r}, a, ar$ with common ratio r.

(iv) Four consecutive terms in a G.P may be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
(here, the common ratio is r^2 not r as above)

Example 2.9

Which of the following sequences are geometric sequences

(i) 5, 10, 15, 20, $\cdots$. (ii) 0.15, 0.015, 0.0015, $\cdots$. (iii) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, 3\sqrt{21}, \cdots$.

Solution

(i) Considering the ratios of the consecutive terms, we see that $\frac{10}{5} \neq \frac{15}{10}$.

Thus, there is no common ratio. Hence it is not a geometric sequence.

(ii) We see that $\frac{0.015}{0.15} = \frac{0.0015}{0.015} = \cdots = \frac{1}{10}$.

Since the common ratio is $\frac{1}{10}$, the given sequence is a geometric sequence.

(iii) Now, $\frac{\sqrt{21}}{\sqrt{7}} = \frac{3\sqrt{7}}{\sqrt{21}} = \frac{3\sqrt{21}}{3\sqrt{7}} = \cdots = \sqrt{3}$. Thus, the common ratio is $\sqrt{3}$.

Therefore, the given sequence is a geometric sequence.

Example 2.10

Find the common ratio and the general term of the following geometric sequences.

(i) $\frac{2}{5}, \frac{6}{25}, \frac{18}{125}, \cdots$. (ii) 0.02, 0.006, 0.0018, $\cdots$.

Solution

(i) Given sequence is a geometric sequence.

The common ratio is given by $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \cdots$.

Thus, $$r = \frac{\frac{6}{25}}{\frac{2}{5}} = \frac{3}{5}.$$

The first term of the sequence is $\frac{2}{5}$. So, the general term of the sequence is

$$t_n = ar^{n-1},\ n = 1, 2, 3, \cdots.$$

$$\Longrightarrow \qquad t_n = \frac{2}{5}\left(\frac{3}{5}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

(ii) The common ratio of the given geometric sequence is

$$r = \frac{0.006}{0.02} = 0.3 = \frac{3}{10}.$$

The first term of the geometric sequence is 0.02

So, the sequence can be represented by

$$t_n = (0.02)\left(\frac{3}{10}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

Example 2.11

The 4th term of a geometric sequence is $\frac{2}{3}$ and the seventh term is $\frac{16}{81}$. Find the geometric sequence.

Solution Given that $t_4 = \frac{2}{3}$ and $t_7 = \frac{16}{81}$.

Using the formula $t_n = ar^{n-1},\ n = 1, 2, 3, \cdots$. for the general term we have,

$$t_4 = ar^3 = \frac{2}{3} \text{ and } t_7 = ar^6 = \frac{16}{81}.$$

Note that in order to find the geometric sequence, we need to find a and r.

By dividing t_7 by t_4 we obtain,

$$\frac{t_7}{t_4} = \frac{ar^6}{ar^3} = \frac{\frac{16}{81}}{\frac{2}{3}} = \frac{8}{27}.$$

Thus, $r^3 = \frac{8}{27} = \left(\frac{2}{3}\right)^3$ which implies $r = \frac{2}{3}$.

Now, $t_4 = \frac{2}{3} \Longrightarrow ar^3 = \left(\frac{2}{3}\right)$.

$$\Longrightarrow \quad a\left(\frac{8}{27}\right) = \frac{2}{3}. \qquad \therefore \quad a = \frac{9}{4}.$$

Hence, the required geometric sequence is $a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},\ ar^n, \cdots$

That is, $\frac{9}{4},\ \frac{9}{4}\left(\frac{2}{3}\right),\ \frac{9}{4}\left(\frac{2}{3}\right)^2,\ \cdots.$

Example 2.12

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture initially, how many bacteria will be present at the end of 14th hour?

Solution Note that the number of bacteria present in the culture doubles at the end of successive hours.

Number of bacteria present initially in the culture $= 30$

Number of bacteria present at the end of first hour $= 2(30)$

Number of bacteria present at the end of second hour $= 2(2(30)) = 30(2^2)$

Continuing in this way, we see that the number of bacteria present at the end of every hour forms a G.P. with the common ratio $r = 2$.

Thus, if t_n denotes the number of bacteria after n hours,

$$t_n = 30\,(2^n) \text{ is the general term of the G.P.}$$

Hence, the number of bacteria at the end of 14^{th} hour is given by $t_{14} = 30\,(2^{14})$.

Example 2.13

An amount ₹500 is deposited in a bank which pays annual interest at the rate of 10% compounded annually. What will be the value of this deposit at the end of 10^{th} year?

Solution

The principal is ₹500. So, the interest for this principal for one year is $500\left(\frac{10}{100}\right) = 50$.

Thus, the principal for the 2nd year = Principal for 1st year + Interest

$$= 500 + 500\left(\frac{10}{100}\right) = 500\left(1 + \frac{10}{100}\right)$$

Now, the interest for the second year $= \left(500\left(1 + \frac{10}{100}\right)\right)\left(\frac{10}{100}\right)$.

So, the principal for the third year $= 500\left(1 + \frac{10}{100}\right) + 500\left(1 + \frac{10}{100}\right)\frac{10}{100}$

$$= 500\left(1 + \frac{10}{100}\right)^2$$

Continuing in this way we see that the principal for the n^{th} year $= 500\left(1 + \frac{10}{100}\right)^{n-1}$.

The amount at the end of $(n-1)^{\text{th}}$ year = Principal for the nth year.

Thus, the amount in the account at the end of 10^{th} year

$$= ₹\ 500\left(1 + \frac{10}{100}\right)^{10} = ₹\ 500\left(\frac{11}{10}\right)^{10}.$$

Remarks

By using the above method, one can derive a formula for finding the total amount for compound interest problems. Derive the formula:

$$A = P(1 + i)^n$$

where A is the amount, P is the principal, $i = \frac{r}{100}$, r is the annual interest rate and n is the number of years.

Example 2.14

The sum of first three terms of a geometric sequence is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Solution We may take the first three terms of the geometric sequence as $\frac{a}{r}, a, ar$.

Then,
$$\frac{a}{r} + a + ar = \frac{13}{12}$$
$$a\left(\frac{1}{r} + 1 + r\right) = \frac{13}{12} \implies a\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12} \qquad (1)$$

Also,
$$\left(\frac{a}{r}\right)(a)(ar) = -1$$
$$\implies a^3 = -1 \qquad \therefore \quad a = -1$$

Substituting $a = -1$ in (1) we obtain,
$$(-1)\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12}$$
$$\implies 12r^2 + 12r + 12 = -13r$$
$$12r^2 + 25r + 12 = 0$$
$$(3r + 4)(4r + 3) = 0$$

Thus, $r = -\frac{4}{3}$ or $-\frac{3}{4}$

When $r = -\frac{4}{3}$ and $a = -1$, the terms are $\frac{3}{4}, -1, \frac{4}{3}$.

When $r = -\frac{3}{4}$ and $a = -1$, we get $\frac{4}{3}, -1, \frac{3}{4}$, which is in the reverse order.

Example 2.15

If a, b, c, d are in geometric sequence, then prove that

$(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$

Solution Given a, b, c, d are in a geometric sequence.

Let r be the common ratio of the given sequence. Here, the first term is a.

Thus, $b = ar, \quad c = ar^2, \quad d = ar^3$

Now, $(b - c)^2 + (c - a)^2 + (d - b)^2$
$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$
$$= a^2[(r - r^2)^2 + (r^2 - 1)^2 + (r^3 - r)^2]$$
$$= a^2[r^2 - 2r^3 + r^4 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$$
$$= a^2[r^6 - 2r^3 + 1] = a^2[r^3 - 1]^2$$
$$= (ar^3 - a)^2 = (a - ar^3)^2 = (a - d)^2$$

Exercise 2.3

1. Find out which of the following sequences are geometric sequences. For those geometric sequences, find the common ratio.

   (i) $0.12,\ 0.24,\ 0.48, \cdots.$ (ii) $0.004,\ 0.02,\ 0.1, \cdots.$ (iii) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \cdots.$

   (iv) $12,\ 1,\ \frac{1}{12}, \cdots.$ (v) $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \cdots.$ (vi) $4, -2, -1, -\frac{1}{2}, \cdots.$

2. Find the 10th term and common ratio of the geometric sequence $\frac{1}{4}, -\frac{1}{2}, 1, -2, \cdots.$

3. If the 4th and 7th terms of a G.P. are 54 and 1458 respectively, find the G.P.

4. In a geometric sequence, the first term is $\frac{1}{3}$ and the sixth term is $\frac{1}{729}$, find the G.P.

5. Which term of the geometric sequence,

   (i) $5,\ 2,\ \frac{4}{5}, \frac{8}{25}, \cdots$, is $\frac{128}{15625}$? (ii) $1,\ 2,\ 4,\ 8, \cdots$, is 1024 ?

6. If the geometric sequences $162, 54, 18, \cdots.$ and $\frac{2}{81}, \frac{2}{27}, \frac{2}{9}, \cdots$ have their n^{th} term equal, find the value of n.

7. The fifth term of a G.P. is 1875. If the first term is 3, find the common ratio.

8. The sum of three terms of a geometric sequence is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

9. If the product of three consecutive terms in G.P. is 216 and sum of their products in pairs is 156, find them.

10. Find the first three consecutive terms in G.P. whose sum is 7 and the sum of their reciprocals is $\frac{7}{4}$

11. The sum of the first three terms of a G.P. is 13 and sum of their squares is 91. Determine the G.P.

12. If ₹1000 is deposited in a bank which pays annual interest at the rate of 5% compounded annually, find the maturity amount at the end of 12 years .

13. A company purchases an office copier machine for ₹50,000. It is estimated that the copier depreciates in its value at a rate of 45% per year. What will be the value of the copier after 15 years?

14. If a, b, c, d are in a geometric sequence, then show that
    $$(a-b+c)(b+c+d) = ab+bc+cd.$$

15. If a, b, c, d are in a G.P., then prove that $a+b,\ \ b+c,\ \ c+d,$ are also in G.P.

2.5 Series

Let us consider the following problem:

A person joined a job on January 1, 1990 at an annual salary of ₹25,000 and received an annual increment of ₹500 each year. What is the total salary he has received upto January 1, 2010?

First of all note that his annual salary forms an arithmetic sequence

$$25000,\ 25500,\ 26000,\ 26500, \cdots,\ (25000 + 19(500)).$$

To answer the above question, we need to add all of his twenty years salary. That is,

$$25000 + 25500 + 26000 + 26500 + \cdots + (25000 + 19(500)).$$

So, we need to develop an idea of summing terms of a sequence.

Definition

An expression of addition of terms of a sequence is called a series.
If a series consists only a finite number of terms, it is called a finite series.
If a series consists of infinite number of terms of a sequence, it is called an infinite series.

Consider a sequence $S = \{a_n\}_{n=1}^{\infty}$ of real numbers. For each $n \in \mathbb{N}$ we define the partial sums by $S_n = a_1 + a_2 +, \cdots + a_n$, $n = 1, 2, 3, \cdots$. Then $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the given sequence $\{a_n\}_{n=1}^{\infty}$.

The ordered pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ is called an infinite series of terms of the sequence $\{a_n\}_1^{\infty}$. The infinite series is denoted by $a_1 + a_2 + a_3 + \cdots$, or simply $\sum_{n=1}^{\infty} a_n$ where the symbol $\sum$ stands for summation and is pronounced as sigma.

Well, we can easily understand finite series (adding finite number of terms). It is impossible to add all the terms of an infinite sequence by the ordinary addition, since one could never complete the task. How can we understand (or assign a meaning to) adding infinitely many terms of a sequence? We will learn about this in higher classes in mathematics. For now we shall focus mostly on finite series.

In this section , we shall study Arithmetic series and Geometric series.

2.5.1 Arithmetic series

An arithmetic series is a series whose terms form an arithmetic sequence.

Sum of first *n* terms of an arithmetic sequence

Consider an arithmetic sequence with first term a and common difference d

given by $a,\ a + d,\ a + 2d, ...,\ a + (n-1)d, \cdots$.

Let S_n be the sum of first n terms of the arithmetic sequence.

Thus, $S_n = a + (a+d) + (a+2d) + \cdots\ (a + (n-1)d)$

$\Longrightarrow\ S_n = na + (d + 2d + 3d + \cdots + (n-1)d)$

$= na + d(1 + 2 + 3 + \cdots + (n-1))$

So, we can simplify this formula if we can find the sum $1 + 2 + \cdots + (n-1)$.

This is nothing but the sum of the arithmetic sequence $1, 2, 3, \cdots, (n-1)$.

So, first we find the sum $1 + 2 + \cdots + (n-1)$ below.

Now, let us find the sum of the first n positive integers.

Let $S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$. (1)

We shall use a small trick to find the above sum. Note that we can write S_n also as

$$S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1. \qquad (2)$$

Adding (1) and (2) we obtain,

$$2S_n = (n+1) + (n+1) + \cdots + (n+1) + (n+1). \qquad (3)$$

Now, how many $(n+1)$ are there on the right hand side of (3)?

There are n terms in each of (1) and (2). We merely added corresponding terms from (1) and (2).

Thus, there must be exactly n such $(n+1)$'s.

Therefore, (3) simplifies to $2S_n = n(n+1)$.

Hence, the sum of the first n positive integers is given by

$$S_n = \frac{n(n+1)}{2}. \qquad \text{So, } 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \qquad (4)$$

This is a useful formula in finding the sums.

Remarks

The above method was first used by the famous German mathematician **Carl Fredrick Gauss,** known as **Prince of Mathematics,** to find the sum of positive integers upto 100. This problem was given to him by his school teacher when he was just five years old. When you go to higher studies in mathematics, you will learn other methods to arrive at the above formula.

Carl Fredrick Gauss
(1777 – 1855)

Now, let us go back to summing first n terms of a general arithmetic sequence. We have already seen that

$$S_n = na + [d + 2d + 3d + \cdots + (n-1)d]$$

$$= na + d[1 + 2 + 3 + \cdots + (n-1)]$$

$$= na + d\frac{n(n-1)}{2} \quad \text{using (4)}$$

$$= \frac{n}{2}[2a + (n-1)d] \qquad (5)$$

Hence, we have

$$S_n = \frac{n}{2}[a + (a + (n-1)d)] = \frac{n}{2}\text{ (first term + last term)}$$
$$= \frac{n}{2}(a + l).$$

The sum S_n of the first n terms of an arithmetic sequence with first term a is given by

(i) $S_n = \frac{n}{2}[2a + (n-1)d]$ if the common difference d is given.

(ii) $S_n = \frac{n}{2}(a + l)$, if the last term l is given.

Example 2.16

Find the sum of the arithmetic series $5 + 11 + 17 + \cdots + 95$.

Solution Given that the series $5 + 11 + 17 + \cdots + 95$ is an arithmetic series.

Note that $a = 5$, $d = 11 - 5 = 6$, $l = 95$.

Now,
$$n = \frac{l-a}{d} + 1$$
$$= \frac{95-5}{6} + 1 = \frac{90}{6} + 1 = 16.$$

Hence, the sum $S_n = \frac{n}{2}[l + a]$

$$S_{16} = \frac{16}{2}[95 + 5] = 8(100) = 800.$$

Example 2.17

Find the sum of the first $2n$ terms of the following series.

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots.$$

Solution We want to find $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to $2n$ terms

$= 1 - 4 + 9 - 16 + 25 - \cdots$ to $2n$ terms

$= (1-4) + (9-16) + (25-36) + \cdots$ to n terms. (after grouping)

$= -3 + (-7) + (-11) + \cdots$ n terms

Now, the above series is in an A.P. with first term $a = -3$ and common difference $d = -4$

Therefore, the required sum $= \frac{n}{2}[2a + (n-1)d]$

$$= \frac{n}{2}[2(-3) + (n-1)(-4)]$$
$$= \frac{n}{2}[-6 - 4n + 4] = \frac{n}{2}[-4n - 2]$$
$$= \frac{-2n}{2}(2n+1) = -n(2n+1).$$

Example 2.18

In an arithmetic series, the sum of first 14 terms is -203 and the sum of the next 11 terms is –572. Find the arithmetic series.

Solution Given that

$$S_{14} = -203$$

$$\Longrightarrow \quad \frac{14}{2}[2a + 13d] = -203$$

$$\Longrightarrow \quad 7[2a + 13d] = -203$$

$$\Longrightarrow \quad 2a + 13d = -29. \qquad (1)$$

Also, the sum of the next 11 terms $= -572$.

Now, $S_{25} = S_{14} + (-572)$

That is, $S_{25} = -203 - 572 = -775$.

$$\Longrightarrow \quad \frac{25}{2}[2a + 24d] = -775$$

$$\Longrightarrow \quad 2a + 24d = -31 \times 2$$

$$\Longrightarrow \quad a + 12d = -31 \qquad (2)$$

Solving (1) and (2) we get, $a = 5$ and $d = -3$.

Thus, the required arithmetic series is $5 + (5-3) + (5 + 2(-3)) + \cdots$.

That is, the series is $5 + 2 - 1 - 4 - 7 - \cdots$.

Example 2.19

How many terms of the arithmetic series $24 + 21 + 18 + 15 + \cdots$, be taken continuously so that their sum is -351.

Solution In the given arithmetic series, $a = 24$, $d = -3$.

Let us find n such that $S_n = -351$

Now, $S_n = \frac{n}{2}[2a + (n-1)d] = -351$

That is, $\frac{n}{2}[2(24) + (n-1)(-3)] = -351$

$$\Longrightarrow \quad \frac{n}{2}[48 - 3n + 3] = -351$$

$$\Longrightarrow \quad n(51 - 3n) = -702$$

$$\Longrightarrow \quad n^2 - 17n - 234 = 0$$

$$(n - 26)(n + 9) = 0$$

$$\therefore \quad n = 26 \text{ or } n = -9$$

Here n, being the number of terms needed, cannot be negative.

Thus, 26 terms are needed to get the sum -351.

Example 2.20

Find the sum of all 3 digit natural numbers, which are divisible by 8.

Solution

The three digit natural numbers divisible by 8 are 104, 112, 120, $\cdots$, 992.

Let S_n denote their sum. That is, $S_n = 104 + 112 + 120 + 128 + , \cdots + 992$.

Now, the sequence 104, 112, 120, $\cdots$, 992 forms an A.P.

Here, $a = 104, \quad d = 8 \quad \text{and} \quad l = 992.$

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{992-104}{8} + 1$$

$$= \frac{888}{8} + 1 = 112.$$

Thus, $S_{112} = \frac{n}{2}[a+l] = \frac{112}{2}[104+992] = 56(1096) = 61376.$

Hence, the sum of all three digit numbers, which are divisible by 8 is equal to 61376.

Example 2.21

The measures of the interior angles taken in order of a polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in the given polygon.

Solution Let n denote the number of sides of the polygon.

Now, the measures of interior angles form an arithmetic sequence.

Let the sum of the interior angles of the polygon be

$$S_n = a + (a+d) + (a+2d) + \cdots + l \text{ , where } a = 85 \text{ and } l = 215.$$

We have, $S_n = \frac{n}{2}[l+a]$ (1)

We know that the sum of the interior angles of a polygon is $(n-2)\times 180^0$.

Thus, $S_n = (n-2)\times 180$

From (1), we have $\frac{n}{2}[l+a] = (n-2)\times 180$

$$\Longrightarrow \quad \frac{n}{2}[215+85] = (n-2)\times 180$$

$$150n = 180(n-2) \quad \Longrightarrow \quad n = 12..$$

Hence, the number of sides of the polygon is 12.

Exercise 2.4

1. Find the sum of the first (i) 75 positive integers (ii) 125 natural numbers.
2. Find the sum of the first 30 terms of an A.P. whose n^{th} term is $3+2n$.
3. Find the sum of each arithmetic series

   (i) $38 + 35 + 32 + \cdots + 2$. (ii) $6 + 5\frac{1}{4} + 4\frac{1}{2} + \cdots$ 25 terms.

4. Find the S_n for the following arithmetic series described.

   (i) $a = 5, \quad n = 30, \quad l = 121$ (ii) $a = 50, \quad n = 25, \quad d = -4$

5. Find the sum of the first 40 terms of the series $1^2 - 2^2 + 3^2 - 4^2 + \cdots$.

6. In an arithmetic series, the sum of first 11 terms is 44 and that of the next 11 terms is 55. Find the arithmetic series.

7. In the arithmetic sequence 60, 56, 52, 48,$\cdots$, starting from the first term, how many terms are needed so that their sum is 368?

8. Find the sum of all 3 digit natural numbers, which are divisible by 9.

9. Find the sum of first 20 terms of the arithmetic series in which 3rd term is 7 and 7th term is 2 more than three times its 3rd term.

10. Find the sum of all natural numbers between 300 and 500 which are divisible by 11.

11. Solve: $1 + 6 + 11 + 16 + \cdots + x = 148$.

12. Find the sum of all numbers between 100 and 200 which are not divisible by 5.

13. A construction company will be penalised each day for delay in construction of a bridge. The penalty will be ₹4000 for the first day and will increase by ₹1000 for each following day. Based on its budget, the company can afford to pay a maximum of ₹1,65,000 towards penalty. Find the maximum number of days by which the completion of work can be delayed

14. A sum of ₹1000 is deposited every year at 8% simple interest. Calculate the interest at the end of each year. Do these interest amounts form an A.P.? If so, find the total interest at the end of 30 years.

15. The sum of first n terms of a certain series is given as $3n^2 - 2n$. Show that the series is an arithmetic series.

16. If a clock strikes once at 1 o'clock, twice at 2 o'clock and so on, how many times will it strike in a day?

17. Show that the sum of an arithmetic series whose first term is a, second term b and the last term is c is equal to $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$.

18. If there are $(2n+1)$ terms in an arithmetic series, then prove that the ratio of the sum of odd terms to the sum of even terms is $(n+1):n$.

19. The ratio of the sums of first m and first n terms of an arithmetic series is $m^2:n^2$ show that the ratio of the m^{th} and n^{th} terms is $(2m-1):(2n-1)$

20. A gardener plans to construct a trapezoidal shaped structure in his garden. The longer side of trapezoid needs to start with a row of 97 bricks. Each row must be decreased by 2 bricks on each end and the construction should stop at 25^{th} row. How many bricks does he need to buy?

2.5.2 Geometric series

A series is a geometric series if the terms of the series form a geometric sequence.

Let $a, ar, ar^2, \cdots, ar^{n-1}, ar^n, \cdots$ be a geometric sequence where $r \neq 0$ is the common ratio. We want to find the sum of the first n terms of this sequence.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ (1)

If $r = 1$, then from (1) it follows that $S_n = na$.

For $r \neq 1$, using (1) we have

$$rS_n = r(a + ar + ar^2 + \cdots + ar^{n-1}) = ar + ar^2 + ar^3 + \cdots + ar^n. \qquad (2)$$

Now subtracting (2) from (1), we get

$$S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n)$$

$$\Longrightarrow \quad S_n(1 - r) = a(1 - r^n)$$

Hence, we have $\quad S_n = \dfrac{a(1-r^n)}{1-r}$, since $r \neq 1$.

The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1} = \dfrac{a(1-r^n)}{1-r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

where a is the first term and r is the common ratio.

Remarks

Actually, if $-1 < r < 1$, then the following formula holds:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$

Note that the sum of infinite number of positive numbers may give a finite value.

Example 2.22

Find the sum of the first 25 terms of the geometric series

$16 - 48 + 144 - 432 + \cdots$.

Solution Here $a = 16$, $r = -\dfrac{48}{16} = -3 \neq 1$. Now, $S_n = \dfrac{a(1-r^n)}{1-r}$, $r \neq 1$.

So, we have $\quad S_{25} = \dfrac{16(1-(-3)^{25})}{1-(-3)} = \dfrac{16(1+3^{25})}{4} = 4(1+3^{25})$.

Example 2.23

Find S_n for each of the geometric series described below:

(i) $a = 2,\ t_6 = 486,\ n = 6$ (ii) $a = 2400,\ r = -3,\ n = 5$

Solution

(i) Here $a = 2,\quad t_6 = 486,\quad n = 6$

Now $t_6 = 2(r)^5 = 486$

$$\Longrightarrow \quad r^5 = 243 \quad \therefore \quad r = 3.$$

$$\text{Now,} \quad S_n = \frac{a(r^n - 1)}{r - 1} \text{ if } r \neq 1$$

$$\text{Thus,} \quad S_6 = \frac{2(3^6 - 1)}{3 - 1} = 3^6 - 1 = 728.$$

(ii) Here $a = 2400,\quad r = -3,\quad n = 5$

$$\text{Thus,} \quad S_5 = \frac{a(r^5 - 1)}{r - 1} \text{ if } r \neq 1$$

$$= \frac{2400[(-3)^5 - 1]}{(-3) - 1}$$

$$\text{Hence,} \quad S_5 = \frac{2400}{4}(1 + 3^5) = 600(1 + 243) = 146400.$$

Example 2.24

In the geometric series $2 + 4 + 8 + \cdots$, starting from the first term how many consecutive terms are needed to yield the sum 1022?

Solution Given the geometric series is $2 + 4 + 8 + \cdots$.

Let n be the number of terms required to get the sum.

Here $a = 2,\quad r = 2,\quad S_n = 1022$.

To find n, let us consider

$$S_n = \frac{a[r^n - 1]}{r - 1} \text{ if } r \neq 1$$

$$= (2)\left[\frac{2^n - 1}{2 - 1}\right] = 2(2^n - 1).$$

But $S_n = 1022$ and hence $2(2^n - 1) = 1022$

$$\Longrightarrow \quad 2^n - 1 = 511$$

$$\Longrightarrow \quad 2^n = 512 = 2^9. \quad \text{Thus,} \quad n = 9.$$

Example 2.25

The first term of a geometric series is 375 and the fourth term is 192. Find the common ratio and the sum of the first 14 terms.

Solution Let a be the first term and r be the common ratio of the given G.P.

Given that $a = 375, \quad t_4 = 192$.

$$\text{Now,} \qquad t_n = ar^{n-1}$$

$$\therefore \qquad t_4 = 375\, r^3 \implies 375\, r^3 = 192$$

$$r^3 = \frac{192}{375} \implies r^3 = \frac{64}{125}$$

$$r^3 = \left(\frac{4}{5}\right)^3 \implies r = \frac{4}{5}, \text{ which is the required common ratio.}$$

$$\text{Now,} \qquad S_n = a\left[\frac{r^n - 1}{r-1}\right] \text{ if } r \neq 1$$

$$\text{Thus,} \qquad S_{14} = \frac{375\left[\left(\frac{4}{5}\right)^{14} - 1\right]}{\frac{4}{5} - 1} = (-1)\times 5 \times 375\left[\left(\frac{4}{5}\right)^{14} - 1\right]$$

$$= (375)(5)\left[1 - \left(\frac{4}{5}\right)^{14}\right] = 1875\left[1 - \left(\frac{4}{5}\right)^{14}\right].$$

Note

In the above example, one can use $S_n = a\left[\frac{1-r^n}{1-r}\right]$ if $r \neq 1$ instead of $S_n = a\left[\frac{r^n - 1}{r-1}\right]$ if $r \neq 1$.

Example 2.26

A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 8 and the sum of the last two terms is 72. Find the series.

Solution Let the sum of the four terms of the geometric series be $a + ar + ar^2 + ar^3$ and $r > 0$

Given that $a + ar = 8$ and $ar^2 + ar^3 = 72$

$$\text{Now,} \qquad ar^2 + ar^3 = r^2(a + ar) = 72$$

$$\implies \qquad r^2(8) = 72 \quad \therefore \quad r = \pm 3$$

Since $r > 0$, we have $r = 3$.

$$\text{Now,} \qquad a + ar = 8 \implies a = 2$$

Thus, the geometric series is $2 + 6 + 18 + 54$.

Example 2.27

Find the sum to n terms of the series $6 + 66 + 666 + \cdots$

Solution Note that the given series is not a geometric series.

We need to find $S_n = 6 + 66 + 666 + \cdots$ to n terms

$$S_n = 6(1 + 11 + 111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{6}{9}(9 + 99 + 999 + \cdots \text{ to } n \text{ terms}) \quad \text{(Multiply and divide by 9)}$$

$$= \frac{2}{3}[(10-1) + (100-1) + (1000-1) + \cdots \text{ to } n \text{ terms}]$$

$$= \frac{2}{3}[(10 + 10^2 + 10^3 + \cdots n \text{ terms}) - n]$$

$$\text{Thus,} \qquad S_n = \frac{2}{3}\left[\frac{10(10^n - 1)}{9} - n\right].$$

Example 2.28

An organisation plans to plant saplings in 25 streets in a town in such a way that one sapling for the first street, two for the second, four for the third, eight for the fourth street and so on. How many saplings are needed to complete the work?

Solution The number of saplings to be planted for each of the 25 streets in the town forms a G.P. Let S_n be the total number of saplings needed.

Then, $S_n = 1+2+4+8+16+\cdots$ to 25 terms.

Here, $a = 1,\quad r = 2,\quad n = 25$

$$S_n = a\left[\frac{r^n-1}{r-1}\right]$$

$$S_{64} = (1)\,\frac{[2^{25}-1]}{2-1}$$

$$= 2^{25}-1$$

Thus, the number of saplings to be needed is $2^{25}-1$.

Exercise 2.5

1. Find the sum of the first 20 terms of the geometric series $\frac{5}{2}+\frac{5}{6}+\frac{5}{18}+\cdots$.

2. Find the sum of the first 27 terms of the geometric series $\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$.

3. Find S_n for each of the geometric series described below.

   (i) $a = 3,\ t_8 = 384,\ n = 8$. (ii) $a = 5,\ r = 3,\ n = 12$.

4. Find the sum of the following finite series

   (i) $1+0.1+0.01+0.001+\cdots+(0.1)^9$ (ii) $1+11+111+\cdots$ to 20 terms.

5. How many consecutive terms starting from the first term of the series

   (i) $3+9+27+\cdots$ would sum to 1092? (ii) $2+6+18+\cdots$ would sum to 728?

6. The second term of a geometric series is 3 and the common ratio is $\frac{4}{5}$. Find the sum of first 23 consecutive terms in the given geometric series.

7. A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 9 and sum of the last two terms is 36. Find the series.

8. Find the sum of first n terms of the series

   (i) $7+77+777+\cdots$. (ii) $0.4+0.94+0.994+\cdots$.

9. Suppose that five people are ill during the first week of an epidemic, and each sick person spreads the contagious disease to four other people by the end of the second week. By the end of 15th week, how many people will be affected by the epidemic?

10. A gardener wanted to reward a boy for his good deeds by giving some mangoes. He gave the boy two choices. He could either have 1000 mangoes at once or he could get 1 mango on the first day, 2 on the second day, 4 on the third day, 8 mangoes on the fourth day and so on for ten days. Which option should the boy choose to get the maximum number of mangoes?

11. A geometric series consists of even number of terms. The sum of all terms is 3 times the sum of odd terms. Find the common ratio.

12. If S_1, S_2 and S_3 are the sum of first n, $2n$ and $3n$ terms of a geometric series respectively, then prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

Remarks

The sum of the first n terms of a geometric series with $a = 1$ and common ratio $x \neq 1$, is given by $1 + x + x^2 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$, $\quad x \neq 1$.

Note that the left hand side of the above equation is a special polynomial in x of degree $n - 1$. This formula will be useful in finding the sum of some series.

2.5.3 Special series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$

We have already used the symbol Σ for summation.

Let us list out some examples of finite series represented by sigma notation.

Sl. No.	Notation	Expansion
1.	$\sum_{k=1}^{n} k$ or $\sum_{j=1}^{n} j$	$1 + 2 + 3 + \cdots + n$
2.	$\sum_{n=2}^{6} (n-1)$	$1 + 2 + 3 + 4 + 5$
3.	$\sum_{d=0}^{5} (d+5)$	$5 + 6 + 7 + 8 + 9 + 10$
4.	$\sum_{k=1}^{n} k^2$	$1^2 + 2^2 + 3^3 + \cdots + n^2$
5.	$\sum_{k=1}^{10} 3 = 3\sum_{k=1}^{10} 1$	$3[1 + 1 + \cdots 10 \text{ terms}] = 30.$

We have derived that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. This can also be obtained using A.P. with $a = 1$, $d = 1$ and $l = n$ as $S_n = \dfrac{n}{2}(a + l) = \dfrac{n}{2}(1 + n)$.

Hence, using sigma notation we write it as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Let us derive the formulae for

$$\text{(i)} \quad \sum_{k=1}^{n}(2k-1), \qquad \text{(ii)} \quad \sum_{k=1}^{n}k^2 \quad \text{and} \quad \text{(iii)} \quad \sum_{k=1}^{n}k^3.$$

Proof:

(i) Let us find $\sum_{k=1}^{n}(2k-1) = 1 + 3 + 5 + \cdots + (2n-1)$.

This is an A.P. consisting of n terms with $a = 1,\ d = 2,\ l = (2n-1)$.

$$\therefore \quad S_n = \frac{n}{2}(1 + 2n - 1) = n^2 \qquad \left(S_n = \frac{n}{2}(a+l)\right)$$

Thus, $\sum_{k=1}^{n}(2k-1) = n^2$ (1)

Remarks

1. The formula (1) can also be obtained by the following method

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}2k - \sum_{k=1}^{n}1 = 2\left(\sum_{k=1}^{n}k\right) - n = \frac{2(n)(n+1)}{2} - n = n^2.$$

2. From (1), $1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2$, since $l = 2n - 1 \Longrightarrow n = \frac{l+1}{2}$.

(ii) We know that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$$\therefore \quad k^3 - (k-1)^3 = k^2 + k(k-1) + (k-1)^2 \quad (\text{ take } a = k \text{ and } b = k-1)$$

$$\Longrightarrow \quad k^3 - (k-1)^3 = 3k^2 - 3k + 1 \qquad (1)$$

When $k = 1$, $\quad 1^3 - 0^3 = 3(1)^2 - 3(1) + 1$

When $k = 2$, $\quad 2^3 - 1^3 = 3(2)^2 - 3(2) + 1$

When $k = 3$, $\quad 3^3 - 2^3 = 3(3)^2 - 3(3) + 1$. Continuing this, we have

when $k = n$, $\quad n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$.

Adding the above equations corresponding to $k = 1, 2, \cdots, n$ column-wise, we obtain

$$n^3 = 3[1^2 + 2^2 + \cdots + n^2] - 3[1 + 2 + \cdots + n] + n$$

Thus, $3[1^2 + 2^2 + \cdots + n^2] = n^3 + 3[1 + 2 + \cdots + n] - n$

$$3\left[\sum_{k=1}^{n}k^2\right] = n^3 + \frac{3n(n+1)}{2} - n$$

Hence, $$\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (2)$$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots n^3$

Let us observe the following pattern.

$$1^3 = 1 = (1)^2$$
$$1^3 + 2^3 = 9 = (1+2)^2$$
$$1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2.$$

Extending this pattern to n terms, we get

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = [1 + 2 + 3 + \cdots + n]^2$$
$$= \left[\frac{n(n+1)}{2}\right]^2$$

Thus, $$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left[\frac{n(n+1)}{2}\right]^2. \qquad (3)$$

(i) The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(ii) The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$.

(iii) The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

(iv) The sum of squares of first n natural numbers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(v) The sum of cubes of the first n natural numbers,

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Example 2.29

Find the sum of the following series

(i) $26 + 27 + 28 + \cdots + 60$ (ii) $1 + 3 + 5 + \cdots$ to 25 terms (ii) $31 + 33 + \cdots + 53$.

Solution

(i) We have $26 + 27 + 28 + \cdots + 60 = (1 + 2 + 3 + \cdots + 60) - (1 + 2 + 3 + \cdots + 25)$

$$= \sum_{1}^{60} n - \sum_{1}^{25} n$$
$$= \frac{60(60+1)}{2} - \frac{25(25+1)}{2}$$
$$= (30 \times 61) - (25 \times 13) = 1830 - 325 = 1505.$$

(ii) Here $n = 25$

$\therefore$ $1 + 3 + 5 + \cdots$ to 25 terms $= 25^2$ $\quad\left(\sum_{k=1}^{n}(2k-1) = n^2 \right)$

$= 625.$

(iii) $31 + 33 + \cdots + 53$

$$= (1 + 3 + 5 + \cdots + 53) - (1 + 3 + 5 + \cdots + 29)$$

$$= \left(\frac{53+1}{2}\right)^2 - \left(\frac{29+1}{2}\right)^2 \qquad \left(1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2 \right)$$

$$= 27^2 - 15^2 = 504.$$

Example 2.30

Find the sum of the following series

(i) $1^2 + 2^2 + 3^2 + \cdots + 25^2$ (ii) $12^2 + 13^2 + 14^2 + \cdots + 35^2$

(iii) $1^2 + 3^2 + 5^2 + \cdots + 51^2$.

Solution

(i) Now, $1^2 + 2^2 + 3^2 + \cdots + 25^2 = \sum_{1}^{25} n^2$

$$= \frac{25(25+1)(50+1)}{6} \qquad \left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \right)$$

$$= \frac{(25)(26)(51)}{6}$$

$\therefore$ $1^2 + 2^2 + 3^2 + \cdots + 25^2 = 5525.$

(ii) Now, $12^2 + 13^2 + 14^2 + \cdots + 35^2$

$$= (1^2 + 2^2 + 3^2 + \cdots + 35^2) - (1^2 + 2^2 + 3^2 + \cdots + 11^2)$$

$$= \sum_{1}^{35} n^2 - \sum_{1}^{11} n^2$$

$$= \frac{35(35+1)(70+1)}{6} - \frac{11(12)(23)}{6}$$

$$= \frac{(35)(36)(71)}{6} - \frac{(11)(12)(23)}{6}$$

$$= 14910 - 506 = 14404.$$

(iii) Now, $1^2 + 3^2 + 5^2 + \cdots + 51^2$

$$= (1^2 + 2^2 + 3^2 + \cdots + 51^2) - (2^2 + 4^2 + 6^2 + \cdots 50^2)$$

$$= \sum_{1}^{51} n^2 - 2^2[1^2 + 2^2 + 3^2 + \cdots + 25^2]$$

$$= \sum_{1}^{51} n^2 - 4\sum_{1}^{25} n^2$$

$$= \frac{51(51+1)(102+1)}{6} - 4 \times \frac{25(25+1)(50+1)}{6}$$

$$= \frac{(51)(52)(103)}{6} - 4 \times \frac{25(26)(51)}{6}$$

$$= 45526 - 22100 = 23426.$$

Example 2.31

Find the sum of the series.

(i) $1^3 + 2^3 + 3^3 + \cdots + 20^3$ (ii) $11^3 + 12^3 + 13^3 + \cdots + 28^3$

Solution

(i) $$1^3 + 2^3 + 3^3 + \cdots 20^3 = \sum_{1}^{20} n^3$$

$$= \left(\frac{20(20+1)}{2}\right)^2 \qquad \text{using } \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$$= \left(\frac{20 \times 21}{2}\right)^2 = (210)^2 = 44100.$$

(ii) Next we consider $11^3 + 12^3 + \cdots + 28^3$

$$= (1^3 + 2^3 + 3^3 + \cdots + 28^3) - (1^3 + 2^3 + \cdots + 10^3)$$

$$= \sum_{1}^{28} n^3 - \sum_{1}^{10} n^3$$

$$= \left[\frac{28(28+1)}{2}\right]^2 - \left[\frac{10(10+1)}{2}\right]^2$$

$$= 406^2 - 55^2 = (406+55)(406-55)$$

$$= (461)(351) = 161811.$$

Example 2.32

Find the value of k, if $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

Solution Note that k is a positive integer.

Given that $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

$$\Longrightarrow \left(\frac{k(k+1)}{2}\right)^2 = 4356 = 6 \times 6 \times 11 \times 11$$

Taking square root, we get $\dfrac{k(k+1)}{2} = 66$

$$\Longrightarrow k^2 + k - 132 = 0 \quad \Longrightarrow (k+12)(k-11) = 0$$

Thus, $k = 11$, since k is positive.

Example 2.33

(i) If $1 + 2 + 3 + \cdots + n = 120$, find $1^3 + 2^3 + 3^3 + \cdots n^3$.

(ii) If $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$, then find $1 + 2 + 3 + \cdots + n$.

Solution

(i) Given $1 + 2 + 3 + \cdots + n = 120$ i.e. $\frac{n(n+1)}{2} = 120$

$$\therefore \quad 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = 120^2 = 14400$$

(ii) Given $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$

$$\Longrightarrow \quad \left(\frac{n(n+1)}{2}\right)^2 = 36100 = 19 \times 19 \times 10 \times 10$$

$$\Longrightarrow \quad \frac{n(n+1)}{2} = 190$$

Thus, $1 + 2 + 3 + \cdots + n = 190$.

Example 2.34

Find the total area of 14 squares whose sides are 11 cm, 12 cm, $\cdots$, 24 cm, respectively.

Solution The areas of the squares form the series $11^2 + 12^2 + \cdots + 24^2$

Total area of 14 squares $= 11^2 + 12^2 + 13^2 + \cdots + 24^2$

$$= (1^2 + 2^2 + 3^3 + \cdots + 24^2) - (1^2 + 2^2 + 3^3 + \cdots + 10^2)$$

$$= \sum_1^{24} n^2 - \sum_1^{10} n^2$$

$$= \frac{24(24+1)(48+1)}{6} - \frac{10(10+1)(20+1)}{6}$$

$$= \frac{(24)(25)(49)}{6} - \frac{(10)(11)(21)}{6}$$

$$= 4900 - 385$$

$= 4515$ sq. cm.

Exercise 2.6

1. Find the sum of the following series.

(i) $1 + 2 + 3 + \cdots + 45$ (ii) $16^2 + 17^2 + 18^2 + \cdots + 25^2$

(iii) $2 + 4 + 6 + \cdots + 100$ (iv) $7 + 14 + 21 \cdots + 490$

(v) $5^2 + 7^2 + 9^2 + \cdots + 39^2$ (vi) $16^3 + 17^3 + \cdots + 35^3$

2. Find the value of k if

(i) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 6084$ (ii) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 2025$

3. If $1 + 2 + 3 + \cdots + p = 171$, then find $1^3 + 2^3 + 3^3 + \cdots + p^3$.

4. If $1^3 + 2^3 + 3^3 + \cdots + k^3 = 8281$, then find $1 + 2 + 3 + \cdots + k$.

5. Find the total area of 12 squares whose sides are 12cm, 13cm, $\cdots$, 23cm. respectively.

6. Find the total volume of 15 cubes whose edges are 16cm, 17cm, 18cm, $\cdots$, 30cm respectively.

Exercise 2.7

Choose the correct answer.

1. Which one of the following is not true?

(A) A sequence is a real valued function defined on $\mathbb{N}$.

(B) Every function represents a sequence.

(C) A sequence may have infinitely many terms.

(D) A sequence may have a finite number of terms.

2. The 8^{th} term of the sequence 1, 1, 2, 3, 5, 8, $\cdots$ is

(A) 25 (B) 24 (C) 23 (D) 21

3. The next term of $\frac{1}{20}$ in the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$ is

(A) $\frac{1}{24}$ (B) $\frac{1}{22}$ (C) $\frac{1}{30}$ (D) $\frac{1}{18}$

4. If a, b, c, l, m are in A.P, then the value of $a - 4b + 6c - 4l + m$ is

(A) 1 (B) 2 (C) 3 (D) 0

5. If a, b, c are in A.P. then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{c}$ (C) $\frac{a}{c}$ (D) 1

6. If the n^{th} term of a sequence is $100n+10$, then the sequence is

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor G.P.

7. If $a_1, a_2, a_3, \cdots$ are in A.P. such that $\frac{a_4}{a_7} = \frac{3}{2}$, then the 13^{th} term of the A.P. is

(A) $\frac{3}{2}$ (B) 0 (C) $12a_1$ (D) $14a_1$

8. If the sequence $a_1, a_2, a_3, \cdots$ is in A.P. , then the sequence $a_5, a_{10}, a_{15}, \cdots$ is

(A) a G.P. (B) an A.P. (C) neither A.P nor G.P. (D) a constant sequence

9. If $k+2$, $4k-6$, $3k-2$ are the three consecutive terms of an A.P, then the value of k is

(A) 2 (B) 3 (C) 4 (D) 5

10. If $a, b, c, l, m. n$ are in A.P., then $3a+7, 3b+7, 3c+7, 3l+7, 3m+7, 3n+7$ form

(A) a G.P. (B) an A.P. (C) a constant sequence (D) neither A.P. nor G.P

11. If the third term of a G.P is 2, then the product of first 5 terms is

(A) 5^2 (B) 2^5 (C) 10 (D) 15

12. If a, b, c are in G.P, then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{b}{c}$ (D) $\frac{c}{b}$

13. If $x,\ 2x+2,\ 3x+3,\cdots$ are in G.P, then $5x,\ 10x+10,\ 15x+15,\cdots$ form

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor a G.P.

14. The sequence $-3, -3, -3,\cdots$ is

(A) an A.P. only (B) a G.P. only (C) neither A.P. nor G.P (D) both A.P. and G.P.

15. If the product of the first four consecutive terms of a G.P is 256 and if the common ratio is 4 and the first term is positive, then its 3rd term is

(A) 8 (B) $\frac{1}{16}$ (C) $\frac{1}{32}$ (D) 16

16. In a G.P, $t_2 = \frac{3}{5}$ and $t_3 = \frac{1}{5}$. Then the common ratio is

(A) $\frac{1}{5}$ (B) $\frac{1}{3}$ (C) 1 (D) 5

17. If $x \neq 0$, then $1 + \sec x + \sec^2 x + \sec^3 x + \sec^4 x + \sec^5 x$ is equal to

(A) $(1+\sec x)(\sec^2 x + \sec^3 x + \sec^4 x)$ (B) $(1+\sec x)(1+\sec^2 x + \sec^4 x)$

(C) $(1-\sec x)(\sec x + \sec^3 x + \sec^5 x)$ (D) $(1+\sec x)(1+\sec^3 x + \sec^4 x)$

18. If the n^{th} term of an A.P. is $t_n = 3 - 5n$, then the sum of the first n terms is

(A) $\frac{n}{2}[1-5n]$ (B) $n(1-5n)$ (C) $\frac{n}{2}(1+5n)$ (D) $\frac{n}{2}(1+n)$

19. The common ratio of the G.P. $a^{m-n},\ a^m,\ a^{m+n}$ is

(A) a^m (B) a^{-m} (C) a^n (D) a^{-n}

20. If $1 + 2 + 3 + \ldots + n = k$ then $1^3 + 2^3 + \cdots + n^3$ is equal to

(A) k^2 (B) k^3 (C) $\frac{k(k+1)}{2}$ (D) $(k+1)^3$

Points to Remember

- A sequence of real numbers is an arrangement or a list of real numbers in a specific order.
- The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence which is nothing but 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference.

  The formula for the general term of an A.P. is $t_n = a + (n-1)d \quad \forall\ n \in \mathbb{N}$.
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, where $r \neq 0$, $n \in \mathbb{N}$ where r is a constant. Here, a_1 is the first term and the constant r is called the common ratio. The formula for the general term of a G.P. is $t_n = ar^{n-1}$, $n = 1, 2, 3, \cdots$.
- An expression of addition of terms of a sequence is called a series. If the sum consists only finite number of terms, then it is called a finite series. If the sum consists of infinite number of terms of a sequence, then it is called an infinite series.
- The sum S_n of the first n terms of an arithmetic sequence with first term a and common difference d is given by $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$, where l is the last term.
- The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

  where a is the first term and r is the common ratio.
- The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
- The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k - 1) = n^2$
- The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$
- The sum of squares of first n natural numbers, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
- The sum of cubes of the first n natural numbers, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Do you know?

A Mersenne number, named after Marin Mersenne, is a positive integer of the form $M = 2^p - 1$, where p is a positive integer. If M is a prime, then it is called a Mersenne prime. Interestingly, if $2^p - 1$ is prime, then p is prime. The largest known prime number $2^{43,112,609} - 1$ is a Mersenne prime.

3

- Introduction
- Polynomials
- Synthetic Division
- GCD and LCM
- Rational Expressions
- Square root
- Quadratic Equations

Al-Khwarizmi

(780-850)

Arab

Al-Khwarizmi's contribution to Mathematics and Geography established the basis for innovation in Algebra and Trigonometry. He presented the first systematic solution of linear and quadratic equations.

He is considered the founder of algebra. His work on arithmetic was responsible for introducing the Arabic numerals based on the Hindu-Arabic numeral system developed in Indian Mathematics, to the Western world.

ALGEBRA

The human mind has never invented a labour-saving machine equal to algebra - Author unknown

3.1 Introduction

Algebra is an important and a very old branch of mathematics which deals with solving algebraic equations. In third century, the Greek mathematician Diophantus wrote a book "Arithmetic" which contained a large number of practical problems. In the sixth and seventh centuries, Indian mathematicians like Aryabhatta and Brahmagupta have worked on linear equations and quadratic equations and developed general methods of solving them.

The next major development in algebra took place in ninth century by Arab mathematicians. In particular, Al-Khwarizmi's book entitled "Compendium on calculation by completion and balancing" was an important milestone. There he used the word aljabra - which was latinized into algebra - translates as competition or restoration. In the 13th century, Leonardo Fibonacci's books on algebra was important and influential. Other highly influential works on algebra were those of the Italian mathematician Luca Pacioli (1445-1517), and of the English mathematician Robert Recorde (1510-1558). In later centuries Algebra blossomed into more abstract and in 19th century British mathematicians took the lead in this effort. Peacock (Britain, 1791-1858) was the founder of axiomatic thinking in arithmetic and algebra. For this reason he is sometimes called the "Euclid of Algebra". DeMorgan (Britain, 1806-1871) extended Peacock's work to consider operations defined on abstract symbols.

In this chapter, we shall focus on learning techniques of solving linear system of equations and quadratic equations.

3.2 System of linear equations in two unknowns

In class IX, we have studied the linear equation $ax + b = 0$, $a \neq 0$, in one unknown x.

Let us consider a linear equation $ax + by = c$, where at least one of a and b is non-zero, in two unknowns x and y. An ordered pair (x_0, y_0) is called a solution to the linear equation if the values $x = x_0$, $y = y_0$ satisfy the equation.

Geometrically, the graph of the linear equation $ax + by = c$ is a straight line in a plane. So each point (x, y) on this line corresponds to a solution of the equation $ax + by = c$. Conversely, every solution (x, y) of the equation is a point on this straight line. Thus, the equation $ax + by = c$ has infinitely many solutions.

A set of finite number of linear equations in two unknowns x and y that are to be treated together, is called a system of linear equations in x and y. Such a system of equations is also called simultaneous equations.

Definition

An ordered pair (x_0, y_0) is called a solution to a linear system in two variables if the values $x = x_0$, $y = y_0$ satisfy all the equations in the system.

A system of linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

in two variables is said to be

(i) consistent if at least one pair of values of x and y satisfies both equations and

(ii) inconsistent if there are no values of x and y that satisfy both equations.

In this section, we shall discuss only a pair of linear equations in two variables.

Remarks

(i) An equation of the form $ax + by = c$ is called linear because the variables are only to the first power, and there are no products of variables in the equation.

(ii) It is also possible to consider linear systems in more than two variables. You will learn this in higher classes.

Let us consider a linear system

$$a_1x + b_1y = c_1 \qquad (1)$$
$$a_2x + b_2y = c_2 \qquad (2)$$

in two variables x and y, where any of the constants a_1, b_1, a_2 and b_2 can be zero with the exception that each equation must have at least one variable in it or simply,

$$a_1^2 + b_1^2 \neq 0 \ , \ a_2^2 + b_2^2 \neq 0.$$

Geometrically the following situations occur. The two straight lines represented by (1) and (2)

(i) may intersect at exactly one point

(ii) may not intersect at any point

(iii) may coincide.

If (i) happens, then the intersecting point gives the unique solution of the system. If (ii) happens, then the system does not have a solution. If (iii) happens, then every point on the line corresponds to a solution to the system. Thus, the system will have infinitely many solutions in this case.

Now, we will solve a system of linear equations in two unknowns using the following algebraic methods (i) the method of elimination (ii) the method of cross multiplication.

3.2.1 Elimination method

In this method, we may combine equations of a system in such a manner as to get rid of one of the unknowns. The elimination of one unknown can be achieved in the following ways.

(i) Multiply or divide the members of the equations by such numbers as to make the coefficients of the unknown to be eliminated numerically equal.

(ii) Then, eliminate by addition if the resulting coefficients have unlike signs and by subtraction if they have like signs.

Example 3.1

Solve $3x - 5y = -16$, $2x + 5y = 31$

Solution The given equations are

$$3x - 5y = -16 \qquad (1)$$

$$2x + 5y = 31 \qquad (2)$$

Note that the coefficients of y in both equations are numerically equal.

So, we can eliminate y easily.

Adding (1) and (2), we obtain an equation

$$5x = 15 \qquad (3)$$

That is, $x = 3$.

Now, we substitute $x = 3$ in (1) or (2) to solve for y.

Substituting $x = 3$ in (1) we obtain, $3(3) - 5y = -16$

$$\Longrightarrow \quad y = 5.$$

Now, (3, 5) a is solution to the given system because (1) and (2) are true when $x = 3$ and $y = 5$ as from (1) and (2) we get, $3(3) - 5(5) = -16$ and $2(3) + 5(5) = 31$.

Note

Obtaining equation (3) in only one variable is an important step in finding the solution. We obtained equation (3) in one variable x by eliminating the variable y. So this method of solving a system by eliminating one of the variables first, is called "method of elimination".

Example 3.2

The cost of 11 pencils and 3 erasers is ₹ 50 and the cost of 8 pencils and 3 erasers is ₹ 38. Find the cost of each pencil and each eraser.

Solution Let x denote the cost of a pencil in rupees and y denote the cost of an eraser in rupees.

Then according to the given information we have

$$11x + 3y = 50 \quad (1)$$

$$8x + 3y = 38 \quad (2)$$

Subtracting (2) from (1) we get, $3x = 12$ which gives $x = 4$.

Now substitute $x= 4$ in (1) to find the value of y. We get,

$$11(4) + 3y = 50 \qquad \text{i.e.,} \quad y = 2.$$

Therefore, $x = 4$ and $y = 2$ is the solution of the given pair of equations.

Thus, the cost of a pencil is ₹ 4 and that of an eraser is ₹ 2.

Note

It is always better to check that the obtained values satisfy the both equations.

Example 3.3

Solve by elimination method $3x + 4y = -25$, $2x - 3y = 6$

Solution The given system is

$$3x + 4y = -25 \quad (1)$$

$$2x - 3y = 6 \quad (2)$$

To eliminate the variable x, let us multiply (1) by 2 and (2) by –3 to obtain

$$(1) \times 2 \implies 6x + 8y = -50 \quad (3)$$

$$(2) \times -3 \implies -6x + 9y = -18 \quad (4)$$

Now, adding (3) and (4) we get, $17y = -68$ which gives $y = -4$

Next, substitute $y = -4$ in (1) to obtain

$$3x + 4(-4) = -25$$

That is, $x = -3$

Hence, the solution is $(-3, -4)$.

Remarks

In Example 3.3, it is not possible to eliminate one of the variables by simply adding or subtracting the given equations as we did in Example 3.1. Thus, first we shall do some manipulations so that coefficients of either x or y are equal except for sign. Then we do the elimination.

Example 3.4

Using elimination method, solve $101x + 99y = 499, \quad 99x + 101y = 501$

Solution The given system of equations is

$$101x + 99y = 499 \qquad (1)$$

$$99x + 101y = 501 \qquad (2)$$

Here, of course we could multiply equations by appropriate numbers to eliminate one of the variables.

However, note that the coefficient of x in one equation is equal to the coefficient of y in the other equation. In such a case, we add and subtract the two equations to get a new system of very simple equations having the same solution.

Adding (1) and (2), we get $\quad 200x + 200y = 1000.$

Dividing by 200 we get, $\quad x + y = 5 \qquad (3)$

Subtracting (2) from (1), we get $\quad 2x - 2y = -2$ which is same as

$$x - y = -1 \qquad (4)$$

Solving (3) and (4), we get $\quad x = 2, \; y = 3.$

Thus, the required solution is (2, 3).

Example 3.5

Solve $3(2x + y) = 7xy; \quad 3(x + 3y) = 11xy$ using elimination method

Solution The given system of equations is

$$3(2x + y) = 7xy \qquad (1)$$

$$3(x + 3y) = 11xy \qquad (2)$$

Observe that the given system is not linear because of the occurrence of xy term.

Also, note that if $x = 0$, then $y = 0$ and vice versa. So, (0, 0) is a solution for the system and any other solution would have both $x \neq 0$ and $y \neq 0$.

Thus, we consider the case where $x \neq 0, \; y \neq 0$.

Dividing both sides of each equation by xy, we get

$$\frac{6}{y} + \frac{3}{x} = 7, \quad \text{i.e.,} \quad \frac{3}{x} + \frac{6}{y} = 7 \qquad (3)$$

and

$$\frac{9}{x} + \frac{3}{y} = 11 \qquad (4)$$

Let $\quad a = \frac{1}{x}$ and $\quad b = \frac{1}{y}$.

Equations (3) and (4) become

$$3a + 6b = 7 \qquad (5)$$

$$9a + 3b = 11 \qquad (6)$$

which is a linear system in a and b.

To eliminate b, we have (6) $\times 2 \Longrightarrow \quad 18a + 6b = 22$ (7)

Subtracting (7) from (5) we get, $\quad -15a = -15$. That is, $a = 1$.

Substituting $a = 1$ in (5) we get, $\quad b = \frac{2}{3}$. Thus, $a = 1$ and $b = \frac{2}{3}$.

When $a = 1$, we have $\frac{1}{x} = 1$. $\quad$ Thus, $x = 1$.

When $b = \frac{2}{3}$, we have $\frac{1}{y} = \frac{2}{3}$. $\quad$ Thus, $y = \frac{3}{2}$.

Thus, the system has two solutions $(1, \frac{3}{2})$ and $(0, 0)$.

Aliter

The given system of equations can also be solved in the following way.

Now, $\quad 3(2x + y) = 7xy$ (1)

$3(x + 3y) = 11xy$ (2)

Now, (2) × 2 – (1) $\Longrightarrow \quad 15y = 15xy$

$\Longrightarrow \quad 15y(1-x) = 0$. Thus, $x = 1$ and $y = 0$

When $x = 1$, we have $y = \frac{3}{2}$ and when $y = 0$, we have $x = 0$

Hence, the two solutions are $(1, \frac{3}{2})$ and $(0, 0)$.

Exercise 3.1

Solve each of the following system of equations by elimination method.

1. $x + 2y = 7, \quad x - 2y = 1$
2. $3x + y = 8, \quad 5x + y = 10$
3. $x + \frac{y}{2} = 4, \quad \frac{x}{3} + 2y = 5$
4. $11x - 7y = xy, \quad 9x - 4y = 6xy$
5. $\frac{3}{x} + \frac{5}{y} = \frac{20}{xy}, \frac{2}{x} + \frac{5}{y} = \frac{15}{xy}, x \neq 0, y \neq 0$
6. $8x - 3y = 5xy, \quad 6x - 5y = -2xy$
7. $13x + 11y = 70, \quad 11x + 13y = 74$
8. $65x - 33y = 97, \quad 33x - 65y = 1$
9. $\frac{15}{x} + \frac{2}{y} = 17, \frac{1}{x} + \frac{1}{y} = \frac{36}{5}, x \neq 0, y \neq 0$
10. $\frac{2}{x} + \frac{2}{3y} = \frac{1}{6}, \frac{3}{x} + \frac{2}{y} = 0, x \neq 0, y \neq 0$

Cardinality of the set of solutions of the system of linear equations

Let us consider the system of two equations

$$a_1x + b_1y + c_1 = 0 \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad (2)$$

where the coefficients are real numbers such that $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$.

Let us apply the elimination method for equating the coefficients of y.

Now, multiply equation (1) by b_2 and equation (2) by b_1, we get,

$$b_2a_1x + b_2b_1y + b_2c_1 = 0 \qquad (3)$$

$$b_1a_2x + b_1b_2y + b_1c_2 = 0 \qquad (4)$$

Subtracting equation (4) from (3), we get

$$(b_2a_1 - b_1a_2)x = b_1c_2 - b_2c_1 \implies x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{provided } a_1b_2 - a_2b_1 \neq 0$$

Substituting the value of x in either (1) or (2) and solving for y, we get

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

Thus, we have

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1},\ a_1b_2 - a_2b_1 \neq 0. \qquad (5)$$

Here, we have to consider two cases.

***Case* (i)** $a_1b_2 - a_2b_1 \neq 0$. That is, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

In this case, the pair of linear equations has a unique solution.

***Case* (ii)** $a_1b_2 - a_2b_1 = 0$. That is, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ if $a_2 \neq 0$ and $b_2 \neq 0$.

In this case, let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$. Then $a_1 = \lambda a_2$, $b_1 = \lambda b_2$

Now, substituting the values of a_1 and b_1 in equation (1) we get,

$$\lambda(a_2x + b_2y) + c_1 = 0 \qquad (6)$$

It is easily observed that both the equations (6) and (2) can be satisfied only if

$$c_1 = \lambda c_2 \implies \frac{c_1}{c_2} = \lambda$$

If $c_1 = \lambda c_2$, any solution of equation (2) will also satisfy the equation (1) and vice versa.

So, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$; then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If $c_1 \neq \lambda c_2$, then any solution of equation (1) will not satisfy equation (2) and vice versa.

Hence, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of linear equations given by (1) and (2) has no solution.

Note

Now, we summarise the above discussion.

For the system of equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0, \text{ where } a_1^2 + b_1^2 \neq 0\ ,\ a_2^2 + b_2^2 \neq 0.$$

(i) If $a_1b_2 - b_1a_2 \neq 0$ or $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, then the system of equations has a unique solution.

(ii) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, then the system of equations has infinitely many solutions.

(iii) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$, then the system of equations has no solution.

3.2.2 Cross multiplication method

While solving a pair of linear equations in two unknowns x and y using elimination method, we utilised the coefficients effectively to get the solution. There is another method called the cross multiplication method, which simplifies the procedure. Now, let us describe this method and see how it works.

Let us consider the system

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

$$a_2x + b_2y + c_2 = 0 \text{ with } a_1b_2 - b_1a_2 \neq 0 \qquad (2)$$

We have already established that the system has the solution

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1},\quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Thus, we can write $\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{1}{a_1b_2 - a_2b_1},\quad \dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$

Let us write the above in the following form

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

The following arrow diagram may be very useful in remembering the above relation.

$$\begin{array}{cccc} & x & y & 1 \\ b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

The arrows between the two numbers indicate that they are multiplied, the second product (upward arrow) is to be subtracted from the first product (downward arrow).

Method of solving a linear system of equations by the above form is called the cross multiplication method.

Note that in the representation $\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$,

$b_1c_2 - b_2c_1$ or $c_1a_2 - c_2a_1$ may be equal to 0 but $a_1b_2 - a_2b_1 \neq 0$.

> Hence, for the system of equations $a_1x + b_1y + c_1 = 0$
> $a_2x + b_2y + c_2 = 0$
>
> (i) if $b_1c_2 - b_2c_1 = 0$ and $a_1b_2 - a_2b_1 \neq 0$, then $x = 0$
>
> (ii) if $c_1a_2 - c_2a_1 = 0$ and $a_1b_2 - a_2b_1 \neq 0$, then $y = 0$

Hereafter, we shall mostly restrict ourselves to the system of linear equations having unique solution and find the solution by the method of cross multiplication.

Example 3.6

Solve

$$2x + 7y - 5 = 0$$
$$-3x + 8y = -11$$

Solution The given system of equations is

$$2x + 7y - 5 = 0$$
$$-3x + 8y + 11 = 0$$

For the cross multiplication method, we write the coefficients as

$$\begin{array}{cccccc} & x & & y & & 1 \\ 7 & & -5 & & 2 & & 7 \\ 8 & & 11 & & -3 & & 8 \end{array}$$

Hence, we get $\dfrac{x}{(7)(11)-(8)(-5)} = \dfrac{y}{(-5)(-3)-(2)(11)} = \dfrac{1}{(2)(8)-(-3)(7)}$.

That is, $\dfrac{x}{117} = \dfrac{y}{-7} = \dfrac{1}{37}$. i.e., $x = \dfrac{117}{37}$, $y = -\dfrac{7}{37}$.

Hence, the solution is $\left(\dfrac{117}{37}, -\dfrac{7}{37}\right)$.

Example 3.7

Using cross multiplication method, solve $3x + 5y = 25$
$7x + 6y = 30$

Solution The given system of equations is $3x + 5y - 25 = 0$
$7x + 6y - 30 = 0$

Now, writing the coefficients for cross multiplication, we get

$$\begin{array}{cccccc} & x & & y & & 1 \\ 5 & & -25 & & 3 & & 5 \\ 6 & & -30 & & 7 & & 6 \end{array}$$

$$\Longrightarrow \frac{x}{-150+150} = \frac{y}{-175+90} = \frac{1}{18-35}. \text{ i.e., } \frac{x}{0} = \frac{y}{-85} = \frac{1}{-17}.$$

Thus, we have $x = 0$, $y = 5$. Hence, the solution is $(0, 5)$.

Note

Here, $\frac{x}{0} = -\frac{1}{17}$ is to mean $x = \frac{0}{-17} = 0$. Thus $\frac{x}{0}$ is only a notation and it is not division by zero. It is always true that division by zero is not defined.

Example 3.8

In a two digit number, the digit in the unit place is twice of the digit in the tenth place. If the digits are reversed, the new number is 27 more than the given number. Find the number.

Solution Let x denote the digit in the tenth place and y denote the digit in unit place.. So, the number may be written as $10x + y$ in the expanded form. (just like 35= 10(3) +5)

When the digits are reversed, x becomes the digit in unit place and y becomes the digit in the tenth place. The changed number, in the expanded form is $10y + x$.

According to the first condition, we have $y = 2x$ which is written as

$$2x - y = 0 \qquad (1)$$

Also, by second condition, we have

$$(10y + x) - (10x + y) = 27$$

That is, $-9x + 9y = 27 \Longrightarrow -x + y = 3 \qquad (2)$

Adding equations (1) and (2), we get $x = 3$.

Substituting $x = 3$ in the equation (2), we get $y = 6$.

Thus, the given number is $(3\times10) + 6 = 36$.

Example 3.9

A fraction is such that if the numerator is multiplied by 3 and the denominator is reduced by 3, we get $\frac{18}{11}$, but if the numerator is increased by 8 and the denominator is doubled, we get $\frac{2}{5}$. Find the fraction.

Solution Let the fraction be $\frac{x}{y}$. According to the given conditions, we have

$$\frac{3x}{y-3} = \frac{18}{11} \quad \text{and} \quad \frac{x+8}{2y} = \frac{2}{5}$$

$$\Longrightarrow 11x = 6y - 18 \quad \text{and} \quad 5x + 40 = 4y$$

So, we have $\qquad 11x - 6y + 18 = 0 \qquad (1)$

$\qquad\qquad\qquad 5x - 4y + 40 = 0 \qquad (2)$

On comparing the coefficients of (1) and (2) with $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$,

we have $\quad a_1 = 11,\ b_1 = -6,\ c_1 = 18\ ;\quad a_2 = 5,\ \ b_2 = -4,\ \ c_2 = 40.$

Thus, $\quad a_1b_2 - a_2b_1 = (11)(-4) - (5)(-6) = -14 \neq 0.$

Hence, the system has a unique solution.

Now, writing the coefficients for the cross multiplication, we have

$$\begin{array}{ccccccc} & x & & y & & 1 & \\ -6 & & 18 & & 11 & & -6 \\ -4 & & 40 & & 5 & & -4 \end{array}$$

$$\Longrightarrow \quad \frac{x}{-240+72} = \frac{y}{90-440} = \frac{1}{-44+30}$$

$$\Longrightarrow \quad \frac{x}{-168} = \frac{y}{-350} = \frac{1}{-14}$$

Thus, $\quad x = \dfrac{168}{14} = 12\ ;\quad y = \dfrac{350}{14} = 25.\quad$ Hence, the fraction is $\dfrac{12}{25}$.

Example 3.10

Eight men and twelve boys can finish a piece of work in 10 days while six men and eight boys can finish the same work in 14 days. Find the number of days taken by one man alone to complete the work and also one boy alone to complete the work.

Solution Let x denote the number of days needed for one man to finish the work and y denote the number of days needed for one boy to finish the work. Clearly, $x \neq 0$ and $y \neq 0$.

So, one man can complete $\dfrac{1}{x}$ part of the work in one day and one boy can complete $\dfrac{1}{y}$ part of the work in one day.

The amount of work done by 8 men and 12 boys in one day is $\dfrac{1}{10}$.

Thus, we have $\quad \dfrac{8}{x} + \dfrac{12}{y} = \dfrac{1}{10} \qquad (1)$

The amount of work done by 6 men and 8 boys in one day is $\dfrac{1}{14}$.

Thus, we have $\quad \dfrac{6}{x} + \dfrac{8}{y} = \dfrac{1}{14} \qquad (2)$

Let $\quad a = \dfrac{1}{x}$ and $b = \dfrac{1}{y}$. Then (1) and (2) give, respectively,

$$8a + 12b = \frac{1}{10} \quad \Longrightarrow \quad 4a + 6b - \frac{1}{20} = 0. \qquad (3)$$

$$6a + 8b = \frac{1}{14} \quad \Longrightarrow \quad 3a + 4b - \frac{1}{28} = 0. \qquad (4)$$

Writing the coefficients of (3) and (4) for the cross multiplication, we have

$$\begin{array}{ccccccc} & a & & b & & 1 & \\ 6 & & -\frac{1}{20} & & 4 & & 6 \\ 4 & & -\frac{1}{28} & & 3 & & 4 \end{array}$$

Thus, we have $\dfrac{a}{-\frac{3}{14}+\frac{1}{5}} = \dfrac{b}{-\frac{3}{20}+\frac{1}{7}} = \dfrac{1}{16-18}$. i.e., $\dfrac{a}{-\frac{1}{70}} = \dfrac{b}{-\frac{1}{140}} = \dfrac{1}{-2}$.

That is, $\quad a = \dfrac{1}{140},\ b = \dfrac{1}{280}$

Thus, we have $\quad x = \dfrac{1}{a} = 140\,,\ \ y = \dfrac{1}{b} = 280.$

Hence, one man can finish the work individually in 140 days and one boy can finish the work individually in 280 days.

Exercise 3.2

1. Solve the following systems of equations using cross multiplication method.

   (i) $3x + 4y = 24,\ 20x - 11y = 47$ (ii) $0.5x + 0.8y = 0.44,\ 0.8x + 0.6y = 0.5$

   (iii) $\dfrac{3x}{2} - \dfrac{5y}{3} = -2,\ \dfrac{x}{3} + \dfrac{y}{2} = \dfrac{13}{6}$ (iv) $\dfrac{5}{x} - \dfrac{4}{y} = -2,\ \dfrac{2}{x} + \dfrac{3}{y} = 13$

2. Formulate the following problems as a pair of equations, and hence find their solutions:

   (i) One number is greater than thrice the other number by 2. If 4 times the smaller number exceeds the greater by 5, find the numbers.

   (ii) The ratio of income of two persons is 9 : 7 and the ratio of their expenditure is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly income.

   (iii) A two digit number is seven times the sum of its digits. The number formed by reversing the digits is 18 less than the given number. Find the given number.

   (iv) Three chairs and two tables cost ₹ 700 and five chairs and three tables cost ₹ 1100. What is the total cost of 2 chairs and 3 tables?

   (v) In a rectangle, if the length is increased and the breadth is reduced each by 2 cm then the area is reduced by $28\ \text{cm}^2$. If the length is reduced by 1 cm and the breadth increased by 2cm , then the area increases by 33cm^2. Find the area of the rectangle.

   (vi) A train travelled a certain distance at a uniform speed. If the train had been 6 km/hr faster, it would have taken 4 hours less than the scheduled time. If the train were slower by 6 km/hr, then it would have taken 6 hours more than the scheduled time. Find the distance covered by the train.

3.3 Quadratic polynomials

A polynomial of degree n in the variable x is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ where $a_0 \neq 0$ and $a_1, a_2, a_3, ..., a_n$ are real constants.

A polynomial of degree two is called a quadratic polynomial and is normally written as $p(x) = ax^2 + bx + c$, where $a \neq 0$, b and c are real constants. Real constants are polynomials of degree zero.

For example, $x^2 + x + 1$, $3x^2 - 1$, $-\frac{3}{2}x^2 + 2x - \frac{7}{3}$ are quadratic polynomials.

The value of a quadratic polynomial $p(x) = ax^2 + bx + c$ at $x = k$ is obtained by replacing x by k in $p(x)$. Thus, the value of $p(x)$ at $x = k$ is $p(k) = ak^2 + bk + c$.

3.3.1 Zeros of a polynomial

Consider a polynomial $p(x)$. If k is a real number such that $p(k) = 0$, then k is called a zero of the polynomial $p(x)$.

For example,

the zeros of the polynomial $q(x) = x^2 - 5x + 6$ are 2 and 3 because $q(2) = 0$ and $q(3) = 0$.

Remarks

A polynomial may not have any zero in real numbers at all. For example, $p(x) = x^2 + 1$ has no zeros in real numbers. That is, there is no real k such that $p(k) = 0$. Geometrically a zero of any polynomial is nothing but the x-coordinate of the point of intersection of the graph of the polynomial and the x-axis if they intersect. (see Fig. 3.1 and Fig. 3.2)

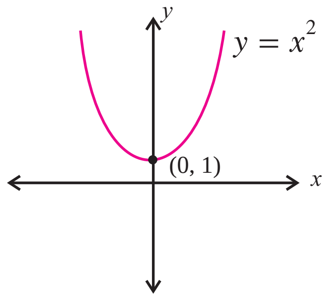


Fig. 3.1

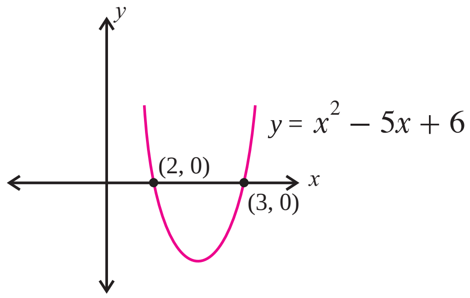


Fig. 3.2

3.3.2 Relationship between zeros and coefficients of a quadratic polynomial

In general, if α and β are the zeros of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then by factor theorem we get, $x - \alpha$ and $x - \beta$ are the factors of $p(x)$.

Therefore, $ax^2 + bx + c = k(x - \alpha)(x - \beta)$, where k is a non zero constant.

$$= k[x^2 - (\alpha + \beta)x + \alpha\beta]$$

Comparing the coefficients of x^2, x and the constant term on both sides, we obtain

$$a = k, \quad b = -k(\alpha+\beta) \quad \text{and} \quad c = k\alpha\beta$$

> The basic relationships between the zeros and the coefficients of $p(x) = ax^2 + bx + c$ are
>
> sum of zeros : $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$.
>
> product of zeros : $\alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$.

Example 3.11

Find the zeros of the quadratic polynomial $x^2 + 9x + 20$, and verify the basic relationships between the zeros and the coefficients.

Solution Let $p(x) = x^2 + 9x + 20 = (x+4)(x+5)$

So, $p(x) = 0 \implies (x+4)(x+5) = 0 \qquad \therefore\ x = -4$ or $x = -5$

Thus, $p(-4) = (-4+4)(-4+5) = 0$ and $p(-5) = (-5+4)(-5+5) = 0$

Hence, the zeros of $p(x)$ are -4 and -5

Thus, sum of zeros $= -9$ and the product of zeros $= 20$ (1)

From the basic relationships, we get

the sum of the zeros $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2} = -\dfrac{9}{1} = -9$ (2)

product of the zeros $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{20}{1} = 20$ (3)

Thus, the basic relationships are verified.

> **Remarks**
>
> To factorize $x^2 + 9x + 20$, one can proceed as follows
>
> 20
>
> 4 5 $\because 4+5=9,\ 4\times5=20$
>
> $\dfrac{4}{1}$ $\dfrac{5}{1}$ constant term
>
> coefficient of x^2
>
> $(x+4)(x+5)$.

> ***Note***
>
> A quadratic polynomial $p(x) = ax^2 + bx + c$ may have atmost two zeros.
>
> Now, for any $a \neq 0$, $a(x^2 - (\alpha+\beta)x + \alpha\beta)$ is a polynomial with zeros α and β. Since we can choose any non zero a, there are infinitely many quadratic polynomials with zeros α and β.

Example 3.12

Find a quadratic polynomial if the sum and product of zeros of it are -4 and 3 respectively.

Solution Let α and β be the zeros of a quadratic polynomial.

Given that $\alpha + \beta = -4$ and $\alpha\beta = 3$.

One of the such polynomials is $p(x) = x^2 - (\alpha+\beta)x + \alpha\beta$

$$= x^2 - (-4)x + 3 = x^2 + 4x + 3$$

Example 3.13

Find a quadratic polynomial with zeros at $x = \frac{1}{4}$ and $x = -1$.

Solution

Let α and β be the zeros of $p(x)$

Using the relationship between zeros and coefficients, we have

$$p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$$
$$= x^2 - \left(\frac{1}{4} - 1\right)x + \left(\frac{1}{4}\right)(-1)$$
$$= x^2 + \frac{3}{4}x - \frac{1}{4}$$

It is a polynomial with zeros $\frac{1}{4}$ and -1.

Aliter The required polynomial is obtained directly as follows:

$$p(x) = \left(x - \frac{1}{4}\right)(x+1)$$
$$= x^2 + \frac{3}{4}x - \frac{1}{4}.$$

Any other polynomial with the desired property is obtained by multiplying $p(x)$ by any non-zero real number.

Note

$4x^2 + 3x - 1$ is also a polynomial with zeros $\frac{1}{4}$ and -1.

Exercise 3.3

1. Find the zeros of the following quadratic polynomials and verify the basic relationships between the zeros and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4x^2 - 4x + 1$ (iii) $6x^2 - 3 - 7x$ (iv) $4x^2 + 8x$

(v) $x^2 - 15$ (vi) $3x^2 - 5x + 2$ (vii) $2x^2 - 2\sqrt{2}x + 1$ (viii) $x^2 + 2x - 143$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeros respectively.

(i) 3, 1 (ii) 2, 4 (iii) 0, 4 (iv) $\sqrt{2}, \frac{1}{5}$

(v) $\frac{1}{3}, 1$ (vi) $\frac{1}{2}, -4$ (vii) $\frac{1}{3}, -\frac{1}{3}$ (viii) $\sqrt{3}, 2$

3.4 Synthetic division

We know that when 29 is divided by 7 we get, 4 as the quotient and 1 as the remainder. Thus, 29 = 4(7) + 1. Similarly one can divide a polynomial $p(x)$ by another polynomial $q(x)$ which results in getting the quotient and remainder such that

$$p(x) = (\text{quotient})q(x) + \text{remainder}$$

That is, $p(x) = s(x)q(x) + r(x)$, where $\deg r(x) < \deg q(x)$.

This is called the Division Algorithm.

If $q(x) = x + a$, then $\deg r(x) = 0$. Thus, $r(x)$ is a constant.

Hence, $p(x) = s(x)(x + a) + r$, where r is a constant.

Now if we put $x = -a$ in the above, we have $p(-a) = s(-a)(-a + a) + r \implies r = p(-a)$.

Thus, if $q(x) = x + a$, then the remainder can be calculated by simply evaluating $p(x)$ at $x = -a$.

Division algorithm :

If $p(x)$ is the dividend and $q(x)$ is the divisor, then by division algorithm we write, $p(x) = s(x)q(x) + r(x)$.

Now, we have the following results.

(i) If $q(x)$ is linear , then $r(x) = r$ is a constant.

(ii) If $\deg q(x) = 1$ (i.e., $q(x)$ is linear), then $\deg p(x) = 1 + \deg s(x)$

(iii) If $p(x)$ is divided by $x + a$, then the remainder is $p(-a)$.

(iv) If $r = 0$, we say $q(x)$ divides $p(x)$ or equivalently $q(x)$ is a factor of $p(x)$.

Remarks

An elegant way of dividing a polynomial by a linear polynomial was introduced by Paolo Ruffin in 1809. His method is known as **synthetic division.** It facilitates the division of a polynomial by a linear polynomial with the help of the coefficients involved.

Paolo Ruffin

(1765-1822, Italy)

Let us explain the method of synthetic division with an example.

Let $p(x) = x^3 + 2x^2 - x - 4$ be the dividend and $q(x) = x + 2$ be the divisor. We shall find the quotient $s(x)$ and the remainder r, by proceeding as follows.

***Step* 1** Arrange the dividend and the divisor according to the descending powers of x and then write the coefficients of dividend in the first row (see figure). Insert 0 for missing terms.

$x^3 + 2x^2 - x - 4$

1 2 –1 –4

***Step* 2** Find out the zero of the divisor.

Step 3 Put 0 for the first entry in the 2nd row.

Complete the entries of the 2nd row and 3rd row as shown below.

–2	1	2	–1	–4
	0	$1\times(-2)$ → –2	$0\times(-2)$ → 0	$-1\times(-2)$ → 2
	1+0 = 1	2+(–2) = 0	–1+0 = –1	–4+2 = –2 ⟵ remainder

Step 4 Write down the quotient and the remainder accordingly. All the entries except the last one in the third row constitute the coefficients of the quotient.

Thus, the quotient is $x^2 - 1$ and the remainder is –2.

Example 3.14

Find the quotient and remainder when $x^3 + x^2 - 7x - 3$ is divided by $x - 3$.

Solution Let $p(x) = x^3 + x^2 - 7x - 3$. The zero of the divisor is 3. So we consider,

$$\begin{array}{c|cccc|l}
3 & 1 & 1 & -7 & -3 & \\
 & 0 & 3 & 12 & 15 & \\
\hline
 & 1 & 4 & 5 & 12 & \longrightarrow \text{Remainder.}
\end{array}$$

$\therefore$ When $p(x)$ is divided by $x - 3$, the quotient is $x^2 + 4x + 5$ and the remainder is 12.

Example 3.15

If the quotient on dividing $2x^4 + x^3 - 14x^2 - 19x + 6$ by $2x + 1$ is $x^3 + ax^2 - bx - 6$. Find the values of a and b, also the remainder.

Solution Let $p(x) = 2x^4 + x^3 - 14x^2 - 19x + 6$.

Given that the divisor is $2x + 1$. Write $2x + 1 = 0$. Then $x = -\frac{1}{2}$

$\therefore$ The zero of the divisor is $-\frac{1}{2}$.

$$\begin{array}{c|ccccc|l}
-\frac{1}{2} & 2 & 1 & -14 & -19 & 6 & \\
 & 0 & -1 & 0 & 7 & 6 & \\
\hline
 & 2 & 0 & -14 & -12 & 12 & \longrightarrow \text{Remainder}
\end{array}$$

So, $2x^4 + x^3 - 14x^2 - 19x + 6 = \left(x + \frac{1}{2}\right)\{2x^3 - 14x - 12\} + 12$

$$= (2x + 1)\frac{1}{2}(2x^3 - 14x - 12) + 12$$

Thus, the quotient is $\frac{1}{2}(2x^3 - 14x - 12) = x^3 - 7x - 6$ and the remainder is 12.

But, given quotient is $x^3 + ax^2 - bx - 6$. Comparing this with the quotient obtained we get, $a = 0$ and $b = 7$. Thus, $a = 0$, $b = 7$ and the remainder is 12.

Exercise 3.4

1. Find the quotient and remainder using synthetic division.
   (i) $(x^3 + x^2 - 3x + 5) \div (x - 1)$ (ii) $(3x^3 - 2x^2 + 7x - 5) \div (x + 3)$
   (iii) $(3x^3 + 4x^2 - 10x + 6) \div (3x - 2)$ (iv) $(3x^3 - 4x^2 - 5) \div (3x + 1)$
   (v) $(8x^4 - 2x^2 + 6x - 5) \div (4x + 1)$ (vi) $(2x^4 - 7x^3 - 13x^2 + 63x - 48) \div (2x - 1)$
2. If the quotient on dividing $x^4 + 10x^3 + 35x^2 + 50x + 29$ by $x + 4$ is $x^3 - ax^2 + bx + 6$, then find a, b and also the remainder.
3. If the quotient on dividing, $8x^4 - 2x^2 + 6x - 7$ by $2x + 1$ is $4x^3 + px^2 - qx + 3$, then find p, q and also the remainder.

3.4.1 Factorization using synthetic division

We have already learnt in class IX, how to factorize quadratic polynomials. In this section, let us learn, how to factorize the cubic polynomial using synthetic division.

If we identify one linear factor of cubic polynomial $p(x)$, then using synthetic division we get the quadratic factor of $p(x)$. Further if possible one can factorize the quadratic factor into two linear factors. Hence the method of synthetic division helps us to factorize a cubic polynomial into linear factors if it can be factorized.

Note

(i) For any polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$.

(ii) $x - a$ is a factor for $p(x)$ if and only if $p(a) = 0$. (Factor theorem)

(iii) $x - 1$ is a factor of $p(x)$ if and only if the sum of coefficients of $p(x)$ is 0.

(iv) $x + 1$ is a factor of $p(x)$ if and only if sum of the coefficients of even powers of x, including constant is equal to sum of the coefficients of odd powers of x.

Example 3.16

(i) Prove that $x - 1$ is a factor of $x^3 - 6x^2 + 11x - 6$.

(ii) Prove that $x + 1$ is a factor of $x^3 + 6x^2 + 11x + 6$.

Solution

(i) Let $p(x) = x^3 - 6x^2 + 11x - 6$.

$p(1) = 1 - 6 + 11 - 6 = 0$. (note that sum of the coefficients is 0)

Thus, $(x - 1)$ is a factor of $p(x)$.

(ii) Let $q(x) = x^3 + 6x^2 + 11x + 6$.

$q(-1) = -1 + 6 - 11 + 6 = 0$. Hence, $x + 1$ is a factor of $q(x)$

Example 3.17

Factorize $2x^3 - 3x^2 - 3x + 2$ into linear factors.

Solution Let $p(x) = 2x^3 - 3x^2 - 3x + 2$

Now, $p(1) = -2 \neq 0$ (note that sum of the coefficients is not zero)

$\therefore$ $(x - 1)$ is not a factor of $p(x)$.

However, $p(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2 = 0$.

So, $x + 1$ is a factor of $p(x)$.

We shall use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -3 & 2 \\ & 0 & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & \boxed{0} \end{array} \longrightarrow \text{Remainder}$$

Thus, $p(x) = (x + 1)(2x^2 - 5x + 2)$

Now, $2x^2 - 5x + 2 = 2x^2 - 4x - x + 2 = (x - 2)(2x - 1)$.

Hence, $2x^3 - 3x^2 - 3x + 2 = (x + 1)(x - 2)(2x - 1)$.

Remarks

To factorize $2x^2 - 5x + 2$, one can proceed as follows

4

$-4 \quad -1$ $\because -4+(-1)=-5,\ -4\times(-1)=4$

$\frac{-4}{2} = \frac{-2}{1}$ $\quad \frac{-1}{2}$ constant term

coefficient of x^2

$(x-2)\,(2x-1)$.

Example 3.18

Factorize $x^3 - 3x^2 - 10x + 24$

Solution Let $p(x) = x^3 - 3x^2 - 10x + 24$.

Since $p(1) \neq 0$ and $p(-1) \neq 0$, neither $x + 1$ nor $x - 1$ is a factor of $p(x)$.

Therefore, we have to search for different values of x by trial and error method.

When $x = 2$, $p(2) = 0$. Thus, $x - 2$ is a factor of $p(x)$.

To find the other factors, let us use the synthetic division.

$$\begin{array}{c|cccc} 2 & 1 & -3 & -10 & 24 \\ & 0 & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \longrightarrow \text{Remainder.}$$

$\therefore$ The other factor is $x^2 - x - 12$.

Now, $x^2 - x - 12 = x^2 - 4x + 3x - 12 = (x - 4)(x + 3)$

Hence, $x^3 - 3x^2 - 10x + 24 = (x - 2)(x + 3)(x - 4)$

Exercise 3.5

1. Factorize each of the following polynomials.

(i) $x^3 - 2x^2 - 5x + 6$	(ii) $4x^3 - 7x + 3$	(iii) $x^3 - 23x^2 + 142x - 120$
(iv) $4x^3 - 5x^2 + 7x - 6$	(v) $x^3 - 7x + 6$	(vi) $x^3 + 13x^2 + 32x + 20$
(vii) $2x^3 - 9x^2 + 7x + 6$	(viii) $x^3 - 5x + 4$	(ix) $x^3 - 10x^2 - x + 10$
(x) $2x^3 + 11x^2 - 7x - 6$	(xi) $x^3 + x^2 + x - 14$	(xii) $x^3 - 5x^2 - 2x + 24$

3.5 Greatest Common Divisor (GCD) and Least Common Multiple (LCM)

3.5.1 Greatest Common Divisor (GCD)

The Highest Common Factor (HCF) or Greatest Common Divisor (GCD) of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

Consider the simple expressions

(i) a^4, a^3, a^5, a^6 (ii) a^3b^4, ab^5c^2, a^2b^7c

In (i), note that a, a^2, a^3 are the divisors of all these expressions. Out of them, a^3 is the divisor with highest power. Therefore a^3 is the GCD of the expressions a^4, a^3, a^5, a^6.

In (ii), similarly, one can easily see that ab^4 is the GCD of a^3b^4, ab^5c^2, a^2b^7c.

If the expressions have numerical coefficients, find their greatest common divisor, and prefix it as a coefficient to the greatest common divisor of the algebraic expressions.

Let us consider a few more examples to understand the greatest common divisor.

Examples 3.19

Find the GCD of the following : (i) 90, 150, 225 (ii) $15x^4y^3z^5$, $12x^2y^7z^2$

(iii) $6(2x^2-3x-2)$, $8(4x^2+4x+1)$, $12(2x^2+7x+3)$

Solution

(i) Let us write the numbers 90, 150 and 225 in the product of their prime factors as

$$90 = 2\times3\times3\times5\ ,\ 150 = 2\times3\times5\times5 \text{ and } 225 = 3\times3\times5\times5$$

From the above 3 and 5 are common prime factors of all the given numbers.

Hence the GCD $= 3\times5 = 15$

(ii) We shall use similar technique to find the GCD of algebraic expressions.

Now let us take the given expressions $15x^4y^3z^5$ and $12x^2y^7z^2$.

Here the common divisors of the given expressions are 3, x^2, y^3 and z^2.

Therefore, GCD $= 3\times x^2\times y^3\times z^2 = 3x^2y^3z^2$

(iii) Given expressions are $6(2x^2-3x-2)$, $8(4x^2+4x+1)$, $12(2x^2+7x+3)$

Now, GCD of 6, 8, 12 is 2

Next let us find the factors of quadratic expressions.

$$2x^2-3x-2 = (2x+1)(x-2)$$

$$4x^2+4x+1 = (2x+1)(2x+1)$$

$$2x^2+7x+3 = (2x+1)(x+3)$$

Common factor of the above quadratic expressions is $(2x+1)$.

Therefore, GCD $= 2(2x+1)$.

3.5.2 Greatest common divisor of polynomials using division algorithm

First let us consider the simple case of finding GCD of 924 and 105.

$924 = 8\times105+84$

$105 = 1\times84+21$,

$84 = 4\times21+0$,

21 is the GCD of 924 and 105

(or)

$$\begin{array}{r|r} & 8 \\ \hline 105 & 924 \\ & 840 \\ \hline & 84 \end{array}\qquad \begin{array}{r|r} & 1 \\ \hline 84 & 105 \\ & 84 \\ \hline & 21 \end{array}\qquad \begin{array}{r|r} & 4 \\ \hline 21 & 84 \\ & 84 \\ \hline & 0 \end{array}$$

Similar technique works with polynomials when they have GCD.

Let $f(x)$ and $g(x)$ be two non constant polynomials with $\deg(f(x)) \geq \deg(g(x))$. We want to find GCD of $f(x)$ and $g(x)$. If $f(x)$ and $g(x)$ can be factored into linear irreducible quadratic polynomials, then we can easily find the GCD by the method which we have learnt above. If the polynomials $f(x)$ and $g(x)$ are not easily factorable, then it will be a difficult problem.

However, the following method gives a systematic way of finding GCD.

Step 1 First, divide $f(x)$ by $g(x)$ to obtain $f(x) = g(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is remainder, so $\deg(g(x)) > \deg(r(x))$

If the remainder $r(x)$ is 0, then $g(x)$ is the GCD of $f(x)$ and $g(x)$.

Step 2 If the remainder $r(x)$ is non-zero, divide $g(x)$ by $r(x)$ to obtain $g(x) = r(x)q(x) + r_1(x)$ where $r_1(x)$ is the remainder. So $\deg r(x) > \deg r_1(x)$.

If the remainder $r_1(x)$ is 0, then $r(x)$ is the required GCD.

Step 3 If $r_1(x)$ is non-zero, then continue the process until we get zero as remainder. The remainder in the last but one step is the GCD of $f(x)$ and $g(x)$.
We write GCD($f(x)$, $g(x)$) to denote the GCD of the polynomials $f(x)$ and $g(x)$

Remarks

Euclid's division algorithm is based on the principle that GCD of two numbers does not change if the small number is subtracted from the larger number. Thus, GCD (252,105) = GCD (147,105) = GCD(42,105) = GCD(63,42) = GCD(21,42) = 21.

Example 3.20

Find the GCD of the polynomials $x^4 + 3x^3 - x - 3$ and $x^3 + x^2 - 5x + 3$.

Solution Let $f(x) = x^4 + 3x^3 - x - 3$ and $g(x) = x^3 + x^2 - 5x + 3$

Here degree of $f(x) >$ degree of $g(x)$. $\therefore$ Divisor is $x^3 + x^2 - 5x + 3$

$$\begin{array}{r|l} & \qquad\qquad x+2 \\ \cline{2-2} x^3+x^2-5x+3 & x^4+3x^3+0x^2-x-3 \\ & x^4+x^3-5x^2+3x \\ \cline{2-2} & \quad 2x^3+5x^2-4x-3 \\ & \quad 2x^3+2x^2-10x+6 \\ \cline{2-2} & \qquad 3x^2+6x-9 \\ & \Longrightarrow \quad x^2+2x-3 \longrightarrow \text{remainder} (\neq 0) \end{array}$$

$$\begin{array}{r|l} & \qquad x-1 \\ \cline{2-2} x^2+2x-3 & x^3+x^2-5x+3 \\ & x^3+2x^2-3x \\ \cline{2-2} & \quad -x^2-2x+3 \\ & \quad -x^2-2x+3 \\ \cline{2-2} & \qquad 0 \longrightarrow \text{remainder} \end{array}$$

Therefore, GCD $(f(x), g(x)) = x^2 + 2x - 3$.

Remarks

The two original expressions have no simple factors (constants). Thus their GCD can have none. Hence, in the above example we removed the simple factor 3 from $3x^2 + 6x - 9$ and took $x^2 + 2x - 3$ as the new divisor.

Example 3.21

Find the GCD of the following polynomials

$3x^4 + 6x^3 - 12x^2 - 24x$ and $4x^4 + 14x^3 + 8x^2 - 8x$.

Solution Let $f(x) = 3x^4 + 6x^3 - 12x^2 - 24x = 3x(x^3 + 2x^2 - 4x - 8)$.

Let $g(x) = 4x^4 + 14x^3 + 8x^2 - 8x = 2x(2x^3 + 7x^2 + 4x - 4)$

Let us find the GCD for the polynomials $x^3 + 2x^2 - 4x - 8$ and $2x^3 + 7x^2 + 4x - 4$

We choose the divisor to be $x^3 + 2x^2 - 4x - 8$.

$$\begin{array}{r|l} & \quad 2 \\ \hline x^3 + 2x^2 - 4x - 8 & 2x^3 + 7x^2 + 4x - 4 \\ & 2x^3 + 4x^2 - 8x - 16 \\ \hline & \quad 3x^2 + 12x + 12 \\ & \quad (x^2 + 4x + 4) \\ & \qquad \hookrightarrow \text{remainder} (\neq 0) \end{array} \qquad \begin{array}{r|l} & \quad x - 2 \\ \hline x^2 + 4x + 4 & x^3 + 2x^2 - 4x - 8 \\ & x^3 + 4x^2 + 4x \\ \hline & \quad -2x^2 - 8x - 8 \\ & \quad -2x^2 - 8x - 8 \\ \hline & \qquad 0 \longrightarrow \text{remainder} \end{array}$$

Common factor of $3x$ and $2x$ is x.

Thus, GCD $(f(x), g(x)) = x(x^2 + 4x + 4)$.

Exercise 3.6

1. Find the greatest common divisor of
   (i) $7x^2yz^4$, $21x^2y^5z^3$ (ii) x^2y, x^3y, x^2y^2
   (iii) $25bc^4d^3$, $35b^2c^5$, $45c^3d$ (iv) $35x^5y^3z^4$, $49x^2yz^3$, $14xy^2z^2$

2. Find the GCD of the following
   (i) $c^2 - d^2$, $c(c - d)$ (ii) $x^4 - 27a^3x$, $(x - 3a)^2$
   (iii) $m^2 - 3m - 18$, $m^2 + 5m + 6$ (iv) $x^2 + 14x + 33$, $x^3 + 10x^2 - 11x$
   (v) $x^2 + 3xy + 2y^2$, $x^2 + 5xy + 6y^2$ (vi) $2x^2 - x - 1$, $4x^2 + 8x + 3$
   (vii) $x^2 - x - 2$, $x^2 + x - 6$, $3x^2 - 13x + 14$ (viii) $x^3 - x^2 + x - 1$, $x^4 - 1$
   (ix) $24(6x^4 - x^3 - 2x^2)$, $20(2x^6 + 3x^5 + x^4)$
   (x) $(a - 1)^5(a + 3)^2$, $(a - 2)^2(a - 1)^3(a + 3)^4$

3. Find the GCD of the following pairs of polynomials using division algorithm.
   (i) $x^3 - 9x^2 + 23x - 15$, $4x^2 - 16x + 12$
   (ii) $3x^3 + 18x^2 + 33x + 18$, $3x^2 + 13x + 10$
   (iii) $2x^3 + 2x^2 + 2x + 2$, $6x^3 + 12x^2 + 6x + 12$
   (iv) $x^3 - 3x^2 + 4x - 12$, $x^4 + x^3 + 4x^2 + 4x$

3.5.3 Least Common Multiple (LCM)

The least common multiple of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder. For example, consider the simple expressions a^4, a^3, a^6.

Now, $a^6,\ a^7,\ a^8, \cdots$ are common multiples of a^3, a^4 and a^6.

Of all the common multiples, the least common multiple is a^6

Hence LCM of a^4, a^3, a^6 is a^6. Similarly, a^3b^7 is the LCM of $a^3b^4,\ ab^5,\ a^2b^7$.

We shall consider some more examples of finding LCM.

Example 3.22

Find the LCM of the following.

(i) 90, 150, 225 (ii) $35a^2c^3b,\ 42a^3cb^2,\ 30ac^2b^3$

(iii) $(a-1)^5(a+3)^2,\ (a-2)^2(a-1)^3(a+3)^4$

(iv) $x^3+y^3,\ x^3-y^3,\ x^4+x^2y^2+y^4$

Solution

(i) Now,

$$90 = 2\times3\times3\times5 = 2^1\times3^2\times5^1$$

$$150 = 2\times3\times5\times5 = 2^1\times3^1\times5^2$$

$$225 = 3\times3\times5\times5 = 3^2\times5^2$$

The product $2^1\times3^2\times5^2 = 450$ is the required LCM.

(ii) Now, LCM of 35, 42 and 30 is $5\times7\times6 = 210$

Hence, the required LCM $= 210\times a^3\times c^3\times b^3 = 210a^3c^3b^3$.

(iii) Now, LCM of $(a-1)^5(a+3)^2,\ (a-2)^2(a-1)^3(a+3)^4$ is $(a-1)^5(a+3)^4(a-2)^2$.

(iv) Let us first find the factors for each of the given expressions.

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$

$$x^4+x^2y^2+y^4 = (x^2+y^2)^2-x^2y^2 = (x^2+xy+y^2)(x^2-xy+y^2)$$

Thus, LCM $= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

$$= (x^3+y^3)(x^3-y^3) = x^6-y^6.$$

Exercise 3.7

Find the LCM of the following.

1. $x^3y^2,\ xyz$
2. $3x^2yz,\ 4x^3y^3$
3. $a^2bc,\ b^2ca,\ c^2ab$
4. $66a^4b^2c^3,\ 44a^3b^4c^2,\ 24a^2b^3c^4$
5. $a^{m+1},\ a^{m+2},\ a^{m+3}$
6. $x^2y+xy^2,\ x^2+xy$
7. $3(a-1),\ 2(a-1)^2,\ (a^2-1)$
8. $2x^2-18y^2,\ 5x^2y+15xy^2,\ x^3+27y^3$
9. $(x+4)^2(x-3)^3,\ (x-1)(x+4)(x-3)^2$
10. $10(9x^2+6xy+y^2),\ 12(3x^2-5xy-2y^2),\ 14(6x^4+2x^3)$.

3.5.4 Relation between LCM and GCD

We know that the product of two positive integers is equal to the product of their LCM and GCD. For example, $21 \times 35 = 105 \times 7$, where LCM (21,35) =105 and GCD (21,35) = 7.

In the same way, we have the following result:

The product of any two polynomials is equal to the product of their LCM and GCD.

That is, $f(x) \times g(x) =$ LCM $(f(x), g(x)) \times$ GCD $(f(x), g(x))$.

Let us justify this result with an example.

Let $f(x) = 12(x^4 - x^3)$ and $g(x) = 8(x^4 - 3x^3 + 2x^2)$ be two polynomials.

Now, $f(x) = 12(x^4 - x^3) = 2^2 \times 3 \times x^3 \times (x-1)$ (1)

Also, $g(x) = 8(x^4 - 3x^3 + 2x^2) = 2^3 \times x^2 \times (x-1) \times (x-2)$ (2)

From (1) and (2) we get,

LCM $(f(x), g(x)) = 2^3 \times 3^1 \times x^3 \times (x-1) \times (x-2) = 24x^3(x-1)(x-2)$

GCD $(f(x), g(x)) = 4x^2(x-1)$

Therefore, LCM $\times$ GCD $= 24x^3(x-1)(x-2) \times 4x^2(x-1)$

$= 96x^5(x-1)^2(x-2)$ (3)

Also, $f(x) \times g(x) = 12x^3(x-1) \times 8x^2(x-1)(x-2)$

$= 96x^5(x-1)^2(x-2)$ (4)

From (3) and (4) we obtain, LCM $\times$ GCD $= f(x) \times g(x)$.

Thus, the product of LCM and GCD of two polynomials is equal to the product of the two polynomials. Further, if $f(x)$, $g(x)$ and one of LCM and GCD are given, then the other can be found without ambiguity because LCM and GCD are unique, except for a factor of –1.

Example 3.23

The GCD of $x^4 + 3x^3 + 5x^2 + 26x + 56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$. Find their LCM.

Solution Let $f(x) = x^4 + 3x^3 + 5x^2 + 26x + 56$ and $g(x) = x^4 + 2x^3 - 4x^2 - x + 28$

Given that GCD $= x^2 + 5x + 7$. Also, we have GCD $\times$ LCM $= f(x) \times g(x)$.

Thus, $\text{LCM} = \dfrac{f(x) \times g(x)}{\text{GCD}}$ (1)

Now, GCD divides both $f(x)$ and $g(x)$.

Let us divide $f(x)$ by the GCD.

	1	−2	8		
1 5 7	1	3	5	26	56
	1	5	7		
		−2	−2	26	
		−2	−10	−14	
			8	40	56
			8	40	56
				0	

When $f(x)$ is divided by GCD, we get the quotient as $x^2 - 2x + 8$.

Now, (1) $\Longrightarrow$ LCM $= (x^2 - 2x + 8) \times g(x)$

Thus, LCM $= (x^2 - 2x + 8)(x^4 + 2x^3 - 4x^2 - x + 28)$.

Note

In the above problem, we can also divide $g(x)$ by GCD and multiply the quotient by $f(x)$ to get the required LCM.

Example 3.24

The GCD and LCM of two polynomials are $x + 1$ and $x^6 - 1$ respectively. If one of the polynomials is $x^3 + 1$, find the other.

Solution Given GCD $= x + 1$ and LCM $= x^6 - 1$

Let $f(x) = x^3 + 1$.

We know that LCM $\times$ GCD $= f(x) \times g(x)$

$$\Longrightarrow g(x) = \frac{\text{LCM} \times \text{GCD}}{f(x)} = \frac{(x^6 - 1)(x + 1)}{x^3 + 1}$$

$$= \frac{(x^3 + 1)(x^3 - 1)(x + 1)}{x^3 + 1} = (x^3 - 1)(x + 1)$$

Hence, $g(x) = (x^3 - 1)(x + 1)$.

Exercise 3.8

1. Find the LCM of each pair of the following polynomials.

   (i) $x^2 - 5x + 6,\ x^2 + 4x - 12$ whose GCD is $x - 2$.

   (ii) $x^4 + 3x^3 + 6x^2 + 5x + 3,\ x^4 + 2x^2 + x + 2$ whose GCD is $x^2 + x + 1$.

   (iii) $2x^3 + 15x^2 + 2x - 35,\ x^3 + 8x^2 + 4x - 21$ whose GCD is $x + 7$.

   (iv) $2x^3 - 3x^2 - 9x + 5,\ 2x^4 - x^3 - 10x^2 - 11x + 8$ whose GCD is $2x - 1$.

2. Find the other polynomial $q(x)$ of each of the following, given that LCM and GCD and one polynomial $p(x)$ respectively.

   (i) $(x + 1)^2(x + 2)^2,\ (x + 1)(x + 2),\ (x + 1)^2(x + 2)$.

   (ii) $(4x + 5)^3(3x - 7)^3,\ (4x + 5)(3x - 7)^2,\ (4x + 5)^3(3x - 7)^2$.

   (iii) $(x^4 - y^4)(x^4 + x^2y^2 + y^4),\ x^2 - y^2,\ x^4 - y^4$.

   (iv) $(x^3 - 4x)(5x + 1),\ (5x^2 + x),\ (5x^3 - 9x^2 - 2x)$.

   (v) $(x - 1)(x - 2)(x^2 - 3x + 3),\ (x - 1),\ (x^3 - 4x^2 + 6x - 3)$.

   (vi) $2(x + 1)(x^2 - 4),\ (x + 1),\ (x + 1)(x - 2)$.

3.6 Rational expressions

A rational number is defined as a quotient $\frac{m}{n}$, of two integers m and $n \neq 0$. Similarly a rational expression is a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is a non zero polynomial.

Every polynomial $p(x)$ is a rational expression, since $p(x)$ can be written as $\frac{p(x)}{1}$ where 1 is the constant polynomial.

However, a rational expression need not be a polynomial, for example $\frac{x}{x^2+1}$ is a rational expression but not a polynomial. Some examples of rational expressions are $2x+7,\ \frac{3x+2}{x^2+x+1},\ \frac{x^3+\sqrt{2}x+5}{x^2+x-\sqrt{3}}$.

3.6.1 Rational expressions in lowest form

If the two polynomials $p(x)$ and $q(x)$ have the integer coefficients such that GCD of $p(x)$ and $q(x)$ is 1, then we say that $\frac{p(x)}{q(x)}$ is a rational expression in its lowest terms.

If a rational expression is not in its lowest terms, then it can be reduced to its lowest terms by dividing both numerator $p(x)$ and denominator $q(x)$ by the GCD of $p(x)$ and $q(x)$.

Let us consider some examples.

Example 3.25

Simplify the rational expressions into lowest forms.

(i) $\frac{5x+20}{7x+28}$ (ii) $\frac{x^3-5x^2}{3x^3+2x^4}$

(iii) $\frac{6x^2-5x+1}{9x^2+12x-5}$ (iv) $\frac{(x-3)(x^2-5x+4)}{(x-1)(x^2-2x-3)}$

Solution

(i) Now, $\frac{5x+20}{7x+28} = \frac{5(x+4)}{7(x+4)} = \frac{5}{7}$

(ii) Now, $\frac{x^3-5x^2}{3x^3+2x^4} = \frac{x^2(x-5)}{x^3(2x+3)} = \frac{x-5}{x(2x+3)}$

(iii) Let $p(x) = 6x^2-5x+1 = (2x-1)(3x-1)$ and

$q(x) = 9x^2+12x-5 = (3x+5)(3x-1)$

Therefore, $\frac{p(x)}{q(x)} = \frac{(2x-1)(3x-1)}{(3x+5)(3x-1)} = \frac{2x-1}{3x+5}$

(iv) Let $f(x) = (x-3)(x^2-5x+4) = (x-3)(x-1)(x-4)$ and

$g(x) = (x-1)(x^2-2x-3) = (x-1)(x-3)(x+1)$

Therefore, $\frac{f(x)}{g(x)} = \frac{(x-3)(x-1)(x-4)}{(x-1)(x-3)(x+1)} = \frac{x-4}{x+1}$

Exercise 3.9

Simplify the following into their lowest forms.

(i) $\dfrac{6x^2+9x}{3x^2-12x}$ (ii) $\dfrac{x^2+1}{x^4-1}$ (iii) $\dfrac{x^3-1}{x^2+x+1}$

(iv) $\dfrac{x^3-27}{x^2-9}$ (v) $\dfrac{x^4+x^2+1}{x^2+x+1}$ (Hint: $x^4+x^2+1=(x^2+1)^2-x^2$)

(vi) $\dfrac{x^3+8}{x^4+4x^2+16}$ (vii) $\dfrac{2x^2+x-3}{2x^2+5x+3}$ (viii) $\dfrac{2x^4-162}{(x^2+9)(2x-6)}$

(ix) $\dfrac{(x-3)(x^2-5x+4)}{(x-4)(x^2-2x-3)}$ (x) $\dfrac{(x-8)(x^2+5x-50)}{(x+10)(x^2-13x+40)}$ (xi) $\dfrac{4x^2+9x+5}{8x^2+6x-5}$

(xii) $\dfrac{(x-1)(x-2)(x^2-9x+14)}{(x-7)(x^2-3x+2)}$

3.6.2 Multiplication and division of rational expressions

If $\dfrac{p(x)}{q(x)}$; $q(x)\neq 0$ and $\dfrac{g(x)}{h(x)}$; $h(x)\neq 0$ are two rational expressions, then

(i) their product $\dfrac{p(x)}{q(x)}\times\dfrac{g(x)}{h(x)}$ is defined as $\dfrac{p(x)\times g(x)}{q(x)\times h(x)}$

(ii) their division $\dfrac{p(x)}{q(x)}\div\dfrac{g(x)}{h(x)}$ is defined as $\dfrac{p(x)}{q(x)}\times\dfrac{h(x)}{g(x)}$.

Thus, $\dfrac{p(x)}{q(x)}\div\dfrac{g(x)}{h(x)} = \dfrac{p(x)\times h(x)}{q(x)\times g(x)}$

Example 3.26

Multiply (i) $\dfrac{x^3y^2}{9z^4}$ by $\dfrac{27z^5}{x^4y^2}$ (ii) $\dfrac{a^3+b^3}{a^2+2ab+b^2}$ by $\dfrac{a^2-b^2}{a-b}$ (iii) $\dfrac{x^3-8}{x^2-4}$ by $\dfrac{x^2+6x+8}{x^2+2x+4}$

Solution

(i) Now, $\dfrac{x^3y^2}{9z^4}\times\dfrac{27z^5}{x^4y^2} = \dfrac{(x^3y^2)(27z^5)}{(9z^4)(x^4y^2)} = \dfrac{3z}{x}$.

(ii) $\dfrac{a^3+b^3}{a^2+2ab+b^2}\times\dfrac{a^2-b^2}{a-b} = \dfrac{(a+b)(a^2-ab+b^2)}{(a+b)(a+b)}\times\dfrac{(a+b)(a-b)}{(a-b)} = a^2-ab+b^2$.

(iii) Now, $\dfrac{x^3-8}{x^2-4}\times\dfrac{x^2+6x+8}{x^2+2x+4} = \dfrac{x^3-2^3}{x^2-2^2}\times\dfrac{(x+4)(x+2)}{x^2+2x+4}$

$$= \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}\times\frac{(x+4)(x+2)}{x^2+2x+4} = x+4.$$

Example 3.27

Divide (i) $\dfrac{4x-4}{x^2-1}$ by $\dfrac{x-1}{x+1}$ (ii) $\dfrac{x^3-1}{x+3}$ by $\dfrac{x^2+x+1}{3x+9}$ (iii) $\dfrac{x^2-1}{x^2-25}$ by $\dfrac{x^2-4x-5}{x^2+4x-5}$

Solution

(i) $\dfrac{4x-4}{x^2-1} \div \dfrac{x-1}{x+1} = \dfrac{4(x-1)}{(x+1)(x-1)} \times \dfrac{(x+1)}{(x-1)} = \dfrac{4}{x-1}$.

(ii) $\dfrac{x^3-1}{x+3} \div \dfrac{x^2+x+1}{3x+9} = \dfrac{(x-1)(x^2+x+1)}{x+3} \times \dfrac{3(x+3)}{x^2+x+1} = 3(x-1)$.

(iii) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5} = \dfrac{(x+1)(x-1)}{(x+5)(x-5)} \times \dfrac{(x+5)(x-1)}{(x-5)(x+1)}$

$= \dfrac{(x-1)(x-1)}{(x-5)(x-5)} = \dfrac{x^2-2x+1}{x^2-10x+25}$.

Exercise 3.10

1. Multiply the following and write your answer in lowest terms.

(i) $\dfrac{x^2-2x}{x+2} \times \dfrac{3x+6}{x-2}$ (ii) $\dfrac{x^2-81}{x^2-4} \times \dfrac{x^2+6x+8}{x^2-5x-36}$

(iii) $\dfrac{x^2-3x-10}{x^2-x-20} \times \dfrac{x^2-2x+4}{x^3+8}$ (iv) $\dfrac{x^2-16}{x^2-3x+2} \times \dfrac{x^2-4}{x^3+64} \times \dfrac{x^2-4x+16}{x^2-2x-8}$

(v) $\dfrac{3x^2+2x-1}{x^2-x-2} \times \dfrac{2x^2-3x-2}{3x^2+5x-2}$ (vi) $\dfrac{2x-1}{x^2+2x+4} \times \dfrac{x^4-8x}{2x^2+5x-3} \times \dfrac{x+3}{x^2-2x}$

2. Divide the following and write your answer in lowest terms.

(i) $\dfrac{x}{x+1} \div \dfrac{x^2}{x^2-1}$ (ii) $\dfrac{x^2-36}{x^2-49} \div \dfrac{x+6}{x+7}$

(iii) $\dfrac{x^2-4x-5}{x^2-25} \div \dfrac{x^2-3x-10}{x^2+7x+10}$ (iv) $\dfrac{x^2+11x+28}{x^2-4x-77} \div \dfrac{x^2+7x+12}{x^2-2x-15}$

(v) $\dfrac{2x^2+13x+15}{x^2+3x-10} \div \dfrac{2x^2-x-6}{x^2-4x+4}$ (vi) $\dfrac{3x^2-x-4}{9x^2-16} \div \dfrac{4x^2-4}{3x^2-2x-1}$

(vii) $\dfrac{2x^2+5x-3}{2x^2+9x+9} \div \dfrac{2x^2+x-1}{2x^2+x-3}$

3.6.3 Addition and subtraction of rational expressions

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are any two rational expressions with $q(x) \neq 0$ and $s(x) \neq 0$, then we define the sum and the difference (subtraction) as

$$\frac{p(x)}{q(x)} \pm \frac{r(x)}{s(x)} = \frac{p(x).s(x) \pm q(x)r(x)}{q(x).s(x)}$$

Example 3.28

Simplify (i) $\dfrac{x+2}{x+3} + \dfrac{x-1}{x-2}$ (ii) $\dfrac{x+1}{(x-1)^2} + \dfrac{1}{x+1}$ (iii) $\dfrac{x^2-x-6}{x^2-9} + \dfrac{x^2+2x-24}{x^2-x-12}$

Solution

(i) $\dfrac{x+2}{x+3}+\dfrac{x-1}{x-2} = \dfrac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)} = \dfrac{2x^2+2x-7}{x^2+x-6}$

(ii) $\dfrac{x+1}{(x-1)^2}+\dfrac{1}{x+1} = \dfrac{(x+1)^2+(x-1)^2}{(x-1)^2(x+1)} = \dfrac{2x^2+2}{(x-1)^2(x+1)}$

$$= \frac{2x^2+2}{x^3-x^2-x+1}$$

(iii) $\dfrac{x^2-x-6}{x^2-9}+\dfrac{x^2+2x-24}{x^2-x-12} = \dfrac{(x-3)(x+2)}{(x+3)(x-3)}+\dfrac{(x+6)(x-4)}{(x+3)(x-4)}$

$$= \frac{x+2}{x+3}+\frac{x+6}{x+3} = \frac{x+2+x+6}{x+3} = \frac{2x+8}{x+3}$$

Example 3.29

What rational expression should be added to $\dfrac{x^3-1}{x^2+2}$ to get $\dfrac{2x^3-x^2+3}{x^2+2}$?

Solution Let $p(x)$ be the required rational expression.

Given that $\dfrac{x^3-1}{x^2+2}+p(x) = \dfrac{2x^3-x^2+3}{x^2+2}$

$$p(x) = \frac{2x^3-x^2+3}{x^2+2} - \frac{x^3-1}{x^2+2}$$

$$= \frac{2x^3-x^2+3-x^3+1}{x^2+2} = \frac{x^3-x^2+4}{x^2+2}$$

Example 3.30

Simplify $\left(\dfrac{2x-1}{x-1}-\dfrac{x+1}{2x+1}\right)+\dfrac{x+2}{x+1}$ as a quotient of two polynomials in the simplest form.

Solution Now, $\left(\dfrac{2x-1}{x-1}-\dfrac{x+1}{2x+1}\right)+\dfrac{x+2}{x+1}$

$$= \left[\frac{(2x-1)(2x+1)-(x+1)(x-1)}{(x-1)(2x+1)}\right]+\frac{x+2}{x+1}$$

$$= \frac{(4x^2-1)-(x^2-1)}{(x-1)(2x+1)}+\frac{x+2}{x+1} = \frac{3x^2}{(x-1)(2x+1)}+\frac{x+2}{x+1}$$

$$= \frac{3x^2(x+1)+(x+2)(x-1)(2x+1)}{(x^2-1)(2x+1)} = \frac{5x^3+6x^2-3x-2}{2x^3+x^2-2x-1}$$

Exercise 3.11

1. Simplify the following as a quotient of two polynomials in the simplest form.

(i) $\dfrac{x^3}{x-2}+\dfrac{8}{2-x}$

(ii) $\dfrac{x+2}{x^2+3x+2}+\dfrac{x-3}{x^2-2x-3}$

(iii) $\dfrac{x^2-x-6}{x^2-9}+\dfrac{x^2+2x-24}{x^2-x-12}$

(iv) $\dfrac{x-2}{x^2-7x+10}+\dfrac{x+3}{x^2-2x-15}$

(v) $\frac{2x^2-5x+3}{x^2-3x+2}-\frac{2x^2-7x-4}{2x^2-3x-2}$ (vi) $\frac{x^2-4}{x^2+6x+8}-\frac{x^2-11x+30}{x^2-x-20}$

(vii) $\left[\frac{2x+5}{x+1}+\frac{x^2+1}{x^2-1}\right]-\left(\frac{3x-2}{x-1}\right)$ (viii) $\frac{1}{x^2+3x+2}+\frac{1}{x^2+5x+6}-\frac{2}{x^2+4x+3}$.

2. Which rational expression should be added to $\frac{x^3-1}{x^2+2}$ to get $\frac{3x^3+2x^2+4}{x^2+2}$?

3. Which rational expression should be subtracted from

$\frac{4x^3-7x^2+5}{2x-1}$ to get $2x^2-5x+1$?

4. If $P=\frac{x}{x+y}$, $Q=\frac{y}{x+y}$, then find $\frac{1}{P-Q}-\frac{2Q}{P^2-Q^2}$.

3.7 Square root

Let $a \in \mathbb{R}$ be a non negative real number. A square root of a, is a real number b such that $b^2=a$. The positive square root of a is denoted by $\sqrt[2]{a}$ or $\sqrt{a}$. Even though both $(-3)^2=9$ and $(+3)^2=9$ are true, the radical sign $\sqrt{\ }$ is used to indicate the positive square root of the number under it. Hence $\sqrt{9}=3$. Similarly, we have $\sqrt{121}=11$, $\sqrt{10000}=100$.

In the same way, the square root of any expression or a polynomial is an expression whose square is equal to the given expression. In the case of polynomials, we take

$$\sqrt{(p(x))^2}=|p(x)|,\ \text{where } |p(x)|=\begin{cases} p(x) & \text{if } p(x)\geq 0 \\ -p(x) & \text{if } p(x)<0 \end{cases}.$$ For example,

$$\sqrt{(x-a)^2}=|(x-a)| \text{ and } \sqrt{(a-b)^2}=|(a-b)|.$$

In general, the following two methods are very familiar to find the square root of a given polynomial (i) factorization method (ii) division method.

In this section, let us learn the factorization method through some examples for both the expressions and polynomials when they are factorable.

3.7.1 Square root by factorization method

Example 3.31

Find the square root of

(i) $121(x-a)^4(x-b)^6(x-c)^{12}$ (ii) $\frac{81x^4y^6z^8}{64w^{12}s^{14}}$ (iii) $(2x+3y)^2-24xy$

Solution

(i) $\sqrt{121(x-a)^4(x-b)^6(x-c)^{12}}=11|(x-a)^2(x-b)^3(x-c)^6|$

(ii) $\sqrt{\frac{81x^4y^6z^8}{64w^{12}s^{14}}}=\left|\frac{9x^2y^3z^4}{8w^6s^7}\right|$

(iii) $\sqrt{(2x+3y)^2-24xy}=\sqrt{4x^2+12xy+9y^2-24xy}=\sqrt{(2x-3y)^2}$
$=|(2x-3y)|$

Example 3.32

Find the square root of (i) $4x^2 + 20xy + 25y^2$ (ii) $x^6 + \frac{1}{x^6} - 2$

(iii) $(6x^2 - x - 2)(3x^2 - 5x + 2)(2x^2 - x - 1)$

Solution

(i) $\sqrt{4x^2 + 20xy + 25y^2} = \sqrt{(2x + 5y)^2} = |(2x + 5y)|$

(ii) $\sqrt{x^6 + \frac{1}{x^6} - 2} = \sqrt{\left(x^3 - \frac{1}{x^3}\right)^2} = \left|\left(x^3 - \frac{1}{x^3}\right)\right|$

(iii) First, let us factorize the polynomials

$6x^2 - x - 2 = (2x + 1)(3x - 2)$; $3x^2 - 5x + 2 = (3x - 2)(x - 1)$ and

$2x^2 - x - 1 = (x - 1)(2x + 1)$

Now, $\sqrt{(6x^2 - x - 2)(3x^2 - 5x + 2)(2x^2 - x - 1)}$

$= \sqrt{(2x + 1)(3x - 2) \times (3x - 2)(x - 1) \times (x - 1)(2x + 1)}$

$= \sqrt{(2x + 1)^2 (3x - 2)^2 (x - 1)^2} = |(2x + 1)(3x - 2)(x - 1)|$

Exercise 3.12

1. Find the square root of the following

(i) $196a^6b^8c^{10}$ (ii) $289(a - b)^4(b - c)^6$ (iii) $(x + 11)^2 - 44x$

(iv) $(x - y)^2 + 4xy$ (v) $121x^8y^6 \div 81x^4y^8$ (vi) $\frac{64(a + b)^4(x - y)^8(b - c)^6}{25(x + y)^4(a - b)^6(b + c)^{10}}$

2. Find the square root of the following:

(i) $16x^2 - 24x + 9$

(ii) $(x^2 - 25)(x^2 + 8x + 15)(x^2 - 2x - 15)$

(iii) $4x^2 + 9y^2 + 25z^2 - 12xy + 30yz - 20zx$

(iv) $x^4 + \frac{1}{x^4} + 2$

(v) $(6x^2 + 5x - 6)(6x^2 - x - 2)(4x^2 + 8x + 3)$

(vi) $(2x^2 - 5x + 2)(3x^2 - 5x - 2)(6x^2 - x - 1)$

3.7.2 Finding the square root of a polynomial by division method

In this method, we find the square root of a polynomial which cannot easily be reduced into product of factors. Also division method is a convenient one when the polynomials are of higher degrees.

One can find the square root of a polynomial the same way of finding the square root of a positive integer. Let us explain this method with the following examples.

To find (i) $\sqrt{66564}$

$$\begin{array}{r|l}
 & \ \ 2\ \ 5\ \ 8 \\ \hline
2 & 6\ \ 65\ \ 64 \\
 & 4 \\ \hline
45 & \ \ 2\ \ 65 \\
 & \ \ 2\ \ 25 \\ \hline
508 & \quad 40\ \ 64 \\
 & \quad 40\ \ 64 \\ \hline
 & \qquad 0
\end{array}$$

(ii) $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1}$

Let $p(x) = 9x^4 + 12x^3 + 10x^2 + 4x + 1$

$$\begin{array}{r|l}
 & 3x^2 + 2x + 1 \\ \hline
3x^2 & 9x^4 + 12x^3 + 10x^2 + 4x + 1 \\
 & 9x^4 \\ \hline
6x^2 + 2x & \quad 12x^3 + 10x^2 \\
 & \quad 12x^3 + 4x^2 \\ \hline
6x^2 + 4x + 1 & \qquad 6x^2 + 4x + 1 \\
 & \qquad 6x^2 + 4x + 1 \\ \hline
 & \qquad\qquad 0
\end{array}$$

Therefore, $\sqrt{66564} = 258$ and $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = |3x^2 + 2x + 1|$

Remarks

(i) While writing the polynomial in **ascending or descending** powers of x, insert zeros for missing terms.

(ii) The above method can be compared with the following procedure.

$$\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = \sqrt{(a+b+c)^2}$$

Therefore, it is a matter of finding the suitable a, b and c.

Now, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$= a^2 + b^2 + 2ab + 2ac + 2bc + c^2$

$= a^2 + (2a+b)b + (2a+2b+c)c$

$= (3x^2)^2 + (6x^2 + 2x)(2x) + (6x^2 + 4x + 1)(1)$

Thus, $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = |3x^2 + 2x + 1|$, where $a = 3x^2$, $b = 2x$ and $c = 1$

Aliter : To find the square root, first write $9x^4 + 12x^3 + 10x^2 + 4x + 1$

$= (mx^2 + nx + l)^2 = m^2x^4 + 2mnx^3 + (n^2 + 2lm)x^2 + 2nlx + l^2$

Compare the coefficients and then find the suitable constants m, n, l.

(iii) It is also quite interesting to note the following :

$25x^4 - 30x^3 + 29x^2 - 12x + 4 = 25x^4 - 30x^3 + 9x^2 + 20x^2 - 12x + 4$

$= (5x^2)^2 + [10x^2 + (-3x)](-3x) + (10x^2 - 6x + 2)2$

$= (5x^2)^2 + [2(5x^2) + (-3x)](-3x) + [2(5x^2) + 2(-3x) + 2]2$

$= a^2 + [2a + (-b)](-b) + [2a + 2(-b) + c]c$

$= a^2 + (-b)^2 + c^2 + 2a(-b) + 2(-b)c + 2ac$

$= (a - b + c)^2$, where $a = 5x^2$, $b = 3x$, $c = 2$

$\therefore \sqrt{25x^4 - 30x^3 + 29x^2 - 12x + 4} = |5x^2 - 3x + 2|$.

Example 3.33

Find the square root of $x^4 - 10x^3 + 37x^2 - 60x + 36$.

Solution Given polynomial is already in descending powers of x.

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\ \hline
x^2 & x^4 - 10x^3 + 37x^2 - 60x + 36 \\
 & x^4 \\ \hline
2x^2 - 5x & \quad -10x^3 + 37x^2 \\
 & \quad -10x^3 + 25x^2 \\ \hline
2x^2 - 10x + 6 & \qquad\qquad 12x^2 - 60x + 36 \\
 & \qquad\qquad 12x^2 - 60x + 36 \\ \hline
 & \qquad\qquad\quad 0
\end{array}
$$

Thus, $\sqrt{x^4 - 10x^3 + 37x^2 - 60x + 36} = |(x^2 - 5x + 6)|$

Example 3.34

Find the square root of $x^4 - 6x^3 + 19x^2 - 30x + 25$

Solution Let us write the polynomial in ascending powers of x and find the square root.

$$
\begin{array}{r|l}
 & 5 - 3x + x^2 \\ \hline
5 & 25 - 30x + 19x^2 - 6x^3 + x^4 \\
 & 25 \\ \hline
10-3x & \quad -30x + 19x^2 \\
 & \quad -30x + 9x^2 \\ \hline
10 - 6x + x^2 & \qquad\qquad 10x^2 - 6x^3 + x^4 \\
 & \qquad\qquad 10x^2 - 6x^3 + x^4 \\ \hline
 & \qquad\qquad\quad 0
\end{array}
$$

Hence, the square root of the given polynomial is $|x^2 - 3x + 5|$

Example 3.35

If $m - nx + 28x^2 + 12x^3 + 9x^4$ is a perfect square, then find the values of m and n.

Solution Arrange the polynomial in descending power of x.

$$9x^4 + 12x^3 + 28x^2 - nx + m.$$

Now,

$$\begin{array}{r|l} & 3x^2 + 2x + 4 \\ \hline 3x^2 & 9x^4 + 12x^3 + 28x^2 - nx + m \\ & 9x^4 \\ \hline 6x^2 + 2x & \quad 12x^3 + 28x^2 \\ & \quad 12x^3 + 4x^2 \\ \hline 6x^2 + 4x + 4 & \qquad 24x^2 - nx + m \\ & \qquad 24x^2 + 16x + 16 \\ \hline & \qquad\qquad 0 \end{array}$$

Since the given polynomial is a perfect square, we must have $n = -16$ and $m = 16$.

Exercise 3.13

1. Find the square root of the following polynomials by division method.

   (i) $x^4 - 4x^3 + 10x^2 - 12x + 9$ (ii) $4x^4 + 8x^3 + 8x^2 + 4x + 1$

   (iii) $9x^4 - 6x^3 + 7x^2 - 2x + 1$ (iv) $4 + 25x^2 - 12x - 24x^3 + 16x^4$

2. Find the values of a and b if the following polynomials are perfect squares.

   (i) $4x^4 - 12x^3 + 37x^2 + ax + b$ (ii) $x^4 - 4x^3 + 10x^2 - ax + b$

   (iii) $ax^4 + bx^3 + 109x^2 - 60x + 36$ (iv) $ax^4 - bx^3 + 40x^2 + 24x + 36$

3.8 Quadratic equations

Greek mathematician Euclid developed a geometrical approach for finding out lengths which in our present day terminology, are solutions of quadratic equations. Solving quadratic equations in general form is often credited to ancient Indian Mathematicians. In fact, Brahma Gupta (A.D 598 - 665) gave an explicit formula to solve a quadratic equation of the form $ax^2 + bx = c$. Later *Sridhar Acharya* (1025 A.D) derived a formula, now known as the quadratic formula, (as quoted by Bhaskara II) for solving a quadratic equation by the method of completing the square.

In this section, we will learn solving quadratic equations, by various methods. We shall also see some applications of quadratic equations.

Definition

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation, whose standard form is $ax^2 + bx + c = 0$, $a \neq 0$.

For example, $2x^2 - 3x + 4 = 0$, $1 - x + x^2 = 0$ are some quadratic equations.

3.8.1 Solution of a quadratic equation by factorization method

Factorization method can be used when the quadratic equation can be factorized into linear factors. Given a product, if any factor is zero, then the whole product is zero. Conversely, if a product is equal to zero, then some factor of that product must be zero, and any factor which contains an unknown may be equal to zero. Thus, in solving a quadratic equation, we find the values of x which make each of the factors zero. That is, we may equate each factor to zero and solve for the unknown.

Example 3.36

Solve $6x^2 - 5x - 25 = 0$

Solution Given $6x^2 - 5x - 25 = 0$.

First, let us find α and β such that $\alpha + \beta = -5$ and $\alpha\beta = 6 \times (-25) = -150$, where -5 is the coefficient of x. Thus, we get $\alpha = -15$ and $\beta = 10$.

Next, $6x^2 - 5x - 25 = 6x^2 - 15x + 10x - 25 = 3x(2x-5) + 5(2x-5)$

$$= (2x-5)(3x+5).$$

Therefore, the solution set is obtained from $2x - 5 = 0$ and $3x + 5 = 0$

Thus, $x = \frac{5}{2}$, $x = -\frac{5}{3}$.

Hence, solution set is $\left\{-\frac{5}{3}, \frac{5}{2}\right\}$.

Example 3.37

Solve $\frac{6}{7x-21} - \frac{1}{x^2-6x+9} + \frac{1}{x^2-9} = 0$

Solution Given equation appears to be a non-quadratic equation. But when we simplify the equation, it will reduce to a quadratic equation.

Now, $$\frac{6}{7(x-3)} - \frac{1}{(x-3)^2} + \frac{1}{(x+3)(x-3)} = 0$$

$$\Longrightarrow \quad \frac{6(x^2-9) - 7(x+3) + 7(x-3)}{7(x-3)^2(x+3)} = 0$$

$$\Longrightarrow \quad 6x^2 - 54 - 42 = 0 \quad \Longrightarrow \quad x^2 - 16 = 0$$

The equation $x^2 = 16$ is quadratic and hence we have two values $x = 4$ and $x = -4$.

$\therefore$ Solution set is $\{-4, 4\}$

Example 3.38

Solve $\sqrt{24 - 10x} = 3 - 4x$, $3 - 4x > 0$

Solution Given $\sqrt{24 - 10x} = 3 - 4x$

Squaring on both sides, we get, $24 - 10x = (3 - 4x)^2$

$$\Longrightarrow 16x^2 - 14x - 15 = 0 \qquad \Longrightarrow 16x^2 - 24x + 10x - 15 = 0$$

$$\Longrightarrow \quad (8x+5)(2x-3)=0 \text{ which gives } x=\frac{3}{2} \text{ or } -\frac{5}{8}$$

When $x=\frac{3}{2}$, $3-4x=3-4\left(\frac{3}{2}\right)<0$ and hence, $x=\frac{3}{2}$ is not a solution of the equation.

When $x=-\frac{5}{8}$, $3-4x>0$ and hence, the solution set is $\left\{-\frac{5}{8}\right\}$.

Remarks

To solve radical equation like the above, we rely on the **squaring property** :

$a=b \Longrightarrow a^2=b^2$. Unfortunately, this squaring property does not guarantee that all solutions of the new equation are solutions of the original equation. For example, on squaring the equation $x=5$ we get $x^2=25$, which in turn gives $x=5$ and $x=-5$. But $x=-5$ is not a solution of the original equation. Such a solution is called an **extraneous** solution.

Thus, the above example shows that when squaring on both sides of a radical equation, the solution of the final equation must be checked to determine whether they are solutions of the original equation or not. This is necessary because no solution of the original equation will be lost by squaring but certain values may be introduced which are roots of the new equation but not of the original equation.

Exercise 3.14

Solve the following quadratic equations by factorization method.

(i) $(2x+3)^2-81=0$ (ii) $3x^2-5x-12=0$ (iii) $\sqrt{5}x^2+2x-3\sqrt{5}=0$

(iv) $3(x^2-6)=x(x+7)-3$ (v) $3x-\frac{8}{x}=2$ (vi) $x+\frac{1}{x}=\frac{26}{5}$

(vii) $\frac{x}{x+1}+\frac{x+1}{x}=\frac{34}{15}$ (viii) $a^2b^2x^2-(a^2+b^2)x+1=0$

(ix) $2(x+1)^2-5(x+1)=12$ (x) $3(x-4)^2-5(x-4)=12$

3.8.2 Solution of a quadratic equation by completing square

From $\left(x+\frac{b}{2}\right)^2=x^2+bx+\left(\frac{b}{2}\right)^2$, note that the last term $\left(\frac{b}{2}\right)^2$ is the square of half the coefficient of x. Hence, the x^2+bx lacks only the term $\left(\frac{b}{2}\right)^2$ of being the square of $x+\frac{b}{2}$. Thus, if the square of half the coefficient of x be added to an expression of the form x^2+bx, the result is the square of a binomial. Such an addition is usually known as completing the square. In this section, we shall find the solution of a quadratic equation by the method of completing the square through the following steps.

***Step* 1** If the coefficient of x^2 is 1, go to step 2. If not, divide both sides of the equation by the coefficient of x^2. Get all the terms with variable on one side of equation.

***Step* 2** Find half the coefficient of x and square it. Add this number to both sides of the equation. To solve the equation, use the square root property:

$$x^2=t \Longrightarrow x=\sqrt{t} \quad \text{or} \quad x=-\sqrt{t} \quad \text{where } t \text{ is non-negative.}$$

Example 3.39

Solve the quadratic equation $5x^2 - 6x - 2 = 0$ by completing the square.

Solution Given quadratic equation is $5x^2 - 6x - 2 = 0$

$$\implies \quad x^2 - \frac{6}{5}x - \frac{2}{5} = 0 \qquad \text{(Divide on both sides by 5)}$$

$$\implies \quad x^2 - 2\left(\frac{3}{5}\right)x = \frac{2}{5} \qquad \left(\frac{3}{5} \text{ is the half of the coefficient of } x\right)$$

$$\implies \quad x^2 - 2\left(\frac{3}{5}\right)x + \frac{9}{25} = \frac{9}{25} + \frac{2}{5} \qquad \left(\text{add } \left(\frac{3}{5}\right)^2 = \frac{9}{25} \text{ on both sides}\right)$$

$$\implies \quad \left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

$$\implies \quad x - \frac{3}{5} = \pm\sqrt{\frac{19}{25}} \qquad \text{(take square root on both sides)}$$

Thus, we have $x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$.

Hence, the solution set is $\left\{\frac{3+\sqrt{19}}{5}, \frac{3-\sqrt{19}}{5}\right\}$.

Example 3.40

Solve the equation $a^2x^2 - 3abx + 2b^2 = 0$ by completing the square

Solution There is nothing to prove if $a = 0$. For $a \neq 0$, we have

$$a^2x^2 - 3abx + 2b^2 = 0$$

$$\implies \quad x^2 - \frac{3b}{a}x + \frac{2b^2}{a^2} = 0 \qquad \implies x^2 - 2\left(\frac{3b}{2a}\right)x = \frac{-2b^2}{a^2}$$

$$\implies \quad x^2 - 2\left(\frac{3b}{2a}\right)x + \frac{9b^2}{4a^2} = \frac{9b^2}{4a^2} - \frac{2b^2}{a^2}$$

$$\implies \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{9b^2 - 8b^2}{4a^2} \qquad \implies \left(x - \frac{3b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\implies \quad x - \frac{3b}{2a} = \pm\frac{b}{2a} \qquad \implies \quad x = \frac{3b \pm b}{2a}$$

Therefore, the solution set is $\left\{\frac{b}{a}, \frac{2b}{a}\right\}$.

3.8.3 Solution of quadratic equation by formula method

In this section, we shall derive the quadratic formula, which is useful for finding the roots of a quadratic equation. Consider a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. We rewrite the given equation as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\implies \quad x^2 + 2\left(\frac{b}{2a}\right)x + \frac{c}{a} = 0 \qquad \implies x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ both sides we get, $x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

That is, $$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\Longrightarrow \quad x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

So, we have $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad (1)$$

The solution set is $\left\{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right\}$.

The formula given in equation (1) is known as quadratic formula.

Now, let us solve some quadratic equations using quadratic formula.

Example 3.41

Solve the equation $\frac{1}{x+1}+\frac{2}{x+2}=\frac{4}{x+4}$, where $x+1\neq 0$, $x+2\neq 0$ and $x+4\neq 0$ using quadratic formula.

Solution Note that the given equation is not in the standard form of a quadratic equation.

Consider $$\frac{1}{x+1}+\frac{2}{x+2} = \frac{4}{x+4}$$

That is, $$\frac{1}{x+1} = 2\left[\frac{2}{x+4}-\frac{1}{x+2}\right] = 2\left[\frac{2x+4-x-4}{(x+4)(x+2)}\right]$$

$$\frac{1}{x+1} = 2\left[\frac{x}{(x+2)(x+4)}\right]$$

$$x^2+6x+8 = 2x^2+2x$$

Thus, we have $x^2-4x-8 = 0$, which is a quadratic equation.

(The above equation can also be obtained by taking LCM)

Using the quadratic formula we get,

$$x = \frac{4\pm\sqrt{16-4(1)(-8)}}{2(1)} = \frac{4\pm\sqrt{48}}{2}$$

Thus, $$x = 2+2\sqrt{3} \text{ or } 2-2\sqrt{3}$$

Hence, the solution set is $\{2-2\sqrt{3}, 2+2\sqrt{3}\}$

Exercise 3.15

1 Solve the following quadratic equations by completing the square .

(i) $x^2+6x-7=0$ (ii) $x^2+3x+1=0$

(iii) $2x^2+5x-3=0$ (iv) $4x^2+4bx-(a^2-b^2)=0$

(v) $x^2-(\sqrt{3}+1)x+\sqrt{3}=0$ (vi) $\frac{5x+7}{x-1}=3x+2$

2. Solve the following quadratic equations using quadratic formula.

(i) $x^2 - 7x + 12 = 0$ (ii) $15x^2 - 11x + 2 = 0$

(iii) $x + \frac{1}{x} = 2\frac{1}{2}$ (iv) $3a^2x^2 - abx - 2b^2 = 0$

(v) $a(x^2 + 1) = x(a^2 + 1)$ (vi) $36x^2 - 12ax + (a^2 - b^2) = 0$

(vii) $\frac{x-1}{x+1} + \frac{x-3}{x-4} = \frac{10}{3}$ (viii) $a^2x^2 + (a^2 - b^2)x - b^2 = 0$

3.8.4 Solution of problems involving quadratic equations

In this section, we will solve some simple problems expressed in words and some problems describing day-to-day life situations involving quadratic equation. First we shall form an equation translating the given statement and then solve it. Finally, we choose the solution that is relevant to the given problem.

Example 3.42

The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

Solution Let x denote the required number. Then its reciprocal is $\frac{1}{x}$

By the given condition, $x + \frac{1}{x} = 5\frac{1}{5} \implies \frac{x^2+1}{x} = \frac{26}{5}$

So, $5x^2 - 26x + 5 = 0$

$$\implies 5x^2 - 25x - x + 5 = 0$$

That is, $(5x - 1)(x - 5) = 0 \implies x = 5$ or $\frac{1}{5}$

Thus, the required numbers are $5, \frac{1}{5}$.

Example 3.43

The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq. cm, then find its base and altitude.

Solution Let the altitude of the triangle be x cm.

By the given condition, the base of the triangle is $(x + 4)$ cm.
Now, the area of the triangle = $\frac{1}{2}(\text{base}) \times \text{height}$

By the given condition $\frac{1}{2}(x + 4)(x) = 48$

$$\implies x^2 + 4x - 96 = 0 \implies (x + 12)(x - 8) = 0$$

$$\implies x = -12 \text{ or } 8$$

But $x = -12$ is not possible (since the length should be positive)

Therefore, $x = 8$ and hence, $x + 4 = 12$.
Thus, the altitude of the triangle is 8cm and the base of the triangle is 12cm.

Example 3.44

A car left 30 minutes later than the scheduled time. In order to reach its destination 150 km away in time, it has to increase its speed by 25 km/hr from its usual speed. Find its usual speed.

Solution Let the usual speed of the car be x km/hr.

Thus, the increased speed of the car is $(x+25)$ km/hr

Total distance $= 150$ km; Time taken $= \dfrac{\text{Distance}}{\text{Speed}}$.

Let T_1 and T_2 be the time taken in hours by the car to cover the given distance in scheduled time and decreased time (as the speed is increased) respectively.

By the given information $T_1 - T_2 = \dfrac{1}{2}$ (30 minutes $= \dfrac{1}{2}$hr)

$$\Longrightarrow \quad \frac{150}{x} - \frac{150}{x+25} = \frac{1}{2} \quad \Longrightarrow \quad 150\left[\frac{x+25-x}{x(x+25)}\right] = \frac{1}{2}$$

$$\Longrightarrow \quad x^2 + 25x - 7500 = 0 \quad \Longrightarrow \quad (x+100)(x-75) = 0$$

Thus, $x = 75$ or -100, but $x = -100$ is not an admissible value.

Therefore, the usual speed of the car is 75 km/hr.

Exercise 3.16

1. The sum of a number and its reciprocal is $\dfrac{65}{8}$. Find the number.
2. The difference of the squares of two positive numbers is 45. The square of the smaller number is four times the larger number. Find the numbers.
3. A farmer wishes to start a 100 sq.m rectangular vegetable garden. Since he has only 30 m barbed wire, he fences the sides of the rectangular garden letting his house compound wall act as the fourth side fence. Find the dimension of the garden.
4. A rectangular field is 20 m long and 14 m wide. There is a path of equal width all around it having an area of 111 sq. metres. Find the width of the path on the outside.
5. A train covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hr more, it would have taken 30 minutes less for the journey. Find the original speed of the train.
6. The speed of a boat in still water is 15 km/hr. It goes 30 km upstream and return downstream to the original point in 4 hrs 30 minutes. Find the speed of the stream.
7. One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their present ages.
8. A chess board contains 64 equal squares and the area of each square is 6.25 cm^2. A border around the board is 2 cm wide. Find the length of the side of the chess board.

9. A takes 6 days less than the time taken by B to finish a piece of work. If both A and B together can finish it in 4 days, find the time that B would take to finish this work by himself.

10. Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels 5 km/hr faster than the second train. If after two hours, they are 50 km apart, find the average speed of each train.

3.8.5 Nature of roots of a quadratic equation

The roots of the equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac > 0$, we get two distinct real roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac = 0$, then the equation has two equal roots $x = \frac{-b}{2a}$.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Therefore there is no real root for the given quadratic equation.

So, evidently the nature of roots depends on the values of $b^2 - 4ac$. The value of the expression $b^2 - 4ac$ discriminates the nature of the roots of $ax^2 + bx + c = 0$ and so it is called the discriminant of the quadratic equation and denoted by the symbol $\triangle$.

Discriminant $\triangle = b^2 - 4ac$	Nature of roots
$\triangle > 0$	Real and unequal
$\triangle = 0$	Real and equal.
$\triangle < 0$	No real roots. (It has imaginary roots)

Example 3.45

Determine the nature of roots of the following quadratic equations

(i) $x^2 - 11x - 10 = 0$ (ii) $4x^2 - 28x + 49 = 0$ (iii) $2x^2 + 5x + 5 = 0$

Solution For $ax^2 + bx + c = 0$, the discriminant, $\triangle = b^2 - 4ac$.

(i) Here, $a = 1$; $b = -11$ and $c = -10$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-11)^2 - 4(1)(-10) = 121 + 40 = 161$$

Thus, $\triangle > 0$. Therefore, the roots are real and unequal.

(ii) Here, $a = 4$, $b = -28$ and $c = 49$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-28)^2 - 4(4)(49) = 0$$

Since $\triangle = 0$, the roots of the given equation are real and equal.

(iii) Here, $a = 2$, $b = 5$ and $c = 5$.

Now, the discriminant $\triangle = b^2 - 4ac$

$$= (5)^2 - 4(2)(5)$$

$$= 25 - 40 = -15$$

Since $\triangle < 0$, the equation has no real roots.

Example 3.46

Prove that the roots of the equation $(a-b+c)x^2 + 2(a-b)x + (a-b-c) = 0$ are rational numbers for all real numbers a and b and for all rational c.

Solution Let the given equation be of the form $Ax^2 + Bx + C = 0$. Then,

$A = a - b + c$, $B = 2(a-b)$ and $C = a - b - c$.

Now, the discriminant of $Ax^2 + Bx + c = 0$ is

$$B^2 - 4AC = [2(a-b)]^2 - 4(a-b+c)(a-b-c)$$

$$= 4(a-b)^2 - 4[(a-b)+c][(a-b)-c]$$

$$= 4(a-b)^2 - 4[(a-b)^2 - c^2]$$

$$\triangle = 4(a-b)^2 - 4(a-b)^2 + 4c^2 = 4c^2, \text{ a perfect square.}$$

Therefore, $\triangle > 0$ and it is a perfect square.

Hence, the roots of the given equation are rational numbers.

Example 3.47

Find the values of k so that the equation $x^2 - 2x(1+3k) + 7(3+2k) = 0$ has real and equal roots.

Solution The given equation is $x^2 - 2x(1+3k) + 7(3+2k) = 0$. (1)

Let the equation (1) be in the form $ax^2 + bx + c = 0$

Here, $a = 1$, $b = -2(3k+1)$, $c = 7(3+2k)$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-2(3k+1))^2 - 4(1)(7)(3+2k)$$

$$= 4(9k^2 + 6k + 1) - 28(3+2k) = 4(9k^2 - 8k - 20)$$

Given that the equation has equal roots. Thus, $\triangle = 0$

$$\Longrightarrow 9k^2 - 8k - 20 = 0$$

$$\Longrightarrow (k-2)(9k+10) = 0$$

Thus, $k = 2, \ -\frac{10}{9}$.

Exercise 3.17

1. Determine the nature of the roots of the equation.

   (i) $x^2 - 8x + 12 = 0$ (ii) $2x^2 - 3x + 4 = 0$

   (iii) $9x^2 + 12x + 4 = 0$ (iv) $3x^2 - 2\sqrt{6}\,x + 2 = 0$

   (v) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$ (vi) $(x - 2a)(x - 2b) = 4ab$

2. Find the values of k for which the roots are real and equal in each of the following equations.

   (i) $2x^2 - 10x + k = 0$ (ii) $12x^2 + 4kx + 3 = 0$

   (iii) $x^2 + 2k(x - 2) + 5 = 0$ (iv) $(k + 1)x^2 - 2(k - 1)x + 1 = 0$

3. Show that the roots of the equation $x^2 + 2(a + b)x + 2(a^2 + b^2) = 0$ are unreal.

4. Show that the roots of the equation $3p^2x^2 - 2pqx + q^2 = 0$ are not real.

5. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + c^2 + d^2 = 0$, where $ad - bc \neq 0$, are equal, prove that $\frac{a}{b} = \frac{c}{d}$.

6. Show that the roots of the equation $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ are always real and they cannot be equal unless $a = b = c$.

7. If the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, then prove that $c^2 = a^2(1 + m^2)$.

3.8.6 Relations between roots and coefficients of a quadratic equation

Consider a quadratic equation $ax^2 + bx + c = 0$, where a , b , c are real numbers and $a \neq 0$. The roots of the given equation are α and β, where

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then, the sum of the roots,

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

and the product of roots,

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2}$$

$$= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Therefore, if α, β are the roots of $ax^2 + bx + c = 0$, then

(i) the sum of the roots, $\alpha + \beta = -\dfrac{b}{a}$

(ii) the product of roots, $\alpha\beta = \dfrac{c}{a}$

Formation of quadratic equation when roots are given

Let α and β be the roots of a quadratic equation.

Then $(x - \alpha)$ and $(x - \beta)$ are factors.

$\therefore \quad (x - \alpha)(x - \beta) = 0$

$\Longrightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$

That is, $x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

> **Note**
> There are infinitely many quadratic equations with the same roots.

Example 3.48

If one of the roots of the equation $3x^2 - 10x + k = 0$ is $\dfrac{1}{3}$, then find the other root and also the value of k.

Solution The given equation is $3x^2 - 10x + k = 0$.

Let the two roots be α and β.

$\therefore \quad \alpha + \beta = \dfrac{-(-10)}{3} = \dfrac{10}{3} \qquad (1)$

Substituting $\alpha = \dfrac{1}{3}$ in (1) we get $\beta = 3$

Also, $\alpha\beta = \dfrac{k}{3}, \quad \Longrightarrow \quad k = 3$

Thus, the other root $\beta = 3$ and the value of $k = 3$.

Example 3.49

If the sum and product of the roots of the quadratic equation $ax^2 - 5x + c = 0$ are both equal to 10, then find the values of a and c.

Solution The given equation is $ax^2 - 5x + c = 0$.

Sum of the roots, $\dfrac{5}{a} = 10, \quad \Longrightarrow \quad a = \dfrac{1}{2}$

Product of the roots, $\dfrac{c}{a} = 10$

$\Longrightarrow \quad c = 10a = 10 \times \dfrac{1}{2} = 5$

Hence, $a = \dfrac{1}{2}$ and $c = 5$

Note

If α and β are the roots of $ax^2+bx+c=0$, then many expressions in α and β like $\alpha^2+\beta^2$, $\alpha^2\beta^2$, $\alpha^2-\beta^2$ etc., can be evaluated using the values of $\alpha+\beta$ and $\alpha\beta$.

Let us write some results involving α and β.

(i) $|\alpha-\beta| = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$

(ii) $\alpha^2+\beta^2 = [(\alpha+\beta)^2-2\alpha\beta]$

(iii) $\alpha^2-\beta^2 = (\alpha+\beta)(\alpha-\beta) = (\alpha+\beta)[\sqrt{(\alpha+\beta)^2-4\alpha\beta}]$ only if $\alpha\geq\beta$

(iv) $\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

(v) $\alpha^3-\beta^3 = (\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

(vi) $\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2-2\alpha^2\beta^2 = [(\alpha+\beta)^2-2\alpha\beta]^2-2(\alpha\beta)^2$

(vii) $\alpha^4-\beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

Example 3.50

If α and β are the roots of the equation $2x^2-3x-1=0$, find the values of

(i) $\alpha^2+\beta^2$ (ii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

(iii) $\alpha-\beta$ if $\alpha>\beta$ (iv) $\left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right)$

(v) $\left(\alpha+\dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha}+\beta\right)$ (vi) $\alpha^4+\beta^4$ (vii) $\dfrac{\alpha^3}{\beta}+\dfrac{\beta^3}{\alpha}$

Solution Given equation is $2x^2-3x-1=0$

Let the given equation be written as $ax^2+bx+c=0$

Then, $a=2$, $b=-3$, $c=-1$. Given α and β are the roots of the equation.

$\therefore\ \alpha+\beta = \dfrac{-b}{a} = \dfrac{-(-3)}{2} = \dfrac{3}{2}$ and $\alpha\beta = -\dfrac{1}{2}$

(i) $$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = \left(\frac{3}{2}\right)^2-2\left(-\frac{1}{2}\right) = \frac{9}{4}+1 = \frac{13}{4}$$

(ii) $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{3}{2}\right)^2-2\left(-\frac{1}{2}\right)}{-\frac{1}{2}} = \frac{13}{4}\times(-2) = -\frac{13}{2}$$

(iii) $$\alpha-\beta = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$= \left[\left(\frac{3}{2}\right)^2-4\times\left(-\frac{1}{2}\right)\right]^{\frac{1}{2}} = \left(\frac{9}{4}+2\right)^{\frac{1}{2}} = \frac{\sqrt{17}}{2}$$

(iv) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \dfrac{\frac{27}{8}+\frac{9}{4}}{\frac{-1}{2}} = -\dfrac{45}{4}$

(v) $\left(\alpha+\dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha}+\beta\right) = \dfrac{(\alpha\beta+1)(1+\alpha\beta)}{\alpha\beta}$

$$= \frac{(1+\alpha\beta)^2}{\alpha\beta} = \frac{\left(1-\frac{1}{2}\right)^2}{-\frac{1}{2}} = -\frac{1}{2}$$

(vi) $\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$= \left(\frac{13}{4}\right)^2 - 2\left(-\frac{1}{2}\right)^2 = \left(\frac{169}{16}-\frac{1}{2}\right) = \frac{161}{16}.$$

(vii) $\dfrac{\alpha^3}{\beta}+\dfrac{\beta^3}{\alpha} = \dfrac{\alpha^4+\beta^4}{\alpha\beta} = \left(\dfrac{161}{16}\right)\left(-\dfrac{2}{1}\right) = -\dfrac{161}{8}.$

Example 3.51

Form the quadratic equation whose roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

Solution Given roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

$\therefore$ Sum of the roots $= 7+\sqrt{3}+7-\sqrt{3} = 14$.

Product of roots $= (7+\sqrt{3})(7-\sqrt{3}) = (7)^2-(\sqrt{3})^2 = 49-3 = 46$.

The required equation is $x^2-(\text{sum of the roots})x+(\text{product of the roots}) = 0$

Thus, the required equation is $x^2-14x+46 = 0$

Example 3.52

If α and β are the roots of the equation

$3x^2-4x+1=0$, form a quadratic equation whose roots are $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$.

Solution Since α,β are the roots of the equation $3x^2-4x+1=0$,

we have $\alpha+\beta = \dfrac{4}{3}$, $\quad \alpha\beta = \dfrac{1}{3}$

Now, for the required equation, the sum of the roots $= \left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right) = \dfrac{\alpha^3+\beta^3}{\alpha\beta}$

$$= \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{\left(\frac{4}{3}\right)^3-3\times\frac{1}{3}\times\frac{4}{3}}{\frac{1}{3}} = \frac{28}{9}$$

Also, product of the roots $= \left(\dfrac{\alpha^2}{\beta}\right)\left(\dfrac{\beta^2}{\alpha}\right) = \alpha\beta = \dfrac{1}{3}$

$\therefore$ The required equation is $x^2-\dfrac{28}{9}x+\dfrac{1}{3} = 0$ or $9x^2-28x+3=0$

Exercise 3.18

1. Find the sum and the product of the roots of the following equations.

   (i) $x^2 - 6x + 5 = 0$ (ii) $kx^2 + rx + pk = 0$

   (iii) $3x^2 - 5x = 0$ (iv) $8x^2 - 25 = 0$

2. Form a quadratic equation whose roots are

   (i) $3, 4$ (ii) $3+\sqrt{7}, 3-\sqrt{7}$ (iii) $\frac{4+\sqrt{7}}{2}, \frac{4-\sqrt{7}}{2}$

3. If α and β are the roots of the equation $3x^2 - 5x + 2 = 0$, then find the values of

   (i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii) $\alpha - \beta$ (iii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

4. If α and β are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of $\alpha^2 + \beta^2$.

5. If α, β are the roots of $2x^2 - 3x - 5 = 0$, form a equation whose roots are α^2 and β^2.

6. If α, β are the roots of $x^2 - 3x + 2 = 0$, form a quadratic equation whose roots are $-\alpha$ and $-\beta$.

7. If α and β are the roots of $x^2 - 3x - 1 = 0$, then form a quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

8. If α and β are the roots of the equation $3x^2 - 6x + 1 = 0$, form an equation whose roots are (i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\alpha^2\beta, \beta^2\alpha$ (iii) $2\alpha+\beta, 2\beta+\alpha$

9. Find a quadratic equation whose roots are the reciprocal of the roots of the equation $4x^2 - 3x - 1 = 0$.

10. If one root of the equation $3x^2 + kx - 81 = 0$ is the square of the other, find k.

11. If one root of the equation $2x^2 - ax + 64 = 0$ is twice the other, then find the value of a

12. If α and β are the roots of $5x^2 - px + 1 = 0$ and $\alpha - \beta = 1$, then find p.

Exercise 3.19

Choose the correct answer.

1. If the system $6x - 2y = 3$, $kx - y = 2$ has a unique solution, then

   (A) $k = 3$ (B) $k \neq 3$ (C) $k = 4$ (D) $k \neq 4$

2. A system of two linear equations in two variables is inconsistent, if their graphs

   (A) coincide (B) intersect only at a point

   (C) do not intersect at any point (D) cut the x-axis

3. The system of equations $x - 4y = 8$, $3x - 12y = 24$

   (A) has infinitely many solutions (B) has no solution

   (C) has a unique solution (D) may or may not have a solution

4. If one zero of the polynomial $p(x) = (k+4)x^2+13x+3k$ is reciprocal of the other, then k is equal to

(A) 2 (B) 3 (C) 4 (D) 5

5. The sum of two zeros of the polynomial $f(x) = 2x^2 + (p+3)x + 5$ is zero, then the value of p is

(A) 3 (B) 4 (C) –3 (D) –4

6. The remainder when $x^2 - 2x + 7$ is divided by $x+4$ is

(A) 28 (B) 29 (C) 30 (D) 31

7. The quotient when $x^3 - 5x^2 + 7x - 4$ is divided by $x–1$ is

(A) $x^2 + 4x + 3$ (B) $x^2 - 4x + 3$ (C) $x^2 - 4x - 3$ (D) $x^2 + 4x - 3$

8. The GCD of $(x^3 + 1)$ and $x^4 - 1$ is

(A) $x^3 - 1$ (B) $x^3 + 1$ (C) $x+1$ (D) $x - 1$

9. The GCD of $x^2 - 2xy + y^2$ and $x^4 - y^4$ is

(A) 1 (B) $x+y$ (C) $x–y$ (D) $x^2 - y^2$

10. The LCM of $x^3 - a^3$ and $(x - a)^2$ is

(A) $(x^3 - a^3)(x + a)$ (B) $(x^3 - a^3)(x - a)^2$

(C) $(x - a)^2(x^2 + ax + a^2)$ (D) $(x + a)^2(x^2 + ax + a^2)$

11. The LCM of a^k, a^{k+3}, a^{k+5} where $k\in\mathbb{N}$ is

(A) a^{k+9} (B) a^k (C) a^{k+6} (D) a^{k+5}

12. The lowest form of the rational expression $\frac{x^2 + 5x + 6}{x^2 - x - 6}$ is

(A) $\frac{x-3}{x+3}$ (B) $\frac{x+3}{x-3}$ (C) $\frac{x+2}{x-3}$ (D) $\frac{x-3}{x+2}$

13. If $\frac{a+b}{a-b}$ and $\frac{a^3 - b^3}{a^3 + b^3}$ are the two rational expressions, then their product is

(A) $\frac{a^2 + ab + b^2}{a^2 - ab + b^2}$ (B) $\frac{a^2 - ab + b^2}{a^2 + ab + b^2}$ (C) $\frac{a^2 - ab - b^2}{a^2 + ab + b^2}$ (D) $\frac{a^2 + ab + b^2}{a^2 - ab - b^2}$

14. On dividing $\frac{x^2 - 25}{x+3}$ by $\frac{x+5}{x^2 - 9}$ is equal to

(A) $(x-5)(x-3)$ (B) $(x-5)(x+3)$ (C) $(x+5)(x-3)$ (D) $(x+5)(x+3)$

15. If $\frac{a^3}{a-b}$ is added with $\frac{b^3}{b-a}$, then the new expression is

(A) $a^2 + ab + b^2$ (B) $a^2 - ab + b^2$ (C) $a^3 + b^3$ (D) $a^3 - b^3$

16. The square root of 49 $(x^2 - 2xy + y^2)^2$ is

(A) $7|x - y|$ (B) $7(x + y)(x - y)$ (C) $7(x + y)^2$ (D) $7(x - y)^2$

17. The square root of $x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$

(A) $|x + y - z|$ (B) $|x - y + z|$ (C) $|x + y + z|$ (D) $|x - y - z|$

18. The square root of $121x^4y^8z^6(l-m)^2$ is

(A) $11x^2y^4z^4|l-m|$ (B) $11x^4y^4|z^3(l-m)|$

(C) $11x^2y^4z^6|l-m|$ (D) $11x^2y^4|z^3(l-m)|$

19. If $ax^2+bx+c=0$ has equal roots, then c is equal

(A) $\frac{b^2}{2a}$ (B) $\frac{b^2}{4a}$ (C) $-\frac{b^2}{2a}$ (D) $-\frac{b^2}{4a}$

20. If $x^2+5kx+16=0$ has no real roots, then

(A) $k>\frac{8}{5}$ (B) $k>-\frac{8}{5}$ (C) $-\frac{8}{5}<k<\frac{8}{5}$ (D) $0<k<\frac{8}{5}$

21. A quadratic equation whose one root is 3 is

(A) $x^2-6x-5=0$ (B) $x^2+6x-5=0$

(C) $x^2-5x-6=0$ (D) $x^2-5x+6=0$

22. The common root of the equations $x^2-bx+c=0$ and $x^2+bx-a=0$ is

(A) $\frac{c+a}{2b}$ (B) $\frac{c-a}{2b}$ (C) $\frac{c+b}{2a}$ (D) $\frac{a+b}{2c}$

23. If α,β are the roots of $ax^2+bx+c=0$ $a\neq 0$, then the wrong statement is

(A) $\alpha^2+\beta^2=\frac{b^2-2ac}{a^2}$ (B) $\alpha\beta=\frac{c}{a}$

(C) $\alpha+\beta=\frac{b}{a}$ (D) $\alpha-\beta=\frac{b^2-4ac}{a}$

24. If α and β are the roots of $ax^2+bx+c=0$, then one of the quadratic equations whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, is

(A) $ax^2+bx+c=0$ (B) $bx^2+ax+c=0$

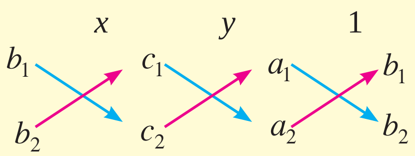
(C) $cx^2+bx+a=0$ (D) $cx^2+ax+b=0$

25. If $b=a+c$, then the equation $ax^2+bx+c=0$ has

(A) real roots (B) no roots (C) equal roots (D) no real roots

Points to Remember

- A set of finite number of linear equations in two variables x and y is called a system of linear equations in x and y. Such a system is also called simultaneous equations.
- Eliminating one of the variables first and then solving a system is called method of elimination.
- The following arrow diagram helps us very much to apply the method of cross multiplication in solving $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.



- A real number k is said to be a zero of a polynomial $p(x)$, if $p(k)=0$.

- The basic relationship between zeros and coefficients of a quadratic polynomial $ax^2 + bx + c = 0$ are,

  Sum of zeros $= -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$

  Product of zeros $= \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

- (i) For any polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$.

  (ii) $x - a$ is a factor for $p(x)$ if and only if $p(a) = 0$.

- GCD of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

- LCM of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder.

- The product of LCM and GCD of any two polynomials is equal to the product of the two polynomials.

- Let $a \in \mathbb{R}$ be a non negative real number. A square root of a, is a real number b such that $b^2 = a$. The square root of a is denoted by $\sqrt[2]{a}$ or $\sqrt{a}$.

- A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a,b,c are real numbers and $a \neq 0$.

- A quadratic equation can be solved by (i) the method of factorization (ii) the method of completing square (iii) using a quadratic formula.

- The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided $b^2 - 4ac \geq 0$.

- A quadratic equation $ax^2 + bx + c = 0$ has

  (i) two distinct real roots if $b^2 - 4ac > 0$

  (ii) two equal roots if $b^2 - 4ac = 0$, and

  (iii) no real roots if $b^2 - 4ac < 0$

Do you know?

Fermat's last theorem: The equation $x^n + y^n = z^n$ has no integer solution when $n > 2$. Fermat wrote, " I have discovered a truely remarkable proof which this margin is too small to contain ". No one was able to solve this for over 300 years until British mathematician Andrew Wiles solved it in 1994. Interestingly he came to know about this problem in his city library when he was a high school student.`

4

MATRICES

Number, place, and combination - the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred - Sylvester

- Introduction
- Formation of Matrices
- Types of Matrices
- Addition, Subtraction and Multiplication of matrices
- Matrix equations

James Joseph Sylvester

(1814-1897)

England

He made fundamental contributions to matrix theory, invariant theory, number theory and combinatorics. He determined all matrices that commute with a given matrix. He introduced many mathematical terms including "discriminant".

In 1880, the Royal Society of London awarded Sylvester the Copley Medal, a highest award for scientific achievement. In 1901, Royal Society of London instituted the Sylvester medal in his memory, to encourage mathematical research.

4.1 Introduction

In this chapter we are going to discuss an important mathematical object called "MATRIX". Here, we shall introduce matrices and study the basics of matrix algebra.

Matrices were formulated and developed as a concept during 18th and 19th centuries. In the beginning, their development was due to transformation of geometric objects and solution of linear equations. However matrices are now one of the most powerful tools in mathematics. Matrices are useful because they enable us to consider an array of many numbers as a single object and perform calculations with these symbols in a very compact form. The " mathematical shorthand" thus obtained is very elegant and powerful and is suitable for various practical problems.

The term "Matrix" for arrangement of numbers, was introduced in 1850 by James Joseph Sylvester. "Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed or produced.

Now let us consider the following system of linear equations in x and y :

$$3x - 2y = 4 \qquad (1)$$

$$2x + 5y = 9 \qquad (2)$$

We already know how to get the solution (2, 1) of this system by the method of elimination (also known as Gaussian Elimination method), where only the coefficients are used and not the variables. The same method can easily be executed and the solution can thus be obtained using matrix algebra.

4.2 Formation of matrices

Let us consider some examples of the ways that matrices can arise.

Kumar has 10 pens. We may express it as (10), with the understanding that the number inside ( ) is the number of pens that Kumar has.

Now, if Kumar has 10 pens and 7 pencils, we may express it as (10 7) with the understanding that the first number inside ( ) is the number of pens while the other one is the number of pencils.

Look at the following information :

Pens and Pencils owned by Kumar and his friends Raju and Gopu are as given below.

Kumar has 10 pens and 7 pencils
Raju has 8 pens and 4 pencils
Gopu has 6 pens and 5 pencils

This can be arranged in tabular form as follows:

	Pens	Pencils
Kumar	10	7
Raju	8	4
Gopu	6	5

This can be expressed in a rectangular array where the entries denote the number of respective items.

$$\text{(i)} \quad \begin{array}{cl} \begin{pmatrix} 10 & 7 \\ 8 & 4 \\ 6 & 5 \end{pmatrix} & \begin{array}{l} \leftarrow \text{first row} \\ \leftarrow \text{second row} \\ \leftarrow \text{third row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{first} & \text{second} \\ \text{column} & \text{column} \end{array} & \end{array}$$

The same information can also be arranged in tabular form as :

	Kumar	Raju	Gopu
Pens	10	8	6
Pencils	7	4	5

This can be expressed in a rectangular array.

$$\text{(ii)} \quad \begin{array}{cl} \begin{pmatrix} 10 & 8 & 6 \\ 7 & 4 & 5 \end{pmatrix} & \begin{array}{l} \leftarrow \text{first row} \\ \leftarrow \text{second row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{first} & \text{second} & \text{third} \\ \text{column} & \text{column} & \text{column} \end{array} & \end{array}$$

In arrangement (i), the entries in the first column represent the number of pens of Kumar, Raju and Gopu respectively and the second column represents the number of pencils owned by Kumar, Raju and Gopu respectively.

Similarly, in arrangement (ii), the entries in the first row represent the number of pens of Kumar, Raju and Gopu respectively. The entries in the second row represent the number of pencils owned by Kumar, Raju and Gopu respectively.

An arrangement or display of numbers of the above kind is called a MATRIX.

Definition

A matrix is a rectangular array of numbers in rows and columns enclosed within square brackets or parenthesis.

A matrix is usually denoted by a single capital letter like $A, B, X, Y, \cdots$. The numbers that make up a matrix are called entries or elements of the matrix. Each horizontal arrangement in a matrix is called a row of that matrix. Each vertical arrangement in a matrix is called a column of that matrix.

Some examples of matrices are

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -8 & 9 \\ 1 & 5 & -1 \end{bmatrix} \text{ and } C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

4.2.1 General form of a matrix

A matrix A with m rows and n columns, is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

where $a_{11}, a_{12}, a_{13}, \dots$ are the elements of the matrix. The above matrix can also be written as $A = [a_{ij}]_{m\times n}$ or $A = (a_{ij})_{m\times n}$, where $i = 1, 2, 3, \dots, m.$ and $j = 1, 2, 3, \dots, n.$

Here, a_{ij} is the element of the matrix lying on the intersection of the i^{th} row and j^{th} column of A.

For example, if $A = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 2 & 1 \\ 7 & 8 & 9 \end{pmatrix}$, then $a_{23} = 1$, the element which occurs in the second row and third column.

Similarly, $a_{11} = 4$, $a_{12} = 5$, $a_{13} = 3$, $a_{21} = 6$, $a_{22} = 2$, $a_{31} = 7$, $a_{32} = 8$ and $a_{33} = 9$.

4.2.2 Order or dimension of a matrix

If a matrix A has m rows and n columns, then we say that the order of A is $m \times n$ (Read as m by n).

The matrix

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ has 2 rows and 3 columns. So, the order of A is 2×3.

Note

In a $m \times n$ matrix, the first letter m always denotes the number of rows and the second letter n always denotes the number of columns.

4.3 Types of matrices

Let us learn certain types of matrices.

(i) Row matrix

A matrix is said to be a **row matrix** if it has only one row. A row matrix is also called as a row vector.

For example, $A = (5 \;\; 3 \;\; 4 \;\; 1)$ and $B = (-3 \;\; 0 \;\; 5)$ are row matrices of orders 1×4 and 1×3 respectively.

In general, $A = (a_{ij})_{1 \times n}$ is a row matrix of order $1 \times n$.

(ii) Column matrix

A matrix is said to be a column matrix if it has only one column. It is also called as a column vector.

For example, $A = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ are column matrices of orders 2×1 and 3×1 respectively.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(iii) Square matrix

A matrix in which the number of rows and the number of columns are equal is said to be a square matrix. For example,

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 5 & -7 \\ 7 & 6 & 1 \end{pmatrix}$ are square matrices of orders 2 and 3 respectively.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

The elements $a_{11}, a_{22}, a_{33}, \cdots, a_{mm}$ are called principal or leading diagonal elements of the square matrix A.

(iv) Diagonal matrix

A square matrix in which all the elements above and below the leading diagonal are equal to zero, is called a diagonal matrix. For example,

$A = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are diagonal matrices of orders 2 and 3 respectively. In general, $A = [a_{ij}]_{m \times m}$ is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

Note

Some of the leading diagonal elements of a diagonal matrix may be zero.

(v) Scalar matrix

A diagonal matrix in which all the elements along the leading diagonal are equal to a non-zero constant is called a scalar matrix. For example,

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \text{ are scalar matrices of orders 2 and 3 respectively.}$$

In general, $A = [a_{ij}]_{m\times m}$ is said to be a scalar matrix if $a_{ij} = \begin{cases} 0, \text{when } & i \neq j \\ k, \text{when } & i = j \end{cases}$ where k is a scalar.

(vi) Unit matrix

A diagonal matrix in which all the leading diagonal entries are 1 is called a unit matrix. A unit matrix of order n is denoted by I_n. For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ are unit matrices of orders 2 and 3 respectively.}$$

In general, a square matrix $A = (a_{ij})_{n\times n}$ is a unit matrix if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Note

A unit matrix is also called an identity matrix with respect to multiplication. Every unit matrix is clearly a scalar matrix. However a scalar matrix need not be a unit matrix. A unit matrix plays the role of the number 1 in numbers.

(vii) Null matrix or Zero-matrix

A matrix is said to be a null matrix or zero-matrix if each of its elements is zero. It is denoted by O. For example,

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ are null matrices of order } 2\times 3 \text{ and } 2\times 2.$$

Note

(i) A zero-matrix need not be a square matrix. (ii) Zero-matrix plays the role of the number zero in numbers. (iii) A matrix does not change if the zero-matrix of same order is added to it or subtracted from it.

(viii) Transpose of a matrix

Definition The transpose of a matrix A is obtained by interchanging rows and columns of the matrix A and it is denoted by A^T (read as A transpose). For example,

$$\text{if} \quad A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}, \quad \text{then } A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$$

In general, if $A = [a_{ij}]_{m\times n}$ then

$$A^T = [b_{ij}]_{n\times m}, \text{ where } b_{ij} = a_{ji}, \text{ for } i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m.$$

Example 4.1

The table shows a five-day forecast indicating high (H) and low (L) temperatures in Fahrenheit. Organise the temperatures in a matrix where the first and second rows represent the High and Low temperatures respectively and identify which day will be the warmest?

Mon	Tue	Wed	Thu	Fri
H 88	H 90	H 86	H 84	H 85
L 54	L 56	L 53	L 52	L 52

Solution The above information can be represented in matrix form as

$$A = \begin{array}{c} \\ H \\ L \end{array}\begin{array}{c} \begin{array}{ccccc} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} & \text{Fri} \end{array} \\ \begin{pmatrix} 88 & 90 & 86 & 84 & 85 \\ 54 & 56 & 53 & 52 & 52 \end{pmatrix} \end{array}. \quad \text{That is , } A = \begin{pmatrix} 88 & 90 & 86 & 84 & 85 \\ 54 & 56 & 53 & 52 & 52 \end{pmatrix}$$

By reading through the first row (High), the warmest day is Tuesday.

Example 4.2

The amount of fat, carbohydrate and protein in grams present in each food item respectively are as follows:

	Item 1	Item 2	Item 3	Item 4
Fat	5	0	1	10
Carbohydrate	0	15	6	9
Protein	7	1	2	8

Use the information to write 3×4 and 4×3 matrices.

Solution The above information can be represented in the form of 3×4 matrix as

$$A = \begin{pmatrix} 5 & 0 & 1 & 10 \\ 0 & 15 & 6 & 9 \\ 7 & 1 & 2 & 8 \end{pmatrix}$$ where the columns correspond to food items. We write

a 4×3 matrix as $B = \begin{pmatrix} 5 & 0 & 7 \\ 0 & 15 & 1 \\ 1 & 6 & 2 \\ 10 & 9 & 8 \end{pmatrix}$ where the rows correspond to food items.

Example 4.3

Let $A = [a_{ij}] = \begin{bmatrix} 1 & 4 & 8 \\ 6 & 2 & 5 \\ 3 & 7 & 0 \\ 9 & -2 & -1 \end{bmatrix}$. Find

(i) the order of the matrix (ii) the elements a_{13} and a_{42} (iii) the position of the element 2.

Solution (i) Since the matrix A has 4 rows and 3 columns, A is of order 4×3.

(ii) The element a_{13} is in the first row and third column. $\therefore\ a_{13} = 8$.

Similarly, $a_{42} = -2$, the element in 4th row and 2nd column.

(iii) The element 2 occurs in 2nd row and 2nd column $\therefore\ a_{22} = 2$.

Example 4.4

Construct a 2×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = |2i - 3j|$

Solution In general a 2×3 matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Now $a_{ij} = |2i - 3j|$ where $i = 1,2$ and $j = 1,2,3$

$$a_{11} = |2(1) - 3(1)| = |-1| = 1, \quad a_{12} = |2(1) - 3(2)| = 4, \quad a_{13} = |2(1) - 3(3)| = 7$$

$$a_{21} = [2(2) - 3] = 1, \quad a_{22} = |2(2) - 3(2)| = 2, \quad a_{23} = |2(2) - 9| = 5$$

Hence the required matrix $A = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 5 \end{pmatrix}$

Example 4.5

If $A = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$, then find A^T and $(A^T)^T$

Solution

$$A = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$$

The transpose A^T of a matrix *A*, is obtained by interchanging rows and columns of the matrix *A*.

Thus, $A^T = \begin{pmatrix} 8 & 1 \\ 5 & -3 \\ 2 & 4 \end{pmatrix}$

Similarly $(A^T)^T$ is obtained by interchanging rows and columns of the matrix A^T.

Hence $(A^T)^T = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$

Note

From the above example, we see that $(A^T)^T = A$. In fact, it is true that $(B^T)^T = B$ for any matrix *B*. Also, $(kA)^T = kA^T$ for any scalar *k*.

Exercise 4.1

1. The rates for the entrance tickets at a water theme park are listed below:

	Week Days rates(₹)	Week End rates(₹)
Adult	400	500
Children	200	250
Senior Citizen	300	400

Write down the matrices for the rates of entrance tickets for adults, children and senior citizens. Also find the dimensions of the matrices.

2. There are 6 Higher Secondary Schools, 8 High Schools and 13 Primary Schools in a town. Represent these data in the form of 3×1 and 1×3 matrices.

3. Find the order of the following matrices.

(i) $\begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & -2 & 6 \\ 6 & -1 & 1 \\ 2 & 4 & 5 \end{pmatrix}$ (iv) $\begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 9 & 7 \\ 6 & 4 \end{pmatrix}$

4. A matrix has 8 elements. What are the possible orders it can have?

5. A matrix consists of 30 elements. What are the possible orders it can have?.

6. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by

(i) $a_{ij} = ij$ (ii) $a_{ij} = 2i - j$ (iii) $a_{ij} = \dfrac{i-j}{i+j}$

7. Construct a 3×2 matrix $A = [a_{ij}]$ whose elements are given by

(i) $a_{ij} = \dfrac{i}{j}$ (ii) $a_{ij} = \dfrac{(i-2j)^2}{2}$ (iii) $a_{ij} = \dfrac{|2i-3j|}{2}$

8. If $A = \begin{pmatrix} 1 & -1 & 3 & 2 \\ 5 & -4 & 7 & 4 \\ 6 & 0 & 9 & 8 \end{pmatrix}$, (i) find the order of the matrix (ii) write down the elements a_{24} and a_{32} (iii) in which row and column does the element 7 occur?

9. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 5 & 0 \end{pmatrix}$, then find the transpose of A.

10. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$, then verify that $(A^T)^T = A$.

4.4 Operation on matrices

In this section, we shall discuss the equality of matrices, multiplication of a matrix by a scalar, addition, subtraction and multiplication of matrices.

Equality of matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal if

(i) they are of the same order and

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, the matrices $\begin{pmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 & 1 \\ 3 & 9 & 5 \end{pmatrix}$ are not equal as the orders of the matrices are different.

Also $\begin{pmatrix} 1 & 2 \\ 8 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 8 \\ 2 & 5 \end{pmatrix}$, since some of the corresponding elements are not equal.

Example 4.6

Find the values of x, y and z if $\begin{pmatrix} x & 5 & 4 \\ 5 & 9 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & z \\ 5 & y & 1 \end{pmatrix}$

Solution As the given matrices are equal, their corresponding elements must be equal. Comparing the corresponding elements, we get $x = 3$, $y = 9$ and $z = 4$.

Example 4.7

Solve : $\begin{pmatrix} y \\ 3x \end{pmatrix} = \begin{pmatrix} 6 - 2x \\ 31 + 4y \end{pmatrix}$

Solution Since the matrices are equal, the corresponding elements are equal.

Comparing the corresponding elements, we get $y = 6 - 2x$ and $3x = 31 + 4y$.

Using $y = 6 - 2x$ in the other equation, we get $3x = 31 + 4(6 - 2x)$

$$3x = 31 + 24 - 8x$$

$\therefore$ $x = 5$ and hence $y = 6 - 2(5) = -4$.

Thus, $x = 5$ and $y = -4$.

Multiplication of a matrix by a scalar

Definition

For a given matrix $A = [a_{ij}]_{m\times n}$ and a scalar (real number) k, we define a new matrix $B = [b_{ij}]_{m\times n}$ where $b_{ij} = ka_{ij}$ for all i and j

Thus, the matrix B is obtained by multiplying each entry of A by the scalar k and written as $B = kA$. This multiplication is called scalar multiplication.

For example, if $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ then $kA = k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$

Example 4.8

If $A = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 6 & -5 \end{pmatrix}$ then find $3A$

Solution The matrix $3A$ is obtained by multiplying every element of A by 3.

$$3A = 3\begin{pmatrix} -1 & 2 & 4 \\ 3 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 3(-1) & 3(2) & 3(4) \\ 3(3) & 3(6) & 3(-5) \end{pmatrix} = \begin{pmatrix} -3 & 6 & 12 \\ 9 & 18 & -15 \end{pmatrix}$$

Addition of matrices

Matrices A and B given below show the marks obtained by 3 boys and 3 girls in the subjects Mathematics and Science respectively.

$$\overset{\text{Mathematics}}{A = \begin{pmatrix} 45 & 72 & 81 \\ 30 & 90 & 65 \end{pmatrix}\begin{matrix}\text{Boys}\\ \text{Girls}\end{matrix}} \qquad \overset{\text{Science}}{B = \begin{pmatrix} 51 & 80 & 90 \\ 42 & 85 & 70 \end{pmatrix}\begin{matrix}\text{Boys}\\ \text{Girls}\end{matrix}}$$

To find the total marks obtained by each student, we shall add the corresponding entries of A and B. We write

$$A+B=\begin{pmatrix}45 & 72 & 81\\ 30 & 90 & 65\end{pmatrix}+\begin{pmatrix}51 & 80 & 90\\ 42 & 85 & 70\end{pmatrix}$$

$$=\begin{pmatrix}45+51 & 72+80 & 81+90\\ 30+42 & 90+85 & 65+70\end{pmatrix}=\begin{pmatrix}96 & 152 & 171\\ 72 & 175 & 135\end{pmatrix}$$

The final matrix shows that the first boy scores a total of 96 marks in Mathematics and Science. Similarly, the last girl scores a total of 135 marks in Mathematics and Science.

Hence, we observe that the sum of two matrices of same order is a matrix obtained by adding the corresponding entries of the given matrices.

Definition

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then the addition of A and B is a matrix $C=[c_{ij}]_{mxn}$ where $c_{ij}=a_{ij}+b_{ij}$ for all i and j.

Note that the operation of addition on matrices is defined as for numbers. The addition of two matrices A and B is denoted by $A+B$. Addition is not defined for matrices of different orders.

Example 4.9

Let $A=\begin{pmatrix}8 & 3 & 2\\ 5 & 9 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & -1\\ 3 & 0\end{pmatrix}$. Find $A+B$ if it exists.

Solution Since A is order of 2×3 and B is of order 2×2, addition of matrices A and B is not possible.

Example 4.10

If $A=\begin{pmatrix}5 & 6 & -2 & 3\\ 1 & 0 & 4 & 2\end{pmatrix}$ and $B=\begin{pmatrix}3 & -1 & 4 & 7\\ 2 & 8 & 2 & 3\end{pmatrix}$, then find $A+B$

Solution Since A and B are of the same order 2×4, addition of A and B is defined.

So, $$A+B=\begin{pmatrix}5 & 6 & -2 & 3\\ 1 & 0 & 4 & 2\end{pmatrix}+\begin{pmatrix}3 & -1 & 4 & 7\\ 2 & 8 & 2 & 3\end{pmatrix}$$

$$=\begin{pmatrix}5+3 & 6-1 & -2+4 & 3+7\\ 1+2 & 0+8 & 4+2 & 2+3\end{pmatrix}$$

Thus, $$A+B=\begin{pmatrix}8 & 5 & 2 & 10\\ 3 & 8 & 6 & 5\end{pmatrix}$$

Negative of a matrix

The negative of a matrix $A=[a_{ij}]_{mxn}$ is denoted by $-A$ and is defined as $-A=(-1)A$. That is, $-A=[b_{ij}]_{mxn}$ where $b_{ij}=-a_{ij}$ for all i and j.

Subtraction of matrices

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then the subtraction $A-B$ is defined as $A-B=A+(-1)B$. That is, $A-B=[c_{ij}]$ where $c_{ij}=a_{ij}-b_{ij}$ for all i and j.

Example 4.11

Matrix A shows the weight of four boys and four girls in kg at the beginning of a diet programme to lose weight. Matrix B shows the corresponding weights after the diet programme.

$$A = \begin{array}{c} \begin{pmatrix} 35 & 40 & 28 & 45 \\ 42 & 38 & 41 & 30 \end{pmatrix} \end{array} \begin{array}{l} \text{Boys} \\ \text{Girls} \end{array}, \quad B = \begin{array}{c} \begin{pmatrix} 32 & 35 & 27 & 41 \\ 40 & 30 & 34 & 27 \end{pmatrix} \end{array} \begin{array}{l} \text{Boys} \\ \text{Girls} \end{array}$$

Find the weight loss of the Boys and Girls.

Solution Weight loss matrix $A - B = \begin{pmatrix} 35 & 40 & 28 & 45 \\ 42 & 38 & 41 & 30 \end{pmatrix} - \begin{pmatrix} 32 & 35 & 27 & 41 \\ 40 & 30 & 34 & 27 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 5 & 1 & 4 \\ 2 & 8 & 7 & 3 \end{pmatrix}.$$

4.5 Properties of matrix addition

(i) Matrix addition is commutative

If A and B are any two matrices of same order, then $A+B = B+A$

(ii) Matrix addition is associative

If A, B and C are any three matrices of same order, then $A + (B + C) = (A + B) + C$

(iii) Existence of additive identity

Null or zero matrix is the additive identity for matrix addition. If A is a matrix of order $m \times n$, then $A + O = O + A = A$, where O is the null matrix of order $m \times n$,

(iv) Existence of additive inverse

For a matrix A, B is called the additive inverse of A if $B + A = A + B = O$.

Since $A + (-A) = (-A) + A = O$, $-A$ is the additive inverse of A.

Note

The additive inverse of a matrix is its negative matrix and it is unique (only one).

Exercise 4.2

1. Find the values of x, y and z from the matrix equation

$$\begin{pmatrix} 5x + 2 & y - 4 \\ 0 & 4z + 6 \end{pmatrix} = \begin{pmatrix} 12 & -8 \\ 0 & 2 \end{pmatrix}$$

2. Solve for x and y if $\begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$

3. If $A = \begin{pmatrix} 2 & 3 \\ -9 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 7 & -1 \end{pmatrix}$, then find the additive inverse of A.

4. Let $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & -1 \\ 4 & 3 \end{pmatrix}$. Find the matrix C if $C = 2A + B$.

5. If $A = \begin{pmatrix} 4 & -2 \\ 5 & -9 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 2 \\ -1 & -3 \end{pmatrix}$ find $6A - 3B$.

6. Find a and b if $a\begin{pmatrix} 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$.

7. Find X and Y if $2X + 3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and $3X + 2Y = \begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}$.

8. Solve for x and y if $\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} + 3\begin{pmatrix} 2x \\ -y \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \end{pmatrix}$.

9. If $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then
verify (i) $A + B = B + A$ (ii) $A + (-A) = O = (-A) + A$.

10. If $A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & -2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, then
verify that $A + (B + C) = (A + B) + C$.

11. An electronic company records each type of entertainment device sold at three of their branch stores so that they can monitor their purchases of supplies. The sales in two weeks are shown in the following spreadsheets.

		T.V.	**DVD**	**Videogames**	**CD Players**
Week I	Store I	30	15	12	10
	Store II	40	20	15	15
	Store III	25	18	10	12
Week II	Store I	25	12	8	6
	Store II	32	10	10	12
	Store III	22	15	8	10

Find the sum of the items sold out in two weeks using matrix addition.

12. The fees structure for one-day admission to a swimming pool is as follows:

Daily Admission Fees in ₹		
Membership	Children	Adult
Before 2.00 p.m.	20	30
After 2.00 p.m.	30	40
Non-Membership		
Before 2.00 p.m.	25	35
After 2.00 p.m.	40	50

Write the matrix that represents the additional cost for non-membership.

4.6 Multiplication of matrices

Suppose that Selvi wants to buy 3 pens and 2 pencils, while Meena needs 4 pens and 5 pencils. Each pen and pencil cost ₹10 and ₹5 respectively. How much money does each need to spend?

Clearly, Since $3\times 10 + 2\times 5 = 40$, Selvi needs ₹ 40.

Since $4\times 10 + 5\times 5 = 65$, Meena needs ₹ 65.

We can also do this using matrix multiplication.

Let us write the above information as follows:

Requirements	Price (in ₹)	Money Needed (in ₹)
$\begin{array}{c} \text{Selvi} \\ \text{Meena} \end{array} \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 5 \end{pmatrix}$	$\begin{pmatrix} 3\times 10 + 2\times 5 \\ 4\times 10 + 5\times 5 \end{pmatrix} = \begin{pmatrix} 40 \\ 65 \end{pmatrix}$

Suppose the cost of each pen and pencil in another shop are ₹8 and ₹4 respectively. The money required by Selvi and Meena will be $3\times 8 + 2\times 4 =$ ₹32 and $4\times 8 + 5\times 4 =$ ₹ 52 . The above information can be represented as

Requirements	Price (in ₹)	Money Needed (in ₹)
$\begin{array}{c} \text{Selvi} \\ \text{Meena} \end{array} \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 3\times 8 + 2\times 4 \\ 4\times 8 + 5\times 4 \end{pmatrix} = \begin{pmatrix} 32 \\ 52 \end{pmatrix}$

Now, the above information in both the cases can be combined in matrix form as shown below.

Requirements	Price (in ₹)	Money needed (in ₹)
$\begin{array}{c} \text{Selvi} \\ \text{Meena} \end{array} \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$	$\begin{pmatrix} 10 & 8 \\ 5 & 4 \end{pmatrix}$	$\begin{pmatrix} 3\times 10 + 2\times 5 & 3\times 8 + 2\times 4 \\ 4\times 10 + 5\times 5 & 4\times 8 + 5\times 4 \end{pmatrix} = \begin{pmatrix} 40 & 32 \\ 65 & 52 \end{pmatrix}$

From the above example, we observe that multiplication of two matrices is possible if the number of columns in the first matrix is equal to the number of rows in the second matrix. Further, for getting the elements of the product matrix, we take rows of the first matrix and columns of the second matrix, multiply them element-wise and sum it.

The following simple example illustrates how to get the elements of the product matrix when the product is defined.

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix}$. Then the product of AB is defined and is given by

$$AB = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix}$$

Step 1 : Multiply the numbers in the first row of A by the numbers in the first column of B, add the products, and put the result in the first row and first column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3) + (-1)5 & \\ & \end{pmatrix}$$

Step 2: Follow the same procedure as in step 1, using the first row of A and second column of B. Write the result in the first row and second column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ & \end{pmatrix}$$

Step 3: Follow the same procedure with the second row of A and first column of B. Write the result in the second row and first column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & \end{pmatrix}$$

Step 4: The procedure is the same for the numbers in the second row of A and second column of B.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & 3(-9)+4(7) \end{pmatrix}$$

Step 5: Simplify to get the product matrix AB

$$\begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & 3(-9)+4(7) \end{pmatrix} = \begin{pmatrix} 1 & -25 \\ 29 & 1 \end{pmatrix}$$

Definition

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then the product matrix AB is defined and is of order $m\times p$. This fact is explained in the following diagram.

$$A_{m\times \boxed{n \quad \text{same} \quad n}}\; B_{\times p}$$

product matrix AB is of order $m\times p$

Example 4.12

Determine whether each matrix product is defined or not. If the product is defined, state the dimension of the product matrix.

(i) $A_{2\times 5}$ and $B_{5\times 4}$ (ii) $A_{1\times 3}$ and $B_{4\times 3}$

Solution

(i) Now, the number of columns in A and the number of rows in B are equal.

So, the product AB is defined.

Also, the product matrix AB is of order 2×4.

(ii) Given that A is of order 1×3 and B is of order 4×3

Now, the number of columns in A and the number of rows in B are not equal.

So, the matrix product AB is not defined.

Example 4.13

Solve $\begin{pmatrix}3 & 2\\ 4 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}8\\ 13\end{pmatrix}$

Solution Given that $\begin{pmatrix}3 & 2\\ 4 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}8\\ 13\end{pmatrix}$

$$\Longrightarrow \quad \begin{pmatrix}3x+2y\\ 4x+5y\end{pmatrix}=\begin{pmatrix}8\\ 13\end{pmatrix}$$

Equating the corresponding elements, we get

$$3x+2y=8 \quad \text{and} \quad 4x+5y=13$$

$$\Longrightarrow 3x+2y-8=0 \quad \text{and} \quad 4x+5y-13=0.$$

Solving the equations by the method of cross multiplication, we get

$$\begin{array}{cccccc} & x & & y & 1 & \\ 2 & & -8 & & 3 & 2\\ 5 & & -13 & & 4 & 5\end{array}$$

$$\Longrightarrow \frac{x}{-26+40}=\frac{y}{-32+39}=\frac{1}{15-8} \quad \Longrightarrow \quad \frac{x}{14}=\frac{y}{7}=\frac{1}{7}$$

Thus, $x=2,\ y=1$

Example 4.14

If $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, then show that $A^2-(a+d)A=(bc-ad)I_2$

Solution Consider $A^2=A\times A$

$$=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix} \qquad (1)$$

Now, $(a+d)A=(a+d)\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$$=\begin{pmatrix}a^2+ad & ab+bd\\ ac+cd & ad+d^2\end{pmatrix} \qquad (2)$$

From (1) and (2) we get,

$$A^2-(a+d)A=\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix}-\begin{pmatrix}a^2+ad & ab+bd\\ ac+cd & ad+d^2\end{pmatrix}$$

$$=\begin{pmatrix}bc-ad & 0\\ 0 & bc-ad\end{pmatrix}=(bc-ad)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Thus, $A^2-(a+d)A=(bc-ad)I_2$

4.7 Properties of matrix multiplication

The matrix multiplication does not retain some important properties enjoyed by multiplication of numbers. Some of such properties are (i) $AB \neq BA$ (in general) (ii) $AB=0$ does not imply that either A or B is a zero-matrix and (iii) $AB=AC$, A is a non-zero matrix, does not imply always that $B=C$.

For example, let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, C = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then,

(i) $AB \neq BA$ (ii) $AD = O$, however, A and D are not zero-matrices and (iii) $AB = AC$, but $B \neq C$. Let us see some properties of matrix multiplication through examples.

(i) Matrix multiplication is not commutative in general

If A and B are two matrices and if AB and BA both are defined, it is not necessary that $AB = BA$.

Example 4.15

If $A = \begin{pmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{pmatrix}$, then find AB and BA if they exist.

Solution The matrix A is of order 3×2 and B is of order 2×3. Thus, both the products AB and BA are defined.

$$\text{Now, } AB = \begin{pmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 72-42 & -24+7 & 16+35 \\ -18+24 & 6-4 & -4-20 \\ 0+18 & 0-3 & 0-15 \end{pmatrix} = \begin{pmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{pmatrix}$$

Similarly,

$$BA = \begin{pmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{pmatrix}\begin{pmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 78 & -69 \\ 50 & -61 \end{pmatrix}. \qquad \text{(Note that } AB \neq BA\text{)}$$

Remarks

Multiplication of two diagonal matrices of same order is commutative. Also, under matrix multiplication unit matrix commutes with any square matrix of same order.

(ii) Matrix multiplication is always associative

For any three matrices A, B and C, we have $(AB)C = A(BC)$, whenever both sides of the equality are defined.

(iii) Matrix multiplication is distributive over addition

For any three matrices A, B and C, we have (i) $A(B+C) = AB + AC$

(ii) $(A+B)C = AC + BC$, whenever both sides of equality are defined.

Example 4.16

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 5 \\ 6 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix}$ verify that $A(B+C) = AB + AC$

Solution Now, $B + C = \begin{pmatrix} -2 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 1 & 10 \end{pmatrix}$

$$\text{Thus, } A(B+C) = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -1 & 6 \\ 1 & 10 \end{pmatrix} = \begin{pmatrix} -1 & 38 \\ 5 & 34 \end{pmatrix} \qquad (1)$$

Now, $AB + AC = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix}$

$$= \begin{pmatrix} -6+12 & 15+14 \\ 2+24 & -5+28 \end{pmatrix} + \begin{pmatrix} 3-10 & 3+6 \\ -1-20 & -1+12 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 29 \\ 26 & 23 \end{pmatrix} + \begin{pmatrix} -7 & 9 \\ -21 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 38 \\ 5 & 34 \end{pmatrix} \qquad (2)$$

From (1) and (2), we have $A(B+C) = AB + AC$

(iv) Existence of multiplicative identity

In ordinary algebra we have the number 1, which has the property that its product with any number is the number itself. We now introduce an analogous concept in matrix algebra.

For any square matrix A of order n, we have $AI = IA = A$, where I is the unit matrix of order n. Hence, I is known as the identity matrix under multiplication.

Example 4.17

If $A = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix}$, then verify $AI = IA = A$, where I is the unit matrix of order 2.

Solution

Now, $AI = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+3 \\ 9+0 & 0-6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix} = A$

Also, $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix} = \begin{pmatrix} 1+0 & 3+0 \\ 0+9 & 0-6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix} = A$

Hence $AI = IA = A$.

(v) Existence of multiplicative inverse

If A is a square matrix of order n, and if there exists a square matrix B of the same order n, such that $AB = BA = I$, where I is the unit matrix of order n, then B is called the multiplicative inverse matrix of A and it is denoted by A^{-1}

Note

(i) Some of the square matrices like $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ do not have multiplicative inverses.

(ii) If B is the multiplicative inverse of A, then A is the multiplicative inverse of B.

(iii) If multiplicative inverse of a square matrix exists, then it is unique.

Example 4.18

Prove that $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ are multiplicative inverses to each other.

Solution Now, $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Also, $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\therefore$ The given matrices are inverses to each other under matrix multiplication.

(vi) Reversal law for transpose of matrices

If A and B are two matrices and if AB is defined , then $(AB)^T = B^T A^T$

Example 4.19

If $A = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & -6 \end{pmatrix}$, then verify that $(AB)^T = B^T A^T$

Solution Now, $AB = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}\begin{pmatrix} 1 & 3 & -6 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{pmatrix}$

Thus, $(AB)^T = \begin{pmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{pmatrix}$ (1)

Now, $B^T A^T = \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix}\begin{pmatrix} -2 & 4 & 5 \end{pmatrix}$

$= \begin{pmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{pmatrix}$ (2)

From (1) and (2), we get $(AB)^T = B^T A^T$.

Exercise 4.3

1. Determine whether the product of the matrices is defined in each case. If so, state the order of the product.

   (i) AB, where $A = [a_{ij}]_{4\times3}$, $B = [b_{ij}]_{3\times2}$ (ii) PQ, where $P = [p_{ij}]_{4\times3}$, $Q = [q_{ij}]_{4\times3}$

   (iii) MN, where $M = [m_{ij}]_{3\times1}$, $N = [n_{ij}]_{1\times5}$ (iv) RS, where $R = [r_{ij}]_{2\times2}$, $S = [s_{ij}]_{2\times2}$

2. Find the product of the matrices, if exists,

   (i) $\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 7 \end{pmatrix}$

   (iii) $\begin{pmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 \\ -3 \end{pmatrix}\begin{pmatrix} 2 & -7 \end{pmatrix}$

3. A fruit vendor sells fruits from his shop. Selling prices of Apple, Mango and Orange are ₹ 20, ₹ 10 and ₹ 5 each respectively. The sales in three days are given below

Day	Apples	Mangoes	Oranges
1	50	60	30
2	40	70	20
3	60	40	10

Write the matrix indicating the total amount collected on each day and hence find the total amount collected from selling of all three fruits combined.

4. Find the values of x and y if $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 9 & 0 \end{pmatrix}$.

5. If $A = \begin{pmatrix} 5 & 3 \\ 7 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} -5 \\ -11 \end{pmatrix}$ and if $AX = C$, then find the values of x and y.

6. If $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ then show that $A^2 - 4A + 5I_2 = O$.

7. If $A = \begin{pmatrix} 3 & 2 \\ 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}$ then find AB and BA. Are they equal?

8. If $A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $C = (2 \quad 1)$ verify $(AB)C = A(BC)$.

9. If $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ verify that $(AB)^T = B^T A^T$.

10. Prove that $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$ are inverses to each other under matrix multiplication.

11. Solve $(x \quad 1)\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ 5 \end{pmatrix} = (\,0\,)$.

12. If $A = \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 6 \\ 3 & -2 \end{pmatrix}$, then prove that $(A+B)^2 \neq A^2 + 2AB + B^2$.

13. If $A = \begin{pmatrix} 3 & 3 \\ 7 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 7 \\ 0 & 9 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$, find $(A+B)C$ and $AC + BC$.

Is $(A+B)C = AC + BC$?

Exercise 4.4

Choose the correct answer.

1. Which one of the following statements is not true?

(A) A scalar matrix is a square matrix

(B) A diagonal matrix is a square matrix

(C) A scalar matrix is a diagonal matrix

(D) A diagonal matrix is a scalar matrix.

2. Matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if

(A) $m < n$ (B) $m > n$ (C) $m = 1$ (D) $m = n$

3. If $\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix} = \begin{pmatrix} 1 & y-2 \\ 8 & 8 \end{pmatrix}$ then the values of x and y respectively are

(A) $-2, 7$ (B) $-\frac{1}{3}, 7$ (C) $-\frac{1}{3}, -\frac{2}{3}$ (D) $2, -7$

4. If $A = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ then $A + B$

(A) $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ (B) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(C) $\begin{pmatrix} -14 \end{pmatrix}$ (D) not defined

5. If a matrix is of order 2×3, then the number of elements in the matrix is

(A) 5 (B) 6 (C) 2 (D) 3

6. If $\begin{pmatrix} 8 & 4 \\ x & 8 \end{pmatrix} = 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ then the value of x is

(A) 1 (B) 2 (C) $\frac{1}{4}$ (D) 4

7. If A is of order 3×4 and B is of order 4×3, then the order of BA is

(A) 3×3 (B) 4×4 (C) 4×3 (D) not defined

8. If $A\times\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$ then the order of A is

(A) 2×1 (B) 2×2 (C) 1×2 (D) 3×2

9. If A and B are square matrices such that $AB = I$ and $BA = I$, then B is

(A) Unit matrix (B) Null matrix

(C) Multiplicative inverse matrix of A (D) $-A$

10. If $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, then the values of x and y respectively, are

(A) $2, 0$ (B) $0, 2$ (C) $0, -2$ (D) $1, 1$

11. If $A = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ and $A + B = O$, then B is

(A) $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$ (C) $\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

12. If $A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$, then A^2 is

(A) $\begin{pmatrix} 16 & 4 \\ 36 & 9 \end{pmatrix}$ (B) $\begin{pmatrix} 8 & -4 \\ 12 & -6 \end{pmatrix}$ (C) $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$

13. A is of order $m \times n$ and B is of order $p \times q$, addition of A and B is possible only if

(A) $m = p$ (B) $n = q$ (C) $n = p$ (D) $m = p, n = q$

14. If $\begin{pmatrix} a & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, then the value of a is

(A) 8 (B) 4 (C) 2 (D) 11

15. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

16. If $A = [a_{ij}]_{2\times 2}$ and $a_{ij} = i + j$, then $A =$

(A) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ (C) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (D) $\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$

17. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the values of a, b, c and d respectively are

(A) $-1, 0, 0, -1$ (B) 1, 0, 0, 1 (C) $-1, 0, 1, 0$ (D) 1, 0, 0, 0

18. If $A = \begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix}$ and $A + B = \begin{pmatrix} -1 & 0 \\ 2 & -4 \end{pmatrix}$ then the matrix $B =$

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$ (C) $\begin{pmatrix} -8 & -2 \\ 1 & -7 \end{pmatrix}$ (D) $\begin{pmatrix} 8 & 2 \\ -1 & 7 \end{pmatrix}$

19. If $\begin{pmatrix} 5 & x & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \end{pmatrix}$, then the value of x is

(A) 7 (B) -7 (C) $\frac{1}{7}$ (D) 0

20. Which one of the following is true for any two square matrices A and B of same order?.

(A) $(AB)^T = A^T B^T$ (B) $(A^T B)^T = A^T B^T$ (C) $(AB)^T = BA$ (D) $(AB)^T = B^T A^T$

Points to Remember

- A matrix is a rectangular array of numbers.
- A matrix having m rows and n columns, is of the order $m \times n$.
- $A = [a_{ij}]_{m\times n}$ is a row matrix if $m = 1$.
- $A = [a_{ij}]_{m\times n}$ is a column matrix if $n = 1$.
- $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.
- $A = [a_{ij}]_{n\times n}$ is diagonal matrix if $a_{ij} = 0$ when $i \neq j$.
- $A = [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = 0$ when $i \neq j$ and $a_{ij} = k$, when $i = j$. (k is a non-zero constant).
- $A = [a_{ij}]$ is unit matrix if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.
- A matrix is said to be a zero matrix if all its elements are zero.
- Two matrices A and B are equal if

  A and B are of same order and their corresponding entries are equal.
- Addition or subtraction of two matrices are possible only when they are of same order.
- Matrix addition is commutative

  That is, $A + B = B + A$, if A and B are matrices of same order.
- Matrix addition is Associative

  That is, $(A + B) + C = A + (B + C)$, if A, B and C are matrices of same order.
- If A is a matrix of order $m\times n$ and B is a matrix of order $n\times p$, then the product matrix AB is defined and is of order $m\times p$.
- Matrix multiplication is not commutative in general. i.e., $AB \neq BA$.
- Matrix multiplication is associative. i.e., $(AB)C = A(BC)$, if both sides are defined.
- $(A^T)^T = A$, $(A + B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$
- Matrices A and B are multiplicative inverses to each other if $AB = BA = I$.
- If $AB = O$, it is not necessary that $A = O$ or $B = O$.

  That is, product of two non-zero matrices may be a zero matrix.

Do you know?

The Abel Prize , which was awarded for the first time in 2003, amounts to One Million US dollar. It is an International Prize awarded by Norwegian Academy of Science and presented annually by the King of Norway to one or more outstanding Mathematicians.

S.R. Srinivasa Varadhan, an Indian-American Mathematician born in Chennai, was awarded the Abel Prize in 2007 for his fundamental contributions to Probability Theory and in particular for creating a unified theory of large deviations.

5

COORDINATE GEOMETRY

No human investigation can be called real science if it cannot be demonstrated mathematically - Leonardo de Vinci

- Introduction
- Section Formula
- Area of Triangle and Quadrilateral
- Straight Lines

Pierre de Fermat
(1601-1665)
France

Together with Rene Descartes, Fermat was one of the two leading mathematicians of the first half of the 17th century. He discovered the fundamental principles of analytical geometry. He discovered an original method of finding the greatest and the smallest ordinates of curved lines.

He made notable contributions to coordinate geometry. Fermat's pioneering work in analytic geometry was circulated in manuscript form in 1636, predating the publication of Descarte's famous "La geometrie".

5.1 Introduction

Coordinate geometry, also known as analytical geometry is the study of geometry using a coordinate system and the principles of algebra and analysis. It helps us to interpret algebraic results geometrically and serves as a bridge between algebra and geometry. A systematic study of geometry using algebra was carried out by a French philosopher and a mathematician Rene Descartes. The use of coordinates was Descartes's great contribution to mathematics, which revolutionized the study of geometry. He published his book "La Geometry" in 1637. In this book, he converted a geometric problem into an algebraic equation, simplified and then solved the equation geometrically. French mathematician Pierre De Fermat also formulated the coordinate geometry at the same period and made great contribution to this field. In 1692, a German mathematician Gottfried Wilhelm Von Leibnitz introduced the modern terms like abscissa and ordinate in coordinate geometry . According to Nicholas Murray Butler, "The analytical geometry of Descartes and the calculus of Newton and Leibntiz have expanded into the marvelous mathematical method".

In class IX, we have studied the basic concepts of the coordinate geometry namely, the coordinate axes, plane, plotting of points in a plane and the distance between two points. In this chapter, we shall study about section formula, area of a triangle, slope and equation of a straight line.

5.2 Section formula

Let us look at the following problem.

Let A and B be two towns. Assume that one can reach town B from A by moving 60km towards east and then 30km towards north . A telephone company wants to raise a relay tower at

P which divides the line joining A and B in the ratio $1:2$ internally. Now, it wants to find the position of P where the relay tower is to be set up.

Choose the point A as the origin. Let $P(x,y)$ be the point. Draw the perpendiculars from P and B to the x-axis, meeting it in C and D respectively. Also draw a perpendicular from P to BD, intersecting at E.

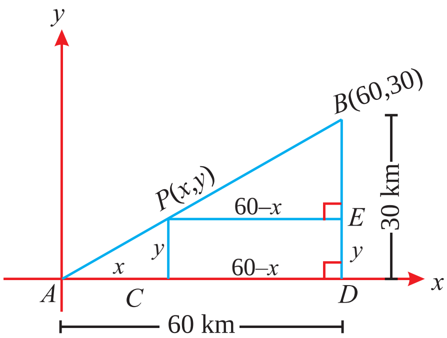


Fig. 5.1

Since ΔPAC and ΔBPE are similar, we have

$$\frac{AC}{PE} = \frac{PC}{BE} = \frac{AP}{PB} = \frac{1}{2}$$

Now $\quad \dfrac{AC}{PE} = \dfrac{1}{2}$

$$\Longrightarrow \quad \frac{x}{60-x} = \frac{1}{2}$$

$$2x = 60 - x$$

Thus, $\quad x = 20.$

Also, $\quad \dfrac{PC}{BE} = \dfrac{1}{2}$

$$\Longrightarrow \quad \frac{y}{30-y} = \frac{1}{2}$$

Thus, $\quad 2y = 30 - y \implies y = 10.$

$\therefore$ The position of the relay tower is at $P(20,10)$.

Taking the above problem as a model, we shall derive the general section formula.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points such that a point $P(x, y)$ divides AB internally in the ratio $l:m$. That is, $\dfrac{AP}{PB} = \dfrac{l}{m}$

From the Fig. 5.2, we get

$$AF = CD = OD - OC = x - x_1$$

$$PG = DE = OE - OD = x_2 - x$$

Also, $\quad PF = PD - FD = y - y_1$

$$BG = BE - GE = y_2 - y$$

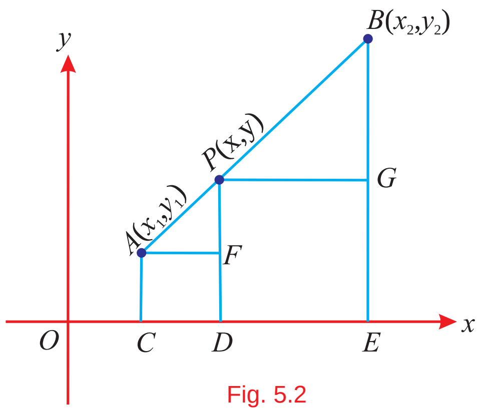


Fig. 5.2

Now, $\quad \Delta AFP$ and ΔPGB are similar.

(Refer chapter 6, section 6.3)

Thus, $\quad \dfrac{AF}{PG} = \dfrac{PF}{BG} = \dfrac{AP}{PB} = \dfrac{l}{m}$

$\therefore \quad \dfrac{AF}{PG} = \dfrac{l}{m}$ and $\dfrac{PF}{BG} = \dfrac{l}{m}$

$$\Longrightarrow \quad \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

$$\Longrightarrow \quad mx - mx_1 = lx_2 - lx$$

$$lx + mx = lx_2 + mx_1$$

$$\Longrightarrow \quad x = \frac{lx_2 + mx_1}{l+m}$$

$$\Longrightarrow \quad \frac{y - y_1}{y_2 - y} = \frac{l}{m}$$

$$\Longrightarrow \quad my - my_1 = ly_2 - ly$$

$$ly + my = ly_2 + my_1$$

$$\Longrightarrow \quad y = \frac{ly_2 + my_1}{l+m}$$

Thus, the point P which divides the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $l : m$ is

$$P\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right)$$

This formula is known as **section formula.**

It is clear that the section formula can be used only when the related three points are collinear.

Results

(i) If P divides a line segment AB joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $l : m$, then the point P is $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}\right)$. In this case $\frac{l}{m}$ is negative.

(ii) **Midpoint of AB**

If M is the midpoint of AB, then M divides the line segment AB internally in the ratio 1:1. By substituting $l = 1$ and $m = 1$ in the section formula, we obtain

the midpoint of AB as $M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$.

The midpoint of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

(iii) **Centroid of a triangle**

Consider a ΔABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let AD, BE and CF be the medians of the ΔABC.

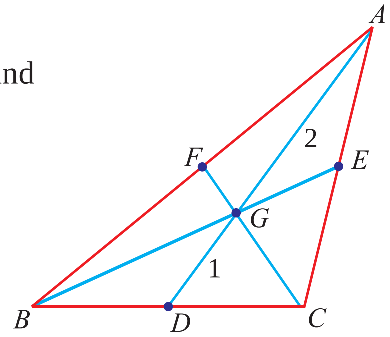


Fig. 5.3

We know that the medians of a triangle are concurrent and the point of concurrency is the centroid.

Let $G(x, y)$ be the centroid of ΔABC.

Now the midpoint of BC is $D\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$

By the property of triangle, the centroid G divides the median AD internally in the ratio $2 : 1$

$\therefore$ By section formula, the centroid

$$G(x, y) = G\left(\frac{2\frac{(x_2 + x_3)}{2} + 1(x_1)}{2 + 1}, \frac{2\frac{(y_2 + y_3)}{2} + 1(y_1)}{2 + 1}\right)$$

$$= G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The centroid of the triangle whose vertices are

$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Example 5.1

Find the midpoint of the line segment joining the points $(3, 0)$ and $(-1, 4)$.

Solution Midpoint $M(x, y)$ of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$M(x, y) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

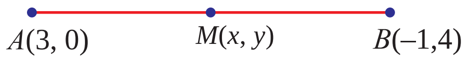


Fig. 5.4

$\therefore$ Midpoint of the line segment joining the points $(3, 0)$ and $(-1, 4)$ is

$$M(x, y) = M\left(\frac{3-1}{2}, \frac{0+4}{2}\right) = M(1, 2).$$

Example 5.2

Find the point which divides the line segment joining the points (3 , 5) and (8 , 10) internally in the ratio 2 : 3.

Solution Let $A(3, 5)$ and $B(8, 10)$ be the given points.

Let the point $P(x,y)$ divide the line AB internally in the ratio 2 :3.

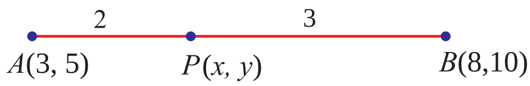


Fig. 5.5

By section formula, $P(x, y) = P\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right)$

Here $x_1 = 3, y_1 = 5, x_2 = 8, y_2 = 10$ and $l = 2, m = 3$

$$\therefore \quad P(x, y) = P\left(\frac{2(8) + 3(3)}{2+3}, \frac{2(10) + 3(5)}{2+3}\right) = P(5, 7)$$

Example 5.3

In what ratio does the point $P(-2, 3)$ divide the line segment joining the points $A(-3, 5)$ and $B(4, -9)$ internally?

Solution Given points are $A(-3, 5)$ and $B(4, -9)$.

Let $P(-2, 3)$ divide AB internally in the ratio $l : m$

By the section formula,

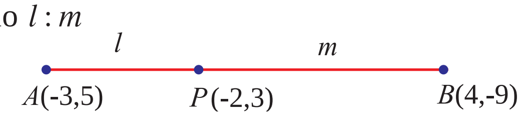


Fig. 5.6

$$P\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right) = P(-2, 3) \quad (1)$$

Here $x_1 = -3, y_1 = 5, x_2 = 4, y_2 = -9$.

$(1) \Longrightarrow \left(\frac{l(4)+m(-3)}{l+m}, \frac{l(-9)+m(5)}{l+m}\right) = (-2, 3)$

Equating the x-coordinates, we get

$$\frac{4l-3m}{l+m} = -2$$

$$\Longrightarrow \qquad 6l = m$$

$$\frac{l}{m} = \frac{1}{6}$$

i.e., $\qquad l : m = 1 : 6$

Hence P divides AB internally in the ratio 1 : 6

Note

(i) In the above example, one may get the ratio by equating y-coordinates also.

(ii) The ratios obtained by equating x-coordinates and by equating y-coordinates are same only when the three points are collinear.

(ii) If a point divides the line segment internally in the ratio $l : m$, then $\frac{l}{m}$ is positive.

(iii) If a point divides the line segment externally in the ratio $l : m$, then $\frac{l}{m}$ is negative.

Example 5.4

Find the points of trisection of the line segment joining $(4, -1)$ and $(-2, -3)$.

Solution Let $A(4,-1)$ and $B(-2,-3)$ be the given points.

Let $P(x,y)$ and $Q(a,b)$ be the points of trisection of AB so that $AP = PQ = QB$

Fig. 5.7

Hence P divides AB internally in the ratio 1 : 2 and Q divides AB internally in the ratio 2 : 1

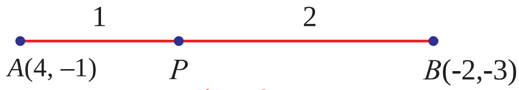


Fig. 5.8

$\therefore$ By the section formula, the required points are

$P\left(\frac{1(-2)+2(4)}{1+2}, \frac{1(-3)+2(-1)}{1+2}\right)$ and

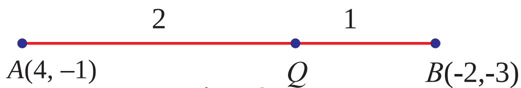


Fig. 5.9

$Q\left(\frac{2(-2)+1(4)}{2+1}, \frac{2(-3)+1(-1)}{2+1}\right)$

$\Longrightarrow \quad P(x,y) = P\left(\frac{-2+8}{3}, \frac{-3-2}{3}\right)$ and $\quad Q(a,b) = Q\left(\frac{-4+4}{3}, \frac{-6-1}{3}\right)$

$\qquad = P\left(2, -\frac{5}{3}\right) \qquad\qquad = Q\left(0, -\frac{7}{3}\right)$.

Note that Q is the midpoint of PB and P is the midpoint of AQ.

Example 5.5

Find the centroid of the triangle whose vertices are $A(4, -6)$, $B(3,-2)$ and $C(5, 2)$.

Solution The centroid $G(x , y)$ of a triangle whose vertices are

(x_1,y_1) , (x_2,y_2) and (x_3,y_3) is given by

$$G(x , y) = G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

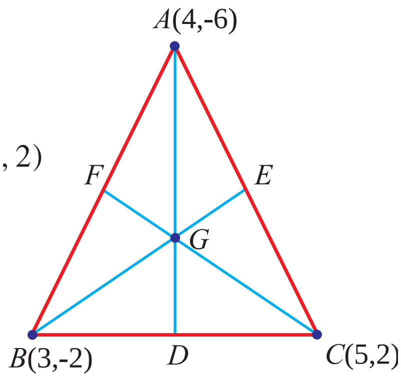


Fig. 5.10

We have $(x_1, y_1) = (4, -6)$, $(x_2, y_2) = (3, -2)$, $(x_3,y_3) = (5, 2)$

$\therefore$ The centroid of the triangle whose vertices are

$(4, -6), (3, -2)$ and $(5, 2)$ is

$$G(x , y) = G\left(\frac{4 + 3 + 5}{3}, \frac{-6 - 2 + 2}{3}\right)$$

$$= G(4, -2).$$

Example 5.6

If $(7, 3), (6, 1), (8, 2)$ and $(p, 4)$ are the vertices of a parallelogram taken in order, then find the value of p.

Solution Let the vertices of the parallelogram be $A(7, 3), B(6, 1), C(8, 2)$ and $D(p, 4)$.

We know that the diagonals of a parallelogram bisect each other.

$\therefore$ The midpoints of the diagonal AC and the diagonal BD coincide.

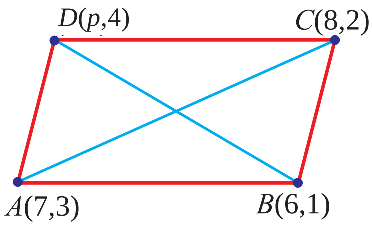


Fig. 5.11

Hence $\left(\frac{7 + 8}{2}, \frac{3 + 2}{2}\right) = \left(\frac{6 + p}{2}, \frac{1 + 4}{2}\right)$

$$\Longrightarrow \left(\frac{6 + p}{2}, \frac{5}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$$

Equating the x-coordinates, we get,

$$\frac{6 + p}{2} = \frac{15}{2}$$

$$\therefore\ p = 9$$

Example 5.7

If C is the midpoint of the line segment joining $A(4 , 0)$ and $B(0 , 6)$ and if O is the origin, then show that C is equidistant from all the vertices of $\triangle OAB$.

Solution The midpoint of AB is $C\left(\frac{4 + 0}{2}, \frac{0 + 6}{2}\right) = C(2, 3)$

We know that the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Distance between $O(0, 0)$ and $C(2, 3)$ is

$$OC = \sqrt{(2 - 0)^2 + (3 - 0)^2} = \sqrt{13}.$$

Distance between $A(4,0)$ and $C(2,3)$,

$$AC = \sqrt{(2-4)^2+(3-0)^2} = \sqrt{4+9} = \sqrt{13}$$

Distance between $B\ (0,6)$ and $C(2,3)$,

$$BC = \sqrt{(2-0)^2+(3-6)^2} = \sqrt{4+9} = \sqrt{13}$$

$\therefore \quad OC = AC = BC$

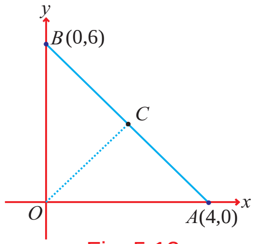


Fig. 5.12

$\therefore$ The point C is equidistant from all the vertices of the $\triangle OAB$.

Note The midpoint C of the hypotenuse, is the circumcentre of the right angled $\triangle OAB$.

Exercise 5.1

1. Find the midpoint of the line segment joining the points

   (i) $(1,-1)$ and $(-5,3)$ (ii) $(0,0)$ and $(0,4)$

2. Find the centroid of the triangle whose vertices are

   (i) $(1,3),(2,7)$ and $(12,-16)$ (ii) $(3,-5),(-7,4)$ and $(10,-2)$

3. The centre of a circle is at $(-6, 4)$. If one end of a diameter of the circle is at the origin, then find the other end.

4. If the centroid of a triangle is at $(1, 3)$ and two of its vertices are $(-7, 6)$ and $(8, 5)$ then find the third vertex of the triangle .

5. Using the section formula, show that the points $A(1,0)$, $B(5,3)$, $C(2,7)$ and $D(-2, 4)$ are the vertices of a parallelogram taken in order.

6. Find the coordinates of the point which divides the line segment joining $(3, 4)$ and $(–6, 2)$ in the ratio $3 : 2$ externally.

7. Find the coordinates of the point which divides the line segment joining $(-3, 5)$ and $(4, -9)$ in the ratio $1 : 6$ internally.

8. Let $A\ (-6,-5)$ and $B\ (-6, 4)$ be two points such that a point P on the line AB satisfies $AP = \frac{2}{9} AB$. Find the point P.

9. Find the points of trisection of the line segment joining the points $A(2,-2)$ and $B(-7, 4)$.

10. Find the points which divide the line segment joining $A(-4\ ,0)$ and $B\ (0,6)$ into four equal parts.

11. Find the ratio in which the x-axis divides the line segment joining the points $(6, 4)$ and $(1,-7)$.

12. In what ratio is the line joining the points $(-5, 1)$ and $(2\ , 3)$ divided by the y-axis? Also, find the point of intersection .

13. Find the length of the medians of the triangle whose vertices are $(1, -1)$, $(0, 4)$ and $(-5, 3)$.

5.3 Area of a triangle

We have already learnt how to calculate the area of a triangle, when some measurements of the triangle are given. Now, if the coordinates of the vertices of a triangle are given, can we find its area ?

Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

Draw the lines AD, BE and CF perpendicular to x-axis.

From the figure, $ED = x_1 - x_2$, $DF = x_3 - x_1$ and

$EF = x_3 - x_2$.

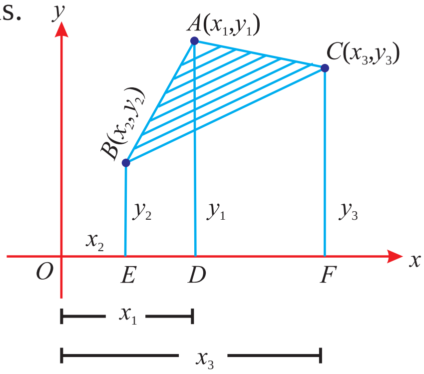


Fig. 5.13

Area of the triangle ABC

= Area of the trapezium $ABED$

+ Area of the trapezium $ADFC$

− Area of the trapezium $BEFC$

$$= \frac{1}{2}(BE + AD)ED + \frac{1}{2}(AD + CF)DF - \frac{1}{2}(BE + CF)EF$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\{x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3\}$$

$\therefore$ Area of the ΔABC is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.sq.units.

> If $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are the vertices of a ΔABC,
> then the area of the ΔABC is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.sq.units.

Note

The area of the triangle can also be written as

$$\frac{1}{2}\{x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2\} \text{ sq.units.}$$

(or) $$\frac{1}{2}\left\{(x_1y_2 + x_2\,y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}\text{sq.units}$$

The following pictorial representation helps us to write the above formula very easily.

Take the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of ΔABC in counter clockwise direction and write them column-wise as shown below.

$$\frac{1}{2}\begin{Bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{Bmatrix}$$

Add the diagonal products x_1y_2, x_2y_3 and x_3y_1 as shown in the dark arrows.

Also add the products x_2y_1, x_3y_2 and x_1y_3 as shown in the dotted arrows and then subtract the latter from the former to get the expression $\frac{1}{2}\left\{(x_1y_2 + x_2\,y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}$

Note

To find the area of a triangle, the following steps may be useful.

(i) Plot the points in a rough diagram.

(ii) Take the vertices in counter clock-wise direction. Otherwise the formula gives a negative value.

(iii) Use the formula, area of the $\Delta ABC = \frac{1}{2}\left\{(x_1y_2 + x_2\,y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}$

5.4 Collinearity of three points

Three or more points in a plane are said to be collinear, if they lie on the same straight line.

In other words, three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear if any one of these points lies on the straight line joining the other two points.

Suppose that the three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear. Then they cannot form a triangle. Hence the area of the $\triangle ABC$ is zero.

i.e., $\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 0$

$$\Longrightarrow \qquad x_1y_2 + x_2y_3 + x_3y_1 = x_2y_1 + x_3y_2 + x_1y_3$$

One can prove that the converse is also true.

Hence the area of $\triangle ABC$ is zero if and only if the points *A*, *B* and *C* are collinear.

5.5 Area of the Quadrilateral

Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $D(x_4,y_4)$ be the vertices of a quadrilateral *ABCD*.

Now the area of the quadrilateral *ABCD* = area of the ΔABD+area of the ΔBCD

$$= \frac{1}{2}\{(x_1y_2 + x_2y_4 + x_4y_1) - (x_2y_1 + x_4y_2 + x_1y_4)\}$$
$$+ \frac{1}{2}\{(x_2y_3 + x_3y_4 + x_4y_2) - (x_3y_2 + x_4y_3 + x_2y_4)\}$$

$\therefore$ Area of the quadrilateral *ABCD*

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}$$

or

$$\frac{1}{2}\{(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\} \text{ sq.units}$$

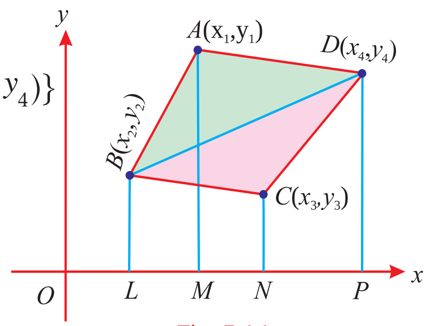


Fig. 5.14

The following pictorial representation helps us to write the above formula very easily.

Take the vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ in counter clockwise direction and write them column-wise as shown below. Follow the same technique as we did in the case of finding the area of a triangle.

$$\frac{1}{2}\left\{\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{array}\right\}.$$

This helps us to get the required expression

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}.$$

Example 5.8

Find the area of the triangle whose vertices are $(1, 2)$, $(-3, 4)$, and $(-5, -6)$.

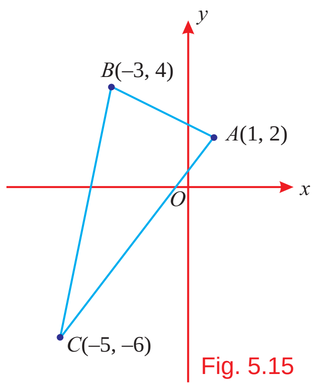


Fig. 5.15

Solution Plot the points in a rough diagram and take them in order.

Let the vertices be $A(1, 2)$, $B(-3, 4)$ and $C(-5, -6)$.

Now the area of $\triangle ABC$ is

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}$$

$$= \frac{1}{2}\{(4 + 18 - 10) - (-6 - 20 - 6)\} \qquad \text{use : } \frac{1}{2}\left\{\begin{array}{cccc} 1 & -3 & -5 & 1 \\ 2 & 4 & -6 & 2 \end{array}\right\}$$

$$= \frac{1}{2}\{12 + 32\} = 22. \text{ sq. units}$$

Example 5.9

If the area of the ΔABC is 68 sq.units and the vertices are $A(6, 7)$, $B(-4, 1)$ and $C(a, -9)$ taken in order, then find the value of a.

Solution Area of $\triangle ABC$ is

$$\frac{1}{2}\{(6 + 36 + 7a) - (-28 + a - 54)\} = 68 \qquad \text{use : } \frac{1}{2}\left\{\begin{array}{cccc} 6 & -4 & a & 6 \\ 7 & 1 & -9 & 7 \end{array}\right\}$$

$$\Longrightarrow (42 + 7a) - (a - 82) = 136$$

$$\Longrightarrow 6a = 12 \qquad \therefore \quad a = 2$$

Example 5.10

Show that the points $A(2, 3)$, $B(4, 0)$ and $C(6, -3)$ are collinear.

Solution Area of the ΔABC is

$$= \frac{1}{2}\{(0 - 12 + 18) - (12 + 0 - 6)\} \qquad \text{use : } \frac{1}{2}\left\{\begin{array}{cccc} 2 & 4 & 6 & 2 \\ 3 & 0 & -3 & 3 \end{array}\right\}$$

$$= \frac{1}{2}\{6 - 6\} = 0.$$

$\therefore$ The given points are collinear.

Example 5.11

If $P(x, y)$ is any point on the line segment joining the points $(a, 0)$ and $(0, b)$, then , prove that $\frac{x}{a} + \frac{y}{b} = 1$, where $a, b \neq 0$.

Solution Now the points (x, y), $(a, 0)$ and $(0, b)$ are collinear.

$\therefore$ The area of the triangle formed by them is zero.

$$\Longrightarrow \qquad ab - bx - ay = 0 \qquad\qquad \text{use: } \frac{1}{2}\begin{Bmatrix} a & 0 & x & a \\ 0 & b & y & 0 \end{Bmatrix}$$

$$\therefore \qquad bx + ay = ab$$

Dividing by ab on both sides, we get,

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \text{where} \qquad a, b \neq 0$$

Example 5.12

Find the area of the quadrilateral formed by the points $(-4, -2)$, $(-3, -5)$, $(3, -2)$ and $(2, 3)$.

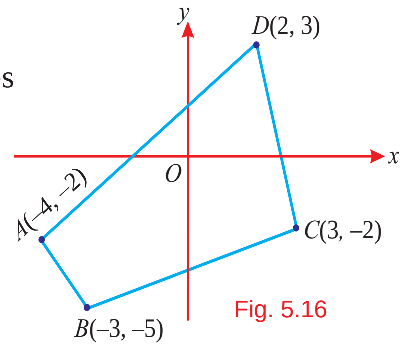


Fig. 5.16

Solution Let us plot the points roughly and take the vertices in counter clock-wise direction.

Let the vertices be

$A(-4, -2)$, $B(-3, -5)$, $C(3, -2)$ and $D(2, 3)$.

Area of the quadrilateral *ABCD*

$$= \frac{1}{2}\{(20 + 6 + 9 - 4) - (6 - 15 - 4 - 12)\}$$

$$= \frac{1}{2}\{31 + 25\} = 28 \text{ sq.units.} \qquad \frac{1}{2}\begin{Bmatrix} -4 & -3 & 3 & 2 & -4 \\ -2 & -5 & -2 & 3 & -2 \end{Bmatrix}$$

Exercise 5.2

1. Find the area of the triangle formed by the points

   (i) $(0, 0)$, $(3, 0)$ and $(0, 2)$ (ii) $(5, 2)$, $(3, -5)$ and $(-5, -1)$

   (iii) $(-4, -5)$, $(4, 5)$ and $(-1, -6)$

2. Vertices of the triangles taken in order and their areas are given below. In each of the following find the value of a.

Vertices	Area (in sq. units)
(i) $(0, 0)$, $(4, a)$, $(6, 4)$	17
(ii) (a, a), $(4, 5)$, $(6, -1)$	9
(iii) $(a, -3)$, $(3, a)$, $(-1, 5)$	12

3. Determine if the following set of points are collinear or not.

(i) (4, 3), (1, 2) and (−2, 1) (ii) (−2, −2), (−6, −2) and (−2, 2)

(iii) $\left(-\frac{3}{2}, 3\right)$, (6, −2) and (−3, 4)

4. In each of the following, find the value of k for which the given points are collinear.

(i) $(k, -1)$, (2, 1) and (4, 5) (ii) $(2, -5)$, $(3, -4)$ and $(9, k)$

(iii) (k, k), $(2, 3)$ and $(4, -1)$

5. Find the area of the quadrilateral whose vertices are

(i) $(6, 9)$, $(7, 4)$, $(4, 2)$ and $(3, 7)$ (ii) $(-3, 4)$, $(-5, -6)$, $(4, -1)$ and $(1, 2)$

(iii) $(-4, 5)$, $(0, 7)$, $(5, -5)$ and $(-4, -2)$

6. If the three points $(h, 0)$, (a, b) and $(0, k)$ lie on a straight line, then using the area of the triangle formula, show that $\frac{a}{h} + \frac{b}{k} = 1$, where $h, k \neq 0$.

7. Find the area of the triangle formed by joining the midpoints of the sides of a triangle whose vertices are $(0, -1)$, $(2, 1)$ and $(0, 3)$. Find the ratio of this area to the area of the given triangle.

5.6 Straight Lines

5.6.1 Angle of Inclination

Let a straight line l intersect the x-axis at A. The angle between the positive x-axis and the line l, measured in counter clockwise direction is called the angle of inclination of the straight line l.

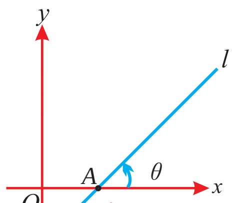


Fig. 5.17

Remarks

If θ is the angle of inclination of a straight line l, then

(i) $0^\circ \leq \theta \leq 180^\circ$

(ii) For horizontal lines, $\theta = 0^\circ$ or 180° and for vertical lines, $\theta = 90^\circ$

(iii) If a straight line initially lies along the x-axis and starts rotating about a fixed point A on the x-axis in the counter clockwise direction and finally coincides with the x-axis, then the angle of inclination of the straight line in the initial position is 0° and that of the line in the final position is 180°.

5.6.2 Slope of a straight line

Definition

If θ is the angle of inclination of a non-vertical straight line l, then $\tan\theta$ is called the Slope or Gradient of the line and is denoted by m.

$\therefore$ The slope of the straight line, $m = \tan\theta$ for $0^\circ \leq \theta \leq 180^\circ$, $\theta \neq 90^\circ$

Remarks

(i) Thus, the slope of x-axis or straight lines parallel to x-axis is zero.

(ii) The slope of y-axis or a straight line parallel to y-axis is not defined because $\tan 90^0$ is not defined. Therefore, whenever we talk about the slope of a straight line, we mean that of a non-vertical straight line.

(iii) If θ is acute, then the slope is positive, whereas if θ is obtuse then the slope is negative.

5.6.3 Slope of a straight line when any two points on the line are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the straight line l whose angle of inclination is θ. Here, $0^\circ \le \theta \le 180^\circ$, $\theta \neq 90^\circ$

Let the straight line AB intersect the x-axis at C.

Now, the slope of the line l is $m = \tan\theta$ (1)

Draw AD and BE perpendicular to x-axis and draw the perpendicular AF line from A to BE.

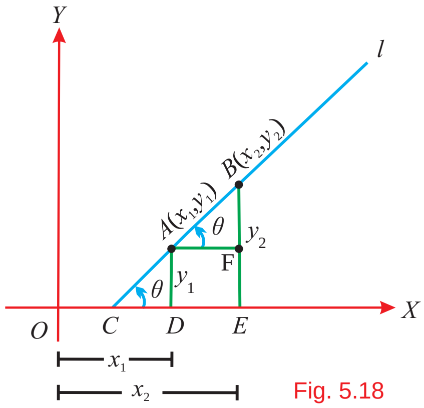


Fig. 5.18

From the figure, we have

$$AF = DE = OE - OD = x_2 - x_1$$

and $\quad BF = BE - EF = BE - AD = y_2 - y_1$

Also, we observe that $\angle DCA = \angle FAB = \theta$

In the right angled ΔABF, we have

$$\tan\theta = \frac{BF}{AF} = \frac{y_2 - y_1}{x_2 - x_1} \quad if\ x_1 \neq x_2 \qquad (2)$$

From (1) and (2), we get the slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

The slope of the straight line joining the points $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ is

$$m = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}} = \frac{\boldsymbol{y_1 - y_2}}{\boldsymbol{x_1 - x_2}} \text{ where } x_1 \neq x_2 \text{ as } \theta \neq 90^\circ.$$

Note

The slope of the straight line joining the points (x_1, y_1) and (x_2, y_2) is also interpreted as

$$m = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}} = \frac{\text{change in } y \text{ coordinates}}{\text{change in } x \text{ coordinates}}.$$

5.6.4 Condition for parallel lines in terms of their slopes

Consider parallel lines l_1 and l_2 whose angles of inclination are θ_1 and θ_2 and slopes are m_1 and m_2 respectively.

Since l_1 and l_2 are parallel, the angles of inclinations θ_1 and θ_2 are equal.

$$\therefore \quad \tan\theta_1 = \tan\theta_2 \quad \Longrightarrow \quad m_1 = m_2$$

$\therefore$ If two non-vertical straight lines are parallel, then their slopes are equal.

The converse is also true. i.e., if the slopes of two lines are equal, then the straight lines are parallel.

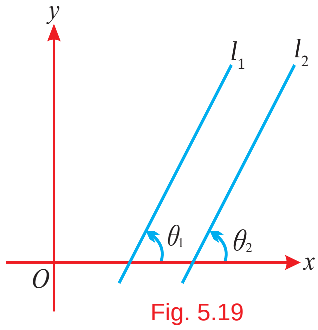


Fig. 5.19

5.6.5 Condition for perpendicular lines in terms of their slopes

Let l_1 and l_2 be two perpendicular straight lines passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively.

Let m_1 and m_2 be their slopes.

Let $C(x_3, y_3)$ be their point of intersection.

The slope of the straight line l_1 is $m_1 = \dfrac{y_3 - y_1}{x_3 - x_1}$

The slope of the straight line l_2 is $m_2 = \dfrac{y_3 - y_2}{x_3 - x_2}$

In the right angled $\triangle ABC$, we have

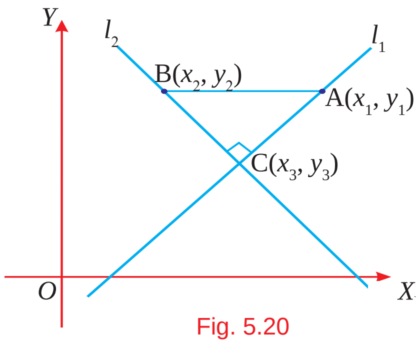


Fig. 5.20

$$AB^2 = AC^2 + BC^2$$

$$\Longrightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\Longrightarrow (x_2 - x_3 + x_3 - x_1)^2 + (y_2 - y_3 + y_3 - y_1)^2$$
$$= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\Longrightarrow \left(x_2 - x_3\right)^2 + \left(x_3 - x_1\right)^2 + 2(x_2 - x_3)(x_3 - x_1) + (y_2 - y_3)^2 + (y_3 - y_1)^2 + 2(y_2 - y_3)(y_3 - y_1)$$
$$= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\Longrightarrow 2(x_2 - x_3)(x_3 - x_1) + 2(y_2 - y_3)(y_3 - y_1) = 0$$

$$\Longrightarrow (y_2 - y_3)(y_3 - y_1) = -(x_2 - x_3)(x_3 - x_1)$$

$$\left(\frac{y_3 - y_1}{x_3 - x_1}\right)\left(\frac{y_3 - y_2}{x_3 - x_2}\right) = -1.$$

$$\Longrightarrow \quad m_1 m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}$$

If two non-vertical straight lines with slopes m_1 and m_2, are perpendicular, then

$$m_1 m_2 = -1.$$

On the other hand, if $m_1 m_2 = -1$, then the two straight lines are perpendicular.

Note

The straight lines x-axis and y-axis are perpendicular to each other. But, the condition $m_1 m_2 = -1$ is not true because the slope of the x-axis is zero and the slope of the y-axis is not defined.

Example 5.13

Find the angle of inclination of the straight line whose slope is $\frac{1}{\sqrt{3}}$.

Solution If θ is the angle of inclination of the line, then the slope of the line is

$$m = \tan\theta \quad \text{where } 0^\circ \leq \theta \leq 180^\circ, \theta \neq 90^\circ.$$

$$\therefore \tan\theta = \frac{1}{\sqrt{3}} \implies \theta = 30^\circ$$

Example 5.14

Find the slope of the straight line whose angle of inclination is 45°.

Solution If θ is the angle of inclination of the line, then the slope of the line is $m = \tan\theta$

Given that $m = \tan 45^\circ \implies m = 1$.

Example 5.15

Find the slope of the straight line passing through the points $(3, -2)$ and $(-1, 4)$.

Solution Slope of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of the straight line passing through the points $(3, -2)$ and $(-1, 4)$ is

$$m = \frac{4 + 2}{-1 - 3} = -\frac{3}{2}.$$

Example 5.16

Using the concept of slope, show that the points $A(5, -2)$, $B(4, -1)$ and $C(1, 2)$ are collinear.

Solution Slope of the line joining the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope of the line AB joining the points $A(5, -2)$ and $B(4 - 1)$ is $m_1 = \frac{-1 + 2}{4 - 5} = -1$

Slope of the line BC joining the points $B(4, -1)$ and $C(1, 2)$ is $m_2 = \frac{2 + 1}{1 - 4} = -1$

Thus, slope of AB = slope of BC.

Also, B is the common point.

Hence, the points A, B and C are collinear.

Example 5.17

Using the concept of slope, show that the points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ taken in order form a parallelogram.

Solution Let $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$ be the given points taken in order.

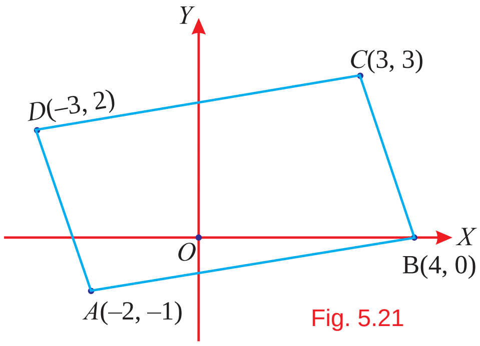


Fig. 5.21

Now the slope of $AB = \dfrac{0+1}{4+2} = \dfrac{1}{6}$

Slope of $CD = \dfrac{2-3}{-3-3} = \dfrac{1}{6}$

$\therefore$ Slope of AB = slope of CD

Hence, AB is parallel to CD. (1)

Now the slope of $BC = \dfrac{3-0}{3-4} = -3$

Slope of $AD = \dfrac{2+1}{-3+2} = -3$

$\therefore$ Slope of BC = slope of AD

Hence, BC is parallel to AD. (2)

From (1) and (2), we see that opposite sides of quadrilateral $ABCD$ are parallel

$\therefore$ $ABCD$ is a parallelogram.

Example 5.18

The vertices of a $\triangle ABC$ are $A(1, 2)$, $B(-4, 5)$ and $C(0, 1)$. Find the slopes of the altitudes of the triangle.

Solution Let AD, BE and CF be the altitudes of a $\triangle ABC$.

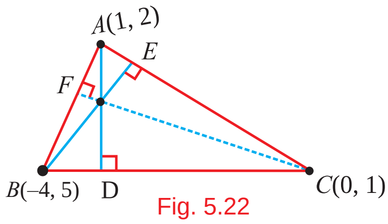


Fig. 5.22

slope of $BC = \dfrac{1-5}{0+4} = -1$

Since the altitude AD is perpendicular to BC,

slope of $AD = 1$ $\quad \because m_1 m_2 = -1$

slope of $AC = \dfrac{1-2}{0-1} = 1$

Thus, slope of $BE = -1$ $\quad \because BE \perp AC$

Also, slope of $AB = \dfrac{5-2}{-4-1} = -\dfrac{3}{5}$

$\therefore$ slope of $CF = \dfrac{5}{3}$ $\quad \because CF \perp AB$

Exercise 5.3

1. Find the angle of inclination of the straight line whose slope is
   (i) 1 (ii) $\sqrt{3}$ (iii) 0

2. Find the slope of the straight line whose angle of inclination is
   (i) 30° (ii) 60° (iii) 90°

3. Find the slope of the straight line passing through the points
   (i) $(3, -2)$ and $(7, 2)$ (ii) $(2, -4)$ and origin
   (iii) $(1+\sqrt{3}, 2)$ and $(3+\sqrt{3}, 4)$

4. Find the angle of inclination of the line passing through the points
   (i) $(1, 2)$ and $(2, 3)$ (ii) $(3, \sqrt{3})$ and $(0, 0)$
   (iii) (a, b) and $(-a, -b)$

5. Find the slope of the line which passes through the origin and the midpoint of the line segment joining the points $(0, -4)$ and $(8, 0)$.

6. The side AB of a square $ABCD$ is parallel to x-axis . Find the
   (i) slope of AB (ii) slope of BC (iii) slope of the diagonal AC

7. The side BC of an equilateral $\triangle ABC$ is parallel to x-axis. Find the slope of AB and the slope of BC.

8. Using the concept of slope, show that each of the following set of points are collinear.
   (i) $(2, 3)$, $(3, -1)$ and $(4, -5)$
   (ii) $(4, 1)$, $(-2, -3)$ and $(-5, -5)$ (iii) $(4, 4)$, $(-2, 6)$ and $(1, 5)$

9. If the points $(a, 1)$, $(1, 2)$ and $(0, b+1)$ are collinear, then show that $\frac{1}{a}+\frac{1}{b}=1$.

10. The line joining the points $A(-2, 3)$ and $B(a, 5)$ is parallel to the line joining the points $C(0, 5)$ and $D(-2, 1)$. Find the value of a.

11. The line joining the points $A(0, 5)$ and $B(4, 2)$ is perpendicular to the line joining the points $C(-1, -2)$ and $D(5, b)$. Find the value of b.

12. The vertices of $\triangle ABC$ are $A(1, 8)$, $B(-2, 4)$, $C(8, -5)$. If M and N are the midpoints of AB and AC respectively, find the slope of MN and hence verify that MN is parallel to BC.

13. A triangle has vertices at $(6, 7)$, $(2, -9)$ and $(-4, 1)$. Find the slopes of its medians.

14. The vertices of a $\triangle ABC$ are $A(-5, 7)$, $B(-4, -5)$ and $C(4, 5)$. Find the slopes of the altitudes of the triangle.

15. Using the concept of slope, show that the vertices $(1\ ,2)$, $(-2\ ,2)$, $(-4\ ,-3)$ and $(-1, -3)$ taken in order form a parallelogram.

16. Show that the opposite sides of a quadrilateral with vertices $A(-2\ ,-4)$, $B(5\ ,-1)$, $C(6\ ,4)$ and $D(-1, 1)$ taken in order are parallel.

5.6.6 Equation of a straight line

Let L be a straight line in the plane. A first degree equation $px + qy + r = 0$ in the variables x and y is satisfied by the x-coordinate and y-coordinate of any point on the line L and any values of x and y that satisfy this equation will be the coordinates of a point on the line L. Hence this equation is called the equation of the straight line L. We want to describe this line L algebraically. That is, we want to describe L by an algebraic equation. Now L is in any one of the following forms:

(i) horizontal line (ii) vertical line (iii) neither vertical nor horizontal

(i) Horizontal line: Let L be a horizontal line.

Then either L is x-axis or L is a horizontal line other than x-axis.

Case (a) If L is x – axis, then a point (x, y) lies on L

if and only if $y = 0$ and x can be any real number.

Thus, $y = 0$ describes x – axis.

$\therefore$ The equation of x-axis is $y = 0$

Case (b) L is a horizontal line other than x-axis.

That is, L is parallel to x-axis.

Now, a point (x, y) lies on L if and only if the y-coordinate must remain a constant and x can be any real number.

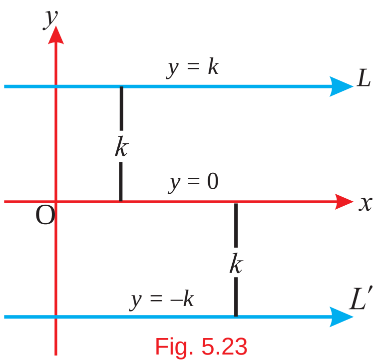


Fig. 5.23

$\therefore$ The equation of a straight line parallel to x-axis is $y = k$, where k is a constant.

Note that if $k > 0$, then L lies above x-axis and if $k < 0$, then L lies below x- axis. If $k = 0$, then L is nothing but the x-axis.

(ii) Vertical line: Let L be a vertical line.

Then either L is y-axis or L is a vertical line other than y-axis.

Case (a) If L is y-axis, then a point (x, y) in the plane lies on L if and only if $x = 0$ and y can be any real number.

Thus $x = 0$ describes y – axis.

$\therefore$ The equation of y-axis is $x = 0$

Case (b) If L is a vertical line other than y-axis, then it is parallel to y-axis.

Now a point (x, y) lies on L if and only if x-coordinate must remain constant and y can be any real number.

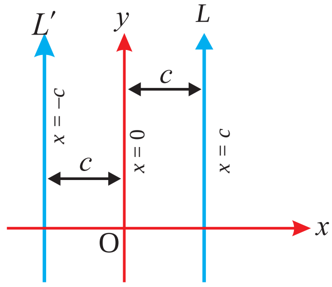


Fig. 5.24

$\therefore$ The equation of a straight line parallel to y-axis is $x = c$, where c is a constant.

Note that if $c > 0$, then L lies to the right y-axis and

if $c < 0$, then L lies to the left of y-axis.

If $c = 0$, then L is nothing but the y-axis.

(iii) Neither vertical nor horizontal: Let L be neither vertical nor horizontal.

In this case how do we describe L by an equation? Let θ denote the angle of inclination. Observe that if we know this θ and a point on L, then we can easily describe L.

> Slope m of a non-vertical line L can be calculated using
>
> (i) $m = \tan\theta$ if we know the angle of inclination θ.
>
> (ii) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ if we know two distinct points (x_1, y_1), (x_2, y_2) on L.
>
> (iii) $m = 0$ if and only if L is horizontal.

Now consider the case where L is not a vertical line and derive the equation of a straight line in the following forms: (a) Slope-Point form (b) Two-Points form (c) Slope-Intercept form (d) Intercepts form

(a) Slope-Point form

Let m be the slope of L and $Q(x_1, y_1)$ be a point on L.

Let $P(x, y)$ be an arbitrary point on L other than Q. Then, we have

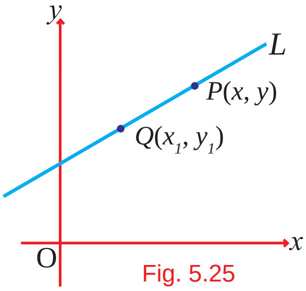


Fig. 5.25

$$m = \frac{y - y_1}{x - x_1} \Leftrightarrow m(x - x_1) = y - y_1$$

Thus, the equation of a straight line with slope m and passing through (x_1, y_1) is

$y - y_1 = m(x - x_1)$ for all points (x, y) on L. (1)

Remarks

(i) Now the first degree equation (1) in the variables x and y is satisfied by the x-coordinate and y-coordinate of any point on the line L. Any value of x and y that satisfies this equation will be the coordinates of a point on the line L. Hence the equation (1) is called the equation of the straight line L.

(ii) The equation (1) says that the change in y-coordinates of the points on L is directly proportional to the change in x-coordinates. The proportionality constant m is the slope.

(b) Two-Points form

Suppose that two distinct points (x_1, y_1), (x_2, y_2) are given on a non-vertical line L.

To find the equation of L, we find the slope of L first and then use (1) .

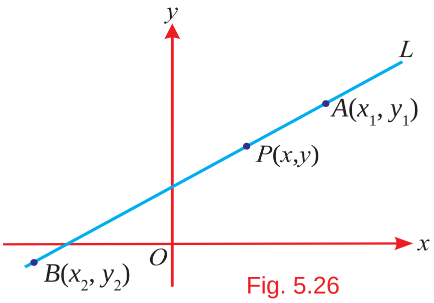


Fig. 5.26

The slope of L is

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 \neq x_1 \text{ as } L \text{ is non-vertical.}$$

Now, the formula (1) gives

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$\Longrightarrow \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \Longrightarrow \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad \text{for all points } (x, y) \text{ on } L \qquad (2)$$

Note

To get the equation of L, we can also use the point (x_2, y_2) instead of (x_1, y_1).

(c) Slope-Intercept form

Suppose that m is the slope of L and c is the y-intercept of L.

Since c is the y-intercept, the point $(0, c)$ lies on L. Now using (1) with $(x_1, y_1) = (0, c)$ we obtain, $y - c = m(x - 0)$

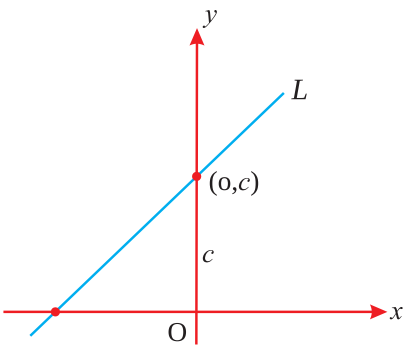


Fig. 5.27

$$\Longrightarrow \quad y = mx + c \quad \text{for all points } (x, y) \text{ on } L. \qquad (3)$$

Thus, $y = mx + c$ is the equation of straight line in the **Slope-Intercept form.**

(d) Intercepts form

Suppose that the straight line L makes non-zero intercepts a and b on the x-axis and on the y-axis respectively.

$\therefore$ The straight line cuts the x-axis at $A(a, 0)$ and the y-axis at $B(0, b)$

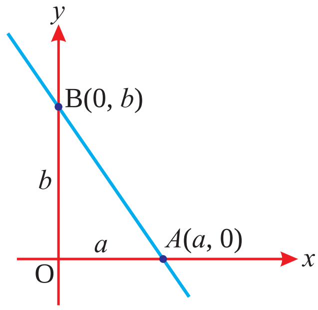


Fig. 5.28

The slope of AB is $m = -\dfrac{b}{a}$.

Now (1) gives, $\quad y - 0 = -\dfrac{b}{a}(x - a)$

$$\Longrightarrow \qquad ay = -bx + ab$$

$$bx + ay = ab$$

Divide by ab to get $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$\therefore$ Equation of a straight line having x-intercept a and y-intercept b is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{for all points } (x, y) \text{ on } L \qquad (4)$$

Note

(i) If the line L with slope m, makes x-intercept d, then the equation of the line is $y = m(x - d)$.

(ii) The straight line $y = mx$ passes through the origin.(both x and y-intercepts are zero for $m \neq 0$).

(iii) Equations (1), (2) and (4) can be simplified to slope-intercept form given by (3).

(iv) Each equation in (1), (2), (3) and (4) can be rewritten in the form
$px + qy + r = 0$ for all points (x, y) on L, which is called the general form of equation of a straight line.

Example 5.19

Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(3, -4)$.

Solution Let L and L' be the straight lines passing through the point $(3, -4)$ and parallel to x-axis and y-axis respectively.

The y-coordinate of every point on the line L is -4.

Hence, the equation of the line L is $\quad y = -4$

Similarly, the x-coordinate of every point on the straight line L' is 3

Hence, the equation of the line L' is $x = 3$.

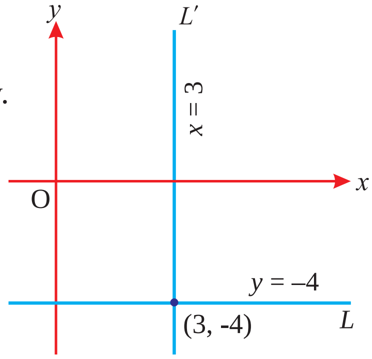


Fig. 5.29

Example 5.20

Find the equation of straight line whose angle of inclination is 45° and y-intercept is $\frac{2}{5}$.

Solution Slope of the line, $m = \tan\theta$

$$= \tan 45^\circ = 1$$

y-intercept is $c = \frac{2}{5}$

By the slope-intercept form, the equation of the straight line is

$$y = mx + c$$

$$y = x + \frac{2}{5} \implies y = \frac{5x+2}{5}$$

$\therefore$ The equation of the straight line is $5x - 5y + 2 = 0$

Example 5.21

Find the equation of the straight line passing through the point $(-2,3)$ with slope $\frac{1}{3}$.

Solution Given that the slope $m = \frac{1}{3}$ and a point $(x_1, y_1) = (-2,3)$

By slope-point formula, the equation of the straight line is

$$y - y_1 = m(x - x_1)$$

$$\implies y - 3 = \frac{1}{3}(x+2)$$

Thus, $x - 3y + 11 = 0$ is the required equation.

Example 5.22

Find the equation of the straight line passing through the points $(-1,1)$ and $(2,-4)$.

Solution Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points.

Here $x_1 = -1$, $y_1 = 1$ and $x_2 = 2$, $y_2 = -4$.

Using two-points formula, the equation of the straight line is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\implies \frac{y-1}{-4-1} = \frac{x+1}{2+1}$$

$$\implies 3y - 3 = -5x - 5$$

Hence, $5x + 3y + 2 = 0$ is the required equation of the straight line.

Example 5.23

The vertices of a $\triangle ABC$ are $A(2, 1)$, $B(-2, 3)$ and $C(4, 5)$. Find the equation of the median through the vertex A.

Solution Median is a straight line joining a vertex and the midpoint of the opposite side.

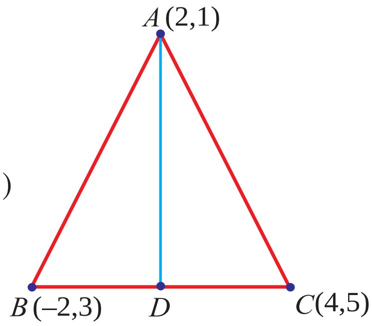


Fig. 5.30

Let D be the midpoint of BC.

$\therefore$ Midpoint of BC is $D\left(\frac{-2+4}{2}, \frac{3+5}{2}\right) = D(1, 4)$

Now the equation of the median AD is

$$\frac{y-1}{4-1} = \frac{x-2}{1-2} \qquad \because (x_1, y_1) = (2,1) \text{ and } (x_2, y_2) = (1,4)$$

$$\frac{y-1}{3} = \frac{x-2}{-1}$$

$\therefore$ $3x + y - 7 = 0$ is the required equation.

Example 5.24

If the x-intercept and y-intercept of a straight line are $\frac{2}{3}$ and $\frac{3}{4}$ respectively, then find the equation of the straight line.

Solution Given that x-intercept of the straight line, $a = \frac{2}{3}$

and the y-intercept of the straight line, $b = \frac{3}{4}$

Using intercept form, the equation of the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{\frac{2}{3}} + \frac{y}{\frac{3}{4}} = 1$$

$$\implies \frac{3x}{2} + \frac{4y}{3} = 1$$

Hence, $9x + 8y - 6 = 0$ is the required equation.

Example 5.25

Find the equations of the straight lines each passing through the point $(6, -2)$ and whose sum of the intercepts is 5.

Solution Let a and b be the x-intercept and y-intercept of the required straight line respectively.

Given that sum of the intercepts, $a + b = 5$

$$\implies \qquad b = 5 - a$$

Now, the equation of the straight line in the intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{a} + \frac{y}{5-a} = 1$$

$$\implies \frac{(5-a)x + ay}{a(5-a)} = 1$$

Thus, $(5-a)x + ay = a(5-a)$ (1)

Since the straight line given by (1) passes through $(6, -2)$, we get,

$$
\begin{aligned}
(5-a)6+a(-2) &= a(5-a)\\
\Longrightarrow \quad a^2-13a+30 &= 0.
\end{aligned}
$$

That is, $\quad (a-3)(a-10) = 0$

$\therefore \quad a = 3$ or $a = 10$

When $a = 3$, (1) $\Longrightarrow (5-3)x+3y = 3(5-3)$

$\Longrightarrow \quad 2x+3y = 6 \qquad (2)$

When $a = 10$, (1) $\Longrightarrow (5-10)x+10y = 10(5-10)$

$\Longrightarrow \quad -5x+10y = -50$

That is, $\quad x-2y-10 = 0. \qquad (3)$

Hence, $2x+3y = 6$ and $x-2y-10 = 0$ are the equations of required straight lines.

Exercise 5.4

1. Write the equations of the straight lines parallel to x- axis which are at a distance of 5 units from the x-axis.

2. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(-5,-2)$.

3. Find the equation of a straight line whose
   (i) slope is -3 and y-intercept is 4.
   (ii) angle of inclination is 60^0 and y-intercept is 3.

4. Find the equation of the line intersecting the y- axis at a distance of 3 units above the origin and $\tan\theta = \frac{1}{2}$, where θ is the angle of inclination.

5. Find the slope and y-intercept of the line whose equation is
   (i) $y = x+1$ (ii) $5x = 3y$ (iii) $4x-2y+1 = 0$ (iv) $10x+15y+6 = 0$

6. Find the equation of the straight line whose
   (i) slope is -4 and passing through $(1, 2)$
   (ii) slope is $\frac{2}{3}$ and passing through $(5, -4)$

7. Find the equation of the straight line which passes through the midpoint of the line segment joining $(4, 2)$ and $(3, 1)$ whose angle of inclination is 30^0.

8. Find the equation of the straight line passing through the points
   (i) $(-2, 5)$ and $(3, 6)$ (ii) $(0, -6)$ and $(-8, 2)$

9. Find the equation of the median from the vertex R in a $\triangle PQR$ with vertices at $P(1, -3)$, $Q(-2, 5)$ and $R(-3, 4)$.

10. By using the concept of the equation of the straight line, prove that the given three points are collinear.

(i) (4, 2), (7, 5) and (9, 7) (ii) (1, 4), (3, −2) and (−3, 16)

11. Find the equation of the straight line whose x and y-intercepts on the axes are given by

(i) 2 and 3 (ii) $-\frac{1}{3}$ and $\frac{3}{2}$ (iii) $\frac{2}{5}$ and $-\frac{3}{4}$

12. Find the x and y intercepts of the straight line

(i) $5x + 3y - 15 = 0$ (ii) $2x - y + 16 = 0$ (iii) $3x + 10y + 4 = 0$

13. Find the equation of the straight line passing through the point (3, 4) and has intercepts which are in the ratio 3 : 2.

14. Find the equation of the straight lines passing through the point (2, 2) and the sum of the intercepts is 9.

15. Find the equation of the straight line passing through the point (5, −3) and whose intercepts on the axes are equal in magnitude but opposite in sign.

16. Find the equation of the line passing through the point (9, −1) and having its x-intercept thrice as its y-intercept.

17. A straight line cuts the coordinate axes at A and B. If the midpoint of AB is (3, 2), then find the equation of AB.

18. Find the equation of the line passing through (22, −6) and having intercept on x-axis exceeds the intercept on y-axis by 5.

19. If $A(3, 6)$ and $C(-1, 2)$ are two vertices of a rhombus $ABCD$, then find the equation of straight line that lies along the diagonal BD.

20. Find the equation of the line whose gradient is $\frac{3}{2}$ and which passes through P, where P divides the line segment joining $A(-2, 6)$ and B (3, −4) in the ratio 2 : 3 internally.

5.7 General Form of Equation of a straight line

We have already pointed out that different forms of the equation of a straight line may be converted into the standard form $ax + by + c = 0$, where a , b and c are real constants such that either $a \neq 0$ or $b \neq 0$.

Now let us find out

(i) the slope of $ax + by + c = 0$

(ii) the equation of a straight line parallel to $ax + by + c = 0$

(iii) the equation of a straight line perpendicular to $ax + by + c = 0$ and

(iv) the point of intersection of two intersecting straight lines.

(i) The general form of the equation of a straight line is $ax+by+c=0$.

The above equation is rewritten as $y = -\dfrac{a}{b}x - \dfrac{c}{b},\ b \neq 0$ (1)

Comparing (1) with the slope-intercept form $y = mx + k$, we get,

$$\text{slope, } m = -\frac{a}{b} \text{ and the } y\text{-intercept } = -\frac{c}{b}$$

$\therefore$ For the equation $ax + by + c = 0$, we have

$$\text{slope } m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} \quad \text{and} \quad \text{the } y\text{-intercept is } -\frac{\text{constant term}}{\text{coefficient of } y}.$$

(ii) Equation of a line parallel to the line $ax+by+c=0$.

We know that two straight lines are parallel if and only if their slopes are equal.

Hence the equations of all lines parallel to the line $ax + by + c = 0$ are of the form

$$ax + by + k = 0, \text{ for different values of } k.$$

(iii) Equation of a line perpendicular to the line $ax+by+c=0$

We know that two non-vertical lines are perpendicular if and only if the product of their slopes is –1.

Hence the equations of all lines perpendicular to the line $ax + by + c = 0$ are

$$bx - ay + k = 0, \text{ for different values of } k.$$

Note

Two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where the coefficients are non-zero,

(i) are parallel if and only if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

(ii) are perpendicular if and only if $a_1a_2 + b_1b_2 = 0$

(iv) The point of intersection of two straight lines

If two straight lines are not parallel, then they will intersect at a point. This point lies on both the straight lines. Hence, the point of intersection is obtained by solving the given two equations.

Example 5.26

Show that the straight lines $3x + 2y - 12 = 0$ and $6x + 4y + 8 = 0$ are parallel.

Solution Slope of the straight line $3x + 2y - 12 = 0$ is $m_1 = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = -\dfrac{3}{2}$

Similarly, the slope of the line $6x + 4y + 8 = 0$ is $m_2 = -\dfrac{6}{4} = -\dfrac{3}{2}$

$\therefore$ $m_1 = m_2$. Hence, the two straight lines are parallel.

Example 5.27

Prove that the straight lines $x + 2y + 1 = 0$ and $2x - y + 5 = 0$ are perpendicular to each other.

Solution Slope of the straight line $x + 2y + 1 = 0$ is $m_1 = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = -\dfrac{1}{2}$

Slope of the straight line $2x - y + 5 = 0$ is $m_2 = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = \dfrac{-2}{-1} = 2$

Product of the slopes $m_1 m_2 = -\dfrac{1}{2} \times 2 = -1$

$\therefore$ The two straight lines are perpendicular.

Example 5.28

Find the equation of the straight line parallel to the line $x - 8y + 13 = 0$ and passing through the point (2, 5).

Solution Equation of the straight line parallel to $x - 8y + 13 = 0$ is $x - 8y + k = 0$

Since it passes through the point (2, 5)

$$2 - 8(5) + k = 0 \implies k = 38$$

$\therefore$ Equation of the required straight line is $x - 8y + 38 = 0$

Example 5.29

The vertices of $\triangle ABC$ are $A(2, 1)$, $B(6, -1)$ and $C(4, 11)$. Find the equation of the straight line along the altitude from the vertex A.

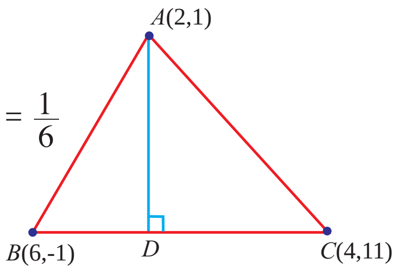


Fig. 5.31

Solution Slope of $BC = \dfrac{11+1}{4-6} = -6$

Since the line AD is perpendicular to the line BC, slope of $AD = \dfrac{1}{6}$

$\therefore$ Equation of AD is $y - y_1 = m(x - x_1)$

$$y - 1 = \frac{1}{6}(x - 2) \implies 6y - 6 = x - 2$$

$\therefore$ Equation of the required straight line is $x - 6y + 4 = 0$

Exercise 5.5

1. Find the slope of the straight line

   (i) $3x + 4y - 6 = 0$ (ii) $y = 7x + 6$ (iii) $4x = 5y + 3$.

2. Show that the straight lines $x + 2y + 1 = 0$ and $3x + 6y + 2 = 0$ are parallel.

3. Show that the straight lines $3x - 5y + 7 = 0$ and $15x + 9y + 4 = 0$ are perpendicular.

4. If the straight lines $\dfrac{y}{2} = x - p$ and $ax + 5 = 3y$ are parallel, then find a.

5. Find the value of a if the straight lines $5x - 2y - 9 = 0$ and $ay + 2x - 11 = 0$ are perpendicular to each other.

6. Find the values of p for which the straight lines $8px+(2-3p)y+1=0$ and $px+8y-7=0$ are perpendicular to each other.

7. If the straight line passing through the points $(h, 3)$ and (4, 1) intersects the line $7x-9y-19=0$ at right angle, then find the value of h.

8. Find the equation of the straight line parallel to the line $3x-y+7=0$ and passing through the point $(1, -2)$.

9. Find the equation of the straight line perpendicular to the straight line $x-2y+3=0$ and passing through the point $(1, -2)$.

10. Find the equation of the perpendicular bisector of the straight line segment joining the points (3, 4) and $(-1, 2)$.

11. Find the equation of the straight line passing through the point of intersection of the lines $2x+y-3=0$ and $5x+y-6=0$ and parallel to the line joining the points (1, 2) and (2, 1).

12. Find the equation of the straight line which passes through the point of intersection of the straight lines $5x-6y=1$ and $3x+2y+5=0$ and is perpendicular to the straight line $3x-5y+11=0$.

13. Find the equation of the straight line joining the point of intersection of the lines $3x-y+9=0$ and $x+2y=4$ and the point of intersection of the lines $2x+y-4=0$ and $x-2y+3=0$.

14. If the vertices of a $\triangle ABC$ are $A(2, -4)$, $B(3, 3)$ and $C(-1, 5)$. Find the equation of the straight line along the altitude from the vertex B.

15. If the vertices of a $\triangle ABC$ are $A(-4,4)$, $B(8,4)$ and $C(8,10)$. Find the equation of the straight line along the median from the vertex A.

16. Find the coordinates of the foot of the perpendicular from the origin on the straight line $3x+2y=13$.

17. If $x+2y=7$ and $2x+y=8$ are the equations of the lines of two diameters of a circle, find the radius of the circle if the point $(0, -2)$ lies on the circle.

18. Find the equation of the straight line segment whose end points are the point of intersection of the straight lines $2x-3y+4=0$, $x-2y+3=0$ and the midpoint of the line joining the points $(3, -2)$ and $(-5, 8)$.

19. In an isosceles $\triangle PQR$, $PQ = PR$. The base QR lies on the x-axis, P lies on the y- axis and $2x-3y+9=0$ is the equation of PQ. Find the equation of the straight line along PR.

Exercise 5.6

Choose the correct answer

1. The midpoint of the line joining $(a, -b)$ and $(3a, 5b)$ is
   (A) $(-a, 2b)$ (B) $(2a, 4b)$ (C) $(2a, 2b)$ (D) $(-a, -3b)$

2. The point P which divides the line segment joining the points $A(1, -3)$ and $B(-3, 9)$ internally in the ratio 1:3 is
   (A) $(2, 1)$ (B) $(0, 0)$ (C) $\left(\frac{5}{3}, 2\right)$ (D) $(1, -2)$

3. If the line segment joining the points $A(3, 4)$ and $B(14, -3)$ meets the x-axis at P, then the ratio in which P divides the segment AB is
   (A) 4 : 3 (B) 3 : 4 (C) 2 : 3 (D) 4 : 1

4. The centroid of the triangle with vertices at $(-2, -5)$, $(-2, 12)$ and $(10, -1)$ is
   (A) $(6, 6)$ (B) $(4, 4)$ (C) $(3, 3)$ (D) $(2, 2)$

5. If $(1, 2)$, $(4, 6)$, $(x, 6)$ and $(3, 2)$ are the vertices of a parallelogram taken in order, then the value of x is
   (A) 6 (B) 2 (C) 1 (D) 3

6. Area of the triangle formed by the points (0,0), $(2, 0)$ and $(0, 2)$ is
   (A) 1 sq. units (B) 2 sq. units (C) 4 sq. units (D) 8 sq. units

7. Area of the quadrilateral formed by the points $(1, 1)$, $(0, 1)$, $(0, 0)$ and $(1, 0)$ is
   (A) 3 sq. units (B) 2 sq. units (C) 4 sq. units (D) 1 sq. units

8. The angle of inclination of a straight line parallel to x-axis is equal to
   (A) 0° (B) 60° (C) 45° (D) 90°

9. Slope of the line joining the points $(3, -2)$ and $(-1, a)$ is $-\frac{3}{2}$, then the value of a is equal to
   (A) 1 (B) 2 (C) 3 (D) 4

10. Slope of the straight line which is perpendicular to the straight line joining the points $(-2, 6)$ and $(4, 8)$ is equal to
    (A) $\frac{1}{3}$ (B) 3 (C) -3 (D) $-\frac{1}{3}$

11. The point of intersection of the straight lines $9x - y - 2 = 0$ and $2x + y - 9 = 0$ is
    (A) $(-1, 7)$ (B) $(7, 1)$ (C) $(1, 7)$ (D) $(-1, -7)$

12. The straight line $4x + 3y - 12 = 0$ intersects the y- axis at
    (A) $(3, 0)$ (B) $(0, 4)$ (C) $(3, 4)$ (D) $(0, -4)$

13. The slope of the straight line $7y - 2x = 11$ is equal to
    (A) $-\frac{7}{2}$ (B) $\frac{7}{2}$ (C) $\frac{2}{7}$ (D) $-\frac{2}{7}$

14. The equation of a straight line passing through the point (2 , –7) and parallel to x-axis is
    (A) $x = 2$ (B) $x = -7$ (C) $y = -7$ (D) $y = 2$

15. The x and y-intercepts of the line $2x - 3y + 6 = 0$, respectively are

(A) 2, 3 (B) 3, 2 (C) $-3, 2$ (D) $3, -2$

16. The centre of a circle is $(-6, 4)$. If one end of the diameter of the circle is at $(-12, 8)$, then the other end is at

(A) $(-18, 12)$ (B) $(-9, 6)$ (C) $(-3, 2)$ (D) $(0, 0)$

17. The equation of the straight line passing through the origin and perpendicular to the straight line $2x + 3y - 7 = 0$ is

(A) $2x + 3y = 0$ (B) $3x - 2y = 0$ (C) $y + 5 = 0$ (D) $y - 5 = 0$

18. The equation of a straight line parallel to y-axis and passing through the point $(-2, 5)$ is

(A) $x - 2 = 0$ (B) $x + 2 = 0$ (C) $y + 5 = 0$ (D) $y - 5 = 0$

19. If the points (2, 5), (4, 6) and (a, a) are collinear, then the value of a is equal to

(A) -8 (B) 4 (C) -4 (D) 8

20. If a straight line $y = 2x + k$ passes through the point (1, 2), then the value of k is equal to

(A) 0 (B) 4 (C) 5 (D) -3

21. The equation of a straight line having slope 3 and y-intercept -4 is

(A) $3x - y - 4 = 0$ (B) $3x + y - 4 = 0$

(C) $3x - y + 4 = 0$ (D) $3x + y + 4 = 0$

22. The point of intersection of the straight lines $y = 0$ and $x = -4$ is

(A) $(0, -4)$ (B) $(-4, 0)$ (C) $(0, 4)$ (D) $(4, 0)$

23. The value of k if the straight lines $3x + 6y + 7 = 0$ and $2x + ky = 5$ are perpendicular is

(A) 1 (B) -1 (C) 2 (D) $\frac{1}{2}$

Points to Remember

- The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- The point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $l : m$ is $\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right)$.
- The point Q which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $l : m$ is $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}\right)$.
- Midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- The area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\sum x_1(y_2 - y_3) = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$
$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}.$$

- Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if

  (i) $x_1y_2 + x_2y_3 + x_3y_1 = x_2y_1 + x_3y_2 + x_1y_3$ (or)

  (ii) Slope of AB = Slope of BC or slope of AC.

- If a line makes an angle θ with the positive direction of x- axis, then the slope $m = \tan\theta$.

- Slope of the non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

- Slope of the line $ax + by + c = 0$ is $m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a}{b}$, $b \neq 0$

- Slope of the horizontal line is 0 and slope of the vertical line is undefined.

- Two lines are parallel if and only if their slopes are equal.

- Two non-vertical lines are perpendicular if and only if the product of their slopes is -1. That is, $m_1 m_2 = -1$.

Equation of straight lines

Sl.No	Straight line	Equation
1.	x-axis	$y = 0$
2.	y-axis	$x = 0$
3.	Parallel to x-axis	$y = k$
4.	Parallel to y-axis	$x = k$
5.	Parallel to $ax+by+c=0$	$ax+by+k = 0$
6.	Perpendicular to $ax+by+c=0$	$bx-ay+k = 0$
	Given	**Equation**
1.	Passing through the origin	$y = mx$
2.	Slope m, y-intercept c	$y = mx + c$
3.	Slope m, a point (x_1, y_1)	$y - y_1 = m(x - x_1)$
4.	Passing through two points (x_1, y_1), (x_2, y_2)	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
5.	x-intercept a , y-intercept b	$\frac{x}{a} + \frac{y}{b} = 1$