Converting from local to global coordinates

Example: Spinning circles at the end of a line segment

Consider a horizontal line segment $\overline{AB}$ as depicted in Figure 1. The line segment has a circle at each end, a blue circle and a pink circle. Each circle rotates independently about line endpoints points. The start of the line segment, i.e., point A is located away from the origin of the world coordinate system (i.e., frame $\mathcal{F}\{0\}$). The rotation radius of the blue circle is 5 and the rotation radius of the pink circle is 2.

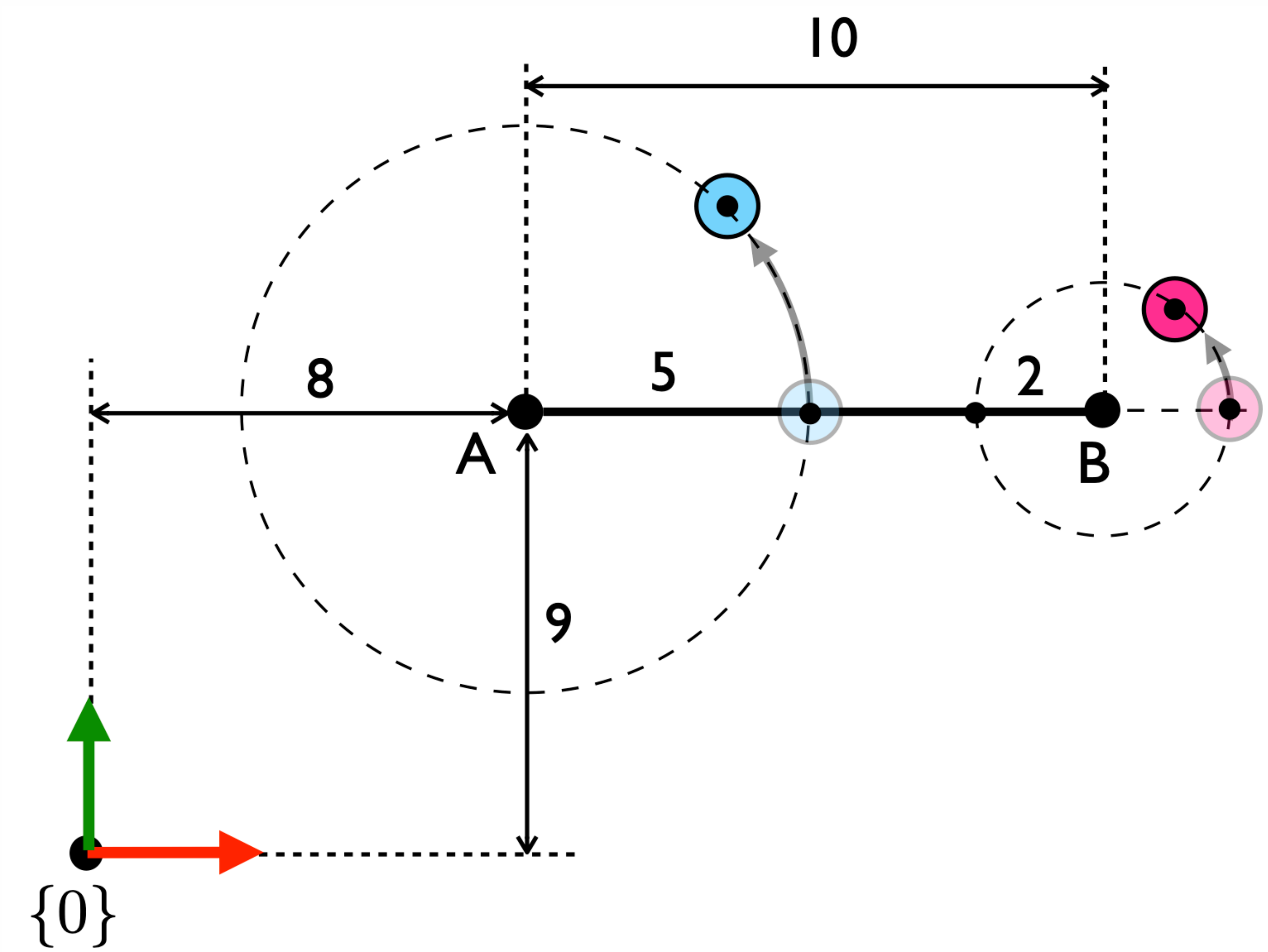

Figure 1: Object with two rotating parts (i.e., blue and pink circles). The blue circle must rotate around point A while the pink circle must rotate around point B. Their frequency of rotation may be different. The world coordinate system is labeled as frame $\mathcal{F}\{0\}$.

Step 1:

Our first step is to choose the best locations and orientations of local coordinate systems for the problem. There can be many different coordinate systems. The ones we will use here will be centered at the points A and B because we want the spheres to rotate around those points. The configuration of local frames shown in Figure 2 is one possibility out of many possible ones.

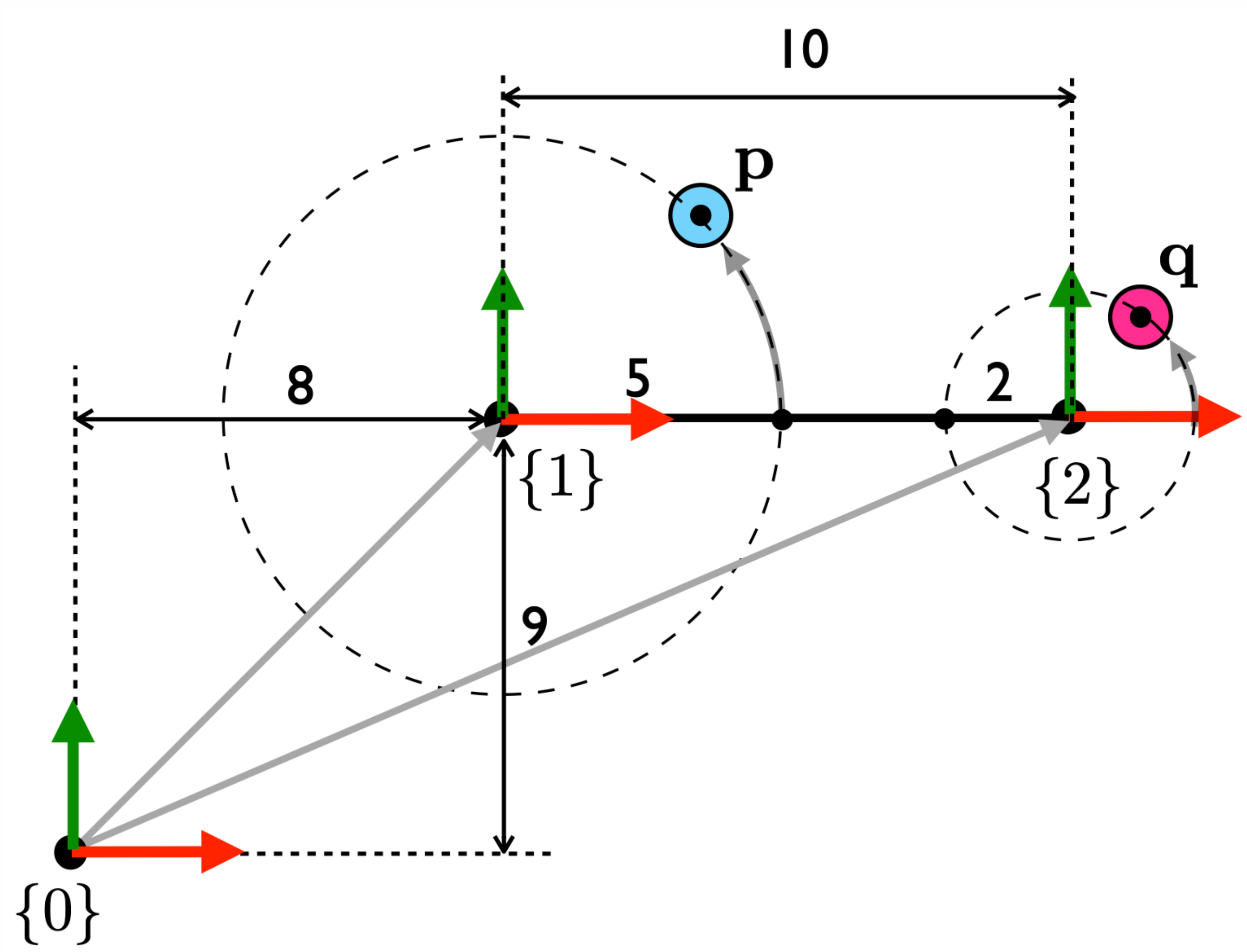

Figure 2: The object's end points are $A = (8, 9)^\intercal$ and $B = (18, 9)^\intercal$. Points $\mathbf{p}$ and $\mathbf{q}$ are the centers of the blue and pink circles, respectively.

In this choice of configuration, all local frames are just translated with respect to the world-coordinate frame (i.e., no rotation). Depending on the application, the local frames may also be rotated.

Step 2:

We create the local-to-global transformation matrices for each local frame.

1. Transformation $\mathcal{F}\{1\} \longrightarrow \mathcal{F}\{0\}$:

$$T_{01} = \begin{bmatrix} R_{01} & \mathbf{t}_{01} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1}$$

In the example described in these notes, there is no rotation between frames $\mathcal{F}\{1\}$ and $\mathcal{F}\{0\}$, i.e., $R_{01} = I$. The origin of $\mathcal{F}\{1\}$ is translated by $\mathbf{t}_{01} = (8, 9)^\mathsf{T}$ w.r.t. frame $\mathcal{F}\{0\}$.

2. Transformation $\mathcal{F}\{2\} \longrightarrow \mathcal{F}\{0\}$:

$$T_{02} = \begin{bmatrix} R_{02} & \mathbf{t}_{02} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 18 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

There is also no rotation between frames $\mathcal{F}\{2\}$ and $\mathcal{F}\{0\}$. The origin of $\mathcal{F}\{1\}$ is translated by $\mathbf{t}_{02} = (18, 9)^\mathsf{T}$ w.r.t. frame $\mathcal{F}\{0\}$.

Step 3:

We build the rotation matrices that will govern the motions of the points $\mathbf{p}$ and $\mathbf{q}$ in their local coordinate systems.

1. Rotation of blue circle about its local origin:

$$R_\theta = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \tag{8}$$

2. Rotation of pink circle about its local origin:

$$R_\phi = \begin{bmatrix} \sin\phi & -\cos\phi \\ \cos\phi & \sin\phi \end{bmatrix} \tag{9}$$

Step 4:

Now, let's try some rotations to see how they work. First, we can rotate the blue circle by angle $\theta = \pi/4$. To do that, we will apply the rotation to the initial location of $\mathbf{p}_{\{1\}} = (5, 0)^\mathsf{T}$ in local coordinates. We will write $\tilde{\mathbf{p}}_{\{1\}}$ to indicate the homogeneous representation of point $\mathbf{p}_{\{1\}}$. The rotation calculation in homogeneous coordinates as follows:

$$\tilde{\mathbf{p}}'_{\{1\}} = \begin{bmatrix} R_\theta & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \tilde{\mathbf{p}}_{\{1\}}. \tag{10}$$

Note that the rotated point $\tilde{\mathbf{p}}'_{\{1\}}$ is written in terms of its local coordinate system (i.e., frame $\mathcal{F}\{1\}$). As a result, the rotated point will not be plotted at its expected location when using library functions such as `plot(x,y)`. Library plotting functions use global (world) coordinates, not local ones. Thus, prior to plotting the rotated point, we must convert its coordinates to global coordinates, i.e.:

$$\tilde{\mathbf{p}}'_{\{0\}} = T_{01}\tilde{\mathbf{p}}'_{\{1\}},$$

$$\begin{bmatrix} x'_p \\ y'_p \\ 1 \end{bmatrix}_{\{0\}} = \underbrace{\begin{bmatrix} R_{01} & \mathbf{t}_{01} \\ \mathbf{0} & 1 \end{bmatrix}}_{\text{local-to-global}} \underbrace{\begin{bmatrix} R_\theta & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}}_{\text{local rotation}} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix}_{\{1\}}. \tag{11}$$

Numerically, the global representation of the rotation of $\tilde{\mathbf{p}}_{\{1\}} = (5, 0, 1)^\intercal$ by an angle angle $\theta = \pi/4$ around point A is:

$$\begin{aligned} \begin{bmatrix} x'_p \\ y'_p \\ 1 \end{bmatrix}_{\{0\}} &= \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\pi/4) & -\cos(\pi/4) & 0 \\ \cos(\pi/4) & \sin(\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}_{\{1\}} \\ &= \begin{bmatrix} 11.5 \\ 12.5 \\ 1 \end{bmatrix}_{\{0\}}, \end{aligned} \tag{12}$$

which are the expected coordinates of the rotated point's position when plotted.

To rotate the pink circle by an angle ϕ around the circles' local frame, we apply a local rotation to $\mathbf{q}_{\{2\}} = (2, 0)^\intercal$, in local coordinates. Here, we use the transformation matrix that converts from frame $\mathcal{F}\{2\}$ to frame $\mathcal{F}\{0\}$, i.e., T_{02}. The equation is given by:

$$\tilde{\mathbf{q}}'_{\{0\}} = T_{02}\tilde{\mathbf{q}}'_{\{2\}},$$

$$\begin{bmatrix} x'_q \\ y'_q \\ 1 \end{bmatrix}_{\{0\}} = \underbrace{\begin{bmatrix} R_{02} & \mathbf{t}_{02} \\ \mathbf{0} & 1 \end{bmatrix}}_{\text{local-to-global}} \underbrace{\begin{bmatrix} R_\phi & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}}_{\text{local rotation}} \begin{bmatrix} x_q \\ y_q \\ 1 \end{bmatrix}_{\{2\}}. \tag{13}$$

After a local rotation by an angle $\phi = \pi/3$ around point B, followed by the change-of-frame transformation, the global coordinates of $\tilde{\mathbf{q}}_{\{2\}} = (2, 0, 1)^\intercal$ are:

$$\begin{aligned} \begin{bmatrix} x'_q \\ y'_q \\ 1 \end{bmatrix}_{\{0\}} &= \begin{bmatrix} 1 & 0 & 18 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\pi/3) & -\cos(\pi/3) & 0 \\ \cos(\pi/3) & \sin(\pi/3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}_{\{2\}} \\ &= \begin{bmatrix} 19.7 \\ 10.0 \\ 1 \end{bmatrix}_{\{0\}}, \end{aligned} \tag{14}$$

which are the expected coordinates of the rotated point's position when plotted.