Composing Transformations

Overview

In these notes, we review the concept of composing functions, the isomorphism between linear transformations and matrices, and that we can compose linear transformations by simply multiplying together the matrices of the transformations.

Isomorphism between linear transformations and matrices

Linear transformations can be expressed in matrix form. Here, we can also say that matrices are representations of linear transformations. This type of equivalence between algebraic structures is called *Isomorphism*.

Let F be a linear transformation that maps a vector $\mathbf{x}$ into a (transformed) vector $F(\mathbf{x})$. Because of the isomorphism between linear transformations and matrices, we know that the matrix of the transformed vector is given by the product of the matrix of the transformation F and the matrix of the original vector $\mathbf{x}$. This equivalence can be written as:

$$[F(\mathbf{x})] = [F] \cdot [\mathbf{x}] \tag{10}$$

For example, a transformation F that rotates the vector $\mathbf{x} = (1, 0)^{\mathsf{T}}$ by an angle $\theta = \frac{\pi}{4}$, can be written in matrix form as:

$$\begin{aligned} [F(\mathbf{x})] &= [F] \cdot [\mathbf{x}] \\ &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{11}$$

Here,

$$[F] = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \tag{12}$$

is called the *matrix of the (linear) transformation*.

Composing transformations by multiplying transformation matrices

In many applications, it is common for transformations to be composed from other transformations. Let F and G be two linear transformations. The composition of F and G is defined by:

$$(FG)(\mathbf{x}) = F(G(\mathbf{x}))$$

Following from the isomorphism in Equation 10, we have:

$$\begin{aligned} [(FG)(\mathbf{x})] &= [F(G(\mathbf{x}))] \\ &= [F] \cdot [G(\mathbf{x})] \\ &= [F] \cdot ([G] \cdot [\mathbf{x}]) \\ &= ([F] \cdot [G]) \cdot [\mathbf{x}]. \end{aligned} \tag{13}$$

Equation 13 tells us that we can compose linear transformations by simply multiplying together the matrices of the transformations. Here, the matrix of the composition transformation is given by:

$$[FG] = [F]\,[G]. \tag{14}$$

For example, to compose the rotation R and scaling S, and apply it to vector $\mathbf{x}$, we do:

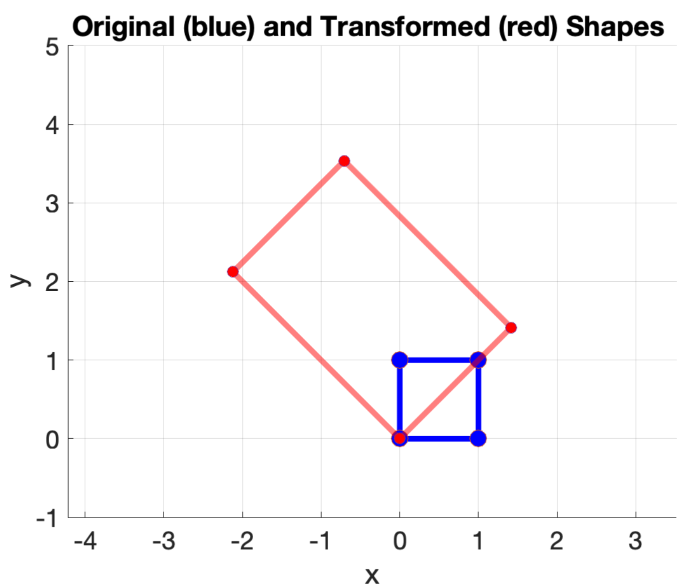
$$\begin{aligned}
[(RS)\,(\mathbf{x})] &= ([R]\,[S])\,[\mathbf{x}] \\
&= \left(\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} && (15) \\
&= \begin{bmatrix} 2\cos\frac{\pi}{4} \\ 2\sin\frac{\pi}{4} \end{bmatrix}. && (16)
\end{aligned}$$

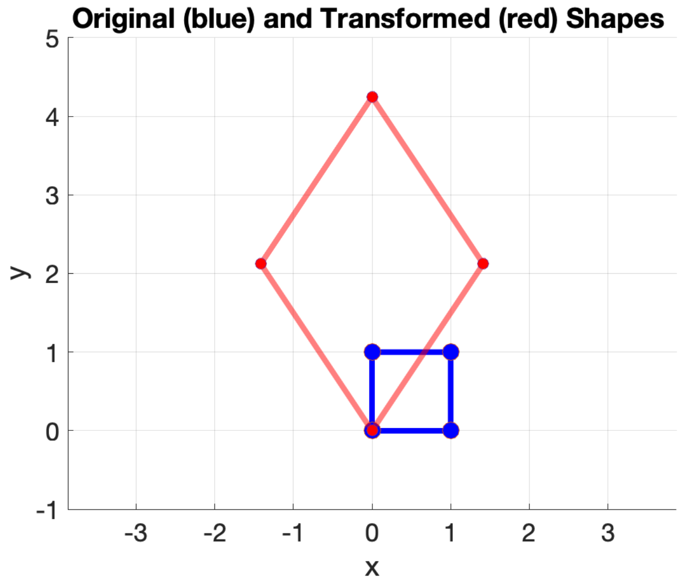
The order of multiplications is important. From matrix algebra, we know that the multiplication of matrices is not commutative, i.e., $AB \neq BA$ (for most cases). The same restriction applies to geometric transformations such as the ones in Equation 16. Indeed, if we switched the other of multiplication in the example in (16), we would have:

$$\begin{aligned}
[(SR)\,(\mathbf{x})] &= ([S]\,[R])\,[\mathbf{x}] \\
&= \left(\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} && (17) \\
&= \begin{bmatrix} 2\cos\frac{\pi}{4} \\ 3\sin\frac{\pi}{4} \end{bmatrix}. && (18)
\end{aligned}$$

As expected, $RS \neq SR$, and the resulting transformed vectors are different.

The following figures show the result of the transformation on a unit square (blue). The red shapes are the transformed shapes. The transformations are RS and SR, respectively.





Again, the resulting shapes are different if we change the order of transformations, i.e., $RS \neq SR$.