Calculus Review: Functions

1 Overview

In mathematics, a function describes the relationship between a quantity that is dependent on another quantity (e.g., position as a function of time, yearly income as a function of worked hours). These quantities are called *variables*, which can be either *dependent* or *independent*.

The dependent variable, which is the value of the function itself, describes the value of the quantity that depends on the other variables. Independent variables do not depend on other variables. Functions can only have one value of the dependent variable for each value of the independent variable. Mathematically, we denote a function as f and its value as $f(x)$. Here, f is the dependent variable and x is the independent variable. Note that it is common for some people to "abuse the notation" a bit and refer to $f(x)$ as a function when this is actually the value of the function at x.

> **Definition**: A function is a mathematical relationship in which the values of a single dependent variable are determined by the values of one or more independent variables. Function means the dependent variable is determined by the independent variable(s).[a].
>
> [a]http://www.columbia.edu/itc/sipa/math/variables.html

2 Characterizing functions in terms of dimensionality

In terms of the dimensionality of the variables, functions can be of the following types:

1. A scalar function of a single scalar variable.
2. A scalar function of multiple scalar variables (or a scalar function of a vector variable).
3. A vector function of a single scalar variable.
4. A vector function of a vector variable.

2.1 A scalar function of a single scalar variable

In scalar functions of a single scalar variable, both the function value, $f(x)$, and its independent variable x are real numbers, i.e., $f(x) \in \mathbb{R}$ and $x \in \mathbb{R}$. Figure 1 shows an example.

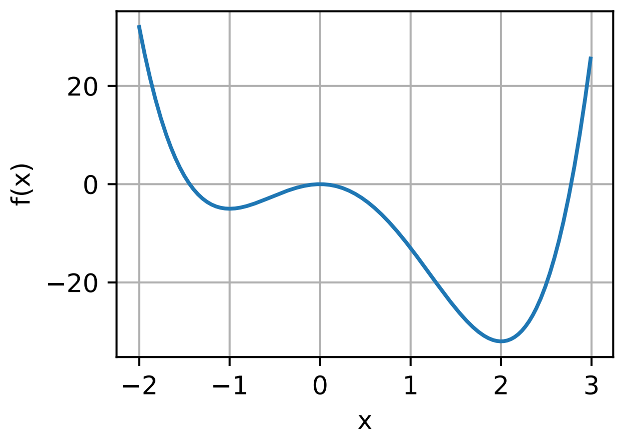

Figure 1: A scalar function of a single scalar variable, $f(x) = 3x^4 - 4x^3 - 12x^2$.

2.2 A scalar function of multiple scalar variables (or a scalar function of a vector variable)

In this case, the function value $f(\mathbf{x})$ is a (single) real number, which is dependent on multiple independent variables $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$, i.e., $f(\mathbf{x}) \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. A bold lowercase letter denotes a vector variable.

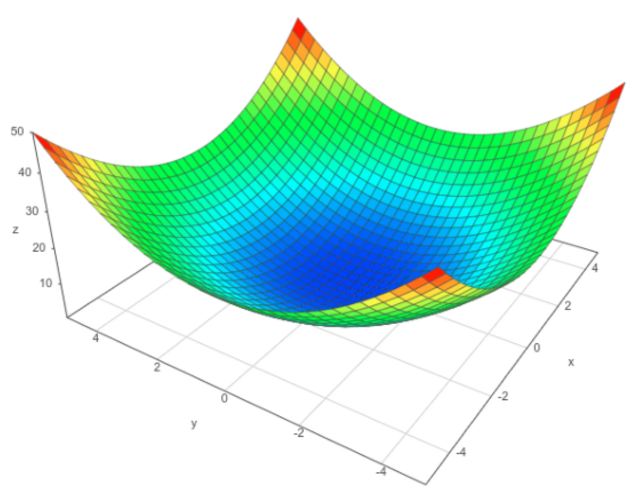

Figure 2: A scalar function of multiple variables (or vector variable), $f(x, y) = x^2 + y^2$.

2.3 A vector function of a single scalar variable

In these types of functions, it is the function value $\mathbf{f}(x)$ that is a vector $\mathbf{f}(x) = (x_1, \ldots, x_n)^\mathsf{T}$ (or multivariate). But, this value $\mathbf{f}(x)$ depends on a single scalar variable, i.e., $\mathbf{f}(x) \in \mathbb{R}^n$ and $x \in \mathbb{R}$. For example, the motion of a particle as a function of time.

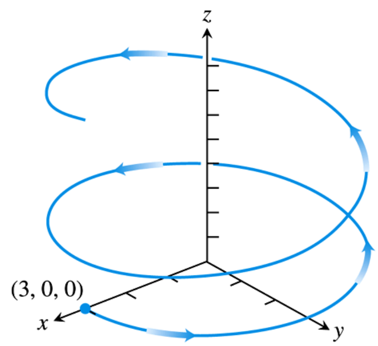

Figure 3: A vector function of a scalar variable. The path of a hang glider with position vector $\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + t^2\,\mathbf{k}$.

2.4 A vector function of a vector variable

For some functions, both the dependent and independent variables are vectors (i.e., multiple dependent variables are dependent of multiple independent variables). In this case, $\mathbf{f}$ is a vector-valued function of a vector of variables, $\mathbf{x}$. Here, $\mathbf{f}\,(\mathbf{x}) = (f_1, f_2, \ldots, f_M)^\mathsf{T}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_N)^\mathsf{T}$, with $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Figure 4 shows an example of the motion of an articulated robot arm. The arm's pose and the 3-D location of its tip are given in terms of a set of joint angles. The figure also shows that a vector function of a vector variable describes a vector field.

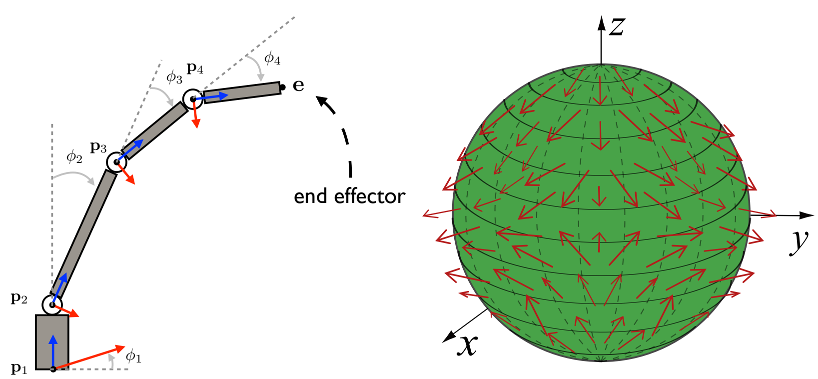

Figure 4: Vector functions of a vector variable. A robot arm and a vector field on a sphere (Figure from: `https://en.wikipedia.org/wiki/Vector_field`).