On the Foundations of Dynamic Games and Probability: Decision Making in Stochastic Extensive Form

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Outline

1. Game theory as a language

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"Doch ein Begriff muss bei dem Worte sein."

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2. Overview

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Why we need an abstract graded language of dynamic games and probability

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Section 1

Game theory as a language

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Why a theory? Language: Confusion and Comparison

Equilibrium, saddle point, Nash equilibrium, subgame-perfect Nash equilibrium, Bayesian Nash equilibrium, perfect equilibrium, Markov perfect equilibrium, perfect Bayesian equilibrium, correlated equilibrium, coarse-correlated equilibrium, Nash equilibrium in stochastic differential games, equilibrium in mean-field games, ...
strategy, pure strategy, mixed strategy, behaviour strategy, randomised strategy, mixed extended strategy, ...
zero-sum game, strategic / normal form game, extensive form game, stochastic game, dynamic game, differential game, stochastic differential game, Dynkin game, timing game, Bayesian game, Stackelberg game, principal-agent problem, interactive decision problem, stochastic optimal control, dynamic programming principle ...

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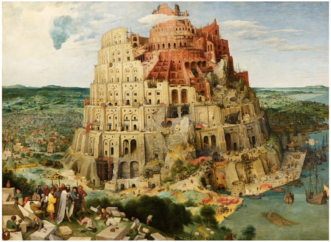

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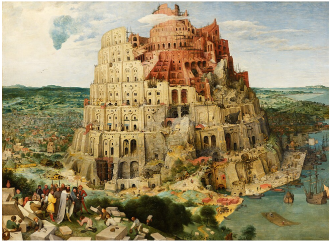

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- When using game-theoretic *words*, they must have a *meaning* in terms of first principles from decision theory.
- Not clear for complex **dynamics** and **uncertainty**: *Confusion*.
- $\longrightarrow$ Abstract **language** for formally *comparing* usages and meanings.

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(3) Economically rational agents' decision making is in strategic **equilibrium**.

What about dynamics and probability?

(4) Dynamic decision making / choice under uncertainty = selecting a **strategy**, i.e. a complete contingent plan of action, alias

$$s\colon \{\textbf{Outside}, , \} \to \{\textbf{enter} , \textbf{enter} , \text{eat pasta}, \text{eat fish}\}.$$

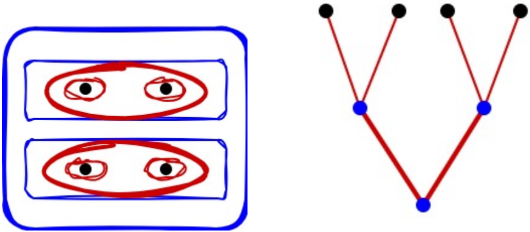

Figure: Choosing a meal, in the refined partitions and tree formulations, where
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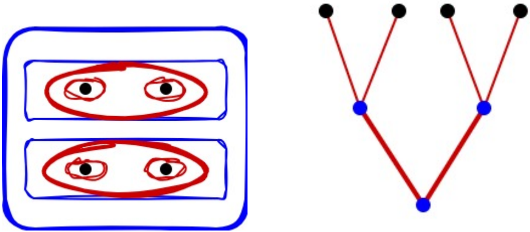

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(5) Representing preference via **expected utility**:

$$\mathrm{EU} = p_1 \cdot u(\text{ pasta}) + p_2 \cdot u(\text{ fish}) + p_3 \cdot u(\text{ pasta}) + p_4 \cdot u(\text{ fish}).$$

Core challenge: Complex dynamics and uncertainty

Figure: A more complex tree. By CMG Lee, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=153344713, 21st Jul 25.

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  - Precisely, even if one is able to distinguish all states, not all sets of states need be measurable.
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- When the vertical structure ("dynamics") is complex, the notion "outcome generated by strategies" easily becomes ill-defined and specifying it gets subtle (e.g. reaction).

Section 2

Overview

—

Why we need an abstract graded language of dynamic games and probability

Technical examples: What do we wish to speak about?

- Construct a — refined partitions / tree-based — extensive form theory incorporating the *measurability* of $g\colon [0,1] \to \mathbb{A}$ describing a choice $g(v)$, based on a signal $v \sim \mathrm{Unif}[0,1]$.

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- Interpret the use of the words "strategy", "equilibrium", "subgame", etc. in terms of game-theoretic first principles for *continuous-time stochastic games*. Applies to important models used in finance, economics, engineering, e.g.:
  - Stochastic differential games:

$$\mathrm{d}\chi_t^s = V(\xi_t^s, \chi_t^s)\,\mathrm{d}\eta_t; \qquad \mathbb{E}_x u_i(\chi^s) \geq \mathbb{E}_x u_i(\chi^{(\tilde{s}^i, s^{-i})}), \quad \forall x,\ i,\ \tilde{s}^i;$$

  - Timing games: $\chi_t^k = 1\{\tau^k > t\}$,

$$u_i(\chi^i, \chi^j) = L_{\tau^i}^i\, 1\{\tau^i < \tau^j\} + M_{\tau^i}^i\, 1\{\tau^i = \tau^j\} + F_{\tau^j}^i\, 1\{\tau^j < \tau^i\}.$$

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- Standard model (dynamic case): **Extensive form** based on **decision trees** / refined partitions (cf. [vNM44, Kuh53, AFR16]). In order to address the mentioned challenges, this theory needs to be generalised — and its limitations be characterised (cf. [Sti92, AFR16]).

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- Aim: *Intrinsic* language of **extensive form characteristics** that also arises *extrinsically* as a limit of strict stochastic extensive form theory.
  - EFC = the flow of information about past choices and exogenous events, along with a set of adapted choices locally available to decision makers

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3. Extensive form characteristics via stochastic decision forests

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Stochastic extensive forms

4. Extensive form characteristics beyond extensive forms

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Continuous-time stochastic timing

Section 3

Extensive form characteristics via stochastic decision forests

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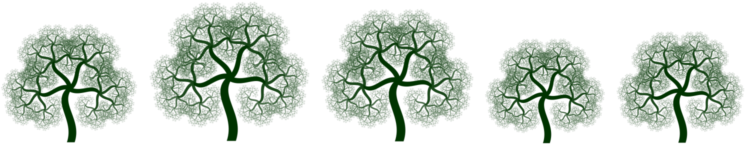
Stochastic extensive forms

Why stochastic decision forests and adapted choices?



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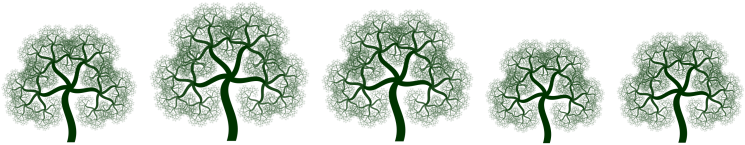

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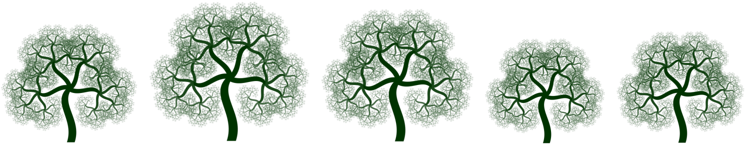

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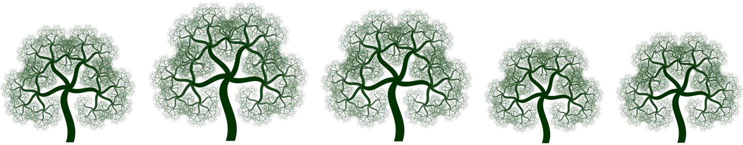

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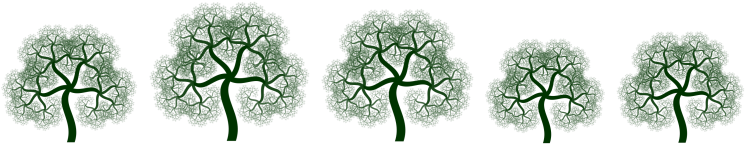

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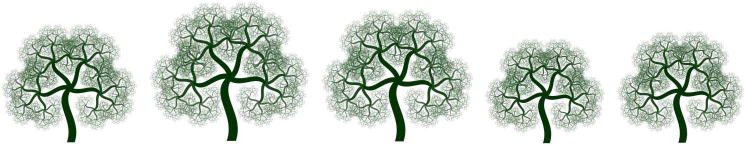

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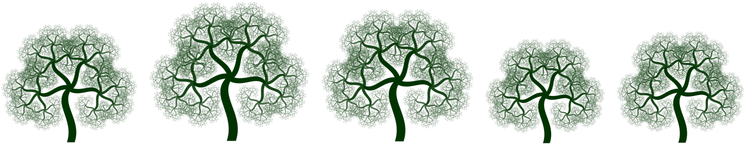

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- Instead of a dynamically acting nature agent $\rightarrow$ a static exogenous scenario space $(\Omega, \mathscr{E})$ plus dynamic updates $\mathscr{F}_x$

What are stochastic decision forests and adapted choices?

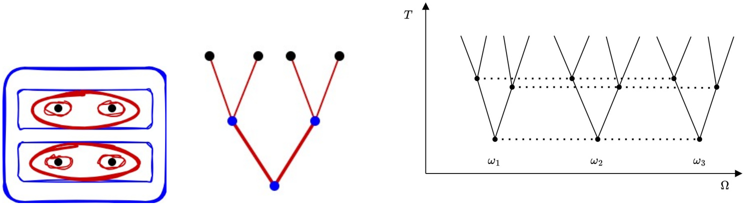

Figure: Decision trees via refined partitions and graphs; a quite simple stochastic decision forest

Main notions	Main results
Decision forest	Forest of decision trees = Decision forest
Stochastic dec. for. (SDF)	Exogenous uncertainty for decision trees ✓, Induced decision tree
Exog. inform. structure	Filtration-like exogenous information ✓
Adapted choice	Exog. measurability + endog. partitions ✓
Action path SDF data	Consistent model based on outcomes $(\omega, f) \in \Omega \times \mathbb{A}^{\mathbb{T}}$, $\mathbb{T}$ time, $\mathbb{A}$ action space ✓

Related to [Har67,68,68a, Wit71, AFR16, HDC22].

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Information	$=$ Endogenous $\oplus$ Exogenous
Strategy	Adaptedness w.r.t. exog. info. $\leftarrow$ choices
Well-posedness (WP)	$\Leftrightarrow$ Well-posedness of sc.-wise ext. forms $\Leftrightarrow$ Simple order-theoretic properties of F $\Rightarrow$ complex dynamic exogenous noise possible
(Random) histories	(Given WP:) $\leftrightarrow$ (Random) moves, and provide a model of "subgames" (given WP)
Action path "SEF"	Covered by the theory of SEF, Well-posedness $\Leftrightarrow$ $\mathbb{T}$ is (ess.ly) well-ordered
Equilibrium	Unifies a large class of equilibrium concepts

Related to [Sel65, Har67,68,68a, AFR16].

Section 4

Extensive form characteristics beyond extensive forms

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Continuous-time stochastic timing

Why we need and how to go beyond extensive forms

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- But otherwise: the model is based on outcomes, strategies induce outcome processes, and preferences are defined on outcome processes!

Preemption games in continuous and discrete time

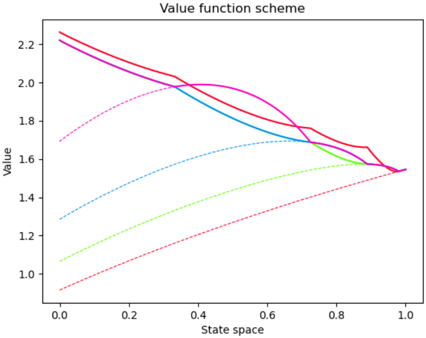

Figure: A symmetric four-player stochastic timing game in continuous time, with underlying diffusion $\mathrm{d}X_t = 0.1(1 - X_t^2)\,\mathrm{d}t + 0.3\sqrt{X_t(1 - X_t)}\,\mathrm{d}B_t$

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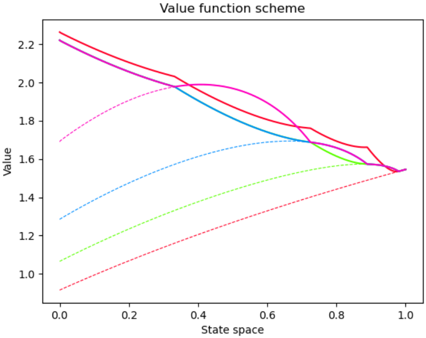

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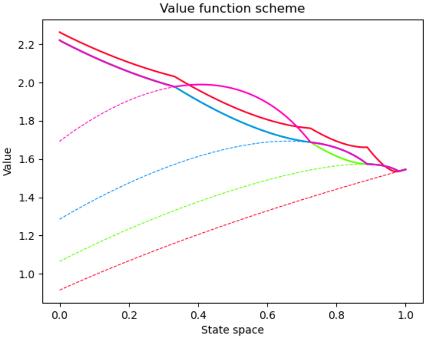

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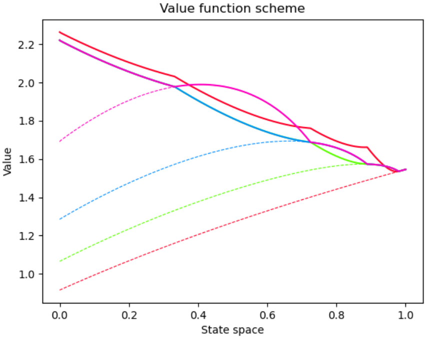

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- Symmetric randomised equilibrium?
- Approaches: additional sharing rules, extended mixed strategies for discrete-time "limit" payoffs (cf. [FT85, Ste16, RS17, Ste18])

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  - $1\{0\}$? Or $1\{(0,0)\}$ and $1\{(0,0),(0,1)\}$ on $\mathbb{R}_+ \times \alpha$, $\alpha \in \mathrm{On}$?

Why do we need vertically extended continuous time?

- Disadvantages of [FT85]-formulation: no intrinsic continuous-time model of outcomes, stacked strategic form; "perfect equilibrium"?
- A key issue is information loss regarding "infinitesimal" reaction:
  - On $k2^{-n}$, $k \in \mathbb{N}$, consider $\xi^n = 1[0, 2^{-n})_{\mathbb{R}_+}$ and ξ^{n-1}. *Limit* $n \to \infty$?
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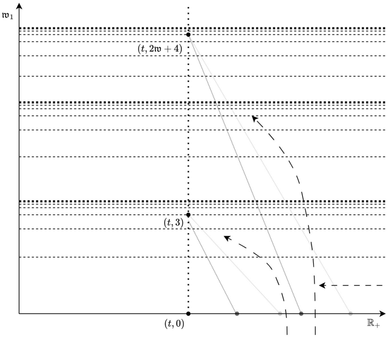

Figure: Vertically extended continuous time and tilting convergence

Stochastic analysis in vertically extended continuous time

Main notions	Main results
$\overline{\mathbb{T}} = [\mathbb{R}_+ \times (\mathfrak{w}_1 + 1)] \cup \{\infty\}$	Smallest complete lattice containing all tilted well-orders embeddable into $\mathbb{R}_+$
$\overline{\mathbb{T}} \leftarrow$ order topology; σ-algebra $\mathscr{P}_{\overline{\mathbb{T}}}$ generated by $\overline{\mathbb{T}} \to \overline{\mathbb{R}_+} \times \{0, \ldots, \alpha\}$, and $\mathscr{B}_{\overline{\mathbb{R}_+} \times \{0,\ldots,\alpha\}}$, $\alpha \in \mathfrak{w}_1$	Continuous $f \colon \overline{\mathbb{T}} \to \mathbb{R}$ mb., Mb. Section and Projection Theorem, Prog. measurability of $[\![\tau]\!]$
Optional times	Characterisation of optional times, Début Theorem, Basic theory of optional / predictable processes
Tilting convergence	Optional processes arise from opt. processes defined on deterministic classical grids by closure under a) continuous binary operations, b) pointwise convergence, and c) tilting convergence.

Example: Timing game in stochastic process form

- $(\Omega, \mathscr{E}, \mathscr{G}, \mathbb{P})$: a completely filtered probability space with time $\overline{\mathbb{T}}$.

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- For $i \in I$, let $\mathscr{F}^i = \mathscr{G} \vee \sigma(\upsilon^i)$, where $\upsilon^i = (\upsilon^{i,n})_{n \in \mathfrak{w}}$ is an i.i.d.-family of Unif$[0,1]$ r.v. independent from $\mathscr{G}$ and υ^j, $j \neq i$.

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- Let $\mathscr{H}^i$ be a *suitably augmented* filtration on W such that $\mathscr{H}^i_t$ contains $\mathscr{F}^i_t \otimes \mathscr{B}_{\mathbb{B}}^{[0,t]_{\mathbb{T}}} \otimes \{\emptyset, \mathbb{B}^{(t,\infty]_{\mathbb{T}}}\}|_W$, $t > 0$, and $\mathscr{H}^i_0 = \mathscr{F}^i_0 \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\}|_W$: basic information structure.

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- Let $\mathscr{M}^i = \mathrm{Prd}(\mathscr{H}^i) \vee \mathrm{Opt}(\mathscr{F}^i \otimes \{\emptyset, \mathbb{B}^{\overline{\mathbb{T}}}\})$: Meyer information structure.

Example: Timing game in stochastic process form

- For any $i \in I$, let $\mathcal{S}_0^i$ be the set of $\mathscr{H}^i$-progressively measurable, processes $s^i : \overline{\mathbb{T}} \times W \to \{0,1\}$, lower semicontinuous from the right, inert above $\mathfrak{w}$, and satisfying both $s^i_t(\omega, h) \leq h^i(t-)$, for all $(t, \omega, h) \in \overline{\mathbb{T}} \times W$, and $s^i_\infty = 0$; then, let $\mathcal{S}^i$ be the set of $s^i \in \mathcal{S}_0^i$ such that any optional time τ^i for i admits $\mathscr{M}^i$-measurable $\tilde{s}^i : \overline{\mathbb{T}} \times W \to \{0,1\}$ with $\tilde{s}^i \in \mathcal{S}_0^i$ and $s^i_{\tau^i} = \tilde{s}^i_{\tau^i}$: strategies for i. Let $\mathcal{S} = \times_{i \in I} \mathcal{S}^i$: strategy profiles.

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Theorem (Well-posedness). For any player $i \in I$, any optional time τ^i for i, any state process $\tilde{\chi}$, there is a unique state process χ within the information set of i given by $(\tau^i, \tilde{\chi})$ such that, for all $k \in I$, all $(t, \omega) \in [\![\tau^i(.,\chi(.)), \infty]\!]$:

$$s_t^k(\omega, \chi(\omega)) = \chi_t^k(\omega).$$

(Related to [Sti92])

Application to a stylised preemption game

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$$s^i_t(\omega, h) = \begin{cases} 1, & \text{if } \left(\pi(t) < \mathfrak{w},\ \upsilon^{i,\pi(t)}(\omega) \geq \frac{\eta^j_{p(t)}}{1+\eta^j_{p(t)}}(\omega)\right), \\ 0, & \text{else.} \end{cases}$$

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Theorem (simplified). s defines an equilibrium. The corresponding action process is a tilting limit of sequences of processes ξ^n along sequences of classical, deterministic grids G_n, which arise from the respective discrete-time equilibrium along G_n, $n \to \infty$.

Comments on the continuous-time preemption example

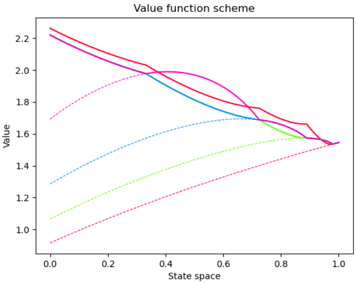

Figure: A symmetric four-player stochastic timing game in continuous time, with underlying diffusion $\mathrm{d}X_t = 0.1(1 - X_t^2)\,\mathrm{d}t + 0.3\sqrt{X_t(1 - X_t)}\,\mathrm{d}B_t$

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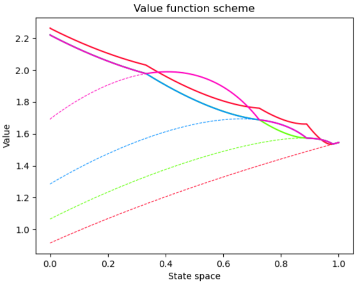

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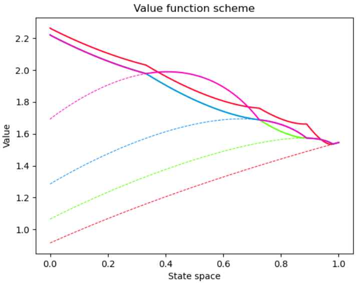

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- *One* strategy process, *derived* counterfactuals, intrinsic model of *timing* outcomes: A vertically-extended-continuous-time model for a vertically-extended-continuous-time problem
- Equilibrium at preemption boundary can be explained as well (though in different form); no tilting limit because both stopping times "evaporate" at infinity; no complete equilibrium convergence (payoffs are discontinuous)

Section 5

Conclusion

Concluding remarks

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- A new perspective on preemption and, more generally, stochastic differential games

Non-exhaustive list of references (for convenience)

For convenience, a non-exhaustive list of the most important references is given.

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A full bibliography can be found in the doctoral dissertation:

- E. Emanuel Rapsch. *On the Foundations of Dynamic Games and Probability: Decision Making in Stochastic Extensive Form*. Doctoral Dissertation, TU Berlin, Berlin 2025.

It is accessible on arXiv under the DOI

https://doi.org/10.48550/arXiv.2508.04752.

- The "Tower of Babel" painting by Pieter Bruegel the Elder, in the collection of KHM Vienna, is in the public domain and has last been downloaded from https://en.wikipedia.org/wiki/The_Tower_of_Babel_(Bruegel)#/media/File:Pieter_Bruegel_the_Elder_-_The_Tower_of_Babel_(Vienna)_-_Google_Art_Project_-_edited.jpg, on 26th August 2025.
- The figure showing the "complex tree" is licensed under CC BY-SA 4.0. The reference is: CMG Lee. (no caption). https://commons.wikimedia.org/w/index.php?curid=153344713, 21st July 2025.
- The logos of TU Berlin and IRTG 2544 are copyrighted by these institutions in 2025, respectively.
- The remaining figures have been generated by the author using the graphics software *draw.io*, version v27.0.9, the *Matplotlib* package in *Python*, and the *tikz* package in LaTeX.

Thank you for your attention!

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Figure: CMG Lee, ibid.; TU Berlin

Section 6

Appendix

Stochastic decision forests

Definition (simplified). A stochastic decision forest on $(\Omega, \mathscr{E})$ is given by a decision forest F, a surjection $\pi \colon F \to \Omega$ whose fibres $\pi^{-1}(\{\omega\}) = T_\omega$ are the connected components of $(F, \supseteq)$, and a set $\mathbb{X}$ such that, if X denotes the set of F's moves:

1. any element $\mathbb{x} \in \mathbb{X}$ is a section of moves, that is, it is a map $\mathbb{x} \colon \Omega \to X$ satisfying $\pi \circ \mathbb{x} = \mathrm{id}_\Omega$;
2. $\mathbb{X}$ induces a covering of F's moves, that is, $\{\mathbb{x}(\omega) \mid \mathbb{x} \in \mathbb{X},\, \omega \in \Omega\} = X$.

The elements of $\mathbb{X}$ are called random moves.

Adapted choices

Let $(F, \pi, \mathbb{X})$ be a stochastic decision forest and $W = \bigcup F$ be the set of outcomes.

Definition (simplified). An *exogenous information structure* is a family $\mathscr{F} = (\mathscr{F}_{\mathbb{x}})_{\mathbb{x}\in\mathbb{X}}$ of sub-σ-algebras of $\mathscr{E}$.

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Definition (simplified; cf. [AFR05]). A *choice* is a non-empty union of nodes. For any subset $c \subseteq W$, let

$$\downarrow c = \{x \in F \mid c \supseteq x\}, \qquad P(c) = \{x \in F \mid \exists y \in \downarrow c \colon \uparrow x = \uparrow y \setminus \downarrow c\}.$$

A choice c is

- *non-redundant* iff for any $\omega \in \Omega$ with $P(c) \cap T_\omega = \emptyset$, we have $c \cap W_\omega = \emptyset$;
- *complete* iff for all random moves $\mathbb{x} \in \mathbb{X}$, $\mathbb{x}^{-1}(P(c)) \in \{\emptyset, \Omega\}$.

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A *reference choice structure* is a family $\mathscr{C} = (\mathscr{C}_{x})_{x \in \mathbb{X}}$ of sets of $\mathscr{C}_{x}$ non-redundant, complete choices c with $x^{-1}(P(c)) = \Omega$, $x \in \mathbb{X}$.

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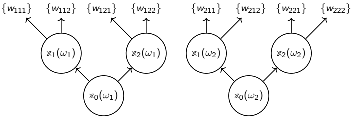
$$\mathbb{x}^{-1}(P(c \cap c')) = \{\omega \in D_{\mathbb{x}} \mid \mathbb{x}(\omega) \in P(c \cap c')\} \in \mathscr{F}_{\mathbb{x}}.$$

Simple examples for stochastic decision forests and adapted choices



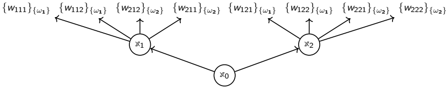


Figure: Left: A simple stochastic decision forest represented as a directed graph. Moves are indicated by circles. — Right: The induced decision tree of random nodes.

Simple examples for stochastic decision forests and adapted choices

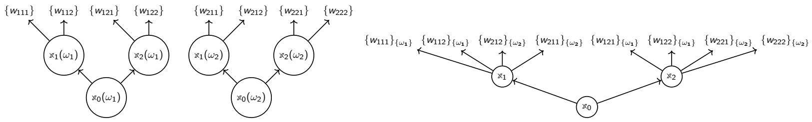


Figure: Left: A simple stochastic decision forest represented as a directed graph. Moves are indicated by circles. — Right: The induced decision tree of random nodes.

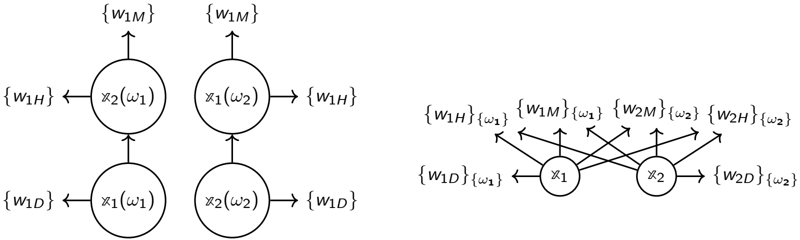


Figure: The absent-minded driver sdf, following [Gil97], and the induced poset of random nodes. Non-minimal elements are indicated by circles, respectively.

Action path stochastic decision forests and adapted choices

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$$c_t(f; g) = \{(\omega, f') \in \mathbb{A}^{\mathbb{T}} \mid f'|_{[0,t)_{\mathbb{T}}} = f|_{[0,t)_{\mathbb{T}}},\, f'(t) = g(\omega)\}, \quad g\colon \Omega \to \mathbb{A}.$$

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- **Theorems** (much simplified). The data induces a stochastic decision forest. $c_t(g)$ is a "suitable" choice. It is $\mathscr{F}$-$\mathscr{C}$-adapted iff g is $\mathscr{G}_t$-measurable.

Application to a stylised preemption game with “diffusion”

Notation: p and π denote the projections from $\overline{\mathbb{T}}$ onto $\overline{\mathbb{R}_+}$ and $\mathfrak{w}_1 + 1$.

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$$s^i_t(\omega, h) = \begin{cases} 1, & \text{if } \left(\pi(t) < \mathfrak{w},\ \upsilon^{i,\pi(t)}(\omega) \geq \frac{p(t)\,\eta^j_{p(t)}}{1+p(t)\,\eta^j_{p(t)}}(\omega) > 0\right) \\ & \text{or } \left(\pi(t) < \mathfrak{w},\ p(t) = 0,\ i = 2\right), \\ 0, & \text{else.} \end{cases}$$

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Theorem (simplified). s defines an equilibrium. On $p(t) > 0$, the corresponding action process is a tilting limit of sequences of processes ξ^n along sequences of classical, deterministic grids G_n, which arise from the respective discrete-time equilibrium along G_n, $n \to \infty$.