Butterworth Filter Design Cheatsheet

CORE CONCEPT

Butterworth = Maximally Flat Magnitude Response - No ripple in passband - Poles evenly distributed on circle in s-plane - At cutoff: $|H(j\omega_C)| = H_{MAX}/\sqrt{2}$ (-3 dB) ALWAYS

PHASE 1: CALCULATE FILTER ORDER

Given Specifications

- H_{MAX}: Maximum passband gain (linear or dB)
- H_{MIN}: Minimum stopband gain (linear or dB)

- ω_C (or f_C): Cutoff frequency
- ω_{MIN} (or f_{MIN}): Stopband frequency where H_{MIN} must be met

Convert Units

dB to Linear:

$$H_{\text{linear}} = 10^{H_{\text{dB}}/20}$$

Hz to rad/s:

$$\omega = 2\pi f$$

Calculate Minimum Order

Low-Pass (stopband ABOVE cutoff):

$$n \geq \frac{1}{2} \cdot \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{\ln(\omega_{MIN}/\omega_C)}$$

High-Pass (stopband BELOW cutoff):

$$n \geq \frac{1}{2} \cdot \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{\ln(\omega_C/\omega_{MIN})}$$

CRITICAL: ALWAYS ROUND UP to nearest integer

Why This Formula Works

Starting from Butterworth magnitude response:

$$|H(j\omega)|^2 = \frac{H_{MAX}^2}{1 + (\omega/\omega_C)^{2n}}$$

At stopband frequency ω_{MIN}:

$$|H(j\omega_{MIN})| = H_{MIN}$$

Substituting and solving:

$$\frac{H_{MAX}^2}{1 + (\omega_{MIN}/\omega_C)^{2n}} = H_{MIN}^2$$

$$(\omega_{MIN}/\omega_C)^{2n} = \frac{H_{MAX}^2}{H_{MIN}^2} - 1$$

Taking logarithm of both sides:

$$2n \ln(\omega_{MIN}/\omega_C) = \ln[(H_{MAX}/H_{MIN})^2 - 1]$$

$$n = \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{2\ln(\omega_{MIN}/\omega_C)}$$

Determine Section Count

- **Even n:** Use $n/2$ second-order sections
- **Odd n:** Use $(n-1)/2$ second-order sections + 1 first-order section

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PHASE 2: CALCULATE BUTTERWORTH POLES

Pole Angles (Normalized $\omega_C = 1$)

For each pole $k = 1, 2, \ldots, n$:

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}$$

Note: k is just an enumeration index (1, 2, 3…). It has NO electrical meaning.

Pole Locations (Normalized)

$$p_k = e^{j\theta_k} = \cos(\theta_k) + j\sin(\theta_k)$$

Properties: - All poles have magnitude $|p_k| = 1$ (on unit circle) - **Only use LEFT half-plane poles** (Real part < 0) - Complex poles always come in conjugate pairs

Denormalize Poles

Scale by actual cutoff frequency:

$$p_{k,\text{actual}} = \omega_C \cdot p_k$$

PHASE 3: EXTRACT SECTION PARAMETERS

Form Quadratic from Conjugate Pair

For poles $p = \alpha + j\beta$ and $p^* = \alpha - j\beta$:

$$(s-p)(s-p^*) = s^2 - (p+p^*)s + p \cdot p^*$$

Since $p + p^* = 2\alpha$ and $p \cdot p^* = \alpha^2 + \beta^2 = |p|^2$:

$$(s-p)(s-p^*) = s^2 - 2\alpha s + (\alpha^2 + \beta^2)$$

For normalized poles ($|p| = 1$):

$$s^2 + as + 1$$

where $a = -2\text{Re}(p) = -2\cos(\theta_k)$

Extract Damping Ratio ζ

Standard second-order form:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2$$

For normalized ($\omega_0 = 1$):

$$s^2 + 2\zeta s + 1$$

Comparing coefficients:

$$2\zeta = a = -2\cos(\theta_k)$$

$$\zeta = -\cos(\theta_k) = \frac{a}{2}$$

Why negative? Poles are in left half-plane, so $\cos(\theta_k) < 0$ for $90° < \theta_k < 270°$.

Calculate Quality Factor Q

$$Q = \frac{1}{2\zeta}$$

Relationships: - Higher $Q \rightarrow$ Lower $\zeta \rightarrow$ More resonance/peaking - Lower $Q \rightarrow$ Higher $\zeta \rightarrow$ More damping

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PHASE 4: UNITY GAIN SALLEN-KEY DESIGN (2nd Order)

Why Unity Gain is ALWAYS Preferred

- **Works for ALL ζ values** (even $\zeta \geq 1$ overdamped)
- Most stable configuration
- Op-amp as simple buffer ($\mu = 1$)
- Zero sensitivity to gain variations

Design Equations - Low-Pass

Given: ω_0 (rad/s) and ζ for the section

Option A - Choose C_1 (typical: 0.01 to 1 F):

$$C_2 = 4\zeta^2 C_1$$

$$R_1 = R_2 = R = \frac{1}{\zeta\omega_0 C_1}$$

Option B - Choose R_2 (typical: 1kΩ to 100kΩ):

$$R_1 = 4\zeta^2 R_2$$

$$C_1 = C_2 = C = \frac{1}{\zeta\omega_0 R_2}$$

Why These Formulas Work

From Sallen-Key transfer function with $\mu = 1$:

$$H(s) = \frac{1}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_2C_1 + R_1C_2)s + 1}$$

Comparing to standard form $s^2 + 2\zeta\omega_0 s + \omega_0^2$:

$$\omega_0 = \frac{1}{\sqrt{R_1R_2C_1C_2}}$$

$$2\zeta\omega_0 = \frac{R_1C_1 + R_2C_1 + R_1C_2}{R_1R_2C_1C_2}$$

For Option A (R1 = R2 = R, choose C1):

$$\omega_0 = \frac{1}{R\sqrt{C_1C_2}}$$

$$2\zeta = \frac{2C_1 + C_2}{\sqrt{C_1C_2}}$$

Solving: $C_2 = 4\zeta^2 C_1$ and $R = \frac{1}{\zeta\omega_0 C_1}$

Design Equations - High-Pass

KEY: Swap ALL R   C from low-pass design

With R_2 chosen:

$$R_1 = 4\zeta^2 R_2$$

$$C_1 = C_2 = C = \frac{1}{\zeta\omega_0 R_2}$$

PHASE 5: FIRST-ORDER SECTIONS (Odd Order Only)

For Real Pole at $s = -\omega_C$

Transfer Function:

$$H(s) = \frac{1}{1 + s/\omega_C} = \frac{\omega_C}{s + \omega_C}$$

Passive RC Design

Circuit: Series R, shunt C to ground

$$\omega_C = \frac{1}{RC}$$

Design: Choose C, then:

$$R = \frac{1}{\omega_C C}$$

Properties: - DC gain = 1 - Slope: -20 dB/decade - No resonance (monotonic response)

Active Design (with gain K)

$$H(s) = \frac{K}{1 + sRC}$$

Use non-inverting op-amp:

$$K = 1 + \frac{R_f}{R_g}$$

PHASE 6: CASCADE ASSEMBLY

Section Ordering - CRITICAL

Rule: Order by Q from LOW to HIGH

Equivalently: Order by ζ from HIGH to LOW

Order	Q	ζ	Reason
First	0.541	0.924	Low Q = high damping = no peaks
Last	1.307	0.383	High Q = resonance peaks = must be last

Why? High-Q sections have resonant peaks. If placed first, they amplify input → clipping. Low-Q sections first reduce signal gradually.

Buffering

Good News: Each Sallen-Key stage naturally has: - High input impedance (op-amp input) - Low output impedance (op-amp output)

Result: No additional buffers needed between stages!

Overall Transfer Function

$$H_{\text{total}}(s) = H_1(s) \times H_2(s) \times \cdots \times H_n(s)$$

Verification: - Multiply all section transfer functions - Check DC gain (set $s = 0$) - Check high-frequency behavior ($s \to \infty$)

PHASE 7: COMPONENT SCALING

When Calculated Values Are Impractical

Frequency Scaling (multiply by k_f)

$$C_{\text{new}} = \frac{C_{\text{old}}}{k_f}$$

$$R_{\text{new}} = R_{\text{old}}$$

(unchanged)

Effect: $\omega_{\text{new}} = k_f \times \omega_{\text{old}}$

Example: To shift cutoff from 1 kHz to 10 kHz, use $k_f = 10$

Impedance Scaling (multiply by k_z)

$$R_{\text{new}} = k_z \times R_{\text{old}}$$

$$C_{\text{new}} = \frac{C_{\text{old}}}{k_z}$$

Effect: $H(s)$ and frequencies unchanged

Example: To increase resistors by factor of 10, use $k_z = 10$

Combined Scaling

$$R_{\text{final}} = k_z \times R_{\text{normalized}}$$

$$C_{\text{final}} = \frac{C_{\text{normalized}}}{k_f \times k_z}$$

Practical Component Ranges

Component	Practical Range	Preferred Range
Resistors	100Ω - 10MΩ	1kΩ - 1MΩ
Capacitors	1pF - 1000 F	100pF - 10 F

UNDERSTANDING TRANSFER FUNCTIONS FROM LINEAR H(s)

Given: Linear Transfer Function

Example: $H(s) = \frac{K}{s+1000}$

Identify Filter Type

From Denominator: - First-order (degree 1): One real pole - Second-order (degree 2): Complex conjugate pair

From Numerator: - Constant → Low-pass - s term → High-pass or Band-pass - s^2 term → High-pass (2nd order)

Extract Pole Locations

For $H(s) = \frac{K}{s+1000}$:

Denominator: $s + 1000 = 0$

Pole: $s = -1000$ (real pole at -1000 rad/s)

Extract Cutoff Frequency

For first-order: $\omega_C = |\text{pole}| = 1000$ rad/s

Extract DC Gain

$$H(0) = \frac{K}{0 + 1000} = \frac{K}{1000}$$

If DC gain should be 1: $K = 1000$

For Second-Order: $H(s) = \frac{K}{s^2+bs+c}$

Find poles: Solve $s^2 + bs + c = 0$

$$s = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

If complex conjugate pair ($b^2 < 4c$):

$$\omega_0 = \sqrt{c}$$

$$\zeta = \frac{b}{2\sqrt{c}}$$

$$Q = \frac{\sqrt{c}}{b}$$

Now use these in Unity Gain design formulas!

POLE ANGLES - COMPLETE DERIVATION

Why Butterworth Poles Are on a Circle

Butterworth magnitude-squared:

$$|H(s)|^2 = \frac{1}{1 + (s/j\omega_C)^{2n}}$$

Poles occur where denominator = 0:

$$1 + (s/j\omega_C)^{2n} = 0$$

$$(s/j\omega_C)^{2n} = -1 = e^{j(2m+1)\pi}$$

$$s/j\omega_C = e^{j(2m+1)\pi/(2n)}$$

$$s = j\omega_C e^{j(2m+1)\pi/(2n)}$$

Since $j = e^{j\pi/2}$:

$$s = \omega_C e^{j[\pi/2 + (2m+1)\pi/(2n)]}$$

For $m = 0, 1, 2, \ldots, n-1$ (using $k = m + 1$):

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}$$

Plugging into Pole Formula

1. **Calculate θ_k for your section's pole**

   Example: For $n = 4$, $k = 1$:

$$\theta_1 = \frac{\pi}{2} + \frac{(2 \cdot 1 - 1)\pi}{2 \cdot 4} = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8} = 112.5°$$

2. **Calculate pole location:**

$$p_1 = \cos(112.5°) + j\sin(112.5°) = -0.383 + j0.924$$

3. **Extract a coefficient:**

$$a = -2\cos(\theta_1) = -2\cos(112.5°) = -2(-0.383) = 0.765$$

4. **Calculate ζ:**

$$\zeta = \frac{a}{2} = \frac{0.765}{2} = 0.383$$

5. **Calculate Q:**

$$Q = \frac{1}{2\zeta} = \frac{1}{2(0.383)} = 1.307$$

6. **Now design Unity Gain section** with $\zeta = 0.383$ and $\omega_0 = \omega_C$

WORKED EXAMPLE: 4th-Order Low-Pass

Given

- $H_{MAX} = 0$ dB $\rightarrow$ 1.0 linear
- $H_{MIN} = -40$ dB $\rightarrow$ 0.01 linear
- $f_C = 2$ kHz $\rightarrow \omega_C = 12566$ rad/s
- $f_{MIN} = 8$ kHz $\rightarrow \omega_{MIN} = 50265$ rad/s

Step 1: Calculate Order

$$n \geq \frac{\ln[(1/0.01)^2 - 1]}{2\ln(50265/12566)} = \frac{\ln(9999)}{2\ln(4)} = \frac{9.210}{2.773} = 3.32$$

Round UP: $n = 4$

Step 2: Calculate Poles

For $n = 4$, $k = 1, 2, 3, 4$:

$$\theta_1 = 112.5°, \quad \theta_2 = 157.5°, \quad \theta_3 = 202.5°, \quad \theta_4 = 247.5°$$

Normalized poles: - $p_1 = -0.383 + j0.924$ - $p_2 = -0.924 + j0.383$ - $p_3 = -0.924 - j0.383$ (conjugate of p_2) - $p_4 = -0.383 - j0.924$ (conjugate of p_1)

Step 3: Form Sections

Section 1 (poles p_1, p_4):

$$s^2 + 0.765s + 1$$

- $\zeta = 0.383$, $Q = 1.307$

Section 2 (poles p_2, p_3):

$$s^2 + 1.848s + 1$$

- $\zeta = 0.924$, $Q = 0.541$

Step 4: Design Unity Gain Sections

Section 1: Choose $C_1 = 0.1\ \ \mathrm{F}$

$$C_2 = 4(0.383)^2(0.1) = 0.0586\ \ \mathrm{F} \approx 0.056\ \ \mathrm{F}$$

$$R = \frac{1}{0.383 \times 12566 \times 0.1 \times 10^{-6}} = 2078\ \Omega \approx 2.2\ \mathrm{k}\Omega$$

Section 2: Choose $C_1 = 0.1\ \ \mathrm{F}$

$$C_2 = 4(0.924)^2(0.1) = 0.341\ \ \mathrm{F} \approx 0.33\ \ \mathrm{F}$$

$$R = \frac{1}{0.924 \times 12566 \times 0.1 \times 10^{-6}} = 861\ \Omega \approx 820\ \Omega$$

Step 5: Cascade (LOW Q → HIGH Q)

1. **Section 2** ($Q = 0.541$): R1=R2=820Ω, C1=0.1 F, C2=0.33 F
2. **Section 1** ($Q = 1.307$): R1=R2=2.2kΩ, C1=0.1 F, C2=0.056 F

KEY FORMULAS SUMMARY

Order:

$$n \geq \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{2\ln(\omega_{MIN}/\omega_C)} \quad \text{(LP)}$$

Pole Angles:

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}$$

Damping:

$$\zeta = -\cos(\theta_k) = \frac{a}{2}, \quad Q = \frac{1}{2\zeta}$$

Unity Gain (choose C_1):

$$C_2 = 4\zeta^2 C_1, \quad R_1 = R_2 = \frac{1}{\zeta\omega_0 C_1}$$

First-Order:

$$H(s) = \frac{1}{1 + sRC}, \quad \omega_C = \frac{1}{RC}$$

DESIGN WORKFLOW

Given Specs → Circuit: 1. Calculate n (round UP) 2. Find pole angles θ_k 3. Extract ζ for each section 4. Use Unity Gain formulas 5. Order sections by Q (low → high) 6. Scale components 7. Cascade with buffers

Given Linear $H(s)$ → Design: 1. Factor denominator → find poles 2. Extract ω_0, ζ from each factor 3. Use Unity Gain formulas 4. Cascade sections

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