Final Quiz Cheatsheet

SECTION 1: TRANSFER FUNCTIONS (NETWORK FUNCTIONS)

Definition

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} \quad \text{or} \quad H(s) = \frac{I_{out}(s)}{I_{in}(s)}$$

Critical: ALL initial conditions must be ZERO when finding H(s)

Types of Network Functions

Type	Definition	Same/Different Ports
Transfer Function	$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$	Different ports
Driving-Point Impedance	$Z(s) = \frac{V(s)}{I(s)}$	Same port
Driving-Point Admittance	$Y(s) = \frac{I(s)}{V(s)} = \frac{1}{Z(s)}$	Same port

Rational Function Form

$$H(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}$$

Poles and Zeros

- **ZEROS:** Roots of numerator $N(s) = 0$ - Frequencies that are **blocked**
- **POLES:** Roots of denominator $D(s) = 0$ - **Natural frequencies** (always in output)

s-Domain Element Impedances

$$Z_R = R \qquad Z_C = \frac{1}{sC} \qquad Z_L = sL$$

Finding H(s) from Circuit - Quick Steps

1. Transform all elements to s-domain
2. **ZERO all initial conditions**
3. Apply circuit analysis (voltage divider, nodal, mesh)
4. Form $H(s) = \frac{\text{Output}}{\text{Input}}$
5. Simplify to rational function

SECTION 2: PARTIAL FRACTION EXPANSION (PFE)

Why PFE?

Convert $V_{out}(s) = H(s) \cdot V_{in}(s)$ into simple fractions for inverse Laplace transform.

Standard Form

$$F(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}$$

Cover-Up Method for Residues

$$k_i = \left[(s - p_i) \cdot F(s)\right]_{s=p_i}$$

Case 1: Distinct Real Poles

Example: $F(s) = \frac{100}{(s+10)(s+100)}$

Setup: $F(s) = \frac{k_1}{s+10} + \frac{k_2}{s+100}$

Find residues:

$$k_1 = \left[(s+10) \cdot \frac{100}{(s+10)(s+100)}\right]_{s=-10} = \frac{100}{-10+100} = \frac{100}{90}$$

$$k_2 = \left[(s+100) \cdot \frac{100}{(s+10)(s+100)}\right]_{s=-100} = \frac{100}{-100+10} = -\frac{100}{90}$$

Time domain: $f(t) = \frac{100}{90}e^{-10t} - \frac{100}{90}e^{-100t}$ for $t \geq 0$

Case 2: Complex Conjugate Poles

Poles: $s = -\alpha \pm j\omega_d$

Time Response:

$$f(t) = 2|k|e^{-\alpha t}\cos(\omega_d t + \angle k)u(t)$$

Common Laplace Transform Pairs

Time Domain	s-Domain
$u(t)$	$\frac{1}{s}$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
$e^{-\alpha t}\cos(\omega t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$e^{-\alpha t}\sin(\omega t)u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$

SECTION 3: OP-AMP FUNDAMENTALS

Ideal Op-Amp Rules

1. **No current flows into either input:** $i_+ = i_- = 0$
2. **Voltage difference is zero (negative feedback):** $V_+ = V_-$

Common Configurations

Inverting Amplifier:

$$H(s) = -\frac{Z_f(s)}{Z_{in}(s)}$$

Non-Inverting Amplifier:

$$H(s) = 1 + \frac{Z_f(s)}{Z_g(s)} = \mu$$

$$\mu = 1 + \frac{R_f}{R_g}$$

Voltage Follower (Buffer):

$$H(s) = 1 \quad (\mu = 1)$$

- High input impedance, low output impedance - Prevents loading between stages

Summing Amplifier:

$$V_{out} = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right)$$

Why Op-Amps in Filter Design?

- Enable **cascading** without loading: $H_{total}(s) = H_1(s) \times H_2(s) \times \cdots$
- Provide **gain** ($\mu > 1$)
- **Buffer** stages (high input Z, low output Z)

SECTION 4: FILTER DESIGN - BUTTERWORTH METHOD

Filter Types by H(s)

Filter	H(s) Numerator	Passes	Blocks
Low-Pass	Constant (K)	Low freq	High freq
High-Pass	s^n	High freq	Low freq
Band-Pass	s	Middle band	Low & High
Band-Stop	$s^2 + \omega_0^2$	Low & High	Middle band

PHASE 1: Calculate Required Filter Order

Given Specifications

- H_{MAX}: Maximum passband gain (linear or dB)
- H_{MIN}: Minimum stopband gain (linear or dB)
- ω_C or f_C: Cutoff frequency
- ω_{MIN} or f_{MIN}: Stopband frequency

Convert dB to Linear

$$H_{\text{linear}} = 10^{H_{\text{dB}}/20}$$

Butterworth Order Formulas **Low-Pass:**

$$n \geq \frac{1}{2} \cdot \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{\ln(\omega_{MIN}/\omega_C)}$$

High-Pass:

$$n \geq \frac{1}{2} \cdot \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{\ln(\omega_C/\omega_{MIN})}$$

ALWAYS ROUND UP to nearest integer

Section Count

- **Odd n:** $(n-1)/2$ second-order + 1 first-order
- **Even n:** $n/2$ second-order sections

PHASE 2: Obtain Butterworth Poles

Pole Angles (Normalized omega_C = 1)

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n$$

Pole Locations

$$p_k = e^{j\theta_k} = \cos(\theta_k) + j\sin(\theta_k)$$

Use only LEFT half-plane poles (Real part < 0)

PHASE 3: Extract Section Parameters

For 2nd-Order Section from Conjugate Pair **Poles:** $p = \alpha + j\beta$ and $p^* = \alpha - j\beta$

Quadratic form:

$$(s - p)(s - p^*) = s^2 - 2\alpha s + (\alpha^2 + \beta^2)$$

For normalized ($|p| = 1$):

$$s^2 + as + 1$$

where $a = -2\cos(\theta_k)$

Extract damping ratio:

$$\zeta_k = \frac{a}{2} = -\cos(\theta_k)$$

Quality factor:

$$Q_k = \frac{1}{2\zeta_k}$$

Quick Reference Table for Damping Ratio and Q

n	Section	Damping Ratio	Q Factor	a coefficient
2	1	0.707	0.707	1.414
3	1	0.500	1.000	1.000
4	1	0.383	1.307	0.765
4	2	0.924	0.541	1.848

PHASE 4: UNITY GAIN METHOD (RECOMMENDED)

Why Unity Gain?

- Works for **ALL damping ratios** (any positive value)
- Most **stable** configuration
- Op-amp as simple buffer ($\mu = 1$)

Design Equations - Low-Pass **Option A:** Choose C_1 (typical: 0.01 to 1 microF)

$$C_2 = 4\zeta^2 C_1$$

$$R_1 = R_2 = R = \frac{1}{\zeta\omega_0 C_1}$$

Option B: Choose R_2 (typical: 1k to 100k ohms)

$$R_1 = 4\zeta^2 R_2$$

$$C_1 = C_2 = C = \frac{1}{\zeta\omega_0 R_2}$$

Design Equations - High-Pass Swap R and C positions from low-pass design.

With R_2 chosen:

$$R_1 = 4\zeta^2 R_2$$

$$C_1 = C_2 = C = \frac{1}{\zeta\omega_0 R_2}$$

PHASE 5: EQUAL ELEMENTS METHOD (BACKUP)

When to Use

- Only if damping ratio is between 0 and 1 (underdamped, Q > 0.5)
- Requires op-amp gain greater than 1

Design Equations - Low-Pass Set $R_1 = R_2 = R$ and $C_1 = C_2 = C$

$$\mu = 3 - 2\zeta$$

Choose C, then:

$$R = \frac{1}{\omega_0 C}$$

Op-amp gain circuit:

$$\mu = 1 + \frac{R_f}{R_g}$$

$$R_f = (\mu - 1) \cdot R_g$$

Validity check: $1 \leq \mu \leq 3$ if and only if damping ratio is between 0 and 1

Design Equations - High-Pass Same as low-pass: $R_1 = R_2 = R$, $C_1 = C_2 = C$, $\mu = 3 - 2\zeta$

PHASE 6: First-Order Sections

For Real Pole at $s = -\omega_C$ **Passive RC Low-Pass:**

$$H(s) = \frac{1}{1 + sRC}, \quad \omega_C = \frac{1}{RC}$$

Choose C, then: $R = \frac{1}{\omega_C C}$

Active Low-Pass (with gain K):

$$H(s) = \frac{K}{1 + sRC}$$

Use non-inverting op-amp: $K = 1 + \frac{R_f}{R_g}$

High-Pass (1st order):

$$H(s) = \frac{sRC}{1 + sRC}$$

Swap R and C positions from low-pass.

PHASE 7: Cascade Assembly

Section Ordering (CRITICAL) **Order by Q: LOW to HIGH**

Place highest damping ratio (lowest Q) first, lowest damping ratio (highest Q) last.

Why? Prevents early clipping from resonant peaks.

Overall Transfer Function

$$H_{\text{total}}(s) = H_1(s) \times H_2(s) \times \cdots \times H_n(s)$$

PHASE 8: Frequency & Impedance Scaling

Frequency Scaling (multiply by k__f)

$$C_{\text{new}} = \frac{C_{\text{old}}}{k_f}, \quad R_{\text{new}} = R_{\text{old}}$$

Effect: All frequencies multiplied by k_f

Impedance Scaling (multiply by k__z)

$$R_{\text{new}} = k_z \times R_{\text{old}}, \quad C_{\text{new}} = \frac{C_{\text{old}}}{k_z}$$

Effect: $H(s)$ and frequencies unchanged

Practical Component Ranges

Component	Practical Range	Preferred Range
Resistors	100 ohm - 10 Mohm	1k ohm - 1 Mohm
Capacitors	1pF - 1000 microF	100pF - 10 microF

SECTION 5: BODE PLOTS

Definition

Bode plots show **magnitude** (in dB) and **phase** versus frequency (log scale).

Axes: - X: $\log_{10}(\omega)$ - Y: $20\log_{10}(|H(j\omega)|)$ dB

Magnitude in dB

$$|H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)|$$

Conversions

dB to Linear:

$$|H| = 10^{(\mathrm{dB}/20)}$$

Linear to dB:

$$\mathrm{dB} = 20\log_{10}|H|$$

Frequency Response from H(s)

1. Substitute $s = j\omega$
2. Calculate magnitude: $|H(j\omega)| = \sqrt{[\mathrm{Re}(H)]^2 + [\mathrm{Im}(H)]^2}$
3. Calculate phase: $\angle H(j\omega) = \arctan[\mathrm{Im}(H)/\mathrm{Re}(H)]$

Sinusoidal Steady-State Response

Input: $v_{in}(t) = A\cos(\omega t + \theta)$

Output: $v_{out}(t) = A \cdot |H(j\omega)|\cos(\omega t + \theta + \angle H(j\omega))$

Key Bode Plot Features

Cutoff Frequency (minus 3 dB point)

$$|H(j\omega_c)| = \frac{|H_{max}|}{\sqrt{2}}$$

Slopes (Asymptotic)

- **1st order:** plus/minus 20 dB/decade per pole/zero
- **2nd order:** plus/minus 40 dB/decade per pole pair
- **nth order:** plus/minus (n times 20) dB/decade

Phase at Cutoff

- **1st order:** plus/minus 45 degrees at cutoff frequency
- **2nd order (Butterworth):** minus 90 degrees at natural frequency

First-Order Filter Forms

Low-Pass:

$$H(s) = \frac{K\omega_c}{s + \omega_c}$$

- DC gain = K - Slope: minus 20 dB/decade above cutoff frequency

High-Pass:

$$H(s) = \frac{Ks}{s + \omega_c}$$

- HF gain = K - Slope: plus 20 dB/decade below cutoff frequency

Second-Order Standard Form

$$H(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Peak magnitude (underdamped):

$$|H(j\omega_0)| = \frac{K}{2\zeta} \quad \text{for } \zeta < \frac{1}{\sqrt{2}}$$

CRITICAL DESIGN CHECKLIST

- Convert all dB to linear
- Round n UP to integer
- Extract all poles (left half-plane only)
- Calculate damping ratio and Q for each section
- Use **Unity Gain Method** (always works)
- Check damping ratio < 1 if using Equal Elements
- Order sections LOW Q to HIGH Q
- Scale components to practical values
- Verify overall $H(s)$ = product of sections

QUICK REFERENCE FORMULAS

Butterworth Order

$$n \geq \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{2\ln(\omega_{MIN}/\omega_C)} \quad \text{(LP)}$$

$$n \geq \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{2\ln(\omega_C/\omega_{MIN})} \quad \text{(HP)}$$

Pole Angles

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}$$

Damping and Q

$$\zeta = -\cos(\theta_k) = \frac{a}{2}, \quad Q = \frac{1}{2\zeta}$$

Unity Gain (Low-Pass)

$$C_2 = 4\zeta^2 C_1, \quad R_1 = R_2 = \frac{1}{\zeta\omega_0 C_1}$$

Equal Elements

$$\mu = 3 - 2\zeta, \quad R = \frac{1}{\omega_0 C}$$

Cover-Up Method

$$k_i = [(s - p_i)F(s)]_{s=p_i}$$

Frequency Response

$$|H(j\omega)| = \sqrt{[\text{Re}]^2 + [\text{Im}]^2}$$

$$\angle H(j\omega) = \arctan[\text{Im}/\text{Re}]$$

COMMON MISTAKES TO AVOID

- Rounding n down instead of up
- Using Equal Elements when damping ratio is 1 or greater
- Forgetting to convert dB to linear
- Wrong frequency ratio (LP vs HP)
- Cascading high-Q sections first
- Forgetting to zero initial conditions for H(s)
- Wrong impedance: C is $\frac{1}{sC}$ NOT sC

PROBLEM-SOLVING WORKFLOW

Given: Filter Specifications to Design Circuit

1. **Calculate n** using Butterworth formula (round UP)
2. **Find poles** using angle formula
3. **Extract damping ratio** for each 2nd-order section
4. **Choose Unity Gain Method** (always works)
5. **Design each section** using Unity Gain equations
6. **Order sections** by Q (low to high)
7. **Scale components** to practical values
8. **Cascade** with op-amp buffers

Given: H(s) to Find Time Response

1. **Factor denominator** to find all poles
2. **Apply PFE** to $V_{out}(s) = H(s) \cdot V_{in}(s)$
3. **Use Cover-Up Method** for residues
4. **Inverse Laplace** each term
5. **Add u(t)** to all terms

Given: Circuit to Find H(s)

1. **Transform to s-domain** (R, 1/(sC), sL)
2. **Zero all initial conditions**
3. **Apply circuit analysis** (nodal, mesh, divider)
4. **Form H(s)** = Output/Input
5. **Simplify** to rational function

END OF CHEATSHEET