Butterworth Filter Design Complete Cheatsheet

Quick Reference: Filter Types & Forms

Filter Type	H(s) Numerator	Passes	Attenuates
Low-Pass	Constant (K)	Low frequencies	High frequencies
High-Pass	s^n	High frequencies	Low frequencies
Band-Pass	s	Middle band	Low & High
Band-Stop	$s^2 + \omega_0^2$	Low & High	Middle band

PHASE 1: CALCULATE REQUIRED FILTER ORDER

Given Specifications (Typical)

- **Filter Type**: LP, HP, BP, or BS
- H_{MAX}: Maximum passband gain (linear or dB)
- H_{MIN}: Minimum stopband gain (linear or dB)
- ω_C **or** f_C: Cutoff/corner frequency
- ω_{MIN} **or** f_{MIN}: Frequency where H_{MIN} must be met

Step 1.1: Convert dB to Linear (if needed)

$$H_{\text{linear}} = 10^{H_{\text{dB}}/20}$$

Step 1.2: Calculate Minimum Order n

Low-Pass Butterworth:

$$n \geq \frac{1}{2} \cdot \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{\ln(\omega_{MIN}/\omega_C)}$$

High-Pass Butterworth:

$$n \geq \frac{1}{2} \cdot \frac{\ln[(H_{MAX}/H_{MIN})^2 - 1]}{\ln(\omega_C/\omega_{MIN})}$$

CRITICAL: Always round UP to nearest integer

Step 1.3: Determine Section Count

Always use as many second order circuits if possible. If an odd number n, then the most second order and then 1 first order.

General formula: - Odd n: $(n-1)/2$ second-order + 1 first-order - **Even n**: $n/2$ second-order

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PHASE 2: OBTAIN BUTTERWORTH POLES

Step 2.1: Calculate Pole Angles

For normalized Butterworth ($\omega_C = 1$):

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n$$

k is simply an index used to enumerate the poles of the normalized Butterworth polynomial. Nothing more. It has no electrical meaning like gain, component value, or stage count.

Step 2.2: Calculate Pole Locations

$$p_k = e^{j\theta_k} = \cos(\theta_k) + j\sin(\theta_k)$$

Only use left half-plane poles (Real part < 0)

PHASE 3: EXTRACT SECTION PARAMETERS

For 2nd-Order Section from Conjugate Pair

Step 3.1: Form quadratic

Poles $p = \alpha + j\beta$ and $p^* = \alpha - j\beta$:

$$(s - p)(s - p^*) = s^2 - 2\alpha s + (\alpha^2 + \beta^2)$$

For normalized: $|p| = 1$, so:

$$s^2 + as + 1$$

where $a = -2\text{Re}(p) = -2\cos(\theta_k)$

Step 3.2: Extract damping ratio ζ

Standard form: $s^2 + 2\zeta\omega_0 s + \omega_0^2$

For normalized ($\omega_0 = 1$): $s^2 + 2\zeta s + 1$

$$\zeta_k = \frac{a}{2} = -\cos(\theta_k)$$

Step 3.3: Calculate Q factor

$$Q_k = \frac{1}{2\zeta_k}$$

Quick Reference: and Q for Common Orders

n	Section	ζ	Q	a coefficient
2	1	0.707	0.707	1.414
3	1	0.500	1.000	1.000
4	1	0.383	1.307	0.765
4	2	0.924	0.541	1.848
5	1	0.309	1.618	0.618
5	2	0.809	0.618	1.618

PHASE 4: DESIGN METHOD SELECTION

Decision Rule: If $\zeta \geq 1$ (or $Q \leq 0.5$), MUST use Unity Gain. Otherwise, use Unity Gain (recommended) or Equal Elements (simpler).

Method Comparison

Criterion	Unity Gain	Equal Elements
Valid ζ range	All ($\zeta \geq 0$)	$0 < \zeta < 1$ only
Op-amp gain μ	1 (buffer)	$3 - 2\zeta$ (1 to 3)
Stability	Excellent	Good
Sensitivity	Low	Medium

PHASE 5A: UNITY GAIN METHOD (RECOMMENDED)

Circuit Description

Op-amp as voltage follower (buffer), non-inverting input connects to RC network, output feeds back directly to inverting input. **Components: 2 resistors (R_1, R_2) and 2 capacitors (C_1, C_2).**

Design Equations

Step 1: Set gain

$$\mu = 1$$

Step 2: Choose starting component

Option A – Choose C_1 (typical: 0.01 to 1 µF)

$$C_2 = (2\zeta)^2 \cdot C_1 = 4\zeta^2 C_1$$

$$R_1 = R_2 = R = \frac{1}{\zeta \omega_0 C_1}$$

Option B – Choose R_2 (typical: 1k to 100k)

$$R_1 = (2\zeta)^2 \cdot R_2 = 4\zeta^2 R_2$$

$$C_1 = C_2 = C = \frac{1}{\zeta \omega_0 R_2}$$

PHASE 5B: EQUAL ELEMENTS METHOD

Circuit Description

Op-amp with gain > 1 (non-inverting configuration). Feedback resistors R_f and R_g set gain. **Components: Equal resistors $R_1 = R_2 = R$ and equal capacitors $C_1 = C_2 = C$ in filter network.**

Design Equations

Step 1: Calculate required gain

$$\mu = 3 - 2\zeta$$

Validity check: $1 \leq \mu \leq 3 \Leftrightarrow 0 < \zeta < 1$. If outside this range, **cannot use this method**.

Step 2: Choose C (typical: 0.01 to 1 µF)

$$R = \frac{1}{\omega_0 C}$$

Step 3: Design op-amp gain circuit

$$\mu = 1 + \frac{R_f}{R_g}$$

$$R_f = (\mu - 1) \cdot R_g$$

Typical: Choose $R_g = 10$ kΩ, then: $R_f = (\mu - 1) \times 10$ kΩ

PHASE 6: FIRST-ORDER SECTIONS

For Real Pole at $s = -\omega_C$

Components: 1 resistor (R) and 1 capacitor (C).

Passive RC Low-Pass:

$H(s) = \frac{1}{1+sRC}, \quad \omega_C = \frac{1}{RC}$

Choose C, solve for R: $R = \frac{1}{\omega_C C}$

Active Low-Pass (with gain K):

$H(s) = \frac{K}{1+sRC}$

Use non-inverting op-amp: $K = 1 + \frac{R_f}{R_g}$

High-Pass (1st order):

$H(s) = \frac{sRC}{1+sRC}$

Swap R and C positions from low-pass.

PHASE 7: CASCADE ASSEMBLY

Step 7.1: Section Ordering (CRITICAL)

Order sections by Q: LOW → HIGH

Place highest ζ (lowest Q) first, lowest ζ (highest Q) last. **Why?** Prevents early clipping from resonant peaks.

Step 7.2: Buffering

Each Sallen-Key stage has high input impedance (op-amp input), so stages naturally don't load each other. No additional buffers needed between stages.

Step 7.3: Overall Transfer Function

$$H_{\text{total}}(s) = H_1(s) \times H_2(s) \times \cdots \times H_n(s)$$

OP-AMP FUNDAMENTALS

Why Use Op-Amps?

- High input impedance (prevents loading previous stages)
- Low output impedance (drives next stage without degradation)
- Enable gain without passive component losses
- Allow cascading of sections without interaction

Key Configurations

1. Voltage Follower (Buffer): $\mu = 1$ exactly. Use for Unity Gain Method. Output connects directly to inverting input. Most stable configuration.

2. Non-Inverting Amplifier: $\mu = 1 + \frac{R_f}{R_g}$. Use for Equal Elements Method when $1 < \mu < 3$. Maintains input signal polarity.

3. Inverting Amplifier: $\mu = -\frac{R_f}{R_{in}}$. NOT typically used in Sallen-Key filters. Can be used for summing (bandstop filters).

4. Summing Amplifier: $V_{out} = -R_f\left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right)$. Required for bandstop (notch) filters to combine LP and HP paths.

Op-Amp Placement

- **Within each Sallen-Key section**: Required (one per 2nd-order section)
- **After passive RC sections**: Required (use buffer)
- **Between active sections**: Usually NOT needed (already buffered)
- **At filter output**: Recommended if driving cables/loads

Critical Op-Amp Specifications

Gain-Bandwidth Product (GBW): $GBW \geq 100 \times f_c \times Q \times \mu$

Slew Rate (SR): $SR \geq 2\pi f_{max} V_{peak}$

Common choices: TL072/LF412 (audio), LM358/TL082 (general), TL071/OPA2134 (high-speed)

Power Supply & Decoupling

Use dual supply (±15V, ±12V, or ±5V typical). **Always add 0.1 µF ceramic capacitor** at each IC between V+ and GND, and between V− and GND. Prevents oscillation and noise.

PHASE 8: FREQUENCY & IMPEDANCE SCALING

When Component Values Are Impractical

Frequency Scaling ($\times k_f$):

$$C_{\text{new}} = \frac{C_{\text{old}}}{k_f}, \quad R_{\text{new}} = R_{\text{old}}$$

Impedance Scaling ($\times k_z$):

$$R_{\text{new}} = k_z \times R_{\text{old}}, \quad C_{\text{new}} = \frac{C_{\text{old}}}{k_z}$$

Combined Scaling:

$$R_{\text{final}} = k_z \times R_{\text{normalized}}, \quad C_{\text{final}} = \frac{C_{\text{normalized}}}{k_f \times k_z}$$

Practical Component Ranges

Component	Practical Range	Preferred Range
Resistors	100Ω - 10MΩ	1kΩ - 1MΩ
Capacitors	1pF - 1000µF	100pF - 10µF

DERIVING MISSING SPECIFICATIONS

If ω_C Given, But ω_{MIN} Not Given

Typical assumption: Transition band = 1 decade

$$\omega_{MIN} = 10 \times \omega_C \text{ (low-pass)}, \quad \omega_{MIN} = 0.1 \times \omega_C \text{ (high-pass)}$$

If H_{MIN} Not Given

Use standard attenuation: -40 dB (0.01) for audio, -60 dB (0.001) for instrumentation

If ζ or Q Given for Section

$$Q = \frac{1}{2\zeta}, \quad \zeta = \frac{1}{2Q}$$

If ω_0 and ζ Given, Find Pole Locations

Complex conjugate poles: $p = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$

Real part: $\sigma = -\zeta\omega_0$, Imaginary part: $\omega_d = \omega_0\sqrt{1-\zeta^2}$ (damped frequency)

Bandwidth Relationships

For 2nd-order: $BW = 2\zeta\omega_0 = \frac{\omega_0}{Q}$

Half-power frequencies:

$$\omega_1 = \omega_0\left(-\zeta + \sqrt{\zeta^2+1}\right), \quad \omega_2 = \omega_0\left(\zeta + \sqrt{\zeta^2+1}\right)$$

BANDPASS AND BANDSTOP FILTER DESIGN

Bandpass Filter (BP)

Concept: Pass middle band, attenuate low and high frequencies.

Design: Cascade lowpass and highpass: $H_{BP}(s) = H_{LP}(s) \times H_{HP}(s)$

Steps: 1. Define: Lower cutoff ω_1 (HP), upper cutoff ω_2 (LP), center $\omega_0 = \sqrt{\omega_1\omega_2}$, bandwidth $BW = \omega_2 - \omega_1$ 2. Design HP section with $\omega_C = \omega_1$ 3. Design LP section with $\omega_C = \omega_2$ 4. Cascade: HP $\rightarrow$ LP (or LP $\rightarrow$ HP)

Key: Total order = (LP order) + (HP order). Use series connection.

Bandstop (Notch) Filter (BS)

Concept: Attenuate middle band, pass low and high frequencies.

Design: Parallel sum: $H_{BS}(s) = H_{LP}(s) + H_{HP}(s)$

Steps: 1. Define: Lower passband edge ω_1 (LP cutoff), upper passband edge ω_2 (HP cutoff), notch center $\omega_0 = \sqrt{\omega_1\omega_2}$ 2. Design LP section with $\omega_C = \omega_1$ (passes below notch) 3. Design HP section with $\omega_C = \omega_2$ (passes above notch) 4. Use **summing amplifier** to combine outputs

Key: Unlike BP (cascade), BS requires **parallel combination** via summing amplifier. Both sections must have same DC/HF gain for flat response.

Comparison: BP vs BS Implementation

Aspect	Bandpass (BP)	Bandstop (BS)
Combination	Cascade (multiply)	Parallel (sum)
Circuit	Series connection	Summing amplifier
Passes	Middle band	Low & High
Attenuates	Low & High	Middle band

QUICK REFERENCE FORMULAS

Order Calculation

LP: $n \geq \frac{\ln[(H_{MAX}/H_{MIN})^2-1]}{2\ln(\omega_{MIN}/\omega_C)}$

HP: $n \geq \frac{\ln[(H_{MAX}/H_{MIN})^2-1]}{2\ln(\omega_C/\omega_{MIN})}$

Pole Angles

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}$$

Damping & Q

$$\zeta = -\cos(\theta_k) = \frac{a}{2}, \quad Q = \frac{1}{2\zeta}$$

Unity Gain

$$C_2 = 4\zeta^2 C_1, \quad R_1 = R_2 = \frac{1}{\zeta\omega_0 C_1}$$

Equal Elements

$$\mu = 3 - 2\zeta, \quad R = \frac{1}{\omega_0 C}$$

CRITICAL CHECKLIST

- ☐ Converted all dB to linear
- ☐ Rounded n UP to integer
- ☐ Extracted all poles (left half-plane only)
- ☐ Grouped poles into sections
- ☐ Calculated ζ and Q for each section
- ☐ Checked $\zeta < 1$ before using Equal Elements
- ☐ Verified $1 \leq \mu \leq 3$ for Equal Elements
- ☐ Ordered sections LOW Q $\rightarrow$ HIGH Q
- ☐ Scaled components to practical values
- ☐ Verified $H_{\text{total}}(s)$ = product of sections

COMMON MISTAKES TO AVOID

Rounding n down instead of up

Using Equal Elements when $\zeta \geq 1$

Forgetting to convert dB to linear

Wrong frequency ratio (LP vs HP)

Cascading high-Q sections first

Using right half-plane poles

Forgetting to denormalize from $\omega_C = 1$

COMPLETE WORKED EXAMPLE

Given Requirements

Low-pass filter: $H_{MAX} = 0$ dB (1.0), $H_{MIN} = -40$ dB (0.01), $f_C = 1$ kHz, $f_{MIN} = 5$ kHz

Solution

Calculate order: $n \geq \frac{\ln[(1/0.01)^2 - 1]}{2\ln(5)} = 2.86 \rightarrow$ **Round up: n = 3**

Sections: $n = 3$ (odd) $\rightarrow$ 1 first-order + 1 second-order

Poles for n = 3: $\theta_1 = 120°$, $\theta_2 = 180°$, $\theta_3 = 240°$

Pole locations: $p_1 = -0.5 + j0.866$, $p_2 = -1$ (real), $p_3 = -0.5 - j0.866$

2nd-order section (p_1, p_3): $s^2 + s + 1 \rightarrow \zeta = 0.5$, $Q = 1$

1st-order section (p_2): $s + 1$

Cascade: 1st-order (Q=0.5) $\rightarrow$ 2nd-order (Q=1)