Final Quiz - Comprehensive Worked Examples

This document contains detailed worked examples covering all major problem types that may appear on the quiz, based on homework assignments, class slides, and typical exam problems.

SECTION 1: TRANSFER FUNCTIONS & CIRCUIT ANALYSIS

EXAMPLE 1.1: Finding H(s) from Op-Amp Circuit (HW07-P1)

Problem: For the inverting op-amp circuit with R1 = 1 kohm, Rf = 2 kohm, C = 0.5 microF in feedback, find H(s) = V2(s)/V1(s).

Solution:

Step 1: Identify impedances - Input impedance: $Z_{in} = R_1 = 1000$ ohm - Feedback impedance (parallel RC):

$$Z_f = R_f \parallel \frac{1}{sC} = \frac{R_f \cdot \frac{1}{sC}}{R_f + \frac{1}{sC}} = \frac{R_f}{1 + sR_f C}$$

Step 2: Apply inverting amplifier formula

$$H(s) = -\frac{Z_f}{Z_{in}} = -\frac{R_f/(1 + sR_f C)}{R_1}$$

Step 3: Substitute values

$$H(s) = -\frac{2000}{1000(1 + s \cdot 2000 \cdot 0.5 \times 10^{-6})}$$

$$H(s) = -\frac{2}{1 + 0.001s} = -\frac{2000}{s + 1000}$$

Answer: $H(s) = -\frac{2000}{s+1000}$ (first-order low-pass with gain 2 and cutoff at 1000 rad/s)

EXAMPLE 1.2: Lead Network Analysis (HW08-P1)

Problem: For the RC network with two series RC branches in voltage divider configuration, find: (a) Input impedance Z(s) = V1(s)/I1(s) (b) Transfer function H(s) = V2(s)/V1(s) in form $H(s) = K\frac{s+\beta}{s+\alpha}$ (c) Design for H(s) with beta = 1000, alpha = 100

Solution:

Part (a): Input impedance (series combination)

$$Z(s) = R_1 + \frac{1}{sC_1} + R_2 \parallel \frac{1}{sC_2}$$

For parallel branch:

$$R_2 \parallel \frac{1}{sC_2} = \frac{R_2}{1 + sR_2 C_2}$$

Part (b): Voltage divider for H(s)

$$H(s) = \frac{Z_{output}}{Z_{total}} = \frac{R_2/(1 + sR_2 C_2)}{R_1 + 1/(sC_1) + R_2/(1 + sR_2 C_2)}$$

After algebraic manipulation:

$$H(s) = K\frac{s + 1/(R_2 C_2)}{s + 1/(R_{eq} C_{eq})}$$

where K depends on component values.

Part (c): Design with C1 = C2 = C

Given: beta = 1000, alpha = 100

From zero: $\beta = \frac{1}{R_2 C} = 1000 \implies R_2 = \frac{1}{1000C}$

From pole: $\alpha = \frac{1}{R_{eq} C} = 100 \implies R_{eq} = \frac{1}{100C}$

Choose C = 1 microF: - $R_2 = 1$ kohm - Need to solve for R1 based on pole constraint

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EXAMPLE 1.3: Series RLC Bandpass (HW07-P3)

Problem: Given R = 20 ohm, L = 1 H, C = 1 microF, find H(s) = V2(s)/V1(s) where V2 is across R.

Solution:

Step 1: Total impedance

$$Z_{total} = R + sL + \frac{1}{sC}$$

Step 2: Voltage divider (output across R)

$$H(s) = \frac{R}{R + sL + \frac{1}{sC}} = \frac{R \cdot sC}{sRLC + s^2 LC + 1}$$

Step 3: Substitute values

$$H(s) = \frac{20s \cdot 10^{-6}}{s \cdot 20 \cdot 1 \cdot 10^{-6} + s^2 \cdot 1 \cdot 10^{-6} + 1}$$

Multiply numerator and denominator by 10^6:

$$H(s) = \frac{20s}{s^2 + 20s + 10^6}$$

Step 4: Find resonant frequency

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{1 \times 10^{-6}}} = 1000 \text{ rad/s}$$

Step 5: At resonance ($\omega = 1000$ rad/s)

$$H(j1000) = \frac{20 \cdot j1000}{(j1000)^2 + 20(j1000) + 10^6} = \frac{j20000}{j20000} = 1$$

Maximum output at resonance!

SECTION 2: PARTIAL FRACTION EXPANSION

EXAMPLE 2.1: Basic PFE with Two Real Poles

Problem: Find inverse Laplace transform of $F(s) = \frac{100000}{(s+100)(s+1000)}$

Solution:

Step 1: Setup PFE

$$F(s) = \frac{k_1}{s + 100} + \frac{k_2}{s + 1000}$$

Step 2: Cover-up method for k1 (pole at s = -100)

$$k_1 = \left[(s + 100) \cdot \frac{100000}{(s + 100)(s + 1000)}\right]_{s=-100}$$

$$k_1 = \frac{100000}{-100 + 1000} = \frac{100000}{900} = 111.11$$

Step 3: Cover-up method for k2 (pole at s = -1000)

$$k_2 = \left[(s + 1000) \cdot \frac{100000}{(s + 100)(s + 1000)}\right]_{s=-1000}$$

$$k_2 = \frac{100000}{-1000 + 100} = \frac{100000}{-900} = -111.11$$

Step 4: Inverse Laplace transform

$$f(t) = 111.11e^{-100t} - 111.11e^{-1000t} \text{ for } t \geq 0$$

Answer: $f(t) = 111.11(e^{-100t} - e^{-1000t})u(t)$

EXAMPLE 2.2: PFE with s Term in Numerator (HW08-P3b)

Problem: Find partial fraction expansion of $H(s) = \frac{10^4 s}{(s+10)(s+1000)}$ in form $K\left(\frac{\alpha}{s+\alpha} - \frac{\beta}{s+\beta}\right)$

Solution:

Step 1: Factor out constant K

$$H(s) = 10^4 \cdot \frac{s}{(s+10)(s+1000)}$$

Step 2: Standard PFE

$$\frac{s}{(s+10)(s+1000)} = \frac{k_1}{s+10} + \frac{k_2}{s+1000}$$

Step 3: Cover-up for k1

$$k_1 = \left[\frac{s}{s+1000}\right]_{s=-10} = \frac{-10}{-10+1000} = \frac{-10}{990} = -0.0101$$

Step 4: Cover-up for k2

$$k_2 = \left[\frac{s}{s+10}\right]_{s=-1000} = \frac{-1000}{-1000+10} = \frac{-1000}{-990} = 1.0101$$

Step 5: Rewrite in requested form

$$H(s) = 10^4(1.0101 \cdot \frac{1}{s+1000} - 0.0101 \cdot \frac{1}{s+10})$$

$$H(s) = 10101\left(\frac{1000}{s+1000} - \frac{10}{s+10}\right)$$

$$H(s) = 10.1\left(\frac{1000}{s+1000} - \frac{10}{s+10}\right)$$

This shows: K = 10.1, alpha = 1000, beta = 10

EXAMPLE 2.3: Repeated Poles (HW06-P4a)

Problem: Given step response $v_2(t) = 24(1 - e^{-50t} - 50te^{-50t})u(t)$, find H(s).

Solution:

Step 1: Take Laplace transform

$$V_2(s) = 24\left[\frac{1}{s} - \frac{1}{s+50} - \frac{50}{(s+50)^2}\right]$$

Step 2: Common denominator

$$V_2(s) = 24 \cdot \frac{(s+50)^2 - s(s+50) - 50s}{s(s+50)^2}$$

$$= 24 \cdot \frac{s^2 + 100s + 2500 - s^2 - 50s - 50s}{s(s+50)^2}$$

$$= 24 \cdot \frac{2500}{s(s+50)^2} = \frac{60000}{s(s+50)^2}$$

Step 3: Find H(s) (input is step, so V1(s) = 1/s)

$$H(s) = \frac{V_2(s)}{V_1(s)} = V_2(s) \cdot s = \frac{60000}{(s+50)^2}$$

This has a repeated pole at s = -50

SECTION 3: SINUSOIDAL STEADY-STATE & FREQUENCY RESPONSE

EXAMPLE 3.1: Finding Output for Sinusoidal Input (HW07-P1)

Problem: Given $H(s) = -\frac{2000}{s+1000}$, find v2(t) for various inputs.

General Method: 1. Substitute $s = j\omega$ 2. Find $|H(j\omega)|$ and $\angle H(j\omega)$ 3. Output: $v_2(t) = A|H(j\omega)|\cos(\omega t + \theta + \angle H(j\omega))$

Case (a): v1(t) = 5cos(10t) V (omega = 10 rad/s, much less than cutoff)

$$H(j10) = -\frac{2000}{j10 + 1000} = -\frac{2000}{1000 + j10}$$

Magnitude:

$$|H(j10)| = \frac{2000}{\sqrt{1000^2 + 10^2}} = \frac{2000}{\sqrt{1000100}} \approx 2.0$$

Phase (negative sign contributes 180 degrees):

$$\angle H(j10) = 180° - \arctan(10/1000) = 180° - 0.57° = 179.4°$$

Answer: $v_2(t) = 10\cos(10t + 179.4°)$ V

Case (b): v1(t) = 5cos(1000t) V (omega = cutoff frequency)

$$H(j1000) = -\frac{2000}{j1000 + 1000} = -\frac{2000}{1000(1 + j)}$$

Magnitude:

$$|H(j1000)| = \frac{2000}{1000\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \approx 1.414$$

Phase:

$$\angle H(j1000) = 180° - 45° = 135°$$

Answer: $v_2(t) = 7.07\cos(1000t + 135°)$ V (This is -3 dB point!)

Case (c): v1(t) = 5cos(100000t) V (omega much greater than cutoff)

$$H(j10^5) = -\frac{2000}{j10^5 + 1000} \approx -\frac{2000}{j10^5}$$

Magnitude:

$$|H(j10^5)| = \frac{2000}{10^5} = 0.02$$

Phase:

$$\angle H(j10^5) = 180° - 90° = 90°$$

Answer: $v_2(t) = 0.1\cos(10^5 t + 90°)$ V

EXAMPLE 3.2: Extracting H(s) from Bode Plot (HW07-P4 & P5)

Problem Type 1: Low-pass $H(s) = K\frac{\alpha}{s+\alpha}$

Given from Bode plot: - DC gain = 6 dB - Phase = -45 degrees at omega = 1000 rad/s

Solution:

Phase is -45 degrees at corner frequency, so: **alpha = 1000 rad/s**

DC gain: $20\log_{10} K = 6$ dB

$$K = 10^{6/20} = 10^{0.3} = 2$$

Answer: $H(s) = \frac{2\cdot 1000}{s+1000} = \frac{2000}{s+1000}$

Problem Type 2: High-pass $H(s) = K\frac{s}{s+\alpha}$

Given from Bode plot: - Phase = 45 degrees at omega = 100 rad/s - High-frequency gain = 0 dB

Solution:

For high-pass, phase = 45 degrees at corner, so: **alpha = 100 rad/s**

High-frequency gain (as omega approaches infinity):

$$\lim_{\omega\to\infty} |H(j\omega)| = K$$

From plot: 0 dB means K = 1

Answer: $H(s) = \frac{s}{s+100}$

SECTION 4: CIRCUIT DESIGN FROM H(s)

EXAMPLE 4.1: First-Order Circuit Design (HW06-P1a)

Problem: Design circuit for $H(s) = \frac{200000}{s+100000}$

Solution:

Step 1: Identify characteristics - First-order low-pass - DC gain K = 2 - Cutoff omega_c = 100000 rad/s = 100 krad/s

Step 2: Choose topology - Non-inverting with RC low-pass

Basic form: $H(s) = \mu \cdot \frac{1/RC}{s+1/RC}$

Step 3: Design RC section for cutoff Choose C = 0.1 microF:

$$\omega_c = \frac{1}{RC} = 100000 \implies R = \frac{1}{100000 \times 0.1 \times 10^{-6}} = 100 \text{ ohm}$$

Step 4: Design op-amp gain Need mu = 2:

$$\mu = 1 + \frac{R_f}{R_g} = 2 \implies R_f = R_g$$

Choose Rg = 10 kohm, then Rf = 10 kohm

Final Design: - Input RC low-pass: R = 100 ohm, C = 0.1 microF - Non-inverting op-amp: Rf = Rg = 10 kohm

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EXAMPLE 4.2: Parallel Summing Design (HW08-P3c)

Problem: Implement $H(s) = 10.1\left(\frac{1000}{s+1000} - \frac{10}{s+10}\right)$ using parallel summing.

Solution:

Step 1: Recognize as difference of two first-order sections

$$H(s) = K_1 \cdot \frac{\alpha_1}{s+\alpha_1} - K_2 \cdot \frac{\alpha_2}{s+\alpha_2}$$

where K1 = 10100, alpha1 = 1000, K2 = 101, alpha2 = 10

Step 2: Design first branch (alpha1 = 1000)

RC section: Choose C11 = 0.1 microF

$$R_{11} = \frac{1}{1000 \times 0.1 \times 10^{-6}} = 10 \text{ kohm}$$

Gain: Need 10100 Use inverting: $\frac{R_f}{R_{in}} = 10100$ (impractical - use cascade instead)

Step 3: Design second branch (alpha2 = 10)

RC section: Choose C21 = 1 microF

$$R_{21} = \frac{1}{10 \times 1 \times 10^{-6}} = 100 \text{ kohm}$$

Step 4: Use summing amplifier to combine with subtraction

The summing stage inverts, so design first stages as inverting to get correct signs.

EXAMPLE 4.3: Integrator and Differentiator (HW08-P4)

Problem: Find H(s) for op-amp integrator and explain name.

Solution:

Circuit: Inverting op-amp with R in input, C in feedback

$$H(s) = -\frac{Z_f}{Z_{in}} = -\frac{1/(sC)}{R} = -\frac{1}{sRC}$$

Time domain relationship:

$$V_{out}(s) = -\frac{1}{sRC}V_{in}(s)$$

$$sV_{out}(s) = -\frac{1}{RC}V_{in}(s)$$

$$\frac{dv_{out}}{dt} = -\frac{1}{RC}v_{in}(t)$$

$$v_{out}(t) = -\frac{1}{RC}\int v_{in}(t)dt$$

This is an integrator!

Differentiator: Swap R and C

$$H(s) = -\frac{R}{1/(sC)} = -sRC$$

This gives: $v_{out}(t) = -RC\frac{dv_{in}}{dt}$

SECTION 5: BUTTERWORTH FILTER DESIGN

EXAMPLE 5.1: Complete 4th-Order Low-Pass (Class 27)

Problem: Design Butterworth low-pass filter: - Hmax = 0 dB (gain = 1.0) - Hmin = -40 dB (gain = 0.01) - fc = 2 kHz - fmin = 8 kHz

Solution:

PHASE 1: Calculate Required Order

Step 1: Convert to linear and rad/s - Hmax = 1.0 (already linear) - Hmin = 0.01 (already linear) - omega_C = 2pi × 2000 = 12566 rad/s - omega_MIN = 2pi × 8000 = 50265 rad/s

Step 2: Apply Butterworth order formula (low-pass)

$$n \geq \frac{\ln[(H_{max}/H_{min})^2 - 1]}{2\ln(\omega_{min}/\omega_C)}$$

$$n \geq \frac{\ln[(1/0.01)^2 - 1]}{2\ln(50265/12566)} = \frac{\ln(10000-1)}{2\ln(4)}$$

$$n \geq \frac{\ln(9999)}{2\ln(4)} = \frac{9.210}{2.773} = 3.32$$

Round UP: n = 4

Step 3: Section breakdown n = 4 (even) → 2 second-order sections, 0 first-order sections

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PHASE 2: Calculate Butterworth Poles

Step 4: Pole angles for n = 4

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n} = \frac{\pi}{2} + \frac{(2k-1)\pi}{8}$$

For k = 1, 2, 3, 4: - theta_1 = pi/2 + pi/8 = 5pi/8 = 112.5 degrees - theta_2 = pi/2 + 3pi/8 = 7pi/8 = 157.5 degrees - theta_3 = pi/2 + 5pi/8 = 9pi/8 = 202.5 degrees - theta_4 = pi/2 + 7pi/8 = 11pi/8 = 247.5 degrees

Step 5: Pole locations (normalized, |p| = 1) - p1 = cos(112.5°) + j sin(112.5°) = -0.383 + j0.924 - p2 = cos(157.5°) + j sin(157.5°) = -0.924 + j0.383 - p3 = cos(202.5°) + j sin(202.5°) = -0.924 - j0.383 - p4 = cos(247.5°) + j sin(247.5°) = -0.383 - j0.924

Use only left-half-plane poles (all have negative real parts)

PHASE 3: Extract Section Parameters

Section 1: Pair p1 and p4 (conjugates)

$$(s-p_1)(s-p_4^*) = s^2 - (p_1+p_4)s + p_1 \cdot p_4$$

$$= s^2 - 2(-0.383)s + (0.383^2 + 0.924^2)$$

$$= s^2 + 0.765s + 1$$

Extract parameters: - Coefficient a = 0.765 - Damping ratio: zeta_1 = a/2 = 0.383 - Quality factor: Q_1 = 1/(2×0.383) = 1.307

Section 2: Pair p2 and p3 (conjugates)

$$s^2 + 1.848s + 1$$

- Coefficient a = 1.848
- Damping ratio: zeta_2 = 0.924
- Quality factor: Q_2 = 0.541

PHASE 4: Design Using Unity Gain Method

Section 1: zeta = 0.383, omega_0 = 12566 rad/s

Choose C1 = 0.1 microF:

$$C_2 = 4\zeta^2 C_1 = 4(0.383)^2(0.1 \text{ microF}) = 0.0586 \text{ microF}$$

Use standard value: **C2 = 0.056 microF** or **0.068 microF**

$$R_1 = R_2 = \frac{1}{\zeta\omega_0 C_1} = \frac{1}{0.383 \times 12566 \times 0.1 \times 10^{-6}}$$

$$= 2078 \text{ ohm} \approx 2.2 \text{ kohm}$$

Section 2: zeta = 0.924, omega_0 = 12566 rad/s

Choose C1 = 0.1 microF:

$$C_2 = 4(0.924)^2(0.1) = 0.341 \text{ microF}$$

Use standard value: **C2 = 0.33 microF**

$$R_1 = R_2 = \frac{1}{0.924 \times 12566 \times 0.1 \times 10^{-6}} = 861 \text{ ohm}$$

Use standard value: **R1 = R2 = 820 ohm**

PHASE 5: Cascade Assembly

Order by Q (LOW to HIGH):

1. **First stage:** Section 2 (Q = 0.541, lower)
    - R1 = R2 = 820 ohm
    - C1 = 0.1 microF, C2 = 0.33 microF
    - Op-amp as buffer (mu = 1)
2. **Second stage:** Section 1 (Q = 1.307, higher)
    - R1 = R2 = 2.2 kohm
    - C1 = 0.1 microF, C2 = 0.056 microF
    - Op-amp as buffer (mu = 1)

Total H(s):

$$H_{total}(s) = H_1(s) \times H_2(s)$$

EXAMPLE 5.2: 3rd-Order Butterworth (Odd Order)

Problem: Design 3rd-order Butterworth low-pass with fc = 1 kHz.

Solution:

Step 1: Section count n = 3 (odd) → 1 first-order + 1 second-order section

Step 2: Pole angles

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{6}$$

For k = 1, 2, 3: - theta_1 = pi/2 + pi/6 = 2pi/3 = 120° → p1 = -0.5 + j0.866 - theta_2 = pi/2 + 3pi/6 = pi = 180° → p2 = -1 (REAL pole) - theta_3 = pi/2 + 5pi/6 = 4pi/3 = 240° → p3 = -0.5 - j0.866

Step 3: Second-order section (p1, p3)

$$s^2 + s + 1$$

- zeta = 0.5
- omega_0 = 2pi × 1000 = 6283 rad/s

Unity Gain Design:

Choose C1 = 0.1 microF:

$$C_2 = 4(0.5)^2(0.1) = 0.1 \text{ microF}$$

$$R_1 = R_2 = \frac{1}{0.5 \times 6283 \times 0.1 \times 10^{-6}} = 3183 \text{ ohm}$$

Use: **R1 = R2 = 3.3 kohm**

Step 4: First-order section (p2 at s = -omega_c)

$$H_{1st}(s) = \frac{1}{1 + s/\omega_c}$$

Passive RC: Choose C = 0.1 microF:

$$R = \frac{1}{6283 \times 0.1 \times 10^{-6}} = 1592 \text{ ohm}$$

Use: **R = 1.6 kohm**

Step 5: Cascade order **First-order first, then second-order** (buffer between stages)

EXAMPLE 5.3: High-Pass 2nd-Order Butterworth (Class 28)

Problem: Design 2nd-order high-pass: - fc = 20 kHz - High-frequency gain = 0 dB - Corner gain = -3 dB (Butterworth characteristic)

Solution:

Step 1: Recognize standard 2nd-order Butterworth - n = 2 → zeta = 0.707 - omega_0 = 2pi × 20000 = 125664 rad/s

Step 2: Unity Gain Method for HIGH-PASS

Key difference: Swap R and C positions from low-pass

Choose R2 = 10 kohm:

$$R_1 = 4\zeta^2 R_2 = 4(0.707)^2(10 \text{ kohm}) = 20 \text{ kohm}$$

$$C_1 = C_2 = C = \frac{1}{\zeta\omega_0 R_2}$$

$$= \frac{1}{0.707 \times 125664 \times 10000} = 1.13 \times 10^{-9} \text{ F}$$

Use: **C1 = C2 = 1.2 nF** or **1.0 nF**

Step 3: Circuit topology Sallen-Key high-pass: - Capacitors in series with input path - Resistors to ground - Op-amp as buffer

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EXAMPLE 5.4: High-Pass Order Calculation (Class 28 Example)

Problem: Design high-pass Butterworth: - Hmax = 20 dB (gain = 10) - Hmin = -40 dB (gain = 0.01) - fc = 10 kHz - fmin = 1 kHz (note: fmin < fc for high-pass!)

Solution:

Step 1: Apply HIGH-PASS order formula (note reversed frequencies!)

$$n \geq \frac{\ln[(H_{max}/H_{min})^2 - 1]}{2\ln(\omega_C/\omega_{min})}$$

Notice: omega_C in numerator for high-pass (opposite of low-pass)

$$n \geq \frac{\ln[(10/0.01)^2 - 1]}{2\ln(62832/6283)} = \frac{\ln(999999)}{2\ln(10)}$$

$$= \frac{13.816}{4.605} = 3.0$$

Exactly n = 3 (no need to round up)

Step 2: Section breakdown n = 3 → 1 first-order + 1 second-order

Then proceed with pole calculations and design as before...

SECTION 6: EQUAL ELEMENTS METHOD

EXAMPLE 6.1: When to Use Equal Elements vs Unity Gain

Problem: Design 2nd-order section with: - omega_0 = 1000 rad/s - zeta = 0.5

Can we use equal elements method?

Solution:

Equal Elements Constraint: For equal elements method (R1 = R2 = R, C1 = C2 = C):

$$\mu = 3 - 2\zeta$$

For zeta = 0.5:

$$\mu = 3 - 2(0.5) = 2$$

This is valid! (mu >= 1 and zeta < 1)

Equal Elements Design:

Choose C = 0.1 microF:

$$R = \frac{1}{\zeta\omega_0 C} = \frac{1}{0.5 \times 1000 \times 0.1 \times 10^{-6}} = 20 \text{ kohm}$$

Gain stage: Need mu = 2

$$\mu = 1 + \frac{R_f}{R_g} = 2 \implies R_f = R_g$$

Choose Rg = 10 kohm, then Rf = 10 kohm

Unity Gain Alternative:

Choose C1 = 0.1 microF:

$$C_2 = 4\zeta^2 C_1 = 4(0.5)^2(0.1) = 0.1 \text{ microF}$$

$$R_1 = R_2 = \frac{1}{\zeta\omega_0 C_1} = 20 \text{ kohm}$$

Both methods work! Equal elements is simpler here.

EXAMPLE 6.2: When Unity Gain is Required

Problem: Design 2nd-order section with: - omega_0 = 1000 rad/s - zeta = 1.5 (overdamped)

Solution:

Check Equal Elements:

$$\mu = 3 - 2(1.5) = 0$$

This fails! (mu must be >= 1)

Must use Unity Gain Method:

Choose C1 = 0.1 microF:

$$C_2 = 4\zeta^2 C_1 = 4(1.5)^2(0.1) = 0.9 \text{ microF}$$

$$R_1 = R_2 = \frac{1}{\zeta\omega_0 C_1} = \frac{1}{1.5 \times 1000 \times 0.1 \times 10^{-6}}$$

$$= 6667 \text{ ohm} \approx 6.8 \text{ kohm}$$

Key Rule: - Equal elements: ONLY for 0 < zeta < 1 (underdamped) - Unity gain: Works for ALL zeta values (including overdamped)

SECTION 7: STATE-VARIABLE FILTER (HW08-P5)

EXAMPLE 7.1: Understanding State-Variable Filter

Problem: For the state-variable filter in HW08-P5, derive H(s) = V2(s)/V1(s).

Solution:

Step 1: Recognize cascaded integrators

The two right-side op-amps are integrators:

$$V_4(s) = -\frac{1}{sRC}V_2(s)$$

$$V_3(s) = -\frac{1}{sRC}V_4(s) = \frac{1}{(sRC)^2}V_2(s) = s^2(RC)^2V_2(s)$$

Step 2: Analyze leftmost op-amp

Non-inverting input (V+): Voltage divider from V1 and V2 feedback

$$V_+(s) = \frac{-sRCR_1V_2(s) + R_2V_1(s)}{R_1 + R_2}$$

Inverting input (V-): Voltage divider from V3 and ground

$$V_-(s) = \frac{s^2(RC)^2R_4V_2(s)}{R_3 + R_4}$$

Step 3: Apply ideal op-amp property (V+ = V-)

$$\frac{-sRCR_1V_2(s) + R_2V_1(s)}{R_1 + R_2} = \frac{s^2(RC)^2R_4V_2(s)}{R_3 + R_4}$$

Step 4: With R4 = R3, solve for H(s)

After algebraic manipulation:

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{R_2(R_3+R_4)}{(R_1+R_2)R_3}}{s^2(RC)^2\frac{R_4}{R_3} + sRC\frac{R_1(R_3+R_4)}{(R_1+R_2)R_3} + 1}$$

With R4 = R3:

$$H(s) = \frac{K}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\left(\frac{s}{\omega_0}\right) + 1}$$

where: - $\omega_0 = \frac{1}{RC}$ - $2\zeta = 2\frac{R_1}{R_1+R_2}$ - $K = \frac{R_2(2R_3)}{(R_1+R_2)R_3} = \frac{2R_2}{R_1+R_2}$

Key Advantage: Can tune omega_0 and zeta independently!

SECTION 8: COMMON MISTAKES TO AVOID

MISTAKE 1: Wrong Butterworth Order Formula

WRONG (Low-Pass):

$$n \geq \frac{\ln[(H_{max}/H_{min})^2 - 1]}{2\ln(\omega_{min}/\omega_C)}$$

with omega_MIN < omega_C (INCORRECT!)

CORRECT (Low-Pass):

$$n \geq \frac{\ln[(H_{max}/H_{min})^2 - 1]}{2\ln(\omega_{min}/\omega_C)}$$

with omega_MIN > omega_C (frequency beyond passband)

CORRECT (High-Pass):

$$n \geq \frac{\ln[(H_{max}/H_{min})^2 - 1]}{2\ln(\omega_C/\omega_{min})}$$

with omega_MIN < omega_C (frequency below passband)

MISTAKE 2: Forgetting to Scale Poles

Problem: 4th-order Butterworth with fc = 2 kHz

WRONG: Use normalized poles directly in H(s)

CORRECT: Scale ALL poles by omega_c - Normalized: p1 = -0.383 + j0.924 - Scaled: p1 = omega_c(-0.383 + j0.924) = 12566(-0.383 + j0.924)

MISTAKE 3: Wrong Cascade Order

WRONG: Place high-Q section first

CORRECT: Always cascade LOW Q to HIGH Q - Prevents overload of high-Q section - Better overall performance

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MISTAKE 4: Confusing zeta and Q

Relationship:

$$Q = \frac{1}{2\zeta}$$

Example: - If zeta = 0.5, then Q = 1.0 (NOT 0.5!) - If Q = 1.307, then zeta = 0.383 (NOT 1.307!)

MISTAKE 5: Using Equal Elements When zeta >= 1

Example: zeta = 1.2 (overdamped)

WRONG: Try equal elements method

$$\mu = 3 - 2(1.2) = 0.6$$

(less than 1, INVALID!)

CORRECT: Must use unity gain method

Rule: Equal elements ONLY for 0 < zeta < 1

SECTION 9: QUICK FORMULAS REFERENCE

First-Order Sections

Low-Pass:

$$H(s) = K\frac{\alpha}{s+\alpha}$$

- DC gain: K - Cutoff: alpha rad/s - Slope: -20 dB/decade

High-Pass:

$$H(s) = K\frac{s}{s+\alpha}$$

- High-frequency gain: K - Cutoff: alpha rad/s - Slope: +20 dB/decade

Second-Order Sections

Standard Form:

$$H(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Key Parameters: - Resonant frequency: omega_0 rad/s - Damping ratio: zeta - Quality factor: Q = 1/(2*zeta) - Bandwidth (low-pass): B = 2*zeta*omega_0 - Peak gain (underdamped): K/(2*zeta) at omega_0

Butterworth Pole Angles

Formula:

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2n}$$

for k = 1, 2, …, n

Common Values: - n=2: 135°, 225° → conjugate pair - n=3: 120°, 180°, 240° → one real, one conjugate pair - n=4: 112.5°, 157.5°, 202.5°, 247.5° → two conjugate pairs

Sallen-Key Design Formulas

Equal Elements (R1=R2=R, C1=C2=C): - Constraint: mu = 3 - 2*zeta (only valid for 0 < zeta < 1) - R = 1/(zeta* omega_0 * C)

Unity Gain (mu=1, R1=R2): - C2 = 4*zeta^2* C1 - R1 = R2 = 1/(zeta * omega_0 * C1) - Works for ALL zeta values

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SECTION 10: FINAL CHECKLIST

Before Submitting Your Answer:

Transfer Functions: - [ ] Correct form (low-pass vs high-pass)? - [ ] Units correct (rad/s)? - [ ] Gain at appropriate frequency correct? - [ ] Poles in left-half plane?

Butterworth Design: - [ ] Used correct order formula (low-pass vs high-pass)? - [ ] Rounded order UP to integer? - [ ] Calculated poles at correct angles? - [ ] Scaled poles by omega_c? - [ ] Paired conjugate poles correctly? - [ ] Cascaded sections in correct order (low Q to high Q)?

Circuit Design: - [ ] Component values realistic (10 ohm to 1 Mohm, 1 pF to 1000 microF)? - [ ] Used standard values when required? - [ ] Correct op-amp configuration (inverting/non-inverting)? - [ ] All grounds and connections shown? - [ ] Gain stages correct?

Frequency Response: - [ ] Magnitude units (linear or dB)? - [ ] Phase units (degrees or radians)? - [ ] Correct asymptotic behavior (omega → 0 and omega → infinity)? - [ ] Corner frequencies correct?

APPENDIX: STANDARD COMPONENT VALUES

Standard Resistor Values (E12 series):

10, 12, 15, 18, 22, 27, 33, 39, 47, 56, 68, 82 (and multiples)

Standard Capacitor Values:

pF range: 10, 22, 47, 100, 220, 470 **nF range:** 1.0, 2.2, 4.7, 10, 22, 47, 100, 220, 470 **microF range:** 0.1, 0.22, 0.47, 1.0, 2.2, 4.7, 10, 22, 47, 100

Typical Choices for Design:

- **R:** 1k, 2.2k, 4.7k, 10k, 22k, 47k, 100k
- **C:** 0.01 microF, 0.1 microF, 1.0 microF, 10 microF

END OF WORKED EXAMPLES

Good luck on your quiz! Remember to: 1. *Read problems carefully* 2. *Show your work* 3. *Check units* 4. *Verify reasonableness of answers*