Complete Linear Circuit Analysis Cheat Sheet

1. Laplace Transform Fundamentals

Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \quad \text{where } s = \sigma + j\omega$$

Common Transform Pairs

Time Domain $f(t)$	s-Domain $F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$

Essential Properties

Linearity:

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

Frequency Shift:

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Time Shift:

$$\mathcal{L}\{f(t-T)u(t-T)\} = e^{-sT}F(s)$$

Differentiation (CRITICAL):

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0^-) - f'(0^-)$$

Integration:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

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2. s-Domain Circuit Elements

Element Impedances

Element	Time Domain	s-Domain Impedance	Initial Condition Model
Resistor	$v = Ri$	$Z_R = R$	None
Capacitor	$i = C\frac{dv}{dt}$	$Z_C = \frac{1}{sC}$	Series: $\frac{v_c(0^-)}{s}$ Parallel: $Cv_c(0^-)$
Inductor	$v = L\frac{di}{dt}$	$Z_L = sL$	Series: $Li_L(0^-)$

Kirchhoff's Laws in s-Domain

- **KVL:** $\sum V_i(s) = 0$ (around closed loops)
- **KCL:** $\sum I_i(s) = 0$ (at nodes)

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3. Circuit Response Components

Total Response Breakdown

$$\text{Total Response} = \text{ZIR} + \text{ZSR}$$

Zero-Input Response (ZIR): - Caused by initial conditions only (stored energy) - No external sources - Purely natural response

Zero-State Response (ZSR): - Caused by external inputs only - All initial conditions = 0 - Contains natural + forced response

Natural vs Forced Response

Natural Response: - From poles of $H(s)$ (network function) - Circuit's inherent modes - Independent of input type - Determined by R, L, C values

Forced Response: - From poles of input signal $V_{in}(s)$ - Same frequency as input - Amplitude scaled by $H(s)$ at input frequency

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4. Network Functions (Transfer Functions)

Definition

$$H(s) = \frac{\text{Output}(s)}{\text{Input}(s)} \quad [\text{with ALL initial conditions} = 0]$$

Types

1. **Transfer Function:** $H(s) = \frac{V_{out}(s)}{V_{in}(s)}$ or $\frac{I_{out}(s)}{I_{in}(s)}$ (different ports)
2. **Driving-Point Impedance:** $Z(s) = \frac{V(s)}{I(s)}$ (same port)
3. **Driving-Point Admittance:** $Y(s) = \frac{I(s)}{V(s)} = \frac{1}{Z(s)}$ (same port)

Rational Function Form

$$H(s) = \frac{N(s)}{D(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}$$

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5. Poles and Zeros

Definitions

ZEROS: Solutions to $N(s) = 0$ (numerator roots) - Frequencies where $H(s) = 0$ - Signals at these frequencies are **blocked** from output - Correspond to **input side** of differential equation

POLES: Solutions to $D(s) = 0$ (denominator roots, characteristic equation) - Frequencies where $H(s) \to \infty$ - Determine **natural response** modes - Always present in output - Correspond to **output side** of differential equation

Pole-Response Relationships

Pole Location	Time Response	Stability
Real: $s = -\alpha$ ($\alpha > 0$)	$ke^{-\alpha t}$ (decay)	Stable
Real: $s = +\alpha$	$ke^{+\alpha t}$ (growth)	Unstable
Complex: $s = -\alpha \pm j\omega_d$	$Ae^{-\alpha t}\cos(\omega_d t + \phi)$	Stable if $\alpha > 0$
Imaginary: $s = \pm j\omega_0$	$A\cos(\omega_0 t + \phi)$	Marginally stable
Origin: $s = 0$	k (constant/step)	Marginally stable

Stability Rule: System is stable if ALL poles have $\text{Re}(s) < 0$ (left half-plane)

Resonance Condition

When **input frequency = pole frequency**: Output term becomes $t \cdot e^{st}$ (grows with time)

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6. Inverse Laplace Transform via Partial Fraction Expansion

Terminology

- **PRF (Proper Rational Function):** Numerator degree < denominator degree
- **IRF (Improper Rational Function):** Numerator degree denominator degree

Case 1: Distinct Real Poles

PFE Setup:

$$F(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}$$

Cover-Up Method for Residue k_i:

$$k_i = [(s - p_i)F(s)]_{s=p_i}$$

Time Domain:

$$f(t) = \sum_{i=1}^{n} k_i e^{p_i t} u(t)$$

Case 2: Complex Conjugate Poles

Poles: $s = -\alpha \pm j\beta$

PFE:

$$F(s) = \frac{k}{s - (-\alpha + j\beta)} + \frac{k^*}{s - (-\alpha - j\beta)} + \text{(other terms)}$$

Time Domain:

$$f(t) = 2|k|e^{-\alpha t}\cos(\beta t + \angle k)u(t) + \text{(other terms)}$$

Alternative (Real Form):

$$f(t) = e^{-\alpha t}[A\cos(\beta t) + B\sin(\beta t)]u(t)$$

where $A = 2\text{Re}\{k\}$, $B = -2\text{Im}\{k\}$

Case 3: Repeated Poles

Pole at $s = p$ with multiplicity m:

PFE:

$$F(s) = \frac{k_m}{(s - p)^m} + \frac{k_{m-1}}{(s - p)^{m-1}} + \cdots + \frac{k_1}{s - p} + \text{(other terms)}$$

Residue Formulas:

$$k_m = [(s - p)^m F(s)]_{s=p}$$

$$k_{m-1} = \left[\frac{d}{ds}(s - p)^m F(s)\right]_{s=p}$$

$$k_{m-j} = \left[\frac{1}{j!}\frac{d^j}{ds^j}(s - p)^m F(s)\right]_{s=p}$$

Time Domain:

$$f(t) = \left[k_m \frac{t^{m-1}}{(m-1)!} + k_{m-1}\frac{t^{m-2}}{(m-2)!} + \cdots + k_1\right]e^{pt}u(t) + \text{(other terms)}$$

Case 4: Improper Rational Functions

If numerator degree denominator degree: 1. Perform polynomial long division: $F(s) = Q(s) + \frac{R(s)}{D(s)}$ 2. Apply PFE to proper fraction $\frac{R(s)}{D(s)}$ 3. Transform polynomial terms: $\mathcal{L}^{-1}\{s^n\} = \delta^{(n)}(t)$ (derivatives of impulse)

Case 5: Delayed Functions

$$\mathcal{L}^{-1}\{F(s)e^{-sT}\} = f(t - T)u(t - T)$$

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7. Initial and Final Value Theorems

Initial Value Theorem (IVT)

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

Requirements: $F(s)$ must be PRF (proper rational function)

Final Value Theorem (FVT)

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Requirements: ALL poles of $sF(s)$ must have $\text{Re}(s) < 0$ (left half-plane)

Warning: FVT is invalid if poles exist at $s = 0$ or in right half-plane

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8. Step-by-Step Problem-Solving Procedures

A. Finding Network Function H(s) from Circuit

1. **Transform** all elements to s-domain: R, $\frac{1}{sC}$, sL
2. **Zero ALL initial conditions** (voltage sources → short, current sources → open)
3. **Transform input sources** to s-domain
4. **Define** input and output variables
5. **Apply circuit analysis:**
    - Voltage/current divider
    - Series/parallel combinations
    - Nodal analysis (KCL)
    - Mesh analysis (KVL)
6. **Form** $H(s) = \frac{\text{Output}(s)}{\text{Input}(s)}$
7. **Simplify** to rational function form

B. Finding Poles and Zeros

1. Express $H(s) = \frac{N(s)}{D(s)}$ in polynomial form
2. **Find Zeros:** Set $N(s) = 0$, solve for s
    - 1st order: $s = -\frac{b}{a}$
    - 2nd order: $s = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
    - Higher: Factor or numerical methods
3. **Find Poles:** Set $D(s) = 0$, solve for s (same methods)
4. **Check discriminant** ($\Delta = b^2 - 4ac$) for 2nd order:
    - $\Delta > 0$: Two real poles
    - $\Delta = 0$: Repeated real pole
    - $\Delta < 0$: Complex conjugate pair

C. Finding Time Response via Inverse Laplace

1. **Calculate** $V_{out}(s) = H(s) \cdot V_{in}(s)$
2. **Check** if PRF or IRF
    - If IRF: Perform long division first
3. **Factor denominator** completely to find ALL poles
4. **Setup PFE** based on pole types (distinct, complex, repeated)
5. **Find residues:**
    - Simple poles: Cover-up method
    - Complex poles: Cover-up, then convert to magnitude/angle
    - Repeated poles: Derivative method
6. **Inverse transform** each term
7. **Add** $u(t)$ to ensure causality
8. **Identify components:**
    - Terms from $H(s)$ poles = Natural response
    - Terms from $V_{in}(s)$ poles = Forced response

D. Converting H(s) to Differential Equation

1. **Start with** $H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{N(s)}{D(s)}$
2. **Cross-multiply:** $V_{out}(s) \cdot D(s) = V_{in}(s) \cdot N(s)$
3. **Replace** s operators:
   - $s \to \frac{d}{dt}$
   - $s^2 \to \frac{d^2}{dt^2}$
   - $s^n \to \frac{d^n}{dt^n}$
4. **Write ODE:**

$$D\left(\frac{d}{dt}\right) v_{out}(t) = N\left(\frac{d}{dt}\right) v_{in}(t)$$

Key: Denominator (poles) → output side (left), Numerator (zeros) → input side (right)

E. Solving Differential Equations with Laplace

1. **Take Laplace transform** of entire equation
2. **Apply differentiation property:** $\mathcal{L}\{f^{(n)}\} = s^n F(s) - s^{n-1} f(0^-) - \cdots - f^{(n-1)}(0^-)$
3. **Substitute** known initial conditions
4. **Solve algebraically** for $F(s)$
5. **Apply inverse Laplace** (PFE + residues)
6. **Verify** using IVT/FVT if applicable

F. Complete Circuit Analysis with Initial Conditions

1. **Analyze** $t < 0$ **circuit:** Find $v_C(0^-)$ and $i_L(0^-)$
2. **Draw s-domain circuit for** $t \geq 0$**:**
   - Replace elements with impedances
   - Add initial condition sources:
     - Capacitor: $\frac{v_c(0^-)}{s}$ in series with $\frac{1}{sC}$
     - Inductor: $Li_L(0^-)$ in series with sL
   - Transform input sources
3. **Apply circuit analysis** (KVL, KCL, nodal, mesh) in s-domain
4. **Solve** for desired $V(s)$ or $I(s)$
5. **Apply inverse Laplace transform**
6. **(Optional) Verify** with IVT: $v(0^+)$ or FVT: $v(\infty)$

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9. Problem-Solving Decision Tree

When given a CIRCUIT:

1. Need $H(s)$? → Procedure A (zero ICs!)
2. Need poles/zeros? → First find $H(s)$, then Procedure B
3. Need time response? → Find $H(s)$, then Procedure C
4. Have initial conditions? → Procedure F

When given H(s):

1. Need poles/zeros? → Procedure B
2. Need differential equation? → Procedure D
3. Need time response? → Procedure C (need input too)

When given DIFFERENTIAL EQUATION:

1. Need $H(s)$? → Laplace transform both sides, solve for Output/Input
2. Need solution? → Procedure E
3. Need poles/zeros? → First get $H(s)$, then Procedure B

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10. Critical Reminders & Common Mistakes

Must Remember

Always zero initial conditions when finding $H(s)$ $H(s)$ describes **ZSR only**, not ZIR **Include** $u(t)$ in all time-domain responses Complex poles **always come in conjugate pairs** Check **FVT applicability** before using (poles must be in left half-plane) **Factor denominator completely** before PFE

Common Errors

Wrong impedances: C is $\frac{1}{sC}$ NOT sC; L is sL NOT $\frac{1}{sL}$ Confusing poles (denominator) with zeros (numerator) Using FVT when poles at origin or in right half-plane Forgetting initial condition sources in s-domain circuit Not checking if PRF before applying IVT Mixing up natural (from $H(s)$ poles) vs forced (from input poles) response Incorrect signs on initial condition sources

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11. Quick Reference Table

s-Domain Impedances

$$Z_R = R \qquad Z_C = \frac{1}{sC} \qquad Z_L = sL$$

Network Function

$$H(s) = \frac{N(s)}{D(s)} = \frac{\text{Output}(s)}{\text{Input}(s)} \quad [\text{ICs} = 0]$$

Poles and Zeros

$$\text{Zeros: } N(s) = 0 \qquad \text{Poles: } D(s) = 0$$

Value Theorems

$$f(0^+) = \lim_{s\to\infty} sF(s) \quad \text{(PRF only)}$$

$$f(\infty) = \lim_{s\to 0} sF(s) \quad \text{(stable only)}$$

Cover-Up Method

$$k_i = [(s-p_i)F(s)]_{s=p_i}$$

Complex Pole Response

$$\text{Poles at } s = -\alpha \pm j\beta \implies f(t) = 2|k|e^{-\alpha t}\cos(\beta t + \angle k)u(t)$$

Differentiation Property

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

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12. Analysis Workflow Summary

Circuit → Network Function: 1. Transform to s-domain 2. Zero ICs 3. Circuit analysis 4. Form $H(s) = \frac{\text{Out}}{\text{In}}$

H(s) → Poles/Zeros: 1. Express as $\frac{N(s)}{D(s)}$ 2. $N(s) = 0 \to$ zeros 3. $D(s) = 0 \to$ poles

s-Domain → Time Domain: 1. Check PRF/IRF 2. Factor denominator 3. Setup PFE 4. Find residues (cover-up) 5. Inverse transform 6. Add $u(t)$

H(s) → Differential Equation: 1. Cross-multiply 2. Replace $s \to \frac{d}{dt}$ 3. Write ODE

System Stability: - Check all poles - All $\text{Re}(s) < 0 \to$ Stable - Any $\text{Re}(s) \geq 0 \to$ Unstable