Comprehensive Laplace Transform Cheat Sheet for Linear Circuit Analysis

Table of Contents

Laplace Transform Fundamentals

Definition

The Laplace Transform converts time-domain functions $f(t)$ into s-domain functions $F(s)$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is a complex variable.

Why Use Laplace Transforms?

- **Converts differential equations $\rightarrow$ algebraic equations**
- **Handles initial conditions automatically**
- **Simplifies circuit analysis dramatically**

Common Laplace Transform Pairs

Time Domain $f(t)$	S-Domain $F(s)$	ROC (Region of Convergence)
$\delta(t)$ (unit impulse)	1	All s
$u(t)$ (unit step)	$\frac{1}{s}$	$\text{Re}(s) > 0$
t (unit ramp)	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
e^{at}	$\frac{1}{s-a}$	$\text{Re}(s) > a$
te^{at}	$\frac{1}{(s-a)^2}$	$\text{Re}(s) > a$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\text{Re}(s) > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\text{Re}(s) > 0$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\text{Re}(s) > -a$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\text{Re}(s) > -a$

Poles and Zeros

- **Poles**: Values of s where $F(s) \to \infty$ (denominator $= 0$)
- **Zeros**: Values of s where $F(s) = 0$ (numerator $= 0$)
- Poles determine the **nature of the time response** (exponential decay, oscillation, etc.)

Six Essential Properties

1. Linearity

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

Use this to: Break complex functions into simpler parts and transform each separately.

2. Frequency Shift (Multiply by e^{-at})

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Use this to: Handle exponential envelopes. Replace s with $s+a$ in $F(s)$.

Example:

$$\mathcal{L}\{e^{-2t}\cos(3t)\} = \frac{s+2}{(s+2)^2+9}$$

3. Time Shift (Delay Property)

$$\mathcal{L}\{f(t-T)u(t-T)\} = e^{-sT}F(s)$$

Use this to: Handle delayed signals. The term e^{-sT} indicates a delay of T seconds.

4. Differentiation Property CRITICAL FOR CIRCUITS

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-)$$

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0^-) - f'(0^-)$$

Use this to: Transform differential equations. Initial conditions appear as source terms!

5. Integration Property

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

Use this to: Handle integrals in your equations (less common in basic circuit analysis).

6. Multiply by t Property

$$\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$$

Use this to: Handle terms multiplied by t, or deal with multiple-order poles in inverse transforms.

Inverse Laplace Transform

Goal

Convert $F(s)$ back to $f(t)$ using **Partial Fraction Expansion (PFE)**.

Terminology

- **PRF (Proper Rational Function)**: Order of numerator < order of denominator
- **IRF (Improper Rational Function)**: Order of numerator $\geq$ order of denominator

General Form After PFE

$$F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}$$

where: - p_i = pole locations (determine exponential rates) - k_i = residues (determine weights/amplitudes)

Case 1: Distinct Real Poles

PFE Form:

$$F(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \cdots + \frac{k_n}{s - p_n}$$

Time Domain:

$$f(t) = k_1 e^{p_1 t} u(t) + k_2 e^{p_2 t} u(t) + \cdots + k_n e^{p_n t} u(t)$$

Finding Residues: Cover-Up Method

The cover-up method is a quick technique for finding residues in partial fraction expansion.

Basic Principle: To find residue k_i for pole at $s = p_i$:

$$k_i = (s - p_i)F(s)\big|_{s=p_i}$$

Step-by-Step Procedure:

1. **Multiply** both sides of $F(s)$ by the factor $(s - p_i)$ corresponding to the pole you're solving for
2. **"Cover up"** (mentally or physically) the factor $(s - p_i)$ in the original denominator
3. **Substitute** $s = p_i$ into what remains
4. **Evaluate** the expression to get k_i

Why It Works: Multiplying by $(s - p_i)$ cancels that factor in the denominator. When we set $s = p_i$, all other fractions in the PFE become zero (they have $(s - p_i)$ in their numerators), leaving only k_i.

Detailed Example:

$$F(s) = \frac{3s + 5}{(s + 1)(s + 2)}$$

Set up PFE:

$$\frac{3s + 5}{(s + 1)(s + 2)} = \frac{k_1}{s + 1} + \frac{k_2}{s + 2}$$

Find k_1 (pole at $s = -1$): 1. Multiply by $(s + 1)$:

$$k_1 + \frac{k_2(s + 1)}{s + 2} = \frac{3s + 5}{s + 2}$$

2. Set $s = -1$ (the second term vanishes):

$$k_1 = \frac{3(-1) + 5}{(-1) + 2} = \frac{2}{1} = 2$$

Find k_2 (pole at $s = -2$): 1. Multiply by $(s + 2)$:

$$\frac{k_1(s + 2)}{s + 1} + k_2 = \frac{3s + 5}{s + 1}$$

2. Set $s = -2$ (the first term vanishes):

$$k_2 = \frac{3(-2) + 5}{(-2) + 1} = \frac{-1}{-1} = 1$$

Result: $f(t) = 2e^{-t}u(t) + e^{-2t}u(t)$

Practical Shortcut (the actual "cover-up"):

For $F(s) = \frac{N(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$

To find k_i: - Cover up $(s - p_i)$ in the denominator - Replace all remaining s with p_i - Calculate the result

Visual Example:

$$F(s) = \frac{10}{(s + 2)(s + 3)(s + 5)}$$

Find k_2 at $s = -3$: - Cover up $(s + 3)$ in denominator: $\frac{10}{(s+2)(s+5)}$ - Substitute $s = -3$: $\frac{10}{(-3+2)(-3+5)} = \frac{10}{(-1)(2)} = -5$

Three-Pole Example:

$$F(s) = \frac{s + 7}{(s + 1)(s + 2)(s + 4)}$$

Find all residues:

k_1 at $s = -1$:

$$k_1 = \frac{(-1)+7}{(-1+2)(-1+4)} = \frac{6}{(1)(3)} = 2$$

k_2 at $s = -2$:

$$k_2 = \frac{(-2)+7}{(-2+1)(-2+4)} = \frac{5}{(-1)(2)} = -\frac{5}{2}$$

k_3 at $s = -4$:

$$k_3 = \frac{(-4)+7}{(-4+1)(-4+2)} = \frac{3}{(-3)(-2)} = \frac{1}{2}$$

Answer:

$$f(t) = \left[2e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}\right]u(t)$$

Important Notes: - Cover-up method **only works for distinct (simple) poles** - For repeated poles, use the derivative method (see Case 3) - For complex poles, you can use cover-up but will get complex residues (see Case 2) - Always verify: multiply out your PFE to check it equals the original $F(s)$

Case 2: Complex Conjugate Poles

PFE Form (poles at $s = -\alpha \pm j\beta$):

$$F(s) = \frac{k}{s-(-\alpha+j\beta)} + \frac{k^*}{s-(-\alpha-j\beta)}$$

where k and k^* are complex conjugates.

Time Domain:

$$f(t) = 2|k|e^{-\alpha t}\cos(\beta t + \angle k)u(t)$$

Steps: 1. Find residue k at one pole using cover-up method 2. Express k in polar form: $k = |k|e^{j\angle k}$ 3. Apply formula above

Alternative Form (using real coefficients):

$$f(t) = e^{-\alpha t}[A\cos(\beta t) + B\sin(\beta t)]u(t)$$

where: - $A = 2\text{Re}\{k\}$ - $B = -2\text{Im}\{k\}$

Case 3: Multiple-Order Poles

PFE Form (pole at $s = p_1$ with multiplicity m):

$$F(s) = \frac{k_{1m}}{(s-p_1)^m} + \frac{k_{1(m-1)}}{(s-p_1)^{m-1}} + \cdots + \frac{k_{11}}{s-p_1} + (\text{other poles})$$

Time Domain:

$$f(t) = \left[k_{1m}\frac{t^{m-1}}{(m-1)!} + k_{1(m-1)}\frac{t^{m-2}}{(m-2)!} + \cdots + k_{11}\right]e^{p_1 t}u(t) + (\text{other terms})$$

Finding Residues - Method 1: Matching Coefficients 1. Find highest-order residue k_{1m} using cover-up method 2. Multiply both sides by $(s-p_1)^m$ 3. Expand and match coefficients of powers of s

Finding Residues - Method 2: n-to-1-Order-Poles For highest order residue:

$$k_{1m} = (s-p_1)^m F(s)\big|_{s=p_1}$$

For next lower order:

$$k_{1(m-1)} = \frac{d}{ds}\left[(s-p_1)^m F(s)\right]\Big|_{s=p_1}$$

And so on with higher derivatives.

Case 4: Improper Rational Functions (IRF)

If numerator order ≥ denominator order:

1. **Perform long division** first
2. Write as: $F(s) = Q(s) + \frac{R(s)}{D(s)}$
3. Apply PFE to the proper fraction $\frac{R(s)}{D(s)}$
4. Inverse transform polynomial terms:
   - $\mathcal{L}^{-1}\{1\} = \delta(t)$ (impulse)
   - $\mathcal{L}^{-1}\{s\} = \delta'(t)$ (derivative of impulse)

Case 5: Delayed Functions (e^{-sT} terms)

$$\mathcal{L}^{-1}\{F(s)e^{-sT}\} = f(t-T)u(t-T)$$

The function is delayed by T seconds and is zero for $t < T$.

Circuit Analysis with Laplace

S-Domain Component Models

Component	Time Domain	S-Domain Impedance	Initial Condition Handling
Resistor	$v(t) = Ri(t)$	$Z_R = R$	None
Inductor	$v(t) = L\frac{di}{dt}$	$Z_L = sL$	Voltage source $Li(0^-)$ in series
Capacitor	$i(t) = C\frac{dv}{dt}$	$Z_C = \frac{1}{sC}$	Voltage source $\frac{v(0^-)}{s}$ in series OR current source $Cv(0^-)$ in parallel

Inductor S-Domain Model (Detailed)

Circuit representation with initial condition:

```
  Li(0-)   +    sL
----[+|-]----[====]----
 (voltage      (impedance)
  source)
```

Equation: $V_L(s) = sLI(s) - Li(0^-)$

Capacitor S-Domain Model (Detailed)

Series representation:

```
 v(0-)/s   +    1/sC
----[+|-]----[====]----
 (voltage      (impedance)
  source)
```

Parallel representation:

```
    1/sC        Cv(0-)
----[====]----[--|>]----
 (impedance)  (current
               source)
```

Equation: $I_C(s) = sCV(s) - Cv(0^-)$

Kirchhoff's Laws in S-Domain

KVL (Kirchhoff's Voltage Law):

$$\sum V_i(s) = 0$$

Sum of voltages around any closed loop equals zero (same as time domain, but with $V(s)$).

KCL (Kirchhoff's Current Law):

$$\sum I_i(s) = 0$$

Sum of currents entering a node equals sum leaving (same as time domain, but with $I(s)$).

Zero-Input Response (ZIR) and Zero-State Response (ZSR)

Total Response:

$$y(t) = y_{ZIR}(t) + y_{ZSR}(t)$$

- **ZIR**: Response due to initial conditions only (no external input)
- **ZSR**: Response due to external input only (zero initial conditions)

Initial and Final Value Theorems

Initial Value Theorem (IVT)

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

Requirements: - $F(s)$ must be a **Proper Rational Function (PRF)** - If IRF, IVT gives ∞ (not useful)

Use: Quickly find initial value without doing full inverse transform.

Final Value Theorem (FVT)

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Requirements: - All poles of $sF(s)$ must have $\mathrm{Re}(s) < 0$ (lie in left half-plane) - If poles on imaginary axis or right half-plane, FVT is **NOT APPLICABLE**

Use: Quickly find steady-state value without doing full inverse transform.

Common Mistake: Applying FVT when poles are at $s = 0$ or in right half-plane. Always check pole locations of $sF(s)$ first!

Step-by-Step Problem Solving Guide

Problem Type 1: Find Laplace Transform of $f(t)$

Steps: 1. **Break down** $f(t)$ into simpler additive terms (use linearity) 2. **Identify** if any special properties apply: - Exponential envelope $\to$ frequency shift property - Time delay $\to$ delay property - Derivative $\to$ differentiation property 3. **Apply** standard transforms from table 4. **Combine** results

Example: Find $\mathcal{L}\{e^{-2t}\sin(3t) + t^2\}$

Solution: - Term 1: $\mathcal{L}\{e^{-2t}\sin(3t)\}$ $\to$ Use frequency shift: replace s with $s+2$ in $\mathcal{L}\{\sin(3t)\}$ - $\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$ - $\mathcal{L}\{e^{-2t}\sin(3t)\} = \frac{3}{(s+2)^2+9}$ - Term 2: $\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$ - **Answer**: $F(s) = \frac{3}{(s+2)^2+9} + \frac{2}{s^3}$

Problem Type 2: Find Inverse Laplace Transform of $F(s)$

Steps: 1. **Check if PRF or IRF** - If IRF: Do long division first 2. **Factor denominator** completely to find all poles 3. **Set up PFE** based on pole types: - Distinct real poles $\to$ simple fractions - Complex conjugate poles $\to$ pair of conjugate fractions - Multiple-order poles $\to$ multiple fractions with increasing powers 4. **Find residues** using cover-up method (or other methods for multiple poles) 5. **Write time-domain expression** using inverse transform formulas 6. **Include** $u(t)$ for causality

Example: Find $\mathcal{L}^{-1}\left\{\frac{2s+10}{s^2+3s+2}\right\}$

Solution: 1. Check: PRF (numerator order 1 < denominator order 2) ✓ 2. Factor: $s^2 + 3s + 2 = (s+1)(s+2)$ 3. PFE: $\frac{2s+10}{(s+1)(s+2)} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$ 4. Find residues: - $k_1 = \frac{2s+10}{s+2}\Big|_{s=-1} = \frac{2(-1)+10}{-1+2} = \frac{8}{1} = 8$ - $k_2 = \frac{2s+10}{s+1}\Big|_{s=-2} = \frac{2(-2)+10}{-2+1} = \frac{6}{-1} = -6$ 5. **Answer**: $f(t) = 8e^{-t}u(t) - 6e^{-2t}u(t)$

Problem Type 3: Solve Differential Equation with Initial Conditions

Steps: 1. **Take Laplace transform** of entire equation (term by term) 2. **Apply differentiation property**: $\mathcal{L}\{f'\} = sF(s) - f(0^-)$ 3. **Substitute** known initial conditions 4. **Solve algebraically** for $F(s)$ 5. **Apply inverse transform** (PFE + cover-up method)

Example: Solve $\frac{dy}{dt} + 3y = e^{-t}$ with $y(0^-) = 2$

Solution: 1. Transform: $\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$ 2. Apply property: $sY(s) - y(0^-) + 3Y(s) = \frac{1}{s+1}$ 3. Substitute: $sY(s) - 2 + 3Y(s) = \frac{1}{s+1}$ 4. Solve: - $(s+3)Y(s) = \frac{1}{s+1} + 2$ - $Y(s) = \frac{1}{(s+1)(s+3)} + \frac{2}{s+3}$ - $Y(s) = \frac{1+2(s+1)}{(s+1)(s+3)} = \frac{2s+3}{(s+1)(s+3)}$ 5. PFE and inverse transform to get $y(t)$

Problem Type 4: Analyze Circuit with Laplace

Steps: 1. **Draw circuit for** $t < 0$: Find initial conditions $i_L(0^-), v_C(0^-)$ 2. **Draw s-domain circuit for $t ≥ 0$**: - Replace R with R - Replace L with sL and series voltage source $Li(0^-)$ - Replace C with $\frac{1}{sC}$ and series voltage source $\frac{v(0^-)}{s}$ - Transform sources to s-domain 3. **Apply circuit analysis** (KVL, KCL, node/mesh analysis) in s-domain 4. **Solve for desired** $V(s)$ **or** $I(s)$ 5. **Apply inverse transform** to get time-domain response 6. **(Optional) Verify** using initial/final value theorems

Example: RL Circuit

Given: $R = 10\Omega$, $L = 1H$, $i_L(0^-) = 0.5A$, step input $v_s(t) = 5u(t)V$

Find: $i_L(t)$ for $t ≥ 0$

Solution: 1. Initial condition: $i_L(0^-) = 0.5A$ 2. S-domain circuit: - Voltage source: $V_s(s) = \frac{5}{s}$ - Series elements: $Li(0^-) = 1(0.5) = 0.5V$, $sL = s$, $R = 10$ 3. KVL: $\frac{5}{s} = 0.5 + sI_L(s) + 10I_L(s)$ 4. Solve: - $\frac{5}{s} - 0.5 = (s+10)I_L(s)$ - $I_L(s) = \frac{5-0.5s}{s(s+10)} = \frac{5-0.5s}{s(s+10)}$ 5. Apply PFE and inverse transform

Problem Type 5: Apply Initial/Final Value Theorems

Steps: 1. **Form** $sF(s)$ 2. **For IVT**: - Check if PRF - Calculate $\lim_{s\to\infty} sF(s)$ 3. **For FVT**: - Check poles of $sF(s)$ (must all be in left half-plane) - If valid, calculate $\lim_{s\to 0} sF(s)$ 4. **(Optional) Verify** by finding $f(t)$ and evaluating at $t = 0^+$ or $t = \infty$

Quick Reference Formulas

Transform Pairs (Most Common)

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$
$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$$
$$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$$
$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Properties (Most Used)

$$\mathcal{L}\{af + bg\} = aF + bG$$
$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$
$$\mathcal{L}\{f'\} = sF(s) - f(0^-)$$
$$\mathcal{L}\{f(t-T)u(t-T)\} = e^{-sT}F(s)$$

Inverse Transforms (By Pole Type)

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = e^{pt}u(t)$$
$$\mathcal{L}^{-1}\left\{\frac{1}{(s-p)^2}\right\} = te^{pt}u(t)$$
$$\mathcal{L}^{-1}\left\{\frac{k}{s+\alpha-j\beta} + \frac{k^*}{s+\alpha+j\beta}\right\} = 2|k|e^{-\alpha t}\cos(\beta t + \angle k)u(t)$$

Value Theorems

$$f(0^+) = \lim_{s\to\infty} sF(s) \quad \text{(PRF only)}$$

$$f(\infty) = \lim_{s\to 0} sF(s) \quad \text{(stable systems only)}$$

Common Mistakes to Avoid

- ☒ **Forgetting initial conditions** when using differentiation property
- ☒ **Not checking if FVT is applicable** (poles in right half-plane)
- ☒ **Missing the $u(t)$ in time-domain answers**
- ☒ **Incorrect sign on initial condition voltage/current sources**
- ☒ **Not factoring denominator completely** before PFE
- ☒ **Using IVT on improper rational functions**
- ☒ **Forgetting to perform long division** for IRFs before PFE
- ☒ **Not checking your answer** with IVT/FVT when possible

This cheat sheet covers all major topics for Laplace transform analysis in linear circuits. Practice each problem type systematically, and always verify your answers when possible!