Butterworth Filter Design - Comprehensive Worked Examples

This document contains detailed worked examples covering all major problem types for Butterworth filter design, based on typical homework and exam problems.

EXAMPLE 1: Basic Low-Pass Filter Design (Given All Specs)

Problem Statement

Design a low-pass Butterworth filter with the following specifications: - Passband: DC to 2 kHz with maximum gain $H_{MAX} = 0$ dB (gain = 1) - Stopband: Above 8 kHz with minimum attenuation $H_{MIN} = -40$ dB (gain = 0.01) - Implement using Sallen-Key topology

Solution

Step 1: Convert specifications to rad/s

$$f_C = 2 \text{ kHz} \rightarrow \omega_C = 2\pi \times 2000 = 12566 \text{ rad/s}$$

$$f_{MIN} = 8 \text{ kHz} \rightarrow \omega_{MIN} = 2\pi \times 8000 = 50265 \text{ rad/s}$$

Step 2: Calculate minimum order

$$n \geq \frac{\ln[(1/0.01)^2 - 1]}{2\ln(50265/12566)} = \frac{\ln(9999)}{2\ln(4)} = \frac{9.210}{2.773} = 3.32$$

Round up: n = 4 (4th order filter)

Step 3: Determine section breakdown - n = 4 (even) → 2 second-order sections - No first-order sections needed

Step 4: Calculate pole angles for n = 4

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{8}$$

For k = 1, 2, 3, 4: - $\theta_1 = \frac{\pi}{2} + \frac{\pi}{8} = \frac{5\pi}{8} = 112.5°$ - $\theta_2 = \frac{\pi}{2} + \frac{3\pi}{8} = \frac{7\pi}{8} = 157.5°$ - $\theta_3 = \frac{\pi}{2} + \frac{5\pi}{8} = \frac{9\pi}{8} = 202.5°$ - $\theta_4 = \frac{\pi}{2} + \frac{7\pi}{8} = \frac{11\pi}{8} = 247.5°$

Step 5: Calculate pole locations - $p_1 = \cos(112.5°) + j\sin(112.5°) = -0.383 + j0.924$ - $p_2 = \cos(157.5°) + j\sin(157.5°) = -0.924 + j0.383$ - $p_3 = -0.924 - j0.383$ (conjugate of p_2) - $p_4 = -0.383 - j0.924$ (conjugate of p_1)

All in left half-plane

Step 6: Form second-order sections

Section 1 (poles p_1, p_4):

$$(s - p_1)(s - p_4) = s^2 - 2(-0.383)s + [(-0.383)^2 + (0.924)^2]$$
$$= s^2 + 0.765s + 1$$

Compare to standard form $s^2 + 2\zeta\omega_0 s + \omega_0^2$: - $\omega_0 = 1$ (normalized) - $2\zeta = 0.765$ - $\zeta = 0.383$ - $Q = \frac{1}{2(0.383)} = 1.307$

Section 2 (poles p_2, p_3):

$$s^2 + 1.848s + 1$$

- $\zeta = 0.924$
- $Q = 0.541$

Step 7: Design Method Selection

Both sections have $0 < \zeta < 1$, so we can use either method. Choose **Unity Gain Method** for both sections (always works).

Step 8: Design Section 1 ($\zeta = 0.383$, $\omega_0 = 12566$ rad/s)

Choose $C_1 = 0.1$ \$ \$F

$$C_2 = 4(0.383)^2(0.1\ \mu\text{F}) = 0.0586\ \mu\text{F} \approx 0.056\ \mu\text{F (standard)}$$

$$R = \frac{1}{0.383 \times 12566 \times 0.1 \times 10^{-6}} = 2078 \text{ ohms} \approx 2.2 \text{ k ohms (standard)}$$

Step 9: Design Section 2 ($\zeta = 0.924$, $\omega_0 = 12566$ rad/s)

Choose $C_1 = 0.1$ $ $F

$$C_2 = 4(0.924)^2(0.1\ \mu\text{F}) = 0.341\ \mu\text{F} \approx 0.33\ \mu\text{F (standard)}$$

$$R = \frac{1}{0.924 \times 12566 \times 0.1 \times 10^{-6}} = 861 \text{ ohms} \approx 820 \text{ ohms (standard)}$$

Step 10: Cascade Order

Section 2 (Q = 0.541, higher ζ) $\rightarrow$ Section 1 (Q = 1.307, lower ζ)

Final Design: - **Stage 1**: $R_1 = R_2 = 820$ ohms, $C_1 = 0.1$ $ $F, $C_2 = 0.33$ $ $F, $\mu = 1$ - **Stage 2**: $R_1 = R_2 = 2.2$ k ohms, $C_1 = 0.1$ $ $F, $C_2 = 0.056$ $ $F, $\mu = 1$

EXAMPLE 2: High-Pass Filter with Gain Requirement

Problem Statement

Design a 3rd-order high-pass Butterworth filter with: - Cutoff frequency: $f_C = 500$ Hz - Stopband frequency: $f_{MIN} = 100$ Hz - Stopband attenuation: $H_{MIN} = -50$ dB - Overall passband gain: K = 10

Solution

Step 1: Verify order requirement

Convert to linear: $H_{MIN} = 10^{-50/20} = 0.00316$

$$n \geq \frac{\ln[(1/0.00316)^2 - 1]}{2\ln(3142/628)} = \frac{\ln(99999)}{2\ln(5)} = \frac{11.51}{3.22} = 3.58$$

Given n = 3, check if sufficient: $3 < 3.58 \rightarrow$ Need n = 4 minimum!

However, if problem states "design 3rd order," proceed with n = 3 (may not meet all specs).

Step 2: Convert to rad/s

$$\omega_C = 2\pi \times 500 = 3142 \text{ rad/s}$$

Step 3: Pole angles for n = 3

$$\theta_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{6}$$

- $\theta_1 = \frac{2\pi}{3} = 120° \rightarrow p_1 = -0.5 + j0.866$
- $\theta_2 = \pi = 180° \rightarrow p_2 = -1$ (real pole)
- $\theta_3 = \frac{4\pi}{3} = 240° \rightarrow p_3 = -0.5 - j0.866$

Step 4: Form sections

Second-order section (p_1, p_3):

$$s^2 + s + 1$$

- $\zeta = 0.5$ - $Q = 1.0$

First-order section (p_2):

$$s + 1$$

Step 5: Design high-pass sections

For high-pass Sallen-Key, we **swap R and C** from low-pass design.

Second-order HP (Unity Gain, $\zeta = 0.5$, $\omega_0 = 3142$ rad/s):

Choose $R_2 = 10$ k ohms

$$R_1 = 4(0.5)^2(10 \text{ k ohms}) = 10 \text{ k ohms}$$

$$C = \frac{1}{0.5 \times 3142 \times 10000} = 63.7 \text{ nF} \approx 68 \text{ nF}$$

So: $R_1 = R_2 = 10$ k ohms, $C_1 = C_2 = 68$ nF

First-order HP ($\omega_C = 3142$ rad/s):

$$H(s) = \frac{sRC}{1 + sRC}$$

Choose $C = 68$ nF:

$$R = \frac{1}{3142 \times 68 \times 10^{-9}} = 4.68 \text{ k ohms} \approx 4.7 \text{ k ohms}$$

Step 6: Add gain of K = 10

Distribute gain equally: $K_1 = K_2 = \sqrt{10} \approx 3.16$ per section

For 2nd-order section, use Equal Elements Method: - Need $\mu = 3.16$ - But Equal Elements requires $\mu = 3 - 2(0.5) = 2$

Alternative: Add gain stage at output:

$$\mu_{final} = 1 + \frac{R_f}{R_g} = 10$$

Choose $R_g = 1$ k ohms $\rightarrow R_f = 9$ k ohms

Final Cascade: 1st-order HP $\rightarrow$ 2nd-order HP $\rightarrow$ Gain stage (non-inverting amp with gain 10)

EXAMPLE 3: Design from Transfer Function H(s)

Problem Statement

Given the transfer function:

$$H(s) = \frac{10000}{(s + 50)(s^2 + 100s + 10000)}$$

Implement this filter using Sallen-Key topology.

Solution

Step 1: Identify structure - Numerator: constant $\rightarrow$ Low-pass filter - Denominator: one real pole + one complex conjugate pair - Order: n = 3

Step 2: Extract section parameters

First-order section: $s + 50$ - Pole at $s = -50$ - $\omega_C = 50$ rad/s

Second-order section: $s^2 + 100s + 10000$ - Compare to $s^2 + 2\zeta\omega_0 s + \omega_0^2$ - $\omega_0^2 = 10000 \rightarrow \omega_0 = 100$ rad/s - $2\zeta\omega_0 = 100 \rightarrow 2\zeta(100) = 100 \rightarrow \zeta = 0.5$ - $Q = 1.0$

Step 3: Design first-order section

$$\omega_C = 50 \text{ rad/s}$$

Choose $C = 1$ \$ \$F:

$$R = \frac{1}{50 \times 1 \times 10^{-6}} = 20 \text{ k ohms}$$

Step 4: Design second-order section (Unity Gain)

$\zeta = 0.5$, $\omega_0 = 100$ rad/s

Choose $C_1 = 1$ \$ \$F:

$$C_2 = 4(0.5)^2(1\ \mu\text{F}) = 1\ \mu\text{F}$$

$$R = \frac{1}{0.5 \times 100 \times 1 \times 10^{-6}} = 20 \text{ k ohms}$$

Step 5: Verify DC gain

Given $H(0) = \frac{10000}{50 \times 10000} = 0.02$

Designed sections both have unity gain → Need gain stage:

$$K = 0.02 \text{ (or can be distributed)}$$

Final Implementation: - 1st-order: $R = 20$ k ohms, $C = 1$ $ $F - 2nd-order: $R_1 = R_2 = 20$ k ohms, $C_1 = C_2 = 1$ $ $F, $\mu = 1$ - Attenuator at input or output to achieve DC gain of 0.02

EXAMPLE 4: Given Bode Plot, Find H(s)

Problem Statement

A Bode magnitude plot shows: - DC gain: 20 dB - -3 dB frequency: 1 kHz - Asymptotic slope: -40 dB/decade starting at 1 kHz - Phase at 1 kHz: -90°

Determine H(s) and identify the filter type.

Solution

Step 1: Extract information - DC gain = 20 dB → Linear: $K = 10^{20/20} = 10$ - $\omega_C = 2\pi \times 1000 = 6283$ rad/s - Slope = -40 dB/dec → 2nd order - Phase = -90° at ω_C → Butterworth 2nd order

Step 2: Identify filter type - Constant at DC, rolls off at high frequency → Low-pass - 2nd order Butterworth → $\zeta = 0.707$

Step 3: Write H(s)

Standard 2nd-order Butterworth LP:

$$H(s) = \frac{K\omega_C^2}{s^2 + \sqrt{2}\omega_C s + \omega_C^2}$$

$$H(s) = \frac{10 \times (6283)^2}{s^2 + \sqrt{2}(6283)s + (6283)^2}$$

$$H(s) = \frac{3.95 \times 10^8}{s^2 + 8886s + 3.95 \times 10^7}$$

Verification: - At DC ($s = 0$): $H(0) = \frac{3.95 \times 10^8}{3.95 \times 10^7} = 10$ - At ω_C ($s = j6283$): $|H(j6283)| = \frac{10}{\sqrt{2}} = 7.07 \rightarrow 20\log(7.07) = 17$ dB (20 - 3 = 17)

EXAMPLE 5: Frequency and Impedance Scaling

Problem Statement

A normalized 2nd-order low-pass Butterworth filter has: - $R = 1$ ohms, $C = 1$ F, $\omega_C = 1$ rad/s

Scale this design to: - New cutoff: $f_C = 10$ kHz - Practical component values: R around 10 k ohms

Solution

Step 1: Determine scaling factors

Frequency scaling:

$$k_f = \frac{\omega_{new}}{\omega_{old}} = \frac{2\pi \times 10000}{1} = 62832$$

Impedance scaling:

$$k_z = \frac{R_{new}}{R_{old}} = \frac{10000}{1} = 10000$$

Step 2: Apply scaling

Resistors:

$$R_{new} = k_z \times R_{old} = 10000 \times 1 = 10 \text{ k ohms}$$

Capacitors:

$$C_{new} = \frac{C_{old}}{k_f \times k_z} = \frac{1}{62832 \times 10000} = 1.59 \times 10^{-9} \text{ F} = 1.59 \text{ nF}$$

Use standard value: $C = 1.5$ nF

Step 3: Verify new cutoff

$$\omega_{C,new} = \frac{1}{R_{new} C_{new}} = \frac{1}{10000 \times 1.5 \times 10^{-9}} = 66667 \text{ rad/s}$$

$$f_{C,new} = \frac{66667}{2\pi} = 10.6 \text{ kHz}$$

(close to 10 kHz target)

EXAMPLE 6: Equal Elements Method Design

Problem Statement

Design a 2nd-order low-pass Butterworth filter using the Equal Elements Method: - Cutoff frequency: $f_C = 5$ kHz - Use $C = 0.01$ \$ \$F

Solution

Step 1: Identify parameters - 2nd-order Butterworth → $\zeta = 0.707$ - $\omega_C = 2\pi \times 5000 = 31416$ rad/s

Step 2: Check if Equal Elements is valid

$$0 < \zeta < 1$$

→ 0.707 is valid

Step 3: Calculate op-amp gain

$$\mu = 3 - 2\zeta = 3 - 2(0.707) = 1.586$$

Step 4: Calculate resistor values

$$R = \frac{1}{\omega_C C} = \frac{1}{31416 \times 0.01 \times 10^{-6}} = 3.18 \text{ k ohms}$$

Use $R_1 = R_2 = 3.3$ k ohms (standard value)

Step 5: Design op-amp gain circuit

$$\mu = 1 + \frac{R_f}{R_g} = 1.586$$

$$\frac{R_f}{R_g} = 0.586$$

Choose $R_g = 10$ k ohms → $R_f = 5.86$ k ohms 5.6 k ohms (standard)

Final Design: - Main circuit: $R_1 = R_2 = 3.3$ k ohms, $C_1 = C_2 = 0.01$ \$ \$F - Gain resistors: $R_f = 5.6$ k ohms, $R_g = 10$ k ohms

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EXAMPLE 7: Bandpass Filter Design

Problem Statement

Design a bandpass filter to pass frequencies from 1 kHz to 5 kHz with: - Minimum attenuation of -40 dB below 200 Hz - Minimum attenuation of -40 dB above 25 kHz

Solution

Step 1: Identify requirements - Lower cutoff: $f_1 = 1$ kHz ($\omega_1 = 6283$ rad/s) - Upper cutoff: $f_2 = 5$ kHz ($\omega_2 = 31416$ rad/s) - Lower stopband: $f_{s1} = 200$ Hz - Upper stopband: $f_{s2} = 25$ kHz

Step 2: Design highpass section (for lower cutoff)

$$n_{HP} \geq \frac{\ln[(1/0.01)^2 - 1]}{2\ln(6283/1257)} = \frac{9.21}{3.22} = 2.86$$

Round up: $n_{HP} = 3$

Design 3rd-order HP with $\omega_C = 6283$ rad/s (from Example 2)

Step 3: Design lowpass section (for upper cutoff)

$$n_{LP} \geq \frac{\ln[(1/0.01)^2 - 1]}{2\ln(157080/31416)} = \frac{9.21}{3.22} = 2.86$$

Round up: $n_{LP} = 3$

Design 3rd-order LP with $\omega_C = 31416$ rad/s: - One 1st-order + one 2nd-order section - Use normalized poles, scale to $\omega_C = 31416$ rad/s

Step 4: Cascade sections

$$H_{BP}(s) = H_{HP}(s) \times H_{LP}(s)$$

Order: HP sections first, then LP sections

Total order: 3 + 3 = 6th order bandpass filter

Step 5: Calculate center frequency and bandwidth

$$\omega_0 = \sqrt{\omega_1 \omega_2} = \sqrt{6283 \times 31416} = 14048 \text{ rad/s}$$

$$f_0 = 2.24 \text{ kHz}$$

$$BW = \omega_2 - \omega_1 = 25133 \text{ rad/s}$$

EXAMPLE 8: Component Value Selection from Specs

Problem Statement

A 2nd-order Sallen-Key low-pass section requires: - $\zeta = 0.383$ - $\omega_0 = 10000$ rad/s - Must use $R_1 = 10$ k ohms

Find R_2, C_1, C_2 using Unity Gain Method.

Solution

Unity Gain equations:

$$R_1 = R_2 = R = \frac{1}{\zeta \omega_0 C_1}$$

$$C_2 = 4\zeta^2 C_1$$

Step 1: Given $R_1 = 10$ k ohms, find C_1

$$10000 = \frac{1}{0.383 \times 10000 \times C_1}$$

$$C_1 = \frac{1}{0.383 \times 10000 \times 10000} = 2.61 \times 10^{-8} \text{ F} = 26.1 \text{ nF}$$

Use standard value: $C_1 = 27$ nF

Step 2: Calculate C_2

$$C_2 = 4(0.383)^2(27 \text{ nF}) = 15.9 \text{ nF}$$

Use standard value: $C_2 = 15$ nF

Step 3: Verify R_2

Since Unity Gain Method requires $R_1 = R_2$:

$$R_2 = 10 \text{ k ohms}$$

Final values: - $R_1 = R_2 = 10$ k ohms - $C_1 = 27$ nF - $C_2 = 15$ nF - $\mu = 1$ (buffer)

EXAMPLE 9: Verifying Filter Performance

Problem Statement

A designed 4th-order Butterworth low-pass filter has: - Section 1: $\zeta_1 = 0.924$, $\omega_0 = 5000$ rad/s - Section 2: $\zeta_2 = 0.383$, $\omega_0 = 5000$ rad/s

Verify that this meets the specification: -40 dB at 20 kHz.

Solution

Step 1: Write section transfer functions

Section 1:

$$H_1(s) = \frac{\omega_0^2}{s^2 + 2(0.924)(5000)s + 5000^2} = \frac{25 \times 10^6}{s^2 + 9240s + 25 \times 10^6}$$

Section 2:

$$H_2(s) = \frac{25 \times 10^6}{s^2 + 3830s + 25 \times 10^6}$$

Overall:

$$H(s) = H_1(s) \times H_2(s)$$

Step 2: Evaluate at $\omega = 2\pi \times 20000 = 125664$ rad/s

For Section 1 at $s = j125664$:

$$|H_1(j\omega)| = \frac{25 \times 10^6}{\sqrt{(25 \times 10^6 - \omega^2)^2 + (9240\omega)^2}}$$

$$= \frac{25 \times 10^6}{\sqrt{(25 \times 10^6 - 1.58 \times 10^{10})^2 + (1.16 \times 10^9)^2}}$$

$$\approx \frac{25 \times 10^6}{1.58 \times 10^{10}} = 0.00158$$

Similarly for Section 2: $|H_2(j\omega)| \approx 0.00158$

Overall magnitude:

$$|H(j\omega)| = 0.00158 \times 0.00158 = 2.5 \times 10^{-6}$$

$$20 \log_{10}(2.5 \times 10^{-6}) = -112 \text{ dB}$$

This exceeds -40 dB requirement (much better attenuation)

EXAMPLE 10: Troubleshooting - Why Unity Gain Required

Problem Statement

Attempt to design a 2nd-order section with $\zeta = 1.2$ using Equal Elements Method. What goes wrong?

Solution

Step 1: Check Equal Elements validity

$$\mu = 3 - 2\zeta = 3 - 2(1.2) = -0.4$$

Problem: $\mu = -0.4 < 1$

This is impossible! A non-inverting amplifier must have $\mu \geq 1$.

Step 2: Why this happens

Equal Elements Method constraint:

$$\mu = 3 - 2\zeta \geq 1$$

$$3 - 1 \geq 2\zeta$$

$$\zeta \leq 1$$

For $\zeta = 1.2 > 1$, the method **cannot work**.

Step 3: Correct approach - Unity Gain Method

With $\zeta = 1.2$, use Unity Gain:

Choose $C_1 = 0.1$ $ $F:

$$C_2 = 4(1.2)^2(0.1) = 0.576\ \mu\text{F}$$

$$R = \frac{1}{1.2 \times \omega_0 \times 0.1 \times 10^{-6}}$$

Lesson: Always check $\zeta < 1$ before using Equal Elements. For overdamped systems ($\zeta \geq 1$), **must use Unity Gain Method**.

PRACTICE PROBLEMS (Try These!)

Problem A

Design a 5th-order low-pass Butterworth filter with $f_C = 3$ kHz to meet -50 dB attenuation at 15 kHz.

Problem B

Given $H(s) = \frac{10^8}{(s+100)(s^2+200s+10000)}$, identify all section parameters and design the circuit.

Problem C

Convert a 3rd-order normalized low-pass filter to a high-pass filter with $f_C = 8$ kHz.

Problem D

Design a bandstop filter to reject 60 Hz $\pm$10 Hz (notch out 50-70 Hz) with -40 dB rejection.

SUMMARY OF KEY TAKEAWAYS

1. **Always round order UP -** Never round down or you won't meet specs
2. **Check** $\zeta < 1$ before using Equal Elements Method
3. **Unity Gain works for everything** - It's the safe choice
4. **Order sections by Q** - Low Q (high ζ) first
5. **Use standard component values** - Nearest E12/E24 series
6. **Verify DC gain** - Make sure overall gain matches requirements
7. **Frequency scaling** - Affects only C (not R)
8. **Impedance scaling** - Affects both R and C oppositely
9. **Bandpass = Cascade** - Multiply transfer functions
10. **Bandstop = Parallel Sum** - Add transfer functions

END OF WORKED EXAMPLES