Fourier coefficients

For $M=0$, we can look at XY model:

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} \vec{s}_i \cdot \vec{s}_j = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j) \quad , \quad \vec{s}_i = (\cos\theta_i, \sin\theta_i)$$

$J \to -J$:

$$\mathcal{H} = J\sum_{\langle ij\rangle} \cos(\theta_i-\theta_j) = -J\sum_{\langle ij\rangle} \cos[\underbrace{(\theta_i+\pi)}_{\substack{\text{half of the spins} \\ \text{flips when interactions change sign}}} - \theta_j]$$

Fourier coefficients:

$$u(\theta) = e^{J\cos\theta}$$

$$\lambda_n(J) = \frac{1}{2\pi}\int_0^{2\pi} e^{J\cos\theta} e^{-in\theta} d\theta = I_n(J)$$

$J \to -J$:

$$\lambda_n(-J) = \frac{1}{2\pi}\int_0^{2\pi} e^{-J\cos\theta} e^{-in\theta} d\theta$$

changing variable $\theta' = \theta + \pi$,

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{J\cos\theta'} e^{-in(\theta'-\pi)} d\theta'$$

$\theta+\pi$; θ

$$= e^{in\pi}\lambda_n(J)$$

$$= (-1)^n \lambda_n(J) \rightarrow$$

for even n ; $\lambda_n(-J) = \lambda_n(J)$
for odd n ; $\lambda_n(-J) = -\lambda_n(J)$

For Ashkin-Teller model at $M=0$;

$$\lambda_{nm}(-J_1, -J_2) = (-1)^{n+m} \lambda_{nm}(J_1, J_2)$$

- for even $n+m$, $\lambda_{nm}(-J) = \lambda_n(J)$
  $(0,0), (1,1), (2,0), \dots$
- for odd $n+m$, $\lambda_{nm}(-J) = -\lambda_n(J)$
  $(0,1), (1,0), \dots$

Under RG;

- for bond-moving:
  when convolving two coefficients, sign properties multiply.

$$(-1)^{n_1+m_1} \times (-1)^{n_2+m_2} = (-1)^{(n_1+n_2)(m_1+m_2)}$$

- for decimation:

  similarly, sign properties multiply.

Villain form

Villain approximation to Hamiltonian:

$$u_V(\theta) = \sum_{p=-\infty}^{\infty} e^{-\frac{J_V}{2}(\theta - 2p\pi)^2}, \quad J_V: \text{effective coupling}$$

$$\lambda_n^V = \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{p=-\infty}^{\infty} e^{\frac{-J_V}{2}(\theta-2\pi p)^2}\right] e^{-in\theta}\, d\theta$$

using Poisson summation formula,
(see Appendix)

$$\sum_{p=-\infty}^{\infty} e^{\frac{-J_V}{2}(\theta-2\pi p)^2} \propto \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{2J_V}} e^{in\theta}$$

this directly gives Fourier coefficients of the Villain form,

$$\lambda_n^V \propto e^{-n^2/2J_V}$$

normalization:

$$\frac{\lambda_n^V}{\lambda_0^V} = \frac{e^{-\frac{n^2}{2J_V}}}{e^0} = e^{-n^2/2J_V}$$

Extension to Ashkin-Teller model at $\mu = 0$:

$$\lambda^{V}_{nm} \propto e^{-\frac{n^2+m^2}{2J_V}}$$

Behavior under $J \to -J$:

$$\lambda^{V}_{nm}(-J) = (-1)^{n+m} e^{-\frac{n^2+m^2}{2|J_V|}}$$

- for even $n+m$, $\lambda^{V}_{nm}(-J) = \lambda^{V}_{nm}(J)$
- for odd $n+m$, $\lambda^{V}_{nm}(-J) = -\lambda^{V}_{nm}(J)$

Appendix

Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \underbrace{\hat{f}}_{\text{Fourier trs. of } f.}(2\pi k)$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

Extended version:

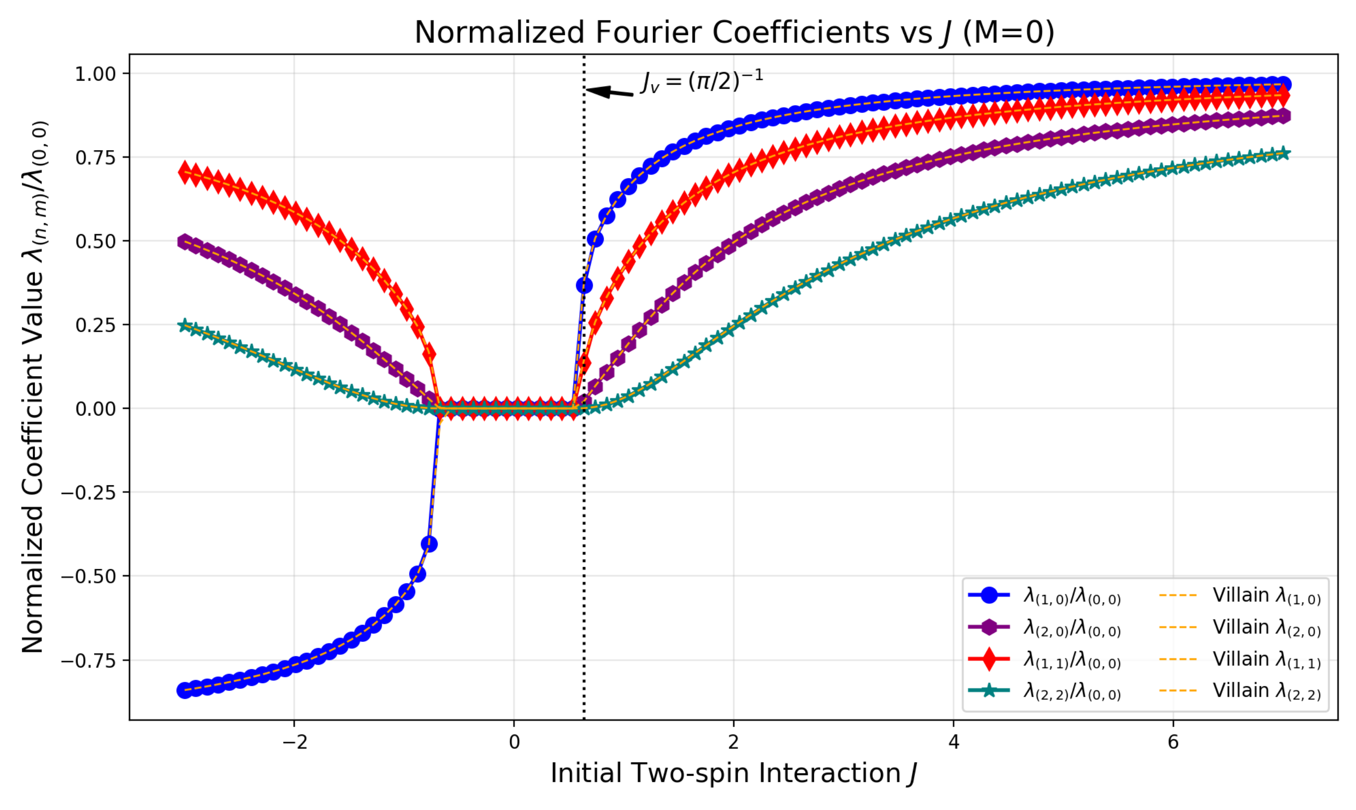
$$\sum_{n=-\infty}^{\infty} f(x+nL) = \frac{1}{L} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{2\pi k}{L}\right) e^{i2\pi k x/L}$$

For Villain form:

$$\sum_{p=-\infty}^{\infty} e^{-\frac{J_V}{2}(\theta-2\pi p)^2} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \sqrt{\frac{2\pi}{J_V}}\, e^{-\frac{n^2}{2J_V}} e^{in\theta}$$

$$= \frac{1}{\sqrt{2\pi J_V}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{2J_V}} e^{in\theta}$$

$$\lambda^{V}_{n} \propto e^{-n^2/2J_V}$$



Normalized Fourier Coefficients vs J (M=0)
$J_v = (\pi/2)^{-1}$
Normalized Coefficient Value $\lambda_{(n,m)}/\lambda_{(0,0)}$
Initial Two-spin Interaction J
$\lambda_{(1,0)}/\lambda_{(0,0)}$
$\lambda_{(2,0)}/\lambda_{(0,0)}$
$\lambda_{(1,1)}/\lambda_{(0,0)}$
$\lambda_{(2,2)}/\lambda_{(0,0)}$
Villain $\lambda_{(1,0)}$
Villain $\lambda_{(2,0)}$
Villain $\lambda_{(1,1)}$
Villain $\lambda_{(2,2)}$