Hamiltonian for $J=0$:

$$\mathcal{H} = -\mu \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \cos(\phi_i - \phi_j)$$

$$u(\theta, \phi) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \lambda_{nm} e^{in\theta} e^{im\phi}$$

Fourier coefficients:

$$\lambda_{nm} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{\mu \cos\theta \cos\phi} e^{-in\theta} e^{-im\phi} \, d\theta \, d\phi$$

Using $\cos\theta \cos\phi = \frac{1}{2}\left[\cos(\theta+\phi) + \cos(\theta-\phi)\right]$,

$$u(\theta, \phi) = e^{\frac{\mu}{2}\left[\cos(\theta+\phi) + \cos(\theta-\phi)\right]}$$

Further using $e^{a\cos\theta} = \sum_{k=-\infty}^{\infty} I_k(a) e^{ik\theta}$

modified Bessel function of the first kind.

$$u(\theta,\phi) = \underbrace{e^{\frac{\mu}{2}\cos(\theta+\phi)}}_{\cong \sum_{p=-\infty}^{\infty} e^{-\frac{\mu_V}{2}\left[(\theta+\phi)-2\pi p\right]^2}} \cdot \underbrace{e^{\frac{\mu}{2}\cos(\theta-\phi)}}_{\cong \sum_{q=-\infty}^{\infty} e^{-\frac{\mu_V}{2}\left[(\theta-\phi)-2\pi q\right]^2}}$$

, μ_V: eff. coupling

$$\cong \sum_{p,q=-\infty}^{\infty} e^{-\mu_V\left[\theta - \pi(p+q)\right]^2 - \mu_V\left[\phi - \pi(p-q)\right]^2}$$

to simplify, $r = p+q$ and $s = p-q$,

$$= \sum_{r,s=-\infty}^{\infty} e^{-\mu_V(\theta-\pi r)^2 - \mu_V(\phi - \pi s)^2}$$

Using Poisson summation formula,

$$u(\theta,\phi) \propto \sum_{n,m=-\infty}^{\infty} e^{-\frac{n^2+m^2}{4\mu_V}} e^{in\theta} e^{im\phi}$$

$$\boxed{\lambda_{nm} \propto e^{-\frac{n^2+m^2}{4\mu_v}}}$$

Behaviour under $\mu \to -\mu$:

$$\lambda_{nm}(-\mu) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{-\mu\cos\theta\cos\phi}\, e^{-in\theta}\, e^{-im\phi}\, d\theta\, d\phi$$

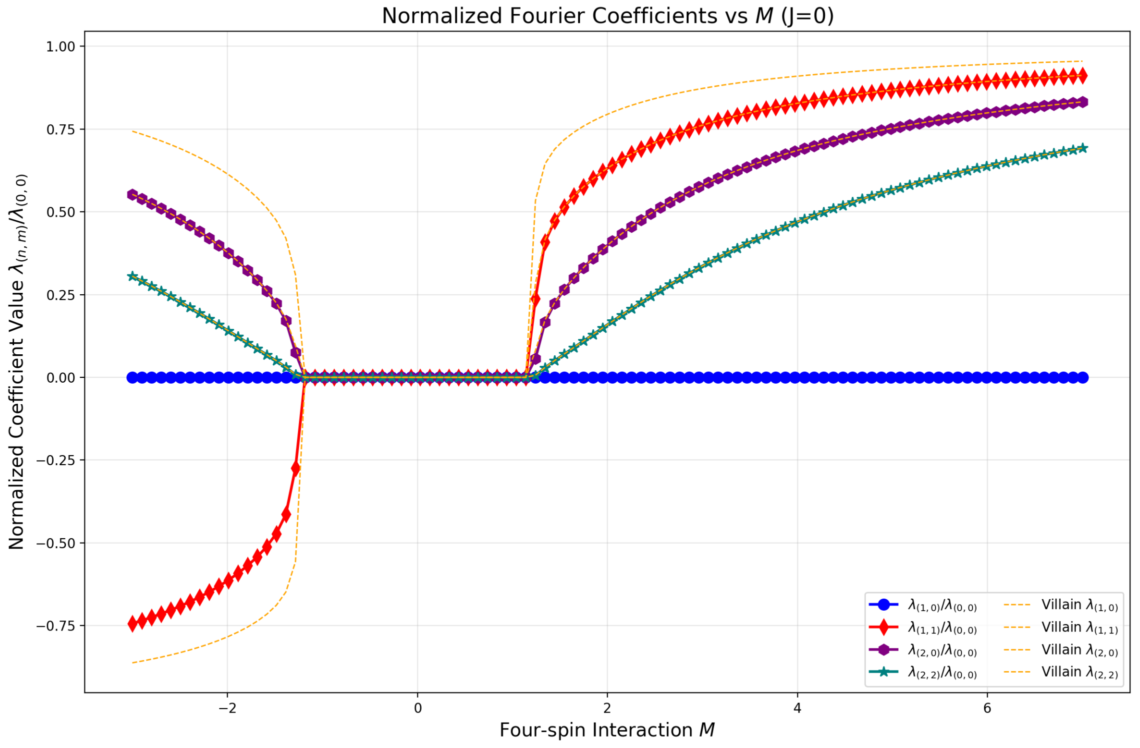
$$\left[\theta' = \theta+\pi \text{ and } \phi' = \phi+\pi\right]$$

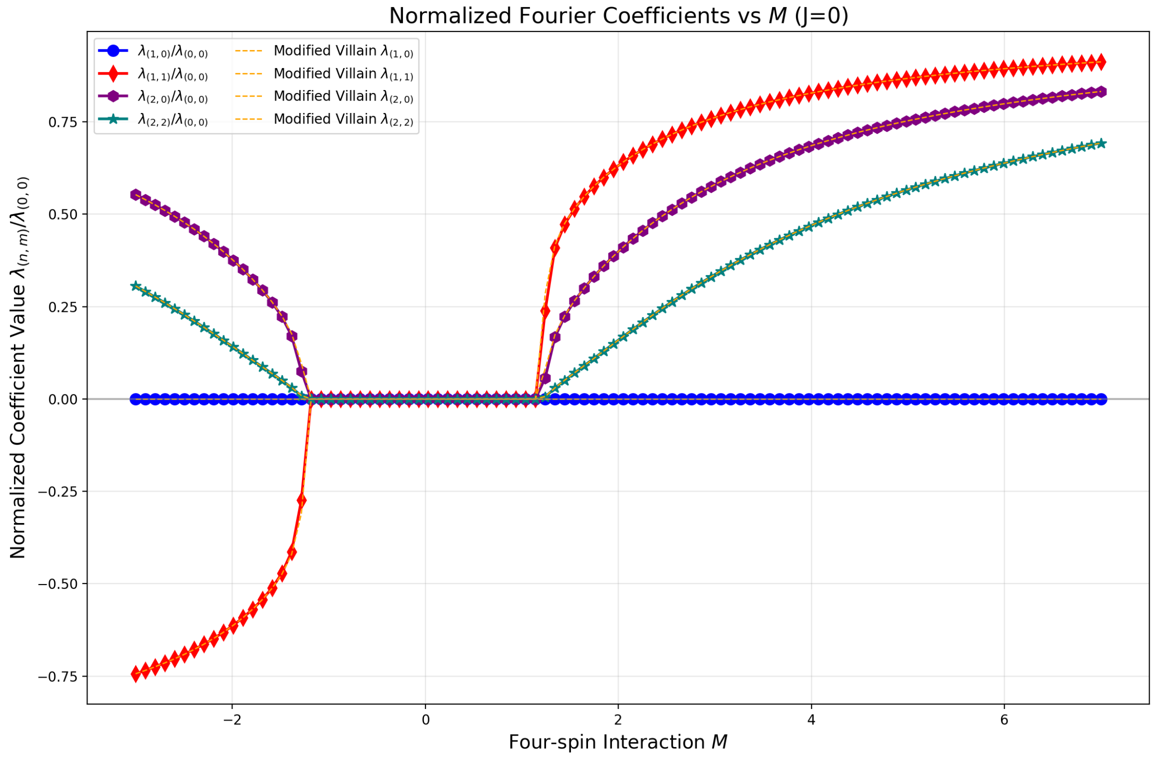
$$= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} e^{\mu\cos\theta'\cos\phi'}\, e^{-in(\theta'-\pi)}\, e^{-im(\phi'-\pi)}\, d\theta' d\phi'$$

$$= e^{-in\pi}\, e^{-im\pi}\, \lambda_{nm}(\mu)$$

$$= (-1)^{n+m}\, \lambda_{nm}(\mu)$$

- for even $n+m$, $\lambda_{nm}(-\mu) = \lambda_{nm}(\mu)$
- for odd $n+m$, $\lambda_{nm}(-\mu) = -\lambda_{nm}(\mu)$


Normalized Fourier Coefficients vs M (J=0)
Normalized Coefficient Value $\lambda_{(n,m)}/\lambda_{(0,0)}$
Four-spin Interaction M
$\lambda_{(1,0)}/\lambda_{(0,0)}$
$\lambda_{(1,1)}/\lambda_{(0,0)}$
$\lambda_{(2,0)}/\lambda_{(0,0)}$
$\lambda_{(2,2)}/\lambda_{(0,0)}$
Villain $\lambda_{(1,0)}$
Villain $\lambda_{(1,1)}$
Villain $\lambda_{(2,0)}$
Villain $\lambda_{(2,2)}$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