Ferromagnetic and Spin-Glass Finite-Tempeature Order but no Antiferromagetic Order in the d=1 Ising Model with Long-Range Power-Law Interactions

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The $d = 1$ Ising ferromagnet and spin glass with long-range power-law interactions $J\,r^{-a}$ is studied for all interaction range exponents a by a renormalization-group transformation that simultaneously projects local ferromagnetism and antiferromagnetism. In the ferromagnetic case, $J > 0$, a finite-temperature ferromagnetic phase occurs for interaction range $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, the phase transition becomes first order, also as predicted by rigorous results. For $a > 2$, the phase transition temperature discontinuously drops to zero and for $a > 2$ there is no ordered phase above zero temperature, also as predicted by rigorous results. At the other end, on approaching $a = 0.74$ from above, namely increasing the range of the interaction, the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before ($a > 0$) the neighbors become equivalent, namely before the interactions become equal for all separations. The critical exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$ are calculated, from a large recursion matrix, varying as a function of a. For the antiferromagnetic case, $J < 0$, all triplets of spins at all ranges have competing interactions and this highly frustrated system does not have an ordered phase. In the spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p), a finite-temperatures spin-glass phase is obtained in the absence of antiferromagnetic phase. A truly unusual phase diagram is obtained. In the spin-glass phase, the signature chaotic behavior under scale change occurs in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, yielding a piecewise chaotic interaction function.

I. ORDERING IN ONE DIMENSION: LONG-RANGE INTERACTIONS

Whereas systems with finite-range interactions do not order above zero temperature in one dimension, certain systems with long-range interactions do order.[1–5] The archetypical example are the Ising ferromagnetic models with power-law interactions, $J\,r^{-a}$. Also as seen below, for antiferromagnetic interactions, the system incorporates saturated frustration and spin-glass ordering without antiferromagnetic ordering, in the absence of quenched randomness.

The model that we study is defined by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{r_1 \neq r_2} J\,|r_1 - r_2|^{-a} s_{r_1} s_{r_2} + H \sum_{r_1} s_{r_1} \qquad (1)$$

where $\beta = 1/k_B T$ is the inverse temperature, r_1 and r_2 designate the sites on the one-dimensional system, at each site r_i there is an Ising spin $s_{r_i} = \pm 1$, and the sums are over all sites in the system. For ferromagnetic and antiferromagnetic systems, the two-spin interactions J are $J = |J| > 0$ and $J = -|J| < 0$, respectively. For the spin-glass system, for each two spins at any range, their interaction is randomly ferromagnetic (with probability $1 - p$) or antiferromagnetic (with probability p). The second term in Eq. (1) is the magnetic-field H term.

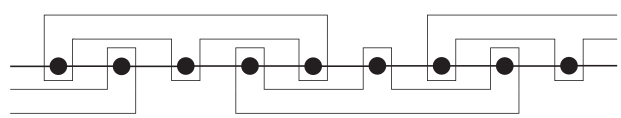

FIG. 1. Renormalization-group cells for $d = 1$. This cell structure projects both local ferromagnetism and antiferromagnetism, and therefore also spin-glass order.

II. METHOD: LONG-RANGE RENORMALIZATION GROUP

We solve this system with Niemeyer and van Leeuwen's two-cell cluster approximation.[6–8] The renormalization-group transformation is constructed by first choosing cells on the $d = 1$ system, as shown in Fig. 1. Each of our cells has three spins. This cell structure projects both local ferromagnetism and antiferromagnetism, and therefore also spin-glass order. Secondly, for each cell, a cell-spin is defined as the sign of the sum of the three spins in the cell,

$$s'_{r'} = signum(s_{r-2} + s_{r-1} + s_r) \qquad (2)$$

where the signum function returns the sign of its argument, primes denote the renormalized system, and $r' = r/b$, where $b = 3$ is the length-rescaling factor. The renormalized interactions are obtained from the conservation of the partition function Z,

$$Z = \sum_{\{s\}} e^{-\beta\mathcal{H}(\{s\})} = \sum_{\{s'\}}\sum_{\{\sigma\}} e^{-\beta\mathcal{H}(\{s'\},\{\sigma\})} = \sum_{\{s'\}} e^{-\beta\mathcal{H}'(\{s'\})} = Z' \,, \quad (3)$$

where the summed variable σ represents, for each cell, the four states that give the same cell-spin value. Thus, the renormalized interactions are obtained from

$$e^{-\beta\mathcal{H}'(\{s'\})} = \sum_{\{\sigma\}} e^{-\beta\mathcal{H}(\{s'\},\{\sigma\})}. \quad (4)$$

The two-cell cluster approximation of Niemeyer and van Leeuwen consists in carrying our this transformation for two cells, including the 6 intracell interactions and the 9 intercell interactions. A recursion relation is obtained for each renormalized interaction,

$$J'_{r'} = \frac{1}{4}\ln\frac{R_{r'}(+1,+1)R_{r'}(-1,-1)}{R_{r'}(+1,-1)R_{r'}(-1,+1)}\,, \qquad H' = \frac{1}{4}\ln\frac{R_1(+1,+1)}{R_1(-1,-1)}\,, \quad (5)$$

where

$$R_{r'}(s'_0, s'_{r'}) = \sum_{\sigma_0,\sigma_{r'}} e^{-\beta\mathcal{H}_{0r'}}, \quad (6)$$

where the unrenormalized two-cell Hamiltonian contains the 6 intracell interactions and the 9 intercell interactions between the 6 spins in cells 0 and r'.

III. RESULTS: FINITE-TEMPERATURE FERROMAGNETIC PHASE IN $d = 1$

The calculated phase diagram of the $d = 1$ long-range ferromagnetic Ising model, with interactions $J\,r^{-a}$, is shown in Fig. 2, in terms of temperature $1/J$ and interaction range a. A finite-temperature ferromagnetic phase occurs for $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, phase transition becomes first order, as predicted by rigorous results [5]. For $a > 2$, the phase transition temperature discontinuously drops to zero and there is no ordered phase above zero temperature, also as predicted by rigorous results.[2, 3] At the other end, on approaching $a = 0.74$ from above, the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before ($a > 0$) the neighbors become equivalent, namely before the interactions become equal for all separations.

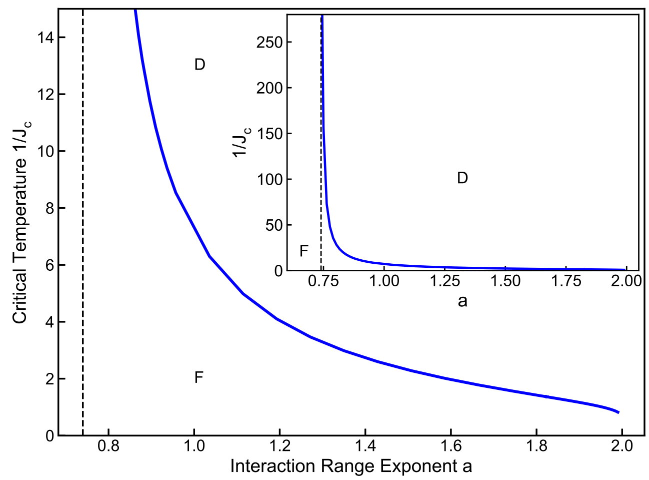

FIG. 2. Calculated phase diagram of the $d = 1$ long-range ferromagnetic Ising model with interactions $J\,r^{-a}$. Ferromagnetic (F) and disordered (D) phases are seen. A finite-temperature ferromagnetic phase occurs for $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, the transition becomes first-order, as predicted by rigorous results [5]. For $a > 2$ the phase transition temperature discontinuously drops to zero and there is no ordered phase above zero temperature, also as predicted by rigorous results [2, 3]. At the other end, on approaching $a = 0.74$ from above, the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before ($a > 0$) the neighbors become equivalent, namely before the interactions become equal for all separations. To the left of the dashed line on this figure is the equivalent-neighbor regime.

The calculated correlation-length critical exponent ν, correlation-function critical exponent η, specific heat critical exponent α, magnetization critical exponents β and δ, the susceptibility critical exponent γ, continuously varying as a function of interaction range a for the finite-temperature ferromagnetic phase transition, are shown in Fig. 3. These critical exponents are calculated, with $H = H' = 0$, from the relations $J'_1, ..., J'_n = funct(J_1, ..., J_n)$ of Eqs. (5,6). Convergence is obtained by calculation up to $n = 20$. The largest (and, as expected, only relevant, namely greater than 1) eigenvalue $\lambda_T = b^{y_T}$ of the derivative matrix of these recursion relations at the critical point gives the correlation-length critical exponent $\nu = 1/y_T$ and the specific heat critical exponent $\alpha = 2 - d/y_T = 2 - 1/y_T$. The magnetization critical exponents $\beta = (d - y_H)/y_T = (1 - y_H)/y_T$ and $\delta = y_H/(d - y_H) = y_H/(1 - y_H)$, the susceptibility critical exponent $\gamma = (2y_H - d)/y_T$, and the correlation-function critical exponent $\eta = 2 + d - y_H = 3 - y_H$ are calculated, at the critical point, with $H = H' = 0$, from $\partial H'/\partial H = b^{y_H}$.[8] Note that at $a = 2$, the magnetization critical exponent $\beta = 0$, which gives a first-order phase transition [9] as the temperature is scanned. At $a = 2$,

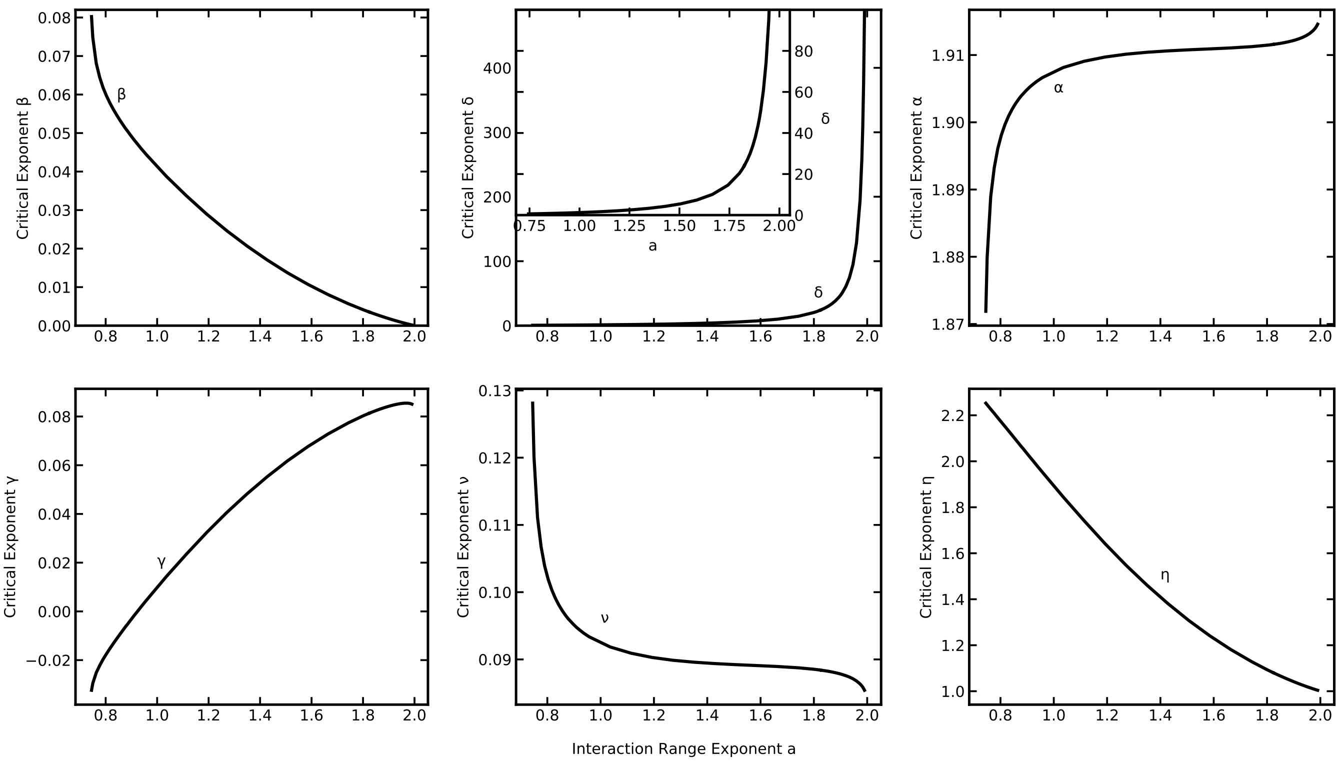

FIG. 3. Correlation-length critical exponent ν, correlation-function critical exponent η, specific heat critical exponent α, magnetization critical exponents β and δ, susceptibility critical exponent γ, as a function of interaction range a for the finite-temperature ferromagnetic phase transition. Note that β reaches 0 and δ diverges to infinity, as expected, as the first-order phase transition as $a = 2$ is reached from below.

the other magnetization critical exponent δ diverges to infitinity, which gives the first-transition as the magnetic field is scanned.

The antiferromagnetic, overly frustrated without randomness, system does not have a finite-temperature phase transition, but the spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p), does have finite-temperature spin-glass phase transitions and chaos inside the spin-glass phase, as seen in Fig. 4.

IV. RESULTS: FINITE-TEMPERATURE SPIN-GLASS PHASE IN $d = 1$

The spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p), does have finite-temperature spin-glass phase transitions and chaos inside the spin-glass phase, as seen in Fig. 4. This truly unusual spin-glass phase diagram, actually does not have an antiferromagnetic phase but has a spin-glass phase. Nevertheless, typical spin-glass system reentrance [10] is seen in this phase diagram, where as temperature is lowered at fixed antiferromagnetic bond concentration p, the ferromagnetic phase appears, but disappears at further lower temperature.

The spin-glass phase shows the chaos under rescaling signature [11–14], in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, as seen in the lower panel of Fig. 4, yielding a piecewise chaotic interaction potential.

For a previous $d = 1$ Ising spin-glass study, with short-range interactions and a zero-temperature spin-glass phase, see [15].

V. CONCLUSION

We have solved the $d = 1$ Ising ferromagnet, antiferromagnet, and spin glass with long-range power-law interactions $J\,r^{-a}$, for all interaction range exponents a by a renormalization-group transformation that simultaneously projects local ferromagnetism, antiferromagnetism, and spin-glass order. In the ferromagnetic case, $J > 0$, a finite-temperature second-order ferromagnetic phase occurs for interaction range $0.74 < a < 2$. The second-order phase transition temperature monotonically decreases between these two limits. At $a = 2$, the phase transition becomes first order, as predicted by rigorous results. For $a > 2$, the phase transition temperature discontinuously drops to zero and for $a > 2$ there is no ordered phase above zero temperature, also as pre-

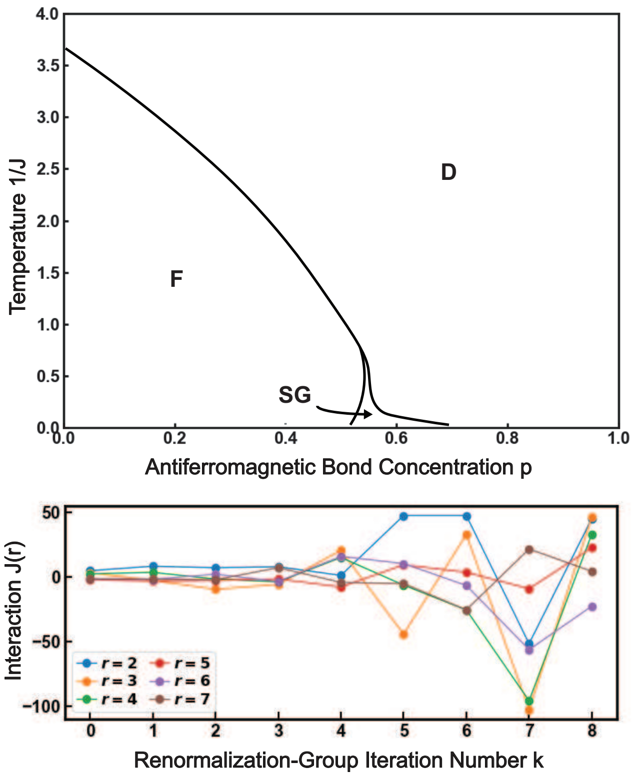

FIG. 4. Calculated finite-temperature phase diagram of the $d = 1$ long-range Ising spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p). Ferromagnetic (F), spin-glass (SG), and disordered (D) phases are seen. This truly unusual spin-glass phase diagram, actually does not have an antiferromagnetic phase but has a spin-glass phase. Bottom panel: Chaos inside the spin-glass phase in $d = 1$. The spin-glass phase shows the chaos under rescaling signature [11–14], in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, as seen in the lower panel of this figure, yielding a piecewise chaotic interaction potential.

dicted by rigorous results. At the other end, on approaching $a = 0.74$ from above, namely increasing the range of the interaction, the phase transition temperature diverges to infinity, meaning that, at all non-infinite temperatures, the system is ferromagnetically ordered. Thus, the equivalent-neighbor interactions regime is entered before ($a > 0$) the neighbors become equivalent, namely before the interactions become equal ($a = 0$) for all separations. The critical exponents $\alpha, \beta, \gamma, \delta, \eta, \nu$ for the second-order phase transitions are calculated, from a large recursion matrix, varying as a function of a.

For the antiferromagnetic case, $J < 0$, all triplets of spins at all ranges have competing interactions and this highly frustrated system does not have an ordered phase.

In the spin-glass system, where all couplings for all separations are randomly ferromagnetic or antiferromagnetic (with probability p), a finite-temperatures spin-glass phase is obtained in the absence of antiferromagnetic phase. A truly unusual phase diagram, with reentrance around the ferromagnetic phase, is obtained. In the spin-glass phase, the signature chaotic behavior under scale change occurs in a richer version than previously: In the long-range interaction of this system, the interactions at every separation become chaotic, yielding a piecewise chaotic interaction function.

ACKNOWLEDGMENTS

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