Exact Renormalization Group with Griffiths Singularities and Spin-Glass Behavior: The Random Ising Chain*

G. Grinstein,† A. N. Berker, J. Chalupa, and Michael Wortis
Department of Physics and Materials Research Laboratory, University of Illinois, Urbana, Illinois 61801
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The quenched random Ising chain is treated by a decimation transformation. An exact recursion relation is derived for the distribution of nearest-neighbor couplings and magnetic fields. Fixed distributions associated with Griffiths singularities and with the spin-glass state are explicitly exhibited.

Recent renormalization group (RG) treatments[1,2] of critical phenomena in magnetic models with quenched impurities indicate that the phase transition in many such systems is sharp and second order. Systems with predominantly ferromagnetic exchange bonds develop a spontaneous magnetization[3] $[\langle S_i\rangle^2]_{\mathrm{av}}$ like that of pure ferromagnets at sufficiently low temperatures. The transition is associated with a critical fixed point[4] of the RG transformation. This fixed point is sometimes but not always that of the corresponding pure system.

One disturbing aspect of this RG picture concerns the so-called Griffiths singularities.[5] These essential singularities in the magnetic field variable h at $h=0$ are known rigorously to be present in certain dilute ferromagnets at all temperatures below the critical temperature of the corresponding pure ferromagnet and above the onset of spontaneous magnetization. While present RG results for dilute magnets are not necessarily incompatible[2] with such behavior, the Griffiths singularities have thus far failed to show up in RG analysis.

In addition, sufficiently strong disorder may produce states with ordering quite different from that of the pure system. In a system coupled by a random mixture of ferromagnetic and antiferromagnetic bonds, ferromagnetism can be replaced by "spin-glass" order. In a spin-glass at $h=0$ the magnetization always vanishes but $[\langle S_i\rangle^2]_{\mathrm{av}}$ develops a nonzero value at sufficiently low temperatures. Edwards and Anderson[6] have described the spin-glass transition in mean field theory. More recently Harris, Lubensky, and Chen,[7] using RG methods, found a spin-glass fixed point in $6-\epsilon$ dimensions.

In this note we point out that both Griffiths singularities and spin-glass behavior may be seen by RG methods in appropriately random one-dimensional Ising models. Nelson and Fisher[8] have made precise the sense in which there is a phase transition in the pure Ising chain at $T=h=0$. The dilute chain is exactly soluble and known to possess Griffiths singularities.[9] The ferromagnetic-antiferromagnetic mixture is trivially soluble at $h=0$; but, it seems not to have been appreciated that it has a spin-glass transition at $T=0$. Neither system has been treated by RG methods. As we show below, the probability distribution for coupling strengths and magnetic fields of the random chain obeys a simple recursion relation under a "decimination" transformation.[8,10] Fixed *distributions* in the random system play the role of fixed *points* in the pure system. We exhibit fixed distributions corresponding to Griffiths singularities and spin-glass behavior.

Recursion relations.—A random Ising chain[11] of N bonds (periodically connected) is described by the reduced Hamiltonian ($S_i=\pm1$),

$$-\beta\mathcal{H}_N=\sum_{i=1}^{N}H_i,\qquad H_i=K_iS_iS_{i+1}+\tfrac{1}{2}(b_i{}^lS_i+b_i{}^rS_{i+1}) \tag{1}$$

($N+1\equiv1$), where for reasons of symmetry we allocate the magnetic field h_i at site i between bonds $i-1$ and i, so $h_i=\frac{1}{2}(b_{i-1}{}^r+b_i{}^l)$. The nearest-neighbor couplings and magnetic fields are assumed to be independent random variables distributed with probabilities $P_1(K)$ and $P_2(h)$, respectively. It is always possible to write[12]

$$P_2(h)=\int db^l\,db^r P_l(b^l)P_r(b^r)\delta(h-\tfrac{1}{2}(b^l+b^r)),$$

making the variables associated with each *bond* independently random. The quenched random free energy per site depends functionally on P_1 and P_2,

$$f=\lim_{N\to\infty}N^{-1}\int\prod_{i=1}^{N}[dK_idh_iP_1(K_i)P_2(h_i)]\ln Z_N. \tag{2}$$

For a particular set of fields and couplings $\{K_i, h_i\}$ the partition function Z_N is given in the usual representation as the 2×2 trace of a product of transfer matrices,

$$Z_N[T] = \mathrm{Tr}\prod_{i=1}^{N} T_i, \quad T_i \equiv T(\rho_i, x_i, y_i, z_i), \tag{3}$$

where

$$T(\rho, x, y, z) \equiv e^{H} = \rho\begin{pmatrix} (yz)^{1/2} & x(y/z)^{1/2} \\ x(y/z)^{-1/2} & (yz)^{-1/2} \end{pmatrix},$$

$x=\rho^{-2}=e^{-2K}$, $y=\exp b^{l}$, and $z=\exp b^{r}$. The "decimation" process[8,10] (summing over alternate spins) now defines a new, renormalized, transfer matrix $T'(\rho_i', x_i', y_i', z_i') = T_{2i-1}T_{2i}$, which preserves the partition function $Z_N[T]=Z_{N/2}[T']$ and induces exact recursion relations of the form[13]

$$x_1' = \left[\frac{(x_2+y_2z_1x_1)(x_1+y_2z_1x_2)}{(y_2z_1+x_1x_2)(1+x_1x_2y_2z_1)}\right]^{1/2}. \tag{4}$$

It is convenient to denote the "active" variables x, y, z collectively by μ ($d\mu \equiv dx\,dy\,dz$). Recursion relations then read $x_1'=X(\mu_1,\mu_2)$, $y_1'=Y(\mu_1,\mu_2)$, $z_1'=Z(\mu_1,\mu_2)$, and $\rho_1'=\rho_1\rho_2R(\mu_1,\mu_2)$. Since the variables μ_i are random independently for each bond i with a probability distribution $\mathcal{P}(\mu)\equiv\mathcal{P}(x, y, z)$, the renormalized variables μ' are distributed according to[1]

$$\mathcal{P}'(x', y', z') = \int (d\mu_1)(d\mu_2)\mathcal{P}(\mu_1)\mathcal{P}(\mu_2)\delta(x'-X)\delta(y'-Y)\delta(z'-Z), \tag{5}$$

which defines the RG transformation $\mathcal{P}'=\mathcal{R}[\mathcal{P}]$.

The initial distribution of x, y, z is

$$\mathcal{P}_0(x, y, z) = \int dK\, db^{l}\, db^{r} P_1(K)P_l(b^{l})P_r(b^{r})\delta(x-e^{-2K})\delta(y-e^{b^{l}})\delta(z-e^{b^{r}}). \tag{6}$$

$\mathcal{P}_0$ changes under successive decimations $\mathcal{P}_n \equiv \mathcal{R}^n[\mathcal{P}_0]$ and in analogy with the pure system[14] the average free energy per site is

$$\bar{f}[P_1, P_2] = \int dK\, P_1(K)K + \sum_{n=0}^{\infty} 2^{-(n+1)}\int (d\mu_1)(d\mu_2)\mathcal{P}_n(\mu_1)\mathcal{P}_n(\mu_2)\ln R(\mu_1,\mu_2). \tag{7}$$

The fixed *distributions* $\mathcal{P}^*=\mathcal{R}[\mathcal{P}^*]$ determine the critical behavior of random models. The special simplicity of one dimension is that, if the variables x, y, z for different bonds are initially uncorrelated, they then remain so under RG iteration. [In higher dimensions the variables $\{\mu_i'\}$ become spatially correlated, even if the variables $\{\mu_i\}$ are not, so the analog of (5) requires[1] the full joint probability distribution $\mathcal{P}(\{\mu_i\}_1^N)$.] Equation (5) is in general intractable. We turn, therefore, to special cases amenable to analytic treatment.

The randomly dilute chain.—In the randomly bond-dilute Ising ferromagnet[9] the nearest-neighbor couplings take the values K_0 (>0) and 0 with probabilities p and $(1-p)$, respectively. We assume that the magnetic field has the uniform value h_0 at each site. According to (6) the initial distribution is[12]

$$\mathcal{P}_0(x, y, z) = [p\,\delta(x-e^{-2K_0}) + (1-p)\delta(x-1)]\delta(y-e^{h_0})\delta(z-e^{h_0}). \tag{8}$$

In zero magnetic field $h_0=0$, we have $y=z=1$ initially and this remains so under iteration. Equation (4) simplifies and one easily shows that as $n\to\infty$, $\mathcal{P}_n$ approaches the fixed distribution $\mathcal{P}^*(x, y, z) = \delta(x-1)\delta(y-1)\delta(z-1)$ characteristic of the *pure* zero-field paramagnet. In one dimension the Griffiths singularities occur at $K_0\to\infty$, $h_0\to 0$ and are extracted most cleanly[9] in the infinite-coupling limit $K_0=\infty$ (h_0 finite). Only $x=0, 1$ then appear in the initial distribution (8), so the recursion relation (4) simplifies to $(1-x_1')=(1-x_1)(1-x_2)$. After n iterations $\mathcal{P}_n$ contains weights p^{2^n} at $x=0$ and $(1-p^{2^n})$ at $x=1$, so the fixed distribution

$$\mathcal{P}^* = \lim_{n\to\infty}\mathcal{P}_n$$

is restricted to $x=1$ ($K=0$). The recursions (5) for this case are simple enough to follow explicitly and one finds

$$\mathcal{P}^*(x, y, z) = \delta(x-1)(1-p)^2\left(\sum_{m_1=0}^{\infty} p^{m_1}\delta\{y-\exp[(2m_1+1)h_0]\}\right)\left(\sum_{m_2=0}^{\infty} p^{m_2}\delta\{z-\exp[(2m_2+1)h_0]\}\right). \tag{9}$$

This is a "high-temperature" ($K=0$, zero coupling) fixed distribution; however, the infinite range of y and z values means that arbitrarily large magnetic fields occur with finite, albeit small, probability. It is the presence in $\mathcal{P}^*$ of a stable spectrum of these large multiples of the initial field h_0 (reflecting the presence of arbitrarily large coupled clusters in the chain) which is responsible for the Griffiths singularities.[9] To make this explicit we calculate the magnetization per site $M(h_0)$ at $K_0=\infty$. Decimation preserves the magnetization per site. Since $K=0$ $(x=1)$ in $\mathcal{P}^*$, the problem reduces to the calculation of the average magnetization of a single uncoupled spin in a random field,

$$M(h_0)=\int(d\mu_1)(d\mu_2)\mathcal{P}^*(\mu_1)\mathcal{P}^*(\mu_2)\tanh[\tfrac{1}{2}(b_1{}^r+b_2{}^l)]$$
$$=(1-p)^2\sum_{n=1}^{\infty}np^{n-1}\tanh nh_0, \tag{10}$$

which is infinitely differentiable but nonanalytic at $h_0=0$, because of the imaginary poles of the hyperbolic tangent.

The fixed distribution (9) is not unique: it depends explicitly on h_0. Indeed, the recursion relations for y' and z' are trivial[15] at $x=1$ $(y_1'=y_1,$ $z_1'=z_2)$, so *any* appropriately normalized $F(x,y,z)=\delta(x-1)f_1(y)f_2(z)$ is a fixed distribution of (5). Most such fixed distributions do not lead to Griffiths singularities: For example, it is straightforward to apply this RG method of calculation to the dilute chain with a finite initial coupling K_0 (finite temperature). The resulting $\mathcal{P}^*$ has the same general structure as (9), only the (still discrete) allowed values of y and z are now bounded above. This $\mathcal{P}^*$ generates the known exact solution[9]; but, all thermodynamic properties are perfectly analytic functions of h_0. The key prerequisite in generating Griffiths singularities is the existence of arbitrarily large magnetic fields in the limiting distribution $\mathcal{P}^*$. Griffiths singularities will, for example, be present in the Ising chain with two randomly mixed couplings[16] $K_0{}^{(1)}=\infty$, $K_0{}^{(2)}$ finite.

Our calculation is not, of course, directly applicable to dimensions $d>1$. However, if a decimation were carried out[10] on a dilute Ising model with $d>1$, it seems likely that the Griffiths singularities would show up by a similar mechanism, a zero-coupling fixed distribution involving large magnetic fields.

The spin-glass.—Suppose the initial Ising couplings K_i are distributed with $P_1(K)$ but all magnetic fields vanish[17] $(h=0, y=z=1)$. It is convenient now to write the transfer matrix in terms of $w=\tanh K=(1-x)/(1+x)$, $-1\leq w\leq 1$. The recursion relation (4) then simplifies to $w_1'=w_1w_2$, so, written in terms of w, the initial distribution $\mathcal{P}_0(w)=\cosh^2 K P_1(K)$ iterates under the RG (5) as

$$\mathcal{P}'(w')=\int_{-1}^{1}dw_1\,dw_2\mathcal{P}(w_1)\mathcal{P}(w_2)\delta(w'-w_1w_2). \tag{11}$$

There are three fixed distributions of (11): $\mathcal{P}_F{}^*(w)=\delta(w-1)$, $\mathcal{P}_p{}^*(w)=\delta(w)$, and $\mathcal{P}_{sg}{}^*(w)=\frac{1}{2}[\delta(w-1)+\delta(w+1)]$. The first of these is the familiar ferromagnetic (or antiferromagnetic[18]) critical fixed point and is accessible only from the two initial distributions $\mathcal{P}_0(w)=\delta(w-1)$ (ferromagnet) and $\mathcal{P}_0(w)=\delta(w+1)$ (antiferromagnet). The second is the normal paramagnetic $(K=0)$ trivial fixed point. Its domain contains all initial distributions $\mathcal{P}_0(w)$ with *any* weight at finite K $(|w|<1)$. $\mathcal{P}_{sg}{}^*$ is the spin-glass (critical) fixed distribution. It is accessible from all $T=0$ $(|w|=1)$ distributions with mixed ferromagnetic and antiferromagnetic couplings $(0<p<1)$, $\mathcal{P}_0(w)=p\delta(w-1)+(1-p)\delta(w+1)$. It is a feature of one dimension that an arbitrarily small concentration of antiferromagnetic bonds suppresses the ferromagnetic transition and leads to spin-glass order at $T=0$.

It is instructive to explore critical behavior near the spin-glass transition by expanding about $\mathcal{P}_{sg}{}^*$. Taking moments of (11) and linearizing about the fixed distribution, one immediately discovers that odd moments are annihilated, while the deviations[3] $\delta[w^{2n}]_{av}\equiv[w^{2n}]_{av}-1$ of even moments from their fixed-point values are all relevant eigenvectors with the common eigenvalue $\Lambda=2=2^1$. We may, therefore, choose a single "reduced temperature" to describe the thermodynamics near the transition. A natural choice is $t\equiv\delta[w^2]_{av}$, which is proportional to $[e^{-2|K|}]_{av}$ near the fixed point, in direct parallel to the RG work on pure chains.[8] Thus, $\delta\bar{f}\sim t^{2-\alpha}$ with $\alpha=1$. Appropriate correlation and susceptibility exponents may be defined from the spin-glass analog of the usual order-parameter correlations[17,19] $G(r)=[\langle S_iS_{i+r}\rangle^2]_{av}=D(r/\xi)/r^{d-2+\eta}$. Near criticality G scales under decimation as $G(r,t)=G(r/2,2t)$, i.e., a function of rt. The spin-glass "susceptibility" $\chi_{sg}\equiv\sum_r G(r,t)$ diverges as t^{-1}. We thus infer directly from RG analysis[17] $\alpha=\nu=\eta=\gamma=1$ in agreement with the scaling laws $d\nu=2-\alpha$ and $\gamma=\nu(2-\eta)$.

Interestingly and perhaps in contrast to higher dimensions, it appears that $\mathcal{P}_{sg}{}^*$ is *not* stable against a uniform magnetic field h_0. We have studied the infinite-coupling problem in finite field with initial distribution[12]

$$\mathcal{P}_0(w,y,z) = [p\delta(w-1)+(1-p)\delta(w+1)]\delta(y-e^{h_0})\delta(z-e^{h_0}).$$

It is not hard to show that the magnetization[20] satisfies $M(h_0)=0$ for *all* h_0. In the RG picture one finds that, as n increases, $\mathcal{P}_n$ equalizes between $w=1$ and $w=-1$; however, the magnetic field distribution spreads indefinitely to $h=\pm\infty$ rather than narrowing to $h=0$.

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[12]Choice of right and left probabilities P_r and P_l is not unique. One may always set $b^l=0$ (or $b^r=0$), so $P_l(b)=\delta(b)$ and $P_r(b)=P_2(b)$. If $P_2(h)=\delta(h-h_0)$, then $P_l=P_r=P_2$ is convenient [e.g., Eq. (8)].

[13]Similar relations for y_1', z_1', and ρ_1' are easy to write down. ρ_1 and ρ_2 do not appear in the recursion relations for x_1', y_1', and z_1'.

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[15]When $K=0$, the fields at *non*integrated spins are unaffected by decimation, so the recursion relations have a fixed ***line***, as found in Ref. 8.

[16]J. Chalupa, unpublished.

[17]Of course, RG methods are unnecessary at $h=0$: $\bar{f}=[K+\ln(1+x)]_{av}$, $[\langle S_iS_{i+r}\rangle]_{av}=[w]_{av}{}^r$, and $[\langle S_iS_{i+r}\rangle^2]_{av}=[w^2]_{av}{}^r$. We merely emphasize the suggestive way these results emerge from the RG.

[18]Because we have chosen an even unit cell (scale factor $b=2$), the antiferromagnet maps to the ferromagnetic fixed point. See Ref. 8.

[19]The *ferromagnetic* correlation length defined from $[\langle S_iS_{i+r}\rangle]_{av}$ *vanishes* at the spin-glass point. The corresponding susceptibility is finite.

[20]In the finite, N-link chain $M_N(h_0)\sim N^{-1/2}$ except in the interval $h_0\sim N^{-1/2}$, where $M_n(h_0)$ passes smoothly through zero at $h_0=0$. Thus,

$$\lim_{N\to\infty}[dM_N/dh_0|_{h_0=0}]$$

is finite and all higher odd derivatives (evaluated at $h_0=0$) diverge as $N\to\infty$, which is connected with the onset of spin-glass order at $h_0=0$.

COMMENTS

Long-Distance Behavior of the Invariant Charge in Non-Abelian Gauge Theories*

Enrico C. Poggio
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
(Received 1 March 1976)

The long-distance behavior of the invariant charge is investigated. For massive quarks, the gluon-quark-antiquark charge is equal to the massless Yang-Mills charge. This result reflects the fact that the infrared limit is reached only as the gluon and quarks approach their mass shell at the same rate. A nonperturbative scheme in terms of the invariant charge, that could explicitly display the infrared structure of non-Abelian theories, is proposed.

In this short note I want to report some interesting results obtained in an attempt to understand the meaning and the behavior of the invariant charge at long distances within the context of quantum chromodynamics (QCD).[1] The case of massive quarks is particularly interesting: Even