Stochastic Adaptive Control

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The deadline for handing in the project report is **23:59 Tuesday, June 2nd, 2025**. The assignment focuses on the state and feedback control for nonlinear state space models. Each student must submit an individual project report, but it is allowed to collaborate with other students in the course. The report must fulfill the following requirements.

1. The length of the report must not exceed 10 pages.
2. The front page of the report must contain the names of any collaborators (make sure they also have your name on their front page).
3. Each question must be answered with a discussion/argumentation, as well as the equations and any central Matlab commands that you use, e.g., Matlab's idare and how you call it. It is not sufficient to only provide the numerical result. You must demonstrate that you understand how to solve the exercise.

Assignment 4

Consider a model of a nuclear fission reactor given by

$$\dot{C}_n(t) = \frac{\rho(t) - \beta}{\Lambda} C_n(t) + \lambda C_p(t), \tag{1a}$$

$$\dot{C}_p(t) = \frac{\beta}{\Lambda} C_n(t) - \lambda C_p(t), \tag{1b}$$

$$\dot{\rho}_{th}(t) = -\kappa H C_n(t), \tag{1c}$$

where C_n is the concentration of neutrons, C_p is the concentration of so-called neutron precursors (essentially, fission products that emit neutrons at a relatively slow rate), and ρ_{th} is the thermal reactivity. The reactivity is given by

$$\rho(t) = \rho_{th}(t) + \rho_{ext}(t), \tag{2}$$

Table 1: Parameter values.

Parameter	κ	λ	Λ	β	H
Value	$5 \cdot 10^{-5}$	3	$5 \cdot 10^{-5}$	0.0065	0.05

where ρ_{ext} is external reactivity which represents the capture of neutrons by a control rod. This is the manipulated input. When the control is fully inserted, it captures as many neutrons as possible and reduces the external reactivity. Conversely, when it is fully removed from the reactor core, it doesn't capture any neutrons, and the reactivity is increased.

The system is in the form

$$\dot{x}(t) = F(x(t)) + G(x(t))u(t), \tag{3}$$

where

$$x(t) = \begin{bmatrix} C_n(t) \\ C_p(t) \\ \rho_{th}(t) \end{bmatrix}, \qquad u(t) = \rho_{ext}(t), \tag{4a}$$

$$F(x(t)) = \begin{bmatrix} \frac{\rho_{th}(t)-\beta}{\Lambda} C_n(t) + \lambda C_p(t) \\ \frac{\beta}{\Lambda} C_n(t) - \lambda C_p(t) \\ -\kappa H C_n(t) \end{bmatrix}, \qquad G(x(t)) = \begin{bmatrix} \frac{1}{\Lambda} C_n(t) \\ 0 \\ 0 \end{bmatrix}. \tag{4b}$$

We discretize these differential equations using Euler's explicit method:

$$x_{t+1} = x_t + F(x_t)\Delta t + G(x_t)u_t \Delta t. \tag{5}$$

Use a time step size[1] of $\Delta t = 10^{-3}$. Finally, the system that we will consider is in the form

$$x_{t+1} = f(x_t) + g(x_t)u_t, \quad f(x_t) = x_t + F(x_t)\Delta t, \quad g(x_t) = G(x_t)\Delta t. \tag{6}$$

You should only consider this system, i.e., it is not necessary to simulate the original differential equations (1).

Question 1.1

Solve the nonlinear optimal control problem using the semilinearization approach from Lecture 13. Use an initial state of $C_n(t_0) = 1000$, $C_p(t_0) = 0$, and $\rho_{th}(t_0) = \beta$ and a prediction and control horizon of $N = 20$ control intervals (i.e., steps in the discrete-time system).

[1]IMPORTANT NOTE: The system is highly stiff, i.e., the state variables change very fast. Therefore, explicit methods are typically not appropriate, and we must choose the time step size very carefully.

The output that must be controlled is C_n (i.e., the first state), and the output equation is in the form

$$y_{o,t} = C_o x_t, \qquad C_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

The subscript "o" indicates that this relates to the output, i.e., the variables that we want to control. They are different from the variables that we will measure in the next question. Use an output weight of 1 and an input weight of 10^{-3} in the controller.

Hint: The Jacobian matrices of F and f are

$$\frac{\partial F}{\partial x}(x(t)) = \begin{bmatrix} \frac{\rho_{th}(t)-\beta}{\Lambda} & \lambda & \frac{1}{\Lambda}C_n(t) \\ \frac{\beta}{\Lambda} & -\lambda & 0 \\ -\kappa H & 0 & 0 \end{bmatrix}, \qquad \frac{\partial f}{\partial x}(x_t) = I + \frac{\partial F}{\partial x}(x_t)\Delta t. \tag{8}$$

Question 1.2

Introduce process noise into the discretized input-affine system (6):

$$x_{t+1} = f(x_t) + g(x_t)u_t + v_t, \quad v_t \sim N(0, R_v), \quad R_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^{-8} \end{bmatrix}. \tag{9}$$

Furthermore, we will measure both the neutron concentration, C_n, and the reactivity, C_p, and the measurement equation is in the form

$$y_t = Cx_t + e_t, \qquad e_t \sim N(0, R_e), \qquad R_e = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{10}$$

Implement an extended Kalman filter for the stochastic model (9)–(10). Use the true initial state as the initial estimate and experiment with the corresponding covariance. What if you choose a different initial state? Is it still able to estimate the reactivity, ρ_{th}?

Hint 1: The measurement equation (10) is linear so you don't need to update it in the extended Kalman filter.

Hint 2: The Jacobian matrix needed in the covariance time update is

$$A_t = \frac{\partial f}{\partial x}(x_t) + \sum_{i=1}^{n_u} \frac{\partial g}{\partial x_i}(x_t)u_t, \qquad \frac{\partial g}{\partial x_i}(x_t) = \frac{\partial G}{\partial x_i}(x_t)\Delta t, \tag{11}$$

where the Jacobian matrices of G are

$$\frac{\partial G}{\partial x_1}(x(t)) = \begin{bmatrix} \frac{1}{\Lambda} \\ 0 \\ 0 \end{bmatrix}, \qquad \frac{\partial G}{\partial x_i}(x(t)) = 0, \qquad i = 2, 3. \tag{12}$$

Question 1.3

Combine the state estimation algorithm and the semilinearization approach from the previous two questions into a feedback control algorithm where you recompute the manipulated inputs, u, every time you get a new measurement, and test it in a closed-loop simulation of the discrete-time input-affine system (9), which includes process and measurement noise. Is the nonlinear feedback control algorithm able to stabilize the system around the setpoint (i.e., around $C_n(t) = 10^4$)?