Knowledge Representation and Reasoning

Exercise Sheet 11

Albert-Ludwigs-Universität Freiburg

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Exercise 11.1

Nonmonotonic Reasoning using Abnormality Predicates

Exercise 11.1 (Nonmonotonic Reasoning using Abnormality Predicates, 4)
Consider the following knowledge base KB and show that $KB \models_{\leq} flies(c) \vee flies(d)$.

$$KB = \{\forall x(Bird(x) \wedge \neg Abnormal(x) \rightarrow flies(x)), Bird(c), Bird(d), \neg flies(c) \vee \neg flies(d)\}$$

Note: The special version of entailment (which is called *minimal entailment* and which is denoted by $\models_{\leq}$) is defined as follows: $KB \models_{\leq} \phi$ holds iff for every interpretation $\mathcal{I}$ such that $\mathcal{I} \models KB$, either $\mathcal{I} \models \phi$ or there is an $\mathcal{I}' \models KB$ such that $\mathcal{I}' < \mathcal{I}$ (with $\mathcal{I}' \leq \mathcal{I}$ iff $Abnormal^{\mathcal{I}'} \subseteq Abnormal^{\mathcal{I}}$). Discuss how this kind of reasoning compares to reasoning under the Closed World Assumption.

The models of *KB* are:

$$\begin{array}{llll}
\mathcal{I}_1: & Bird^{\mathcal{I}_1} = \{c,d\}, & flies^{\mathcal{I}_1} = \emptyset, & Abnormal^{\mathcal{I}_1} = \{c,d\} \\
\mathcal{I}_2: & Bird^{\mathcal{I}_2} = \{c,d\}, & flies^{\mathcal{I}_2} = \{c\}, & Abnormal^{\mathcal{I}_2} = \{d\} \\
\mathcal{I}_3: & Bird^{\mathcal{I}_3} = \{c,d\}, & flies^{\mathcal{I}_3} = \{c\}, & Abnormal^{\mathcal{I}_3} = \{c,d\} \\
\mathcal{I}_4: & Bird^{\mathcal{I}_4} = \{c,d\}, & flies^{\mathcal{I}_4} = \{d\}, & Abnormal^{\mathcal{I}_4} = \{c\} \\
\mathcal{I}_5: & Bird^{\mathcal{I}_5} = \{c,d\}, & flies^{\mathcal{I}_5} = \{d\}, & Abnormal^{\mathcal{I}_5} = \{c,d\}
\end{array}$$

$\mathcal{I}_2, \ldots, \mathcal{I}_5$ all satisfy $flies(c) \vee flies(d)$.

$\mathcal{I}_1 \not\models flies(c) \vee flies(d)$, but $\mathcal{I}_2 < \mathcal{I}_1$ (and $\mathcal{I}_4 < \mathcal{I}_1$).

Thus, $KB \models_{\leq} flies(c) \vee flies(d)$.

Minimal entailment (less formal): $KB \models_{\leq} \phi$ iff $\mathcal{I}' \models \phi$ for all interpretations $\mathcal{I}'$ that are minimal under $\leq$ in $\{\mathcal{I} \models KB\}$.

The set of formulas which we consider true under minimal entailment is determined by the minimization of abnormality predicates (we only need to look at the least abnormal models).

This is similar to the closed world assumption, where we assume formulas that can not be inferred from the KB to be false. This can be done, e.g., by minimizing *all* predicates.

As it turns out, there is a formal relationship between a certain formalization of the closed world assumption and *circumscription*[1], of which minimal entailment is the semantical characterization.

Note: None of this is relevant for the exam.

[1] https://www.sciencedirect.com/science/article/abs/pii/0004370285900554

Exercise 11.2

LTL Satisfiability

Exercise 11.2 (LTL Satisfiability, 1+1+1+1)
Give satisfying interpretations for the following LTL formulas or prove that they don't exist.

(a) $GFA \wedge GF\neg A$

(b) $G(A \wedge X\neg A)$

(c) $G(AUB) \wedge F(\neg A \wedge \neg B)$

(d) $G(AUB) \wedge G\neg A$

(a) $\mathcal{I} = \langle (q_1, q_2, q_1, q_2, \ldots), \tau \rangle$ with $\tau(q_1) = \{A\}$ and $\tau(q_2) = \emptyset$.

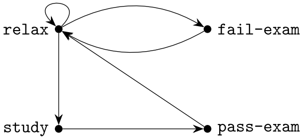
(b) To satisfy $G(A \wedge X\neg A)$ each state must satisfy A, while its successor must not satisfy A. This is not possible, since the same holds for that successor.

(c) $G(AUB)$ implies that every state satisfies A or B (or both). Thus, it is impossible to also satisfy $F(\neg A \wedge \neg B)$ by the same interpretation.

(d) $\mathcal{I} = \langle (q_1, q_1, q_1, \ldots), \tau \rangle$ with $\tau(q_1) = \{B\}$.

Exercise 11.3 (Transition Systems and Fairness, 1+1+2)

Consider the following transition system $\mathcal{T}$ with $q_i =$ `relax` and $\forall q : \tau(q) = \{q\}$.



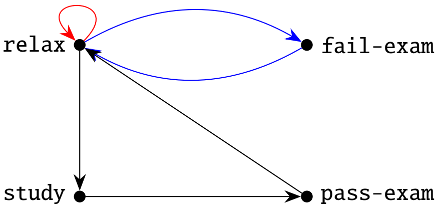
(a) Show that $F\texttt{study} \lor F\texttt{fail-exam}$ is satisfiable but not valid.

(b) Come up with a fairness condition ψ that only uses the propositional variable `relax` such that $\mathcal{T} \models_\psi F\texttt{study} \lor F\texttt{fail-exam}$.

(c) Prove that strong fairness implies weak fairness by showing that for arbitrary LTL formulae φ and ψ, the formula $FG\varphi \rightarrow GF\psi$ follows from $GF\varphi \rightarrow GF\psi$.

Exercise 11.3 (a)

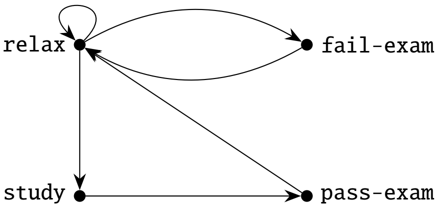
Show that $F\texttt{study} \lor F\texttt{fail-exam}$ is satisfiable but not valid



- The path $(\texttt{relax}, \texttt{fail-exam}, \texttt{relax}, \texttt{fail-exam}, \ldots)$ satisfies $F\texttt{study} \lor F\texttt{fail-exam}$.
- The path $(\texttt{relax}, \texttt{relax}, \texttt{relax}, \ldots)$ does not satisfy $F\texttt{study} \lor F\texttt{fail-exam}$, therefore the formula is not valid in $\mathcal{T}$.

Exercise 11.3 (b)

Come up with a fairness condition ψ such that $\mathcal{T} \models_{\psi} F\texttt{study} \lor F\texttt{fail-exam}$



- To prevent the infinite loop in `relax` we introduce the unconditional fairness assumption $\psi = GF\neg relax$.
- Under this assumption, one must always stop relaxing at some point and start studying or fail the next exam. Thus $\mathcal{T} \models_{\psi} F\texttt{study} \lor F\texttt{fail-exam}$ holds.

Exercise 11.3 (c)

Prove that strong fairness implies weak fairness

- We first show that $FG\varphi \rightarrow GF\varphi$ is valid:
  - For an interpretation $\mathcal{I} = \langle (q_1, q_2, \ldots), \tau \rangle$ with $\mathcal{I} \models FG\varphi$, there exists an $i \geq 1$ s.t. $\tau(q_j) \models \varphi$ for all $j \geq i$.
  - It therefore also holds, that for every $i \geq 1$ there is a $j \geq i$ with $\tau(q_j) \models \varphi$. This translates to $\mathcal{I} \models GF\varphi$.
- Applying this observation $FG\varphi \rightarrow GF\varphi$ under the strong fairness assumption $GF\varphi \rightarrow GF\psi$, we can infer $FG\varphi \rightarrow GF\psi$, i.e. weak fairness.

Exercise 11.4 (Complexity of LTL satisfiability, 6)
In the lecture, we proved the PSPACE-hardness of LTL satisfiability via reduction from Corridor Tiling. Prove that satisfiability is still PSPACE-hard for a version of LTL without the X operator, i. e., where all formulas are of the form

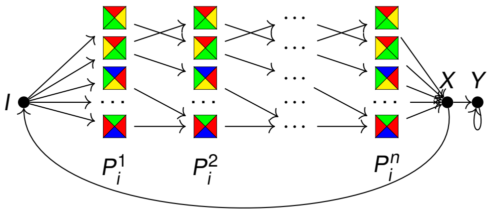
$$\varphi ::= a \mid G\varphi \mid F\varphi \mid \varphi U \varphi \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

Hint: It is sufficient to slightly modify the proof from the lecture. You only need to modify the formulas φ_1, φ_2, and φ_3, and the valuations $\tau(P_i^j)$ of the transition system that is constructed from the corridor tiling instance.

Corridor Tiling:

Given a set of tiles $\mathfrak{C} \subseteq C^4$, two colours $c_1^\star, c_2^\star \in C$, and a natural number n in **unary** encoding, decide whether there is a valid $(n \times m)$ tiling T for some m such that $\forall i \in \{1, \ldots, n\} : S(T(i,1)) = c_1^\star \wedge N(T(i,m)) = c_2^\star$.

In the lecture, we used the following construction:



$\tau(P_i^j) = \{n_{N(t_i)}, s_{S(t_i)}\}$
$\tau(X) = \tau(I) = \emptyset$
$\tau(Y) = \{Y\}$

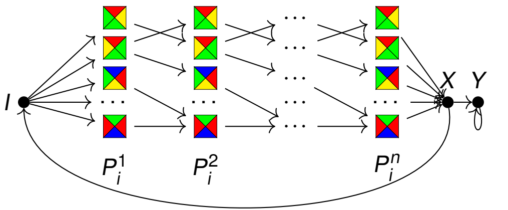
$\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$ with

- $\phi_1 = \bigwedge_{i=1}^{n} X^i s_{c_1^\star}$
- $\phi_2 = F(\bigwedge_{i=1}^{n} X^i n_{c_2^\star})$
- $\phi_3 = \bigwedge_{j=1}^{|C|} G(n_{c_j} \rightarrow X^{n+2}(s_{c_j} \vee Y))$

with $X^i = \underbrace{XX \ldots X}_{i \text{ times}}$

A path that satisfies ϕ corresponds to a solution to the tiling problem.

We get rid of the X operators by encoding the tile columns in our valuation:



$\tau(P_i^j) = \{n_{N(t_i)}, s_{S(t_i)}, x_j\}$
$\tau(X) = \{x_0\}, \tau(I) = \emptyset$
$\tau(Y) = \{Y\}$

$\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$ with

- $\phi_1 = \bigwedge_{i=1}^{n} (\neg x_i) U (x_i \wedge s_{c_1^\star})$
- $\phi_2 = F\Big(x_0 \wedge \bigwedge_{i=1}^{n} \big((\neg x_i)\ U\ (x_i \wedge n_{c_2^\star})\big)\Big)$
- $\phi_3 = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{|C|} G\Big((x_i \wedge n_{c_j}) \rightarrow (\neg x_0) U \Big(x_0 \wedge (\neg x_i) U ((x_i \wedge s_{c_j}) \vee Y)\Big)\Big)$

A path that satisfies ϕ corresponds to a solution to the tiling problem.