Foundations of Deep Learning, Winter Term 2021/22

Week 3: MLPs and Backpropagation

MLPs and Backpropagation

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Overview of Week 3

1. Recap of Multilayer Perceptrons
2. Recap of Chain Rule of Calculus
3. Calculating Gradients with Backpropagation
4. Backprop for a Linear Layer in Matrix Form
5. Full Matrix Form of Backpropagation Equations
6. Further Reading, Summary of the Week, References

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Recap of Multilayer Perceptrons

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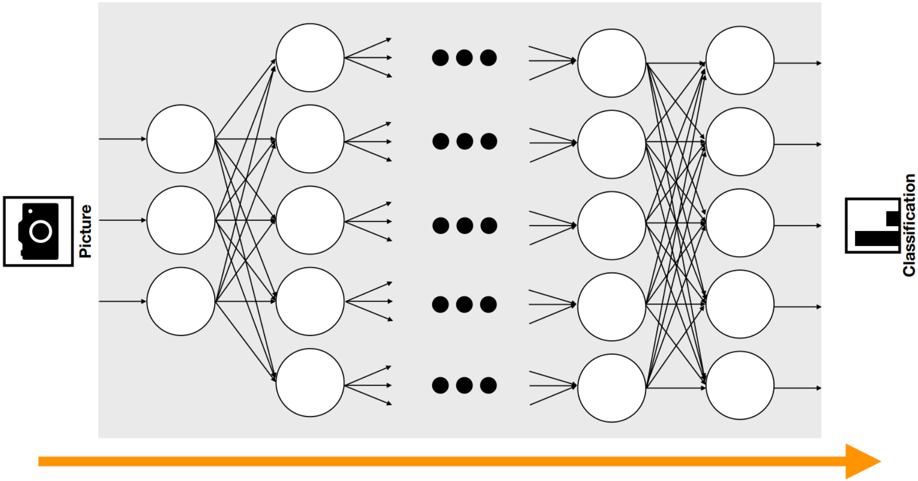
Computation is Performed Layer-by-Layer



Layer-by-Layer Computations

- Layer 1 pre-activations:
$$\mathbf{z}^{(1)} = \mathbf{W}^{(1)^\mathsf{T}}\mathbf{x} + \mathbf{b}^{(1)}$$
- Layer 1 activations:
$$\mathbf{h}^{(1)} = g^{(1)}(\mathbf{z}^{(1)})$$
- Layer i pre-activations:
$$\mathbf{z}^{(i)} = \mathbf{W}^{(i)^\mathsf{T}}\mathbf{h}^{(i-1)} + \mathbf{b}^{(i)}$$
- Layer i activations:
$$\mathbf{h}^{(i)} = g^{(i)}(\mathbf{z}^{(i)})$$
- Overall network output as one big nested function (network with one hidden layer):
$$\hat{\mathbf{y}} = g^{(2)}(\mathbf{W}^{(2)^\mathsf{T}} g^{(1)}(\mathbf{W}^{(1)^\mathsf{T}}\mathbf{x} + \mathbf{b}^{(1)}) + \mathbf{b}^{(2)})$$

Questions to Answer for Yourself / Discuss with Friends

- Repetition: What are typical examples of activation functions?
- Repetition: What is the purpose of using non-linear activation functions?

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Recap of Chain Rule of Calculus

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Chain Rule

The chain rule computes derivatives for compositions of functions by using their individual derivatives and the product of their functions as below.

For two functions $g(x)$ and $f(y) = f(g(x))$, the chain rule states:

$$(f \circ g)'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

For $y = g(x)$ and $z = f(g(x)) = f(y)$:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial f(g(x))}{\partial g(x)}\frac{\partial g(x)}{\partial x}$$

Let $y = g(x) = \sin(x)$ and $z = f(\sin(x)) = \ln(y)$. Then:

$$\frac{\partial z}{\partial x} = \frac{\partial \ln(\sin(x))}{\partial \sin(x)}\frac{\partial \sin(x)}{\partial x} = \frac{1}{\sin(x)} \cdot \cos(x)$$

Chain Rule

As a generalization of the scalar case, consider $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$. If $\mathbf{y} = g(\mathbf{x})$ and $z = f(\mathbf{y})$, then

$$\frac{\partial z}{\partial x_i} = \sum_j \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$

We can write this in a more compact way using vector notation as:

$$\nabla_{\mathbf{x}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right)^T \nabla_{\mathbf{y}} z$$

where $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ is the $n \times m$ Jacobian matrix of g, and with $\nabla_{\mathbf{y}} z$ being the gradient of z with respect to the vector $\mathbf{y}$. This can also be generalized to tensors, see chapter $6.5.2$ in the book.

Partial Derivative Example

Consider the vector $\mathbf{y} \in \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$, and $z = f(\mathbf{y})$, then

$$\nabla_{\mathbf{y}} z = \begin{bmatrix} \frac{\partial z}{\partial y_1} \\ \frac{\partial z}{\partial y_2} \\ \vdots \\ \frac{\partial z}{\partial y_n} \end{bmatrix}$$

is the gradient of z with respect to the vector $\mathbf{y}$.
For $f(\mathbf{y}) = \dfrac{1}{2}\|\mathbf{y}\|_2^2$, the derivative of z with respect to y would be:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Partial Derivative Example 2

Consider the vectors $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ a function $g : \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbf{y} = g(\mathbf{x})$, then

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix}$$

is the $n \times m$ Jacobian matrix of g.

Questions to Answer for Yourself / Discuss with Friends

- Application of what you just learned:
  What is the derivative of the ReLU activation function?

- Application of what you just learned:
  What is the derivative of the logistic sigmoid activation function?

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Calculating Gradients with Backpropagation

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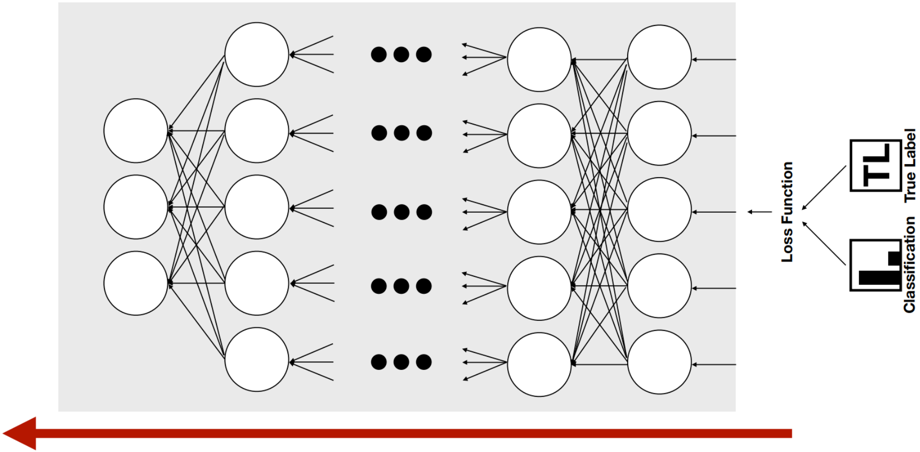
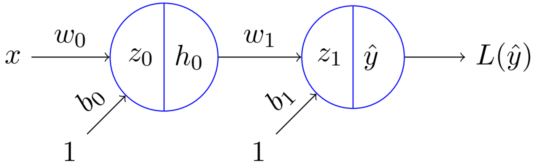
Backpropagation: Information Flow Illustration



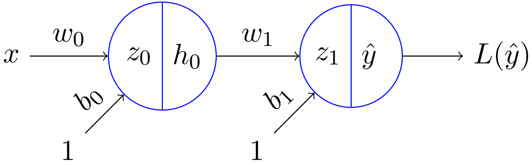
Calculating Partial Derivatives (cont.)

- We will look at how to derive the gradients in the context of a simple example network:



- We would like to know how the change in any of the weights and biases influences the loss, so we calculate $\frac{\partial L}{\partial w}$ and $\frac{\partial L}{\partial b}$ for all weights and biases in the network.

Calculating Partial Derivatives (cont.)



$$
\begin{aligned}
\hat{y} &= g_1(z_1) \\
z_1 &= w_1 h_0 + b_1 \\
h_0 &= g_0(z_0) \\
z_0 &= w_0 x + b_0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial L}{\partial \hat{y}} \\
\frac{\partial L}{\partial z_1} &= \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z_1} \\
\frac{\partial L}{\partial h_0} &= \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial h_0} \\
\frac{\partial L}{\partial z_0} &= \frac{\partial L}{\partial h_0} \frac{\partial h_0}{\partial z_0}
\end{aligned}
$$

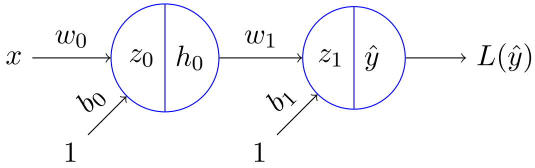
$$
\begin{aligned}
\frac{\partial L}{\partial w_1} &= \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial w_1} \\
\frac{\partial L}{\partial b_1} &= \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial b_1} \\
\frac{\partial L}{\partial w_0} &= \frac{\partial L}{\partial z_0} \frac{\partial z_0}{\partial w_0} \\
\frac{\partial L}{\partial b_0} &= \frac{\partial L}{\partial z_0} \frac{\partial z_0}{\partial b_0}
\end{aligned}
$$

Calculating Partial Derivatives (cont.)

Derivative of the activation function w.r.t its activation $\frac{\partial h}{\partial z} = h'(z)$ depends on which activation we use:

- linear activation: $h(z) = z \rightarrow h'(z) = 1$
- logistic sigmoid activation: $h(z) = 1/(1 + \exp(-z)) \rightarrow h'(z) = h(z)(1 - h(z))$
- hyperbolic tangent sigmoid activation: $h(z) = \text{tanh}(z) \rightarrow h'(z) = 1 - h(z)^2$
- ReLU activation: $h(z) = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases} \rightarrow h'(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$

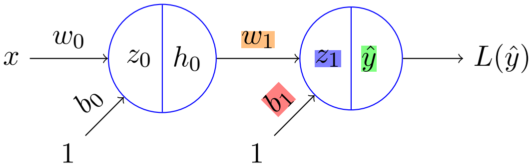
Reconsider 2-Layer MLP as an Example



For each pattern $\mathbf{x}_n$ in training set, perform forward pass:

- hidden layer:
  - $z_0 = xw_0 + b_0$
  - $h_0 = g_0(z_0) = \text{ReLU}(z_0)$
- output layer:
  - $z_1 = h_0 w_1 + b_1$
  - $\hat{y} = g_1(z_1) = z_1$
- g_0 being a ReLU, and g_1 being a linear activation function
- Consider squared error loss: $L = \frac{1}{2}(\hat{y} - y)^2$

Reconsider 2-Layer Example (cont.)



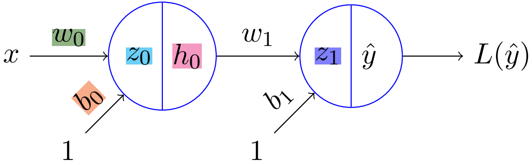
Forward pass:

$$
\begin{aligned}
L &= \frac{1}{2}(\hat{y} - y)^2 \\
\hat{y} &= g_1(z_1) && = z_1 \\
z_1 &= w_1 h_0 + b_1
\end{aligned}
$$

Backward pass:

$$
\begin{aligned}
\frac{\partial L}{\partial \hat{y}} &= \hat{y} - y \\
\frac{\partial L}{\partial z_1} &= \frac{\partial L}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial z_1} && = (\hat{y} - y) \cdot g_1'(z_1) && = (\hat{y} - y) \cdot 1 \\
\frac{\partial L}{\partial w_1} &= \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial w_1} && = \frac{\partial L}{\partial z_1} h_0 \\
\frac{\partial L}{\partial b_1} &= \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial b_1} && = \frac{\partial L}{\partial z_1} \cdot 1
\end{aligned}
$$

Reconsider 2-Layer Example (cont.)



Forward pass:

$$h_0 = g_0(z_0) = \begin{cases} z_0 & \text{if } z_0 > 0 \\ 0 & \text{if } z_0 \leq 0 \end{cases}$$

$$z_0 = w_0 x + b_0$$

Backward pass:

$$\frac{\partial L}{\partial h_0} = \frac{\partial L}{\partial z_1} \frac{\partial z_1}{\partial h_0} = \frac{\partial L}{\partial z_1} w_1$$

$$\frac{\partial L}{\partial z_0} = \frac{\partial L}{\partial h_0} \frac{\partial h_0}{\partial z_0} = \begin{cases} \frac{\partial L}{\partial h_0} & \text{if } z_0 > 0 \\ 0 & \text{if } z_0 \leq 0 \end{cases}$$

$$\frac{\partial L}{\partial w_0} = \frac{\partial L}{\partial z_0} \frac{\partial z_0}{\partial w_0} = \frac{\partial L}{\partial z_0} x$$

$$\frac{\partial L}{\partial b_0} = \frac{\partial L}{\partial z_0} \frac{\partial z_0}{\partial b_0} = \frac{\partial L}{\partial z_0} \cdot 1$$

Generic MLP Learning Algorithm Using Backpropagation

- Generic MLP learning algorithm:
  - choose an initial weight vector $\boldsymbol{w}$
  - initialize minimization approach
  - **while** error did not converge **do**
    - **for all** $(\mathbf{x}, y) \in \mathcal{D}$ **do**
      - apply $\mathbf{x}$ to network and calculate the network output (forward pass)
      - calculate $\frac{\partial L_n}{\partial w}$ and $\frac{\partial L_n}{\partial b}$ for all weights and biases (backward pass)
    - **end for**
    - calculate total gradients $\frac{\partial L}{\partial w}$ and $\frac{\partial L}{\partial b}$ for all weights and biases, summing over all training patterns
    - perform one update step of the minimization approach
  - **end while**
- Learning by epoch: All training patterns are considered for each update step of function minimization

Questions to Answer for Yourself / Discuss with Friends

- Repetition: What do the computed gradients represent?
- Repetition: Why do we have to compute the gradients?

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Backprop for a Linear Layer in Matrix Form

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Acknowledgment

Example adopted from notes [Johnson, 2017] by Justin Johnson.

Setup and Forward Pass

We will derive the equations for backpropagating through a linear layer using minibatches.

- Input $\mathbf{X}$ $(N \times D)$, weight matrix $\mathbf{W}$ $(D \times M)$, output $\mathbf{Y}$ $(N \times M)$
- Note that data points are stored in the rows of $\mathbf{X}$, just as in PyTorch
- Consider $N = 2, D = 2, M = 3 \rightarrow$ forward pass:

$$\mathbf{X} = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \qquad \mathbf{W} = \begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,3} \\ w_{2,1} & w_{2,2} & w_{2,3} \end{pmatrix}$$

$$\begin{aligned} \mathbf{Y} &= \mathbf{XW} \\ &= \begin{pmatrix} x_{1,1}w_{1,1} + x_{1,2}w_{2,1} & x_{1,1}w_{1,2} + x_{1,2}w_{2,2} & x_{1,1}w_{1,3} + x_{2,2}w_{2,3} \\ x_{2,1}w_{1,1} + x_{2,2}w_{2,1} & x_{2,1}w_{1,2} + x_{2,2}w_{2,2} & x_{2,1}w_{1,3} + x_{2,2}w_{2,3} \end{pmatrix} \end{aligned}$$

- Assume derivative of loss L w.r.t. output Y, $\frac{\partial L}{\partial Y}$ $(N \times M)$, has already been computed:

$$\frac{\partial L}{\partial \mathbf{Y}} = \begin{pmatrix} \frac{\partial L}{\partial y_{1,1}} & \frac{\partial L}{\partial y_{1,2}} & \frac{\partial L}{\partial y_{1,3}} \\ \frac{\partial L}{\partial y_{2,1}} & \frac{\partial L}{\partial y_{2,2}} & \frac{\partial L}{\partial y_{2,3}} \end{pmatrix}$$

Setup and Forward Pass

- Goal during backward pass: use $\frac{\partial L}{\partial \mathbf{Y}}$ to compute $\frac{\partial L}{\partial \mathbf{X}}$ $(N \times D)$ and $\frac{\partial L}{\partial \mathbf{W}}$ $(D \times M)$
- Using the chain rule:

$$\frac{\partial L}{\partial \mathbf{X}} = \frac{\partial L}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{X}} \qquad \frac{\partial L}{\partial \mathbf{W}} = \frac{\partial L}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}}{\partial \mathbf{W}}$$

- The terms $\frac{\partial \mathbf{Y}}{\partial \mathbf{X}}$ and $\frac{\partial \mathbf{Y}}{\partial \mathbf{W}}$ are called *Jacobian matrices* (they are actually 4d-tensors here). What's their shape, e.g. for $\frac{\partial \mathbf{Y}}{\partial \mathbf{X}}$? $N \times M \times N \times D$.
- For a network with $N = 64$ and $M = D = 4096$, how much memory would we need if we want to store $\frac{\partial Y}{\partial X}$ in 32-bit floating point precision?
  $64 \cdot 4096 \cdot 64 \cdot 4096 \cdot 4 = 274,877,906,944$ bytes, which is roughly 256 GB $\rightarrow$ want to avoid having to store this!

One Element at a Time

- We can get around having to form the Jacobians by proceeding one element at a time. Let's work out the case for $\frac{\partial L}{\partial X}$:

$$\frac{\partial L}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\partial L}{\partial x_{1,1}} & \frac{\partial L}{\partial x_{1,2}} \\ \frac{\partial L}{\partial x_{2,1}} & \frac{\partial L}{\partial x_{2,2}} \end{pmatrix}$$

- One element at a time, let's first look at $\frac{\partial L}{\partial x_{1,1}}$. Using the chain rule, we compute it as:

$$\begin{aligned} \frac{\partial L}{\partial x_{1,1}} &= \mathrm{Tr}\left(\left[\frac{\partial L}{\partial \mathbf{Y}}\right]^{\intercal} \frac{\partial \mathbf{Y}}{\partial x_{1,1}}\right) = \frac{\partial L}{\partial y_{1,1}} \frac{\partial y_{1,1}}{\partial x_{1,1}} + \frac{\partial L}{\partial y_{1,2}} \frac{\partial y_{1,2}}{\partial x_{1,1}} + \frac{\partial L}{\partial y_{1,3}} \frac{\partial y_{1,3}}{\partial x_{1,1}} \\ &= \frac{\partial L}{\partial y_{1,1}} w_{1,1} + \frac{\partial L}{\partial y_{1,2}} w_{1,2} + \frac{\partial L}{\partial y_{1,3}} w_{1,3} \end{aligned}$$

One Element at a Time

$$\frac{\partial L}{\partial x_{1,1}} = \frac{\partial L}{\partial y_{1,1}} w_{1,1} + \frac{\partial L}{\partial y_{1,2}} w_{1,2} + \frac{\partial L}{\partial y_{1,3}} w_{1,3}$$
$$= \operatorname{Tr}\left(\begin{pmatrix} \frac{\partial L}{\partial y_{1,1}} & \frac{\partial L}{\partial y_{2,1}} \\ \frac{\partial L}{\partial y_{1,2}} & \frac{\partial L}{\partial y_{2,2}} \\ \frac{\partial L}{\partial y_{1,3}} & \frac{\partial L}{\partial y_{2,3}} \end{pmatrix} \begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,3} \\ 0 & 0 & 0 \end{pmatrix} \right)$$

- Doing this for the other elements of $\frac{\partial L}{\partial \mathbf{X}}$, the final matrix is:

$$\frac{\partial L}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\partial L}{\partial y_{1,1}} w_{1,1} + \frac{\partial L}{\partial y_{1,2}} w_{1,2} + \frac{\partial L}{\partial y_{1,3}} w_{1,3} & \frac{\partial L}{\partial y_{1,1}} w_{2,1} + \frac{\partial L}{\partial y_{1,2}} w_{2,2} + \frac{\partial L}{\partial y_{1,3}} w_{2,3} \\ \frac{\partial L}{\partial y_{2,1}} w_{1,1} + \frac{\partial L}{\partial y_{2,2}} w_{1,2} + \frac{\partial L}{\partial y_{2,3}} w_{1,3} & \frac{\partial L}{\partial y_{2,1}} w_{2,1} + \frac{\partial L}{\partial y_{2,2}} w_{2,2} + \frac{\partial L}{\partial y_{2,3}} w_{2,3} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial L}{\partial y_{1,1}} & \frac{\partial L}{\partial y_{1,2}} & \frac{\partial L}{\partial y_{1,3}} \\ \frac{\partial L}{\partial y_{2,1}} & \frac{\partial L}{\partial y_{2,2}} & \frac{\partial L}{\partial y_{2,3}} \end{pmatrix} \begin{pmatrix} w_{1,1} & w_{2,1} \\ w_{1,2} & w_{2,2} \\ w_{1,3} & w_{2,3} \end{pmatrix} = \boxed{\frac{\partial L}{\partial \mathbf{Y}} \mathbf{W}^{\intercal}}$$

One Element at a Time

- This holds for any values of N, D, and M
- Using the same strategy, one component at a time, for $\frac{\partial L}{\partial \mathbf{W}}$, we get:

$$\frac{\partial L}{\partial \mathbf{W}} = \boxed{\mathbf{X}^{\intercal} \frac{\partial L}{\partial \mathbf{Y}}}$$

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Full Matrix Form of Backpropagation Equations

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2-Layer MLP in Matrix Notation

Consider a two-layer MLP with D inputs, M hidden units, and K output units, where we calculate the forward pass for n data points in parallel, using matrix notation for everything.

Forward pass:

- hidden layer:
  - $\mathbf{Z}_0 = \mathbf{X}\mathbf{W}_0 + \mathbf{B}_0$
  - $\mathbf{H}_0 = g_0(\mathbf{Z}_0)$
- output layer:
  - $\mathbf{Z}_1 = \mathbf{H}_0\mathbf{W}_1 + \mathbf{B}_1$
  - $\hat{\mathbf{Y}} = g_1(\mathbf{Z}_1)$
- g_0 being a ReLU, and g_1 being a linear activation function, both applied element-wise
- dimensions: $\mathbf{X} : n \times D, \mathbf{W}_0 : D \times M, \mathbf{B}_0 : n \times M, \mathbf{Z}_0 : n \times M, \mathbf{H}_0 : n \times M, \mathbf{W}_1 : M \times K, \mathbf{B}_1 : n \times K, \mathbf{Z}_1 : n \times K, \hat{\mathbf{Y}} : n \times K$

Mean squared error loss: $L = \frac{1}{2n} \sum_{i,j} \left(\hat{\mathbf{Y}}_{ij} - \mathbf{Y}_{ij} \right)^2$

2-Layer MLP in Matrix Notation (cont.)

Backward pass starts with calculating the error gradient:

$$\frac{\partial L}{\partial \hat{\mathbf{Y}}} = \frac{1}{n}(\hat{\mathbf{Y}} - \mathbf{Y})$$

Gradients for the output layer:

$$\begin{aligned}
\frac{\partial L}{\partial \mathbf{Z}_1} &= \frac{\partial L}{\partial \hat{\mathbf{Y}}} \frac{\partial \hat{\mathbf{Y}}}{\partial \mathbf{Z}_1} = \frac{\partial L}{\partial \hat{\mathbf{Y}}} \odot g_1'(\mathbf{Z}_1) = \frac{1}{n}(\hat{\mathbf{Y}} - \mathbf{Y}) \odot \mathbf{1}_{n\times K} = \frac{1}{n}(\hat{\mathbf{Y}} - \mathbf{Y}) \\
\frac{\partial L}{\partial \mathbf{W}_1} &= \frac{\partial L}{\partial \mathbf{Z}_1} \frac{\partial \mathbf{Z}_1}{\partial \mathbf{W}_1} = \mathbf{H}_0^\intercal \frac{\partial L}{\partial \mathbf{Z}_1} \\
\frac{\partial L}{\partial \mathbf{B}_1} &= \frac{\partial L}{\partial \mathbf{Z}_1} \frac{\partial \mathbf{Z}_1}{\partial \mathbf{B}_1} = \mathbf{1}_{n\times n} \frac{\partial L}{\partial \mathbf{Z}_1}
\end{aligned}$$

To calculate $\frac{\partial L}{\partial \mathbf{W}_1}$ we need the activations $\mathbf{H}_0$ we calculated in the forward pass. The Hadamard product $a \odot b$ denotes element-wise multiplication of two matrices. $\mathbf{1}_{n\times n}$ is a $n \times n$ matrix where every element is equal to 1.

2-Layer MLP in Matrix Notation (cont.)

Gradients for the hidden layer:

$$
\begin{aligned}
\frac{\partial L}{\partial \mathbf{H}_0} &= \frac{\partial L}{\partial \mathbf{Z}_1}\frac{\partial \mathbf{Z}_1}{\partial \mathbf{H}_0} = \frac{\partial L}{\partial \mathbf{Z}_1}\mathbf{W}_1^\intercal \\
\frac{\partial L}{\partial \mathbf{Z}_0} &= \frac{\partial L}{\partial \mathbf{H}_0}\frac{\partial \mathbf{H}_0}{\partial \mathbf{Z}_0} \\
&= \frac{\partial L}{\partial \mathbf{H}_0} \odot g_0'(\mathbf{Z}_0) = \left(z_{ij}'\right) \text{ with } z_{ij}' = \begin{cases} \left(\frac{\partial L}{\partial \mathbf{H}_0}\right)_{ij} & \text{if } z_{ij} > 0 \\ 0 & \text{if } z_{ij} \leq 0 \end{cases} \\
\frac{\partial L}{\partial \mathbf{W}_0} &= \frac{\partial L}{\partial \mathbf{Z}_0}\frac{\partial \mathbf{Z}_0}{\partial \mathbf{W}_0} = X^\intercal \frac{\partial L}{\partial \mathbf{Z}_0} \\
\frac{\partial L}{\partial \mathbf{B}_0} &= \frac{\partial L}{\partial \mathbf{Z}_0}\frac{\partial \mathbf{Z}_0}{\partial \mathbf{B}_0} = \mathbf{1}_{n\times n}\frac{\partial L}{\partial \mathbf{Z}_0}
\end{aligned}
$$

Questions to Answer for Yourself / Discuss with Friends

- Transfer: Can you write the matrix notation for a 2-layer MLP with logistic sigmoid activation functions for both g_0 and g_1?

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6. Further Reading, Summary of the Week, References

Foundations of Deep Learning, Winter Term 2021/22

Week 3: MLPs and Backpropagation

Further Reading, Summary of the Week, References

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Summary by Learning Goals

Having heard this week's lectures, you can now ...

- explain how to calculate partial derivatives for MLP weights using backpropagation
- use matrix form of backpropagation to calculate gradients for mini-batches

Further Reading

- Resources for this lecture: Deep Learning Book [Goodfellow et al., 2016, chapter 6]
- Stanford Course on Deep Learning [Johnson, 2017]

References

Goodfellow, I., Bengio, Y., Courville, A. (2016)
Deep Learning
MIT Press.
https://www.deeplearningbook.org/

Johnson, J. (2017)
Backpropagation for a Linear Layer
Stanford University CS231n
PDF 2017: cs231n.stanford.edu/handouts/linear-backprop.pdf
Course website: http://cs231n.stanford.edu/
Course notes: https://cs231n.github.io/
Lecture 2: https://cs231n.github.io/optimization-2