Knowledge Representation and Reasoning

Exercise Sheet 8

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17th December 2021

Exercise 8.1 (a)

Expressiveness of $\mathcal{ALCF}$

Exercise 8.1 (EXPRESSIBILITY AND COMPLEXITY, 3 + 3)

(a) You are asked to show that concept constructors are not just syntactic sugar. In particular, if one adds the functionality concept constructor — $(\leq 1r)$ — to $\mathcal{ALC}$ one gets the language $\mathcal{ALCF}$. Show that $\mathcal{ALCF}$ is indeed more expressive than $\mathcal{ALC}$ by proving that the concept $(\leq 1r)$ cannot be expressed in $\mathcal{ALC}$.
Hint: Assume that there is an $\mathcal{ALC}$ concept C which is equivalent to $(\leq 1r)$. Provide an appropriate interpretation $\mathcal{I}$ which satisfies $(\leq 1r)$ and by assumption also C, i.e., $(\leq 1r)^{\mathcal{I}} = C^{\mathcal{I}} \neq \emptyset$. Then construct another interpretation $\mathcal{I}'$ from $\mathcal{I}$ as follows:

$$\begin{aligned}
\mathcal{D}^{\mathcal{I}'} &= \mathcal{D}^{\mathcal{I}} \times \mathbb{N} & \\
A^{\mathcal{I}'} &= \{(d,i) \mid d \in A^{\mathcal{I}}, i \in \mathbb{N}\} & \text{for all primitive concepts } A \\
s^{\mathcal{I}'} &= \{((d,i),(e,j)) \mid (d,e) \in s^{\mathcal{I}}, i,j \in \mathbb{N}\} & \text{for all primitive roles } s
\end{aligned}$$

For this new interpretation, show that $C^{\mathcal{I}'} \neq (\leq 1r)^{\mathcal{I}'}$, contradicting our assumption.

We consider the following interpretation $\mathcal{I}$:

$$\mathcal{D}^{\mathcal{I}} = \{a\} \qquad r^{\mathcal{I}} = \{(a,a)\} \qquad (\leq 1r)^{\mathcal{I}} = \{a\}$$

Assuming C is an $\mathcal{ALC}$ concept expressing $(\leq 1r)$, we also have $C^{\mathcal{I}} = \{a\}$.

Exercise 8.1 (a)

Expressiveness of $\mathcal{ALCF}$

Our interpretation $\mathcal{I}$ was given as:

$$\mathcal{D}^{\mathcal{I}} = \{a\} \qquad r^{\mathcal{I}} = \{(a,a)\} \qquad (\leq 1 r)^{\mathcal{I}} = \{a\} \qquad C^{\mathcal{I}} = \{a\}$$

We now construct the interpretation $\mathcal{I}'$ as follows:

$$\mathcal{D}^{\mathcal{I}'} = \mathcal{D}^{\mathcal{I}} \times \mathbb{N} = \{(a,1),(a,2),\dots\}$$

$$r^{\mathcal{I}'} = \{((d,i),(e,j)) \mid (d,e) \in r^{\mathcal{I}}, i,j \in \mathbb{N}\} = \{\langle (a,1),(a,1)\rangle, \langle (a,1),(a,2)\rangle, \dots, \langle (a,2),(a,1)\rangle, \langle (a,2),(a,2)\rangle, \dots, \dots\}$$

$$(\leq 1 r)^{\mathcal{I}'} = \emptyset$$

We will now show the following contradiction to our assumption:

$$C^{\mathcal{I}'} = \{(d,i) \mid d \in C^{\mathcal{I}}, i \in \mathbb{N}\} = \{(a,1),(a,2),\dots\} \neq \emptyset = (\leq 1 r)^{\mathcal{I}'}$$

For primitive concepts, $C^{\mathcal{I}'} = \{(d,i) \mid d \in C^{\mathcal{I}}, i \in \mathbb{N}\}$ holds by definition.

We show that it holds for arbitrary $\mathcal{ALC}$ concepts C.

Exercise 8.1 (a)

Expressiveness of $\mathcal{ALCF}$

Let the interpretation $\mathcal{I}'$ be defined as follows:

$$\mathcal{D}^{\mathcal{I}'} = \mathcal{D}^{\mathcal{I}} \times \mathbb{N}$$

$$A^{\mathcal{I}'} = \{(d,i) \mid d \in A^{\mathcal{I}}, i \in \mathbb{N}\} \qquad \text{for all primitive concepts } A$$

$$s^{\mathcal{I}'} = \{((d,i),(e,j)) \mid (d,e) \in s^{\mathcal{I}}, i,j \in \mathbb{N}\} \qquad \text{for all primitive roles } s$$

We show via structural induction that for an arbitrary $\mathcal{ALC}$ concept C also:

$$C^{\mathcal{I}'} = \{(d,i) \mid d \in C^{\mathcal{I}}, i \in \mathbb{N}\}$$

Base case: For primitive C, the identity holds by definition.

Assuming the identity already holds for two concepts A and B, we have:

$$\begin{aligned}
(A \sqcap B)^{\mathcal{I}'} &= A^{\mathcal{I}'} \cap B^{\mathcal{I}'} = \{(d,i) \mid d \in A^{\mathcal{I}}, i \in \mathbb{N}\} \cap \{(d,i) \mid d \in B^{\mathcal{I}}, i \in \mathbb{N}\} \\
&= \{(d,i) \mid d \in (A^{\mathcal{I}} \cap B^{\mathcal{I}}), i \in \mathbb{N}\} = \{(d,i) \mid d \in (A \sqcap B)^{\mathcal{I}}, i \in \mathbb{N}\} \\
(\neg A)^{\mathcal{I}'} &= \mathcal{D}^{\mathcal{I}'} \setminus A^{\mathcal{I}'} = (\mathcal{D}^{\mathcal{I}} \times \mathbb{N}) \setminus \{(d,i) \mid d \in A^{\mathcal{I}}, i \in \mathbb{N}\} \\
&= \{(d,i) \mid d \in (D^{\mathcal{I}} \setminus A^{\mathcal{I}}), i \in \mathbb{N}\} = \{(d,i) \mid d \in (\neg A)^{\mathcal{I}}, i \in \mathbb{N}\}
\end{aligned}$$

Exercise 8.1 (a)

Expressiveness of $\mathcal{ALCF}$

Let the interpretation $\mathcal{I}'$ be defined as follows:

$$\mathcal{D}^{\mathcal{I}'} = \mathcal{D}^{\mathcal{I}} \times \mathbb{N}$$

$$A^{\mathcal{I}'} = \{(d,i) \mid d \in A^{\mathcal{I}}, i \in \mathbb{N}\} \qquad \text{for all primitive concepts } A$$

$$s^{\mathcal{I}'} = \{((d,i),(e,j)) \mid (d,e) \in s^{\mathcal{I}}, i,j \in \mathbb{N}\} \qquad \text{for all primitive roles } s$$

We show via structural induction that for an arbitrary $\mathcal{ALC}$ concept C also:

$$C^{\mathcal{I}'} = \{(d,i) \mid d \in C^{\mathcal{I}}, i \in \mathbb{N}\}$$

Base case: For primitive C, the identity holds by definition.

Assuming the identity already holds for two concepts A and B, we have:

$$(\forall r.A)^{\mathcal{I}'} = \{d \in \mathcal{D}^{\mathcal{I}'} : r^{\mathcal{I}'}(d) \subseteq A^{\mathcal{I}'}\} = \{d \in (\mathcal{D}^{\mathcal{I}} \times \mathbb{N}) : r^{\mathcal{I}'}(d) \subseteq \{(e,i) \mid e \in A^{\mathcal{I}}, i \in \mathbb{N}\}\}$$
$$= \{d \in \mathcal{D}^{\mathcal{I}} : r^{\mathcal{I}}(d) \subseteq A^{\mathcal{I}}\} \times \mathbb{N} = \{(d,i) \mid d \in (\forall r.A)^{\mathcal{I}}, i \in \mathbb{N}\}$$

We can ignore the cases $A \sqcup B$ and $\exists r.A$ since they are equivalent to the already proven cases $\neg(\neg A \sqcap \neg B)$ and $\neg\forall r.\neg B$. $\square$

(b) Given the TBox $\mathcal{T}$, determine the description logic used to describe the concepts. What is the complexity class of the satisfiability problem of this logic? Propose how $\mathcal{T}$ could be expressed in a less complex DL.

- $\mathcal{T} = \{A \doteq \exists r.(\forall s.C \sqcup \exists s.\neg C), B \doteq (\geq 1s), C \doteq B \sqcap \forall r.\neg(\neg A \sqcup \neg B)\}$

We have a cyclic TBox and number restrictions.
⇒ Logic $\mathcal{ALCN}$, concept satisfiability EXPTIME-complete.

We can remove cycles by transforming $A \doteq \exists r.(\forall s.C \sqcup \exists s.\neg C)$ to $A \doteq \exists r$.
⇒ Logic $\mathcal{ALCN}$, concept satisfiability PSPACE-complete.

We can remove number restrictions by transforming $B \doteq (\geq 1s)$ to $B \doteq \exists s$.
⇒ Logic $\mathcal{ALC}$, concept satisfiability PSPACE-complete.

We can finally remove negations, disjunctions in C, obtaining the TBox

$$\mathcal{T}' = \{A \doteq \exists r, B \doteq \exists s, C \doteq B \sqcap \forall r.(A \sqcap B)\}$$

The resulting logic is $\mathcal{FL}^-$, concept satisfiability is trivial.

Complexity Navigator: `http://www.cs.man.ac.uk/~ezolin/dl/`.

Exercise 8.2 (Divine Reasoning, $2 + 3 + 1$)

Consider the following ontology, describing a fraction of the ancient Greek pantheon:

$$\begin{aligned}
\texttt{Immortal} &\sqsubseteq \texttt{Living_Entity} \\
\texttt{Titan} &\sqsubseteq \texttt{Immortal} \\
\texttt{Olympian} &\doteq \texttt{Immortal} \sqcap \neg\texttt{Titan} \sqcap \\
&\quad\ \exists\texttt{has-parent}.\texttt{Titan} \\
\texttt{Demigod} &\doteq \texttt{Living_Entity} \sqcap \\
&\quad\ \exists\texttt{has-parent}.\texttt{Olympian} \sqcap \\
&\quad\ \exists\texttt{has-parent}.\neg\texttt{Immortal}
\end{aligned}$$

RHEA:	Titan
ZEUS:	Immortal
ZEUS:	¬Titan
DANAE:	¬Immortal
THESEUS:	Living_Entity
(ZEUS, RHEA):	has-parent
(THESEUS, ZEUS):	has-parent
(THESEUS, DANAE):	has-parent

(a) Based on the given ABox, check whether any of the individuals `RHEA`, `ZEUS`, `DANAE` and `THESEUS` are instances of the concepts `Olympian`, ¬`Olympian`, `Demigod` or ¬`Demigod`.

(b) Using the tableau algorithm, show that it is generally possible for a single individual to be an `Olympian` as well as a `Demigod`.

(c) Construct a new axiom to add to the TBox which makes it impossible to simultaneously be an `Olympian` and a `Demigod`. You should not modify any of the existing axioms or introduce any new roles or concepts.

Exercise 8.2 (a)

Divine Reasoning – Instances of Olympian and Demigod

$$
\begin{aligned}
\texttt{Immortal} &\sqsubseteq \texttt{Living\_Entity} \\
\texttt{Titan} &\sqsubseteq \texttt{Immortal} \\
\texttt{Olympian} &\doteq \texttt{Immortal} \sqcap \neg\texttt{Titan} \sqcap \\
&\quad \exists\texttt{has-parent.Titan} \\
\texttt{Demigod} &\doteq \texttt{Living\_Entity} \sqcap \\
&\quad \exists\texttt{has-parent.Olympian} \sqcap \\
&\quad \exists\texttt{has-parent.}\neg\texttt{Immortal}
\end{aligned}
$$

RHEA:	Titan
ZEUS:	Immortal
ZEUS:	¬Titan
DANAE:	¬Immortal
THESEUS:	Living_Entity
(ZEUS, RHEA):	has-parent
(THESEUS, ZEUS):	has-parent
(THESEUS, DANAE):	has-parent

We can make the following inferences:

RHEA: ¬Olympian (bc. RHEA: Titan)

ZEUS: ∃has-parent.Titan (bc. RHEA: Titan)
ZEUS: Olympian (bc. ZEUS: Immortal, ZEUS: ¬Titan and ZEUS: ∃has-parent.Titan)

DANAE: ¬Olympian (bc. DANAE: ¬Immortal)

THESEUS: ∃has-parent.Olympian (bc. ZEUS: Olympian)
THESEUS: ∃has-parent.¬Immortal (bc. DANAE: ¬Immortal)
THESEUS: Demigod (bc. THESEUS: Living_Entity, THESEUS: ∃has-parent.Olympian and THESEUS: ∃has-parent.¬Immortal)

Exercise 8.2 (b)

Divine Reasoning – Olympian and Demigod at the same time? (Normalization)

We start by normalization and unfolding (and abbreviation for the sake of readability):

$\mathtt{Im} \doteq \mathtt{LE} \sqcap \mathtt{Im}^*$

$\mathtt{Ti} \doteq \mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*$

$\mathtt{Ol} \doteq \mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*) \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)$

$\mathtt{De} \doteq \mathtt{LE} \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*) \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)) \sqcap \exists \mathrm{hp}.\neg(\mathtt{LE} \sqcap \mathtt{Im}^*)$

Next we transform the concept descriptions in question into NNF:

$\mathtt{Ol} \doteq \mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)$

$\mathtt{De} \doteq \mathtt{LE} \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)) \sqcap \exists \mathrm{hp}.(\neg\mathtt{LE} \sqcup \neg\mathtt{Im}^*)$

Note: We applied distributivity to transform $\mathit{LE} \sqcap \mathit{Im}^* \sqcap (\neg\mathit{LE} \sqcup \neg\mathit{Im}^* \sqcup \neg\mathit{Ti}^*) \equiv \mathit{LE} \sqcap \mathit{Im}^* \sqcap \neg\mathit{Ti}^*$*. While this property was not introduced in the lecture, it can easily be proven via set theory.*

In conjunction we get:

$$\mathtt{Ol} \sqcap \mathtt{De} \equiv \mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*) \sqcap$$
$$\exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathrm{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)) \sqcap \exists \mathrm{hp}.(\neg\mathtt{LE} \sqcup \neg\mathtt{Im}^*)$$

Exercise 8.2 (b)

Divine Reasoning – `Olympian` and `Demigod` at the same time? (Tableau)

$x : \mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*) \sqcap \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)) \sqcap \exists \mathtt{hp}.(\neg\mathtt{LE} \sqcup \neg\mathtt{Im}^*)$

$x : \mathtt{LE}, x : \mathtt{Im}^*, x : \neg\mathtt{Ti}^*$

$x : \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)$

$x : \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \neg\mathtt{Ti}^* \sqcap \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*))$

$x : \exists \mathtt{hp}.(\neg\mathtt{LE} \sqcup \neg\mathtt{Im}^*)$

$x\ hp\ y$

$y : \mathtt{LE}, y : \mathtt{Im}^*, y : \mathtt{Ti}^*$

$x\ hp\ z$

$z : \mathtt{LE}, z : \mathtt{Im}^*, z : \neg\mathtt{Ti}^*$

$z : \exists \mathtt{hp}.(\mathtt{LE} \sqcap \mathtt{Im}^* \sqcap \mathtt{Ti}^*)$

$x\ hp\ u$

$u : \neg\mathtt{LE} \sqcup \neg\mathtt{Im}^*$

$z\ hp\ v$

$v : \mathtt{LE}, v : \mathtt{Im}^*, v : \mathtt{Ti}^*$

$u : \neg LE$	$u : \neg\mathtt{Im}^*$
$\times$	$\times$

Both branches are closed (one would have been enough).

Exercise 8.2 (b)

Divine Reasoning – Olympian and Demigod at the same time? (Model)

From the Tableau we can construct the following model $\mathcal{I}$ with $\mathcal{D} = \{x, y, z, u, v\}$:

$$\begin{aligned}
\texttt{Living_Entity}^{\mathcal{I}} &= \{x, y, z, v\} \\
\texttt{Immortal}^{*\mathcal{I}} &= \{x, y, z, v\} \\
\texttt{Titan}^{*\mathcal{I}} &= \{y, v\} \\
\texttt{has-parent}^{\mathcal{I}} &= \{(x, y), (x, z), (x, u), (z, v)\}
\end{aligned}$$

$$\begin{aligned}
\texttt{Immortal}^{\mathcal{I}} &= \{x, y, z, v\} \\
\texttt{Titan}^{\mathcal{I}} &= \{y, v\} \\
\texttt{Olympian}^{\mathcal{I}} &= \{x, z\} \\
\texttt{Demigod}^{\mathcal{I}} &= \{x\}
\end{aligned}$$

In this model, x is an Olympian as well as a Demigod.

Exercise 8.2 (c)

Christmas Reasoning – Prohibiting Olympian Demigods

Looking at the model constructed in (b), we see that the individual *x* needed three parents to satisfy all constraints: One parent has to be an Olympian, one has to be a Titan (which cannot be the same as the Olympian, since Olympians cannot be Titans) and one has to be mortal (which disqualifies both the Olympian and the Titan).

Idea: A reasonable additional restriction might be limiting living enitites to at most two parents:

$$\texttt{Living\_Entity} \sqsubseteq \leq 2\texttt{has-parent}$$

As an additional exercise, you could prove that this is indeed the case by repeating the tableau construction with this addition to the TBox – which should now lead to clashes. To deal with the cardinality restriction you will need the $\mathcal{ALCQ}$-tableau rules from exercise 7.