Knowledge Representation and Reasoning

Exercise Sheet 4

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel, Gregor Behnke, Thorsten Engesser, Rolf-David Bergdoll, Leonardo Mieschendahl, Johannes Herrmann

19th November 2021

Exercise 4.1 – Abduction Example

Exercise 4.1 (Abduction Example, 2+2+2)
Let a propositional background theory T be given as follows:

$$\begin{aligned} T = \{&\neg(\mathit{cloudless\text{-}last\text{-}night} \wedge \mathit{rained\text{-}last\text{-}night}),\\ &\mathit{rained\text{-}last\text{-}night} \rightarrow \mathit{grass\text{-}is\text{-}wet},\\ &\mathit{rained\text{-}last\text{-}night} \leftrightarrow \neg \mathit{road\text{-}is\text{-}dry},\\ &\mathit{sprinkler\text{-}was\text{-}on} \rightarrow \mathit{grass\text{-}is\text{-}wet},\\ &\mathit{grass\text{-}is\text{-}wet} \rightarrow \mathit{grass\text{-}is\text{-}shiny\text{-}and\text{-}cold},\\ &\mathit{grass\text{-}is\text{-}wet} \rightarrow \mathit{shoes\text{-}are\text{-}wet},\\ &\mathit{shoes\text{-}have\text{-}been\text{-}washed} \rightarrow \mathit{shoes\text{-}are\text{-}wet}\} \end{aligned}$$

Furthermore, let $O = \{\mathit{shoes\text{-}are\text{-}wet}, \mathit{road\text{-}is\text{-}dry}, \mathit{grass\text{-}is\text{-}shiny\text{-}and\text{-}cold}\}$ be a set of observations and $H = \{\mathit{cloudless\text{-}last\text{-}night}, \mathit{rained\text{-}last\text{-}night}, \mathit{sprinkler\text{-}was\text{-}on}, \mathit{shoes\text{-}have\text{-}been\text{-}washed}\}$ be a set of hypotheses.

(a) Decide the abduction existence problem for the instance given by T, O, and H. Provide at least one explanation E or prove that there exist none.

(b) Which of the hypotheses in H are relevant, given T, O and H?

(c) Which of the hypotheses in H are necessary, given T, O and H?

Exercise 4.1 (a)

Decide the abduction existence problem for (T, O, H)

The **abduction existence** problem, as defined in the lecture:

Instance: Given a **propositional background theory** T, a set of propositional atoms called **observations** O, and a set of propositional atoms called **hypotheses** H.

Question: Is there an explanation E, i.e. a subset $E \subseteq H$ such that $T \cup E$ is satisfiable and $T \cup E \models \bigwedge O$?

$$\begin{aligned}
T = \{&\neg(\textit{cloudless-last-night} \wedge \textit{rained-last-night}),\\
&\textit{rained-last-night} \rightarrow \textit{grass-is-wet},\\
&\textit{rained-last-night} \leftrightarrow \neg\textit{road-is-dry},\\
&\textit{sprinkler-was-on} \rightarrow \textit{grass-is-wet},\\
&\textit{grass-is-wet} \rightarrow \textit{grass-is-shiny-and-cold},\\
&\textit{grass-is-wet} \rightarrow \textit{shoes-are-wet},\\
&\textit{shoes-have-been-washed} \rightarrow \textit{shoes-are-wet}\}
\end{aligned}$$

$$\begin{aligned}
O = \{&\textit{shoes-are-wet},\\
&\textit{road-is-dry},\\
&\textit{grass-is-shiny-and-cold}\}\\
H = \{&\textit{cloudless-last-night},\\
&\textit{rained-last-night},\\
&\textit{sprinkler-was-on},\\
&\textit{shoes-have-been-washed}\}
\end{aligned}$$

Exercise 4.1 (a)

Decide the abduction existence problem for (T, O, H) – Explanation

Consider explanation $E = \{\mathit{cloudless\text{-}last\text{-}night}, \mathit{sprinkler\text{-}was\text{-}on}\}$

$$\begin{aligned}
T = \{&\neg(\mathit{cloudless\text{-}last\text{-}night} \land \mathit{rained\text{-}last\text{-}night}),\\
&\mathit{rained\text{-}last\text{-}night} \rightarrow \mathit{grass\text{-}is\text{-}wet},\\
&\mathit{rained\text{-}last\text{-}night} \leftrightarrow \neg \mathit{road\text{-}is\text{-}dry},\\
&\mathit{sprinkler\text{-}was\text{-}on} \rightarrow \mathit{grass\text{-}is\text{-}wet},\\
&\mathit{grass\text{-}is\text{-}wet} \rightarrow \mathit{grass\text{-}is\text{-}shiny\text{-}and\text{-}cold},\\
&\mathit{grass\text{-}is\text{-}wet} \rightarrow \mathit{shoes\text{-}are\text{-}wet},\\
&\mathit{shoes\text{-}have\text{-}been\text{-}washed} \rightarrow \mathit{shoes\text{-}are\text{-}wet}\}
\end{aligned}$$

$$\begin{aligned}
O = \{&\mathit{shoes\text{-}are\text{-}wet},\\
&\mathit{road\text{-}is\text{-}dry},\\
&\mathit{grass\text{-}is\text{-}shiny\text{-}and\text{-}cold}\}\\
H = \{&\mathit{cloudless\text{-}last\text{-}night},\\
&\mathit{rained\text{-}last\text{-}night},\\
&\mathit{sprinkler\text{-}was\text{-}on},\\
&\mathit{shoes\text{-}have\text{-}been\text{-}washed}\}
\end{aligned}$$

Exercise 4.1 (a)

Decide the abduction existence problem for (T, O, H) – Explanation

Consider explanation $E = \{\textit{cloudless-last-night}, \textit{sprinkler-was-on}\}$

$$
\begin{aligned}
T = \{&\neg(\textit{cloudless-last-night} \land \textit{rained-last-night}),\\
&\textit{rained-last-night} \rightarrow \textit{grass-is-wet},\\
&\textit{rained-last-night} \leftrightarrow \neg\textit{road-is-dry},\\
&\textit{sprinkler-was-on} \rightarrow \textit{grass-is-wet},\\
&\textit{grass-is-wet} \rightarrow \textit{grass-is-shiny-and-cold},\\
&\textit{grass-is-wet} \rightarrow \textit{shoes-are-wet},\\
&\textit{shoes-have-been-washed} \rightarrow \textit{shoes-are-wet}\}
\end{aligned}
$$

$$
\begin{aligned}
O = \{&\textit{shoes-are-wet},\\
&\textit{road-is-dry},\\
&\textit{grass-is-shiny-and-cold}\}\\
H = \{&\textit{cloudless-last-night},\\
&\textit{rained-last-night},\\
&\textit{sprinkler-was-on},\\
&\textit{shoes-have-been-washed}\}
\end{aligned}
$$

- *cloudless-last-night* implies $\neg$*rained-last-night*, which in turn entails *road-is-dry*.

Exercise 4.1 (a)

Decide the abduction existence problem for (T, O, H) – Explanation

Consider explanation $E = \{\textit{cloudless-last-night}, \textit{sprinkler-was-on}\}$

$$
\begin{aligned}
T = \{&\neg(\textit{cloudless-last-night} \land \textit{rained-last-night}),\\
&\textit{rained-last-night} \rightarrow \textit{grass-is-wet},\\
&\textit{rained-last-night} \leftrightarrow \neg\textit{road-is-dry},\\
&\textit{sprinkler-was-on} \rightarrow \textit{grass-is-wet},\\
&\textit{grass-is-wet} \rightarrow \textit{grass-is-shiny-and-cold},\\
&\textit{grass-is-wet} \rightarrow \textit{shoes-are-wet},\\
&\textit{shoes-have-been-washed} \rightarrow \textit{shoes-are-wet}\}
\end{aligned}
$$

$$
\begin{aligned}
O = \{&\textit{shoes-are-wet},\\
&\textit{road-is-dry},\\
&\textit{grass-is-shiny-and-cold}\}\\
H = \{&\textit{cloudless-last-night},\\
&\textit{rained-last-night},\\
&\textit{sprinkler-was-on},\\
&\textit{shoes-have-been-washed}\}
\end{aligned}
$$

- *cloudless-last-night* implies $\neg$*rained-last-night*, which in turn entails *road-is-dry*.
- *sprinkler-was-on* implies *grass-is-wet*, which entails *shoes-are-wet* and *grass-is-shiny-and-cold*.

Exercise 4.1 (a)

Decide the abduction existence problem for (T, O, H) – Explanation

Consider explanation $E = \{\textit{cloudless-last-night}, \textit{sprinkler-was-on}\}$

$$
\begin{aligned}
T = \{&\neg(\textit{cloudless-last-night} \land \textit{rained-last-night}),\\
&\textit{rained-last-night} \rightarrow \textit{grass-is-wet},\\
&\textit{rained-last-night} \leftrightarrow \neg\textit{road-is-dry},\\
&\textit{sprinkler-was-on} \rightarrow \textit{grass-is-wet},\\
&\textit{grass-is-wet} \rightarrow \textit{grass-is-shiny-and-cold},\\
&\textit{grass-is-wet} \rightarrow \textit{shoes-are-wet},\\
&\textit{shoes-have-been-washed} \rightarrow \textit{shoes-are-wet}\}
\end{aligned}
$$

$$
\begin{aligned}
O = \{&\textit{shoes-are-wet},\\
&\textit{road-is-dry},\\
&\textit{grass-is-shiny-and-cold}\}\\
H = \{&\textit{cloudless-last-night},\\
&\textit{rained-last-night},\\
&\textit{sprinkler-was-on},\\
&\textit{shoes-have-been-washed}\}
\end{aligned}
$$

- *cloudless-last-night* implies ¬*rained-last-night*, which in turn entails *road-is-dry*.
- *sprinkler-was-on* implies *grass-is-wet*, which entails *shoes-are-wet* and *grass-is-shiny-and-cold*.
- $T \cup E$ is satisfiable, e.g. set *rained-last-night* $\mapsto$ **F** and all other variables **T**.

Exercise 4.1 (b)

Which hypotheses are relevant?

A hypothesis is relevant iff it is contained in at least one explanation.

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A hypothesis is relevant iff it is contained in at least one explanation.

- As shown in (a), $E = \{\textit{cloudless-last-night}, \textit{sprinkler-was-on}\}$ is an explanation, thus *cloudless-last-night* and *sprinkler-was-on* are relevant.
- Because of $\textit{rained-last-night} \leftrightarrow \neg\textit{road-is-dry}$, the hypothesis *rained-last-night* conflicts with the observation *road-is-dry*. Therefore *rained-last-night* cannot be part of any explanation and is therefore not relevant.
- The hypotheses *cloudless-last-night* and *sprinkler-was-on* validify the theory and explain all observation without imposing a truth value for *shoes-have-been-washed*. Consequently, $E' = \{\textit{cloudless-last-night}, \textit{sprinkler-was-on}, \textit{shoes-have-been-washed}\}$ is also an acceptable explanation, which entails that *shoes-have-been-washed* is relevant.

The relevant hypotheses are *cloudless-last-night*, *sprinkler-was-on* and *shoes-have-been-washed*.

Exercise 4.1 (c)

Which hypotheses are necessary?

A hypothesis is necessary iff it is contained in all explanations.

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Which hypotheses are necessary?

A hypothesis is necessary iff it is contained in all explanations.

- Since we know of an explanation that only contains *cloudless-last-night* and *sprinkler-was-on*, we only have to consider those hypotheses.
- The observation *road-is-dry* can only be explained by ¬*rained-last-night*, which in turn can only be deduced by the hypothesis *cloudless-last-night*. Thus, *cloudless-last-night* has to be included in every explanation and is therefore necessary.
- The observation *grass-is-shiny-and-cold* can only be explained by *grass-is-wet*. *grass-is-wet* is implied by *sprinkler-was-on* but also by *rained-last-night*. However, we already concluded that *rained-last-night* cannot hold true, because of the observation *road-is-dry*. Hence, *sprinkler-was-on* is necessary, as it is the only possible explanation for *grass-is-shiny-and-cold* in conjunction with *road-is-dry*.

The necessary hypotheses are *cloudless-last-night* and *sprinkler-was-on*.

Exercise 4.2 – Abduction Complexity

Proof of Σ_2^P-completeness for abduction relevance

Instance: Given a *propositional background theory* T, a set of propositional atoms called *observations* O, a set of propositional atoms called *hypotheses* H, and a hypothesis $h \in H$.

Question: Is h a *relevant* hypothesis, i.e, is there an explanation E (meaning a subset $E \subseteq H$ such that $T \cup E$ is satisfiable and $T \cup E \models \bigwedge O$) such that $h \in E$?

Exercise 4.2 (Abduction Complexity, 6)

Prove that the abduction relevance problem is Σ_2^P-complete. *Hint:* To show Σ_2^P-hardness, try to modify the Σ_2^P-hardness proof of the abduction existence problem.

Exercise 4.2

Proof of Σ_2^P-completeness for abduction relevance – Membership in Σ_2^P

Membership in Σ_2^P is again straight forward:

Guess an explanation $E \subseteq H$ which includes h and use a SAT-oracle to verify the that $T \cup E$ satisfiable and that $\bigwedge T \wedge \bigwedge E \rightarrow \bigwedge O$ is valid.

For Σ_2^P-hardness we slightly modify the reduction from $\mathrm{QBF}_{2,\exists}$ shown in the lecture...

Exercise 4.2

Proof of Σ_2^P-completeness for abduction relevance – Σ_2^P-hardness

Given QBF $\exists x_1 \dots \exists x_n \forall y_1 \dots \forall y_m \varphi$, we again introduce the atoms x_i' for each x_i and the observation atom s. We now also add an additional atom h and add it only to the hypotheses:

$T = \{x_i \leftrightarrow \neg x_i' \mid 1 \leq i \leq n\} \cup \{\varphi \rightarrow s\}$
$H = \{x_i, \dots, x_n\} \cup \{x_i', \dots, x_n'\} \cup \{h\}$
$O = \{s\}$

Here, h is relevant iff the QBF is true:

- As for the original proof, if the QBF is true, we can construct an explanation E from the corresponding truth assignment. Since h does not appear in the theory, we can simply add it to E to get another acceptable explanation $E' = E \cup \{h\}$. Thus, h is relevant.
- Conversely, assume that h is relevant, which means that there exists an explanation E that contains h. From this explanation we can extract the truth assignment for the QBF as in the original proof, simply ignoring the h.

$\rightsquigarrow$ Abduction relevance is Σ_2^P-complete.