MT Coursera Week 04

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§1 Working With Quantifiers 1

§1.1 Negating statements that have quantifiers

A positive statement is one that contains no negation symbols, or one with the negation symbols far inside the function.

Example 1.1 (Positive statement)

Let $A(x)$ be some property of x. For example, x is a real root of the equation $x^2 + 2x + 1 = 0$.

We shall show that $\not[\forall x A(x)]$ is equivalent to $\exists x\,[\neg A(x)]$.

Assume $\neg\,[\forall x A(x)]$. If it is not the case that for all x, $A(x)$, then at least one x must fail to satisfy $A(x)$. So, for at least one x, $\neg A(x)$ is true. In symbols, $\exists x\,[\neg A(x)]$.

Then, assume $\exists x\,[\neg A(x)]$. Then there is an x for which $A(x)$ is false. Then $A(x)$ cannot be true for all x. In other words, $\forall x A(x)$ must be false, or $\neg\,[\forall x A(x)]$.

This proves both implications and establishes the claim.

Similarly, the statement "it is not the case that all motorists run red lights" $\iff$ the statement "there is a motorist who does not run red lights" logically speaking.

Let's look at an eariler example, "all domestic cars are badly made."

Example 1.2 (All domestic cars are badly made)

Mathematically, we can let C be the set of all cars, $D(x)$ means x is domestic, and $M(x)$ means x is badly made.

With this notation, the sentence becomes

$$(\forall x \in C)\,[D(x) \implies M(x)]\,.$$

Then, the negation is

$$(\exists x \in C)\,[D(x) \not\Longrightarrow M(x)]\,.$$

We know that $[D(x) \not\Longrightarrow M(X)]$ is equivalent to $D(x) \wedge \neg M(x)$. So,

$$\neg\,(\forall x \in C)\,[D(x) \implies M(x)]$$

is equivalent to

$$(\exists x \in C)\,[D(x) \wedge \neg M(x)]\,.$$

§2 Working With Quantifiers 2

Let's look at another example, specifically, the statement that "all prime numbers are odd".

Example 2.1

Let $P(x)$: "x is prime", and $Q(x)$: x is odd. Or, symbolically, $\forall x\,[P(x) \implies Q(x)]$. When we negate this, we have

$$\begin{aligned} \neg\forall x\,[P(x) \implies Q(x)] &\iff \exists x\,[P(x) \not\Longrightarrow Q(x)] \\ &\iff \exists x\,[P(x) \land \neg Q(x)] \end{aligned}$$

or that "there is a prime that is not odd".

Then, our process is simple, as we just need to find a prime that is not, which is 2. Therefore, we have proven this statement.

§2.1 Domain of Quantification

Suppose a sentece is written as the following.

$$\forall x\,[x > 0 \implies \exists y(xy = 1)]\,.$$

In this case, we have no idea whether this statement is true or false, as we have no idea what the variable x denotes. Any **quantifier** only tells us something if we know the **variable**.

Therefore, associated with any quantifier, we have what's called the **domain of quantification**, which basically tells us "what does the x denote?"

Example 2.2

To make the statement more explicit, we can write it as

$$(\forall x \in \mathbb{Q})\,[x > 0 \implies \exists y(xy = 1)]\,.$$

Although we did not explicitly denote the domain for the variable y, since we already have done so for x, it is usually enough to assume that they have the same domain of quantification. Or, that the statement is true $\forall y \in \mathbb{Q}$. Of course, if we want to be more explicit, we can write this as

$$(\forall x \in \mathbb{Q})\,[x > 0 \implies (\exists y \in \mathbb{Q})(xy = 1)]\,.$$

Mathematicians sometimes omit the quantifier; observe the equation

$$x \geq 0 \implies \sqrt{x} \geq 0.$$

What that means is (something like):

$$(\forall x \in \mathbb{R})x \geq 0 \implies \sqrt{x} \geq 0.$$

We assume the above domain as it is the most reasonable and common assumption for this specific statement. This is known as **implicit quantification**, and although not recommended, the professionals do this all the time.

Remark 2.3. Another important thing to keep in mind is that we cannot distribute for all or existence statements. For example,

$$\forall x\,[E(x) \vee O(x)]$$

is true, while

$$\forall x E(x) \vee \forall x O(x)$$

is false.