XM452 Lecture Notes 6

Erdaifu Luo

26 March 2023

§27 Magic Squares (Intro)

§27.1 Magic Squares

Definition 27.1 (Magic Squares)**.** An $n \times n$ square with n^2 entries is a **magic square** if the sum of the entries in any row or column is always the same.

Example 27.2

The following magic square is the Albrecht Durer engraving "Melancolia"

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

§27.2 The Uniform Step Method

First, place the number 0 through $5^2 - 1 = 24$ in the cells.

Place 0 in any cell (x_0, y_0), for example, take $x_0 = 4$ and $y_0 = 3$,

			0	

Then take $(x_1, y_1) = (x_0 + 1, y_0 + 2) \pmod 5$. Basically, our step is to go right by 1 and up by 2, and we haave to take mod 5 since we may run out of cells.

Thus, $x_1 = 5$, and $y - 1 = 3 + 2 = 5$. Thus, $(5, 5)$ contains 1.

				1
			0	

Also, $x_2 = 5 + 1 = 1$, $y_2 = 5 + 2 = 2 \pmod 5$. So $(1, 2)$ contains 2.
Continuing, j is in the cell:

$$x_j \equiv x_{j-1} + 1 \equiv \cdots \equiv x_0 + j \equiv \boxed{4 + j} \pmod 5$$

$$y_j \equiv y_{j-1} + 2 \equiv \cdots \equiv y_0 + 2j \equiv \boxed{3 + 2j} \pmod 5.$$

But note $(x_5, y_5) \equiv (x_0 + 5, y_0 + 10) \equiv (x_0, y_0) \pmod 5$, and you can see that this overlaps with the number 0, so this only works for $0, 1, 2, 3, 4$.

				1
	3			
			0	
2				
		4		

Instead, we put the 5 in the cell one to the right and three up (mod 5) from the cell containing 0. To place $6, 7, 8, 9$, continue going 1 to the right and 2 up.

	7			1
	3		9	
6			0	
2		8		
		4		5

But note that if we continue any further, 10 would be assigned to the same cell as 9.

Definition 27.3 (The Uniform Step Method)**.** Let a, b, c, d, e, f be integers. The **uniform step method** puts the n^2 numbers $j = 0, 1, 2, \ldots, n^2 - 1$ in the cells with coordinates (x_j, y_j) where

$$x_j = a + cj + e \left\lceil \frac{j}{n} \right\rceil \pmod n,$$

$$y_j = b + dj + f \left\lceil \frac{j}{n} \right\rceil \pmod n.$$

Here, $\left\lceil \frac{j}{n} \right\rceil$ means the largest integer $\leq \frac{j}{n}$. E. g. $\lceil \frac{3}{2} \rceil = 1$.

Example 27.4

Let $a = 4$, $b = 3$, $c = 1$, $d = 2$, $e = 1$, $f = 3$. These fill the square

19	7	20	13	1
10	3	16	9	22
6	24	12	0	18
2	15	8	21	14
23	11	4	17	5

§27.3 Filled vs. Magic Squares

Definition 27.5 (Filled)**.** The uniform step method is said to **fill** an $n \times n$ square if each of the n^2 cells in the square has exactly 1 entry.

Example 27.6

$n = 3$. Let $a = 3$, $b = 1$, $c = 1$, $d = -1$, $e = 1$, $f = 2$.

$$x_j = 3 + j + \left\lceil \frac{j}{3} \right\rceil \pmod 3,$$

$$y_j = 1 - j + \left\lceil 2\frac{j}{3} \right\rceil \pmod 3.$$

This produces

1, 3, 8		
	2, 4, 6	
		0, 5, 7

which is magic, yet isn't filled.

§28 Filling Squares

The uniform step method sometimes fills $n \times n$ squares. But it does not always fill them, and if it does, we have not shown that the square is always magic.

In this, we'll show a condition on the uniform step method that gurantees that all n^2 cells are filled. A needed preliminary theorem is

Theorem 28.1

Let j be an integer and $0 \leq j \leq n^2 - 1$. Then there are unique integers u and v such that $j = vn + u$.

Note that u and v are determined by the conditions

$$j \equiv u \pmod n, \left\lceil \frac{j}{n} \right\rceil = v.$$

Proof. Let u be the unique integer between 0 and $n - 1$ such that $j \equiv u \pmod n$. Thus there is an integer v such that $j = vn + u$. Now, all we have to verify is that the conditions for v matches up.

Since $0 \leq u \leq n - 1$ and $0 \leq j \leq n^2 - 1$, some arithmetic shows that $0 \leq v \leq n - 1$.

Now we must show that u and v are unique. To show that, suppose $j = vn + u$ where $0 \leq u \leq n - 1$, and $0 \leq v \leq n - 1$.

Then, $j \equiv u \pmod n$, so u is uniquely determined. But then

$$v = \frac{j \cdot u}{n} \leq \frac{j}{n} < \frac{j + (n - u)}{n} = v + 1$$

so v is uniquely determined by $v = \left\lceil \frac{j}{n} \right\rceil$. □

Now remember what the uniform step method is: for all j between 0 and $n^2 - 1$, we put j in the cell (x_j, y_j) where

$$x_j = a + cj + e\left\lceil \frac{j}{n} \right\rceil \pmod{n},$$

$$y_j = b + dj + f\left\lceil \frac{j}{n} \right\rceil \pmod{n}.$$

for some choice of a, b, c, d, e, f.

Theorem 28.2

The uniform step method fills every cell if $(cf - de, n) = 1$.

Proof. We need to show that if $(cf - de, n) = 1$, then no cell is assigned two numbers. The proof is by contradiction.

Assume there are two integers $j_1 \neq j_2$, both between 0 and $n^2 - 1$ for which

$$x_{j_1} \equiv x_{j_2}, y_{j_1} \equiv y_{j_2} \pmod{n}.$$

By the previous theorem we can write

$$j_1 = v_1 n + u_1, j_2 = v_2 n + u_2$$

for unique integers $0 \leq u_1,\ v_1,\ u_2,\ v_2 \leq n - 1$. So since

$$a + cj_1 + e\left\lceil \frac{j_1}{n} \right\rceil \equiv a + cj_2 + e\left\lceil \frac{j_2}{n} \right\rceil \pmod{n}$$
$$b + dj_1 + f\left\lceil \frac{j_1}{n} \right\rceil \equiv b + dj_2 + f\left\lceil \frac{j_2}{n} \right\rceil \pmod{n}$$

we can write

$$a + cu_1 + ev_1 \equiv a + cu_2 + ev_2 \pmod{n}$$
$$b + du_1 + fv_1 \equiv b + du_2 + fv_2 \pmod{n}$$

We can subtract off the a and b terms to get

$$cu_1 + ev_1 \equiv cu_2 + ev_2 \pmod{n}$$
$$du_1 + fv_1 \equiv du_2 + fv_2 \pmod{n}$$

If we already know u_2 and v_2 and want to solve for u_1 and v_1, a previous theorem says that there is a unique solution mod n if

$$(cf - de, 1) = 1.$$

Clearly $u_1 = u_2$ and $v_1 = v_2$ is a solution, so it is unique mod n. Therefore

$$u_1 \equiv u_2, v_1 \equiv v_2 \pmod{n}$$

which means that $u_1 = u_2$ and $v_1 = v_2$ (since $0 \leq u_1, v_1, u_2, v_2 \leq n - 1$).

Therefore, $j_1 = v_1 n + u_1 = v_2 n + u_2 = j_2$, which contradicts our assumption that $j_1 \neq j_2$, so every number is placed in a different cell. □

§29 Producing Magic Squares

§29.1 Prerequisites

We know that when the uniform step method fills all n^2 cells in an $n \times n$ square. When does the method gives us a magic square however?

To do this, we will need the following lemma:

Lemma 29.1

Let q, r, and s be integers and $(q, n) = (r, n) = 1$. Then there are exactly n integers in the range $0 \leq j \leq n^2 - 1$ that satisfies the congruence

$$q \cdot j + r \left\lceil \frac{j}{n} \right\rceil \equiv s \pmod{n}$$

and the sum of these integers is

$$\frac{n(n^2-1)}{2}.$$

We first require a more careful definition.

Definition 29.2 (Magic)**.** Suppose n^2 different integers are put in cells of a $n \times n$ square (not necessarily filling it). If the sum of the entries of each column is always the same, we say that the square is *column* magic.

If the sum of the entries of each row is always the same, we say that the square is *row* magic.

If it is both row and column magic, we say it is a *magic square*. The common sums in each of these cases is called the *magic sum*.

§29.2 Theorem for Magic Squares

Theorem 29.3

Suppose that the numbers $0, \dots, n^2 - 1$ are put in an $n \times n$ square using the uniform step method. Meaning that the number j is put in the cell $(x_j, y_j$ where

$$x_j \equiv a + cj + e \left\lceil \frac{j}{n} \right\rceil \pmod{n},$$

$$y_j \equiv b + dj + f \left\lceil \frac{j}{n} \right\rceil \pmod{n},$$

where a, b, c, d, e, $f \in \mathbb{Z}$. Then,

(1) If $(c, n) = (e, n) = 1$, then the square is column magic.

(2) If $(d, n) = (f, n) = 1$, then the square is row magic.

(3) If $(c, n) = (e, n) = (d, n) = (f, n) = 1$, then the square is magic. In each of these cases, the magic sum is

$$\frac{n(n^2 - 1)}{2}.$$

Note that in this theorem, it doesn't matter what a and b are. That is, it doesn't matter where the initial number $j = 0$ is placed.

Proof. Suppose $(c, n) = (e, n) = 1$. We will show that this square is column magic.

So the number j is put in the cell (x_j, y_j). It lies in the kth column if and only if $x_j = k$. That is,

$$a + cj + e \left\lceil \frac{j}{n} \right\rceil \equiv k \pmod{n}$$

or

$$cj + e \left\lceil \frac{j}{n} \right\rceil \equiv k - a \pmod{n}.$$

Notice that this is the equation described in the lemma, which states that there are exactly n integers in the range $0 \le j \le n^2 - 1$ with this property and their sum is

$$\frac{n(n^2 - 1)}{2}.$$

Thus, the sum of every column is the above and the square is column magic. The statement implying that the square is row magic is proved in the same way. The third part of the theorem follows from the first two parts. $\square$

§30 The Lemma Behind Uniform Step Method

Let's restate the lemma,

Lemma 30.1

Let q, r, and s be integers and $(q, n) = (r, n) = 1$. Then there are exactly n integers in the range $0 \leq j \leq n^2 - 1$ that satisfies the congruence

$$q \cdot j + r \left\lceil \frac{j}{n} \right\rceil \equiv s \pmod{n}$$

and the sum of these integers is

$$\frac{n(n^2 - 1)}{2}.$$

Proof. By a theorem proved a couple of lectures ago, if $0 \leq j \leq n^2 - 1$, then there are unique integers u and v between 0 and $n - 1$ such that

$$j = vn + u.$$

Conversely, given integers u and v satisfying these properties, the number $j = vn + u$ must satisfy

$$0 \leq j \leq n^2 - 1.$$

Thus, j determines the numbers u and v and conversely. The congruence in the statement of the theoreme can be written in terms of u and v:

$$qu + rv \equiv s \pmod{n}.$$

So we need to show that if $(q, n) = (r, n) = 1$, then there are exactly n pairs of integers u and v (between 0 and $n - 1$) that satisifes the congruence

$$qu + rv \equiv s \pmod{n}.$$

Since $(r, n) = 1$, a previous theorem tells us that for a given integer u between 0 and $n - 1$, there is a unique solution v to the equation

$$rv \equiv s - qu \pmod{n.}$$

This equation produces an unique v mod n, but if we stipuate that v lies between 0 and $n - 1$, v is an unique integer. Now there are n integers u between 0 and $n - 1$, and each produces an integer v between 0 and $n - 1$ so that

$$rv \equiv s - qu \pmod{n}$$

which is the same as

$$qu + rv \equiv s \pmod{n}.$$

Thus there are n pairs (u, v) between 0 and $n - 1$ that satisfy this relationship, as claimed.

As observed before, this means that there are n numbers j between 0 and $n^2 - 1$ which satisfy

$$q \cdot j + r \left\lceil \frac{j}{n} \right\rceil \equiv s \pmod{n}$$

since

$$j \equiv u \pmod{n}, \left\lceil \frac{j}{n} \right\rceil \equiv v \pmod{n}.$$

Let these values of j be $j_0, j_1, \dots, j_{n-1}$ and the corresponding values of u and v be $u_0, v_0, \dots, u_{n-1}, v_{n-1}$.

We need to show that

$$j_0 + j_1 + \cdots + j_{n-1} = \frac{n(n^2-1)}{2}.$$

Now we already observed that the values of $u_0, \dots, u_{n-1}$ take on each value between 0 and $n-1$ exactly once.

Moreover, for any value of v, there is a unique u such that

$$qu + rv \equiv s \pmod n$$

where $0 \le u \le n-1$ since $(q,u) = 1$.

Thus every v between 0 and $n-1$ has a corresponding u in the range 0 to $n-1$. Thus the v_j's in the above list also take on every value between 0 and $n-1$ exactly once.

We can therefore compute the sum

$$\begin{aligned}\sum_{k=0}^{n-1} j_k &= \sum_{k=0}^{n-1} (nv_k + u_k) \\ &= n\sum_{k=0}^{n-1} v_j + \sum_{k=0}^{n-1} u_k.\end{aligned}$$

But each of the sums $\sum_{k=0}^{n-1} v_j$ and $\sum_{k=0}^{n-1} u_k$ are equal to the sum

$$0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

because the u's and the v's each take on every value between 0 and $n-1$ exactly once. Thus,

$$\begin{aligned}\sum_{k=0}^{n-1} j_k &= \sum_{k=0}^{n-1} (nv_k + u_k) \\ &= n \cdot \frac{n(n-1)}{2} + \frac{n(n-1)}{2} \\ &= \frac{n(n-1)}{2} \cdot (n+1) \\ &= \frac{n(n^2-1)}{2}\end{aligned}$$

□

§31 Diabolic Squares

An $n \times n$ square has two main diagonals: One from the upper left to the lower right, given by cells (x, y) satisfying

$$x + y = n + 1.$$

Similarily, one from the upper right to the lower left, which is simply

$$y = x.$$

Equivalently, these diagonals have the mod n equations

$$x + y \equiv 1 \pmod{n}$$

and

$$x \equiv y \pmod{n}.$$

There are also broken diagonals ($2n - 2$ of them) which satisfy the equations

$$y + x \equiv k \pmod{n}$$

or

$$y - x \equiv k \pmod{n}.$$

The first set are called **positive diagonals** and the second set are called **negative diagonals**.

Definition 31.1. Suppose that we enter integers in the cell of an $n \times n$ square (not necessarily filling it).

If the sum on the positive diagonals is always the same, we say the square is magic in the positive diagonals.

If the sum on the negative diagonals is always the same, we say the square is magic in the negative diagonals.

If the square is magic in both diagonals, we say it's diabolic, and the sum of the result are called diabolic sums.

Theorem 31.2

Suppose the numbers $0, 1, \ldots, n^2 - 1$ are put in an $n \times n$ square according to

$$x_j \equiv a + cj + e \left\lceil \frac{j}{n} \right\rceil \pmod{n},$$

$$y_j \equiv b + dj + f \left\lceil \frac{j}{n} \right\rceil \pmod{n}.$$

If $(c + d, n) = (e + f, n) = 1$, then the square is magic in the positive diagonals.

If $(c - d, n) = (e - f, n) = 1$, then the square is magic in the negative diagonals.

If $(c + d, n) = (e + f, n) = (c - d, n) = (e - f, n) = 1$, then it is a diabolic square with diabolic sum

$$\frac{n(n^2 - 1)}{2}.$$

Proof. Suppose $(c + d, n) = (e + f, n) = 1$. We will show the square is magic in the positive diagonals.

The kth positive diagonal has the equation

$$x + y \equiv k \pmod{n}$$

so j occurs on the kth positive diagonal if and only if

$$x_j + y_j \equiv k \pmod{n}.$$

From the formula for the uniform step method, the above is equivalent to

$$(c + d) \cdot j + (e + f) \left\lceil \frac{j}{n} \right\rceil \equiv k - a - b \pmod{n}.$$

But the lemma from lesson 30 says that the number of such integer is n, and their sum is $\frac{n(n^2-1)}{2}$. So every positive diagonal has n elements and has the sum shown above. The theorem about negative diagonals is proven the exact same way, same for the third condition. $\square$