MT Coursera Week 03

Erdaifu Luo

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§1 Quantifiers

There exists two types of quantifiers: **there exists**, and **for all**. These are all we need to look at, due to the special nature of mathematica truths, and the majority of mathematical theorems will have one of two forms:

1. There is an object x having property P.
2. For all objects x property P holds.

Example 1.1 (There is an object x having property P)
Some examples are

1. The equation $x^2 + 2x + 1 = 0$ has a real root.
2. There is a real number x such that $x^2 + 2x + 1 = 0$.

Instead of there is, mathematicians often state "there exists", which can be represented by the symbol $\exists$. The symbol is called the existential quantifier.

Therefore, the above statement could be represented by

$$\exists x \left[x^2 + 2x + 1 = 0\right].$$

To prove the statement, find an actual x that solves the equation, such as $x = -1$.

Example 1.2
Another example for an existential statement is "$\sqrt{2}$ is not rationa".

This could be rewritten as "There exists rational numbers p, q such that $\sqrt{2} = \frac{p}{q}$".

To make it more specific, we can write it in the following way.

$$(\exists p \in \mathbb{N})\,(\exists q \in \mathbb{N})\left[\sqrt{2} = \frac{p}{q}\right]$$

where $\mathbb{N}$ denotes the set of natural numbers.

Or, the statement

$$(\exists p, q \in \mathbb{N})\left[\sqrt{2} = \frac{p}{q}\right]$$

represents the same thing.

Now, let's tackle the second statement, "for all objects x property P holds". The phrase "for all" is also known as the **universal quantifier**, and is represented by the symbol $\forall$.

Example 1.3 (For all objects x property P holds)
The statement "The square of any real number is greater than or equal to zero", could be written as

$$\forall x \left(x^2 \geq 0\right).$$

The statement could also be phrased as "for all x, x^2 is greater than or equal to 0". Simimarily, using set theory, we could represent this as

$$(\forall x \in \mathbb{R}) \left(x^2 \geq 0\right).$$

In mathematics, quantifiers are often combined. If one wants to write "there is no largest natural number", it would be represented as

$$(\forall m \in \mathbb{N}) (\exists n \in \mathbb{N}) (n > m).$$

The order of the quantifiers is important. For example, the equation below,

$$(\exists n \in \mathbb{N}) (\forall m \in \mathbb{N}) (n > m)$$

would mean "there is a natural number bigger than all natural numbers", which does not make sense.