XM452 Lecture 8

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§37 Decimal Expansions of Rational Numbers

When there is a digit that repeat in a continued fraction, we represent it using a bar on top of the repeating part. For example,

$$\frac{1}{3} = 0.\overline{3}.$$

The digits under the bar are called the **period** of the decimal expansion, and the number of these digits is the **length** of the period. If the expansion stops, then we say the expansion **terminates**.

Theorem 37.1

Let m and n be positive integers. Then $\frac{m}{n}$ has a decimal expansion that either terminates or is periodic.

Proof. We view a terminating decimal as a periodic decimal with period $\overline{0}$. So we just need to show that every positive rational number has a periodic decimal expansion.

By the devision algorithm,

$$m = qn + a_1, \quad 0 \leq a_1 \leq n-1$$

Now in the decimal expansion of $\frac{m}{n}$, let the digits to the right of the decimal point be $0.q_1q_2q_3q_4\ldots$.

To find q_1, notice that

$$10a_1 = q_1 n + a_2, \quad 0 \leq a_2 \leq n-1.$$

Similarily,

$$\begin{aligned}
10a_2 &= q_2 n + a_3, \quad 0 \leq a_3 \leq n-1 \\
10a_3 &= q_3 n + a_4, \quad 0 \leq a_4 \leq n-1 \\
&\vdots \\
10a_j &= q_j n + a_{j+1}, \quad 0 \leq a_{j+1} \leq n-1
\end{aligned}$$

tWe observe that we only need to show that $a_j = a_k$ for some $j \neq k$ in this process. For if this is true then $10a_j = 10a_k$, which would imply $q_j = q_k$ with raminders $a_{j+1} = a_{k+1}$, then the same thing will happen to these reaminders, so that we have

$$q_j = q_k, q_{j+1} = q_{k+1}.$$

Then that we have a periodic function.

Now it surely is true that $a_j = a_k$, for some $j \neq k$ because all the a_j's are between 0 and $n-1$, or there are only infinitely many possibilities. Thus the a_s's cannot be different. $\square$

Notice that by the proof of this theorem, $\frac{m}{n}$ has period having length $\leq n$. We will understand more about the length by the next theorem.

Definition 37.2. A decimal expansion is called **purely periodic** if the period starts with the first digit to the right of the decimal point.

So, $\frac{1}{3} = .33333\overline{3}$ is purely periodic.

§38 Rational and Irrational Numbers

In the last lecture, we stated the following theorema bout purely periodic decimal expansions. In this lecture we prove the theorem, and then discuss certain properties of irrational numbers.

Theorem 38.1

Suppose $(m,n) = (10,n) = 1$, where m and n are positive integers. Then $\frac{m}{n}$ has a purely periodic decimal expansion of period length $ord_n(10)$.

Proof. Recall the notation we used. The digits to the right of the decimal point are denoted $0.q_1q_2q_3\ldots$ and we have integers $0 \leq a_j \leq n-1$ defined by the division algorithm

$$10a_1 \equiv q_1 n + a_2$$
$$\vdots$$
$$10a_j \equiv q_j n + a_{j+1}$$

We will show that $a_1 = a_{1+s}$ for some s. This will imply that the block of digits are

$$10a_1 = a_2 \pmod{n}$$
$$\vdots$$
$$10a_j = a_{j+1} \pmod{n}$$

So for all $t > 0$, $a_{1+t} \equiv 10a_t \equiv 10^2 a_{t-1} \equiv \cdots \equiv 10^t a_1 \pmod{n}$.

Let $s = ord_n(10)$. Then, by taking $t = s$ and the definition of order,

$$a_{1+s} \equiv 10^s a_1 \equiv a_1 \pmod{n}.$$

But both a_1 and a_{1+s} are integers between 0 and $n-1$. Thus,

$$a_1 = a_{1+s}$$

as claimed.

To show that $s = ord_n(10)$ is the length of the period, call the length r. Then we know

$$r \leq s$$

because $.q_1q_2\ldots q_s$ repeats. The first and second complete periods are therefore

$$.q_1q_2\cdots q_r, \quad .q_{r+1}q_{r+2}\cdots q_{2r}.$$

From the above requations, we see

$$\frac{a_1}{n} = .\overline{q_1 q_2 \dots q_r} = .\overline{q_{r+1} q_{r+2} \dots q_{2r}} = \frac{a_{r+1}}{n}.$$

Now we use this to compute. Since $m \equiv a_1 \pmod{n}$,

$$10^r m \equiv 10^r a_1 \equiv a_{1+r} \equiv a_1 \equiv m \pmod{n}.$$

Since $(m, n) = 1$, we can divide by m and get

$$10^r \equiv 1 \pmod{n}.$$

This implies, since $(10, n) = 1$, that $s = ord_n(10) \mid r$. This means that $s \leq r$. We already knew $r \leq s$, so we must have $r = s$. $\square$

We now use what we know about the decimal expansions of ratioanl numbers to prove that certain numbers are irrational.

For example, $0.1011011101111\dots$ is irrational since it is not periodic.

The following result says that if one has an irrational number than one can produce infinitely many more.

Theorem 38.2

If α is irrational, and a and b are rational, with $b \neq 0$, then

$$a + b\alpha, \quad \frac{1}{\alpha}$$

are also irrational.

Proof. Let $x = a + b\alpha$. Then $b\alpha = x - a$, so that $\alpha = \frac{x-a}{b}$. Thus if x is rational, so is $x - a$ since a is rational and so is $\frac{x-a}{b}$ since b is a nonzero rational. The sums, differences, and quotients of rationals are rational.

Similarly, if $y = \frac{1}{\alpha}$, then $\alpha = \frac{1}{y}$, so that if y were rational, then α would be as well. $\square$

Theorem 38.3

Suppose a and b are rational and α is irrational. If $a + b\alpha = 0$ then $a = b = 0$.

Proof. If $b \neq 0$, then by the above result, $a + b\alpha$ is irrational. But 0 is not irrational. Thus $b = 0$, since this means that $a = 0$ as well. $\square$

§39 Continued Fractions: Geometric Preliminaries

Definition 39.1. Let P be a point in the plane with coordinates (b, a) and suppose that (b, c) is the intersection of a line L with the vertical line $x = b$. If $a > c$ we say that P is **over** or above L; if $a < c$ we say that P is **under** or **below** L.

Definition 39.2. Let P have coordinate (b, a) with $b \neq 0$. We say that the quotient $\frac{a}{b}$ is the **slope** of P. (Actually, it is the slope of the line through the origin and P).

Theorem 39.3

Let L be a line through the origin with slope α. If $P = (b, a)$ is a point in the first quadrant $(b, a > 0)$, then P is above L if and only if the slope of P is greater than α. P is on L iff the slope of P is equal to α.

Proof. The line L is given by the equation $y = \alpha x$. So the intersection with the line $x = b$ is the point $(b, \alpha b)$. Since $b > 0$,

$$\begin{aligned} b > b\alpha \quad &\Longleftrightarrow \frac{a}{b} > \alpha \\ a = b\alpha \quad &\Longleftrightarrow \frac{a}{b} = \alpha \\ a < b\alpha \quad &\Longleftrightarrow \frac{a}{b} < \alpha. \end{aligned}$$

□

The closer the slope of a point P is to the slope of a line L through the origin, the closer the distance that point is from L. We make that precise:

Theorem 39.4

Let L be the line given by the equation $y = \alpha x$ with slope $\alpha > 0$. Suppose (q, p) is a point with integral coordinates in the first quadrant and has the property that if (n, m) is another point with integral coordinates in the first quadrant with $0 < n \leq q$, then the distance from (p, q) to L is less than or equal to the distance from (n, m) to L. Then we have the relation between slopes:

$$\left|\alpha - \frac{p}{q}\right| \leq \left|\alpha - \frac{m}{n}\right|.$$

Proof. If $\frac{p}{q} = \alpha$, then clearly $P = (q, b)$ lies on L and so the theorem is obvious. So we assume (q, p) does not lie on L. Let d be the distance from $P = (q, p)$ to L. So $d > 0$. The distance d is measured along the perpendicular from P to L. P is either above or below L, and assume for now that P is below L.

Let L_1 be the line through P that is parallel to L. So the distance from L_1 to L is d. Let L_2 be the line parallel to L that is a distance d from L that lies above L. Let Q and R be the intersections of the line $x = q$ with L_2 and L_1 respectively. Also let A and B be the points on L such that AP and BQ are perpendicular.

Then, we see that the coordinates of Q are $(q, 2q\alpha - p)$. So the slopes of OP and OQ are $\frac{p}{q}$ and $\frac{(2q\alpha - p)}{q} = 2\alpha - \frac{p}{q}$, respectively. Now suppose that

$$\left|\alpha - \frac{p}{q} > \left|\alpha - \frac{m}{n}\right|\right|.$$

Now, one of $\alpha - \frac{m}{n}$ or $\frac{m}{n} - \alpha$ is positive and the other is negative, so

$$\alpha - \frac{p}{q} > \alpha - \frac{m}{n}, \quad a - \frac{p}{q} > \frac{m}{n} - \alpha.$$

From the frist of these we see that sope of $C = \frac{m}{n} > \frac{p}{q} =$ slope of OP.

And from the second, slope of $OQ = 2\alpha - \frac{p}{q} > \frac{m}{n} =$ slope of C. Thus, C is above OP and below OQ, and not on either of those lines.

C is above OP and below OQ and not on either of these lines. Since $0 < n \leq q$, C is actually inside the triangle OPQ (or on the edge PQ). Thus the distance from C to L is less than d. This is a contradiction to our hypothesis, hence we conconlude the equation in the theorem. □

Definition 39.5. The "set of points closest to a line L" is the set of all points (q, p), (q and p integers, $q \geq 1$) with the property that if (n, m) is any other integral coordinate point with $0 < n \leq q$, then the distance from (q, p) to L is less than the distance from (n, m) to L.

This will be helpful in finding "good fractional approximations" to real numbers. For example if $\alpha = \sqrt{3}$, so $L = \{y = \sqrt{3}x\}$, then the first several elements of the set of points closest to L are

$$\{(1, 2), (3, 5), (4, 7), (11, 19), (15, 26)\}.$$

§40 The Continued Fraction Algorithm

Theorem 40.1

If a line L passes through the origin, and points U_1 and U_2 are on opposite sides of L, there is a unique integer a such that either $U_1 + aU_2$ is on L or $U_1 + aU_2$ and $U_1 + (a + 1)U_2$ are on opposite sides of L.

$U_1 + aU_2$ is closer to L than U_2, and if it is not on L, it is on the same side of L as U_1. If U_1 is closer to L than U_2 then $a = 0$, otherwise $a \geq 1$.

Definition 40.2. Let $\alpha > 0$ be a real number. The following process is the **continued fraction algorithm for approximating alpha**: Let L be the line $y = \alpha x$. Let $V_{-2} = (1, 0)$, $V_{-1} = (0, 1)$. Let $V_0 = V_{-2} + a_0 V_{-1} = (1, a_0)$, where a_0 is the unique integer such that $V_{-2} + a_0 V_{-1}$ is either on L or on the same side of L as V_{-2} but $V_{-2} + (a_0 + 1)V_{-1}$ is on the opposite side of L from V_{-2}. If V_0 is on L we terminate the process. If V_0 is not on L then the process is repeated.

In general, if we already have V_{n-2} and V_{n-1} (and V_{n-1} is not on L), then we let

$$V_n = V_{n-2} + a_n V_{n-1},$$

where a_n is the unique integer such that $V_{n-2} + a_n V_{n-1}$ is either on L or on the same side of L as V_{n-2}, but $V_{n-2} + (a_n + 1)V_{n-1}$ is either on L or the same side of L as V_{n-2}, but $V_{n-2} + (a_n + 1)V_{n-1}$ is on the opposite side of L from V_{n-2}. If V_n is on L the process stops. Otherwise, it is continued.

The numbers $a_0, a_1, a_2, \ldots$ and the points $V_{-2}, V_{-1}, V_0, \ldots$ are said to be given by the continued fraction algorithm for approximating α.

If $V_n = (q_n, q_n)$ has slope $\frac{p_n}{q_n}$ which as we will see gives better and better approximations to α.

Theorem 40.3

For $n \geq 1$ and $a_n \geq 1$ (a_0 may be 0). Furthermore,

$$V_{-2}, V_{-1}, V_0, \ldots$$

are successively closer to L and $V_0, V_2, V_4, \ldots$ are all below L while $V_1, V_3, V_5, \ldots$ are all above L (unless the process terminates for some V on L).

Proof. This follows immediately from the last theorem and the defintiion of the V_n's.

Notice that if α is irrational, the process never terminates, because if V_n lies on L, the point V_n would have the same slope as L, or α. But V_n has integral coordinates, so its slope is rational. □

§41 More on the Continued Fraction Algorithm

We first compute the first few terms of the continued fraction algorithm when $\alpha = \sqrt{3}$. We begin with $V_{-2} = (1, 0)$, which is below the line L: $y = \sqrt{3}x$, and $V_{-1} = (0, 1)$, which is above L.

Since $V_{-2} + 1V_{-1} = (1, 1)$ is below L and $V_{-2} + 2V_{-1} = (1, 2)$ is above L, we have $a_0 = 1$ and $V_0 = (1, 1)$.

Also, $V_{-1} + 1V_0 = (1, 2)$ is above L and $V_{-1} + 2V_0 = (2, 3)$ is below L so that $a_1 = 1$ as well, and $V_1 = (1, 2)$.

Continuing, $V_0 + 2V_1 = (3, 5)$ is below L while $V_0 + 3V_1 = (4, 7)$ is above, so $a_2 = 2$ and $V_2 = (3, 5)$.

Continuing this way, we get $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 2, a_5 = 1$. Similarily, we also have $V_0 = (1, 1)$, $V_1 = (1, 2)$, $V_2 = (3, 5)$, $V_3 = (4, 7)$, $V_4 = (11, 19)$, $V_5 = (15, 26)$.

So, if we write $V_i = (p_i, q_i)$, so that slope $V_1 = \frac{p_1}{q_1}$, we have slope $V_0 = 1$, slope $V_1 = 2$, slope $V_2 = \frac{5}{3}$, slope $V_3 = \frac{7}{4}$, slope $V_4 = \frac{19}{11}$, slope $V_5 = \frac{25}{15}$, giving us our approximation.

Lemma 41.1

Let U_1, U_2, be two points with non-negative coordinates, with neither at the origin. Suppose further that they have different slopes. If $a > 0$, then the slope of $U_1 + aU_2 = U_3$ is strictly between the slopes of U_1 and U_2.

Lemma 41.2

Let $x_0, x_1, \ldots$ be a sequence of real numbers. Let $U_{-2} = (1, 0)$, $U_{-1} = (0, 1)$.

$$U_0 \equiv U_{-2} + x_0 U_{-1}, \quad U_1 = U_{-1} + x_1 U_0, \ldots, U_n = U_{n-2} + x_n U_{n-1}.$$

Let (s_n, r_n) be the coordinates of U_n. Then for all $u \geq 0$,

$$r_{n-1}s_{n-2} - r_{n-2}s_{n-1} = (-1)^n.$$