Hopf Fibrations

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- To be more specific, the similarity between a space E and a product space $B \times F$ is defined with the continuous map

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- The map π is known as the **projection** of the bundle, the space E is known as the **total space**, B as the **base space**, and F as the **fiber**.

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Definition (Fiber Bundles)

A fiber bundle is a structure (E, B, π, F) as defined above. We shall assume that the base space B is connected. We require for every $x \in B$, $\exists U \subseteq B$ such that U is an open neighborhood, such that there is a homeomorphism

$$\phi : \pi^{-1}(U) \to U \times F.$$

Or, for any $p \in B$, the preimage $\pi^{-1}(\{p\})$ is homeomorphic to F and is called the fiber over p.

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Let π equals a fiber bundle of F over B, meaning that $E = B \times F$ and $\pi : E \to B$. Here, E is not just *locally* a product, it is *globally* one. Any fiber bundles like this is called a trivial bundle.

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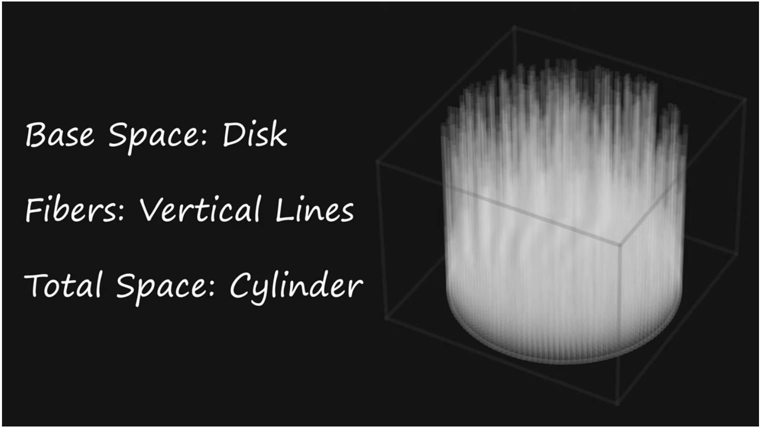
It would be much helpful for us to see a visualization, since our ultimate goal is understanding the Hopf Fibration.

Here are some trivial fiber bundles with the base space and fibers.



Credit: *Richard Behiel*

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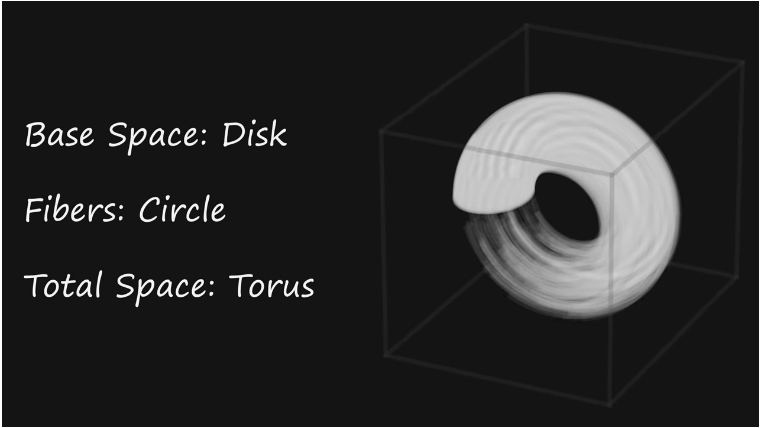

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$$S^1 \hookrightarrow S^3 \xrightarrow{p} S^2.$$

- This means that the fiber space S^1 (a circle) is embedded into the space S^3 (a hypersphere), and then $p : S^3 \to S^2$ projects S^3 onto the base space S^2.

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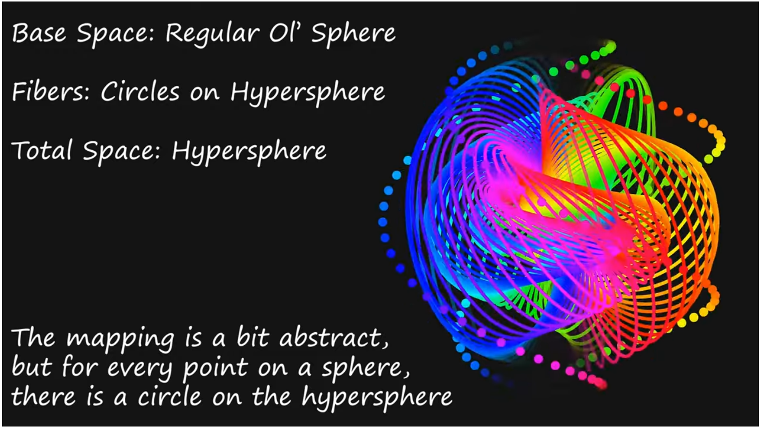
- Some unique properties are that none of the circles intersect, and one of the projected fibers is a line, which is thought of as a circle through infinity.
- We won't go over how to construct the Hopf fibration, although it is quite simple with the fiber bundle structure, but **we will visualize it**.



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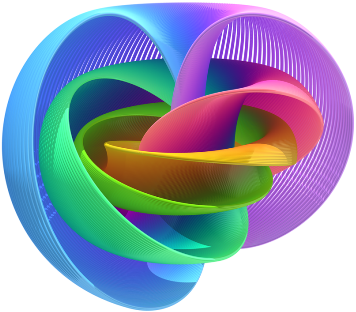
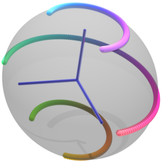
Visualization





Video

The link leads you to a simulation you can mess around in.
https://philogb.github.io/page/hopf/

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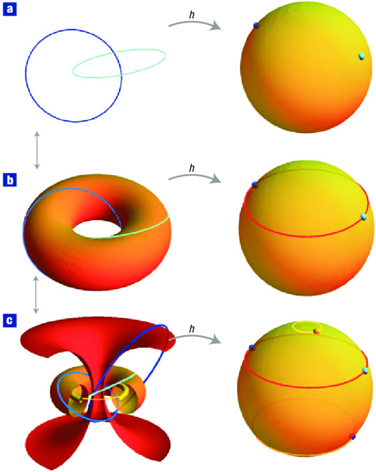

Credit: *William Irvine*

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- One last thing is that the Hopf fibration appears frequently in fields like physics. In fact, it is utilized in

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    - Qubits: Two level quantum systems
    - Mechanics: Harmonic oscillator
    - General Relativity: Taub-NUT space
    - Twistor Theory: Robinson congruences
    - Wignerism: Helicity Representations
    - Magnetic monopoles
    - Dirac equation
    - Gauge symmetry

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Sources:

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- https://youtu.be/PYR9worLEGo
- https://youtu.be/dkyvZo68IoM
- https://arxiv.org/abs/1808.08271