XM452 Lecture Notes 9

Erdaifu Luo

29 April 2023

§42 Estimating Closeness of the approximation by the Continued Fraction Algorithm

Theorem 42.1

The coordinates of the nth stage in the continued fraction algorithm V_n, which are given by

$$q_n = q_{n-2} + a_n q_{n-1}, \quad p_n = p_{n-2} + a_n p_{n-1}$$

are integers. Furthermore,

$$p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = (-1)^n$$

Thus p_n and q_n are relatively prime.

Finally,

$$1 = q_0 \leq q_1 < \cdots < q_n,$$

and if α is irrational, so that the algorithm does not terminate, then $\lim_{n\to\infty} q_n = +\infty$.

Proof. By lemma 1 of the last lecture, the slope of V_2 is between the slope of V_0 and V_1. That is,

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_1}{q_1}.$$

If we now apply lemma 1 to V_1, V_2, V_3, we get

$$\frac{p_2}{q_2} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Inserting this into the above, we get

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Applying lemma 1 to V_2, V_3, and V_4 and inserting it into this equation, we get

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Continuing in this way we get the sequences of inequalities in the theorem except that α has not been placed in this yet.

But we know from a previous theorem that the even V_k's are all below L and the odd V_k's are all above L. Comparing slopes, this says that

$$\frac{p_{2k}}{q_{2k}} < \alpha$$

for all k, and

$$\alpha < \frac{p_{2j+1}}{q_{2j+1}}$$

for all j. □

Theorem 42.2

Suppose α is irrational and $n > 0$. Then,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2} \leq \frac{1}{q_n^2}.$$

Furthermore,

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \alpha.$$

Proof. By the last theorem, α is strictly between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$. Thus,

$$\begin{aligned}\left|\alpha - \frac{p_n}{q_n}\right| &< \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| \\ &= \left|\frac{p_{n+1}q_n - p_nq_{n+1}}{q_nq_{n+1}}\right| \\ &= \left|\frac{(-1)^{n+2}}{q_nq_{n+1}}\right| \\ &= \frac{1}{q_nq_{n+1}}.\end{aligned}$$

□

Recall our approximation to $\sqrt{3}$.

For example, we had $\frac{p_4}{q_4} = \frac{19}{11}$, $\frac{p_5}{q_5} = \frac{26}{15}$.

This theorem says that $\frac{26}{15}$ is within $\frac{1}{15^2} = \frac{1}{225}$ of $\sqrt{3}$.

It turns out that the next approximation is $V_6 = (41, 71)$ and so $\frac{p_6}{q_6} = \frac{71}{41}$ which this theorem says is within $\frac{1}{41^2}$ of $\sqrt{3}$.

Moreover, it also says that the rougher approximation $\frac{p_5}{q_5} = \frac{26}{15}$ is within $\frac{1}{q_5q_6} = \frac{1}{15\cdot 41} = \frac{1}{615}$ of $\sqrt{3}$.

§43 Computing the Integral Coefficients

The continued fraction expansion depends on finding the values a_n.

In the continued fraction algorithm, let d_n be the distance from the point $V_n = (q_n, p_n)$ to the line $L : y = \alpha x$.

By definition, one sees quickly that

$$d_n = d_{n-2} - a_nd_{n-1}.$$

Moreover, since V_n is closer to L than V_{n-1}, we have

$$d_{n-1} > d_{n-2} - a_n d_{n-1} \geq 0.$$

and so

$$1 > \frac{d_{n-2}}{d_{n-1}} - a_n \geq 0.$$

Or,

$$a_n \leq \frac{d_{n-2}}{d_{n-1}}$$

but

$$a_n + 1 > \frac{d_{n-2}}{d_{n-1}}.$$

Thus we have

$$a_n = \left[\frac{d_{n-2}}{d_{n-1}}\right].$$

To say this more easily, let

$$\alpha_n = \frac{d_{n-2}}{d_{n-1}},$$

where α_n is a real number, Then,

$$a_n = [\alpha_n].$$

Moreover, by dividing both sides of the first equation by d_{n-1}, we have that

$$\frac{d_{n-2}}{d_{n-1}} = a_n + \frac{d_n}{d_{n-1}}.$$

If V_n is not on L, so that $d_n \neq 0$, then

$$\alpha_n = a_n + \frac{d_n}{d_{n-1}} = a_n + \frac{1}{\alpha_{n+1}}$$

Thus if we can find an expression for α_0, we can recursively compute the a_n's and the α_n's from the previous equations. This is answered by the following theorem:

Theorem 43.1

$\alpha_0 = \alpha$. Thus the numbers a_n may be computed from the following formulas: For $n \geq 0$,

$$a_n = [\alpha_n]$$

and if α_n is not an integer, α_{n+1} can be found form the formula

$$\alpha_n = a_n + \frac{d_n}{d_{n-1}} = a_n + \frac{1}{\alpha_{n+1}}.$$

Proof. From the above arguments, we only need to check that $\alpha_0 = \alpha$. By definition,

$$\alpha_0 = \frac{d_{-2}}{d_{-1}}.$$

where d_{-2} is the distance from $V_{-2} = (1, 0)$ to $L : y = \alpha x$ and d_{-1} is the distance from $V_{-1} = (0, 1)$ to L. □

§44 Computing the Integral Coefficients

Theorem 44.1

For each $n \geq 0$, we have

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_{n-1} + \cfrac{1}{\alpha_n}}}}}}.$$

The approximation $\frac{p_n}{q_n}$ is given by

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\vdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}.$$