Conservative level set methods for incompressible air/water flow

Chris Kees, Matthew Farthing, Clint Dawson, and Serge Prudhomme

Computational Mechanics Brown Bag Seminar
U.S. Army Engineer Research and Development Center

Nov 20, 2008

Context

- “Particle Scale Distribution of Soil Moisture”, Military Engineering Basic Research Program 6.1 (GSL), FY2.

Context

- “Particle Scale Distribution of Soil Moisture”, Military Engineering Basic Research Program 6.1 (GSL), FY2.
- “High Fidelity Vesself Effects”, Navigation Systems Research Program (CHL), FY3.

Context

- “Particle Scale Distribution of Soil Moisture”, Military Engineering Basic Research Program 6.1 (GSL), FY2.
- “High Fidelity Vesself Effects”, Navigation Systems Research Program (CHL), FY3.
- Both these projects have at their core solving moving boundary problems for the incompressible Navier-Stokes equations.

Particle Scale Project Objectives and Products

- The overall objective is to develop a better understanding of variably saturated granular/porous media by combining multi-scale theoretical and computational model with data.

Particle Scale Project Objectives and Products

- The overall objective is to develop a better understanding of variably saturated granular/porous media by combining multi-scale theoretical and computational model with data.
  - Assemble particle scale, macroscale, and multiscale mathematical models of air/water flow in static granular materials.

Particle Scale Project Objectives and Products

- The overall objective is to develop a better understanding of variably saturated granular/porous media by combining multi-scale theoretical and computational model with data.
  - Assemble particle scale, macroscale, and multiscale mathematical models of air/water flow in static granular materials.
  - Develop tools for simulating at both scales (“classical” continuum models, pore network, lattice Boltzmann).

Particle Scale Project Objectives and Products

- The overall objective is to develop a better understanding of variably saturated granular/porous media by combining multi-scale theoretical and computational model with data.
  - Assemble particle scale, macroscale, and multiscale mathematical models of air/water flow in static granular materials.
  - Develop tools for simulating at both scales (“classical” continuum models, pore network, lattice Boltzmann).
  - Develop tools for estimation of macroscale constitutive relations (homogenization/upscaling).

Particle Scale Project Objectives and Products

- The overall objective is to develop a better understanding of variably saturated granular/porous media by combining multi-scale theoretical and computational model with data.
  - Assemble particle scale, macroscale, and multiscale mathematical models of air/water flow in static granular materials.
  - Develop tools for simulating at both scales ("classical" continuum models, pore network, lattice Boltzmann).
  - Develop tools for estimation of macroscale constitutive relations (homogenization/upscaling).
  - Develop tools for multiscale simulation (variational multiscale methods).

High Fidelity Vessels Project Objectives and Products

- The overall objective is to develop accurate and robust numerical models of breaking waves and moving vessels primarily for locally adaptive, unstructured tetrahedral meshes

High Fidelity Vessels Project Objectives and Products

- The overall objective is to develop accurate and robust numerical models of breaking waves and moving vessels primarily for locally adaptive, unstructured tetrahedral meshes
  - Develop Eulerian free surface tools: level set/volume-of-fluid (free surface evolution) and eikonal equations (geometry/boundary conditions)

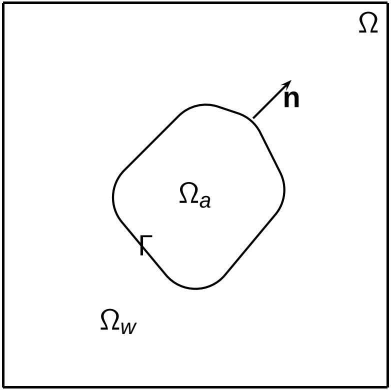
High Fidelity Vessels Project Objectives and Products

- The overall objective is to develop accurate and robust numerical models of breaking waves and moving vessels primarily for locally adaptive, unstructured tetrahedral meshes
  - Develop Eulerian free surface tools: level set/volume-of-fluid (free surface evolution) and eikonal equations (geometry/boundary conditions)
  - Develop two-phase (air/water) Navier-Stokes capability

High Fidelity Vessels Project Objectives and Products

- The overall objective is to develop accurate and robust numerical models of breaking waves and moving vessels primarily for locally adaptive, unstructured tetrahedral meshes
  - Develop Eulerian free surface tools: level set/volume-of-fluid (free surface evolution) and eikonal equations (geometry/boundary conditions)
  - Develop two-phase (air/water) Navier-Stokes capability
  - Develop three-phase (air/water/solid) capability

Two phases separated by a sharp interface



Flow equations and jump conditions

- In Ω_a and Ω_w we assume:

$$\nabla \cdot \mathbf{v}_\alpha = 0 \tag{1}$$

$$\frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \boldsymbol{\sigma}_\alpha) = \rho_\alpha \mathbf{g} \tag{2}$$

where $\alpha = w, a$ and

$$\boldsymbol{\sigma}_\alpha = -p_\alpha I + \rho_\alpha \nu_\alpha \left(\nabla \mathbf{v}_\alpha + \nabla \mathbf{v}_\alpha^t \right) \tag{3}$$

Flow equations and jump conditions

- In Ω_a and Ω_w we assume:

$$\begin{aligned} \nabla \cdot \mathbf{v}_\alpha &= 0 && (1) \\ \frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \boldsymbol{\sigma}_\alpha) &= \rho_\alpha \mathbf{g} && (2) \end{aligned}$$

where $\alpha = w, a$ and

$$\boldsymbol{\sigma}_\alpha = -p_\alpha I + \rho_\alpha \nu_\alpha \left(\nabla \mathbf{v}_\alpha + \nabla \mathbf{v}_\alpha^t \right) \quad (3)$$

- We don't allow the interface to absorb mass:

$$\mathbf{v}_a \cdot \mathbf{n}_a + \mathbf{v}_w \cdot \mathbf{n}_w = 0 \text{ on } \Gamma \quad (4)$$

Flow equations and jump conditions

- In Ω_a and Ω_w we assume:

$$\nabla \cdot \mathbf{v}_\alpha = 0 \quad (1)$$

$$\frac{\partial \rho_\alpha \mathbf{v}_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha - \boldsymbol{\sigma}_\alpha) = \rho_\alpha \mathbf{g} \quad (2)$$

where $\alpha = w, a$ and

$$\boldsymbol{\sigma}_\alpha = -p_\alpha I + \rho_\alpha \nu_\alpha \left(\nabla \mathbf{v}_\alpha + \nabla \mathbf{v}_\alpha^t \right) \quad (3)$$

- We don't allow the interface to absorb mass:

$$\mathbf{v}_a \cdot \mathbf{n}_a + \mathbf{v}_w \cdot \mathbf{n}_w = 0 \text{ on } \Gamma \quad (4)$$

- We do allow the interface to resist normal stress:

$$\boldsymbol{\sigma}_a \cdot \mathbf{n}_a + \boldsymbol{\sigma}_w \cdot \mathbf{n}_w = -\mathbf{f}_\Gamma \text{ on } \Gamma \quad (5)$$

Continuum surface force approximation

- We enforce the jump conditions through a singular source term in the momentum equation.

Continuum surface force approximation

- We enforce the jump conditions through a singular source term in the momentum equation.
- We represent Γ as $\phi(x,t) = 0$ (i.e. the zero level set of some function ϕ).

Continuum surface force approximation

- We enforce the jump conditions through a singular source term in the momentum equation.
- We represent Γ as $\phi(x, t) = 0$ (i.e. the zero level set of some function ϕ).
- If $x(t)$ is a particle path then it remains on Γ so

$$\frac{d\phi(x(t), t)}{dt} = \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \frac{dx}{dt} = \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \mathbf{v} = 0 \qquad (6)$$

Continuum surface force approximation

- We enforce the jump conditions through a singular source term in the momentum equation.
- We represent Γ as $\phi(x,t) = 0$ (i.e. the zero level set of some function ϕ).
- If $x(t)$ is a particle path then it remains on Γ so

$$\frac{d\phi(x(t),t)}{dt} = \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \frac{dx}{dt} = \frac{\partial \phi}{\partial t} + \nabla \phi \cdot \mathbf{v} = 0 \qquad (6)$$

- This is an equation for the level set function ϕ that we extend to all of Ω using the fluid velocity.

Two-phase Navier-Stokes with Surface Tension

$$\nabla \cdot \mathbf{v} = 0 \tag{7}$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \otimes \mathbf{v} - \rho \nu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^t\right)\right] = \rho \mathbf{g} - \nabla p + \mathbf{f}_\Gamma \tag{8}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \tag{9}$$

$$\mathbf{f}_\Gamma(\phi) = \delta(\phi)\sigma_{aw}\kappa(\phi)\mathbf{n} \tag{10}$$

$$\rho(\phi) = H(\phi)\rho_w + \left[1 - H(\phi)\right]\rho_a \tag{11}$$

$$\nu(\phi) = H(\phi)\nu_w + \left[1 - H(\phi)\right]\nu_a \tag{12}$$

$$\mathbf{n} = \frac{\nabla \phi}{\|\nabla \phi\|} \tag{13}$$

$$\kappa = -\nabla \cdot \mathbf{n} \tag{14}$$

Two-phase Flow, cont'd

- $H(\phi)$ is the Heaviside function

$$H(\phi) = \begin{cases} 0 & \phi < 0 \\ 1/2 & \phi = 0 \\ 1 & \phi > 0 \end{cases} \tag{15}$$

Two-phase Flow, cont'd

- $H(\phi)$ is the Heaviside function

$$H(\phi) = \begin{cases} 0 & \phi < 0 \\ 1/2 & \phi = 0 \\ 1 & \phi > 0 \end{cases} \tag{15}$$

- $\delta(\phi)$ is the distributional derivative of H, the Dirac delta function

$$\int_{-\infty}^{\infty} \delta(\phi) f d\phi = f(0) \tag{16}$$

The Level-Set Conservation Problem

- The integral form of conservation of mass (or volume for incompressible fluids) over an interval $[t_n, t_{n+1}]$ is

$$\int_{\Omega} \rho_a H(\phi) dV \Big|_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} \int_{\partial\Omega} \rho_a H(\phi) \mathbf{v} \cdot \mathbf{n} dS = 0 \qquad (17)$$

The Level-Set Conservation Problem

- The integral form of conservation of mass (or volume for incompressible fluids) over an interval $[t_n, t_{n+1}]$ is

$$\int_{\Omega} \rho_a H(\phi) dV \Big|_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} \int_{\partial\Omega} \rho_a H(\phi) \mathbf{v} \cdot \mathbf{n} dS = 0 \tag{17}$$

- Unfortunately this is not enforced by standard discretizations of

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \tag{18}$$

or

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}) = 0 \tag{19}$$

The Level-Set Conservation Problem

- The integral form of conservation of mass (or volume for incompressible fluids) over an interval $[t_n, t_{n+1}]$ is

$$\int_{\Omega} \rho_a H(\phi) dV \Big|_{t_n}^{t_{n+1}} + \int_{t_n}^{t_{n+1}} \int_{\partial\Omega} \rho_a H(\phi) \mathbf{v} \cdot \mathbf{n} dS = 0 \qquad (17)$$

- Unfortunately this is not enforced by standard discretizations of

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0 \qquad (18)$$

or

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{v}) = 0 \qquad (19)$$

- But equation 17 doesn't uniquely define ϕ so we can't use it in a level set approach

Example: Vortex in a Box

Example: Bubble in a Box

Methods for Maintaining Conservation

- Compute a more accurate solution.

Methods for Maintaining Conservation

- Compute a more accurate solution.
- Compute the fluid mass or volume in each element and reconstruct Γ at each time step (volume-of-fluid methods).

Methods for Maintaining Conservation

- Compute a more accurate solution.
- Compute the fluid mass or volume in each element and reconstruct Γ at each time step (volume-of-fluid methods).
- Hybrid methods

Methods for Maintaining Conservation

- Compute a more accurate solution.
- Compute the fluid mass or volume in each element and reconstruct Γ at each time step (volume-of-fluid methods).
- Hybrid methods
  1. Particle/Level Set Methods

Methods for Maintaining Conservation

- Compute a more accurate solution.
- Compute the fluid mass or volume in each element and reconstruct Γ at each time step (volume-of-fluid methods).
- Hybrid methods
  1. Particle/Level Set Methods
  2. Level Set/Volume-of-Fluid Methods

Methods for Maintaining Conservation

- Compute a more accurate solution.
- Compute the fluid mass or volume in each element and reconstruct Γ at each time step (volume-of-fluid methods).
- Hybrid methods
  1. Particle/Level Set Methods
  2. Level Set/Volume-of-Fluid Methods
- Conservative level set methods: mimic volume conservation with a smoothed Heaviside function and identify Γ with $H = 1/2$.

A New Conservative Level Set Method?

- Try to "minimally" correct the level set equation to obtain conservation, which is a "low order" property: Find ϕ' such that

$$\int_{\Omega} \left[H(\phi + \phi') - H^{n+1} \right] dV = 0 \qquad (20)$$

A New Conservative Level Set Method?

- Try to "minimally" correct the level set equation to obtain conservation, which is a "low order" property: Find ϕ' such that
$$\int_{\Omega} \left[H(\phi + \phi') - H^{n+1} \right] dV = 0 \tag{20}$$
- Problems:

A New Conservative Level Set Method?

- Try to “minimally” correct the level set equation to obtain conservation, which is a “low order” property: Find ϕ' such that

$$\int_\Omega \left[H(\phi + \phi') - H^{n+1} \right] dV = 0 \tag{20}$$

- Problems:
  1. Such a ϕ' may not exist.

A New Conservative Level Set Method?

- Try to "minimally" correct the level set equation to obtain conservation, which is a "low order" property: Find ϕ' such that

$$\int_{\Omega} \left[H(\phi + \phi') - H^{n+1} \right] dV = 0 \qquad (20)$$

- Problems:
  1. Such a ϕ' may not exist.
  2. If it does, it's not unique.

A New Conservative Level Set Method?

- Try to "minimally" correct the level set equation to obtain conservation, which is a "low order" property: Find ϕ' such that

$$\int_\Omega \left[H(\phi + \phi') - H^{n+1} \right] dV = 0 \tag{20}$$

- Problems:
  1. Such a ϕ' may not exist.
  2. If it does, it's not unique.
  3. The equation nonlinear and non-differentiable.

A New Conservative Level Set Method?

- Try to "minimally" correct the level set equation to obtain conservation, which is a "low order" property: Find ϕ' such that

$$\int_{\Omega} \left[H(\phi + \phi') - H^{n+1} \right] dV = 0 \tag{20}$$

- Problems:
  1. Such a ϕ' may not exist.
  2. If it does, it's not unique.
  3. The equation nonlinear and non-differentiable.
- Sounds like a great idea!

Variational Problem

$$\min_{\phi_c \in \mathcal{A}} I(\phi_c) \tag{21}$$

where

$$I(\phi_c) = \frac{1}{2}\int_\Omega \|\nabla \phi\|^2 dV \tag{22}$$

$$\mathcal{A} = \{\phi_c \in V_\phi | J(\phi_c) = 0\} \tag{23}$$

$$J(\phi_c) = \int_\Omega [H(\phi_{n+1} + \phi_c) - H_{n+1}]\, dV \tag{24}$$

A nonlinear elliptic equation

Find $\phi' \in H^1(\Omega)$ such that

$$\int_\Omega \left\{ \left[H(\phi_{h,n+1} + \phi^c) - \bar{H}_{n+1} \right] w + \epsilon \nabla \phi^c \cdot \nabla w \right\} dV = 0 \quad \forall w \in H^1(\Omega) \tag{25}$$

where $\bar{H}^{n+1}$ is piecewise constant and satsifies

$$\int_\Omega \left(\bar{H}_{n+1} - H(\phi_{h,n}) \right) dV + \int_{t_n}^{t_{n+1}} \int_{\partial\Omega} H(\phi) \mathbf{v} \cdot \mathbf{n} dS = 0 \tag{26}$$

Results: Vortex in a Box

Results: Bubble in a Box

Summary

- We formulated a mass/volume correction equation that can be added to the standard level set methods to enforce conservation.

Summary

- We formulated a mass/volume correction equation that can be added to the standard level set methods to enforce conservation.
- It applies to higher order and discontinuous finite elements as well.

Summary

- We formulated a mass/volume correction equation that can be added to the standard level set methods to enforce conservation.
- It applies to higher order and discontinuous finite elements as well.
- To do: local (embarassingly parallel) version, complete the theory for the correction equation and the nonlinear solver.