Mixed methods on the cheap (sometimes)

brown-bag lunch seminar

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Variably Saturated Flow in Porous Media

$$
\begin{aligned}
m_t + \nabla \cdot (\rho \mathbf{q}) &= b, \text{ for } \mathbf{x}, t \in \Omega \times [0, T] \\
m &= \rho\theta \\
\mathbf{q} &= -k_r \boldsymbol{K}_s \left(\nabla\psi - \rho \mathbf{g}_u\right) \\
\boldsymbol{\sigma} &= \rho \mathbf{q}
\end{aligned} \tag{1}
$$

with

$$
\psi = \psi^b, \text{ on } \Gamma_D, \quad \boldsymbol{\sigma} \cdot \mathbf{n} = q^b, \text{ on } \Gamma_N
$$

Nonlinear, Scalar PDE

Rewrite as generic nonlinear advection-diffusion equation for convenience ...

$$
\begin{aligned}
m_t + \nabla \cdot \boldsymbol{\sigma} &= b && (2)\\
\boldsymbol{\sigma} &= \mathbf{f} - \boldsymbol{a}\nabla\psi && (3)
\end{aligned}
$$

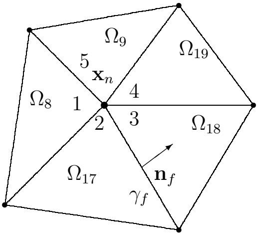
or

$$
m_t + \nabla \cdot (\mathbf{f} - \boldsymbol{a}\nabla\psi) = b \tag{4}
$$

Locally Conservative Methods

Local conservation has long been a motivating factor in method development (FVM, MFEM, MPFA, CV-MFEM, ...)

$$\mathcal{E}(n) = \{8, 9, 17, 18, 19\}$$
$$e^* = \{1, 5, 2, 3, 4\}$$



$$e_\ell(f) = 17,\ e_r(f) = 18$$
$$e^*_\ell(f) = 2,\ e^*_r(f) = 3$$

Element-based conservation

$$\int_{\Omega_e} (\hat{m}_t - b)\,\mathrm{d}x + \int_{\partial\Omega_e} \boldsymbol{\sigma}_h \cdot \mathbf{n}\,\mathrm{d}s = 0$$

Standard Mixed FEM formulation

Find $(\psi_h, \boldsymbol{\sigma}_h)$ in $(W_h, \mathbf{W}_h)$ such that

$$\begin{aligned} \int_\Omega \hat{m}_t w_h \,\mathrm{d}x + \int_\Omega \nabla \cdot \boldsymbol{\sigma}_h w_h \,\mathrm{d}x &= \int_\Omega b w_h \,\mathrm{d}x, \ \forall w_h \in W_h \\ \int_\Omega \boldsymbol{a}^{-1}(\boldsymbol{\sigma}_h - \mathbf{f}) \cdot \mathbf{w}_h + \int_{\partial\Omega} \psi_h \mathbf{w}_h \cdot \mathbf{n} \,\mathrm{d}s &= \int_\Omega \psi_h \nabla \cdot \mathbf{w}_h \,\mathrm{d}x, \ \forall \mathbf{w}_h \in \mathbf{W}_h \end{aligned}$$

Here $W_h \in L_2(\Omega)$, and $\mathbf{W}_h \in H(\mathrm{div}, \Omega)$

Mixed Hybrid Version

Find $(\psi_h, \boldsymbol{\sigma}_h, \Lambda_h)$ in $(W_h, \widetilde{\mathbf{W}}_h, L_h)$ such that

$$\begin{aligned}
\sum_e \int_{\Omega_e} \hat{m}_t w_h \, \mathrm{d}x + \int_{\Omega_e} \nabla \cdot \boldsymbol{\sigma}_h w_h \, \mathrm{d}x &= \int_{\Omega_e} b w_h \, \mathrm{d}x, \\
\sum_e \int_{\Omega_e} \boldsymbol{a}^{-1}(\boldsymbol{\sigma}_h - \mathbf{f}) \cdot \mathbf{w}_h + \int_{\partial\Omega_e} \Lambda_h \mathbf{w}_h \cdot \mathbf{n} \, \mathrm{d}s &= \int_{\Omega_e} \psi_h \nabla \cdot \mathbf{w}_h \, \mathrm{d}x, \\
\sum_e \int_{\partial\Omega_e} \boldsymbol{\sigma}_h \cdot \mathbf{n}_e \mu_h \, \mathrm{d}s &= 0, \ \forall \mu_h \in L_h,
\end{aligned}$$

Local Representation for RT0

For RT0 on simplices, $W_h(\Omega_e) = P^0$, $\widetilde{\mathbf{W}}_h(\Omega_e) = P^0 \oplus \mathbf{x}P^0$

$$\int_{\gamma_f} \boldsymbol{\sigma}_{h,f} \cdot \mathbf{n}_f \, \mathrm{d}s \tag{5}$$

for $\gamma_f \in \partial\Omega_e$ can be used as the degree of freedom along with the local basis

$$\mathbf{N}_{e,i_f} = \frac{1}{n_d|\Omega_e|}\left(\mathbf{x} - \mathbf{x}_{n,i_f}\right), \; i_f = 1, \ldots, n_d + 1 \tag{6}$$

L_h is spanned by constant functions on the faces γ_f in $\mathcal{M}^h$

Standard CG Formulation

find $\psi_h \in V_h \in H^1(\Omega)$

$$\int_\Omega \hat{m}_t w_h \,\mathrm{d}x - \int_\Omega (\mathbf{f} - \boldsymbol{a}\nabla\psi_h)\cdot\nabla w_h \,\mathrm{d}x + \int_{\Gamma_N} \sigma^b w_h \,\mathrm{d}s - \int_\Omega b w_h \,\mathrm{d}x = 0, \ \forall w_h \in W_h \tag{7}$$

P^1 nonconforming approximation

P^1 nonconforming approximation, find $\psi_h \in V_h^{nc}$

$$\begin{aligned} \int_\Omega \hat{m}_t w_h \,\mathrm{d}x \quad &- \quad \int_\Omega (\mathbf{f} - \boldsymbol{a}\nabla\psi_h)\cdot\nabla w_h \,\mathrm{d}x + \int_{\Gamma_N} \sigma^b w_h \,\mathrm{d}s \\ &- \int_\Omega b w_h \,\mathrm{d}x = 0 \forall w_h \in W_h^{nc} \end{aligned} \tag{8}$$

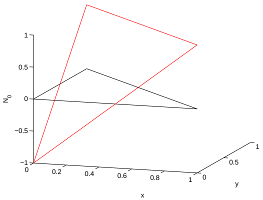
with trial space

$$\begin{aligned} V_h^{nc} \quad = \quad & \{v : v|_{\Omega_e} \in P^1(\Omega_e),\ \forall \Omega_e \in T_h; v \text{ cont. at } \bar{\mathbf{x}}_f,\ \forall \gamma_f \in \Gamma_I; \\ & v = \psi^b \text{ at } \bar{\mathbf{x}}_f,\ \forall \gamma_f \in \Gamma_D\} \end{aligned} \tag{9}$$

P^1 NC shape functions

Locally use the Crouzeix-Raviart space with basis

$$
\begin{aligned}
N_i &= n_d \left(\frac{1}{n_d} - \lambda_i \right), \\
\lambda_i &= 1 - \frac{(\mathbf{x} - \mathbf{x}_i) \cdot \mathbf{n}_i}{(\mathbf{x}_{i+1} - \mathbf{x}_i) \cdot \mathbf{n}_i} \\
\nabla N_i &= |\gamma_i| \mathbf{n}_i / |\Omega_e|
\end{aligned}
$$



$i = 1, \ldots, n_d + 1$

P^1 NC velocity

Define local velocity approximation

$$\hat{\boldsymbol{\sigma}}_{h,e} = \bar{\mathbf{f}}_e - \bar{\boldsymbol{a}}_e \nabla \psi_h + \frac{\bar{d}_e}{n_d}(\mathbf{x} - \bar{\mathbf{x}}_e) + \mathbf{c}_e \tag{10}$$

where $\bar{\boldsymbol{a}}_e$ and $\bar{\mathbf{f}}_e$ represent averages (componentwise) over Ω_e

$$\bar{d}_e = \frac{1}{|\Omega_e|} \int_{\Omega_e} (b - \hat{m}_t)\, \mathrm{d}x = \bar{b}_e - \bar{\hat{m}}_{t,e} \tag{11}$$

$\boldsymbol{\sigma}_{h,e}$ is in the lowest order Raviart-Thomas space on Ω_e. Local conservation

$$\begin{aligned}\int_{\Omega_e} \hat{m}_t \,\mathrm{d}x + \int_{\Omega_e} \nabla \cdot \hat{\boldsymbol{\sigma}}_{h,e} \,\mathrm{d}x - \int_{\Omega_e} b \,\mathrm{d}x &= |\Omega_e| \left(\bar{\hat{m}}_{t,e} - \bar{b}_e\right) + \bar{d}_e |\Omega_e| \\ &= 0\end{aligned}$$

P^1 NC velocity (cont'd.)

The piecewise constant $\mathbf{c}_e$ serves to enforce continuity at element interfaces and requires, in general, the solution of a local $n_d \times n_d$ system on each element, Chou and Tang(2000)

$$
\begin{aligned}
\boldsymbol{B}_e^{nc}\mathbf{c}_e &= \boldsymbol{\eta}_e \\
B_{e,ij} &= |\partial\Omega_{e,i}|n^j_{e,i}, \; i,j = 1,\ldots,n_d \\
\eta_{e,i} &= \int_{\Omega_e} b w_{h,i}\,\mathrm{d}x - \frac{|\Omega_e|}{n_d+1}\bar{b}_e - \int_{\Omega_e} \hat{m}_t w_{h,i}\,\mathrm{d}x + \frac{|\Omega_e|}{n_d+1}\bar{\hat{m}}_t
\end{aligned}
\tag{12}
$$

P^1 NC (cont'd)

eqn (8) and eqn (10) (with $\mathbf{c}_e = 0$) yield solutions equivalent to a MHFEM discretization with the correct L_2 projections and assumptions on the problem data, Marini(1985), Arbogast and Chen(1995), Chen(1996).

In essence, piecewise constant approximations are needed for $\boldsymbol{a}$ and $\mathbf{f}$ in the mixed formulation and b, $\hat{m}_t$ in the NC formulation.

- Can also view $(\psi_h, \boldsymbol{\sigma}_h)$ as the solution to a finite volume "box scheme."
- In general, we would like to keep a consistent mass integral because it's less distributed in this case
- If a consistent mass integral is used or source term, we can still recover a locally conservative $\boldsymbol{\sigma}_h$ by solving appropriate element problems for $\mathbf{c}_e = 0$.

Comparisons

We perform a series of numerical experiments to evaluate the accuracy of the CG VPP algorithm and the effectiveness of multiscale stabilization in controlling over/undershoot.

Abbrev.	Definition
CG	conforming Galerkin approximation,
CG-S	multiscale stabilized CG with shockcapturing ($\nu_c = 0.1^\dagger$)
CG-V	lumped CG approximation with vertex quadrature
NC	P^1 nonconforming approach

$\dagger$ $\nu_c = 0.5$ for Problem V.

Linear, elliptic problems

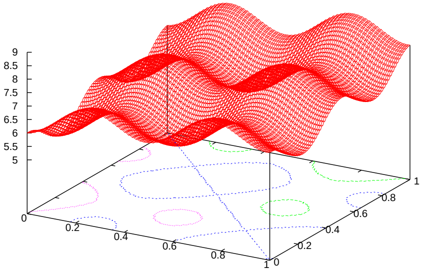
Smooth analytical solutions and domain properties

$$\begin{aligned} u(\mathbf{x}) &= \sin^2(2\pi x_1) + \cos^2(2\pi x_2) + x_1 + x_2 + 5 \\ a_{ij}(\mathbf{x}) &= (5 + x_i^2)\delta_{ij} \end{aligned} \tag{13}$$

for $n_d = 2$ and

$$\begin{aligned} u(\mathbf{x}) &= \sum_{i=1}^{3} x_i^2 \\ a_{ij}(\mathbf{x}) &= (5 + x_i^2 x_{i+1})\delta_{ij} \end{aligned} \tag{14}$$

for $n_d = 3$.



Spatial Error

Table: $\varepsilon_{u,2}$, Problem I

	level	h	N_{dof}	$\varepsilon_{u,2}$	rate
CG	4	0.0442	1089	9.36×10^{-4}	1.98
NC	4	0.0442	3136	9.86×10^{-4}	1.98
CG	5	0.0221	4225	2.35×10^{-4}	1.99
NC	5	0.0221	12416	2.48×10^{-4}	1.99

Table: $\varepsilon_{\sigma,2}$ and ε_{mc}, Problem I

	level	$\varepsilon_{\sigma,2}$	rate	ε_{mc}
CG-PE	4	0.111	0.989	0.628
CG-LN	4	0.110	0.984	0
CG-SW	4	0.111	1.00	0
NC	4	0.110	0.980	0
CG-PE	5	0.0554	0.997	0.205
CG-LN	5	0.0553	0.996	0
CG-SW	5	0.0554	1.00	10^{-6}
NC	5	0.0553	0.994	0

Boundary layer example

Van Genuchten Mualem *p-s-k* relations, $n_{vg} = 4.264$, $\alpha_{vg} = 5.47$ [1/m], $K_s = 5.04\boldsymbol{I}$ [m/d].

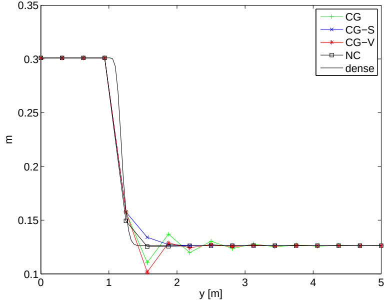


Table: CPU overhead

Method	Level	Its.	CPU [s]
LN	4	-	1.35×10^{-3}
CG-V-SW	4	250	3.33×10^{-2}
CG-S-SW	4	595	6.67×10^{-2}
LN	5	-	5.52×10^{-3}
CG-V-SW	5	83	3.33×10^{-2}
CG-S-SW	5	207	1.17×10^{-1}

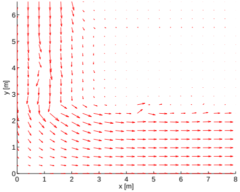
Spatial Error

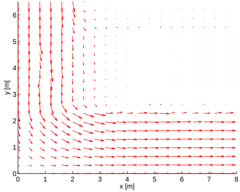
Method	Level†	N_{dof}	$\varepsilon_{\psi,\infty}$	ε_{mc}	$\varepsilon_{\sigma_1,\infty}$	$\varepsilon_{\sigma_2,\infty}$
CG-PE	4	289	8.72×10^{-2}	2.26×10^{-3}	4.40×10^{-3}	2.19×10^{0}
CG-LN	4	289	8.72×10^{-2}	1.35×10^{-8}	4.37×10^{-4}	1.89×10^{-1}
CG-S-LN	4	289	4.36×10^{-2}	3.00×10^{-7}	3.90×10^{-4}	3.65×10^{-1}
CG-S-SW	4	289	4.36×10^{-2}	9.97×10^{-7}	2.10×10^{-3}	6.91×10^{-1}
CG-V-LN	4	289	1.75×10^{-1}	9.33×10^{-9}	2.29×10^{-4}	9.84×10^{-2}
CG-V-SW	4	289	1.75×10^{-1}	9.93×10^{-7}	6.99×10^{-4}	2.49×10^{-1}
NC	4	800	9.49×10^{-2}	0	5.21×10^{-8}	1.66×10^{-4}
CG-PE	5	1089	3.28×10^{-2}	7.17×10^{-4}	2.22×10^{-3}	1.61×10^{0}
CG-LN	5	1089	3.28×10^{-2}	6.10×10^{-8}	3.29×10^{-4}	1.71×10^{-1}
CG-S-LN	5	1089	2.30×10^{-2}	8.88×10^{-8}	3.39×10^{-4}	3.58×10^{-1}
CG-S-SW	5	1089	2.30×10^{-2}	9.97×10^{-7}	1.11×10^{-3}	3.98×10^{-1}
CG-V-LN	5	1089	5.58×10^{-2}	4.92×10^{-9}	2.54×10^{-4}	1.12×10^{-1}
CG-V-SW	5	1089	5.58×10^{-2}	9.98×10^{-7}	4.99×10^{-4}	1.45×10^{-1}
NC	5	3136	2.32×10^{-2}	0	2.71×10^{-7}	1.55×10^{-5}

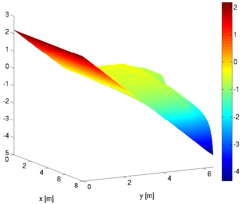
† $h = 0.319$ on level 4, and $h = 0.160$ on level 5

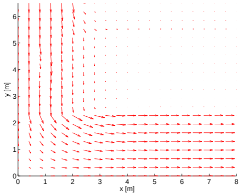
Block heterogeneous example

Constant recharge into two-dimensional domain, n_{vg} ranges from 1.632 to 5.



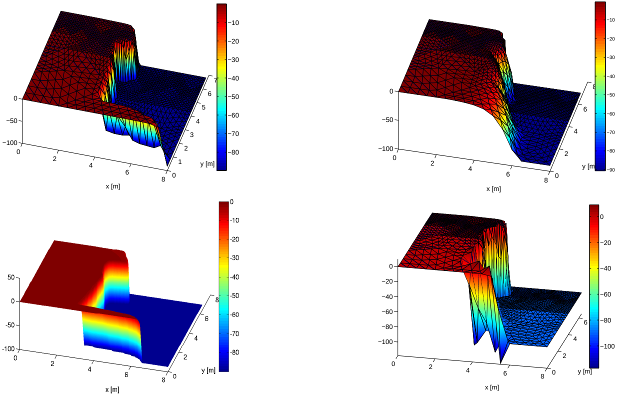






Block heterogeneous example (transient)

Infiltration into two-dimensional domain, n_{vg} ranges from 1.632 to 5.



Revisiting P^1 nonconforming behavior for sharp fronts

At least two options to improve NC. Subgrid viscosity stabilization Alaoui and Ern (2006), and/or local refinement