A Fractional Step θ-Method for Time Dependent Viscoelastic Fluid Flow

John Chrispell

March, 2008

Advisors: Dr. Lea Jenkins and Dr. Vincent Ervin

CLEMSON
UNIVERSITY

Play Fluid Movie

A pool filled with non-newtonian fluid

Motivation

The Time Dependent Johnson-Segalman Model
for Viscoelastic fluid flow:

$$
\begin{aligned}
\boldsymbol{\sigma} + \lambda \left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) \right) - 2\alpha \mathbf{d}(\boldsymbol{u}) &= 0 \ \text{ in } \Omega \\
Re \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \right) + \nabla p - 2(1-\alpha) \nabla \cdot \mathbf{d}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{\sigma} &= \mathbf{f} \ \text{ in } \Omega \\
\nabla \cdot \boldsymbol{u} &= 0, \ \text{ in } \ \Omega \\
\boldsymbol{u} &= 0, \ \text{ on } \ \partial\Omega \\
\boldsymbol{u}(x,0) &= \boldsymbol{u}_0 \ \text{ in } \ \Omega \\
\boldsymbol{\sigma}(x,0) &= \boldsymbol{\sigma}_0 \ \text{ in } \ \Omega
\end{aligned}
$$

where

$$
g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) = \frac{1-a}{2} \left(\boldsymbol{\sigma} \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \boldsymbol{\sigma} \right) - \frac{1+a}{2} \left(\boldsymbol{\sigma} \nabla \boldsymbol{u} + \boldsymbol{\sigma} \nabla \boldsymbol{u}^T \right)
$$

$$
\mathbf{d}(\boldsymbol{u}) = \frac{1}{2} \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \right)
$$

Outline

The Fractional Step θ-Method

Setup for Viscoelastic Fluid Flow

Idea of the Analysis

Numerical Results full method

Error Estimates for Stress, and Stokes

Idea of Proof

Contraction Problem

Summary

The Fractional Step Method

Abstract form:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + F(u,x,t) &= 0 && \text{in } \Omega \times (0,T] \\
\text{subject to} \quad u(x,t) &= 0 && x \in \partial\Omega \times (0,T) \\
u(x,0) &= u_0(x) && x \in \Omega
\end{aligned}
$$

Additively split F:

$$F(u,x,t) = {}^1F(u,x,t) + {}^2F(u,x,t)$$

For example (convection diffusion):

$$
\begin{aligned}
{}^1F(u,x,t) &= -\nabla \cdot k\nabla u + \frac{c}{2}u - f \\
{}^2F(u,x,t) &= \mathbf{b} \cdot \nabla u + \frac{c}{2}u
\end{aligned}
$$

Specialized Solution Techniques

Stabilize Hyperbolic Problems with Streamline Upwinding

Variational formulation viscoelasticity: $(\boldsymbol{\sigma}, \boldsymbol{\tau} + \delta \boldsymbol{u} \cdot \nabla \boldsymbol{\tau})$

Here $(\boldsymbol{\sigma}, \mathbf{w}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{w} \, dA$ and δ is a small positive constant.

Specialized Solution Techniques

Stabilize Hyperbolic Problems with Streamline Upwinding

Variational formulation viscoelasticity: $(\boldsymbol{\sigma}, \boldsymbol{\tau} + \delta \boldsymbol{u} \cdot \nabla \boldsymbol{\tau})$

Here $(\boldsymbol{\sigma}, \mathbf{w}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{w} \, dA$ and δ is a small positive constant.

Decouple the unknowns

For viscoelasticity velocity and pressure are separated from stress.

Stokes like problem	constitutive eqn.
$\boldsymbol{u}, p$	$\boldsymbol{\sigma}$

The Fractional Step Method

Specialized Solution Techniques

Stabilize Hyperbolic Problems with Streamline Upwinding

Variational formulation viscoelasticity: $(\boldsymbol{\sigma}, \boldsymbol{\tau} + \delta \boldsymbol{u} \cdot \nabla \boldsymbol{\tau})$

Here $(\boldsymbol{\sigma}, \mathbf{w}) = \int_\Omega \boldsymbol{\sigma} : \mathbf{w} \, dA$ and δ is a small positive constant.

Decouple the unknowns

For viscoelasticity velocity and pressure are separated from stress.

Stokes like problem	constitutive eqn.
$\boldsymbol{u}, p$	$\boldsymbol{\sigma}$

Only Linear Systems to be Solved

Viscoelastic Fluids

$$
\begin{aligned}
\boldsymbol{\sigma} + \lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla\boldsymbol{u})\right) - 2\alpha\mathbf{d}(\boldsymbol{u}) &= 0 \ \text{ in } \Omega \\
Re\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p - 2(1-\alpha)\nabla\cdot\mathbf{d}(\boldsymbol{u}) - \nabla\cdot\boldsymbol{\sigma} &= \mathbf{f} \ \text{ in } \Omega \\
\nabla\cdot\boldsymbol{u} &= 0, \ \text{ in } \ \Omega \\
\boldsymbol{u} &= 0, \ \text{ on } \ \partial\Omega \\
\boldsymbol{u}(x,0) &= \boldsymbol{u}_0 \ \text{ in } \ \Omega \\
\boldsymbol{\sigma}(x,0) &= \boldsymbol{\sigma}_0 \ \text{ in } \ \Omega
\end{aligned}
$$

where

$$
g_a(\boldsymbol{\sigma}, \nabla\boldsymbol{u}) = \frac{1-a}{2}\left(\boldsymbol{\sigma}\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\boldsymbol{\sigma}\right) - \frac{1+a}{2}\left(\boldsymbol{\sigma}\nabla\boldsymbol{u} + \boldsymbol{\sigma}\nabla\boldsymbol{u}^T\right)
$$

$$
\mathbf{d}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)
$$

Viscoelastic Fluids

$$
\begin{aligned}
\boldsymbol{\sigma} + \lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\sigma} + g_a\left(\boldsymbol{\sigma}, \nabla\boldsymbol{u}\right)\right) - 2\alpha\mathbf{d}(\boldsymbol{u}) &= 0 \text{ in } \Omega \\
Re\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p - 2\left(1-\alpha\right)\nabla\cdot\mathbf{d}(\boldsymbol{u}) - \nabla\cdot\boldsymbol{\sigma} &= \mathbf{f} \text{ in } \Omega \\
\nabla\cdot\boldsymbol{u} &= 0, \text{ in } \Omega \\
\boldsymbol{u} &= 0, \text{ on } \partial\Omega \\
\boldsymbol{u}(x,0) &= \boldsymbol{u}_0 \text{ in } \Omega \\
\boldsymbol{\sigma}(x,0) &= \boldsymbol{\sigma}_0 \text{ in } \Omega
\end{aligned}
$$

where

$$
g_a\left(\boldsymbol{\sigma}, \nabla\boldsymbol{u}\right) = \frac{1-a}{2}\left(\boldsymbol{\sigma}\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\boldsymbol{\sigma}\right) - \frac{1+a}{2}\left(\boldsymbol{\sigma}\nabla\boldsymbol{u} + \boldsymbol{\sigma}\nabla\boldsymbol{u}^T\right)
$$

$$
\mathbf{d}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)
$$

Assumption:

$$
\textbf{Slow flow} \Longrightarrow \boldsymbol{u}\cdot\nabla\boldsymbol{u} = 0
$$

Viscoelastic Fluids

$$
\begin{aligned}
\boldsymbol{\sigma} + \lambda\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla\boldsymbol{u})\right) - 2\alpha\mathbf{d}(\boldsymbol{u}) &= 0 \ \text{ in } \Omega \\
Re\frac{\partial \boldsymbol{u}}{\partial t} + \nabla p - 2(1-\alpha)\nabla\cdot\mathbf{d}(\boldsymbol{u}) - \nabla\cdot\boldsymbol{\sigma} &= \mathbf{f} \ \text{ in } \Omega \\
\nabla\cdot\boldsymbol{u} &= 0, \ \text{ in } \ \Omega \\
\boldsymbol{u} &= 0, \ \text{ on } \ \partial\Omega \\
\boldsymbol{u}(x,0) &= \boldsymbol{u}_0 \ \text{ in } \ \Omega \\
\boldsymbol{\sigma}(x,0) &= \boldsymbol{\sigma}_0 \ \text{ in } \ \Omega
\end{aligned}
$$

where

$$
g_a(\boldsymbol{\sigma}, \nabla\boldsymbol{u}) = \frac{1-a}{2}\left(\boldsymbol{\sigma}\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\boldsymbol{\sigma}\right) - \frac{1+a}{2}\left(\boldsymbol{\sigma}\nabla\boldsymbol{u} + \boldsymbol{\sigma}\nabla\boldsymbol{u}^T\right)
$$

$$
\mathbf{d}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right)
$$

Assumption:

Slow flow $\Longrightarrow \boldsymbol{u}\cdot\nabla\boldsymbol{u} = 0$

Note: $a = 1$ **gives Oldroyd B-type Constitutive equation**

Split for Viscoelasticity

Constitutive equation:

$$\begin{aligned} F_{\boldsymbol{\sigma}_1} &:= \omega\boldsymbol{\sigma} \\ F_{\boldsymbol{\sigma}_2} &:= (1-\omega)\boldsymbol{\sigma} + \lambda\left(\boldsymbol{u}\cdot\nabla\boldsymbol{\sigma} + g_a(\boldsymbol{\sigma},\nabla\boldsymbol{u})\right) - 2\alpha\mathbf{d}(\boldsymbol{u}) \end{aligned}$$

Conservation of Momentum:

$$\begin{aligned} F_{\boldsymbol{u}_1} &:= -2(1-\alpha)\nabla\cdot\mathbf{d}(\boldsymbol{u}) - \nabla\cdot\boldsymbol{\sigma} - f \\ F_{\boldsymbol{u}_2} &:= 0 \end{aligned}$$

The viscoelastic model is

$$\begin{aligned} \lambda\frac{\partial\boldsymbol{\sigma}}{\partial t} + F_{\boldsymbol{\sigma}_1} + F_{\boldsymbol{\sigma}_2} &= \mathbf{0} \\ Re\frac{\partial\boldsymbol{u}}{\partial t} + \nabla p + F_{\boldsymbol{u}_1} + F_{\boldsymbol{u}_2} &= \mathbf{0} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned}$$

Split for Viscoelasticity

Constitutive equation:

$$\begin{aligned} F_{\boldsymbol{\sigma}_1} &:= \omega \boldsymbol{\sigma} \\ F_{\boldsymbol{\sigma}_2} &:= (1-\omega)\boldsymbol{\sigma} + \lambda \left(\boldsymbol{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\boldsymbol{\sigma}, \nabla \boldsymbol{u}) \right) - 2\alpha \mathbf{d}(\boldsymbol{u}) \end{aligned}$$

Conservation of Momentum:

$$\begin{aligned} F_{\boldsymbol{u}_1} &:= -2(1-\alpha)\nabla \cdot \mathbf{d}(\boldsymbol{u}) - \nabla \cdot \boldsymbol{\sigma} - f \\ F_{\boldsymbol{u}_2} &:= 0 \end{aligned}$$

The viscoelastic model is

$$\begin{aligned} \lambda \frac{\partial \boldsymbol{\sigma}}{\partial t} + F_{\boldsymbol{\sigma}_1} + F_{\boldsymbol{\sigma}_2} &= \mathbf{0} \\ Re \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p + F_{\boldsymbol{u}_1} + F_{\boldsymbol{u}_2} &= \mathbf{0} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

What have we done?

- decoupled the velocity and pressure from the stress.
- linearized the computational equations.

θ-Method Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\boldsymbol{u}_1}^{(n+\theta)} &= -F_{\boldsymbol{u}_2}^{(n)} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

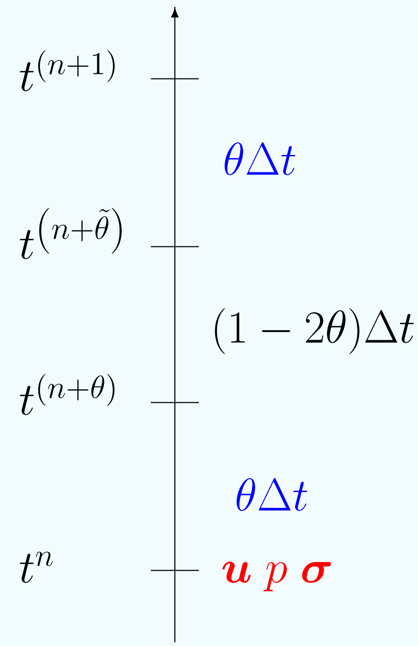
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F_{\boldsymbol{u}_2}^{\left(n+\tilde{\theta}\right)} &= -F_{\boldsymbol{u}_1}^{(n+\theta)} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_2}^{\left(n+\tilde{\theta}\right)} = -F_{\boldsymbol{\sigma}_1}^{(n+\theta)}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



θ-Method Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\boldsymbol{u}_1}^{(n+\theta)} &= -F_{\boldsymbol{u}_2}^{(n)} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

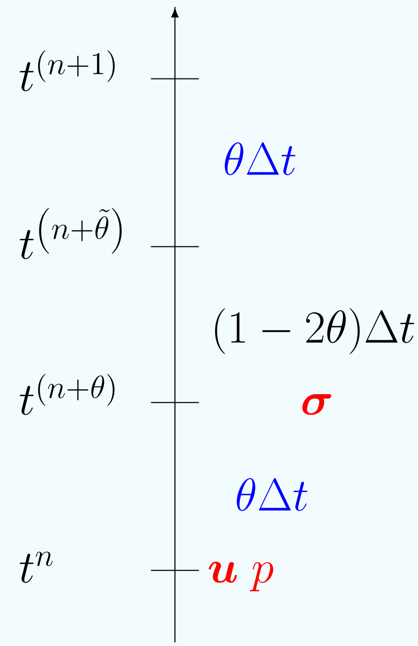
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F_{\boldsymbol{u}_2}^{\left(n+\tilde{\theta}\right)} &= -F_{\boldsymbol{u}_1}^{(n+\theta)} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_2}^{\left(n+\tilde{\theta}\right)} = -F_{\boldsymbol{\sigma}_1}^{(n+\theta)}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



θ-Method Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\boldsymbol{u}_1}^{(n+\theta)} &= -F_{\boldsymbol{u}_2}^{(n)} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

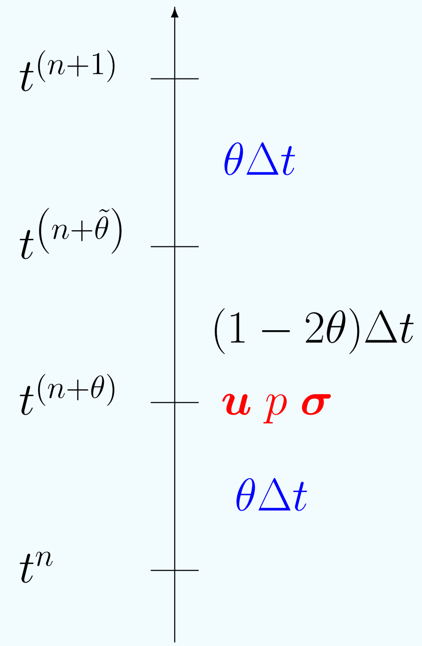
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F_{\boldsymbol{u}_2}^{\left(n+\tilde{\theta}\right)} &= -F_{\boldsymbol{u}_1}^{(n+\theta)} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_2}^{\left(n+\tilde{\theta}\right)} = -F_{\boldsymbol{\sigma}_1}^{(n+\theta)}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



θ-Method Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F^{(n+\theta)}_{\boldsymbol{\sigma}_1} = -F^{(n)}_{\boldsymbol{\sigma}_2}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F^{(n+\theta)}_{\boldsymbol{u}_1} &= -F^{(n)}_{\boldsymbol{u}_2} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

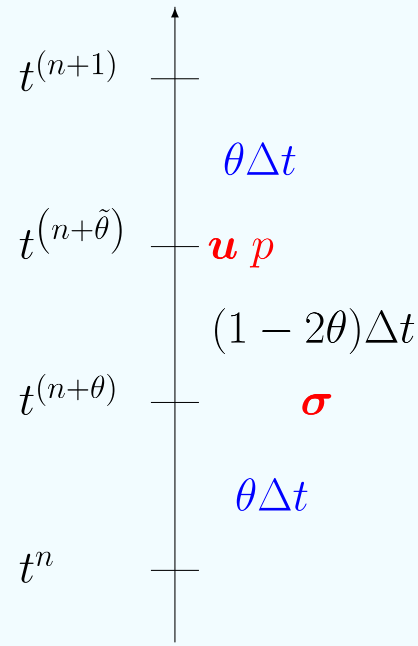
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F^{\left(n+\tilde{\theta}\right)}_{\boldsymbol{u}_2} &= -F^{(n+\theta)}_{\boldsymbol{u}_1} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F^{\left(n+\tilde{\theta}\right)}_{\boldsymbol{\sigma}_2} = -F^{(n+\theta)}_{\boldsymbol{\sigma}_1}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



θ-Method Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\boldsymbol{u}_1}^{(n+\theta)} &= -F_{\boldsymbol{u}_2}^{(n)} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

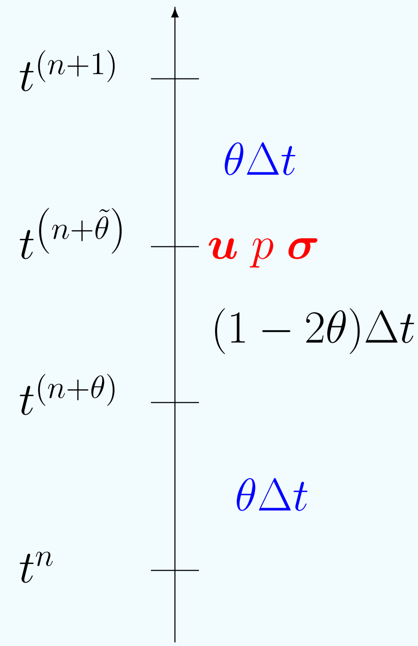
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F_{\boldsymbol{u}_2}^{\left(n+\tilde{\theta}\right)} &= -F_{\boldsymbol{u}_1}^{(n+\theta)} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_2}^{\left(n+\tilde{\theta}\right)} = -F_{\boldsymbol{\sigma}_1}^{(n+\theta)}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\boldsymbol{u}_1}^{(n+\theta)} &= -F_{\boldsymbol{u}_2}^{(n)} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

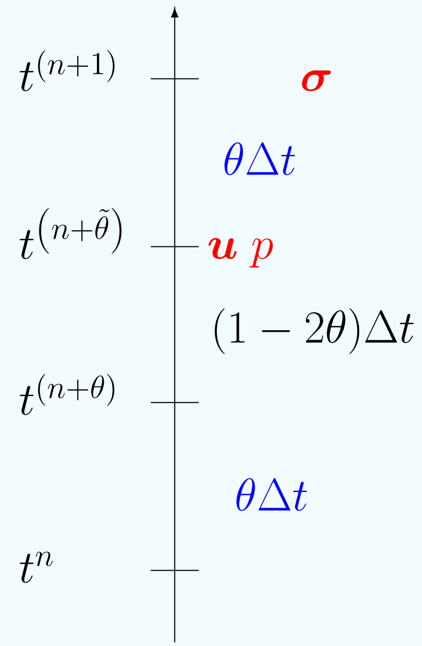
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F_{\boldsymbol{u}_2}^{\left(n+\tilde{\theta}\right)} &= -F_{\boldsymbol{u}_1}^{(n+\theta)} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_2}^{\left(n+\tilde{\theta}\right)} = -F_{\boldsymbol{\sigma}_1}^{(n+\theta)}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



θ-Method Viscoelasticity

Step 1a: (Update the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{(n+\theta)} - \boldsymbol{\sigma}^{(n)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_1}^{(n+\theta)} = -F_{\boldsymbol{\sigma}_2}^{(n)}$$

Step 1b: (Solve for velocity and pressure.)

$$\begin{aligned} Re \frac{\boldsymbol{u}^{(n+\theta)} - \boldsymbol{u}^{(n)}}{\theta \Delta t} + \nabla p^{(n+\theta)} + F_{\boldsymbol{u}_1}^{(n+\theta)} &= -F_{\boldsymbol{u}_2}^{(n)} \\ \nabla \cdot \boldsymbol{u}^{(n+\theta)} &= 0 \end{aligned}$$

Step 2a: (Update the velocity and pressure.)

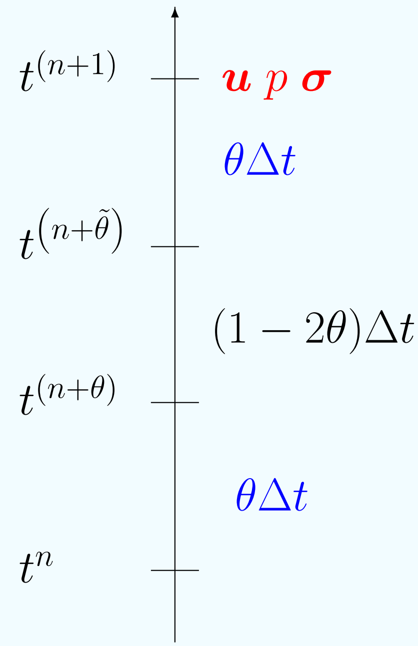
$$\begin{aligned} Re \frac{\boldsymbol{u}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{u}^{(n+\theta)}}{\theta \Delta t} + \nabla p^{\left(n+\tilde{\theta}\right)} + F_{\boldsymbol{u}_2}^{\left(n+\tilde{\theta}\right)} &= -F_{\boldsymbol{u}_1}^{(n+\theta)} \\ \nabla \cdot \boldsymbol{u}^{\left(n+\tilde{\theta}\right)} &= 0 \end{aligned}$$

Step 2b: (Solve for the stress.)

$$\lambda \frac{\boldsymbol{\sigma}^{\left(n+\tilde{\theta}\right)} - \boldsymbol{\sigma}^{(n+\theta)}}{\theta \Delta t} + F_{\boldsymbol{\sigma}_2}^{\left(n+\tilde{\theta}\right)} = -F_{\boldsymbol{\sigma}_1}^{(n+\theta)}$$

Step 3a and 3b: (Repeat Step 1a and 1b)

with $(n) = \left(n + \tilde{\theta}\right)$ and $(n+\theta) = (n+1)$



Optimal θ Value

Second order w.r.t. Δt:

Taylor series during analysis:

The first order terms in the expansions (the coefficients of Δt) all reduce to a multiple of:

$$2\theta^2 \; - \; 4\theta \; + \; 1 \, ,$$

and this has roots of $\theta \; = \; 1 \pm \sqrt{2}/2$.

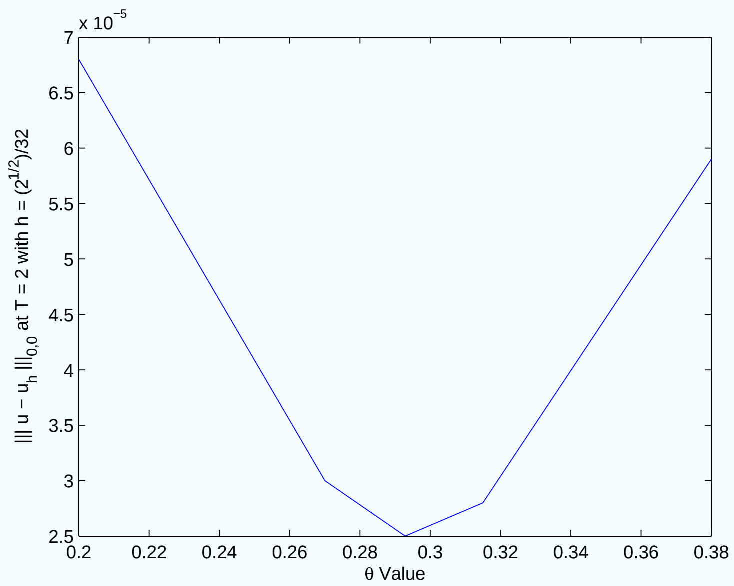
Optimal choice:

For θ in $(0, 1/2)$

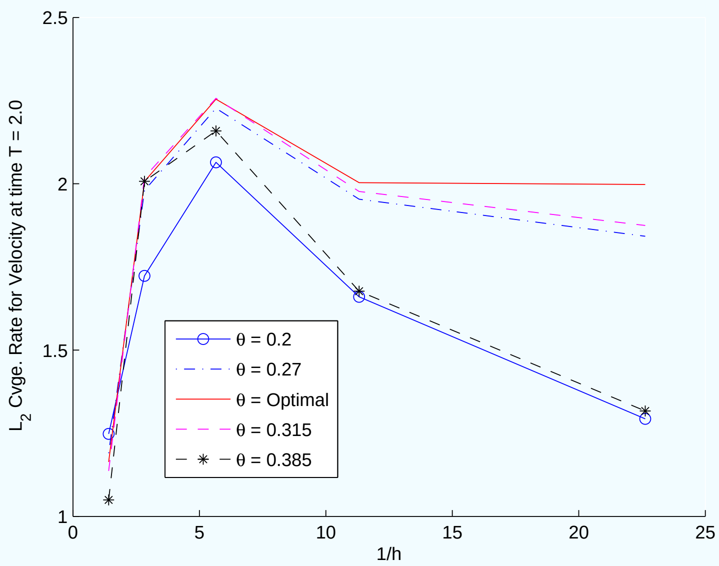
$$\theta = 1 - \frac{\sqrt{2}}{2} \qquad \Longrightarrow \qquad \text{error is } O\left((\Delta t)^2\right).$$

Optimal θ Value

Error and experimental convergence rate plots.



(a) Error $|||\boldsymbol{u} - \boldsymbol{u}_h|||_{0,0}$ as a function of θ.



(b) Experimental convergence rates.

Error Analysis

Define the following notation:

$$\tilde{\boldsymbol{u}}_h \;:=\; \text{discrete approximation using true } \boldsymbol{\sigma},$$
$$\tilde{\boldsymbol{\sigma}}_h \;:=\; \text{discrete approximation using true } \boldsymbol{u}, \text{ and } p.$$

Define following norms:

$$\|v\|_{0,k} := \left(\int_0^T \|v(\cdot,t)\|_k^2 \, dt\right)^{1/2}, \quad |||v|||_{0,k} := \left(\sum_{n=1}^{N} \Delta t \, \|v^n\|_k^2\right)^{1/2},$$

$$\|v\|_{\infty,k} := \sup_{0<t<T} \|v(\cdot,t)\|_k \,, \quad |||v|||_{\infty,k} := \max_{1<n<N} \|v^n\|_k \,.$$

Error Analysis

Fully decouple the analysis into two distinct pieces.

Error Analysis

Fully decouple the analysis into two distinct pieces.

Analyze Stokes problem for $\tilde{\boldsymbol{u}}_h$ and $\tilde{p}_h$:

- Consider only Steps 1b, 2a, and 3b of the method.
- $\tilde{\boldsymbol{u}}_h$ is the θ-method approximation assuming a known true value for $\boldsymbol{\sigma}$.
- Obtain a priori error estimate

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} = O(\Delta t^2, h^2),$$

using Taylor-Hood element pair.

Error Analysis

Fully decouple the analysis into two distinct pieces.

Analyze Stokes problem for $\tilde{\boldsymbol{u}}_h$ and $\tilde{p}_h$:

- Consider only Steps 1b, 2a, and 3b of the method.
- $\tilde{\boldsymbol{u}}_h$ is the θ-method approximation assuming a known true value for $\boldsymbol{\sigma}$.
- Obtain a priori error estimate

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} = O(\Delta t^2, h^2),$$

using Taylor-Hood element pair.

Analyze constitutive equation for $\tilde{\boldsymbol{\sigma}}_h$:

- Consider only Steps 1a, 2b, and 3a of the method.
- $\tilde{\boldsymbol{\sigma}}_h$ is the θ-method approximation assuming known true values for $\boldsymbol{u}$ and p.
- Obtain a priori error estimates

$$|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0} = O(\Delta t^2, \Delta t\delta, \delta h, \delta, h),$$

using piecewise linear elements.

Error Analysis

Fully decouple the analysis into two distinct pieces.

Analyze Stokes problem for $\tilde{\boldsymbol{u}}_h$ and $\tilde{p}_h$:

- Consider only Steps 1b, 2a, and 3b of the method.
- $\tilde{\boldsymbol{u}}_h$ is the θ-method approximation assuming a known true value for $\boldsymbol{\sigma}$.
- Obtain a priori error estimate

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} = O(\Delta t^2, h^2),$$

using Taylor-Hood element pair.

Analyze constitutive equation for $\tilde{\boldsymbol{\sigma}}_h$:

- Consider only Steps 1a, 2b, and 3a of the method.
- $\tilde{\boldsymbol{\sigma}}_h$ is the θ-method approximation assuming known true values for $\boldsymbol{u}$ and p.
- Obtain a priori error estimates

$$|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0} = O(\Delta t^2, \Delta t\delta, \delta h, \delta, h),$$

using piecewise linear elements.

Bring the two distinct pieces together.

For a bound on the full approximation technique for viscoelastic fluid flow.

- Induction argument
- Triangle inequality

$$\begin{aligned}\left\|\boldsymbol{u}^\theta - \boldsymbol{u}_h^\theta\right\|^2 &\leq \left(\left\|\boldsymbol{u}^\theta - \tilde{\boldsymbol{u}}_h^\theta\right\| + \left\|\tilde{\boldsymbol{u}}_h^\theta - \boldsymbol{u}_h^\theta\right\|\right)^2 \\ &\leq 2\left\|\boldsymbol{u}^\theta - \tilde{\boldsymbol{u}}_h^\theta\right\|^2 + 2\left\|\tilde{\boldsymbol{u}}_h^\theta - \boldsymbol{u}_h^\theta\right\|^2\end{aligned}$$

Note: $\tilde{\boldsymbol{u}}_h^\theta - \boldsymbol{u}_h^\theta$ is in the approximation space.

Error Analysis

Walk Forward in Time.

For any step we get the bound: $\left\|\hat{\boldsymbol{u}}_h^n\right\|^2 \leq C + B\left\|\hat{\boldsymbol{u}}_h^{n-1}\right\|.$

Assuming that $\left\|\hat{\boldsymbol{u}}_h^0\right\|^2 = 0$ gives

$$
\begin{aligned}
\left\|\hat{\boldsymbol{u}}_h^0\right\|^2 &= 0, \\
\\
\left\|\hat{\boldsymbol{u}}_h^1\right\|^2 &= C + B\left\|\hat{\boldsymbol{u}}_h^0\right\| \\
&= C, \\
\\
\left\|\hat{\boldsymbol{u}}_h^2\right\|^2 &= C + B\left\|\hat{\boldsymbol{u}}_h^1\right\| \\
&= C(1+B), \\
\\
\left\|\hat{\boldsymbol{u}}_h^3\right\|^2 &= C + B\left\|\hat{\boldsymbol{u}}_h^2\right\|^2 \\
&= C + BC(1+B) \\
&= C(1+B+B^2), \\
&\vdots \\
\left\|\hat{\boldsymbol{u}}_h^n\right\|^2 &= C + B\left\|\hat{\boldsymbol{u}}_h^{n-1}\right\|^2 \\
&= C(1+B+B^2+\ldots+B^{n-1}),
\end{aligned}
$$

Example Problem

Let $\Omega = (0,1) \times (0,1)$, and set

$$Re = 1, \quad \alpha = 1/2, \quad \lambda = 2, \quad \text{and} \;\; a = 1.$$

Use the (optimal) value of $\theta = 1 - \sqrt{2}/2 \approx 0.29289$ and set $\omega = 1/2$.

For the true solution:

$$\boldsymbol{u} = \begin{pmatrix} e^{\left(x+y-\frac{1}{2}t\right)}(x^2 - x)(y^2 - y) \\ -e^{(x+y-t)}(x^2 - x)(y^2 - y) \end{pmatrix}, \tag{1}$$

$$p = \cos(2\pi x)(y^2 - y), \tag{2}$$

$$\boldsymbol{\sigma} = 2\alpha \mathbf{d}(\boldsymbol{u}). \tag{3}$$

Remark:

- A RHS function is added to the constitutive equation.
- f in momentum is calculated using (1)-(3).

Observed Convergence Rates

Move to Full Table

δ	$\|\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|\|_{0,1}$	$\|\|\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|\|\|_{0,0}$
0	2	2
$O(h)$	1	1
$O(h^{3/2})$	$3/2$	$3/2$
$O(h^2)$	2	2

Note: Convergence rate for $|||p-p_h|||_{0,0}=1$ independent of δ.

$|||\boldsymbol{u}-\tilde{\boldsymbol{u}}_h|||_{0,1}=O(\Delta t^2,h^2)$, $\quad |||\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}_h|||_{0,0}=O(\Delta t^2,\Delta t\delta,\delta h,\delta,h)$

Error Constitutive Equation

Theorem 1 (Assuming u and p are known)**.** *For a sufficiently smooth solution $\boldsymbol{\sigma}$, $\boldsymbol{u}$, p such that*

$$\|\boldsymbol{u}\|_\infty, \|\boldsymbol{u}_t\|_\infty, \|\boldsymbol{u}_{tt}\|_\infty, \|\nabla\boldsymbol{u}\|_\infty, \|(\nabla\boldsymbol{u})_t\|_\infty, \text{ and } \|(\nabla\boldsymbol{u})_{tt}\|_\infty \le M, \quad \forall\, t \in [0,T],$$

$\Delta t \le Ch^2$, *the fractional step θ-method approximation, $\tilde{\boldsymbol{\sigma}}_h$ given by Step 1a, Step 2b, and Step 3a converges to $\boldsymbol{\sigma}$ on the interval $(0,T]$ as $\Delta t, h \to 0$, and satisfies the error estimates:*

$$|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{\infty,0} \le F_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + Ch^{m+1}\, |||\boldsymbol{\sigma}|||_{\infty,m+1}, \tag{4}$$

$$|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0} \le F_{\boldsymbol{\sigma}}(\Delta t, h, \delta) + Ch^{m+1}\, |||\boldsymbol{\sigma}|||_{0,m+1}, \tag{5}$$

where

$$\begin{aligned} F_{\boldsymbol{\sigma}}(\Delta t, h, \delta) \; := \; & C(\Delta t)^2\Big(\|\boldsymbol{\sigma}_{ttt}\|_{0,0} + \|\boldsymbol{\sigma}_{tt}\|_{0,1} + \|\boldsymbol{\sigma}_t\|_{0,1} + \|\boldsymbol{\sigma}\|_{0,1} \\ & \quad + \|\boldsymbol{\sigma}_{tt}\|_{0,0} + \|\boldsymbol{\sigma}_t\|_{0,0} + \|\boldsymbol{\sigma}\|_{0,0} + C_T\Big) \\ & + C(\Delta t)\delta\left(\|\boldsymbol{\sigma}\|_{0,1} + \|\boldsymbol{\sigma}_t\|_{0,1} + \|\boldsymbol{\sigma}\|_{0,0} + \|\boldsymbol{\sigma}_t\|_{0,0} + C_T\right) \\ & + C\left(h^{m+1} + h^m + \delta h^m\right) |||\boldsymbol{\sigma}|||_{0,m+1} \\ & + Ch^{m+1}\, \|\boldsymbol{\sigma}_t\|_{0,m+1} + \; C\delta\, |||\boldsymbol{\sigma}_t|||_{0,0}. \end{aligned} \tag{6}$$

Using piecewise continuous linear elements for $\tilde{\boldsymbol{\sigma}}_h$:

$|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0} = O(\Delta t^2, \Delta t\delta, \delta h, \delta, h)$ Move to Results

Outline of Proof

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1.

Step 2.

Step 3.

Step 4.

Step 5.

Outline of Proof

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt:

$$\tilde{\boldsymbol{\sigma}}_h^{(n)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+1)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{\left(n+\tilde{\theta}\right)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}.$$

Step 2.

Step 3.

Step 4.

Step 5.

Outline of Proof

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt:

$$\tilde{\boldsymbol{\sigma}}_h^{(n)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+1)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{\left(n+\tilde{\theta}\right)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3.

Step 4.

Step 5.

Outline of Proof

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt:

$$\tilde{\boldsymbol{\sigma}}_h^{(n)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+1)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{\left(n+\tilde{\theta}\right)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3. Sum the linear combinations from $n = 0$ to $n = l - 1$, making note that each telescopes. Add the linear combinations together forming a single expression.

Step 4.

Step 5.

Outline of Proof

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt:

$$\tilde{\boldsymbol{\sigma}}_h^{(n)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+1)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{\left(n+\tilde{\theta}\right)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3. Sum the linear combinations from $n = 0$ to $n = l - 1$, making note that each telescopes. Add the linear combinations together forming a single expression.

Step 4. Apply suitable inequalities/estimates to the terms in the single expression.

Step 5.

Outline of Proof

Corresponds Step 1a, Step 2b, and Step 3a of the algorithm.

Step 1. Linear combinations of variational formulations are used to obtain *unit strides* of size Δt:

$$\tilde{\boldsymbol{\sigma}}_h^{(n)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+1)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n-\theta)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{\left(n+\tilde{\theta}\right)}, \qquad \tilde{\boldsymbol{\sigma}}_h^{(n+\theta-1)} \text{ to } \tilde{\boldsymbol{\sigma}}_h^{(n+\theta)}.$$

Step 2. Evaluate the true solution at the midpoint of each *unit stride* and subtract it from each linear combination.

Step 3. Sum the linear combinations from $n = 0$ to $n = l - 1$, making note that each telescopes. Add the linear combinations together forming a single expression.

Step 4. Apply suitable inequalities/estimates to the terms in the single expression.

Step 5. Apply Gronwall's lemma and the triangle inequality to obtain error estimates for

$$\left\|\boldsymbol{\sigma}^{(l)} - \tilde{\boldsymbol{\sigma}}_h^{(l)}\right\| + \left\|\boldsymbol{\sigma}^{(l-\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(l-\theta)}\right\| + \left\|\boldsymbol{\sigma}^{(l-1+\theta)} - \tilde{\boldsymbol{\sigma}}_h^{(l-1+\theta)}\right\|.$$

Error Stress

Observed Convergence Rates

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$\left(1, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$
0	$\vert\vert\vert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\vert\vert\vert_{0,0}$	2.1235e-1	6.6773e-2	1.9191e-2	5.0437e-3	1.2830e-3
	Cvge. Rate	-	1.7	1.8	1.9	2.0
$\frac{h}{\sqrt{2}}$	$\vert\vert\vert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\vert\vert\vert_{0,0}$	2.0070e-1	8.4563e-2	3.7449e-2	1.6980e-2	8.0428e-3
	Cvge. Rate	-	1.2	1.2	1.1	1.1
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$\vert\vert\vert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\vert\vert\vert_{0,0}$	2.0174e-1	7.3678e-2	2.3575e-2	6.9629e-3	2.0645e-3
	Cvge. Rate	-	1.5	1.6	1.8	1.8
$\left(\frac{h}{\sqrt{2}}\right)^{2}$	$\vert\vert\vert\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\vert\vert\vert_{0,0}^2$	2.0346e-1	6.9501e-2	2.0245e-2	5.3281e-3	1.3546e-3
	Cvge. Rate	-	1.5	1.8	1.9	2.0

$$|||\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h|||_{0,0} = O(\Delta t^2, \Delta t\delta, \delta h, \delta, h)$$

Theorem 2 (Assuming $\boldsymbol{\sigma}$ is known)**.** *For a sufficiently smooth solutions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, p such that $\|\boldsymbol{\sigma}\|_\infty \le M, \ \forall\, t \in [0,T]$, and $\Delta t \le Ch^2$, the fractional step θ-method approximation, $\tilde{\boldsymbol{u}}_h$ given by Step 1b, Step 2a, and Step 3b converges to $\boldsymbol{u}$ on the interval $(0,T]$ as $\Delta t, h \to 0$, and satisfies the error estimates:*

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{\infty,0} \le F_{\boldsymbol{u}}(\Delta t, h, \delta) + Ch^{k+1}\, |||\boldsymbol{u}|||_{\infty,k+1}\,, \tag{7}$$

and

$$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} \le F_{\boldsymbol{u}}(\Delta t, h, \delta) + Ch^{k}\, |||\boldsymbol{u}|||_{0,k+1}\,, \tag{8}$$

where

$$\begin{aligned} F_{\boldsymbol{u}}(\Delta t, h, \delta) \ :=\ & Ch^{k+1}\,\|\boldsymbol{u}_t\|_{0,k+1} + Ch^k\,|||\boldsymbol{u}|||_{0,k+1} + Ch^{q+1}\,|||p|||_{0,q+1} \\ & + C(\Delta t)^2\,\|\boldsymbol{u}_{ttt}\|_{0,0} + C(\Delta t)^2\,\|\boldsymbol{u}_{tt}\|_{0,1} + C(\Delta t)^2\,\|\mathbf{f}_{tt}\|_{0,0} \\ & + C(\Delta t)^2 C_T. \end{aligned} \tag{9}$$

Using piecewise continuous quadratic elements for $\tilde{\boldsymbol{u}}_h$,
and piecewise continuous linear elements for $\tilde{p}_h$:

$|||\boldsymbol{u} - \tilde{\boldsymbol{u}}_h|||_{0,1} = O(\Delta t^2, h^2)$ Move to Results

Error Stokes Problem

Table 1: Convergence rates at $T = 2$

$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$	$\left(\frac{1}{64}, \frac{\sqrt{2}}{128}\right)$
$\lvert\lvert\lvert \boldsymbol{u} - \tilde{\boldsymbol{u}}_h \rvert\rvert\rvert_{0,1}$	4.4196e-2	1.1485e-2	2.9707e-3	7.5759e-4	1.9142e-4	4.8129e-5
Cvge. Rate	-	1.9	2.0	2.0	2.0	2.0
$\lvert\lvert\lvert \boldsymbol{u} - \tilde{\boldsymbol{u}}_h \rvert\rvert\rvert_{\infty,0}$	1.4734e-3	1.7996e-4	2.5014e-5	4.1759e-6	8.5692e-7	2.1636e-7
Cvge. Rate	-	3.0	2.8	2.6	2.3	2.0
$\lvert\lvert\lvert p - \tilde{p}_h \rvert\rvert\rvert_{0,0}$	1.0859e-1	6.6842e-3	1.5033e-3	3.9097e-4	1.2086e-4	4.7884e-5
Cvge. Rate	-	4.0	2.2	1.9	1.7	1.3
$\lvert\lvert\lvert p - \tilde{p}_h \rvert\rvert\rvert_{\infty,0}$	8.4003e-2	4.9703e-3	1.1343e-3	3.2878e-4	1.2797e-4	6.0659e-5
Cvge. Rate	-	4.1	2.1	1.8	1.4	1.1

$$\lvert\lvert\lvert \boldsymbol{u} - \tilde{\boldsymbol{u}}_h \rvert\rvert\rvert_{0,1} = O(\Delta t^2, h^2)$$

Move to Theorem

Inflow :

$$\boldsymbol{u} = \left(1 - e^{-t}\right)\begin{pmatrix} \frac{1}{32}\left(1 - y^2\right) \\ 0 \end{pmatrix}, \text{ and } \boldsymbol{\sigma} \text{ set accordingly.}$$

Outflow :

$$\boldsymbol{u} = \left(1 - e^{-t}\right)\begin{pmatrix} 2\left(\frac{1}{16} - y^2\right) \\ 0 \end{pmatrix}.$$

Solid walls: No slip B.C. for $\boldsymbol{u}$.

Bottom: Symmetry condition along the bottom of the computational domain.

L was set to $1/4$ in shown computations.

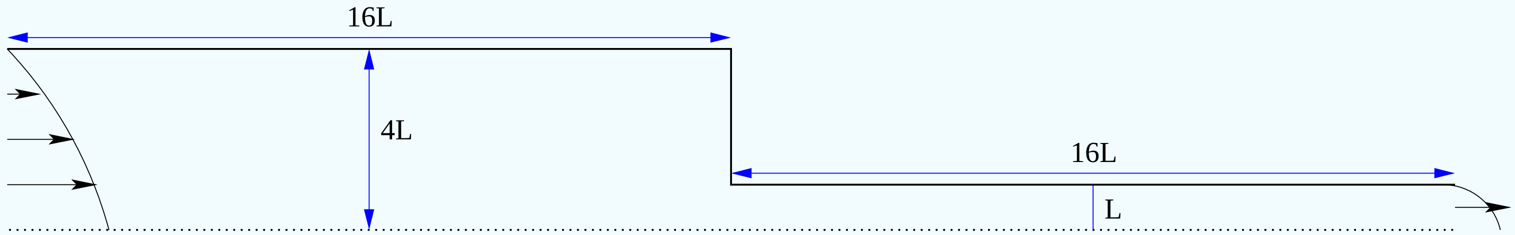
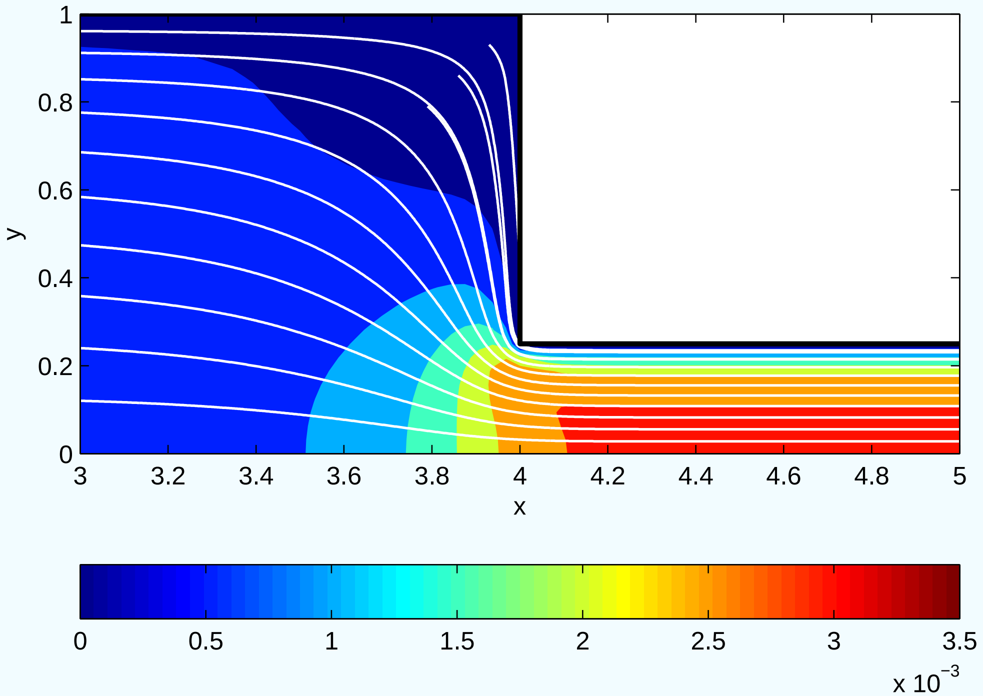


Figure 1: Plot of 4:1 contraction domain geometry.

4:1 Contraction Animation



$$\Delta x_{\min} = 0.0625, \quad \Delta y_{\min} = 0.015625, \quad T = 2.5$$

$$Re = 1, \quad \lambda = 2, \quad a = 1, \quad \alpha = 8/9, \quad \delta = (2/\Delta y_{\min})^2$$

Summary

Fractional Step θ-Method

- For appropriate choices of θ second order temporal convergence is achieved.
- Allows for decoupling of operators:
  - Convection from Diffusion
  - Stress from Pressure/Velocity.
  - Linear from Nonlinear

Results in:

- **smaller systems to solve**
- **application of specialized solution techniques**

*References

[1] J.C. Chrispell, V.J. Ervin, and E.W. Jenkins. A fractional step θ-method for convection-diffusion problems. *J. Math. Anal. Appl.*, 333(1):204–218, 2007.

[2] V.J. Ervin and N. Heuer. Approximation of time-dependent, viscoelastic fluid flow: Crank-Nicolson, finite element approximation. *Numer. Methods Partial Differential Equations*, 20(2):248–283, 2004.

[3] V.J. Ervin and W.W. Miles. Approximation of time-dependent viscoelastic fluid flow: SUPG approximation. *SIAM J. Numer. Anal.*, 41(2):457–486 (electronic), 2003.

[4] R. Glowinski and J.F. Périaux. Numerical methods for nonlinear problem in fluid dynamics. In *Supercomputing*, pages 381–479. North-Holland, Amsterdam, 1987.

[5] P. Saramito. A new θ-scheme algorithm and incompressible FEM for viscoelastic fluid flows. *RAIRO Modél. Math. Anal. Numér.*, 28(1):1–35, 1994.

[6] R. Sureshkumar, M.D. Smith, R.C. Armstrong, and R.A. Brown. Linear stability and dynamics of viscoelastic flows using time-dependent numerical simulations. *J. Non-Newt. Fluid Mech.*, 82:57–104, 1999.

[7] S. Turek. A comparative study of time-stepping techniques for the incompressible Navier-Stokes equations: from fully implicit non-linear schemes to semi-implicit projection methods. *Internat. J. Numer. Methods Fluids*, 22(10):987–1011, 1996.

Table 2: Convergence rates at $T = 2$

$\delta \downarrow$	$(\Delta t, h) \rightarrow$	$\left(\frac{1}{2}, \frac{\sqrt{2}}{4}\right)$	$\left(\frac{1}{4}, \frac{\sqrt{2}}{8}\right)$	$\left(\frac{1}{8}, \frac{\sqrt{2}}{16}\right)$	$\left(\frac{1}{16}, \frac{\sqrt{2}}{32}\right)$	$\left(\frac{1}{32}, \frac{\sqrt{2}}{64}\right)$
0	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	4.7608e-2	1.2323e-2	3.2034e-3	8.3187e-4	2.1793e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	6.5569e-2	1.9248e-2	5.0845e-3	1.3000e-3	3.2981e-4
	Cvge. Rate	-	1.8	1.9	2.0	2.0
$\frac{h}{\sqrt{2}}$	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	5.0150e-2	1.6193e-2	6.4636e-3	2.9147e-3	1.3989e-3
	Cvge. Rate	-	1.6	1.3	1.1	1.1
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	8.1922e-2	3.5905e-2	1.5791e-2	7.3743e-3	3.5789e-3
	Cvge. Rate	-	1.2	1.2	1.1	1.0
$\left(\frac{h}{\sqrt{2}}\right)^{\frac{3}{2}}$	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	4.8620e-2	1.3135e-2	3.6334e-3	1.0192e-3	2.9513e-4
	Cvge. Rate	-	1.9	1.9	1.8	1.8
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	7.1941e-2	2.3392e-2	6.8309e-3	1.9976e-3	6.0598e-4
	Cvge. Rate	-	1.6	1.8	1.8	1.7
$\left(\frac{h}{\sqrt{2}}\right)^{2}$	$\lvert\lvert\lvert \boldsymbol{u} - \boldsymbol{u}_h \rvert\rvert\rvert_{0,1}$	4.8022e-2	1.2507e-2	3.2644e-3	8.4818e-4	2.2193e-4
	Cvge. Rate	-	1.9	1.9	1.9	1.9
	$\lvert\lvert\lvert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rvert\rvert\rvert_{0,0}$	6.8104e-2	2.0308e-2	5.3614e-3	1.3680e-3	3.4645e-4
	Cvge. Rate	-	1.7	1.9	2.0	2.0

Move to: Full Method Summary