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Coastal and Hydraulics Laboratory

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Finite Element Methods for Variable Density Flows

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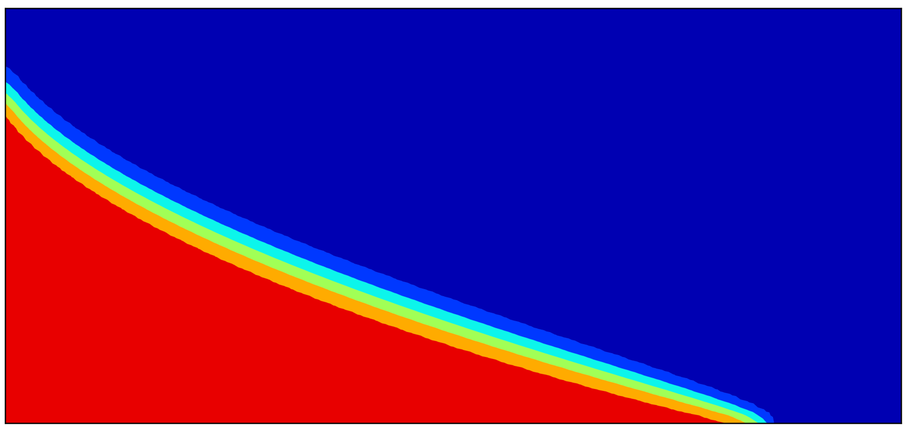

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Finite Element Methods for Variable Density Flows

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Abstract: Saltwater intrusion into coastal freshwater aquifers is an ongoing problem that will continue to impact coastal freshwater resources as coastal populations increase. To effectively model saltwater intrusion, the impacts of salt content on fluid density must be accounted for to properly model saltwater/freshwater transition zones and sharp interfaces. We present a model for variable density fluid flow and solute transport where a conforming finite element method discretization with a locally conservative velocity post-processing method is used for the flow model and a variational multi-scale stabilized conforming finite element method is used for transport. This formulation provides a consistent velocity and performs well even in advection-dominated problems that can occur in saltwater intrusion modeling. The physical model is presented as well as the formulation of the numerical model and solution methods. The model is tested against several 2D and 3D numerical and experimental benchmark problems and the results are presented to verify the code.

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Preface

This report is a product of the User Productivity Enhancement, Technology Transfer, and Training (PETTT) Program in the Environmental Quality Modeling (EQM) computational technology area of the Department of Defense High Performance Computing Modernization Office and of the Multiscale and Fluid-Structure Interaction work unit of the ERDC Military Engineering 6.2 program. The report was prepared by LTC Timothy Povich and Dr. Clint Dawson of the Institute for Computational Engineering and Science at the University of Texas, Austin and by Drs. Christopher E. Kees and Matthew W. Farthing of the Hydrologic Systems Branch. General supervision was provided by Dr. William D. Martin, Director, CHL; Dr. Charles A. Randall was the project manager for this eort. Dr. David A. Horner was the Technical Director. COL Kevin J. Wilson was Commander and Executive Director of the Engineer Research and Development Center. Dr. Jeffrey P. Holland was Director.

1 Introduction

2 Variable Density Incompressible Navier-Stokes Time Stepping Techniques

2.1 Preliminaries

2.1.1 Notation

Find the solutions at time t^{k+1}

$$(\rho_h^{k+1}, \mathbf{u}_h^{k+1}, p_h^{k+1}) \in W_h \times \mathbf{X}_h \times M_h \tag{1}$$

2.2 Penalty-like Perturbation of Continuous Equations

The standard approach to creating splittings has been to think about the splitting operator as a projection scheme, where we have the two sequences of velocities $\{\mathbf{u}^k\}$ and $\{\tilde{\mathbf{u}}^k\}$ and the pressure increment $\{\phi^k\}$ that represent the standard Helmholtz decomposition of L^2 velocity fields into solenoidal and irrotational components

$$\tilde{\mathbf{u}}^k = \mathbf{u}^k + \frac{\rho}{\tau}\nabla\phi^k. \tag{2}$$

Thus we can view $\mathbf{u}^k$ as the projection of our velocity field $\tilde{\mathbf{u}}^k$ onto the divergence free subset of velocity fields. This works just fine in the case of constant density but for variable density it is not so simple as we cannot just pull the density out of the divergence and end up with a variable coefficient laplacian equation to solve each time step.

$$-\nabla\cdot\left(\frac{1}{\rho^{k+1}}\nabla\phi\right) = F, \qquad \partial_n\phi|_{\partial\Omega} = 0 \tag{3}$$

This can be badly conditioned and rather more difficult to assemble and solve than the constant coefficient version. Also, here it is clear to see why uniform lower bounds on the density ρ^{k+1} must be maintained.

We instead look at the constant density projection scheme in terms of an ϵ perturbation of the original system. Thus the formerly stated projection part becomes a penalty-like adjustment that is easily generalized to the variable density framework.

The incremental pressure correction algorithm, an improvement on the original Chorin/Themam algorithm for constant density, can be expressed solely in terms of the non-solenoidal velocity $\tilde{\mathbf{u}}^k$ and pressure p^k in the form

$$\begin{cases} \rho\left(\frac{\tilde{\mathbf{u}}^{k+1}-\tilde{\mathbf{u}}^k}{\tau}+\tilde{\mathbf{u}}^k\cdot\nabla\tilde{\mathbf{u}}^{k+1}\right)-\mu\Delta\tilde{\mathbf{u}}^{k+1}+\nabla\left(p^k+\phi^k\right)=\mathbf{f}^{k+1}, & \tilde{\mathbf{u}}^k\big|_{\partial\Omega}=0 \\ \nabla\cdot\tilde{\mathbf{u}}^{k+1}-\frac{\tau}{\rho}\Delta\phi^{n+1}, & \partial_{\mathbf{n}}\phi|_{\partial\Omega}=0 \\ p^{k+1}=p^k+\phi^{k+1} \end{cases} \tag{4}$$

This can be seen as a discrete version of the following system

$$\begin{cases} \rho\left(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla p-\mu\Delta\mathbf{u}=\mathbf{f}, & \mathbf{u}|_{\partial\Omega}=0 \\ \nabla\cdot\mathbf{u}-\frac{\varepsilon}{\rho}\Delta\phi=0, & \partial_{\mathbf{n}}\phi|_{\partial\Omega}=0 \\ \varepsilon p_t=\phi \end{cases} \tag{5}$$

where we have replaced difference quotients with time derivatives and substituted $\varepsilon=\tau$ for the remaining τ's and recognized $p^k+\phi^k=p^k+\left(p^k-p^{k-1}\right)=2p^k-p^{k-1}\approx p^{k+1}$ as a second order extrapolation of pressure to time t^{k+1}.

This is a second order $\mathcal{O}(\varepsilon^2)$ perturbation of the constant density incompressible navier stokes equations. Simpler versions of this discrete system were observed by Rannacher in **add citation here** to be nothing more than penalities on the divergence of velocity in the momentum equation in a norm resembling the H^{-1} norm. Hense the term penalty-like algorithms. As described in Guermond and Salgado (2009), the system (**5**) is the starting point for the algorithms described below.

2.3 BDF1 Incremental Rotational Scheme

2.3.1 Time-stepping technique

We proceed in a three step update scheme. First we update the density, second the velocity, and third the pressure. So given

$$(\rho_h^k,\mathbf{u}_h^k,p_h^k)\in W_h\times\mathbf{X}_h\times M_h \tag{6}$$

we update to obtain

$$(\rho_h^{k+1},\mathbf{u}_h^{k+1},p_h^{k+1})\in W_h\times\mathbf{X}_h\times M_h. \tag{7}$$

Density Update

We first solve the hyperbolic system with a monotone preserving scheme such as subgrid viscosity, edge stabilization or entropy viscosity using a DG or CG solver. The equation of update is

$$\frac{\rho^{k+1}-\rho^{k}}{\tau^{k+1}} + \nabla\cdot\left(\rho_h^{k+1}\mathbf{u}^k\right) - \frac{\rho^{k+1}}{2}\nabla\cdot\mathbf{u}^k = 0 \tag{8}$$

where the last term is a consistent stabilization which leads to unconditional stability of the scheme. Our choice of hyperbolic solver scheme doesn't matter as long as it satisfies the following stability hypothesis:

$$\chi \leq \min_{\mathbf{x}\in\overline{\Omega}} \rho_h^{k+1}(\mathbf{x}), \qquad \max_{\mathbf{x}\in\overline{\Omega}} \rho_h^{k+1}(\mathbf{x}) \leq c_\rho \tag{9}$$

for all $k \geq 1$.

Thus we obtain the weak solution $\rho_h^{k+1} \in W_h$ that satifies the above requirements.

Velocity Update

Once we have the density ρ_h^{k+1} we can now solve for the velocity. It turns out that we do not explicitly require this velocity to be divergence free, but simply penalize the divergence using the pressure rotational update. We define our update terms

$$\begin{aligned}
\rho_h^* &= \frac{1}{2}\left(\rho_h^{k+1}-\rho_h^k\right)\\
\delta p_h^k &= p_h^k - p_h^{k-1}\\
p_h^\# &= p_h^k + \delta p_h^k = 2p_h^k - p_h^{k-1}
\end{aligned}$$

so that $p_h^\#$ is a second order extrapolation of pressure to time t^{k+1}. Next we solve for $\mathbf{u}_h^{k+1} \in \mathbf{X}_h$ that satisfies the following system for all $\mathbf{v}_h \in \mathbf{X}_h$,

$$\begin{aligned}
&\left\langle \frac{\rho_h^*\mathbf{u}_h^{k+1} - \rho_h^k\mathbf{u}_h^k}{\tau^{k+1}}, \mathbf{v}_h \right\rangle + \left\langle \rho_h^{k+1}\mathbf{u}_h^k\cdot\nabla\mathbf{u}_h^{k+1}, \mathbf{v}_h\right\rangle\\
&\quad + \left\langle \frac{1}{2}\nabla\cdot\left(\mathbf{u}_h^k\right)\mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle + \mu\left\langle \nabla\mathbf{u}_h^{k+1}, \nabla\mathbf{v}_h\right\rangle\\
&\qquad + \left\langle \nabla p_h^\#, \mathbf{v}_h \right\rangle = \left\langle \mathbf{f}^{k+1}, \mathbf{v}_h\right\rangle.
\end{aligned} \tag{10}$$

Pressure Update

Now that we have ρ_h^{k+1} and $\mathbf{u}_h^{k+1}$, we solve for the pressure increment $\phi_h^\flat \in M_h$ which then allows us to solve for the pressure $p_h^{k+1} \in M_h$. Recalling that $\chi \leq \min\limits_{\mathbf{x}\in\overline{\Omega}} \rho_h^{k+1}(\mathbf{x})$ (note that we will often choose it to be the minimum), we let $\phi_h^\flat \in M_h$ be the weak solution of

$$\Delta\phi^\flat = \frac{\chi}{\tau^{k+1}}\nabla\cdot\mathbf{u}^{k+1}, \qquad \partial_{\mathbf{n}}\phi^\flat\big|_{\partial\Omega} = 0 \tag{11}$$

mainly it solves

$$\left\langle \nabla\phi_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau^{k+1}}\left\langle \mathbf{u}_h^{k+1}, \nabla r_h \right\rangle \tag{12}$$

for all $r_h \in M_h$. Then we update the pressure as

$$p^{k+1} = \phi^\flat + p^k - \mu\nabla\cdot\mathbf{u}^{k+1}, \tag{13}$$

or in other words, we solve for $p_h^{k+1} \in M_h$ such that for all $r_h \in M_h$

$$\left\langle p_h^{k+1}, r_h \right\rangle = \left\langle \phi_h^\flat + p_h^k, r_h \right\rangle + \mu\left\langle \mathbf{u}_h^{k+1}, \nabla r_h \right\rangle. \tag{14}$$

The last term involving the divergence of velocity makes this the rotational form and leaving it off is the standard form. In standard form, it is simple enough to just add p_h^{k+1} and $\phi_h^\flat$ to update p_h^{k+1} instead of solving the linear system.

2.4 BDF2 Incremental Rotational Scheme

References

Guermond, J.-L. and A. Salgado (2009). A splitting method for incompressible flows with variable density based on a pressure poisson equation. *Journal of Computational Physics 228*(8), 2834–2846.