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Finite Element Methods for Variable Density Flows

S. R. Patty, M. W. Farthing and C. E. Kees

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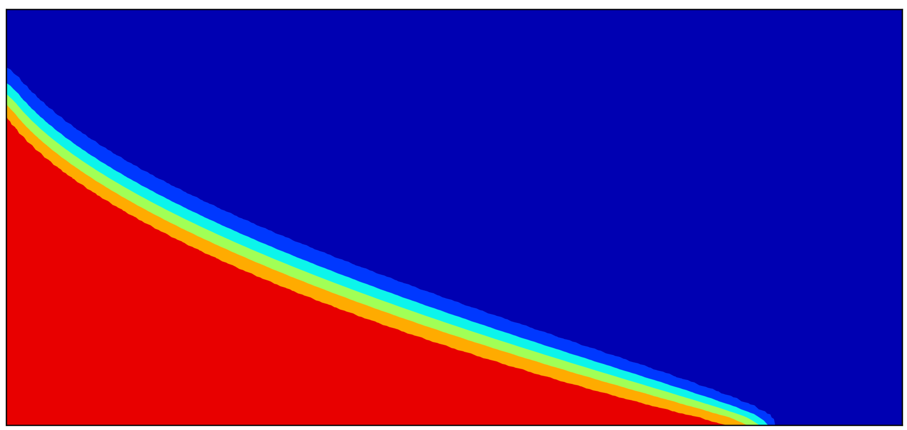

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Preface

This report is a product of the User Productivity Enhancement, Technology Transfer, and Training (PETTT) Program in the Environmental Quality Modeling (EQM) computational technology area of the Department of Defense High Performance Computing Modernization Office and of the Multiscale and Fluid-Structure Interaction work unit of the ERDC Military Engineering 6.2 program. The report was prepared by LTC Timothy Povich and Dr. Clint Dawson of the Institute for Computational Engineering and Science at the University of Texas, Austin and by Drs. Christopher E. Kees and Matthew W. Farthing of the Hydrologic Systems Branch. General supervision was provided by Dr. William D. Martin, Director, CHL; Dr. Charles A. Randall was the project manager for this eort. Dr. David A. Horner was the Technical Director. COL Kevin J. Wilson was Commander and Executive Director of the Engineer Research and Development Center. Dr. Jeffrey P. Holland was Director.

1 Introduction

2 Variable Density Incompressible Navier-Stokes Time Stepping Techniques

The system of equations treated here are

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \rho \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - \mu \Delta \mathbf{u} = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1}$$

with initial and boundary conditions

$$\begin{cases} \rho(\mathbf{x},0) = \rho_0(\mathbf{x}), & \rho(\mathbf{x},t)|_{\partial\Omega^{in}} = a(\mathbf{x},t) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{u}(\mathbf{x},t)|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

where the inflow boundary, $\partial\Omega^{in} = \{\mathbf{x} \in \partial\Omega \mid \mathbf{u} \cdot \mathbf{n} < 0\}$ where $\mathbf{n}$ is the outward unit normal vector. Under the restrictive assumption that $\mathbf{u}(\mathbf{x},t)|_{\partial\Omega} = 0$, we have $\partial\Omega^{in} = \emptyset$.

2.1 Preliminaries

2.1.1 Notation

Find the solutions at time t^{k+1}

$$(\rho_h^{k+1}, \mathbf{u}_h^{k+1}, p_h^{k+1}) \in W_h \times \mathbf{X}_h \times M_h \tag{3}$$

2.2 Penalty-like Perturbation of Continuous Equations

The standard approach to creating splittings has been to think about the splitting operator as a projection scheme, where we have the two sequences of velocities $\{\mathbf{u}^k\}$ and $\{\tilde{\mathbf{u}}^k\}$ and the pressure increment $\{\phi^k\}$ that represent the standard Helmholtz decomposition of L^2 velocity fields into solenoidal and irrotational components

$$\tilde{\mathbf{u}}^k = \mathbf{u}^k + \frac{\rho}{\tau} \nabla \phi^k. \tag{4}$$

Thus we can view $\mathbf{u}^k$ as the projection of our velocity field $\tilde{\mathbf{u}}^k$ onto the divergence free subset of velocity fields. This works just fine in the case of constant density but for variable density it is not so simple as we cannot just pull the density out of the divergence and end up with a variable coefficient laplacian equation to solve each time step.

$$-\nabla \cdot \left(\frac{1}{\rho^{k+1}} \nabla \phi \right) = F, \qquad \partial_n \phi|_{\partial\Omega} = 0 \tag{5}$$

This can be badly conditioned and rather more difficult to assemble and solve than the constant coefficient version. Also, here it is clear to see why uniform lower bounds on the density ρ^{k+1} must be maintained.

We instead look at the constant density projection scheme in terms of an ϵ perturbation of the original system. Thus the formerly stated projection part becomes a penalty-like adjustment that is easily generalized to the variable density framework.

The incremental pressure correction algorithm, an improvement on the original Chorin/Themam algorithm for constant density, can be expressed solely in terms of the non-solenoidal velocity $\tilde{\mathbf{u}}^k$ and pressure p^k in the form

$$\begin{cases} \rho \left(\frac{\tilde{\mathbf{u}}^{k+1} - \tilde{\mathbf{u}}^k}{\tau} + \tilde{\mathbf{u}}^k \cdot \nabla \tilde{\mathbf{u}}^{k+1} \right) - \mu \Delta \tilde{\mathbf{u}}^{k+1} + \nabla \left(p^k + \phi^k \right) = \mathbf{f}^{k+1}, & \tilde{\mathbf{u}}^k \big|_{\partial\Omega} = 0 \\ \nabla \cdot \tilde{\mathbf{u}}^{k+1} - \frac{\tau}{\rho} \Delta \phi^{n+1}, & \partial_{\mathbf{n}} \phi|_{\partial\Omega} = 0 \\ p^{k+1} = p^k + \phi^{k+1} \end{cases} \tag{6}$$

This can be seen as a discrete version of the following system

$$\begin{cases} \rho \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - \mu \Delta \mathbf{u} = \mathbf{f}, & \mathbf{u}|_{\partial\Omega} = 0 \\ \nabla \cdot \mathbf{u} - \frac{\varepsilon}{\rho} \Delta \phi = 0, & \partial_{\mathbf{n}} \phi|_{\partial\Omega} = 0 \\ \varepsilon p_t = \phi \end{cases} \tag{7}$$

where we have replaced difference quotients with time derivatives and substituted $\varepsilon = \tau$ for the remaining τ's and recognized $p^k + \phi^k = p^k + \left(p^k - p^{k-1} \right) = 2p^k - p^{k-1} \approx p^{k+1}$ as a second order extrapolation of pressure to time t^{k+1}.

This is a second order $\mathcal{O}(\varepsilon^2)$ perturbation of the constant density incompressible navier stokes equations. Simpler versions of this discrete system were observed by Rannacher in Rannacher (1992) to be nothing more than penalities on the divergence of velocity in the momentum equation in a norm resembling the H^{-1} norm. Hense the term penalty-like algorithms. As described in Guermond and Salgado (2009), the system (7) is the starting point for the algorithms described below.

2.3 BDF1 Incremental Rotational Scheme

add in introduction of how toget from penalty equations to BDF1 algorithm

2.3.1 Time-stepping technique

We proceed in a three step update scheme. First we update the density, second the velocity, and third the pressure. So given

$$(\rho_h^k, \mathbf{u}_h^k, p_h^k) \in W_h \times \mathbf{X}_h \times M_h \tag{8}$$

we update to obtain

$$(\rho_h^{k+1}, \mathbf{u}_h^{k+1}, p_h^{k+1}) \in W_h \times \mathbf{X}_h \times M_h. \tag{9}$$

Density Update

We first solve the hyperbolic system with a monotone preserving scheme such as subgrid viscosity, edge stabilization or entropy viscosity using a DG or CG solver. The equation of update is

$$\frac{\rho^{k+1} - \rho^k}{\tau} + \nabla \cdot \left(\rho_h^{k+1}\mathbf{u}^k\right) - \frac{\rho^{k+1}}{2}\nabla \cdot \mathbf{u}^k = 0 \tag{10}$$

where the last term is a consistent stabilization which leads to unconditional stability of the scheme. Our choice of hyperbolic solver scheme doesn't matter as long as it satisfies the following stability hypothesis:

$$\chi \leq \min_{\mathbf{x}\in\overline{\Omega}} \rho_h^{k+1}(\mathbf{x}), \qquad \max_{\mathbf{x}\in\overline{\Omega}} \rho_h^{k+1}(\mathbf{x}) \leq c_\rho \tag{11}$$

for all $k \geq 1$.

Thus we obtain the weak solution $\rho_h^{k+1} \in W_h$ that satifies the above requirements.

Velocity Update

Once we have the density ρ_h^{k+1} we can now solve for the velocity. It turns out that we do not explicitly require this velocity to be divergence free, but simply penalize the divergence using the pressure rotational update. We define our update terms

$$
\begin{aligned}
\rho_h^* &= \frac{1}{2}\left(\rho_h^{k+1} + \rho_h^k\right)\\
\delta p_h^k &= p_h^k - p_h^{k-1}\\
p_h^{\#} &= p_h^k + \delta p_h^k = 2p_h^k - p_h^{k-1}
\end{aligned}
$$

so that $p_h^{\#}$ is a second order extrapolation of pressure to time t^{k+1}. Next we solve for $\mathbf{u}_h^{k+1} \in \mathbf{X}_h$ that satisfies the following system for all $\mathbf{v}_h \in \mathbf{X}_h$,

$$
\begin{aligned}
&\left\langle \frac{\rho_h^*\mathbf{u}_h^{k+1} - \rho_h^k\mathbf{u}_h^k}{\tau}, \mathbf{v}_h \right\rangle + \left\langle \rho_h^{k+1}\mathbf{u}_h^k \cdot \nabla\mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle + \mu\left\langle \nabla\mathbf{u}_h^{k+1}, \nabla\mathbf{v}_h \right\rangle\\
&\qquad + \left\langle \frac{1}{2}\nabla\cdot\left(\rho_h^{k+1}\mathbf{u}_h^k\right)\mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle + \left\langle \nabla p_h^{\#}, \mathbf{v}_h \right\rangle = \left\langle \mathbf{f}^{k+1}, \mathbf{v}_h \right\rangle.
\end{aligned} \tag{12}
$$

We note that Equation (12) can be rewritten in a more recognizable format that emphasizes the consistent stabilizing term as a scaled version of conservation of mass, namely

$$
\begin{aligned}
&\left\langle \rho_h^k\frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\tau}, \mathbf{v}_h \right\rangle + \left\langle \rho_h^{k+1}\mathbf{u}_h^k \cdot \nabla\mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle + \mu\left\langle \nabla\mathbf{u}_h^{k+1}, \nabla\mathbf{v}_h \right\rangle + \left\langle \nabla p_h^{\#}, \mathbf{v}_h \right\rangle\\
&\qquad + \frac{1}{2}\left\langle \left(\frac{\rho_h^{k+1} - \rho_h^k}{\tau} + \nabla\cdot\left(\rho_h^{k+1}\mathbf{u}_h^k\right)\right)\mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle = \left\langle \mathbf{f}^{k+1}, \mathbf{v}_h \right\rangle.
\end{aligned} \tag{13}
$$

Pressure Update

Now that we have ρ_h^{k+1} and $\mathbf{u}_h^{k+1}$, we solve for the pressure increment $\phi_h^\flat \in M_h$ which then allows us to solve for the pressure $p_h^{k+1} \in M_h$. Recalling that $\chi \leq \min_{\mathbf{x}\in\overline{\Omega}} \rho_h^{k+1}(\mathbf{x})$ (note that we will often choose it to be the minimum), we let $\phi_h^\flat \in M_h$ be the weak solution of

$$
\Delta\phi^\flat = \frac{\chi}{\tau}\nabla\cdot\mathbf{u}^{k+1}, \qquad \partial_{\mathbf{n}}\phi^\flat\big|_{\partial\Omega} = 0 \tag{14}
$$

mainly it solves

$$
\left\langle \nabla\phi_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau}\left\langle \mathbf{u}_h^{k+1}, \nabla r_h \right\rangle \tag{15}
$$

for all $r_h \in M_h$. Then we update the pressure as

$$
p^{k+1} = \phi^\flat + p^k - \mu\nabla\cdot\mathbf{u}^{k+1}, \tag{16}
$$

or in other words, we solve for $p_h^{k+1} \in M_h$ such that for all $r_h \in M_h$

$$\left\langle p_h^{k+1}, r_h \right\rangle = \left\langle \phi_h^\flat + p_h^k, r_h \right\rangle + \mu \left\langle \mathbf{u}_h^{k+1}, \nabla r_h \right\rangle . \qquad \textbf{(17)}$$

The last term involving the divergence of velocity makes this the rotational form and leaving it off is the standard form. In standard form, it is simple enough to just add p_h^{k+1} and $\phi_h^\flat$ to update p_h^{k+1} instead of solving the linear system.

2.3.2 Summary of BDF1 Constant Time Step Incremental Rotational Algorithm

Given $(\rho^k, \mathbf{u}^k, p^k) \in W_h \times \mathbf{X}_h \times M_h$ where the spaces $\mathbf{X}_h \times M_h$ satisfy the standard LBB inf-sup conditions, for instance a Taylor-Hood spaces $\mathbb{P}^2 \times \mathbb{P}^1$. First, solve for $\rho^{k+1} \in W_h$ such that

$$\frac{\rho^{k+1} - \rho^k}{\tau} + \nabla \cdot \left(\rho_h^{k+1} \mathbf{u}^k \right) - \frac{\rho^{k+1}}{2} \nabla \cdot \mathbf{u}^k = 0. \qquad \textbf{(18)}$$

using a monotone preserving hyperbolic transport method. Second, find $\mathbf{u}_h^{k+1} \in \mathbf{X}_h$ such that for all $\mathbf{v}_h \in \mathbf{X}_h$,

$$\left\langle \frac{\frac{1}{2}\left(\rho_h^{k+1} + \rho_h^k\right) \mathbf{u}_h^{k+1} - \rho_h^k \mathbf{u}_h^k}{\tau}, \mathbf{v}_h \right\rangle + \left\langle \rho_h^{k+1} \mathbf{u}_h^k \cdot \nabla \mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle + \mu \left\langle \nabla \mathbf{u}_h^{k+1}, \nabla \mathbf{v}_h \right\rangle$$
$$+ \left\langle \frac{1}{2} \nabla \cdot \left(\rho_h^{k+1} \mathbf{u}_h^k \right) \mathbf{u}_h^{k+1}, \mathbf{v}_h \right\rangle + \left\langle \nabla \left(2p_h^k - p_h^{k-1} \right), \mathbf{v}_h \right\rangle = \left\langle \mathbf{f}^{k+1}, \mathbf{v}_h \right\rangle . \qquad \textbf{(19)}$$

Finally we find $\phi_h^\flat \in M_h$ such that for all $r_h \in M_h$,

$$\left\langle \nabla \phi_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau} \left\langle \mathbf{u}_h^{k+1}, \nabla r_h \right\rangle \qquad \textbf{(20)}$$

where $\chi \leq \min_{\mathbf{x} \in \overline{\Omega}} \rho_0(\mathbf{x})$ and update the pressure $p_h^{k+1} \in M_h$ such that for all $r_h \in M_h$

$$\left\langle p_h^{k+1}, r_h \right\rangle = \left\langle \phi_h^\flat + p_h^k, r_h \right\rangle + \mu \left\langle \mathbf{u}_h^{k+1}, \nabla r_h \right\rangle . \qquad \textbf{(21)}$$

Thus we have solved for $(\rho^{k+1}, \mathbf{u}^{k+1}, p^{k+1})$ and can move on to the next time step.

2.3.3 Error analysis of BDF1 Constant Time Step Incremental Rotational Algorithm

The stability of the above algorithm is proved in Guermond and Salgado (2009) and the error analysis was carried out in Guermond and Salgado (2011). We refer the reader to those articles for details and give the main results contained in Theorem 4.2 of Guermond and Salgado (2011).

We define the Stokes projection, $(\Pi_h \mathbf{u}(t), \Pi_h p(t)) \in \mathbf{X}_h \times M_h$, of the solution $(\mathbf{u}(t), p(t))$ to (1)-(2) to be the pair that solves

$$\begin{cases} \mu \left\langle \nabla \Pi_h \mathbf{u}(t), \nabla \mathbf{v}_h \right\rangle + \left\langle \nabla \Pi_h p(t), \mathbf{v}_h \right\rangle = \mu \left\langle \nabla \mathbf{u}(t), \nabla \mathbf{v}_h \right\rangle - \left\langle p(t), \nabla \cdot \mathbf{v}_h \right\rangle, & \forall \mathbf{v}_h \in \mathbf{X}_h \\ \left\langle \Pi_h \mathbf{u}(t), \nabla r_h \right\rangle = 0, & \forall r_h \in M_h \end{cases} \tag{22}$$

Theorem 1. *Assume that the true solution to (1) and (2) satisfies (with $l \geq 1$)*

$$\boldsymbol{u} \in W^{2,\infty}\left(\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{l+1}(\Omega)\right), \qquad p \in W^{1,\infty}\left(H^l(\Omega)\right) \tag{23}$$

Let $(\boldsymbol{u}_h)_\tau$ be the solutions of velocity as described in Subsection 2.3.2. Then if one of either

$$\|p_h^0 - \Pi_h p^0\|_{L^2(\Omega)} \leq c h^{\frac{l+1}{2}} \tag{24}$$

or

$$\|p_h^0\|_{H^1(\Omega)} \leq c \quad \text{and} \quad \left\langle \boldsymbol{u}_h^0, \nabla r_h \right\rangle = 0 \;\; \forall r_h \in M_h \tag{25}$$

is satisfied, then

$$\| \left(\boldsymbol{u}\right)_\tau - \left(\boldsymbol{u}_h\right)_\tau \|_{\ell^\infty(\mathbf{L}^2(\Omega))} \leq c\left(\tau + h^{\min(l+1,m)}\right) \tag{26}$$

and

$$\| \left(\boldsymbol{u}\right)_\tau - \left(\boldsymbol{u}_h\right)_\tau \|_{\ell^2(\mathbf{H}^1(\Omega))} \leq c\left(\tau + h^{\min(l,m)}\right) \tag{27}$$

Remark. Here $m > 0$ is a constant that deals with the coupling between velocity and our solution of density transport. In particular it depends on the method we use to solve the mass conservation. The details are described in Section 4.2 of Guermond and Salgado (2011) where they conjecture that one can obtain $m = 1$ by approximating the mass conservation with a linear first-order viscosity or first-order level set or phase field approach. They further conjecture that $m > 1$ if one uses a nonlinear stabilization technique like discontinuous Galerkin with limiters or entropy viscosity.

2.3.4 BDF1 Variable Time Step Scheme

In the case of variable time step $\tau^k := t^k - t^{k-1}$, there is only one change for BDF1. The choice of pressure for the momentum equations is augmented to reflect the second order extrapolation in variable time steps:

$$p_h^\sharp := p_h^k + \frac{\tau^{k+1}}{\tau^k}\delta p_h^k = p_h^k + \frac{\tau^{k+1}}{\tau^k}\left(p_h^k - p_h^{k-1}\right). \tag{28}$$

Everything else is kept the same as above. The convergence results have not been proved for the case of variable time stepping but it is reasonable to suspect that they still hold under a standard CFL conditions augmented with a maximally allowed time step, ie

$$\tau^{k+1} \leq \min\left\{ c\frac{\min_{T\in\mathcal{T}_h}(h_T/p_T)}{\|\mathbf{u}^k\|_{L^\infty(\Omega)}}, \tau^{max} \right\}, \qquad c < 1. \tag{29}$$

This time stepping controls the conservation of mass equation since the physical velocity being used is $\mathbf{u}^k$, and the conservation of momentum equations since we are using the semi-implicit convection term $\mathbf{u}^k \cdot \nabla \mathbf{u}^{k+1}$, thus again the physical velocity is $\mathbf{u}^k$. It is common to choose in practice the values $c = 0.33$ or $c = 0.5$.

2.4 BDF2 Incremental Rotational Scheme

References

Guermond, J.-L. and A. Salgado (2009). A splitting method for incompressible flows with variable density based on a pressure poisson equation. *Journal of Computational Physics 228*(8), 2834–2846.

Guermond, J.-L. and A. J. Salgado (2011). Error analysis of a fractional time-stepping technique for incompressible flows with variable density. *SIAM Journal on Numerical Analysis 49*(3), 917–944.

Rannacher, R. (1992). *On Chorin's projection method for the incompressible Navier-Stokes equations*. Springer.