Some Thoughts on the Homogenization of Granular Media:

Kinematics of Granular Media

John F. Peters

US Army Engineering Research and Development Center, Vicksburg, MS 39180-6199

John.f.peters@erdc.usace.army.mil

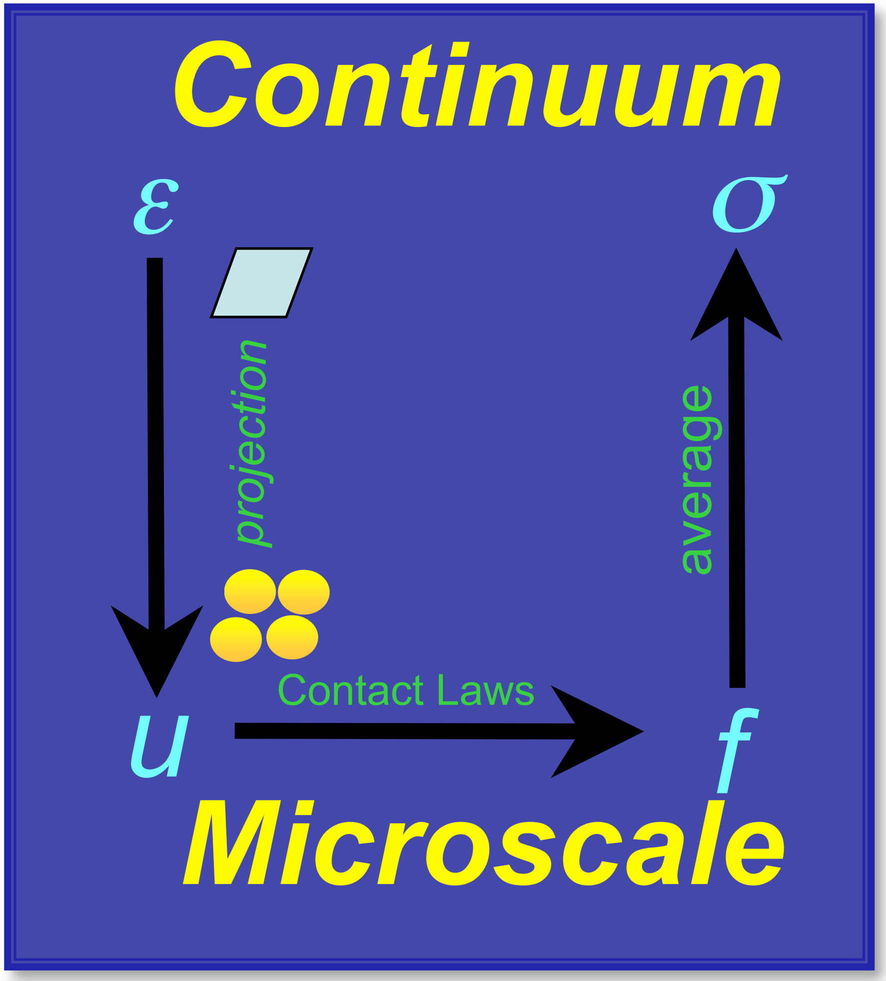
April 16, 2008

Motivations

- Continuum models do not lead to well-posed problems.
  - Non-local models needed.
- Discrete element models have issues of scale dependence.
  - Need to devise coarsened media
- Micromechanical theories require phenomenological relationships.
  - Where is the micromechanics?

Remark on Micromechanics

- Simple projection schemes force all particle centers to move compatibly with the continuum.
- Projecting continuum motions to microscale ignores important instability modes.
- Alternative bridging methods must be developed for DEM to make full use of microscale capabilities.



How are kinematics of continuum and micro-scale to be linked?

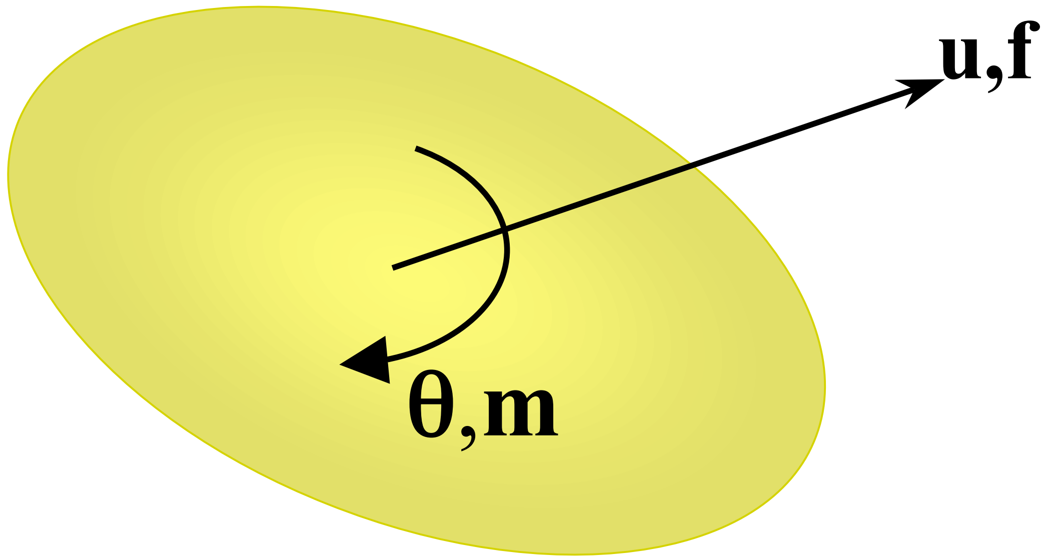
Deformation in a Particulate Medium

- From the concept of symmetry of physical laws the energy state of a medium is invariant to rigid-body motion.
  - Rigid-body motion is equivalent to motion of the observer.

- Deformation, therefore, consists of all motion that are orthogonal to rigid-body motion.

- The goal is to construct “standard” modes that can be linked to continuum deformation.

Degrees of Freedom



$$N = 6N^p$$

Kinematics

$$\Delta u_i^c = u_i^{bc} - u_i^{ac}$$

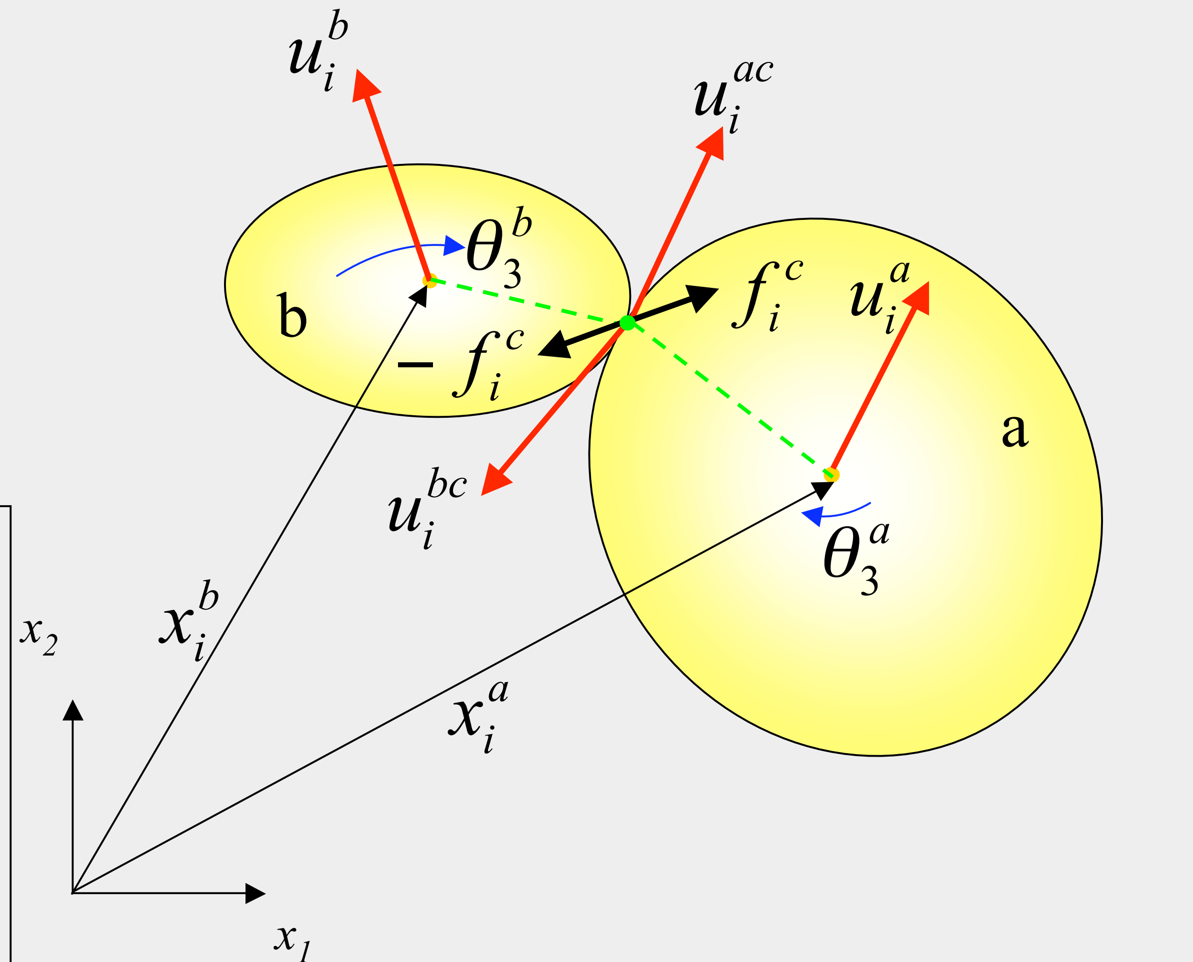
$$W^c = \Delta u_i^c f_i^c$$

Rigid-Body Motion

$$W^c = 0 \quad \forall c$$

$$\theta_3^a = \theta_3^b = \omega_3$$

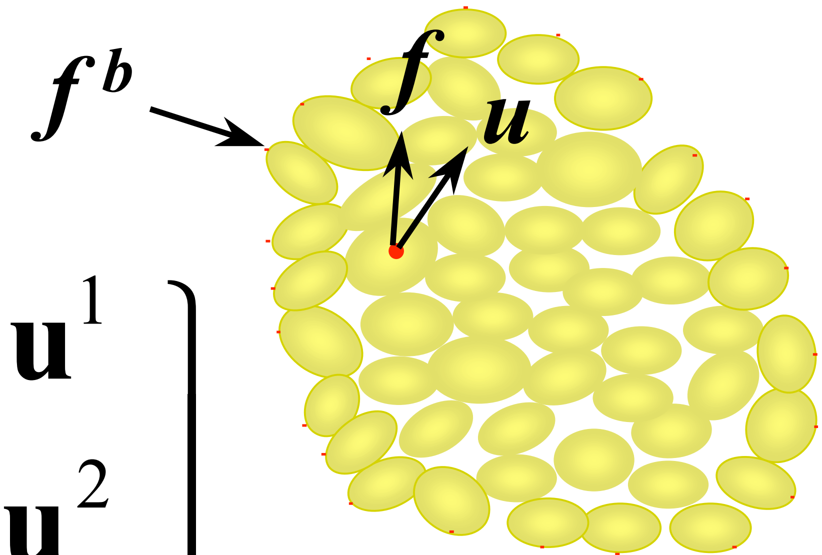
$$u_i^a - u_i^b = \omega_3 (r_k^a - r_k^b) e_{i3k}$$

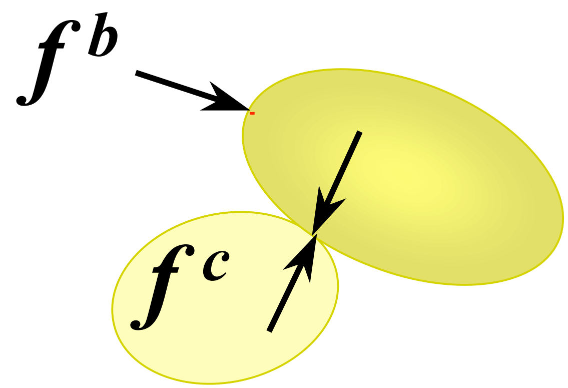


$$\Delta u_i^c = u_i^b - u_i^a + e_{i3k}\left(\theta_3^b r_k^{bc} - \theta_3^a r_k^{ac}\right)$$

The Medium

$$\mathbf{f} = \begin{Bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^{N^p} \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \vdots \\ \mathbf{u}^{N^p} \end{Bmatrix}$$

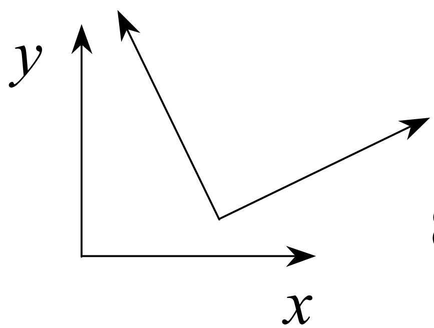




Rigid Body Modes and the Null Space

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ -y^1 & x^1 & 1 & \cdots & -y^N & x^N & 1 & \cdots & -y^{N^p} & x^{N^p} & 1 \end{bmatrix}$$

(for two dimensions)



$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u} \quad \Rightarrow \quad \mathbf{B}\mathbf{m} = \mathbf{0}$$

$$\underbrace{\mathbf{u}_c = \mathbf{M}\mathbf{u}}_{\text{From definition}} \quad \Rightarrow \quad \mathbf{M}\mathbf{m} = \mathbf{0}$$

Deformation of REV

$$\mathbf{v} = \chi\varepsilon + \mathbf{v}'$$

$$R^2 = (\mathbf{v}')^T \mathbf{v}' = (\mathbf{v} - \chi\varepsilon)^T (\mathbf{v} - \chi\varepsilon)$$

$$\varepsilon = (\chi^T \chi)^{-1} \chi^T \mathbf{v} = \mathbf{B}\mathbf{v}$$

$$\mathbf{B}\mathbf{v}' = \mathbf{0}$$

Velocity Variation

$$\mathbf{v}' = \sum_{p=1}^{N'} v^p \phi^p; \quad (\phi^p)^T \phi^m = 0, p \neq m, \quad (\phi^p)^T \phi^p = 1$$

$$\mathbf{B}\phi^p = 0$$

$$\mathbf{R}\phi^p = 0 \text{ (rigid-body modes)}$$

$$\mathbf{v} = \chi\varepsilon + \sum_{p=1}^{N^p} v^p \phi^p$$

$$v^p = (\phi^p)^T \mathbf{v}$$

Remark: v^p represent *internal variables.*

Projecting the Continuum Modes

$$v_x = v_x^o + \frac{\partial v_x}{\partial x} x + \frac{\partial v_x}{\partial y} y$$

$$v_y = v_y^o + \frac{\partial v_y}{\partial x} x + \frac{\partial v_y}{\partial y} y$$

$$\omega_z = \omega_z^o + \frac{\partial \omega_z}{\partial x} x + \frac{\partial \omega_z}{\partial y} y$$

Linear Projection

$$\varepsilon_{xx} = \frac{\partial v_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v_y}{\partial y}, \quad \varepsilon_{xy} = \frac{\partial v_x}{\partial y}, \quad \varepsilon_{yx} = \frac{\partial v_y}{\partial x}$$

$$\gamma_{xy} = \tfrac{1}{2}(\varepsilon_{xy} + \varepsilon_{yx}), \quad \omega_{xy} = \tfrac{1}{2}(\varepsilon_{xy} - \varepsilon_{yx})$$

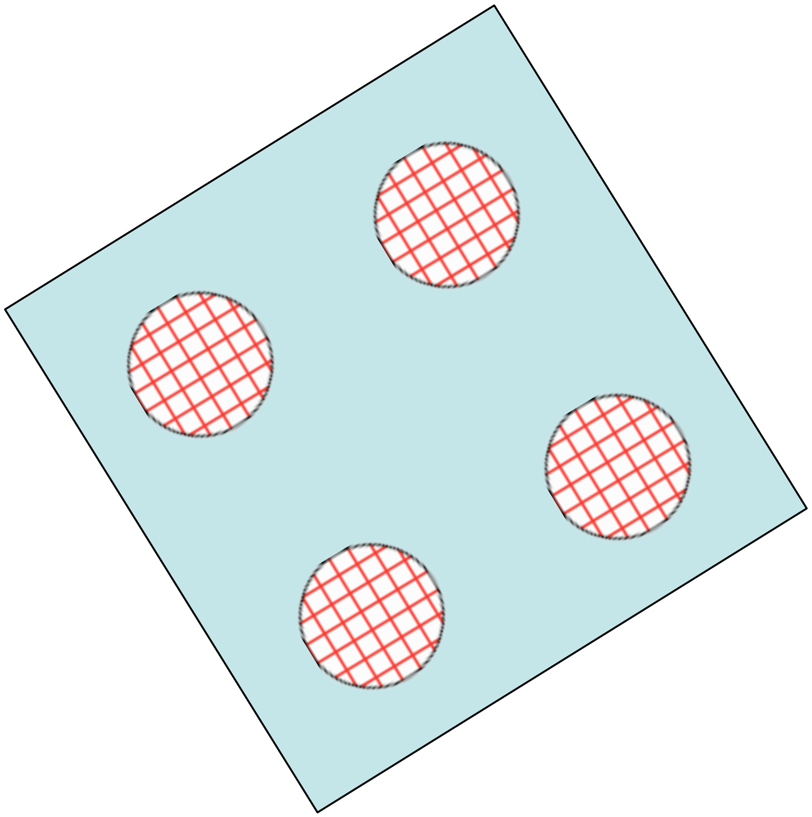
or

$\varepsilon_{xy} = \gamma_{xy} + \omega_{xy}$ and $\varepsilon_{yx} = \gamma_{xy} - \omega_{xy}$

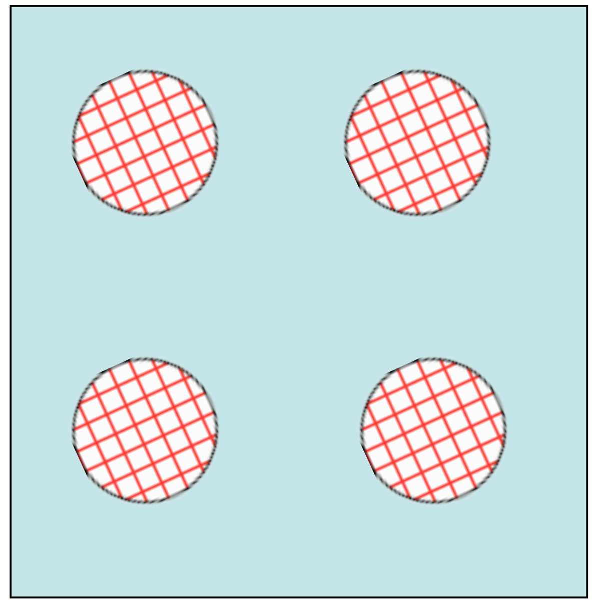
$$\begin{array}{cc} & \begin{array}{ccccccccc} v_x^o & v_y^o & \omega^o & \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} & \omega_{xy} & \kappa_{zx} & \kappa_{zy} \end{array} \\ \begin{Bmatrix} v_x \\ v_y \\ \omega_z \end{Bmatrix} = & \begin{bmatrix} 1 & 0 & y & x & 0 & y & 0 & 0 & 0 \\ 0 & 1 & -x & 0 & y & x & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & x & y \end{bmatrix} \end{array}$$

$$\{v\} = [\chi]\{\varepsilon\}$$

Two Independent Rotations



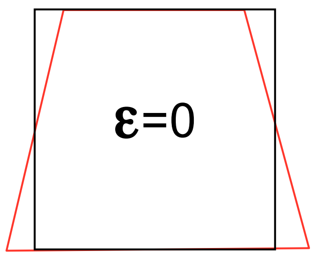
Rigid-Body Rotation (ω^o)



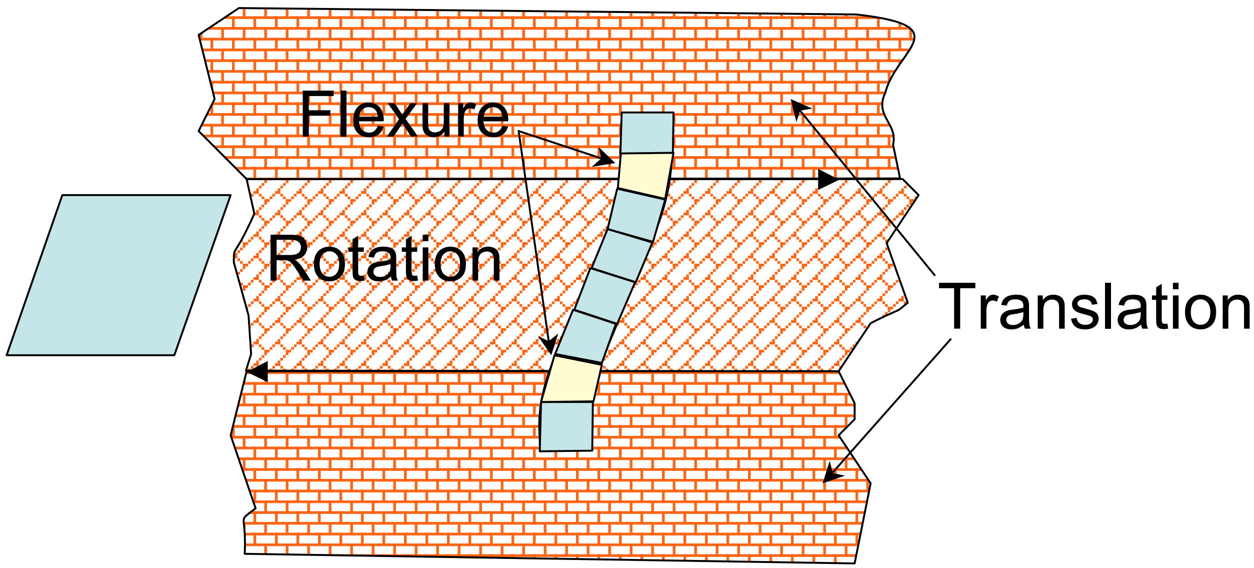
Cosserat Rotation
(Incompatible, ω_{xy})

Remark on Flexure Modes

- The bi-linear (xy) mode is a flexure mode that is also a zero-strain mode.

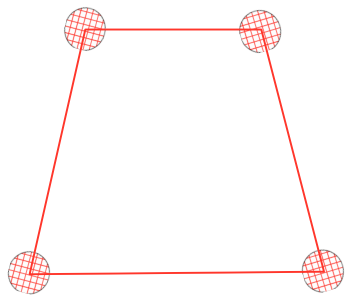


Flexure mode

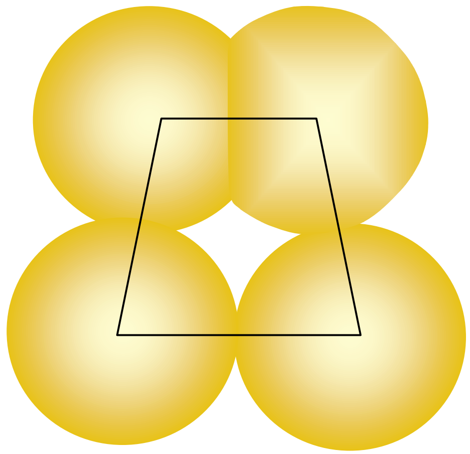


Shear Band

Rotations and Flexture



Gradient in rotations caused by compatible motion



Force comes from normal contact motions not from particle rotation

Building the Higher Modes

$$\chi^1 \equiv (1,1,1,\cdots 1)^T, \quad \chi^x \equiv (x_1, x_2, x_3 \cdots x_N)^T, \quad \chi^y \equiv (y_1, y_2, y_3 \cdots y_N)^T$$

$$\psi^{x^2} \equiv (x_1^2, x_2^2, x_3^2 \cdots x_N^2)^T, \quad \psi^{xy} \equiv (x_1 y_1, x_2 y_2, x_3 y_3 \cdots x_N y_N)^T, \quad \psi^{y^2} \equiv (y_1^2, y_2^2, y_3^2 \cdots y_N^2)^T, \cdots$$

Apply Gram-Schmidt process

$$\phi^1 = a\chi^1 + b\chi^x + c\chi^y + \psi^{x^2}$$

$$\chi^1 \bullet \phi^1 = a\chi^1 \bullet \chi^1 + \cancelto{0}{b\chi^1 \bullet \chi^x} + \cancelto{0}{c\chi^1 \bullet \chi^y} + \chi^1 \bullet \psi^{x^2} = 0 \Rightarrow a = -\frac{\chi^1 \bullet \psi^{x^2}}{\chi^1 \bullet \chi^1}$$

$$\phi^2 = a\chi^1 + b\chi^x + c\chi^y + d\phi^1$$

$$\vdots$$

Extracting Modes from DEM

$$
\begin{Bmatrix} \begin{Bmatrix} v_x^1 \\ v_y^1 \\ \omega_z^1 \end{Bmatrix} \\ \begin{Bmatrix} v_x^2 \\ v_y^2 \\ \omega_z^2 \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} v_x^N \\ v_y^N \\ \omega_z^N \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \chi^1 \\ \chi^2 \\ \vdots \\ \chi^N \end{bmatrix} \{\varepsilon\} + \alpha_1 \begin{Bmatrix} \begin{Bmatrix} \phi_1^1 \\ 0 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} \phi_2^1 \\ 0 \\ 0 \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} \phi_N^1 \\ 0 \\ 0 \end{Bmatrix} \end{Bmatrix} + \beta_1 \begin{Bmatrix} \begin{Bmatrix} 0 \\ \phi_1^1 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ \phi_2^1 \\ 0 \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} 0 \\ \phi_N^1 \\ 0 \end{Bmatrix} \end{Bmatrix} + \gamma_1 \begin{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \phi_1^1 \end{Bmatrix} \\ \begin{Bmatrix} 0 \\ 0 \\ \phi_2^1 \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} 0 \\ 0 \\ \phi_N^1 \end{Bmatrix} \end{Bmatrix} + \cdots
$$

$$\mathbf{v} = \boldsymbol{\chi}\boldsymbol{\varepsilon} + \sum_{i=1}^{N} \alpha_i \boldsymbol{\phi}^i$$

$$\mathbf{v} = \boldsymbol{\chi}\boldsymbol{\varepsilon} + \sum_{i=1}^{N} \alpha_i \boldsymbol{\phi}^i$$

$$\boldsymbol{\chi}^{\mathrm{T}}\mathbf{v} = \boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi}\boldsymbol{\varepsilon} + \cancelto{0}{\sum_{i=1}^{N} \alpha_i \boldsymbol{\chi}^{\mathrm{T}} \boldsymbol{\phi}^i}$$

$$\boldsymbol{\varepsilon} = (\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\chi})^{-1}\boldsymbol{\chi}^{\mathrm{T}}\mathbf{v}$$

$$(\boldsymbol{\phi}^i)^T \mathbf{v} = \cancelto{0}{(\boldsymbol{\phi}^i)^T \boldsymbol{\chi}\boldsymbol{\varepsilon}} + \sum_{i=1}^{N} \alpha_i (\boldsymbol{\phi}^i)^T \boldsymbol{\phi}^i$$

$$\alpha_i = \frac{(\boldsymbol{\phi}^i)^T \mathbf{v}}{(\boldsymbol{\phi}^i)^T \boldsymbol{\phi}^i}$$

Power

$$\frac{\mathbf{f}^T\mathbf{v}}{V_{REV}} = \frac{\mathbf{f}^T\chi\varepsilon}{V_{REV}} + \sum_p v^p \frac{\mathbf{f}^T\phi^p}{V_{REV}}$$

$$\frac{\mathbf{f}^T\mathbf{v}}{V_{REV}} = \sigma^T\varepsilon + \sum_p v^p f^p$$

f^p is the force resisting diffusive motion.

Consistent Stress

$$\int_{REV} \sigma^T \varepsilon \, dV = \mathbf{f}^T \mathbf{v}$$

$$V_{REV} \; \sigma^T \varepsilon = \mathbf{f}^T \mathbf{v}$$

$$\varepsilon = (\chi^T \chi)^{-1} \chi^T \mathbf{v} \Rightarrow \boxed{\sigma = \frac{1}{V_{REV}} \chi^T \mathbf{f}}$$

Kinetic Energy

$$\mathbf{p} = \mathbf{M}\mathbf{v} = \chi\mathbf{p}^{\varepsilon} + \sum_{p} p^{p}\phi^{p}$$

$$K_E = \tfrac{1}{2}\mathbf{v}^T\mathbf{p} = \tfrac{1}{2}\varepsilon^T\mathbf{p}^{\varepsilon} + \tfrac{1}{2}\sum_{p} v^{p} p^{p}$$

Note: Rigid-body motions not included in K_E.

Free Energy

$$\psi = \sum_p \psi^p \phi^p$$

Energy Pathway

It is difficult to measure dissipation

- K_E is an internal energy similar to heat.
- K_E is generated, then dissipated. So rate of K_E is equivalent to dissipation of energy.
- It is postulated that evolution relationships can be most easily observed through monitoring the energy pathway:

$$\sigma^T \dot{\varepsilon} \rightarrow \dot{\psi} \rightarrow \dot{K}_E$$

Conclusions

- Strain and equilibrium arise from rigid-body modes (null space); a continuum is not necessary.

- Stress and strain can take several equivalent forms, *all of which are equivalent to contact motion and force.*

- Constitutive equations can be defined from micromechanical laws through this equivalence.

- Internal variables related to inelastic micromechanical quantities and are not simply deviations from strain.

- The theory can be cast in form of irreversible thermodynamics, including evolution of micromechanical variables.


Thank You