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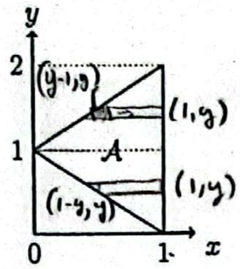
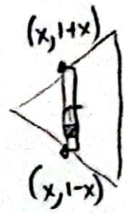
3. Two continuous random variables, X and Y, are uniformly distributed over the triangular shaped area $\mathcal{A}$ shown in the figure below.



$$f_{X,Y}(x,y) = \begin{cases} 1, & (x,y)\in A \\ 0, & o.w \end{cases}$$



(a) (4 points) Find $f_Y(y)$.
(b) (7 points) Find $\text{cov}(X,Y)$.

The random variable T is defined as $T = |X - Y|$.

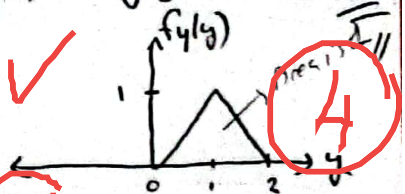
(c) (2 points) Define the range of T values which have non-zero probability.
(d) (4 points) Find $P(\{T \le t\} \cap \{X \ge Y\})$.
(e) (4 points) Find $P(\{T \le t\} \cap \{X < Y\})$.
(f) (4 points) Find the cumulative distribution function of T, $F_T(t)$.

(a) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

$0 \le y \le 1 \rightarrow f_Y(y) = \int_{1-y}^{1} f_{X,Y}(x,y)dx = \int_{1-y}^{1} 1\,dx = x\Big|_{1-y}^{1} = (1)-(1-y) = y$

$1 \le y \le 2 \rightarrow f_Y(y) = \int_{y-1}^{1} f_{X,Y}(x,y)dx = \int_{y-1}^{1} 1\,dx = x\Big|_{y-1}^{1} = (1)-(y-1) = 2-y$

$$f_Y(y) = \begin{cases} y, & y\in[0,1] \\ 2-y, & y\in[1,2] \\ 0, & o.w \end{cases}$$



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(b) $\text{cov}(X,Y) = E[(X-E[X])(Y-E[Y])] = \iint g(x,y) f_{X,Y}(x,y)\,dx\,dy$

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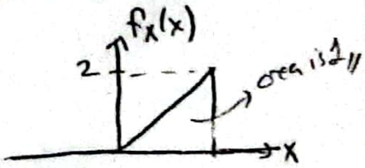
$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_{1-x}^{1+x} f_{X,Y}(x,y)dy = \int_{1-x}^{1+x} 1\,dy = y\Big|_{1-x}^{1+x} = (1+x)-(1-x) = 2x$ where $x\in[0,1]$



1 $E[Y] = 1$, $E[X] = \int_0^1 x \cdot f_X(x)dx = \int_0^1 2x^2 dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$ 1

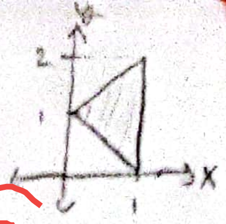
then;

2 $\text{Cov}(X,Y) = \int_0^1\int_{(1-x)}^{1+x} (x-\frac{2}{3})(y-1)\,dy\,dx = \int_0^1\int_{1-x}^{1+x} (xy - x - \frac{2}{3}y + \frac{2}{3})dy\,dx = \int_0^1 \left[\frac{1}{2}xy^2 - xy - \frac{y^2}{3} + \frac{2}{3}y\right]_{y=1-x}^{1+x} dx$

$\text{cov}(X,Y) = \int_0^1 \left[\left(x^3(\frac{1}{2}) + x^2(-\frac{1}{3}) + x(-\frac{1}{2}) + \frac{1}{3}\right) - \left(x^3(\frac{1}{2}) + x^2(-\frac{1}{3}) + x(-\frac{1}{2}) + \frac{1}{3}\right)\right]dx$

$\text{cov}(X,Y) = \int_0^1 0\cdot dx = 0 \rightarrow \boxed{\text{Cov}(X,Y) = 0}$

(c) $T = |X-Y|$ find range of T, obviously $T \geq 0$ //



for x, $y_{max}(x) = (1+x)$, $y_{min}(x) = (1-x)$

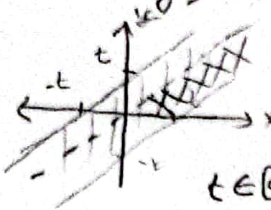
if max is used; $T = |1|$, if min is used $T = |2x-1|$

for $x \in [0,1]$, $2x-1 \leq 1 \rightarrow T_{max}$ is $1 \rightarrow \boxed{T \in [0,1]}$ // 2

(d) find $P(T \leq t, X \geq Y)$;

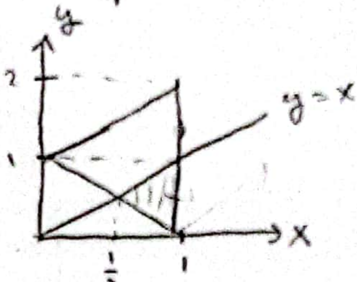
$$P(T \leq t, X \geq Y) = P(\underbrace{-t \leq X-Y \leq t}_{\text{I}}, \underbrace{X > Y}_{\text{II}})$$

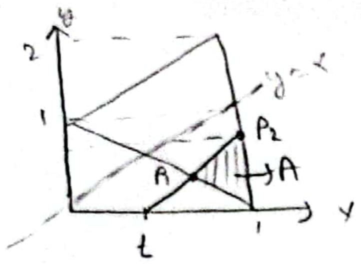
I: $y \leq x+t$, $y \geq x-t$, assume $t > 0$ since $P(T<0) = 0$ from part c //



$t \in [0,1]$

II $\rightarrow y < x$

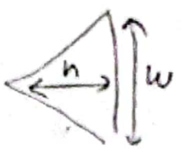




P_1: $x - t = 1 - x$

$x = \left(\frac{t+1}{2}\right)$ $y = \frac{1}{2} - \frac{t}{2} \rightarrow \left(\left(\frac{t+1}{2}\right), \left(\frac{1-t}{2}\right)\right)$

P_2: $(1, 1-t)$



$h = \left(\frac{1-t}{2}\right)$

$w = (1-t)$

$$A = \frac{1}{2}\left(\frac{1-t}{2}\right)(1-t) = \left(\frac{t^2-2t+1}{4}\right)$$

$A_{max} = A|_{t=0} = \frac{1}{4} \rightarrow$ then $P(T \leq t, X \geq Y) = \begin{cases} 0, & t<0 \\ \left(\frac{1}{4}\right) - \left(\frac{t^2-2t+1}{4}\right), & 0 \leq t \leq 1 \\ \frac{1}{4}, & 1 < t \end{cases}$ | since $f_{X,Y} = 1$

$$\boxed{P(T \leq t, X \geq Y) = \begin{cases} 0, & t<0 \\ \frac{t}{2} - \frac{t^2}{4}, & 0 \leq t \leq 1 \\ \frac{1}{4}, & \text{o.w} \end{cases}}$$ // 4

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(e) $P(T \leq t) = P((T \leq t) \cap \Omega) = P((T \leq t) \cap (X \geq Y \cup X < Y)) = P(T \leq t, X \geq Y) + P(T \leq t, X < Y)$

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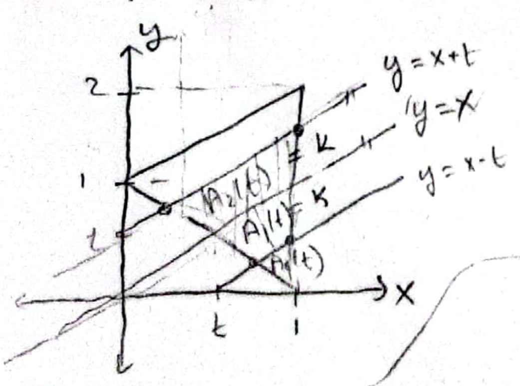
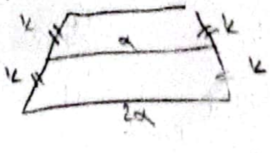
$$P(T \leq t) = P(-t \leq X - Y \leq t)$$





$3P_{A_1}(t) = P_{A_2}(t)$

$$\boxed{P_{A_2}(t) = \begin{cases} 0, & t<0 \\ \cancel{\frac{3t}{2}} - \frac{3t^2}{4}, & 0 \leq t \leq 1 \\ \frac{3}{4}, & \text{o.w} \end{cases}}$$ //

(f) $P(T \leq t) = P_{A_2}(t) + P_{A_1}(t)$ as shown in "e". Thus

$$F_T(t) = P(T \leq t) = \begin{cases} 0, & t<0 \\ \cancel{2t - t^2}, & 0 \leq t \leq 1 \\ 1, & \text{o.w} \end{cases}$$ //