CSE 321

Introduction to Algorithm Design

HW 1

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Q1) I assumed like:

$$T_4(n) < T_6(n) < T_1(n) < T_2(n) < T_5(n) < T_3(n)$$

a) $T_4 = O(T_6)$

$$\lim_{n\to\infty} \frac{\ln^2 n}{\sqrt[3]{n}} = \lim_{n\to\infty} \frac{2\ln(n) \cdot 1/n}{\frac{1}{3}\cdot n^{-2/3}} = \lim_{n\to\infty} \frac{2\ln(n)}{1/3 \cdot n^{1/3}}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n}}{\frac{1}{9}\cdot n^{-2/3}} = \lim_{n\to\infty} \frac{2}{\frac{1}{9}\cdot n^{1/3}} \Longrightarrow \frac{\text{Number}}{\text{Infinity}} \Longrightarrow 0$$

b) $T_6 = O(T_1)$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{3n^4+3n^3+1} = \lim_{n\to\infty} \frac{\cancel{n^{1/3}}(1)}{\cancel{n^{1/3}}(3n^{11/3}+3n^{8/3}+n^3)} \Rightarrow \frac{\text{Number}}{\text{Infinity}} \Rightarrow 0$$

c) $T_1 = O(T_2)$

$$\lim_{n\to\infty} \frac{3n^4+3n^3+1}{3^n} = \lim_{n\to\infty} \frac{12n^3+9n^2}{\ln 3 \cdot 3^n} = \lim_{n\to\infty} \frac{36n^2+18n}{\ln^2 3 \cdot 3^n}$$

$$= \lim_{n\to\infty} \frac{72n+18}{\ln^3 3 \cdot 3^n} = \lim_{n\to\infty} \frac{72}{\ln^4 3 \cdot 3^n} \Rightarrow \frac{\text{Number}}{\text{Infinity}} \Rightarrow 0$$

d) $T_2 = O(T_5)$

$$\lim_{n \to \infty} \frac{3^n}{2^{2n}} = \lim_{n \to \infty} \frac{3^n}{4^n} \Longrightarrow 0$$

e) $T_5 = O(T_3)$

$$\lim_{n \to \infty} \frac{2^{2n}}{(n-2)!} = \lim_{n \to \infty} \frac{4^n}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n} = \lim_{n \to \infty} \frac{1}{\sqrt{2\pi n} \cdot \left(\frac{n}{4e}\right)^n}$$

$$\Rightarrow \frac{\text{Number}}{\text{Infinity}} \Longrightarrow 0$$

Stirling's Approximation

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

Q2)

a) This algorithm calculates the average of the largest and the smallest element of the list and returns the closest element to the average.

plum = the smallest element

watermelon = the biggest element

orange = the closest element to the average.

orangeTime = variable to make the loop continue

fruit = each element in fruits.

fruits = array.

b)

i. This scenario occurs whether the smallest element is at the last place. It is $O(n)$.

j. This scenario occurs whether the smallest element is at the first place. (Array is not shifted). It is $\Omega(n)$

k. Since the best and the worst case are the same, average will be equal to them. It is $\theta(n)$.

Q3)

a) $\sum_{i=0}^{n-1} (i^2+1)^2 = \sum_{i=0}^{n-1} i^4+2i^2+1 = f(n)$

$$\int_0^{n-2} (x^4+2x^2+1)\,dx \le f(x) \le \int_1^{n} (x^4+2x^2+1)\,dx$$

$$= (x^4+2x^2+1)\Big|_0^{n-2} \le f(x) \le (x^4+2x^2+1)\Big|_1^{n}$$

$$= \frac{(n-2)^5}{5} + \frac{2(n-2)^3}{3} + n-2 \le f(x) \le \frac{n^5}{5} + \frac{2n^3}{3} + n - \left(\frac{1}{5}+\frac{2}{3}+1\right)$$

$\Longrightarrow f(n) \in \theta(n^5)$

<u>code</u>

```
for (int i=0; i<n; i++) {
    sum += (i*i+1)*(i*i+1)
}
```

b) $\sum_{i=2}^{n-1} \log i^2 = f(n)$ (I will assume log as ln not to work with constants)

$$\int_2^{n-2} \log x^2 dx \leq f(n) \leq \int_3^{n} \log x^2 dx$$

$$\int \log x^2 dx \Rightarrow \begin{array}{l} \log x^2 = u \\ dx = dv \\ x = v \\ \frac{2}{x} dx = du \end{array} \Longrightarrow u.v - \int v.du$$

$$= \log x^2 . x - \int x . \frac{2}{x} dx$$

$$= \log x^2 . x - 2x = x(\log x^2 - 2)$$

$$\underbrace{(n-2)}_{n}\underbrace{(\log(n-2)^2-2)}_{\log n} - (2(\log 4 - 2)) \leq f(n) \leq n(\log n^2 - 2) - (3(\log 9 - 2))$$

$$\Longrightarrow f(n) \in \Theta(n \log n)$$

c) $\sum_{i=1}^{n} (i+1).2^{i-1} = f(n)$

$$\int (x+1).2^{x-1}\,dx \implies \begin{array}{l} x+1 = u \\ 2^{x-1} = dv \\ dx = du \\ \dfrac{2^{x-1}}{\ln 2} = v \end{array} \implies u.v - \int v.du$$

$$(x+1).\frac{2^{x-1}}{\ln 2} - \int \frac{2^{x-1}}{\ln 2}.dx = (x+1).\frac{2^{x-1}}{\ln 2} - \frac{2^{x-1}}{\ln^2 2}$$

$$\int_{1}^{n-1} (i+1).2^{i-1}\,di \leq f(n) \leq \int_{2}^{n+1} (i+1).2^{i-1}.\,di$$

$$\underbrace{n.\frac{2^{n-2}}{\ln 2} - \frac{2^{n-2}}{\ln 2}}_{n.2^n} - \left(2.\frac{1}{\ln 2} - \frac{1}{\ln^2 2}\right) \leq f(n) \leq (n+2).\frac{2^n}{\ln 2} - \frac{2^n}{\ln 2} - \left(3.\frac{2}{\ln 2} - \frac{2}{\ln^2 2}\right)$$

$f(n) \in \theta(n.2^n)$

d) $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i+j)$

$$\sum_{i=0}^{n-1} i^2 + \frac{(i-1)(i-2)}{2} = \sum_{i=0}^{n-1} i^2 + \frac{i^2+3i+2}{2} = f(n)$$

$$= \int_0^{n-2} \left(i^2 + \frac{i^2+3i+2}{2}\right) di \leq f(x) \leq \int_1^{n} \left(i^2 + \frac{i^2+3i+2}{2}\right) di$$

$$= \frac{i^3}{3} + \frac{i^3}{6} + \frac{3i^2}{4} + i \Big|_0^{n-2} \leq f(x) \leq \frac{i^3}{3} + \frac{i^3}{6} + \frac{3i^2}{4} + i \Big|_1^{n}$$

$$= \frac{(n-2)^3}{3} + \frac{(n-2)^3}{6} + \frac{3(n-2)^2}{4} + n - 2 \leq f(x) \leq \frac{n^3}{3} + \frac{n^3}{6} + \frac{3n^2}{4} + n - \left(\frac{1}{3} + \frac{1}{6} + \frac{3}{4} + 1\right)$$

$$\Longrightarrow f(n) \in \Theta(n^3)$$

Code

```
for (int i=0; i<n; i++){
    for (int j=0; j<i; j++) {
            sum+= i+j;
    }
}
```

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Q4)
```
int fun(int n){
    int count = 0;
    for(int i=n; i>0; i/=2){
        for(int j=0; j<i; j++){
            count += 1;
        }
    }
    return count;
}
```

Count will be like $n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} \dots$

So,

$$n \underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)}_{\frac{1-r^n}{1-r} \Rightarrow r = 1/2} = n \cdot \frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{2}}$$

$$= \underbrace{2n \cdot \left(1 - \left(\frac{1}{2}\right)^n\right)}_{\theta(n)}$$

So, summation representation will be,

$$\sum_{i=0}^{n} n \cdot \left(\frac{1}{2}\right)^i$$

Q5)

a) $n^3 \in o(3^{2n})$

$$\lim_{n\to\infty} \frac{n^3}{3^{2n}} = \lim_{n\to\infty} \frac{3n^2}{\ln 9 . 9^n} = \lim_{n\to\infty} \frac{6n}{\ln^2 9 . 9^n} = \lim_{n\to\infty} \frac{6}{\ln^3 9 . 9^n}$$

$$\Rightarrow \frac{\text{Number}}{\text{Infinity}} \Longrightarrow 0$$ This one is TRUE.

b) $n \in o(\log \log n)$

$$\lim_{n\to\infty} \frac{n}{\log(\log n)} = \lim_{n\to\infty} \frac{1}{\frac{\frac{1}{n.\log n}}{\log n . \ln 10}} \Longrightarrow \frac{1}{\frac{\overbrace{\frac{1}{\infty}}^{0}}{\infty}} \Rightarrow \frac{1}{0}$$

$$\Longrightarrow \infty$$ this one is FALSE.

c) $n^2 \log^2 n \in O(n!)$

$$\lim_{n\to\infty} \frac{n^2 \log^2 n}{\sqrt{2\pi n} . \left(\frac{n}{e}\right)^n}$$ $\longrightarrow$ Stirling's Approximation

n^n has the biggest growth value.

this one is TRUE.

d) $\sqrt{10n^2+7n+3} \in \Theta(n)$

$$\lim_{n\to\infty} \frac{n}{\sqrt{10n^2+7n+3}} = \lim_{n\to\infty} \frac{n}{|\sqrt{10}\,n| + \sqrt{7n} + \sqrt{3}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\sqrt{10} + \underbrace{\sqrt{7}\,n^{-1/2}}_{0} + \underbrace{\sqrt{3}\,n^{-1}}_{0}\right)}$$

$= \frac{1}{\sqrt{10}}$ This one is TRUE