# Error Analysis

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*http* : *//teacher.nsrl.rochester.edu/phy_labs/appendixb/appendixb.html*

## 1 Introduction

The knowledge we have of the physical world is obtained by doing experiments and making measurements. It is important to understand how to express such data and how to analyze and draw meaningful conclusions from it.

In doing this it is crucial to understand that all measurements of physical quantities are subject to uncertainties. It is never possible to measure anything exactly. It is good, of course, to make the error as small as possible but it is always there. And in order to draw valid conclusions the error must be indicated and dealt with properly.

Take the measurement of a person's height as an example. Assuming that her height has been determined to be $5'8''$, how accurate is our result?

Well, the height of a person depends on how straight she stands, whether she just got up (most people are slightly taller when getting up from a long rest in horizontal position), whether she has her shoes on, and how long her hair is and how it is made up. These inaccuracies could all be called errors of definition. A quantity such as height is not exactly defined without specifying many other circumstances.

Even if you could precisely specify the "circumstances", your result would still have an error associated with it. The scale you are using is of limited accuracy; when you read the scale, you may have to estimate a fraction between the marks on the scale, etc.

If the result of a measurement is to have meaning it cannot consist of the measured value alone. An indication of how accurate the result is must be included also. Indeed, typically more effort is required to determine the error or uncertainty in a measurement than to perform the measurement itself. Thus, the result of any physical measurement has two essential components: (1) A numerical value (in a specified system of units) giving the best estimate possible of the quantity measured, and (2) the degree of uncertainty associated with this estimated value. For example, a measurement of the width of a table would yield a result such as $(95.3 \pm 0.1)cm$.

## 2 Significant Figures

The significant figures of a (measured or calculated) quantity are the meaningful digits in it. There are conventions which you should learn and follow for how to express numbers so as to properly indicate their significant figures.

- Any digit that is not zero is significant. Thus 549 has three significant figures and 1.892 has four significant figures.

- Zeros between non zero digits are significant. Thus 4023 has four significant figures.

- Zeros to the left of the first non zero digit are not significant. Thus 0.000034 has only two significant figures. This is more easily seen if it is written as $3.4 \times 10^{-5}$.

- For numbers with decimal points, zeros to the right of a non zero digit are significant. Thus 2.00 has three significant figures and 0.050 has two significant figures. For this reason it is important to keep the trailing zeros to indicate the actual number of significant figures.

- For numbers without decimal points, trailing zeros may or may not be significant. Thus, 400 indicates only one significant figure. To indicate that the trailing zeros are significant a decimal point must be added. For example, 400. has three significant figures, and has one significant figure.

- Exact numbers have an infinite number of significant digits. For example, if there are two oranges on a table, then the number of oranges is $2.000\ldots$ . Defined numbers are also like this. For example, the number of centimeters per inch (2.54) has an infinite number of significant digits, as does the speed of light ($299792458 m/s$).

There are also specific rules for how to consistently express the uncertainty associated with a number. In general, the last significant figure in any result should be of the same order of magnitude (i.e.. in the same decimal position) as the uncertainty. Also, the uncertainty should be rounded to one or two significant figures. Always work out the uncertainty after finding the number of significant figures for the actual measurement.
For example,

- $9.82 \pm 0.02$

- $10.0 \pm 1.5$

- $4 \pm 1$

The following numbers are all incorrect.

- $9.82 \pm 0.02385$ is wrong but $9.82 \pm 0.02$ is fine.

- $10.0 \pm 2$ is wrong but $10.0 \pm 2.0$ is fine.

- $4 \pm 0.5$ is wrong but $4.0 \pm 0.5$ is fine.

In practice, when doing mathematical calculations, it is a good idea to keep one more digit than is significant to reduce rounding errors. But in the end, the answer must be expressed with only the proper number of significant figures. After addition or subtraction, the result is significant only to the place determined by the largest last significant place in the original numbers. For example, $89.332 + 1.1 = 90.432$ should be rounded to get 90.4 (the tenths place is the last significant place in 1.1). After multiplication or division, the number of significant figures in the result is determined by the original number with the smallest number of significant figures. For example, $(2.80)(4.5039) = 12.61092$ should be rounded off to 12.6 (three significant figures like 2.80).
Refer to any good introductory chemistry textbook for an explanation of the methodology for working out significant figures.

# 3   The Idea of Error

The concept of error needs to be well understood. What is and what is not meant by "error"?
A measurement may be made of a quantity which has an accepted value which can be looked up in a handbook (e.g.. the density of brass). The difference between the measurement and the accepted value is not what is meant by error. Such accepted values are not "right" answers. They are just measurements made by other people which have errors associated with them as well.
Nor does error mean "blunder". Reading a scale backwards, misunderstanding what you are doing or elbowing your lab partner's measuring apparatus are blunders which can be caught and should simply be disregarded.
Obviously, it cannot be determined exactly how far off a measurement is; if this could be done, it would be possible to just give a more accurate, corrected value.
Error, then, has to do with uncertainty in measurements that nothing can be done about. If a measurement is repeated, the values obtained will differ and none of the results can be preferred over the others. Although it is not possible to do anything about such error, it can be characterized. For instance, the repeated measurements may cluster tightly together or they may spread widely. This pattern can be analyzed systematically.

# 4   Classification of Error

Generally, errors can be divided into two broad and rough but useful classes: systematic and random.

Systematic errors are errors which tend to shift all measurements in a systematic way so their mean value is displaced. This may be due to such things as incorrect calibration of equipment, consistently improper use of equipment or failure to properly account for some effect. In a sense, a systematic error is rather like a blunder and large systematic errors can and must be eliminated in a good experiment. But small systematic errors will always be present. For instance, no instrument can ever be calibrated perfectly.

Other sources of systematic errors are external effects which can change the results of the experiment, but for which the corrections are not well known. In science, the reasons why several independent confirmations of experimental results are often required (especially using different techniques) is because different apparatus at different places may be affected by different systematic effects. Aside from making mistakes (such as thinking one is using the x10 scale, and actually using the x100 scale), the reason why experiments sometimes yield results which may be far outside the quoted errors is because of systematic effects which were not accounted for.

Random errors are errors which fluctuate from one measurement to the next. They yield results distributed about some mean value. They can occur for a variety of reasons.

- They may occur due to lack of sensitivity. For a sufficiently a small change an instrument may not be able to respond to it or to indicate it or the observer may not be able to discern it.

- They may occur due to noise. There may be extraneous disturbances which cannot be taken into account.

- They may be due to imprecise definition.

- They may also occur due to statistical processes such as the roll of dice.

Random errors displace measurements in an arbitrary direction whereas systematic errors displace measurements in a single direction. Some systematic error can be substantially eliminated (or properly taken into account). Random errors are unavoidable and must be lived with. Many times you will find results quoted with two errors. The first error quoted is usually the random error, and the second is called the systematic error. If only one error is quoted, then the errors from all sources are added together. (In quadrature as described in the section on propagation of errors.)

A good example of "random error" is the statistical error associated with sampling or counting. For example, consider radioactive decay which occurs randomly at a some (average) rate. If a sample has, on average, 1000 radioactive decays per second then the expected number of decays in 5 seconds would be 5000. A particular measurement in a 5 second interval will, of course, vary from this average but it will generally yield a value within $5000 \pm$ ___. Behavior like this, where the error,

$$\Delta n = \sqrt{n_{expected}}, \tag{1}$$

is called a Poisson statistical process. Typically if one does not know it is assumed that,

$$x_{measured} = x_{expected},$$

in order to estimate this error.

## 4.1   Estimating Uncertainty by One Measurement

Assume that you have measured a physical quantity only once. What is the error in this measurement?

It can be found by a simple reasoning that the maximum fallibility would be as amount of the accuracy in the measurement tool. Say you've measured a length with a millimetric ruler, then you would made an error of $1mm$ at the very most. This means that the true length might be

$0.5mm$ less or $0.5mm$ more than your measurement. In other words single measurements have an error half of the accuracy of the measurement tool.

$$\text{For a ruler:} \quad \Delta x = 0.05cm.$$

Let's discuss an example. Say you try to measure the length of a pencil with a millimetric ruler and the accuracy of this ruler is $0.1cm$. Therefore if you measured the length as $18.3cm$ then you could make an error of $0.1cm$ in total; meaning the true value of this one-time measurement would lie between $(18.3 + 0.05)cm$ and $(18.3 - 0.05)cm$. This means that the measurement has an uncertainty of $0.05cm$. Thus you should consider $(18.30 \pm 0.05)cm$ as your measurement.

If you would measured the length with a vernier and read $18.27cm$, since the accuracy is $0.01cm$ in it, you would made an error of $0.01cm$. Then you should consider $(18.270 \pm 0.005)cm$ as your measurement.

But true way to be "sure" that the measurements are reliable in an experiment is to repeat the experiment as many times as possible and perform an error analysis on "the mean value" of these repeated data to report. Let's dig in how to do so.

## 4.2   Mean Value

Suppose an experiment were repeated many, say N, times to get,

$$x_1, \ x_2, \ x_3, \ ..., \ x_N,$$

N measurements of the same quantity, x. If the errors were random then the errors in these results would differ in sign and magnitude. So if the average or mean value of our measurements were calculated,

$$\bar{x} = \frac{x_1 + x_2 + x_3 + ... + x_N}{N} = \frac{1}{N}\sum_{i=1}^{N} x_i, \tag{2}$$

some of the random variations could be expected to cancel out with others in the sum. This is the best that can be done to deal with random errors: repeat the measurement many times, varying as many "irrelevant" parameters as possible and use the average as the best estimate of the true value of x. (It should be pointed out that this estimate for a given N will differ from the limit as the true mean value; though, of course, for larger N it will be closer to the limit.) In the case of the previous example: measure the height at different times of day, using different scales, different helpers to read the scale, etc.

Doing this should give a result with less error than any of the individual measurements. But it is obviously expensive, time consuming and tedious. So, eventually one must compromise and decide that the job is done. Nevertheless, repeating the experiment is the only way to gain confidence in and knowledge of its accuracy. In the process an estimate of the deviation of the measurements from the mean value can be obtained.

## 4.3   Measuring Error

There are several different ways the distribution of the measured values of a repeated experiment such as discussed above can be specified.

### Maximum Error

The maximum and minimum values of the data set, $x_{max}$ and $x_{min}$, could be specified. In these terms, the quantity,

$$\Delta x_{max} = \frac{x_{max} - x_{min}}{2} \tag{3}$$

is the maximum error. And virtually no measurements should ever fall outside $\bar{x} \pm \Delta x_{max}$.

**Probable Error**

The probable error, $\Delta x_{prob}$, specifies the range $\bar{x} \pm \Delta x_{prob}$ which contains 50% of the measured values.

**Average Deviation**

The average deviation is the average of the deviations from the mean,

$$\Delta x_{av} = \frac{1}{N} \sum_k |x_k - \bar{x}| \tag{4}$$

For a Gaussian distribution of the data, about 58% will lie within $\bar{x} \pm \Delta x_{av}$.

**Standard Deviation**

For the data to have a Gaussian distribution means that the probability of obtaining the result x is,

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}. \tag{5}$$

where $x_0$ is most probable value and $\sigma$, which is called the standard deviation, determines the width of the distribution. Because of the law of large numbers this assumption will tend to be valid for random errors. And so it is common practice to quote error in terms of the standard deviation of a Gaussian distribution fit to the observed data distribution. This is the way you should quote error in your reports. It is just as wrong to indicate an error which is too large as one which is too small. In the measurement of the height of a person, we would reasonably expect the error to be $1/4''$ if a careful job was done, and maybe $\pm 3/4''$ if we did a hurried sample measurement. Certainly saying that a person's height is $5'8.250'' \pm 0.002''$ is ridiculous (a single jump will compress your spine more than this) but saying that a person's height is $5'8'' \pm 6''$ implies that we have, at best, made a very rough estimate!

## 4.4 Standard Deviation

The mean is the most probable value of a Gaussian distribution. In terms of the mean, the standard deviation of any distribution is,

$$\sigma_x = \sqrt{\frac{1}{N} \sum_k (x_k - \bar{x})^2}. \tag{6}$$

The quantity $\sigma_x^2$, the square of the standard deviation, is called the variance. There is another definition of the standard deviation which is,

$$\sigma_x = \sqrt{\frac{1}{N-1} \sum_k (x_k - \bar{x})^2}. \tag{7}$$

The reason behind the necessity of second definition for standard deviation is not to have large-enough number of measurements. Since all of the measurements contains some (positive or negative) error their mean value will also have some error. If one could repeat the measurement many many more times, than the mean value of x would converge to the true value of x. But to perform enough number of repeats has a high cost and is not time-sufficient. So one should stop repeating the measurements at a logical number[1] and find/define other methods to minimize the errors in measurements and calculations. Since there is not enough number of repeats, the mean value of x is bigger than the real mean value of x. Thus, $(x_k - \bar{x})^2$ as calculated is always a little bit smaller than $(x_k - \bar{x}_{true})^2$, the quantity really wanted. In the theory of probability (that is, using the

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[1]This number is up to the experimental setup, the nature of the experiment itself and the experience of the experimentalist.

assumption that the data has a Gaussian distribution), it can be shown that this underestimate is corrected by using N-1 instead of N.

Meaning of the standard deviation is that if one could made one more measurement of x then (this is also a property of a Gaussian distribution) it would have some 68% probability of lying within $\bar{x} \pm \sigma_x$. Note that this means that about 30% of all experiments will disagree with the accepted value by more than one standard deviation!

However, we are also interested in the error of the mean, which is smaller than $\sigma_x$ if there were several measurements. An exact calculation yields,

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \sqrt{\frac{1}{N(N-1)} \sum_k (x_k - \bar{x})^2}, \tag{8}$$

for the standard error of the mean. This means that, for example, if there were 20 measurements, the error on the mean itself would be 4.47 times smaller then the error of each measurement. The number to report for this series of N measurements of x is $\bar{x} \pm \sigma_{\bar{x}}$. The meaning of this is that if the N measurements of x were repeated there would be a 68% probability the new mean value of would lie within $\bar{x} \pm \sigma_{\bar{x}}$ (that is between $\bar{x} + \sigma_{\bar{x}}$ and $\bar{x} - \sigma_{\bar{x}}$). Note that this also means that there is a 32% probability that it will fall outside of this range. This means that out of 100 experiments of this type, on the average, 32 experiments will obtain a value which is outside the standard errors. For a Gaussian distribution there is a 5% probability that the true value is outside of the range $\bar{x} \pm 2\sigma_{\bar{x}}$, i.e. twice the standard error, and only a 0.3% chance that it is outside the range of $\pm 3\sigma_{\bar{x}}$.

## Examples

Suppose the number of cosmic ray particles passing through some detecting device every hour is measured nine times and the results are those in the Table (1). Thus we have $\bar{x} = 900/9 = 100$ and $\sigma_x^2 = 1500/8 = 188$ or $\sigma_x = 14$. Then the probability that one more measurement of x will lie within $100 \pm 14$ is 68%.

| $i$ | $x_i$ | $(x_i - \bar{x})^2$ |
|---|---|---|
| 1 | 80 | 400 |
| 2 | 95 | 25 |
| 3 | 100 | 0 |
| 4 | 110 | 100 |
| 5 | 90 | 100 |
| 6 | 115 | 225 |
| 7 | 85 | 225 |
| 8 | 120 | 400 |
| 9 | 105 | 25 |
| Sum | 900 | 1500 |

Table 1: Number of cosmic ray particles passing through a detector device every hour.

The value to be reported for this series of measurements is $100 \pm (14/3)$ or $100 \pm 5$. If one were to make another series of nine measurements of x there would be a 68% probability the new mean would lie within the range $100 \pm 5$.

Random counting processes like this example obey a Poisson distribution for which $\sigma_x = \sqrt{\bar{x}}$. So one would expect the value of $\sigma_x$ to be 10. This is somewhat less than the value of 14 obtained above; indicating either the process is not quite random or, what is more likely, more measurements are needed.

The same error analysis can be used for any set of repeated measurements whether they arise from random processes or not. For example in the Atwood's machine experiment to measure g you are

asked to measure time five times for a given distance of fall s. The mean value of the time is,

$$\bar{t} = \frac{1}{5} \sum_i t_i \tag{9}$$

and the standard error of the mean is,

$$\sigma_{\bar{t}} = \sqrt{\frac{\sum_i (t_i - \bar{t})^2}{N(N-1)}} \tag{10}$$

where N = 5.

For the distance measurement you will have to estimate $\Delta s$, the precision with which you can measure the drop distance (probably of the order of 2-3 mm).

## 5 Propagation of Errors

Frequently, the result of an experiment will not be measured directly. Rather, it will be calculated from several measured physical quantities (each of which has a mean value and an error). What is the resulting error in the final result of such an experiment? For instance, what is the error in $Z = A + B$ where $A$ and $B$ are two measured quantities with errors $\Delta A$ and $\Delta B$ respectively?

A first thought might be that the error in $Z$ would be just the sum of the errors in $A$ and $B$. After all,

$$(A + \Delta A) + (B + \Delta B) = (A + B) + (\Delta A + \Delta B) \tag{11}$$

and

$$(A - \Delta A) + (B - \Delta B) = (A + B) - (\Delta A + \Delta B). \tag{12}$$

But this assumes that, when combined, the errors in $A$ and $B$ have the same sign and maximum magnitude; that is that they always combine in the worst possible way. This could only happen if the errors in the two variables were perfectly correlated, (i.e.. if the two variables were not really independent).

If the variables are independent then sometimes the error in one variable will happen to cancel out some of the error in the other and so, on the average, the error in $Z$ will be less than the sum of the errors in its parts. A reasonable way to try to take this into account is to treat the perturbations in $Z$ produced by perturbations in its parts as if they were "perpendicular" and added according to the Pythagorean theorem,

$$\Delta Z = \sqrt{(\Delta A)^2 + (\Delta B)^2}. \tag{13}$$

That is, if $A = 100 \pm 3$ and $B = 6 \pm 4$ then $Z = 106 \pm 5$ since .

This idea can be used to derive a general rule. Suppose there are two measurements, $A$ and $B$, and the final result is $Z = F(A, B)$ for some function $F$. If $A$ is perturbed by $\Delta A$ then $Z$ will be perturbed by

$$\left(\frac{\partial F}{\partial A}\right) \Delta A, \tag{14}$$

where $\frac{\partial F}{\partial A}$ is the derivative of $F$ with respect to $A$ with $B$ held constant. Similarly the perturbation in $Z$ due to a perturbation in $B$ is,

$$\left(\frac{\partial F}{\partial B}\right) \Delta B. \tag{15}$$

Combining these by the Pythagorean theorem yields

$$\Delta Z = \sqrt{\left(\frac{\partial F}{\partial A}\right)^2 (\Delta A)^2 + \left(\frac{\partial F}{\partial B}\right)^2 (\Delta B)^2} \tag{16}$$

7

In the example of $Z = A + B$ considered above,

$$\left(\frac{\partial F}{\partial A}\right) = 1, \qquad \left(\frac{\partial F}{\partial B}\right) = 1, \tag{17}$$

so this gives the same result as before. Similarly if $Z = A - B$ then,

$$\left(\frac{\partial F}{\partial A}\right) = 1, \qquad \left(\frac{\partial F}{\partial B}\right) = -1, \tag{18}$$

which also gives the same result. Errors combine in the same way for both addition and subtraction. However, if $Z = AB$ then,

$$\left(\frac{\partial F}{\partial A}\right) = B, \qquad \left(\frac{\partial F}{\partial B}\right) = A, \tag{19}$$

so

$$\Delta Z = \sqrt{B^2(\Delta A)^2 + A^2(\Delta B)^2}. \tag{20}$$

Thus,

$$\frac{\Delta Z}{Z} = \frac{\Delta Z}{AB} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{DeltaB}{B}\right)^2} \tag{21}$$

or the fractional error in Z is the square root of the sum of the squares of the fractional errors in its parts. (You should be able to verify that the result is the same for division as it is for multiplication.) For example,

$$(100.0 \pm 0.3)(6.0 \pm 0.4) = 600 \pm \sqrt{\left(\frac{100.0}{0.3}\right)^2 + \left(\frac{6.0}{0.4}\right)^2} = 600 \pm 40 \tag{22}$$

It should be noted that since the above applies only when the two measured quantities are independent of each other it does not apply when, for example, one physical quantity is measured and what is required is its square. If $Z = A^2$ then the perturbation in Z due to a perturbation in A is,

$$\Delta Z = \left(\frac{\partial Z}{\partial A}\right)\Delta A = 2A\Delta A. \tag{23}$$

Thus, in this case,

$$Z = (A \pm \Delta A)^2 = A^2 \pm 2A\Delta A = A^2\left(1 \pm 2\frac{\Delta A}{A}\right) \tag{24}$$

and not $A^2\left(1 \pm \frac{\Delta A}{A}\right)$ as would be obtained by misapplying the rule for independent variables. For example,

$$(10 \pm 1)^2 = 100 \pm 20 \neq 100 \pm 14. \tag{25}$$

If a variable $Z$ depends on one or two variables ($A$ and $B$) which have independent errors ($\Delta A$ and $\Delta B$) then the rule for calculating the error in $Z$ is tabulated in Table (2) for a variety of simple relationships. These rules may be compounded for more complicated situations.[2, 3]

| $Z(A, B)$ | $\Delta Z$ |
|---|---|
| $Z = A + B$ | $(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$ |
| $Z = A - B$ | $(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$ |
| $Z = AB$ | $(\frac{\Delta Z}{Z})^2 = (\frac{\Delta A}{A})^2 + (\frac{\Delta B}{B})^2$ |
| $Z = \frac{A}{B}$ | $(\frac{\Delta Z}{Z})^2 = (\frac{\Delta A}{A})^2 + (\frac{\Delta B}{B})^2$ |
| $Z = A^n$ | $\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}$ |
| $Z = ln(A)$ | $\Delta Z = \frac{\Delta A}{A}$ |
| $Z = e^A$ | $\frac{\Delta Z}{Z} = \Delta A$ |

Table 2: Error calculation rules for some simple relations.

# 6 Examples

## 6.1 Uncertainty in Single Measurements

Consider the single time measurement taken on airtable of frequency $40Hz$. Accuracy of this machine is,

$$accuracy = \frac{1}{f} = 0.025 sec. \qquad (26)$$

Then the error in this measurement is,

$$\Delta t = \frac{accuracy}{2} \approx 0.013 sec. \qquad (27)$$

## 6.2 Simple Error Calculation

Calculate standard deviation, true standard deviation and standard error of mean value for given data in Table (1).

- Mean Value:

$$
\begin{aligned}
\bar{x} &= \frac{1}{N} \sum_i x_i \\
&= \frac{1}{9}\{80 + 95 + 100 + 110 + 90 + 115 + 85 + 120 + 105\} \\
&= 100
\end{aligned}
$$

- Standard Deviation:

$$
\begin{aligned}
\sigma &= \sqrt{\frac{1}{N} \sum_k (x_k - \bar{x})^2} \\
&= \sqrt{\frac{(80 - 100)^2 + (95 - 100)^2 + (100 - 100)^2 + \cdots + (120 - 100)^2 + (105 - 100)^2}{9}} \\
&\approx 12.9 \qquad (28)
\end{aligned}
$$

- True Standard Deviation:

$$
\begin{aligned}
\sigma_x &= \sqrt{\frac{1}{N-1}\sum_k (x_k - \bar{x})^2} \\
&= \sqrt{\frac{(80-100)^2 + (95-100)^2 + (100-100)^2 + \cdots + (120-100)^2 + (105-100)^2}{9-1}} \\
&\approx 13.69
\end{aligned}
\tag{29}
$$

- Standard Error of Mean Value:

$$
\begin{aligned}
\sigma_{\bar{x}} &= \frac{\sigma_x}{\sqrt{N}} \\
&= \frac{12.9}{\sqrt{9-1}} \\
&\approx 4.56
\end{aligned}
\tag{30}
$$

$\Rightarrow$ All in all, you should report that

- the result you get in the counting cosmic ray particles experiment is $100.0 \pm 4.6$,
- if one takes another measurement, that measurement is within $100.0 \pm 13.7$ with 68% probability and
- if one takes another 9 measurements it's mean value is 68% likely to be within $100.0 \pm 4.6$.

## 6.3   Propagation of errors

Consider a measurement of Experiment 1 taken as $x = 5cm$ and $t = 0.2sec$ with an airtable of frequency, $f = 50Hz$. The accuricies are as follows,

$$
\text{accuracy of ruler} = 0.01cm, \qquad \text{accuracy of timer} = \frac{1}{f} = 0.02sec.
\tag{31}
$$

Thus, errors in distance and time measurements are,

$$
\Delta x = 0.005cm, \qquad \Delta t = 0.01sec.
\tag{32}
$$

Recall that the subject is "*the motion with constant velocity*" in Experiment 1. Thus the speed is,

$$
V = \frac{x}{t} = \frac{5cm}{0.2sec} = 25cm/sec.
\tag{33}
$$

According to Eqn (16), the error in speed is,

$$
\begin{aligned}
\Delta V &= \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 (\Delta x)^2 + \left(\frac{\partial V}{\partial t}\right)^2 (\Delta t)^2} \\
&= \sqrt{\left(\frac{1}{t}\right)^2 (\Delta x)^2 + \left(-\frac{x}{t^2}\right)^2 (\Delta t)^2} \\
&= \sqrt{\frac{1}{(0.2sec)^2}(0.05cm)^2 + \frac{(5cm)^2}{(0.2sec)^4}(0.01sec)^2} \\
&\approx 0.35cm/sec.
\end{aligned}
\tag{34}
$$

Therefore the speed value for this measurement is $(25.00 \pm 0.35)cm/sec$.

# 7 Experimental Part

We'll perform inclined plane experiment on airtable several times. First let's recall the theoretical background of inclined plane.
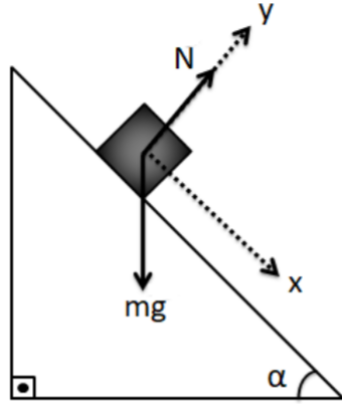
## 7.1 Theory



Figure 1: Force Diagram for a body on an inclined plane[1].

There are only two forces on the mass on a frictionless inclined plane shown in Figure (1), which are normal force and its own weight.

$$m\vec{g} = mg\sin\alpha(\hat{\mathbf{i}}) + mg\cos\alpha(-\hat{\mathbf{j}}), \qquad (35)$$
$$\vec{N} = N\hat{\mathbf{j}}. \qquad (36)$$

Then the net force on it is,

$$\vec{F_{net}} = m\vec{g} + \vec{N} = mg\sin\alpha\hat{\mathbf{i}} + (-mg\cos\alpha + N)\hat{\mathbf{j}}. \quad (37)$$

According to Newton's 2nd law of motion, net force on a body is equal to its mass times acceleration.

$$\vec{F_{net}} = m\vec{a} \qquad (38)$$

If Eqn (37) and Eqn (38) are evaluated together, one will get,

$$m(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}) = mg\sin\alpha\hat{\mathbf{i}} + (-mg\cos\alpha + N)\hat{\mathbf{j}}. \qquad (39)$$

Since cartesian unitary vectors are linearly independent, one can split Eqn(39) into two independent equations for each direction as follows,

$$ma_x = mg\sin\alpha, \qquad (40)$$
$$ma_y = -mg\cos\alpha + N. \qquad (41)$$

Because there is no motion in $y$-direction, the velocity and the acceleration in that direction is zero, meaning the acceleration of the system is,

$$\vec{a} = a_x\hat{\mathbf{i}}. \qquad (42)$$

Then one can infer that $mg\sin\alpha = N$ by Eqn (41). On the other hand, using Eqn (40) one can get the acceleration of the system as,

$$a = g\sin\alpha. \qquad (43)$$

## 7.2 Procedure

### 7.2.1 Experimental Procedure

1. Turn the airtable into an inclined plane by placing its cylindirical part under its back foot.

2. Turn on the lab table and then the airtable.

3. By using only compressor's pedal, make sure the airtable is working.

4. If the airtable is working; turn it off. If not; contact with your lab instructor.

5. Place the carbon paper into the airtable and experiment sheet onto the carbon paper.

6. By using a folded paper, immobilize one of the pucks at one of the down-corners inside of the airtable.

7. Place the other puck at the upmost part of the experiment paper.

8. Using both of the pedals, run the airtable and let the puch slide down.

9. Move the experiment paper away from the airtable on the experiment table and label it as "#1".

10. Repeat steps 7&9, nine more times with a clean-new experiment paper each time and label them with increasing numbers.

### 7.2.2   Analysis Procedure

1. Determine a total interval number and write it down on Eqn (45).

2. Take the experiment paper #1.

3. Label the dots on it one by one beginning with the first dot as 0 until you reach the n_th dot.

4. Measure the total distance between 0th and n_th dots and put it in the cell on Table (3).

5. Calculate the total time elapsed inbetween 0th and n_th dots and write it on Table (3).

6. Using the distance-time formula for the motion with constant acceleration with vanishing initial location and velocity,

$$x(t) = \frac{1}{2}at^2, \tag{44}$$

calculate the experimental acceleration value and note it on Table (3).

7. Repeat the steps 2-6 with other experiment papers.

8. Derive the formula of error in acceleration using Eqn (16) and Eqn (44). Write your derivation in Section (7.3) with details and explanations.

9. Choose a row from Table (3) and make an elaborate example calculation for the error in acceleration using the formula that you have derived at step 8 for that row in Section 7.4.

10. Calculate the error in other experimental acceleration values using the formula you have derived in Section 7.3 and write your results on Table (3).

11. Write the tool you used for distance measurements and its accuracy in Section 7.4. Calculate and write the error in single distance measurement in same section. Use this value to complete the measured values in 1st column of Table (3).

12. Write the tool you used for time measurements and its accuracy in Section (7.4). Calculate and write the error in single time measurement in same section. Use this value to complete the measured values in 2nd column of Table (3).

At the end, one row of Table (3) should look like in Figure (2).

Table 3:

| $i$ | $x \pm \Delta x \, (cm)$ | $t \pm \Delta t \, (sec)$ | $a_{exp} \pm \Delta a_{exp} \, (cm/sec^2)$ |
|---|---|---|---|
| 1 | $15.20 \pm 0.05$ | $0.400 \pm 0.013$ | $190.00 \pm 12.37$ |
| | | | |

Figure 2: Screenshot of Table (3) after measurements are written.

13. Calculate the sum and the mean value of experimental acceleration and write them in Eqn (48) and Eqn (49).

14. Copy the experimental acceleration values from Table (3) to Table (4) without their error parts.

15. Calculate deviation square values for each measurement and write them on Table (4).

16. Calculate standard deviation with both formulas and standard error of mean. And write them in Section 7.4 with details.

## 7.3 Derivation of the formula of error in acceleration

## 7.4  Data & Results

Table 3:

| $i$ | $x \pm \Delta x\,(cm)$ | $t \pm \Delta t\,(sec)$ | $a_{exp} \pm \Delta a_{exp}\,(cm/sec^2)$ |
|---|---|---|---|
| 1 | $15.20 \pm 0.05$ | $0.400 \pm 0.013$ | $190.00 \pm 12.37$ |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |
| 8 | | | |
| 9 | | | |
| N=10 | | | |

Number of intervals on experiment papers:

$$n = \text{_____} \tag{45}$$

Distance measurement tool: _____.

Accuracy of distance measurement tool: _____.

Error for distance measurements:

$$\Delta x = \text{_____} \tag{46}$$

Frequency of airtable: _____.

Accuracy of time: _____.

Error for time measurements:
$$\Delta t = \text{_____}. \tag{47}$$

Example calculation for $\Delta a_{exp}$ (Using _____ row from Table (3)):

The sum of all $a_{exp}$ values:
$$\sum_{i=1}^{N} (a_{exp})_i = \text{_____}cm/sec^2 \tag{48}$$

Mean value for $a_{exp}$:
$$\bar{a}_{exp} = \text{_____}cm/sec^2 \tag{49}$$

Table 4:

| $i$ | $a_{exp}(cm/sec^2)$ | $((a_{exp})_i - \bar{a}_{exp})^2 (cm^2/sec^4)$ |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| Sum | | |
| Mean | | |

The sum of deviation-squares:

$$\sum_{i=1}^{N} ((a_{exp})_i - \bar{a}_{exp})^2 = \text{_____}cm^2/sec^4 \qquad (50)$$

- Standard deviation (with Eqn (6)):

$$\sigma = \underline{\hspace{5cm}} \tag{51}$$

Calculation:

- Standard deviation (with Eqn (7)):

$$\sigma = \underline{\hspace{5cm}} \tag{52}$$

Calculation:

- Standard error for $\bar{a}_{exp}$:

$$\sigma_{\bar{a}_{exp}} = \underline{\hspace{5cm}} \tag{53}$$

Calculation:

# 8 Conclusions

# 9   Notes

# References

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