Middle East Technical University | Department of Computer Engineering

CENG371

Scientific Computing

Fall 2022-2023

Homework 3

Due: December 30th, 2023, 23:55

Question 1 (40 points)

Compute the singular value decomposition (you can use built-in algorithms, such as MATLAB's `svd`) of a grayscale image of your choice, $I \in \mathbb{R}^{m \times n}$ where $r = \min(m, n) > 100$.

a) **(15 pts)** Compute low-rank approximations, I_k, $k \in \{1, 2, \ldots, r\}$ of this image by discarding the smallest k singular values. In your reports, include I, $I_{r/2}$, and I_{r-10}, and make a comparison between them (2-3 sentences).

b) **(15 pts)** Denote the singular values by $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_r \,|\, \sigma_i < \sigma_{i+1}\}$, and let $S(k) = \sqrt{\sum_{i=1}^{k} \sigma_i^2}$. Plot $S(k)$ and the errors $\|I - I_k\|_F$ where $\|.\|_F$ is the Frobenius norm in a single log-plot (you can use `loglog` in MATLAB). Include the plots in your reports and reflect on your observations.

c) **(10 pts)** Based on your observations from part a) and part b), suggest a use case for the low-rank approximation scheme.

Question 2 (60 points)

Let $N(t) = 0.3 + 2t - 1.2t^2 + 0.5t^3 = \sum_{k=0}^{3} c_k t^k$ denote the quantity of an observable N as a function of time t. Let $N_{obs}(t)$ denote the observed **noisy** values of this system, given by

$$N_{obs}(t) = \sum_{k=0}^{3} c_k (t + \varepsilon)^k, \quad \varepsilon \sim \mathcal{N}(0, 0.01)$$

where $\mathcal{N}(\mu, \sigma)$ is the Gaussian distribution with mean μ and standard deviation σ, and $\varepsilon \sim \mathcal{P}$ means ε is sampled from the probability distribution $\mathcal{P}$. In MATLAB, you can simply use `randn*0.01` to sample the random variables. We can generate a set of observations as

$$O_n = \left\{ (t, N_{obs}(t)) \,\middle|\, t \in \left\{ \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n-1}{n+1}, \frac{n}{n+1} \right\} \right\}.$$

For this problem, you will acquire approximations $\hat{c}_k^{(n)}$ to the constants c_k **from a set of observations**.

a) **(10 pts)** Generate sets observations of this system for $n = \{5, 10, 100\}$. For each set of observations, construct the corresponding $A_n \in \mathbb{R}^{nx4}$ and $b_n \in \mathbb{R}^n$ such that $A_n \hat{c}_k^{(n)} = b_n$ states the problem to be solved. At this step, assume you know the actual c_k values while generating observations via $N_{obs}(t)$.

b) **(30 pts)** Now assume we do not know what the c_k are, and we want to estimate them from the observations alone. Using any of the methods we have seen in this course, acquire approximations for c_k. In your reports, refer to the name of the related code file and explain your solution strategy.

c) **(20 pts)** How does the value n affect the quality of the approximations for c_k? Explain by discussing the average errors ($\|c_k - \hat{c}_k^{(n)}\|_2$) over multiple runs for each n.

Note: Checking the closely related example in the lecture notes (Part 8) should be very helpful for this question.

Regulations

1. Make sure that you reflect **your own reasoning** in a clean and concise manner.
2. Your submission should include a single PDF and your `.m` files.
3. Submission will be done via odtuclass.
4. **Late Submission:** Accepted with a penalty of $-5 \times (day)^2$.