5

Stress-Strain Material Laws

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§5.1. Introduction

Recall from the previous Lecture the following connections between various quantities that appear in continuum structural mechanics:

$$\boxed{\text{internal forces} \Rightarrow \text{ stresses } \stackrel{\text{MP}}{\Rightarrow} \text{ strains } \Rightarrow \text{ displacements} \Rightarrow \text{size \& shape changes}} \tag{5.1}$$

$$\boxed{\text{displacements } \Rightarrow \text{ strains } \stackrel{\text{MP}}{\Rightarrow} \text{ stresses } \Rightarrow \text{ internal forces}} \tag{5.2}$$

Of these, we have studied mechanical stresses in Lecture 1 and strains in Lecture 4. How are they linked? Through the *material properties* of the structural body. This is pictured by the 'MP' symbol above the appropriate arrow connectors. Material behavior is mathematically characterized by the so-called *constitutive equations*, also called *material laws*.

§5.2. Constitutive Equations

In this Lecture we will restrict detailed examination of constitutive behavior to elastic isotropic materials. More complex behavior (for example: orthotropy, plasticity, viscoelasticity, and fracture) are studied in senior and graduate level courses in Structural and Solid mechanics.

§5.2.1. Material Behavior Assumptions

There is a very wide range of materials used for structures, with drastically different behavior. In addition the same material can go through different response regimes: elastic, plastic, viscoelastic, cracking and localization, fracture. As noted above, we will restrict our attention to a very specific material class and response regime by making the following behavioral assumptions.

1. *Macroscopic Model*. The material is mathematically modeled as a *continuum* body. Features at the meso, micro and nano levels: crystal grains, molecules, and atoms, are ignored.
2. *Elasticity*. This means the stress-strain response is *reversible* and consequently the material has a preferred *natural* state. This state is assumed to be taken in the *absence of loads* at a *reference temperature*. By convention we will say that the material is then *unstressed* and *undeformed*. On applying loads, and possibly temperature changes, the material develops nonzero stresses and strains, and moves to occupy a *deformed* configuration.
3. *Linearity*. The relationship between strains and stresses is linear. Doubling stresses doubles strains, and viceversa.
4. *Isotropy*. The properties of the material are independent of direction. This is a good assumption for materials such as metals, concrete, plastics, etc. It is not adequate for heterogenous mixtures such as composites or reinforced concrete, which are *anisotropic* by nature. The substantial complications introduced by anisotropic behavior justifies its exclusion from an introductory treatment.
5. *Small Strains*. Deformations are considered so small that *changes of geometry are neglected* as the loads are applied. Violation of this assumption requires the introduction of nonlinear relations between displacements and strains. This is necessary for highly deformable materials such as rubber (more generally, polymers). Inclusion of nonlinear behavior significantly complicates the constitutive equations and is therefore left for advanced courses.

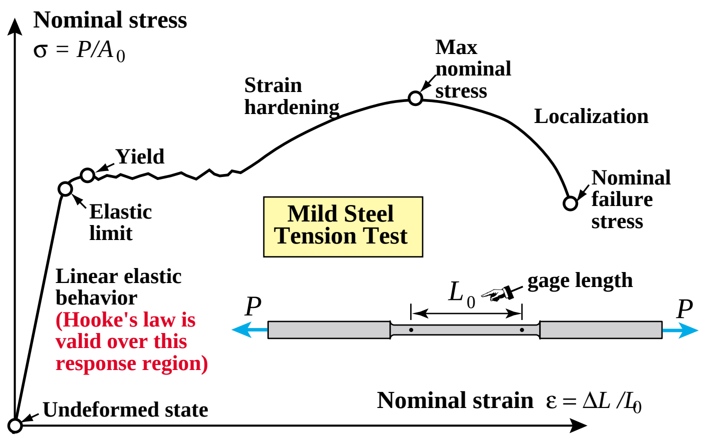

FIGURE 5.1. Typical tension test behavior of mild steel, which displays a well defined yield point and extensive yield region.

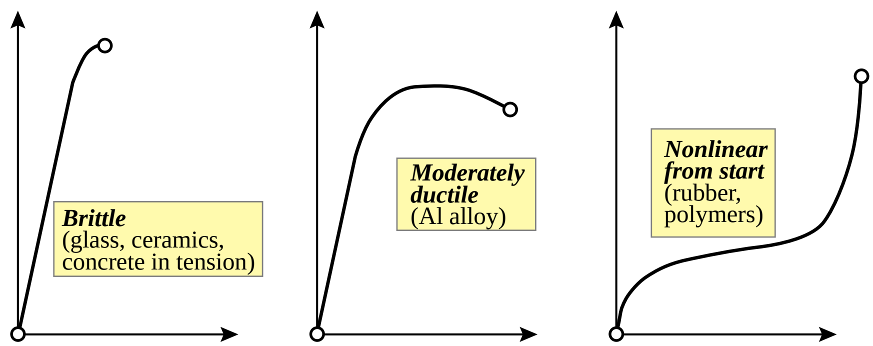

FIGURE 5.2. Three material response “flavors” as displayed in a tension test.

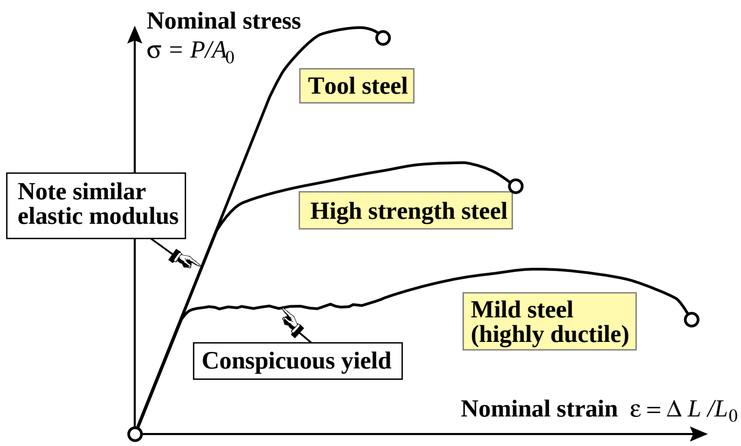

FIGURE 5.3. Different steel grades have approximately the same elastic modulus, but very different post-elastic behavior.

§5.2.2. The Tension Test Revisited

The first acquaintance of an engineering student with lab-controlled material behavior is usually through *tension tests* carried out during the first Mechanics, Statics and Structures sophomore course. Test results are usually displayed as axial nominal strain versus axial nominal strain, as illustrated in Figure 5.1 for the test of a mild steel specimen up to failure. Several response regions are indicated there: linearly elastic, yield, strain hardening, localization and failure. These are discussed in the aforementioned course, and studied further in the senior course on Aerospace Materials. It is sufficient to note here that we shall be concerned with the *linearly elastic region* that occurs before yield. In that region the 1D Hooke's law is assumed to hold.

Material behavior may depart significantly from that shown in Figure 5.1. Three distinct flavors: brittle, moderately ductile and nonlinear-from-start, are shown schematically in Figure 5.2 . Brittle materials such as glass, rock, ceramics, concrete-under-tension, etc., exhibit primarily linear behavior up to near failure by fracture. Metallic alloys used in aerospace, such as Aluminum and Titanium alloys, display moderately ductile behavior, without a well defined yield point and yield region: the stress-strain curve gradually turns down finally dropping to failure. Some materials, such as rubbers and polymers, exhibit strong nonlinear behavior from the start. Although such materials may be elastic there is no easily identifiable *linearly* elastic region.

Even for a well known material such as steel, the tension test behavior can vary significantly depending on combination with other components. Figure 5.3 sketches the response of mild steel with high-strength steel used in critical structural components, and with tool steel. Mild steel is highly ductile and clearly exhibits an extensive yield region. Hi-strength steel is less ductile and does not show a well defined yield point. Tool steel has little ductility, and its behavior displays features associated with brittle materials. The trade off between ductility and strength is typical. Note, however, that all three grades of steel have *approximately the same elastic modulus*, which is the slope of the stress-strain line in the linear region of the tension test.

§5.3. Characterizing a Linearly Elastic Isotropic Material

For an *isotropic* material in the linearly elastic region of its response, *four* numerical properties are sufficient to establish constitutive equations. Those equations are associated with the well known Hooke's law, originally enunciated by Robert Hooke by 1660 for a spring, and later expressed in terms of stresses and strains once those concepts appeared in the XIX Century. These four properties: E, ν, G and α, are tabulated in Figure 5.4.

§5.3.1. Determination Of Elastic Modulus and Poisson's Ratio

The experimental determination of the elastic modulus E and Poisson's ratio ν makes use of a uniaxial tension test specimen such as the one pictured in Figure 5.5. See slides for operational details.

§5.3.2. Determination Of Shear Modulus

The experimental determination of the shear modulus G makes use of a torsion test specimen such as the one pictured in Figure 5.6. See slides for operational details.

E **Elastic modulus, a.k.a. Young's modulus**
Physical dimension: stress=force/area (e.g. ksi)

ν **Poisson's ratio**
Physical dimension: dimensionless (just a number)

G **Shear modulus, a.k.a. modulus of rigidity**
Physical dimension: stress=force/area (e.g. MPa)

α **Coefficient of thermal expansion**
Physical dimension: 1/degree (e.g., 1/°C)

E, ν **and** G **are not independent. They are linked by**

$E = 2G\,(1+\nu)$, $G = E/(2(1+\nu))$, $\nu = E/(2G)-1$

FIGURE 5.4. Four properties that fully characterize the thermomechanical response of an isotropic material in the linearly elastic range.

§5.3.3. Determination Of Thermal Expansion Coefficient

The experimental determination of the thermal expansion coefficient α can be made byt using a uniaxial tension test specimen such as the one pictured in Figure 5.7. See slides for operational details.

§5.4. Hooke's Law in 1D

Once the values of E, ν, G and α are experimentally determined (for widely used structural materials they can be simply read off a manual), they can be used to construct thermoelastic constitutive equations that link stresses and strains as described in the following subsections.

§5.4.1. Elastic Modulus And Poisson's Ratio In 1D Stress State

The one-dimensional Hooke's law relates 1D normal stress to 1D extensional strain through two material parameters introduced previously: the modulus of elasticity E, also called Young's modulus and Poisson's ratio ν. The modulus of elasticity connects axial stress σ to axial strain ϵ:

$$\sigma = E\,\epsilon, \quad E = \frac{\sigma}{\epsilon}, \quad \epsilon = \frac{\sigma}{E}. \tag{5.3}$$

Poisson's ratio ν is defined as ratio of lateral strain to axial strain:

$$\nu = \left| \frac{\text{lateral strain}}{\text{axial strain}} \right| = -\frac{\text{lateral strain}}{\text{axial strain}}. \tag{5.4}$$

The $-$ sign is introduced for convenience so that ν comes out positive. For structural materials ν lies in the range $0.0 \le \nu < 0.5$. For most metals $\nu \approx 0.25$–0.35. For concrete and ceramics, $\nu \approx 0.10$. For cork $\nu \approx 0$. For rubber, $\nu \approx 0.5$ to 3 places. A material for which $\nu = 0.5$ is called *incompressible*.

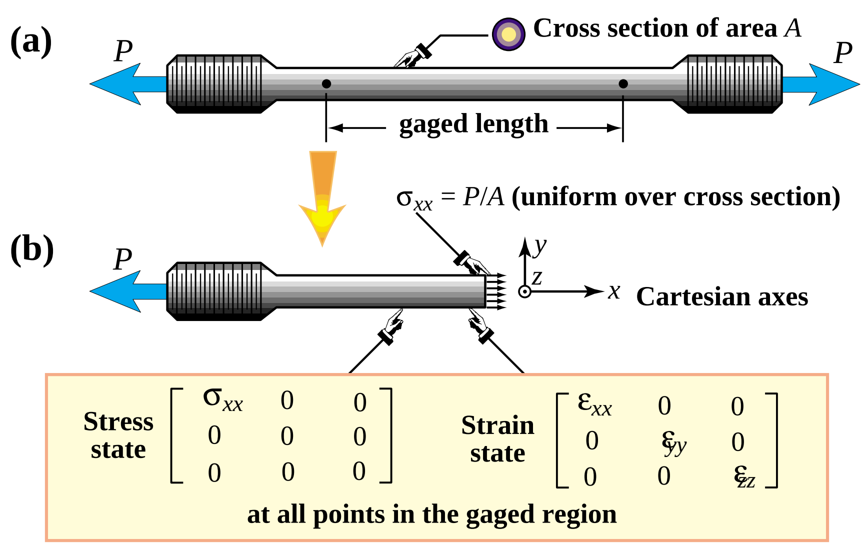

FIGURE 5.5. Speciment for determination of elastic modulus E and Poisson's ratio ν in the linearly elastic response region of an isotropic material, using an uniaxial tension test.

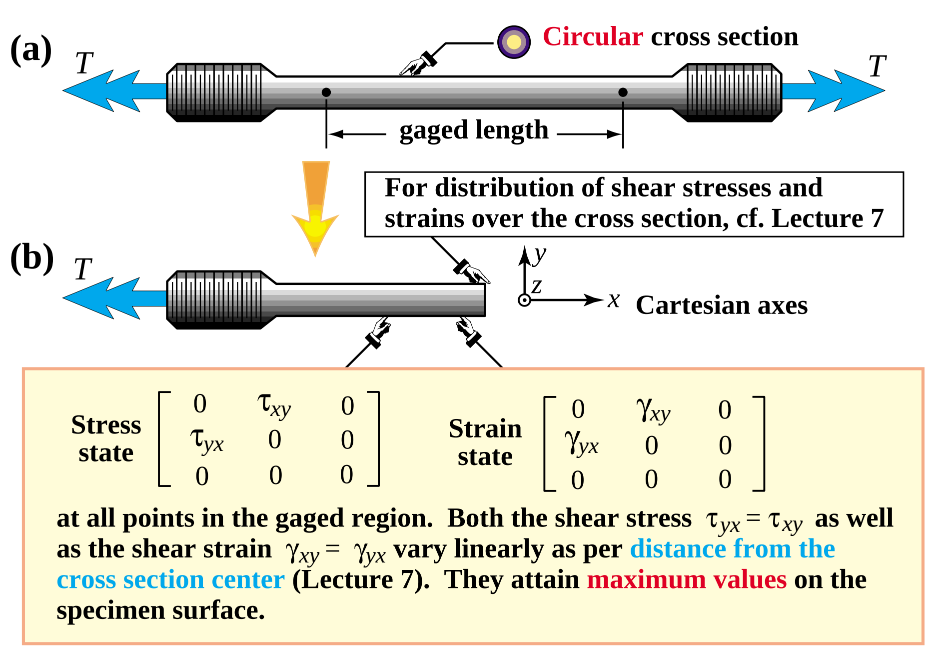

FIGURE 5.6. Speciment for determination of shear modulus G in the linearly elastic response region of an isotropic material, using a torsion test.

§5.4.2. Shear Modulus In 1D Stress State

The shear modulus G connects a shear strain γ to the corresponding shear stress τ:

$$\tau = G\,\gamma, \qquad G = \frac{\tau}{\gamma}, \qquad \gamma = \frac{\tau}{G}. \tag{5.5}$$

"Corresponding" means that if $\gamma = \gamma_{xy}$, say, then $\tau = \tau_{xy}$, and similarly for the other shear components. The shear modulus is usually obtained from a torsion test. It turns out that the

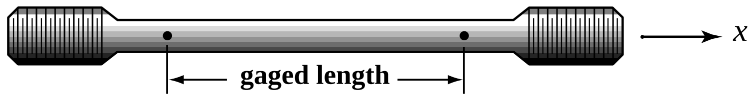

FIGURE 5.7. Speciment for determination of coefficient of thermal expansion α by heating a tension test specimen and allowing it to expand freely.

3 material properties E, ν and G for an elastic isotropic material are not independent, but are connected by the relations

$$G = \frac{E}{2(1+\nu)}, \quad E = 2(1+\nu)\,G, \quad \nu = \frac{E}{2G} - 1. \tag{5.6}$$

which means that if two of them are known by measurement, the third one can be obtained from the relations (5.6). In practice the three properties are often measured independently, and the (approximate) verification of (5.6) gives an idea of "how isotropic" the material is.

§5.4.3. Thermal Strains In 1D Stress State

A temperature change of ΔT with respect to a *base* or *reference* level produces a thermal strain

$$\epsilon_T = \alpha\,\Delta T, \tag{5.7}$$

in which α is the coefficient of thermal dilatation, measured in $1/^\circ F$ or $1/^\circ C$. This is typically positive: $\alpha > 0$ and very small: $\alpha << 1$, of order 10^{-6} for most structural materials. To combine mechanical and thermal effects in 1D, the strains are *superposed*:

$$\epsilon = \epsilon_M + \epsilon_T = \frac{\sigma}{E} + \alpha\,\Delta T, \tag{5.8}$$

The last expression is valid if the material is linearly elastic and obeys the 1D Hooke's law.

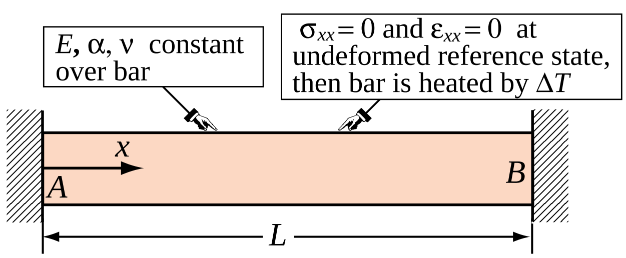

FIGURE 5.8. Heated bar precluded from axial expansion. This bar will develop a *compressive* axial stress called a thermal stress.

Example 5.1. The bar AB shown in Figure 5.8 is precluded from extending axially. It has elastic modulus E and coefficient of dilatation $\alpha > 0$. The stress σ is zero when the bar is at the reference temperature T_{ref}. Find which axial stress σ develops if the temperature changes to $T = T_{ref} + \Delta T$.

Since the bar length cannot change, the combined axial strain must be zero:

$$\epsilon_{xx} = \epsilon = \frac{\sigma}{E} + \alpha\,\Delta T = 0, \tag{5.9}$$

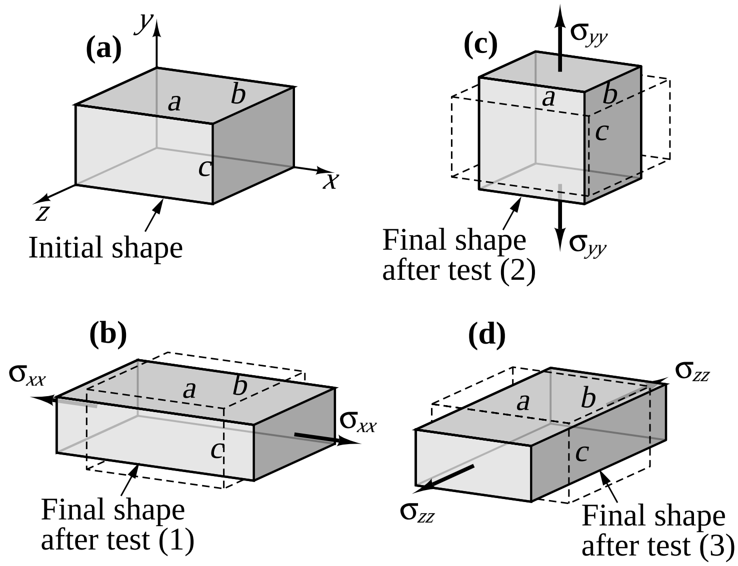

FIGURE 5.9. Derivation of three-dimensional Generalized Hooke's Law for normal stresses and strains. Three tension tests are assumed to be carried out along $\{x, y, z\}$, respectively, and strains superposed.

Solving for σ gives

$$\sigma = -E\,\alpha\,\Delta T. \tag{5.10}$$

Since E and α are positive, a rise in temperature, i.e., $\Delta T > 0$, will produce a negative axial stress, and the bar will be in *compression*. This is an example of the so-called *thermally induced stress* or simply *thermal stress*. It is the reason behind the use of expansion joints in pavements, rails and bridges. The effect is important in orbiting vehicles such as satellites, which undergo extreme (and cyclical) temperature changes from full sun to Earth shade.

§5.5. Generalized Hooke's Law in 3D

§5.5.1. Strain-To-Stress Relations

We now generalize the foregoing equations to the three-dimensional case, still assuming that the material is elastic and isotropic. Condider a cube of material aligned with the axes $\{x, y, z\}$, as shown in Figure 5.9. Imagine that three"tension tests", labeled (1), (2) and (3) respectively, are conducted along x, y and z, respectively. Pulling the material by applying σ_{xx} along x will produce normal strains

$$\epsilon_{xx}^{(1)} = \frac{\sigma_{xx}}{E}, \qquad \epsilon_{yy}^{(1)} = -\frac{\nu\,\sigma_{xx}}{E}, \qquad \epsilon_{zz}^{(1)} = -\frac{\nu\,\sigma_{xx}}{E}. \tag{5.11}$$

Next, pull the material by σ_{yy} along y to get the strains

$$\epsilon_{yy}^{(2)} = \frac{\sigma_{yy}}{E}, \qquad \epsilon_{xx}^{(2)} = -\frac{\nu\,\sigma_{yy}}{E}, \qquad \epsilon_{zz}^{(2)} = -\frac{\nu\,\sigma_{yy}}{E}. \tag{5.12}$$

Finally pull the material by σ_{zz} along z to get

$$\epsilon_{zz}^{(3)} = \frac{\sigma_{zz}}{E}, \qquad \epsilon_{xx}^{(3)} = -\frac{\nu\,\sigma_{zz}}{E}, \qquad \epsilon_{yy}^{(3)} = -\frac{\nu\,\sigma_{zz}}{E}. \tag{5.13}$$

In the general case the cube is subjected to *combined* normal stresses σ_{xx}, σ_{yy} and σ_{zz}. Since we assumed that the material is linearly elastic, the combined strains can be obtained by *superposition* of the foregoing results:

$$\begin{aligned}
\epsilon_{xx} &= \epsilon_{xx}^{(1)} + \epsilon_{xx}^{(2)} + \epsilon_{xx}^{(3)} = \frac{\sigma_{xx}}{E} - \frac{\nu\,\sigma_{yy}}{E} - \frac{\nu\,\sigma_{zz}}{E} = \frac{1}{E}\left(\sigma_{xx} - \nu\,\sigma_{yy} - \nu\,\sigma_{zz}\right).\\
\epsilon_{yy} &= \epsilon_{yy}^{(1)} + \epsilon_{yy}^{(2)} + \epsilon_{yy}^{(3)} = -\frac{\nu\,\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} - \frac{\nu\,\sigma_{zz}}{E} = \frac{1}{E}\left(-\nu\,\sigma_{xx} + \sigma_{yy} - \nu\,\sigma_{zz}\right).\\
\epsilon_{zz} &= \epsilon_{zz}^{(1)} + \epsilon_{zz}^{(2)} + \epsilon_{zz}^{(3)} = -\frac{\nu\,\sigma_{xx}}{E} - \frac{\nu\,\sigma_{yy}}{E} + \frac{\sigma_{zz}}{E} = \frac{1}{E}\left(-\nu\,\sigma_{xx} - \nu\,\sigma_{yy} + \sigma_{zz}\right).
\end{aligned} \tag{5.14}$$

The shear strains and stresses are connected by the shear modulus as

$$\gamma_{xy} = \gamma_{yx} = \frac{\tau_{xy}}{G} = \frac{\tau_{yx}}{G}, \quad \gamma_{yz} = \gamma_{zy} = \frac{\tau_{yz}}{G} = \frac{\tau_{zy}}{G}, \quad \gamma_{zx} = \gamma_{xz} = \frac{\tau_{zx}}{G} = \frac{\tau_{xz}}{G}. \tag{5.15}$$

The three equations in (5.14), plus the three in (5.15), may be collectively expressed in matrix form as

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}. \tag{5.16}$$

§5.5.2. Stress-To-Strain Relations

To get stresses if the strains are given, the most expedient method is to invert the matrix equation (**?**). This gives

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}. \tag{5.17}$$

Here $\hat{E}$ is an "effective" modulus modified by Poisson's ratio:

$$\hat{E} = \frac{E}{(1-2\nu)(1+\nu)} \tag{5.18}$$

The six relations in (5.17) written out in long form are

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\,\epsilon_{xx} + \nu\,\epsilon_{yy} + \nu\,\epsilon_{zz}\right],\\
\sigma_{yy} &= \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\,\epsilon_{xx} + (1-\nu)\,\epsilon_{yy} + \nu\,\epsilon_{zz}\right],\\
\sigma_{zz} &= \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\,\epsilon_{xx} + \nu\,\epsilon_{yy} + (1-\nu)\,\epsilon_{zz}\right],\\
\tau_{xy} &= G\,\gamma_{xy}, \quad \tau_{yz} = G\,\gamma_{yz}, \quad \tau_{zx} = G\,\gamma_{zx}.
\end{aligned} \tag{5.19}$$

The combination

$$\sigma_{av} = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \tag{5.20}$$

is called the *mean stress*, or *average stress*. The negative of σ_{av} is the *pressure*: $p = -\sigma_{av}$.

The combination $\epsilon_v = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ is called the *volumetric strain*, or *dilatation*. The negative of ϵ_v is known as the *condensation*. Both pressure and volumetric strain are *invariants*, that is, their value does not change if axes $\{x, y, z\}$ are rotated. An important relation between pressure and volumetric strain can be obtained by adding the first three equations in (**?**), which upon simplification and accounting for (5.20) and $p = -\sigma_{av}$ relates pressure and volumetric strain as

$$p = -\frac{E}{3(1-2\nu)}\epsilon_v = -K\,\epsilon_v. \tag{5.21}$$

This coefficient K is called the *bulk modulus*. If Poisson's ratio approaches $\frac{1}{2}$, which happens for near incompressible materials, $K \to \infty$.

Remark 5.1. In the solid mechanics literature p is also defined (depending on author's preferences) as $p = \sigma_{av} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$, which is the negative of the above one. If so, $p = +K\,\epsilon_v$. The definition $p = -\sigma_{av}$ is the most common one in fluid mechanics.

§5.5.3. Thermal Effects in 3D

To incorporate the effect of a temperature change ΔT with respect to a base or reference temperature, add $\alpha\,\Delta T$ to the three normal strains in (5.14)

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\big(\quad \sigma_{xx} - \nu\,\sigma_{yy} - \nu\,\sigma_{zz}\big) + \alpha\,\Delta T,\\
\epsilon_{yy} &= \frac{1}{E}\big(-\nu\,\sigma_{xx} + \sigma_{yy} - \nu\,\sigma_{zz}\big) + \alpha\,\Delta T,\\
\epsilon_{zz} &= \frac{1}{E}\big(-\nu\,\sigma_{xx} - \nu\,\sigma_{yy} + \sigma_{zz}\big) + \alpha\,\Delta T.
\end{aligned} \tag{5.22}$$

No change in the shear strain-stress relation is needed because if the material is linearly elastic and isotropic, a temperature change only produces normal strains. The stress-to-strain matrix relation (5.16) expands to

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} + \alpha\,\Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{5.23}$$

Inverting this relation provides the stress-strain relations that account for a temperature change:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 & 0 & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} - \frac{E\,\alpha\,\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5.24}$$

in which $\hat{E}$ is defined in (5.18). Note that if all mechanical normal strains ϵ_{xx}, ϵ_{yy}, and ϵ_{zz} vanish, but $\Delta T \neq 0$, the normal stresses given by (5.24) are nonzero. Those are called *initial thermal stresses*, and are important in engineering systems exposed to large temperature variations, such as rails, turbine engines, satellites or reentry vehicles.

§5.6. Generalized Hooke's Law in 2D

Two specializations of the foregoing 3D equations to two dimensions are of interest in the applications: *plane stress* and *plane strain*. Plane stress is more important in Aerospace structures, which tend to be thin, so in this course more attention is given to that case. Both specializations are reviewed next.

§5.6.1. Plane Stress

In this case all stress components with a z component are assumed to vanish. For a linearly elastic isotropic material, the strain and stress matrices take on the form

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}, \qquad \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.25}$$

Note that the ϵ_{zz} strain, often called the *transverse strain* or *thickness strain* in applications, in general will be nonzero because of Poisson's ratio effect. The strain-stress equations are easily obtained by going to (5.14) and (5.15) and setting $\sigma_{zz} = \tau_{yz} = \tau_{zx} = 0$. This gives

$$\begin{gathered} \epsilon_{xx} = \frac{1}{E}\left(\ \sigma_{xx} - \nu\,\sigma_{yy}\right), \quad \epsilon_{yy} = \frac{1}{E}\left(-\nu\,\sigma_{xx} + \sigma_{yy}\right), \quad \epsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right), \\ \gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \gamma_{zx} = 0. \end{gathered} \tag{5.26}$$

The matrix form, omitting known zero components, is

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}. \tag{5.27}$$

Inverting the matrix composed by the first, second and fourth rows of the above relation gives the stress-strain equations

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \tilde{E} & \tilde{E}\,\nu & 0 \\ \tilde{E}\,\nu & \tilde{E} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}. \tag{5.28}$$

in which $\tilde{E} = E/(1-\nu^2)$. Written in long hand,

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\,\epsilon_{yy}), \quad \sigma_{yy} = \frac{E}{1-\nu^2}(\epsilon_{yy} + \nu\,\epsilon_{xx}), \quad \tau_{xy} = G\,\gamma_{xy}. \tag{5.29}$$

§5.6.2. Plane Strain

In this case all strain components with a z component are assumed to vanish. For a linearly elastic isotropic material, the strain and stress matrices take on the form

$$\begin{bmatrix} \epsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \epsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \tag{5.30}$$

Note that the σ_{zz} stress, which is called the *transverse stress* in applications, in general will not vanish. The strain-to-stress relations can be easily obtained by setting $\epsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$ in (5.19). This gives

$$\begin{aligned} \sigma_{xx} &= \frac{E}{(1-2\nu)(1+\nu)}\left[(1-\nu)\,\epsilon_{xx} + \nu\,\epsilon_{yy}\right], \\ \sigma_{yy} &= \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\,\epsilon_{xx} + (1-\nu)\,\epsilon_{yy}\right], \\ \sigma_{zz} &= \frac{E}{(1-2\nu)(1+\nu)}\left[\nu\,\epsilon_{xx} + \nu\,\epsilon_{yy}\right], \\ \tau_{xy} &= G\,\gamma_{xy}, \quad \tau_{yz} = 0, \quad \tau_{zx} = 0. \end{aligned} \tag{5.31}$$

which in matrix form, with the zero components removed, is

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \hat{E}\,(1-\nu) & \hat{E}\,\nu & 0 \\ \hat{E}\,\nu & \hat{E}\,(1-\nu) & 0 \\ \hat{E}\,\nu & \hat{E}\,\nu & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}. \tag{5.32}$$

Inverting the system provided by extracting the first, second and fourth rows of (5.32) gives the stress-to-strain relations, which are omitted for simplicity.

The effect of temperature changes can be incorporated in both plane stress and plane strain relations without any difficulty.

Backup Material Only — Not Covered In Lecture

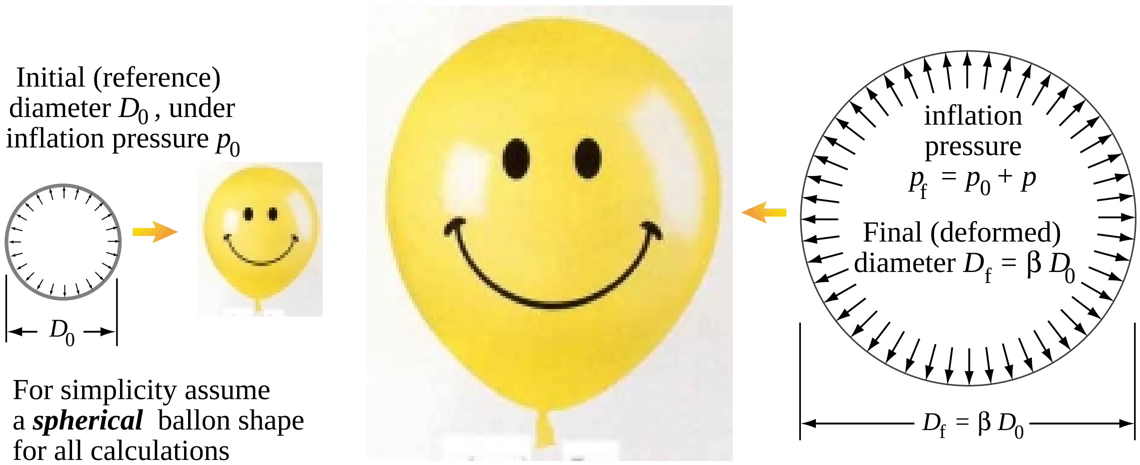

FIGURE 5.10. Inflating balloon example problem.

§5.7. Example: An Inflating Balloon

This is a generalization of Problem 3 of Recitation #2. The main change is that all data is expressed and kept in variable form until the problem is solved. Specific numbers are plugged in at the end.

This will be the only problem in the course where some features of *nonlinear mechanics* appear. These come in by writing the governing equations in both the initial and final geometries, without linearization.

§5.7.1. Strains and Stresses in Balloon Wall

The problem is depicted in Figure 5.10. A spherical rubber balloon has initial diameter D_0 under inflation pressure p_0. This is called the *initial configuration*. The pressure is increased by p so that the final pressure is $p_f = p_0 + p$. The balloon assumes a spherical shape with final diameter $D_f = \beta\, D_0$, in which $\beta > 1$. This will be called the *final configuration*. The initial wall thickness is $t_0 << D_0$ and the final thickness is t_f. Since the balloon geometry is assumed to remain *spherical* for simplicity, we can apply to both configurations the stress formulas for the thin-wall spherical vessel derived in Lecture 3.

The strains (but not stresses) are assumed to be *zero* in the initial configuration. The average circumferential extensional strain assumed in the final configuration depends on whether we take the Lagrangian or the Eulerian strain measure, which are designated by ϵ^L_{av} and ϵ^E_{av}, respectively. Obviously

$$\epsilon^L_f = \frac{\pi(D_f - D_0)}{\pi D_0} = \beta - 1, \qquad \epsilon^E_f = \frac{\pi(D_f - D_0)}{\pi D_f} = \frac{\beta - 1}{\beta}, \tag{5.33}$$

Since the balloon is assumed to remain spherical and its thickness is very small compared to its diameter, the above strains hold at all points of the balloon wall, and are the *same* in any direction tangent to the sphere. If we choose the sphere normal as local z axis, the wall is in a *plane stress* state.

Next we introduce material laws. We will assume that rubber obeys the two-dimensional, plane stress generalized Hooke's law (5.29) with respect to the Eulerian strain measure, with effective modulus of elasticity E and Poisson's ratio ν.[1] Setting $\epsilon_{xx} = \epsilon_{yy} = \epsilon_f^E$ and $\gamma_{xy} = 0$ therein and accounting for the initial stress σ_0, we obtain the inplane normal stress in the final configuration:

$$\sigma_{xx} = \sigma_{yy} = \sigma_f = \sigma_0 + \frac{E}{1-\nu^2}(\epsilon_f^E + \nu\epsilon_f^E) = \sigma_0 + \frac{E}{1-\nu}\,\epsilon_f^E = \sigma_0 + \frac{E}{1-\nu}\,\frac{\beta-1}{\beta}. \qquad (5.34)$$

The normal inplane wall stress is the same in all directions, so it is called simply σ_0 and σ_f, for initial and final configurations, respectively. The inplane shear stress τ vanishes in all directions.

Assume D_0, t_0, E and ν are given as data. An interesting question: what is the relation between p (the excess or gage pressure) and the diameter $D_f = \beta D_0$? And, is there a maximum pressure that will cause the balloon to burst?

§5.7.2. When Will the Balloon Burst?

To relate p and β it is necessary to express the wall stresses σ_0 and σ_f in terms of geometry and internal pressure. This is provided by equation (3.10) in Lecture 3, derived for a thin-wall spherical vessel. In that equation replace p, R and t by quantities in the initial and final configurations:

$$\sigma_0 = \frac{p_0\,R_0}{2t_0} = \frac{p_0\,D_0}{4t_0}, \qquad \sigma_f = \frac{p_f\,R_f}{2\,t_f} = \frac{(p_0+p)\,D_f}{4\,t_f} = \frac{(p_0+p)\,\beta\,D_0}{4\,t_f}. \qquad (5.35)$$

All quantities in the above expressions are known in terms of the data, except t_f. A kinematic analysis beyond the scope of this course shows that

$$t_f = \left[1 + 2\nu\left(\frac{1}{\beta^2} - 1\right)\right] t_0. \qquad (5.36)$$

We can check (5.36) by inserting two limit values of Poisson's ratio:

$\nu = 0$: $t_f = t_0$. This is correct since the thickness does not change.

$\nu = 1/2$: $t_f = t_0/\beta^2$. Is this correct? If $\nu = 1/2$ the material is *incompressible* and does not change volume. The initial and final volume of the thin-wall spherical balloon are $V_0 = \pi D_0^2\,t_0$ and $V_f = \pi D_f^2\,t_f = \pi\beta^2\,D_0^2\,t_f$, respectively. On setting $V_0 = V_f$ and solving for t_f we get $t_f = t_0/\beta^2$.

To obtain p in terms of β, replace (5.36) into (5.35), equate this to (5.34) and solve for p. The result provided by *Mathematica* is

$$p = \frac{4Et_0(1-\beta)(2\nu + \beta^2(1-2\nu)) + D_0 p_0 \beta(1-\nu)(4\nu + \beta^2(2-\beta-4\nu))}{D_0\beta^4(1-\nu)} \qquad (5.37)$$

[1] This is a *very rough approximation* since constitutive equations for rubber (and polymers in general) are highly nonlinear. But getting closer to reality would take us into the realm of nonlinear elasticity, which is a graduate-level topic.

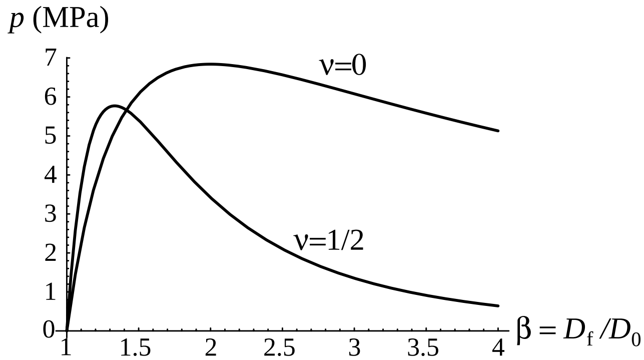

FIGURE 5.11. Inflating pressure (in MPa) versus diameter expansion ratio $\beta = D_f / D_0$ for a balloon with $E = 1900$ MPa, $D_0 = 50$ mm, $p_0 = 0$ MPa, $t_0 = 0.18$ mm, $1 \le \beta \le 4$ and Poisson's ratios $\nu = 0$ and $\nu = \frac{1}{2}$.

This expression simplifies considerably in the two Poisson's ratio limits:

$$p|_{\nu=0} = \frac{4E\, t_0\, (\beta - 1) + D_0\, p_0\, (2 - \beta)\, \beta}{D_0\, \beta^2} \tag{5.38}$$

$$p|_{\nu=1/2} = \frac{8E\, t_0\, (\beta - 1) + D_0\, p_0\, (2 - \beta^3)\, \beta}{D_0\, \beta^4} \tag{5.39}$$

Pressure versus diameter ratio curves given by (5.38) and (5.39) are plotted in Figure 5.11 for the numerical values indicated there. Those values correspond to the data used in Problem 3 of Recitation 2, in which $\nu = 1/2$ was specified from the start.

Rubber (and, in general, polymer materials) are nearly incompressible; for example $\nu \approx 0.4995$ for rubber. Consequently, the response depicted in Figure 5.11 for $\nu = 1/2$ is more physically relevant than the other one.

Do the response plots in Figure 5.11 tell you when an inflating balloon is about to collapse? Yes. This is the matter of a (optional) Homework Exercise.