University of Pennsylvania
The Wharton School

Fall 2011   FNCE 601   Franklin Allen

Corporate Finance

BULK PACK
QUARTER 1

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COURSE SCHEDULE AND DUE DATES
QUARTER 1

Week 1:

Th: 9/8   Section 1: Introduction
Section 2: The Objective Function for Corporations

Week 2:

Tu: 9/13   Section 2 (cont.)

Th: 9/15   Section 3: Calculating Present Values

Week 3:

Tu: 9/20   **PROBLEM SET 1 DUE (**Hand in at the start of class)
Section 4: The Valuation of Bonds and Stocks

Th: 9/22   Section 5: A Comparison of Investment Criteria

Week 4:

Tu: 9/27   **PROBLEM SET 2 DUE**
Section 5 (cont.)
Section 6: Practical Aspects of the NPV Rule

Th: 9/29   Section 6 (cont.)

Week 5:

Tu: 10/4   **PROBLEM SET 3 DUE**
Section 7: Measuring Risk
Section 7(i): Statistics Review
Section 7(ii): A Theory of Asset Pricing: Introduction

Th: 10/6   Section 7(iii): Mean-Standard Deviation Analysis
Section 7(iv): Diversification and the Measurement of Risk

Week 6:

Tu: 10/11   **TOLLHOUSE COOKIE CASE DUE BY 9am – UPLOAD TO WEBCAFE**
Section 8: The Capital Asset Pricing Model

Th: 10/13   Section 8 (cont.)

Week 7:

Tu: 10/18   **PROBLEM SET 4 DUE**
Section 9: Capital Budgeting and Risk

Th: 10/20   Exam week for MBA Core – No class

Tu: 10/25   Exam week for MBA Core - No class

QUARTER 2
Week 1:
Th: 10/27   Review session during regular class

Mon: 10/31   **MIDTERM EXAM** 6:15-7:45pm (rooms announced later)

UNIVERSITY OF PENNSYLVANIA
Wharton School

FNCE 601
Corporate Finance
Fall 2011

Problem Set 1
(Due September 20, 2011)

<u>Section 2</u>

1.   Lara Beal has $8,000 in cash now. She can lend and borrow at the bank at 15 percent per period. Her only investment possibility other than the bank is a project that costs $5,000 now in her Youth and has a payoff of $6,000 in cash one period from now in her Old Age. What is the most she can consume in her Old Age?

2.   Tsuyoshi Kondo has no cash of his own. He can lend and borrow at the bank at 18 percent per period. An investment that costs $12,000 now in his Youth and has a payoff of $15,000 one period from now in his Old Age is the only investment available to him. What is the most he can consume now in his Youth?

3.   Chioma Muhammad can lend and borrow at the bank at 25 percent per period. She has $10,000 in cash now. Her only investment possibility costs $16,000 now in her Youth and has a payoff of $25,000 in her Old Age. What is the most she can consume in her Youth?

4.   Consider a firm owned by two shareholders, Sarah Dandridge and Peter Burke. Each owns fifty percent of the firm's equity. The firm has a project available to it that costs $50,000 and has a payoff of $70,000. The opportunity cost of capital for both shareholders is 15%. Sarah has wealth now of $30,000 and Peter has wealth now of $50,000. What is the most each can consume now given they can lend and borrow as much as they want at 15%?

5.   Which of the following assumptions is necessary for a diverse group of shareholders to always all agree on their company's capital budgeting decisions?

        A.   They can all lend at the same rate but cannot borrow.
        B.   Their lending rate is less than their borrowing rate.
        C.   They all can lend and borrow at the same rate.
        D.   None of the above.

6.   Consider a world with two points in time, t_0 and t_1. Cynthia Sopwith inherits $2.5 million at t_0. She has two projects she can invest this inheritance in at t_0. Project A costs $450,000 at t_0 and yields $575,000 at t_1. Project B costs $950,000 at t_0 and yields $1,060,000 at t_1. She can also lend and borrow at a bank at an interest rate of 15 percent. Cynthia is only interested in consumption at t_1; she does not get any utility from consumption at t_0.

(a)   Which projects should Cynthia undertake?

(b)   What is the most Cynthia can consume at t_1? How much does she need to lend or borrow at the bank to achieve this level of consumption?

(c)   How would your answers to (a) and (b) be changed if Cynthia could lend to the bank at 10 percent and borrow from the bank at 20 percent?

Section 3

7.   If the present value of $320 paid one year from now at date 1 is $290, what is the one-year discount factor?

8.   If the present value of $460 paid one year from now at date 1 is $430, what is the one-year discount rate?

9.   If the 1-year rate of interest, r_1, is 6% per year and the 2-year rate of interest, r_2, is 7% per annum what is the reinvestment rate of interest, $r_{1,2}$, during the second year (i.e., between dates 1 and 2) that can be locked in now?

10.   How can a firm lock in a reinvestment rate $r_{2,3}$ now for the third year (i.e., between dates 2 and 3) using a 2-year zero coupon bond and a 3-year zero coupon bond?

11.   Suppose in question 10 the rate on the 2-year zero coupon bond is $r_2 = 7\%$ and the rate on the 3-year zero coupon bond is $r_3 = 8\%$. What is the reinvestment rate $r_{2,3}$?

12.   How can a profit be made if the rate on a 1-year zero coupon bond is $r_1 = 5\%$ and the rate on a 2-year zero coupon bond is $r_2 = 2\%$ if it is possible to invest in cash as well as buy and sell the bonds?

13.   Consider a perpetuity that pays $60 a year each year with the first payment being a year from now. The discount rate is 8%. What is the PV of the perpetuity?

14. Consider the perpetuity in question 13 except now the first payment is *today* rather than a year from now. What is the PV of the perpetuity now?

15. Consider a growing perpetuity that has an initial payment *today* of $100. After that the annual payment increases by 5% a year forever. The discount rate is 10%. What is the PV of the growing perpetuity?

16. (2 points) What is the formula for finding the value of a t-year growing annuity with growth rate g with the first payment of C one period from now?

$$A. \quad \frac{C}{r}\left(1 - \frac{1}{(1+r)^t}\right)$$

$$B. \quad \frac{C}{r-g}\left(1 - \frac{(1+g)^t}{(1+r)^t}\right)$$

$$C. \quad \frac{C}{r-g}\left(1 - \frac{1}{(1+r)^t}\right)$$

17. Tom Turnip has just taken out a $190,000 mortgage at an interest rate of 7%. If the mortgage calls for 15 equal annual installments, what is the amount of each installment?

18. (2 points) Suppose you have just won the lottery and must choose one of the following (guaranteed) payoffs. Which one would you choose? The interest rate is 7%; ignore tax consequences.

   A. $100,000 paid today
   B. $140,000 paid five years from today
   C. $50,000 paid one year from today and $68,000 paid four years from today
   D. $14,000 paid per year for ten years, with the first year's payment made today

19. Suppose the continuously compounded rate of interest is 5% per year. What is the equivalent annual rate where the interest is paid once at the end of the year?

20.   Sonia Sopwith has set her heart on buying a car. She plans to do this 5 years from now at which time she expects the car to cost $9,475. The rate of interest at the bank she uses is expected to remain constant at 8 percent per year throughout the next five years. Two years from now, her mother has promised to give $2,500 to her which she can put in the bank to help pay for the car. Assuming she saves the same amount out of her salary each year, how much must she save at the end of years 1-5 if she wishes to have enough money to buy the car five years from now?

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Problem Set 2
(Due September 27, 2011)

Section 4

1.   Consider a firm which with certainty will pay dividends of $5 at date 1, $6 at date 2 and
      $7 at date 3 and nothing thereafter. The current stock price is $15 per share. Investors
      can borrow and lend as much as they want at 6% per period. Is an arbitrage opportunity
      available? Explain how a profit can be made buying or selling 1 share.

2.   A bond with par value $1,000 and an annual coupon of $70 matures in six years. The
      first coupon is due one year from now. The current price of the bond is $1050. Using a
      financial calculator, find the yield on the bond.

3. $Div_1 = \$10$; $Div_2 = \$20$; $P_0 = \$60$ and $r = 0.08$. What is P_2?

4.   Company A's dividends are expected to grow at a constant rate of $g = 4\%$ $Div_1 = \$25$ and
      $r = 0.06$. What is the current price of the stock?

5.   Company C's dividends are expected to grow at a constant rate of g. The company <u>has
      just paid</u> a dividend D. What is the current price of the stock?

      A. $P_0 = \dfrac{D(1+r)}{r - g}$

      B. $P_0 = \dfrac{D(1+g)}{r - g}$

      C. $P_0 = \dfrac{D}{r - g}$

6. Company D's dividends are expected to grow at a constant rate of g. It is <u>just</u> <u>about</u> <u>to</u> <u>pay</u> a dividend D. What is the current price of the stock?

A. $P_0 = \dfrac{D(1+g)}{r-g}$

B. $P_0 = D + \dfrac{D(1+g)}{r-g}$

C. $P_0 = \dfrac{D}{r-g}$

7. A stock is just about to pay a dividend of $2. In 1 year's time and each year thereafter the firm is expected to pay a dividend which is 5% higher than the previous year's dividend. The market capitalization rate of similar stocks is 10%. What is the price of the stock today? What will be the price of the stock tomorrow just after the $2 dividend has been paid (i.e., after it has gone ex-dividend)?

8. A stock has a current price of $85 per share. The dividend one year from now is expected to be $6 and will grow thereafter by 4% per year. What is the stock's market capitalization rate?

9. Consider a stock with a current price of $45 per share. The market capitalization rate on similar stocks is 11%. The dividend one year from now is expected to be $3 and the dividend two years from now is expected to be $4. After that the dividend is expected to grow at a constant rate. What is this constant dividend growth rate?

Examination Question - Section 4

10. (a) Ivan Smith is enthusiastic about the prospects for Ippoxy Incorporated. The firm pays dividends on an annual basis. The next dividend, which is expected to be $1.50 per share, will be paid one year from now. Ivan forecasts that dividends will grow at 25 percent for three years after this, then at 15 percent for four years and 4 percent forever thereafter. The discount rate is 8 percent per year. What is the value of a share of stock based on these assumptions?

(b) Ivan is also enthusiastic about Outlandish Incorporated. The firm pays dividends on an annual basis with the next dividend to be paid one year from now. The firm has <u>just</u> <u>paid</u> a dividend of $0.80 per share. Ivan expects that dividends will remain at this level for the next ten years. They will then increase to $1.00 for three years and then revert to $0.80 forever thereafter. The discount rate is 6 percent per year. What is the value of a share of stock based on these assumptions?

Problem Set 3
(Due October 4, 2011)

Section 5

1. What is/are the IRR(s) of the following project?

C_0	C_1	C_2	C_3
(100)	40	40	40

2. What is/are the IRR(s) of the following project?

C_0	C_1	C_2	C_3
(1000)	1250	1700	(2000)

3. Project A has an IRR of 16 percent and Project B an IRR of 14 percent. The cost of capital is 8 percent. The projects are mutually exclusive. Is it possible to conclude the firm should choose A?

4. Projects A and B have the following cash flows:

	C_0	C_1	C_2	C_3	C_4
A	(100)	60	50	40	30
B	(80)	40	50	0	0

(i) If a company uses the payback rule with a cutoff period of 2 years, which projects would it accept?

(ii) Are these good decisions if the opportunity cost of capital is 10%?

5.   A company has a limit of $200 million to invest and has the following possible investment opportunities. The cost of capital is r = 0.08.

Project	Investment Millions	NPV Millions	IRR %
A	100	140	15
B	20	20	15
C	50	65	43
D	50	-10	5
E	150	30	10
F	40	32	50
G	20	18	30

Given the limited amount of investment, what is the maximum NPV that the company can obtain?

Examination Question - Section 5

6.   Consider the following two mutually exclusive investments.

Year	A ($'s)	B ($'s)
0	-100	-100
1	50	70
2	70	75
3	40	10

Calculate the IRR for each. For what range(s) of discount rates, if any, will the simple IRR rule for mutually exclusive projects (accept the project with the highest IRR) conflict with the NPV rule?

Section 6

7.   Which of the following statements is/are correct?

A.   Sunk costs are not relevant for the decision to accept or reject the continuation of a project and should therefore be ignored

B.   The investment decision should be based on average rather than incremental payoffs

C.   In capital budgeting, it is important always to deduct the project's share of non-incremental overhead expenses

D.   When calculating cash flows it is important to deduct a realistic forecast of depreciation; do not accept the accountant's figure at face value

E.   Interest and dividend payments should be deducted when calculating cash flows

F.   It is important to consider all incidental effects, such as the effect of a railroad branch line on the traffic it brings to the main line

G.   None of the above

8. Suppose the rate of inflation is anticipated to be 45% and the nominal discount rate is 68%, what is the real discount rate?

9. Machines A and B cost the same, but have different useful lives and operating costs. A lasts four years and B lasts five years. Cash outflows are:

t =	0	1	2	3	4	5
Machine A	-10	-3	-3	-3	-3	
					(Replace)	
Machine B	-10	-4	-4	-4	-4	-4
						(Replace)

Which machine has the lowest equivalent annual cost at an opportunity cost of capital of 10%?

10. Emerson Mechanical is building a new plant to make machine tools in Indiana. The initial working capital needs of the plant to finance inventory, payroll and so forth are $15M. Each year these needs are expected to increase by 6 percent. After five years the plant will be closed and the working capital recovered. In undertaking a capital budgeting exercise to see whether or not to build the plant, what are the cash flows that should be included to take account of the working capital?

Examination Questions - Section 6

11. Eugene Rastignac is thinking of leaving his current job on 12/31/X0 in order to set up a mushroom farm which he intends to run for 5 years. The land needed for the project can be purchased on 12/31/X0 for $500,000 (nominal) and it is expected it can be sold on 12/31/X5 for $500,000 (real). The land cannot be depreciated; gains and losses on the land are not subject to taxation. The equipment needed can be purchased on 12/31/X0 for $150,000 (nominal). It has an expected life of five years and its salvage value is expected to be $25,000 (real). The equipment is depreciated on a straight line basis. The output of the farm is expected to be 0.85 tons of mushrooms per year. The price of mushrooms on 12/31/X0 is expected to be $200,000 per ton (nominal) and is expected to grow at 4 percent per year in real terms. The total annual costs at 12/31/X0 prices are $50,000 (nominal) and are expected to increase at 8 percent per year in real terms. The total working capital needs are expected to remain constant at $100,000 (nominal) throughout the five years. The working capital is recovered at the end of the project. The rate of inflation over the next five years is expected to be 5 percent per year. The nominal discount rate for this type of project is 10 percent per year. The tax rate for the farm is 34 percent. Assume all cash flows occur at the end of the year and all losses can be offset against other income. What is the NPV of the project on 12/31/X0?

12.   Station WJXT is considering the replacement of its old fully depreciated sound mixer. This sound mixer has zero after-tax cash flows associated with it. Two new models are available. Mixer X has a cost of $216,600, a five year expected life, and real after-tax cash flows (including the tax shield from depreciation) of $68,200 per year. Mixer Y has a cost of $345,000, a ten year life, and real after-tax cash flows (including the tax shield from depreciation) of $83,400 per year. No new technological developments are expected. The real discount rate is 12 percent. Should WJXT replace the old mixer, and, if so, with X or Y?

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Problem Set 4
(Due October 18, 2011)

Sections 7 and 8

1.   Which of the following statements is/are correct?

     A.   Mean and variance have the same units
     B.   Variance is a measure of dispersion
     C.   If two random variables have a positive covariance it means they tend to move in the same direction
     D.   If two random variables have a zero covariance they tend to move in opposite directions
     E.   Correlation coefficients lie only between 0 and 1

2.   The stocks A and B have the following distributions of returns.

	A	B	Probability
State 1	5	10	0.1
State 2	6	7	0.2
State 3	2	3	0.1
State 4	9	8	0.3
State 5	7	7	0.3

     (i)   What are the means, variances and standard deviations of stocks A and B?
     (ii)   What are the covariance and correlation between stocks A and B?
     (iii)   Consider a portfolio with $\pi = 0.50$ in A and $1 - \pi = 0.5$ in B. What are the mean, variance and standard deviation of this portfolio?

3.   Suppose there are four stocks X_i, i = 1,…, 4. Each stock pays off 0 with probability 0.5 and 2 with probability 0.5. The payoffs on all four stocks are independent.
     (i) What are the mean, variance and standard deviation of payoffs of a portfolio consisting of ½ a share of stock X_1 and ½ a share of stock X_2?
     (ii) What are the mean, variance and standard deviation of payoffs of a portfolio consisting of ¼ of a share of each of the four stocks?

4.   Which of the following is/are correct?

        A.   Diversification reduces risk because prices of stocks do not usually move exactly
              together
        B.   The risk that can be potentially eliminated by diversification is called market risk
        C.   The risk of a well-diversified portfolio depends on the market risk of the
              securities included in the portfolio
        D.   The sensitivity of an investment's return to market movements is usually called its
              beta
        E.   The expected return on a portfolio is a weighted average of the expected returns
              on the individual securities
        F.   The standard deviation of returns on a portfolio is equal to the weighted average
              of the expected returns on the individual securities if these returns are completely
              uncorrelated
        G.   None of the above

5.   Use EXCEL to answer this question. Stock X has mean return 0.10 and standard
      deviation 0.3. Stock Y has mean return 0.05 and standard deviation 0.1. The correlation
      between them is +0.2.
      (i)   Plot the portfolio locus for these two stocks in mean standard-deviation space
            with the mean on the vertical axis and the standard deviation on the horizontal
            axis. Specifically, trace the risk-return tradeoff given by combining these two
            stocks in varying amounts. Let π represent the proportion of wealth invested in X
            and 1-π the proportion in Y. Allow π to vary between -0.2 and 1.2 in steps of 0.1.
            Attach a copy of your plot of the portfolio locus and the data it is based on with
            your answers.
      (ii)   What investments in X and Y do the portfolios with $\pi < 0$ and $\pi > 1$ correspond
            to?

6.   Tim Turner has 60 percent of his money in Microsoft stock and the remainder in AT&T.
      On past evidence, the standard deviation is 34% for Microsoft and 28% for AT&T.
      Suppose the correlation between Microsoft and AT&T is 0.46. What is the standard
      deviation of Mr. Turner's portfolio?

7.   What is the standard deviation of a well-diversified portfolio of stocks with an average
      beta of 0.64 if the standard deviation of the market is 0.30?

8.   Suppose a well-diversified portfolio has a beta of 0.43. The risk free rate is 5 percent and
      the expected return on the market portfolio is 13 percent. What is the expected return on
      the portfolio?

9.   Bernard Bontebok borrows $5,000 at 8% and invests this together with $5,000 of his own money in a portfolio of common stocks. If the stock portfolio has a standard deviation of 24%, what is the standard deviation of Mr. Bontebok's investment?

10.   Please follow the steps to download data from Yahoo and use the real time data to calculate the beta of several stocks.

Step 1: Download stock price data from Yahoo finance
    1 Go to http://finance.yahoo.com/
    2 Type in the ticker of the following stocks (one at a time), click "Get Quotes"
      Whole Foods Market (ticker: WFM)
      Ford (ticker: F)
      Oracle (ticker: ORCL)
      Goldman Sachs (ticker: GS)
      Estee Lauder (ticker: EL)
      Amazon (ticker: AMZN)
    3 Go to "historical prices" from the menu at the left hand side.
    4 Change the Start Date to Aug 30 2010, and the End Date to Aug 31 2011, choose "daily", then click "get price"
    5 Scroll down the webpage and click "download to spreadsheet" and save the file in your computer.
    6 Use the downloaded data to calculate daily returns from the "adjusted close" (The adjusted close price adjusts for dividend and splits. Most of the time it's the same as the "close price".)

 Step 2: Download market index data from Yahoo finance
      The above stocks are listed on NYSE or NASDAQ. Theoretically, we should use a value weighted market index including all the stocks listed in NYSE and NASDAQ. However, calculating this market index takes some time. To make things simpler, here we use the S&P 500 index as a proxy for the market.
      The ticker for S&P 500 is ^GSPC on yahoo finance. You can download and calculate the daily market return in the same way as in Step 1.

Step 3: Download the risk free returns from the Federal Reserve
    1 Go to http://www.federalreserve.gov/releases/h15/data.htm
    2 Find **Treasury constant maturities,** download the 1-month treasury rate.
    3   Note that the unit here is expressed in annualized percentage terms. You need to convert it to daily returns (assume there are 252 trading days in one year).

Step 4: Calculate the market beta of the stocks in Step 1. (There are several functions you can use in Excel to complete the task, SLOPE and LINEST are two most commonly used ones.)

Examination Question - Section 8

11.   (Note: The three parts of this question are separate. You should not use information
from one part in any other part.)

(a) Tony Simpson has $5,000 of his own money. He intends to borrow $3,000 to use
together with his own $5,000 to give a total of $8,000 to invest in the market portfolio.
The assumptions of the Capital Asset Pricing Model are satisfied. The expected return
on the market portfolio is 17 percent and the risk free rate is 8 percent. The standard
deviation of the market portfolio is 5 percent. What are the expected return and standard
deviation of his portfolio?

(b) Suppose the market portfolio has an expected return of 0.22 and a standard deviation
of 0.19. The risk free rate is 0.09. How could you construct a portfolio having an
expected return of 0.15? What would be the standard deviation of this portfolio?
(Assume the Capital Asset Pricing Model holds).

(c) You have identified three potential investments with the following characteristics:

Investment	E(r)	Beta	Unique Risk
Riskless Asset A	8%	0.0	None
Security B	16%	0.8	None
Security C	25%	1.2	None

(i)   Plot expected returns against betas for these three securities. Do they all lie on
the security market line? Is an arbitrage opportunity available?
(ii)   Give a zero-investment, zero-risk portfolio with positive expected return that has
either +$1 or -$1 invested in C. What is the expected return on this portfolio?

FNCE 601

Fall 2011   THE TOLLHOUSE COOKIE CASE   Franklin Allen

Instructions: This case will count for 4% of the course grade. It is due on **Tuesday, October 11 at 9:00am**. It should be uploaded to the FNCE 601 Webcafe site by clicking on the icon "Turn-In Folders – Case I" and following the instructions there. The case should be done in groups of three people or less. Each group should include **one** EXCEL file containing the spreadsheets that are developed to solve the case and a brief one-page write-up that explains carefully any assumptions that you make in order to be able to solve the problem (cut and paste the write-up from Word if necessary). Make sure you include all team members' full names in the file.

The Tollhouse Company is thinking of building a bakery to produce a new type of chocolate chip cookie that is free of both cholesterol and saturated fat and has 2 calories per cookie. The new cookies are to be called "Dream On". The bakery is expected to last for 25 years. Its initial cost is $85 million. This cost can be depreciated over 15 years using straight line depreciation. The salvage value of the equipment in the bakery after 15 years is $2 million in real terms. After 15 years the bakery will be renovated in order to ensure it can continue producing. The old machines will be sold and replaced with new machines. The cost of the renovation will be $20 million in real terms and can be depreciated (again using straight line depreciation) over the remaining 10 years of the bakery's life. The salvage value of the equipment after the remaining 10 years will be $1 million in real terms. The land the bakery is built on could be rented out for $1.25 million a year in real terms for 25 years.

The bakery will be able to produce 25 million packets of cookies a year. The price of a packet of cookies is currently $2.75. It is expected to grow at a rate of 5% per year in real terms for the first 2 years, then at 2% per year in real terms for 4 years and finally at 0% per year in real terms thereafter for the remainder of the bakery's life. The firm expects to be able to sell all the packets of Dream On that it can produce. The basic ingredients for a package of cookies currently cost $0.75. These costs are expected to grow by 1% per year in real terms through the lifetime of the project. The labor required to operate the bakery is expected to cost a total of $11 million dollars in nominal terms during the first year and this is expected to increase at 4% per year in real terms thereafter. The total working capital requirement initially is $18 million and this is expected to increase at 3% per year in real terms.

The rate of inflation is expected to be 3% per year for the bakery's life. The firm's total tax rate including local taxes is 36%. The firm expects to make substantial profits on its other operations so that it can offset any losses on the bakery for tax purposes. Its opportunity cost of capital for projects of this type is 11% per year in nominal terms.

Construct two spreadsheets in EXCEL to find the NPV of the bakery. One spreadsheet should be in nominal terms and one spreadsheet should be in real terms. The two spreadsheets should give the same value for the NPV. Should the firm build the bakery?

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FORMULA SHEET FOR MIDTERM EXAM

Present value formulas:

Basic formula:

$$PV = \sum_{t=1}^{n} \frac{C_t}{(1 + r_t)^t}$$

Perpetuity:

$$PV = \frac{C}{r}$$

Growing perpetuity:

$$PV = \frac{C}{r - g}$$

Discount factor:

$$DF(t\ years,\ r\%) = \frac{1}{(1 + r)^t}$$

Future value factor:

$$FV(t\ years,\ r\%) = (1 + r)^t$$

Annuity factor:

$$AF(t\ years,\ r\ \%) = \frac{1}{r}\left(1 - \frac{1}{(1 + r)^t}\right)$$

Growing annuity:

$$PV = \frac{C}{r - g}\left(1 - \frac{(1 + g)^t}{(1 + r)^t}\right)$$

Interest r per year
 compounded q times
 a year for t years:

$$C_t = PV\left(1 + \frac{r}{q}\right)^{qt}$$

Continuously compounded:

$$PV = e^{-r_c t} C_t$$

Continuous/annual
 relationship

$$r_c = \log_e(1 + r);\ r = e^{r_c} - 1$$

Reinvestment rate:

$$(1 + r_{1,2}) = \frac{(1 + r_2)^2}{(1 + r_1)}$$

Basic stock valuation formula:

$$P_0 = \sum_{t=1}^{\infty} \frac{DIV_t}{(1 + r)^t}$$

Constant growth case:

$$P_0 = \frac{DIV_1}{r - g}$$

Internal Rate of Return:

$$C_0 + \frac{1}{1 + IRR} C_1 + \frac{1}{(1 + IRR)^2} C_2 + \ldots + \frac{1}{(1 + IRR)^t} C_t = 0$$

Profitability Index: $\quad PI = \frac{NPV\ of\ project}{Initial\ Investment}$

Payback: $\quad$ Payback = Period initial investment is recovered within

Inflation:

$$Real\ discount\ rate = \frac{1 + nominal\ rate}{1 + inflation\ rate} - 1$$

$$Nominal\ cash\ flow = Real\ cash\ flow \times (1 + inflation\ rate)^t$$

$$Real\ cash\ flow = Nominal\ cash\ flow\ /\ (1 + inflation\ rate)^t$$

Statistics formulas:
Expectation:

$$EX = \sum_{i=1}^{n} p_i\, x_i$$

$$E[f(X)] = \sum_{i=1}^{n} p_i\, f(x_i)$$

If a is constant:

$$E(aX) = aE(X)$$

$$E(X + Y) = E(X) + E(Y)$$

Variance:

$$Var\ X = \sigma_X^2 = E\left[(X - EX)^2\right] = \sum_{i=1}^{n} p_i (X_i - EX)^2$$

Standard deviation:
$$SD_X = \sigma_X = \sqrt{\sigma_X^2}$$

Covariance:
$$Cov(X, Y) = \sigma_{XY} = E\left[(X - EX)(Y - EY)\right] = \sum_{i=1}^{n} p_i(x_i - EX)(y_i - EY)$$

Correlation coefficient:
$$\rho_{XY} = Corr(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{Cov(X, Y)}{SD_X SD_Y}$$

Beta:
$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{Cov(Stock\ i, M)}{Var\ M}$$

Portfolio formulas:

Mean and Variance of a portfolio with π in A and $1 - \pi$ in B (i.e. two assets)

Mean:
$$Mean_{port} = \pi EA + (1 - \pi)EB$$

Variance:
$$Var_{port} = \pi^2 Var\ A + (1 - \pi)^2 Var\ B + 2\pi(1 - \pi)Corr(A, B)SD_A SD_B$$

Mean and Variance of a portfolio with π_i in X_i, (i.e. N assets)

Mean:
$$Mean_{port} = \pi_1 EX_1 + \pi_2 EX_2 + \ldots + \pi_N EX_N$$

Variance:
$$Var_{port} = \sum_{i=1}^{N} \pi_i^2 Var\ X_i + \sum_{i=1\ i \neq j}^{N} \sum_{j=1}^{N} \pi_i \pi_j Cov(X_i, X_j)$$

For a well-diversified portfolio:
$$SD_{port} = \beta_{port} SD_M$$

Beta:
$$\beta_{port} = \pi_1 \beta_1 + \pi_2 \beta_2 + \ldots + \pi_N \beta_N$$

For portfolios on the Capital Market Line: $\beta_{port} = \pi$; $SD_{port} = \pi SD_M$

CAPM:
$$r = r_F + \beta(r_M - r_F)$$

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Example Midterm Exam 1

Instructions: You have 1 hour 30 minutes to answer all four questions. All four questions carry equal weight. If anything is unclear, please state carefully any assumptions you make.

1.   Consider a world with two points in time t_0, t_1. Nelly Brown has just inherited $5M. She has only two projects she can invest in. Project A costs $3M at t_0 and pays off $3.8M at t_1. Project B costs $2M at t_0 and pays off $2.5M at t_1. She can lend and borrow at a bank at an interest rate of 20 percent. Nelly is only interested in consumption today at t_0; she does not get any utility from consumption at t_1.

    a.   What is the most Nelly can consume at t_0? How does she achieve this (i.e. what projects does she do and how much does she borrow or lend at the bank)?

    b.   How would your answers to (a) be changed if Nelly could lend to the bank at a rate of 15 percent but could borrow at 30 percent?

2.   Suppose you have decided to start saving money to take a long-awaited world-cruise, which you want to take 5 years from today. You estimate the amount you will have to pay at that time will be $6000. The savings account you established for your trip offers 16% per annum interest compounded quarterly.

    a.   How much will you have to deposit each year (at year-end) to have your $6000 if your first deposit is made 1 year from today and the final deposit is made on the day the cruise departs?

    b.   How much will you have to deposit each year if your first deposit is made 1 year from today and the final deposit is made 1 year before the cruise departs?

    c.   How much would you have to deposit each year if your first deposit is made now and the final deposit is made 1 year before the cruise departs?

3.   The International Tractor Corporation is considering purchasing personal computers. It can either buy Apple PC's or IBM PC's. The Apple PC costs $1200 and is expected to last for five years. Annual maintenance costs are $200 per year paid at year's end. The machines are expected to have no salvage value. The IBM PC costs $1800 and is expected to last six years and has annual maintenance costs of $250. It is expected to have a salvage value of $300. The firm estimates its workload is such that it should either buy 400 Apple PC's or 300 IBM PC's. There is expected to be no technological progress. International Tractor uses straight line depreciation. Both maintenance costs and depreciation are tax-deductible. Its tax rate is 40%. Its discount rate for this type of investment is 8 percent. Should the firm buy Apple PC's or IBM PC's?

4.   a.   The respective values of the beta coefficients for the returns on three securities x,y,z are 1.0, 0.5, and 0.8 when the variance on the market portfolio is equal to 0.0625.

              i.   What are the respective values of the covariance between each security's return and the returns on the market portfolio?

              ii.   What would be the value of beta for a portfolio, P, composed equally of the three securities?

              iii.   What would be the value of the covariance between the returns on the equally-weighted portfolio P and the return on the market portfolio?

       b.   Joe Smith lives in a country where the assumptions of the capital asset pricing model are satisfied. There are only two risky securities, Acorn and Bethlehem. Joe has 40 percent of his portfolio in the riskless asset, 35 percent, in Acorn and 25 percent in Bethlehem. If the total value in the entire economy of Acorn is $3.5 million, what is the total value of Bethlehem?

       c.   The risk-free rate, r_f, is 10 percent and the risk premium on the market, $(r_M - r_f)$, is expected to be 8.25 percent. Sally Smith is less risk-averse than the average. She can tolerate beta values up to 1.5. Nevertheless she dislikes holding individually risky stocks and prefers to borrow and put her holdings in a widely diversified indexed mutual fund with a beta of 1.0. If she has $10,000 of her own money to invest in the fund, how much additional money should she borrow to get a portfolio with a beta of 1.5? What is her expected rate of return?
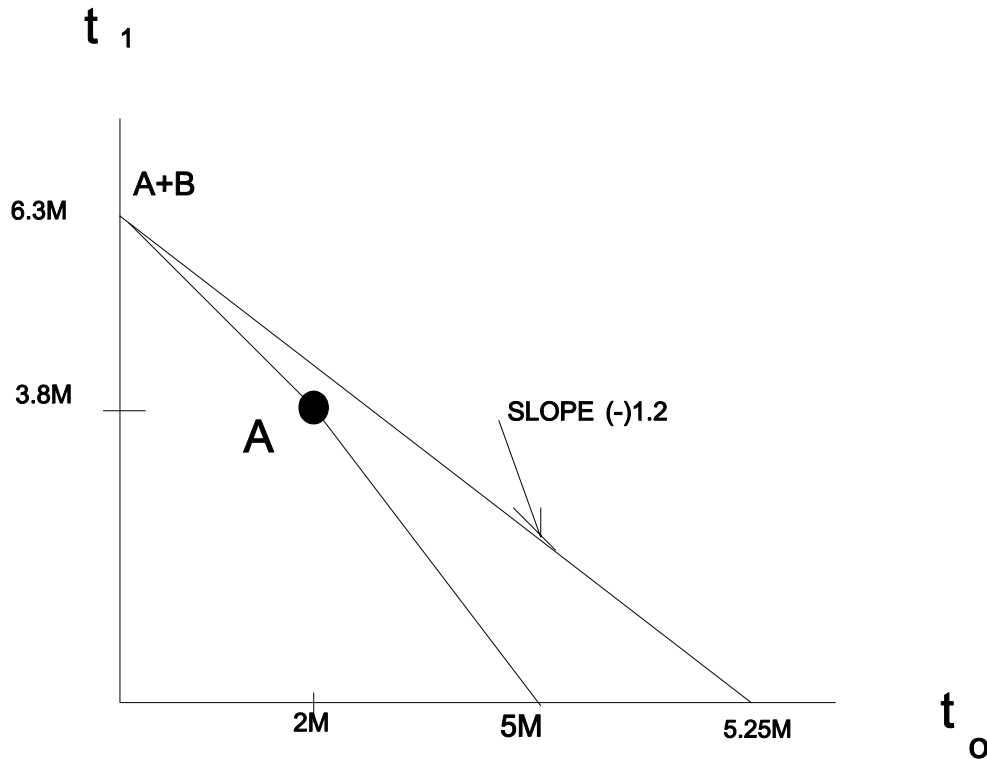
Example Midterm Exam 1   Franklin Allen

Solutions

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1. a. (15 points)

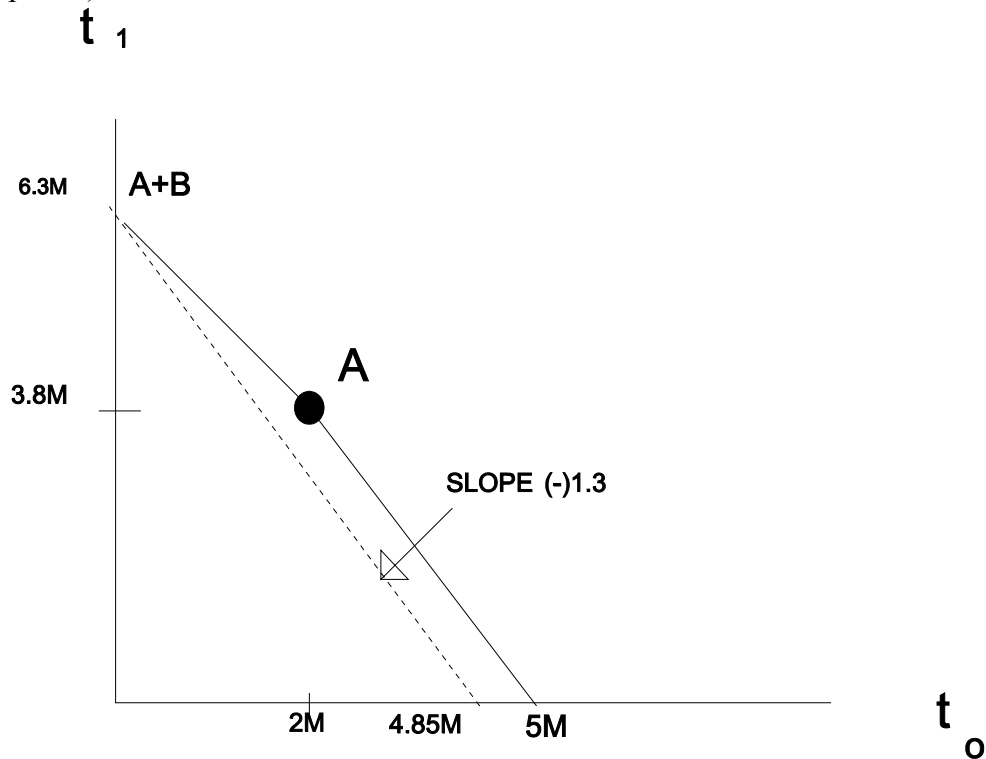$$NPV_A = -3 + \frac{3.8}{1.2} = 0.167 \quad > 0$$

$$NPV_B = -2 + \frac{2.5}{1.2} = 0.0833 \quad > 0$$

Therefore, do both projects.

Hence, maximum possible consumption at t_o : $\frac{6.3}{1.2} = 5.25M$

Therefore, do projects A and B, borrow 5.25M from the bank and use next period's cash flow, 6.3M, to pay off the loan.

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1 b. (10 points)

t_1

6.3M   A+B

3.8M

A

SLOPE (-)1.3

2M   4.85M   5M t_0

Since Nelly is only interested in consumption today, it is sufficient to look at the projects using the borrowing rate only. Using 30 percent:

$$NPV_A = -3 + \frac{3.8}{1.3} = -0.077$$

$$NPV_B = -2 + \frac{2.5}{1.3} = -0.077$$

Therefore, she shouldn't do either project. She should just consume 5 M today and not lend or borrow at the bank.

2.

t =	0	1	2	3	4	5
Cruise						$6,000
a.		C	C	C	C	C
b.		C	C	C	C	
c.	C	C	C	C	C	

2a. (8 points)

Annual interest rate = $(1 + (0.16/4))^4 - 1$

$$= 16.99\% \text{ or } 17\%$$

PV of cost of cruise = $6000 x (DF 5 yrs, 17%) = $6000 x (0.456)

$$= \$2,736$$

If the person saves C every year, then

C x AF(5 yrs., 17%) = $2,736

$C = \dfrac{2,736}{3.199} = \$855.27$

Therefore, you must save $855.27 to finance your trip.

2b. (8 points)

In this case

C x AF(4 yrs., 17%) = $2,736

$C = \dfrac{2736}{2.743} = 997.45$

Therefore, you must save $997.45 every year.

2c. (9 points)

In this case

C + C x AF(4 yrs., 17%) = $2,736

C x [1 + 2.743] = $2,736

$C = \dfrac{2,736}{3.743} = \$730.96$

Therefore, you must save $730.96 every year.

3. (25 points)

Apple PC's

Cost per machine:	0	1	2	...	5
Initial cost of purchase	-1200				

Depreciation tax shield

$$\frac{1200 \times 0.4}{5} = 96$$

Annual maintenance cost after tax

$$-0.6 \times 200 = -120$$

PV of costs = -1200 + ((96 - 120)xAF(5 yr.,8%)) = -1200 - (24x3.993)
= -1295.83

Equivalent Annual Cost (EAC) = $\dfrac{-1295.83}{AF(5yrs,8\%)}$ = -$324.53

EAC for 400 Apple PC's: -324.53 x 400 = -$129,810.36.

IBM PC's

	0	1	2	. . .	6
Initial Cost	-1800				
Salvage					300

Depreciation tax shield

$$\frac{(1800 - 300) \times 0.4}{6} = 100$$

Annual maintenance costs after tax

$$250 \times 0.6 = -150$$

PV costs = -1800 + 300 x DF(6 yrs.,8%) + (100 - 150) x AF(6 yrs.,8%)

= -1800 + 300*(0.630) - 50*4.623 = -1842.15

EAC = -1842.15/AF(6 yr., 8%) : -398.475

EAC for 300 IBM's : -398.475*300 = -$119,542.5

Therefore, buy IBM PC's since they cost less.

4 (a)(i) (3 points)
From the definition of β,

$$\beta = \frac{Cov(asset,M)}{Var(M)}$$

Therefore, $Cov(asset,M) = \beta \times Var(M)$

Using this formula for assets X, Y and Z

$Cov(X, M) = 1 \times 0.0625 = 0.0625$

$Cov(Y, M) = 0.5 \times 0.0625 = 0.03125$

$Cov(Z, M) = 0.8 \times 0.0625 = 0.05.$

(ii) (3 points) Using the formula for the beta of a portfolio

$$\beta_{PORT} = \pi_X \beta_X + \pi_Y \beta_Y + \pi_Z \beta_Z$$

For an equally weighted portfolio, $\pi_X = \pi_Y = \pi_Z = 1/3$

Therefore, $\beta_{PORT} = (1/3) \times 1 + (1/3) \times 0.5 + (1/3) \times 0.8 = 0.767$

(iii) (3 points) Using the definition of β

$Cov(PORT,M) = \beta_{PORT} \times Var(M) = 0.767 \times 0.0625 = 0.0479167$

(b) (8 points) Since everybody holds the market portfolio in a CAPM world, everybody must hold Acorn to Bethlehem in the ratio

$35/25 = 1.4$

Therefore, $\dfrac{\text{Total value of Acorn}}{\text{Total value of Bethlehem}} = 1.4$

Therefore, Total value of Bethlehem $= \dfrac{3.5M}{1.4} = \$2.5M$

(c) (8 points) She should borrow $5000 and invest it together with her own $10,000 in the widely diversified indexed mutual fund to get a portfolio with beta of 1.5. She is effectively mixing the risk-free asset and the market with 1.5 times her wealth in the market so the beta is 1.5. Then, using the CAPM,

$ER_{PORT} = 0.10 + 1.5 \times 0.0825 = 0.22375$

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Example Midterm Exam 2

Instructions: You have 1 hour 30 minutes to answer all four questions. All four questions carry equal weight. If anything is unclear, please state carefully any assumptions you make.

1.   Consider a world with two periods $t = 0, 1$, perfect capital markets and no uncertainty. Kristine and James have each inherited 50% of the shares of the Nared Corporation from their parents who founded the company. This is the only wealth they have, i.e. the only payments they receive at dates 0 and 1 are from the Nared Corporation. Suppose initially that they can lend and borrow as much as they like at the bank at 11 percent. The Nared Corporation has $10 million on hand to undertake investments. It has 4 projects available to it. The costs at date 0 and payoffs at date 1 are shown below.

	Cost at t=0	Payoff at t=1
Project A	$2,150,000	$2,420,000
Project B	$1,140,000	$1,360,000
Project C	$2,470,000	$2,700,000
Project D	$2,560,000	$2,735,000

(a) What rule should Kristine and James tell the managers of the firm to follow?
(b) Which projects should the Nared Corporation undertake?
(c) Suppose that at date 0 the Nared Corporation pays out all that is left over from the $10 million after undertaking the projects you answered in part (b). At date 1 it pays out all the payoffs from these projects.
      (i) How much can Kristine consume at date 1 if she consumes $2 million at date 0?
      (ii) How much can James consume at date 1 if he consumes $5 million at date 0?
(d) Suppose Kristine and James can now borrow from the bank at 14 percent and they can lend to the bank at 8 percent.
      (i) Which projects would Kristine like the firm to undertake assuming she still wants to consume $2 million at date 0?
      (ii) Which projects would James like the firm to undertake assuming he still wants to consume $5 million at date 0?

2.

(a)   Maribelle has just finished college. She is planning her financial future. She intends to do an MBA starting 4 years from now. She will have to pay tuition of $25,000 for the first year and $27,000 in the second year. Tuition is paid at the start of each year she is attending. Finally, she wants to retire 40 years from now and have a pension of $50,000 a year for twenty years with the first payment being 40 years from now. She can borrow and lend as much as she likes at 8 percent. What constant amount does she have to save at the end of each year starting 1 year from now during the time she is working (i.e., she saves at the end of years 1-4 and at the end of years 7-40) to cover all her anticipated expenses?

(b)   Jerome is valuing the WJ Company. It pays dividends once every six months. It has just paid a dividend of $6.30. For the next five years the dividend is expected to grow by 6% every six months. After that it is expected to grow at 2% every six months in perpetuity. What is the value of the stock if the opportunity cost of capital is 10 percent compounded annually?

3. The current date is 12/31/X7 and the Cancan Corporation is thinking of buying a new plant. The plant will be ready to start producing tomorrow, i.e. on 1/1/X8. The plant will produce for five years. The cost of the building and new equipment is $10 million (real) and is to be paid today on 12/31/X7. The salvage value five years from now is $2 million (real). The plant and equipment can be depreciated over the five years using straight line depreciation. In addition to the new equipment purchased as part of the initial $10 million outlay, the firm will use some fork lift trucks which it already owns. If the project was not undertaken these fork lift trucks could be sold now for $650,000 (real). These fork lift trucks have already been depreciated to zero so all of the $650,000 (real) would be taxable. If the fork lift trucks are used in the plant they will have no salvage value in 5 years time. The labor for the plant is expected to cost $400,000 (real) per year for each of the five years the plant is in production. The raw materials cost $2 million (real) per year. The plant will produce 265,000 units of output the first year it is in production and this will grow by 5 percent per year for the five years. The price of each unit of the output can be sold for is $20 (real) for each of the five years the plant is in production. All costs and revenues can be assumed to occur at the end of each year. The firm has other profitable ongoing operations. Its tax rate is 38 percent. Its opportunity cost of capital for this kind of project is 12 percent (nominal). The anticipated inflation rate is 3 percent per year. What is the NPV of the plant?

4.

(a) There are two risky assets X and Y and a risk free asset Z. The means and standard deviations are as follows.

Asset	Mean	Standard Deviation
X	0.15	0.20
Y	0.11	0.30
Z	0.07	0

The correlation of X and Y is -0.1. What is the mean and standard deviation of a portfolio with 30 percent in X, 45 percent in Y and 25 percent in Z?

(b) Suppose the capital asset pricing model holds. The risk free rate is 8 percent. The expected return on the market portfolio is 16 percent. The standard deviation of the market portfolio is 35 percent. Nola is willing to hold a portfolio with a standard deviation of up to 45 percent but no more than this.

   (i) Given this, what is the highest expected return Nola can obtain?
   (ii) What does the portfolio that allows this expected return consist of if Nola has wealth of $10,000?

(c) You have discovered three well-diversified portfolios with the following characteristics.

Portfolio	Expected Return	Beta
A	12%	0.2
B	17%	1.0
C	25%	2.5

   (i) Do these three securities lie on the security market line?
   (ii) Give a zero-investment, zero-risk portfolio with positive expected return that has either +$1 or -$1 invested in C.
   (iii) What is the expected return on this portfolio?

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Example Midterm Exam 2   Franklin Allen
Solutions

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1. (25 points total)

Project	t=0 cost($)	t=1 payoff($)	Return(%)	NPV at 11%($)
A	2,150,000	2,420,000	12.6	30,180
B	1,140,000	1,360,000	19.3	85,225
C	2,470,000	2,700,000	9.3	-37,568
D	2,560,000	2,735,000	6.8	-96,036

(a) (1 point) Maximize NPV.

(b) (4 points) Choose A, B they have a higher return than 11% or equivalently are positive NPV at 11% discount rate.

(c) (i) (6 points) If the Nared Corporation funds Projects A and B, at date 0

Total amount remaining = 10,000,000 - (2,150,000+1,140,000) = $6,710,000

This amount is paid out to shareholders. Given she owns 50% of the shares, at date 0

Kristine receives = $0.5 \times 6,710,000 = \$3,355,000$

Given she consumes $2,000,000 at date 0,

Kristine puts in bank = 3,355,000 - 2,000,000 = $1,355,000

Hence, taking account of the both her 50% share of the firm and her bank deposit,

Kristine's date 1 consumption = $0.5(2,420,000 + 1,360,000) + 1,355,000 \times 1.11$
$= \underline{\$3,394,050}$

(ii) (6 points) James receives the same payment of $3,355,000 from Nared since he also owns 50% of the shares. Since he consumes $5,000,000 at date 0

James borrows from bank = 5,000,000 - 3,355,000 = $1,645,000
James' date 1 consumption = $0.5(2,420,000 + 1,360,000) - 1,645,000 \times 1.11$
$= \underline{\$64,050}$

(d) (i) (4 points) Since Kristine is a lender, the relevant rate for her is the lending rate of 8 percent. She therefore wishes the firm to undertake A, B and C since these are the projects with a higher return than 8 percent.

(ii) (4 points) Since James is a borrower, the relevant rate for him is the borrowing rate of 14 percent. He therefore wishes the firm to undertake B.

2. (a) 15 points

t =	0	1...	4	5	6	7...	40	...	59
Tuition			25,000	27,000					
Pension							50,000 →		50,000
Savings		S →	S				S →		S

Equating savings with the amount needed gives the equation:

$$S \times AF(4yrs,8\%) + S \times AF(34yrs,8\%) \times DF(6yrs,8\%)$$

$$= 25{,}000 \times DF(4yrs,8\%) + 27{,}000 \times DF(5yrs,8\%) + 50{,}000 \times AF(20yrs,8\%) \times DF(39yrs,8\%)$$

$$\underline{S = \$5{,}762}$$

(b) (10 points)

One period = 6 months

Interest rate per six months $= (1.10)^{0.5} - 1 = 0.0488$

t= just paid	0	1	2...	10	11...	∞
DIV. 6.3		6.3×1.06 = 6.678	6.3×1.06^2	6.3×1.06^{10} = 11.282	11.282×1.02 = 11.508	

$$PV = 6.678 \times GAF(10 \text{ periods}, g = 6\%, r = 4.88\%) + \frac{1}{1.0488^{10}} \frac{11.508}{(0.0488 - 0.02)}$$

$$\underline{= 314.85}$$

3. (25 points)

Work in real terms except for depreciation
Figures in millions of dollars

$$\text{Real discount rate} = \frac{1.12}{1.03} - 1 = 0.0874$$

t=	0	1…	5	Total in t=0 $'s
Plant cost	-10			-10
Salvage			2	$2/1.0874^5 = 1.316$
Nom. Depr. (see note below)		$(10-2\times1.03^5)/5$ $= 1.536$ →		$1.53629\times AF(5yrs,12\%)$ $= 5.538$
Tax shield				$0.38\times5.538 = 2.104$
Opportunity cost fork lift trucks	-0.62×0.65 $= -0.403$			-0.403
Units of output		0.265 → g=5%		
Gross revenue at 20 per unit				$0.265\times20\times$ GAF(5yrs,g=5%,r=8.74%) $= 22.752$
Labor		0.4 →		
Raw materials		2.0 →		
Total gross cost				$2.4\times AF(5yrs,8.74\%)$ $= 9.399$
Net after tax operating income				$0.62(22.752-9.399) = 8.279$

$$NPV = -10 + 1.316 + 2.104 - 0.403 + 8.279 = \underline{1.296}$$

Note: Although we are working in real terms we can use a short cut method to calculate the nominal depreciation. We know it is equivalent to discount nominal cash flows at a nominal rate and real cash flows at a real discount rate. Hence when we have a line like nominal depreciation we can discount it to $t = 0$ at the nominal rate and add it in with the other numbers (real and nominal are the same at $t = 0$). This saves converting to real and then discounting at the real rate.

4. (a) (8 points)

Means add: $Er_{port} = 0.3\times0.15 + 0.45\times0.11 + 0.25\times0.07 = \underline{0.112 \text{ or } 11.2\%}$

Variances don't add: covariances matter

$$Var_{port} = \pi_X^2 VarX + \pi_Y^2 VarY + \pi_Z^2 VarZ$$
$$+ 2\pi_X\pi_Y Cov(X,Y) + 2\pi_X\pi_Z Cov(X,Z) + 2\pi_Y\pi_Z Cov(Y,Z)$$

Since Z is risk free Var Z = 0 and Cov(X,Z) = Cov(Y,Z) = 0 this simplifies to

$$Var_{port} = \pi_X^2 VarX + \pi_Y^2 VarY + 2\pi_X\pi_Y Cov(X,Y)$$

$$= 0.3^2 \times 0.2^2 + 0.45^2 \times 0.3^2 + 2 \times 0.3 \times 0.45(-0.1) \times 0.2 \times 0.3 = 0.0202$$

$$SD_{port} = (0.0202)^{0.5} = \underline{0.1421}$$

(b) (8 points)
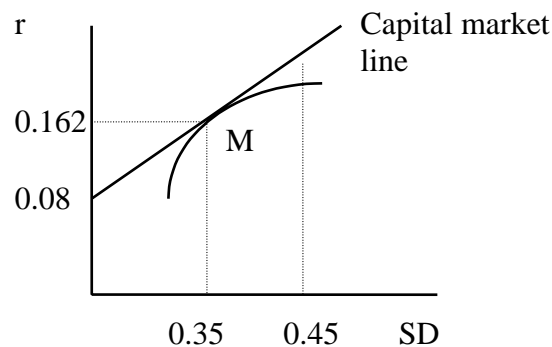
(i) (5 points) For portfolios along the capital market line which consist of the risk free and M

$$SD_{port} = \pi\sigma_M$$

In this case we know SD_{port} and σ_M and need to find π so

$\pi = 0.45/0.35 = 1.286$

Since $\pi=\beta=1.286$ for portfolios along the capital market line it follows using the CAPM that

Expected return = 0.08+1.286(0.16-0.08) = $\underline{0.183}$

(ii) (3 points) To construct a portfolio on the CML with a SD of 45 percent it is necessary to borrow 0.286×10,000 = $2,860 and put 1.286×10,000 = $12,860 in the market.

(c) (9 points)

(i) (2 points)

r

0.25

Security market line

C

0.17

B

0.12

A

0.2   1.0   2.5   β
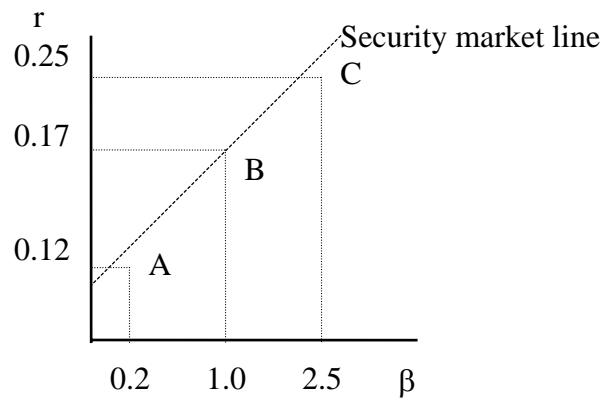
The three portfolios do not lie on a straight line and therefore cannot lie on the security market line. This can be seen directly by considering the fact that the slope from A to B = 0.05/0.8 = 0.0625 is not equal to the slope from B to C = 0.08/1.5 = 0.05333.

(ii) (5 points) Since the three portfolios don't lie on the security market line there will be an arbitrage opportunity. Try $\pi_C = -1$ first.

Zero investment: $\pi_A + \pi_B + \pi_C = 0$

Zero risk: $\pi_A \beta_A + \pi_B \beta_B + \pi_C \beta_C = 0$

Substituting in for the β's and putting $\pi_C = -1$ gives two equations in two unknowns:

$\pi_A + \pi_B - 1 = 0$

$0.2\pi_A + \pi_B - 2.5 = 0$

Solving simultaneously gives $\pi_A = -1.875$ and $\pi_B = 2.875$.

(iii) (2 points)

Expected return = -1.875(0.12) + 2.875(0.17) - 0.25 = 0.01375

If you had started with $\pi_C = 1$ then you would have obtained -0.01375 as the answer and should then have just switched the signs of $\pi_A,$ π_B and π_C.

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Example Midterm Exam 3

Instructions: You have 1 hour 30 minutes to answer all four questions. All four questions carry equal weight. If anything is unclear, please state carefully any assumptions you make.

1.   (a)   Nicole wishes to save for a pension for her retirement. She plans on retiring 20 years from now. Her desired income for the 25 years following that is $30,000 per year with all pension withdrawals being made at the end of the year and the first being 21 years from now. She also wishes to buy a house 10 years from now which will require $100,000. She can borrow or lend as much as she likes from her bank at 8% per year. What is the constant amount she needs to deposit in the bank as savings each year if the first deposit is made 1 year from now and the last 15 years from now? (Note that all figures are in nominal terms.)

      (b)   Takuma is valuing the stock of the Paloma Company which manufactures machine tools. The firm is *just about to pay* a dividend of $2 per share. This is expected to grow at 15% a year for five years, 10% a year for the ten years after that before finally settling down to a growth rate of 5% per year for ever after that. The market capitalization rate of similar stocks is 8%. What is the present value of each share?

2.   (a) Consider the following project.

C_0	C_1	C_2	C_3	C_4	C_5	C_6
-2,500	2,700	2,900	-3,000	2,600	2,800	-6,000

      (i)   What is/are the internal rates of return between 0% and 100%?
      (ii)   If the opportunity cost of capital is 5% should the project be undertaken?
      (iii) If the opportunity cost of capital is 10% should the project be undertaken?

      (b) Consider the following two mutually exclusive projects.

	C_0	C_1	C_2	C_3
A	-1,000	700	650	-100
B	-1,500	400	400	1,000

For which range of opportunity costs of capital between 0% and 50% should A be chosen?

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3.   The Salinas Corporation is considering whether or not to build a plant for manufacturing furniture. The current date is 12/31/X0. The initial outlay today will be $10 million (real). The plant will start production on 1/1/X1. It will produce furniture for 5 years. At the end of that time the salvage value of the plant is expected to be $5 million (nominal). Each year the corporation estimates it will be able to produce and sell 1,000 sofas. The sofas are priced at $5,000 (real) each and this price is expected to remain constant in real terms throughout the time the plant is producing. The raw materials are expected to cost a total of $500,000 (real) each year that furniture is made. The total cost of labor is estimated to be $600,000 (real) on 12/31/X1. It is expected to grow at 3% per year in real terms for the foreseeable future. The land the plant is to be built on could be rented out for $300,000 (real) each year with the rent being paid at the *beginning* of each year (i.e., the first rental payment is 12/31/X0). The corporation's opportunity cost of capital for this type of project is 12% (nominal). The expected rate of inflation is expected to be 5% throughout the period under consideration. The relevant tax rate for the corporation is 35%. Depreciation for tax purposes is nominal and is straight line during the five years the plant is in production. The firm has profitable ongoing operations. Assume all cash flows occur at the end of each year except where otherwise stated. What is the NPV of the plant?

4.

(a)   Consider two risky stocks A and B. Stock A has an expected return of 20% and a standard deviation of 25%. Stock B has an expected return of 16% and a standard deviation of 13%. The correlation of A and B is +0.3. What is the expected return and standard deviation of a portfolio with 60% in A and 40% in B?

(b)   Jamie has wealth of $10,000 to invest. The market portfolio consists of three stocks with 20% being Stock A, 40% being Stock B and the remainder Stock C. The expected return on this market portfolio is 14%. The risk free rate is 5%. The CAPM holds. Suppose Jamie's optimal portfolio has an expected return of 23%, what are the dollar values of her investments in A, B, C and the risk free asset?

(c)   Suppose the assumptions of the CAPM are satisfied. The risk free rate is 5%. The expected return on the market portfolio is 16%. You have discovered two well-diversified portfolios with no unique risk which have the following characteristics.

Portfolio	Expected Return	Beta
A	8%	0.2
B	17%	1.2

(i) Do these portfolios lie on the security market line?
(ii) Give a zero-investment, zero-risk portfolio with positive expected return that has either +$1 or -$1 invested in the risk free asset.
(iii) What is the expected return on this portfolio?

Example Midterm Exam 3

Solutions

_____

1. (25 points total)
   a) (10 points)

   $$\text{PV(Pension)} = \sum_{i=1}^{25} \frac{30,000}{(1+r)^{20+i}} = \frac{30,000}{(1+r)^{20}} \times \text{AF(T=25, r=8\%)}, \text{ where AF(25, 8\%)} =$$

   10.675;

   $$\text{PV(Deposit)} = \sum_{j=1}^{15} \frac{D}{(1+r)^{j}} = D \times \text{AF(T=15, r=8\%)}, \text{ where AF(15,8\%)} = 8.559; \text{ and}$$

   $$\text{PV(House)} = \frac{100,000}{(1+r)^{10}} = \frac{100,000}{(1+8\%)^{10}} = 46,319.35. \text{ We know that the annual deposit has}$$

   to satisfy the following equation:

   PV(Deposit) = PV(House) + PV(Pension).

   Hence we have:

   $$D = \frac{\text{PV(House)} + \text{AF(25,8\%)} \cdot 30,000/(1+8\%)^{20}}{\text{AF(15,8\%)}}$$

   $$= \frac{46,319.35 + 10.675 \cdot 30,000/(1+8\%)^{20}}{8.559} = \$13,439.47$$

   b) (15 points)
   From the dividend stream, we have the following equation for PV(stock):

   P_0 = DIV0+GA1(Yr.1~5, g=15%)+GA2(Yr.6~15, g=10%)+GP(Yr.16~∞, g=5%).

   We also have GA1=DIV1·GAF1(T=5, r=8%, g=15%), where DIV1=2×(1+15%)=2.3;

   $$\text{GA2} = \text{DIV1'} \cdot \text{GAF2(T=10, r=8\%, g=10\%)}, \text{ where DIV1'} = \frac{2 \times (1+15\%)^{5} \times (1+10\%)}{(1+8\%)^{5}} =$$

   $$3.011; \text{ and GP} = \text{DIV1''} \cdot \text{GPF(r=8\%, g=5\%)}, \text{ where DIV1''} = \frac{2 \times 1.15^{5} \times 1.1^{10} \times (1+5\%)}{(1+8\%)^{15}}$$

   = 3.451.

   Therefore, P_0 = 2 + 12.121 + 30.322 + 115.123 = \$159.57
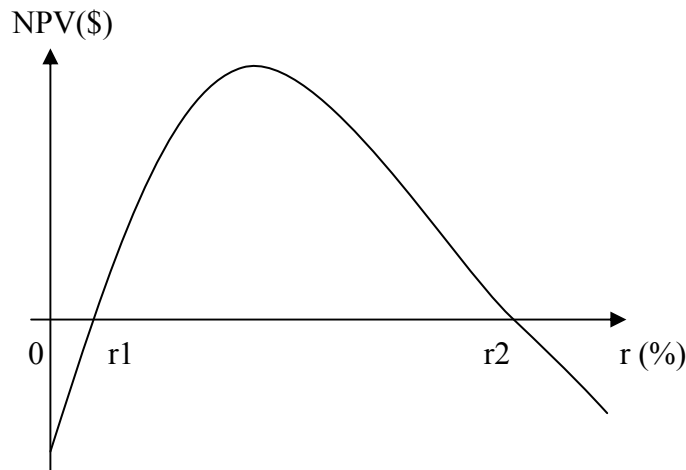
2. (25 points total)
   a) (12 points)
      i) (6 points)   From the information given on cash flow, we can define NPV for the
      project, as a function of rates of return, r:

      $$NPV(r) = -2500 + \frac{2700}{(1+r)} + \frac{2900}{(1+r)^2} - \frac{3000}{(1+r)^3} + \frac{2600}{(1+r)^4} + \frac{2800}{(1+r)^5} - \frac{6000}{(1+r)^6}$$

      We find that there are two IRRs for the project, and can be best described by the
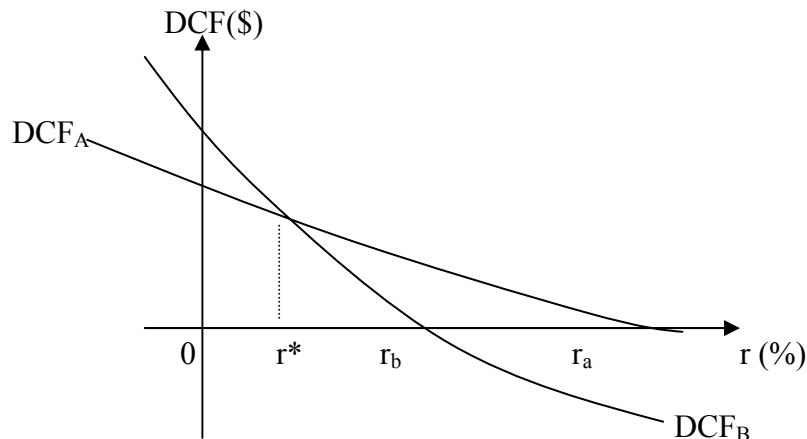   following graph:

      From the graph, IRR1 = r1 = 5.51%, and IRR2 = r2 = 53.41%.

      ii) (3 points) If r = 5%, NPV (5%) = - $34.09, so we should reject the project.
      iii) (3 points) If r = 10%, NPV (10%) = $224.87, so the project should be taken.

   b) (13 points)
      We can plot both DCF(r)'s on the same graph:

We know that r* satisfies the equation: $DCF_A(r^*) = DCF_B(r^*)$. Solving this equation with one unknown, we obtain that $r^* = 2.09\%$. To do this on your calculator, find the IRR on the incremental cash flows $C_A - C_B$. To see why this works, consider the case where we have three dates 0, 1, and 2. We need

$$C_{A0} + \frac{C_{A1}}{1+r^*} + \frac{C_{A2}}{(1+r^*)^2} = C_{B0} + \frac{C_{B1}}{1+r^*} + \frac{C_{B2}}{(1+r^*)^2} \quad \text{or equivalently}$$

$$C_{A0} - C_{B0} + \frac{C_{A1} - C_{B1}}{1+r^*} + \frac{C_{A2} - C_{B2}}{(1+r^*)^2} = 0$$

so we find r* by solving for IRR for $C_{At} - C_{Bt}$.

From the graph, we observe that when $r > r^*$, $DCF_A(r^*) > DCF_B(r^*)$ so for these opportunity costs of capital, $NPV_A > NPV_B$. But this does not imply that we should choose project A for all $r > r^*$. In fact, calculating IRR for A yields that when $r > 17.93\%$, $NPV_A < 0$, and it should be rejected. To summarize, when $2.09\% < r < 17.93\%$, project A should be taken.

3. (25 points total)
   General Information: $r = 12\%$, $t = 35\%$, $\pi = 5\%$.

   A. The "REAL" approach – all numbers are REAL and in Thousands

Item/Year	12/31/X0	12/31/X1	12/31/X2	12/31/X3	12/31/X4	12/31/X5
Plant	(10000)					3917.63[a]
Deprec. Tax Shield[1]		333.33	317.46	302.34	287.95	274.23
Revenue		5000.00	5000.00	5000.00	5000.00	5000.00
Cost (Material)		500.00	500.00	500.00	500.00	500.00
Cost (Labor)[2]		600.00	618.00	636.54	655.64	675.30
After-tax Profit[3]		2535.00	2523.30	2511.25	2498.83	2486.06
After-tax Rent Loss	(195.00)	(195.00)	(195.00)	(195.00)	(195.00)	
Cash Flow	(10195)	2673.33	2645.76	2618.59	2591.78	6677.92
DCF[4]	(10195)	2506.25	2325.38	2157.65	2002.09	4836.13
NPV	3632.50					

Notes:
1. To calculate annual (real) deprec. tax savings(DTS), we first calculate the *nominal* figure, $DTS(Nom) = t \times [10000 - 5000] \div 5 = \$350$. Then we convert this number into real by multiplying the Inflation factor, $1/(1+\pi)$, every year.
2. The growth rate 3% is real

3. After-tax profit is given by: (Revenue – Total Cost) · (1-t).
4. The **real** discount factor is given by adjusting the **nominal** factor using the inflation factor:

$$RDF = \frac{1}{(1+NDF)/(1+\pi)} = \frac{1}{(1+12\%)/(1+5\%)} = 0.9375.$$

a. This is the *real salvage value* $= 5000/(1+\pi)^5 = 3917.63$.

B. The "NOMINAL" approach – all numbers are NOMINAL and in Thousands

Item/Year	12/31/X0	12/31/X1	12/31/X2	12/31/X3	12/31/X4	12/31/X5
Plant	(10000)					5000.00
Deprec. Tax Shield[1]		350.00	350.00	350.00	350.00	350.00
Revenue[2]		5250.00	5512.50	5788.13	6077.53	6381.41
Cost (Material)[2]		525.00	551.25	578.81	607.75	638.14
Cost (Labor)[2,3]		630.00	681.35	736.87	796.93	861.88
After-tax Profit		2611.75	2781.94	2907.09	3037.35	3172.90
After-tax Rent Loss	(195.00)	(204.75)	(214.99)	(225.74)	(237.02)	
Cash Flow	(10195)	2807.00	2916.95	3031.35	3150.33	8522.90
DCF[4]	(10195)	2506.25	2325.38	2157.65	2002.09	4836.13
NPV	3632.50					

Notes:
1. The nominal DTS should be a stream of constants. See note A.1. for calculation.
2. All (real) figures need to be adjusted for inflation.
3. The nominal growth factor is given by the product of real growth factor and inflation factor.
4. Use nominal discount factor, 12%.

4. (25 points total)
   a) (5 points)   Given the information on return/risk for each stock, we can calculate the return and SD of the portfolio:

   $$R(P) = \pi_A \cdot r_A + \pi_B \cdot r_B = 0.6 \cdot 20\% + 0.4 \cdot 16\% = 18.4\%;$$

   $$VAR(P) = \pi_A^2 \cdot Var(A) + \pi_B^2 \cdot Var(B) + 2\pi_A \cdot \pi_B \cdot Cov(A,B)$$
   $$= 0.6^2 \cdot 0.25^2 + 0.4^2 \cdot 0.13^2 + 2 \cdot 0.6 \cdot 0.4 \cdot 0.3 \cdot 0.25 \cdot 0.13$$
   $$= 0.0299;$$

   $$SD(P) = \sqrt{VAR(P)} = \sqrt{0.0299} = 17.29\%.$$

b) (8 points)   By the CAPM, we have $r = r_f + \beta(r_M - r_f)$, from which we can solve for β:

$$23\% = 5\% + \beta \cdot (14\% - 5\%) \qquad \rightarrow \qquad \beta = \frac{18\%}{9\%} = 2 = \pi_M.$$

Therefore, Jamie *borrows* $10,000 and invest a total $20,000 into the market portfolio. From the given positions in the stocks, we can calculate the dollar amount invested in each stock:

Stock A: $20,000 \times 20\% = \$4,000$;
Stock B: $20,000 \times 40\% = \$8,000$;
Stock C: $20,000 \times 40\% = \$8,000$.

Finally, his position in riskless asset is <u>short</u> (borrow) $10,000.

c) (12 points)
i) (3 points)   Again, by the CAPM, we have $r = r_f + \beta(r_M - r_f)$. Then the <u>Security Market Line</u> (SML) is given by $r = 0.05 + 0.11\beta$.

Port. A: $8\% > 5\% + 0.2 \cdot 11\% = 7.2\%$,
Port. B: $17\% < 5\% + 1.2 \cdot 11\% = 18.2\%$.

Therefore, Portfolio A lies *above* the SML, while Portfolio B lies *below* it.

ii) (8 points)   The desired portfolio must satisfy the following conditions:
Zero - risk: $\pi_A \cdot \beta_A + \pi_B \cdot \beta_B + \pi_f \cdot \beta_f = 0$;
Zero – investment: $\pi_A + \pi_B + \pi_f = 0$.

We first "guess" that $\pi_f = 1$ (and in dollar amount, <u>long</u> $1), and solve for π_A and π_B , then we check if the expected return on the portfolio is *positive* or not.

By the zero – investment condition, we have $\pi_A + \pi_B = -1$; from the zero – risk condition, we have $\pi_A \cdot 0.2 + \pi_B \cdot 1.2 = 0$. Solving these two equations we have $\pi_A = -1.2$(*short* $1.2), $\pi_B = 0.2$(*long* $0.2). Then the return on this portfolio is:

$$R(P) = \pi_A \cdot r_A + \pi_B \cdot r_B + \pi_f \cdot r_f = -1.2 \times 8\% + 0.2 \times 17\% + 1 \times 5\% = -0.012 < 0!$$

So we should revise our initial "guess" and let $\pi_f = -1$(and in dollar amount, <u>short/borrow</u> $1). Following the same procedure, we have $\pi_A + \pi_B = 1$ and $\pi_A \cdot 0.2 + \pi_B \cdot 1.2 = 0$. Therefore, we obtain $\pi_A = 1.2$(*long* $1.2), $\pi_B = -0.2$(*short* $0.2).

Remark: In fact, from our answers in part i), A is above the SML, while B is below the line, we know that our investment strategy of "Long A" (the undervalued portfolio) and "Short B"(the overvalued portfolio) has to be correct. This also tells us why our initial guess is wrong.

iii)   (1 point)
From the optimal portfolio we constructed in ii), we have:

$$R(P) = \pi_A \cdot r_A + \pi_B \cdot r_B + \pi_f \cdot r_f = 1.2 \times 8\% - 0.2 \times 17\% - 1 \times 5\% = 0.012 > 0.$$

This means, for *every* dollar we borrow, we can earn a profit of \$0.012 with certainty.

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Example Midterm Exam 4

Instructions: You have 1 hour 30 minutes to answer all four questions. All four questions carry equal weight. If anything is unclear, please state carefully any assumptions you make.

1.   Florencia has no funds of her own. However, she can lend and borrow as much as she likes at the bank at 7% interest. She has access to some entrepreneurial projects. The cost of these projects at date $t = 0$ and the revenues they generate at date $t = 1$ are known with certainty and are listed below.

Project	Cost at $t = 0$	Revenue at $t = 1$
A	23,000	24,840
B	125,670	145,777
C	320,000	339,200
D	225,000	270,000

(i)   Which projects should she undertake?

(ii)   What is the most she can consume at $t = 0$ and how much does she need to lend or borrow in total (including the funding of the projects) at the bank to do this?

(iii)   How would your answers to (i)-(ii) be changed if Florencia can lend to the bank at 5% and borrow from the bank at 9%?

2.(a)   Bill and Jane are married with one child. The current date is the start of the college year. All tuition is paid at the start of the year. If their child were to attend college starting now they would have to pay a total of $35,000 now for the year. This cost is expected to grow at 4% per year for the foreseeable future. They anticipate their child will go to college 7 years from now and will attend for four years. They also wish to purchase a house for holidays in the mountains 10 years from now for $350,000. They can lend and borrow as much as they like at 5% per year from their bank. They wish to save S each year with the first deposit now and the last nine years from today. How large should S be in order for them to be able to send their child to college and afford the holiday house in the mountains?

(b)   Shaohuan is valuing the FLR Company. Dividends are paid each year. He anticipates that the first dividend will be $4.30 and will be paid one year from now. This is expected to grow at 25% for the following five years. After that the growth rate is expected to slow down to 15% for the next five years. After that the growth rate of dividends is expected to be 3% forever. What is the value of the stock if the appropriate discount rate is 6%?

3.   The PYW Corporation is a trucking company that moves household goods long distance. It is thinking of replacing part of its current fleet of trucks. These are currently all Volvos. The firm has narrowed the choice to two models. Both are capable of carrying the same amount of goods. The first is one made by Volvo. This model costs $150,000 (nominal) initially. It is expected to have a life of 5 years with a salvage value of $20,000 (real). The maintenance costs are expected to be $20,000 (real) before tax each year.   The second model is one made by Mercedes. This costs $250,000 (nominal) but lasts for 8 years with a salvage value of $35,000 (real). The maintenance costs are expected to be $25,000 (real) before tax each year. The firm uses straight line depreciation and has a total tax rate of 40%. It is anticipated that it will be profitable for the foreseeable future. The firm has a discount rate for this type of project of 12% (nominal). The rate of inflation is expected to be 4% per year. Assume all cash flows occur at the end of each year except where otherwise stated. Which type of trucks should be purchased assuming that similar trucks will be available in the foreseeable future?

4.   (The two parts are separate so you cannot use information from one in the other.)

(a)   Portfolio A has 20 stocks in it and is well diversified. All stocks have equal weights in the portfolio. The first five stocks have a beta of 0.6. The next five stocks have a beta of 0.8. The remaining 10 stocks have a beta of 1.2. The standard deviation of the market portfolio is 25%. What is the standard deviation of portfolio A?

(b)   Suppose the assumptions of the CAPM are satisfied. You have discovered four well-diversified portfolios with no unique risk.   They have the following characteristics. Three of them are correctly priced while one is mispriced.

Portfolio	Expected Return	Beta
A	17%	1.2
B	20%	1.5
C	24%	1.7
D	29%	2.4

(i) Which three portfolios are correctly priced and which is mispriced?
(ii) Give a zero-investment, zero-risk portfolio with positive expected return that has either +$1 or -$1 invested in the mispriced security and which excludes the correctly priced security with the highest expected return.
(iii) What is the expected return on this portfolio?

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Example Midterm Exam 5

Instructions: You have 1 hour 30 minutes to answer all four questions. All four questions carry equal weight. If anything is unclear, please state carefully any assumptions you make.

1.(a)   Consider a world with three dates t = 0, 1, and 2, perfect capital markets and no uncertainty. Haiman has just inherited $0.5 million. She has decided to set up a corporation called *Bubble.com*. This will manufacture and sell kits of liquid soap and wire frames that will allow children and adults to blow soap bubbles. These kits will be marketed over the internet. Suppose initially she can lend and borrow at the bank at 6 percent per period. *Bubble.com* has the following projects available.

Project	Cost at t = 0	Revenue at t = 1	Revenue at t = 2
Large bubble kit	$325,000	$130,000	$260,000
Medium bubble kit	$255,000	$128,000	$140,000
Small bubble kit	$430,000	$220,000	$270,000

(i)   What simple instructions can Haiman give to ensure that the managers she hires create as much wealth as possible for her?

(ii)   Which projects should the managers choose to undertake?

(iii)   What is the most Haiman can consume at date t = 2 given she wishes to consume $0.3 million at date t = 0 and $0.2 million at date t = 1?

(b)   Consider a project with the following cash flows.

C_0	C_1	C_2	C_3
-850	+5,000	-1,400	-1,500

(i)   What, if any, is/are the IRR(s) of this project between 0 and 50 percent?

(ii)   What is the NPV of the project at 10 percent?

2.(a)   Srikant is planning his retirement. The rate of interest that he can lend and borrow at the bank is 6 percent. He would like to retire 20 years from now. He currently has $125,000 in the bank. He intends to buy a car 3 years from now. He estimates it will cost $55,000 then. He would like to buy his mother a house 10 years from now. He estimates it will cost $230,000 then. Srikant wants to have a fixed pension of $100,000 a year with the first payment being 21 years from now and the last being 40 years from now. What is the constant amount he needs to save each year assuming the first time he puts away money is 1 year from now and the last time is 20 years from now?

(b)   Sandy is valuing the Halloween Company. It is just about to pay a dividend of $10 per share. This dividend is expected to grow at 15 percent per year for five years and then at 10 percent for the following five years. After that it will grow at 4 percent in perpetuity. What is the current value of a share if the opportunity cost of capital for this type of firm is 12 percent?

3.   Over the last five years the Billagong Company has spent $25 million developing a new product called bings. It is considering whether to build a plant in California that will manufacture bings. The current date is 12/31/X0. The cost of the plant is $10 million to be paid now. It will take one year to build. The plant will start producing on 1/1/X2 and the first revenues and costs will be received and paid on 12/31/X2. The plant is expected to produce for three years. It will produce 3 million bings a year. The plant can be depreciated over the three years it is in production. The plant has zero salvage value. The company can sell each bing for $5 and the raw materials will cost $2 per bing in all three years bings are produced. The total labor costs for the first year of production will be $1.5 million (i.e. on 12/31/X2) and these are expected to grow at a rate of 4 percent per year. The land the plant will be built on could be rented out for $500,000 a year with the rent being paid each year at the beginning of the year. The firm already owns some of the machines that it will use for producing the bings. These cost $1 million ten years ago. The $10 million cost of the plant mentioned initially does not include the cost of these machines. These machines are currently fully depreciated and have no market value at all. All figures are in nominal terms and are stated in before tax terms unless otherwise indicated. The firm uses straight-line depreciation and has a tax rate of 35 percent. It has profitable ongoing operations and an opportunity cost of capital for this type of project of 10 percent. What is the NPV of the plant?

4.   (Parts (a), (b) and (c) are separate so you cannot use information from one in any other.)

(a)   Suppose the capital asset pricing model holds. The market portfolio has an expected return of 15 percent and a standard deviation of 20 percent. The risk free rate is 5 percent.

        (i) How would you construct a portfolio on the capital market line (i.e. consisting of the market portfolio and the risk free asset) with a standard deviation of 65 percent?

        (ii) What is the expected return on this portfolio?

(b)   Suppose there are only two risky firms in an economy. The total value of the shares of firm A is $5 billion and the total value of the shares of firm B is $7 billion. For firm A the standard deviation of the returns on its shares is 20 percent and the expected return is 12 percent. For firm B the standard deviation is 30 percent and the expected return is 16 percent. There is also a risk free asset with a rate of return of 5 percent and total value of $5 billion. The assumptions of the CAPM are satisfied. What is the expected return on a portfolio consisting of 50 percent in the risk free asset and fifty percent in the market portfolio?

(c)   Suppose the assumptions of the CAPM are satisfied.   You have discovered three well-diversified portfolios with no unique risk.   They have the following characteristics.

Portfolio	Expected Return	Beta
A	6%	0
B	14%	0.8
C	22%	1.2

        (i) Can these three investments lie on the security market line?
        (ii) Give a zero-investment, zero-risk portfolio with positive expected return that has either +$1 or -$1 invested in portfolio A.
        (iii) What is the expected payoff on this portfolio?

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UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

LECTURE NOTES

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

QUARTER 1 - WEEK 1

Th: 9/8/11

Section 1: Introduction

Read Chapter 1 BMA

Purpose of the Course

The purpose of the course is to give you a framework for thinking about how a firm

should make investment and financing decisions to create value for its shareholders. In order to

do this we will not only need to look at the firm but also to consider how financial markets work

and how investors in those markets should make decisions. By the end of the course you should

have a framework for thinking about business problems.

Reading List

The reading list contains the details of the course.

Study Methods

The material for this course is not the sort that most can absorb easily, especially to start with.

You'll find it helpful at the beginning of the course to read the chapters from the book before we

go through it in class. That way, hopefully, you'll get more from the lectures. As the term

progresses you may find you don't need to do this as much, but at least to start with you should

do so. After we've been through the material in class you should again read the lecture notes and

relevant chapters. You should make sure you understand them, and this may require reading the

material slowly a number of times--it may be necessary to reread certain passages over and over.

You should not get depressed if you can't understand it the first time through. The thing to do is

to try to understand each individual step in an argument one at a time, and then once you've done

that, you'll find you have a better grasp of the idea as a whole. One thing you may find helpful is

to form study groups and discuss the material together.

Once you've been to class and read through the material, the only way you can be sure you understand it is to do the problems. Practically every week we will be doing a problem set or case to be handed in and graded. The problem sets will consist of questions to make sure you have read the material and questions from old exams so that when you get to the midterm and final, you'll know what to expect. It's very important that you do the assignments. The problem sets are the minimum number of problems you should do. If you want to do well in this course, you should also look at the problems in the backs of the chapters. The special solutions manual that came shrunk wrap with the book contains fully worked out solutions and you should go through these, preferably after you have attempted the questions.

For those of you who do not have much previous experience of finance I would recommend buying Barron's Dictionary of Financial Terms. There is also a glossary at the end of the book in Brealey, Myers, and Allen that explains many terms.

It's up to you whether you bring lecture notes to class. Some people like to make notes to themselves in the margins. The notes are what I lecture from so some people like to be able to look back if they missed something. Other people like to sit back and listen knowing that they don't have to take notes themselves. Sometimes I'll ask questions. Don't look down at the notes to try and find the answer – try to think it through. If you find yourself getting bored in class or falling asleep try leaving the notes at home and start taking them yourself. This might not be the answer but it's worth a try.

I hope you'll ask questions during lectures. If you don't understand something, it's likely there are a lot of other people in the room who also haven't understood it. If after you've studied

the material and put some time and effort in you still don't understand something, come and see

me or one of the TA's. We are here to help you.

I am going to start by covering some of the background knowledge that you'll find

helpful for the course. I will review some basic material on microeconomics, algebra and

statistics. The course will be much easier to understand if you know this background material.

Review of Background Material

Microeconomics

You need to know about indifference curves and utility maximization subject to a budget

constraint.

If we have two commodities, apples and bananas say, we can represent a person's

preference for these commodities by indifference curves
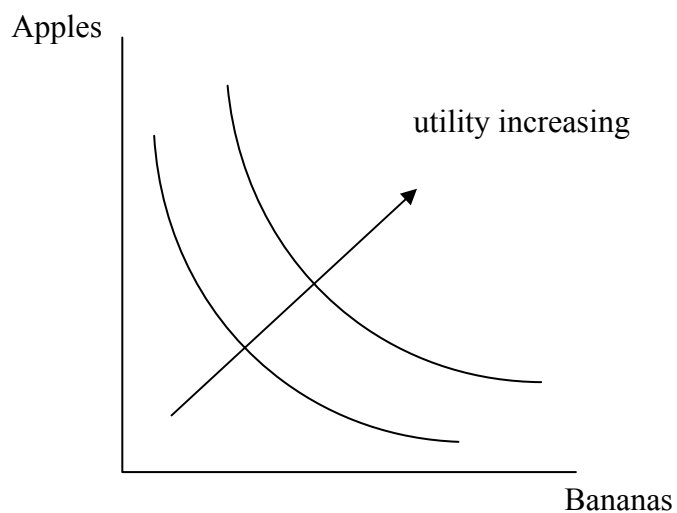
An indifference curve is the locus of combinations of apples and bananas such that the person is indifferent. It is assumed more is preferred to less, so moving in a northeasterly direction utility is increasing.

We are all constrained by a budget constraint: we can't spend more than our income on apples and bananas. It must be the case that

$$P_A A + P_B B \leq I$$

where P_A, P_B are the prices of apples and bananas respectively, A and B are the quantities purchased, and I is income.

This can be represented on our usual diagram.

$$A = -\frac{P_B}{P_A} B + \frac{I}{P_A}$$
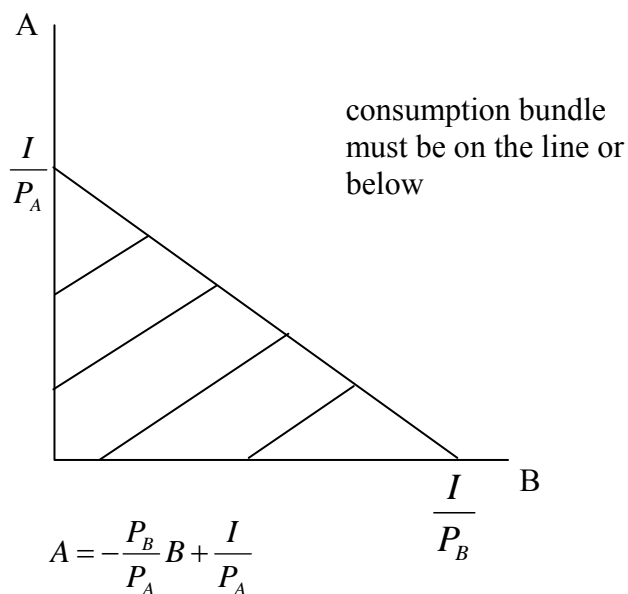
If we add indifference curves to this diagram of the budget constraint we can use it to find the combination of A and B a person will choose if he maximizes his utility.
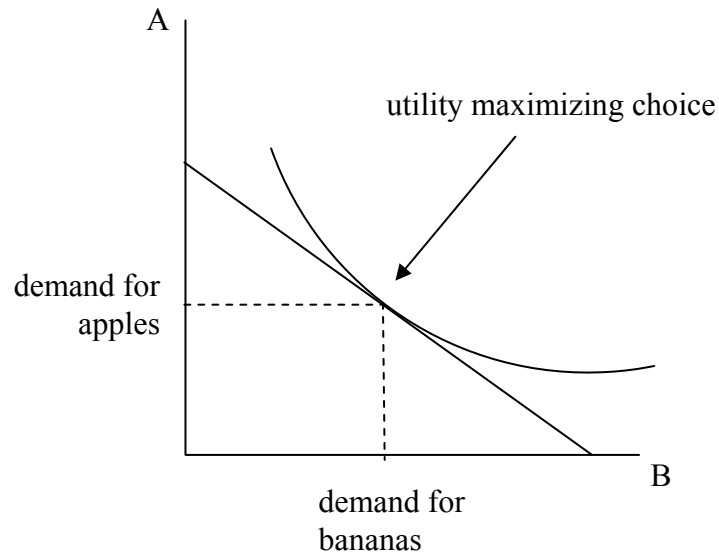
In the next section we will be using these concepts a lot, so if you are at all shaky on them, you should review them as soon as possible.

Future Values and Present Values

One of the most important ideas in finance is present value. If you put $1 in the bank today at 10% interest, then in a year's time you'll have $1.10. This involves going forward through time.

$$FV = 1 \times 1.10 = \$1.10$$

An equivalent notion is to go backwards through time and say that the present value of $1.10 in one year's time if the interest rate is 10% is $1 now, i.e.,

$$PV = \frac{1.10}{1 + 0.10} = \$1$$

In general, C_1 one year from now if the interest rate is r has

$$PV = \frac{C_1}{1+r}$$

Suppose you put $1 in the bank today at 10% interest, then in two years time you'll have

$$FV = \$1 \times 1.10^2 = \$1.21$$

Alternatively we can say that the present value of $1.21 in two year's time if the interest rate is 10% is $1 now, i.e.,

$$PV = \frac{1.21}{(1+0.10)^2} = \$1$$

In general, C_2 two years from now if the interest rate is r has

$$PV = \frac{C_2}{(1+r)^2}$$

Similarly, C_t t years from now has

$$PV = \frac{C_t}{(1+r)^t}$$

Suppose next we have a stream of cash flows $C_1, C_2, C_3, \ldots, C_T$. We can represent this on a time line:

Date	0	1	2	3	…	T

Cash Flows: C_1 C_2 C_3 C_T
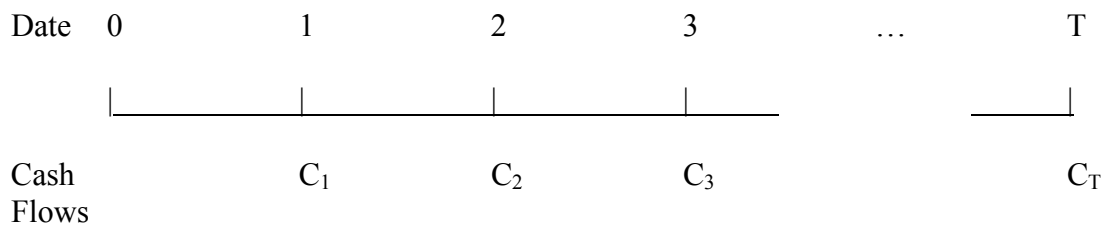
The equivalent in terms of today's money of this stream of cash flows is

$$PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \ldots + \frac{C_T}{(1+r)^T}$$

Summing Geometric Series: Perpetuities

Fairly soon we'll be talking a lot about PV's, and one of the things it will be useful to know is how to sum a geometric series.

A special case of a sequence of cash flows that is of interest is when the cash flow at every date is the same, i.e. $C_t = C$ for every date t and this stream goes on forever. This is called a perpetuity.

Date	0	1	2	3	…	∞

Cash Flows

| | | C | C | C | … | |

(1)

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + ...$$

Multiplying (1) by $(1 + r)$ gives:

(2)

$$(1 + r)PV = C + \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + ...$$

Subtracting (1) from (2):

$$rPV = C$$

$$PV = \frac{C}{r}$$

As an example, suppose C = $20 and r = 0.10 then PV = 20/0.10 = $200.

Another way to see what is going on here is to consider the case where you put $200 in the bank at 10% forever and receive $20 per year forever. A stream of $20 forever is therefore equivalent to $200 today. In general, if you put PV in the bank you receive C = rPV forever so PV = C/r.

Now that's the last time I'm going to go through the algebra of that or any other derivation of a formula on the board. In the future I'll just refer you back to this lecture for perpetuities and give you the general outline for other cases. However at this point you should memorize the fact that for perpetuities you just divide by the interest rate. There will be derivations of formulas in the appendix for Section 3 and in the solutions to Problem Set 2. If at any point you find yourself thinking where does that formula come from you can check it out.

Statistics

We won't be using the statistics for some time and when we get to the point we do need it, I'll review it. For those of you who want some more time to refresh your memory, I'll just list the basic concepts you'll need to understand:

Random Variable

Probability

Expectation

The Mean or Average

Variance

Standard deviation

Covariance

Correlation

That's the basic background knowledge you'll need. If you are at all doubtful about any of the concepts, review them soon.

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UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

LECTURE NOTES

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

QUARTER 1 - WEEKS 1 AND 2

Th: 9/08/11 and Tu: 9/13/11

PART II. INVESTMENT DECISIONS

Section 2: The Objective Function for Corporations

Read Chapter 1 in BMA

Motivation Example

Suppose you are at a Ford shareholders' meeting. Three of the shareholders there have very different ideas about what the firm should do.

Old lady:	Wants money now. Wants Ford to invest in small cars and hybrids since this would yield a quick profit.
Little Boy's trust fund representative	Wants money a long way in the future. Wants Ford to invest in fuel cell technology for electric cars.
Pension fund	Wants money in the medium term. Thinks oil prices will return to the levels of the 1990's and early 2000's some time in that period. Recommends that Ford invest in large cars.

What should Ford do?

We shall soon be in a position to answer this question.

Net Present Value

Suppose that a firm has a project. To keep it very simple, assume the project has a cost now and generates a (net) revenue one year from now. The net present value or the NPV as it is known is net amount generated by the project all evaluated in terms of today's dollars.

$$NPV = - Cost + PV(Revenue)$$

We can think of cost as a negative cash flow and denote it C_0 since it occurs now, at time 0, i.e.,

$$C_0 = - \text{Cost}$$

It's important to note that C_0 is a negative number usually. If something costs \$2M, then C_0 =

-2M.

Also suppose the revenue one year from now is C_1

$$PV(\text{Re venue}) = \frac{C_1}{1+r}$$

Substituting we have

$$NPV = C_0 + \frac{C_1}{1+r}$$

What NPV is measuring is the net amount of wealth in terms of dollars today that the project

creates. If the PV of revenues is greater than cost then wealth is created.

Now, the NPV rule states that if a project has a positive NPV, we should accept it; if it's

negative we should reject it.

NPV Rule: Accept project if NPV > 0.

For example, suppose $C_0 = - 90$, $C_1 = 110$, and r = 10%. Should we accept? Yes, since

NPV = +10. Suppose $C_0 = - 105$. Should we accept? No, since NPV = -5. Using the NPV rule

always leads to an increase in NPV and hence is equivalent to maximizing NPV.

In this section we're going to look at the NPV rule for an individual. Then we'll show

why it's a valid rule for a corporation even though the corporation may be owned by many

different types of people. We're going to do this by considering a simple example.

Example

Bill Ross has inherited $1M. He has developed a real aversion to work, which he completely detests. He therefore plans to use his inheritance to finance himself for the rest of his life. For simplicity we'll divide his life into two periods, youth and old age. Also, we're going to assume that there is only one financial institution, a bank, which lends and borrows at a rate of 20%, so that for every dollar deposited in youth $1.20 is received in old age.

Bank Alone

Assuming this bank is the only opportunity open to Bill, what can he do?
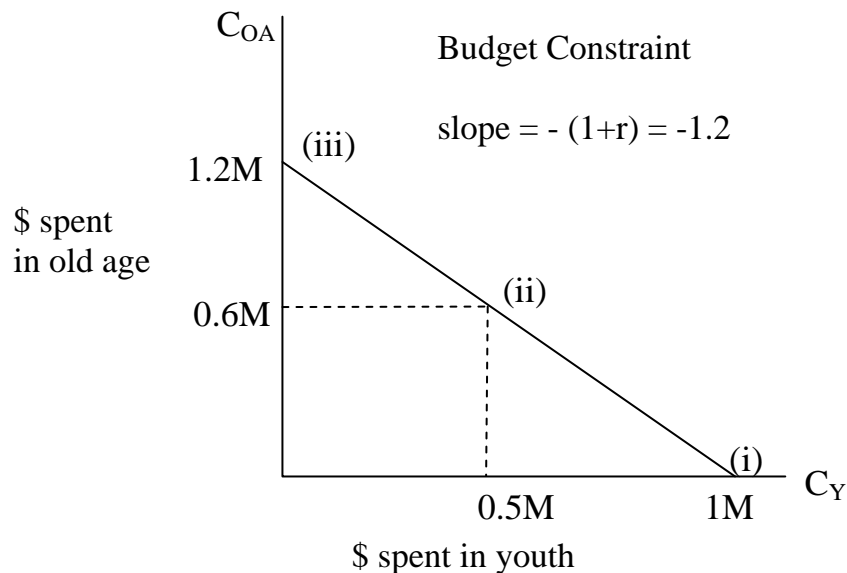
(i)   He could go on a fantastic trip around the world, spend the whole $1M, and then live in poverty with nothing for his old age. He is at ($1M, 0) on the diagram.

(ii)   He could spend $0.5M in his youth, have a moderate lifestyle, put $0.5M in the bank, and still have $0.6M for his old age. He is at ($0.5M, $0.6M).

(iii) He could put all his money in the bank for his old age and spend nothing in his youth, so that he can take an even better trip around the world in his

old age. In this case he gets (0, $1.2M).

(iv)   If we consider all the other possibilities, we get a straight line between (i) and

(iii).

He can thus consume anywhere on the straight line between $1M in youth and $1.2M in

old age. Analytically we have

$$C_{OA} = 1.2M - 1.2 C_Y$$

The basic point is that the bank allows him to transform $1 today into $1.20 next period

and vice-versa. He can move up or down the line by putting more or less money in the bank.

Projects Alone

We're next going to look at the case where there is no bank and there are only productive

opportunities or projects that allow him to transfer wealth from his youth to his old age.

Bill fancies himself as an entrepreneur and sits down to work out what investments he

can make. He ranks them in terms of profitability with the most profitable being first and the

least profitable last.

Project A

Bill is a wine lover. He estimates that a small vineyard that has recently come on the

market will cost him $50,000 now and will yield him $200,000 for his old age. This is the best

project he can think of. Hence, if he invests just $50,000 in this project he can consume $1M -

$50,000 = $950,000 now and he can consume the $200,000 project A generates in his old age.

He is at the point A in the diagram and consumes ($0.95M, $0.2M).
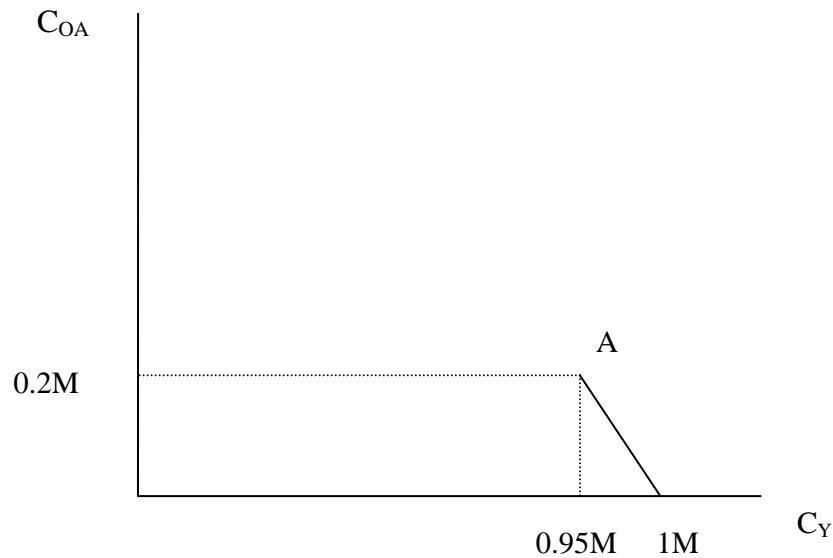
Project B

Bill is also a gourmet. The next best project he can think of is to run a restaurant in the town he lives in. He reckons for a $100,000 outlay now he can get $140,000 in his old age. If he just undertakes this and project A, then he can consume $950,000 - $100,000 = $850,000 now and $200,000 + $140,000 = $340,000 that the two projects generate in his old age. He is at the point A + B in the diagram below and consumes ($0.85M, $0.34M).

C_{OA}

A+B

0.34M

A

0.2M

0.85M   0.95M 1M C_Y
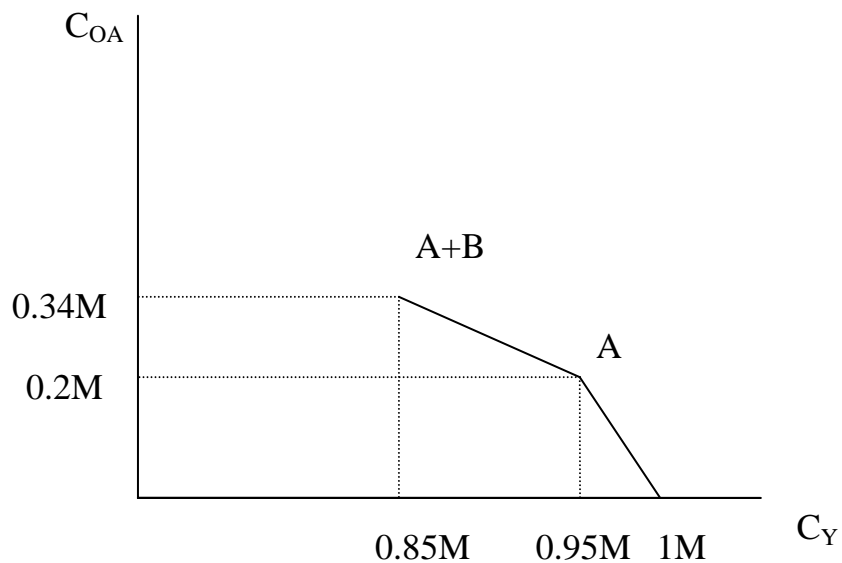
Project C and so on

There are a number of other projects he thinks of. We can trace out a curve, called the

production possibilities curve, to approximate these series of short straight lines as shown below.

With projects alone he can consume anywhere along the curve.

C_{OA}

C

A+B

Production
possibilities
curve

0.34M

A

0.2M

0.85M   0.95M   1 C_Y
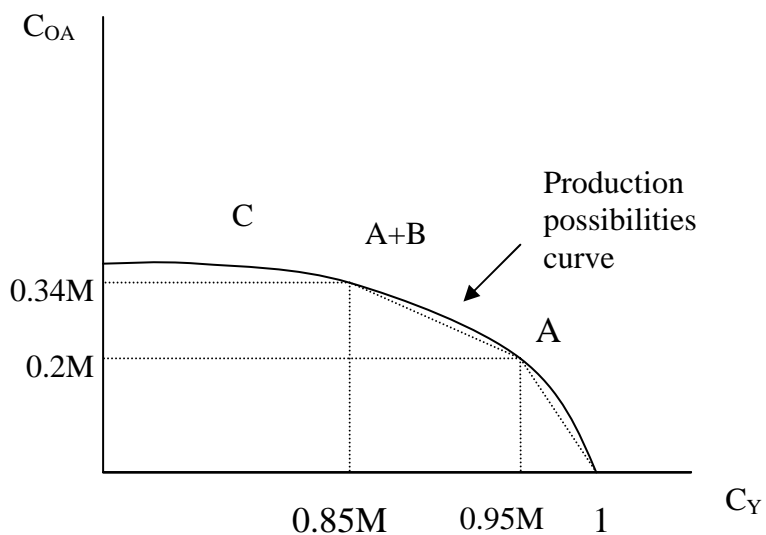
Projects and the Bank

We now consider what he can do if we take account of both his projects and the possibility of borrowing and lending at the bank.

Project A

Suppose he just undertakes the first project, A. He has $950,000 now and $200,000 later on--what can he do?

(i)   He could simply consume $950,000 now and $200,000 later on without going to the bank. He consumes at ($0.95M, $0.2M).

(ii)   Alternatively, he can use all the money he receives to have a marvelous time. He has $950,000 now and $200,000 later on, which he can use to repay a loan that he spends now. How much can he borrow to repay with the $200,000 later on; i.e., what is the present value of $200,000 at 20%?

$$PV(200{,}000_{OA} \text{ at } 20\%) = \frac{200{,}000}{1.20} = 166{,}667$$

Hence,

Total possible in consumption youth = 950,000 + 166,667

= $1.1167M
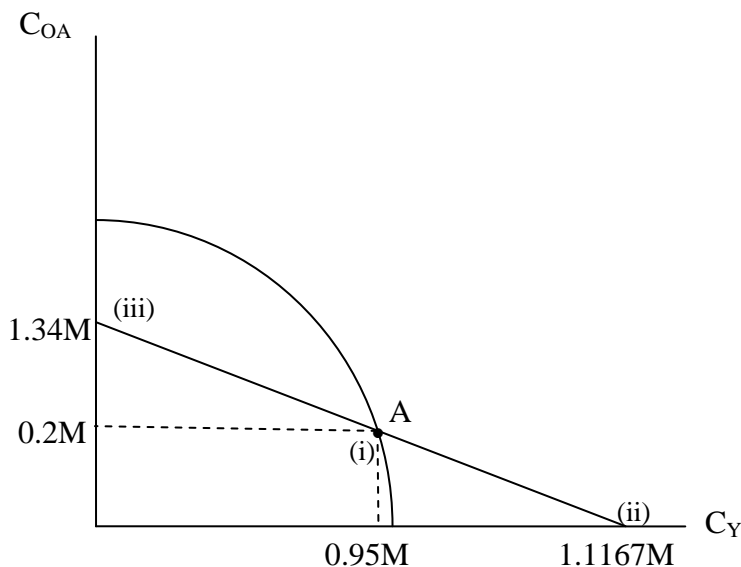
(iii)   Alternatively he can plan to spend it all next period.

$$\text{Total in old age} = 1.2 \times 950{,}000 + 200{,}000$$

$$= \$1.34M$$

As before, we can go on doing this until we trace out a straight-line budget constraint as before.

The equation for this line is

$$C_{OA} = 1.34M - 1.2\, C_Y$$

The slope is again - 1.2 since the interest rate is 20%. This follows from the structure of the

problem since all values in Old Age are multiplied by 1.20 compared to their values in Youth.

The bank always allows Bill to move up and down a straight line with slope -1.2.

Project B

Now suppose he undertakes the first and second projects, A and B, so that he produces at

A + B. We can go through the same calculations again and get another straight line representing

his consumption possibilities.
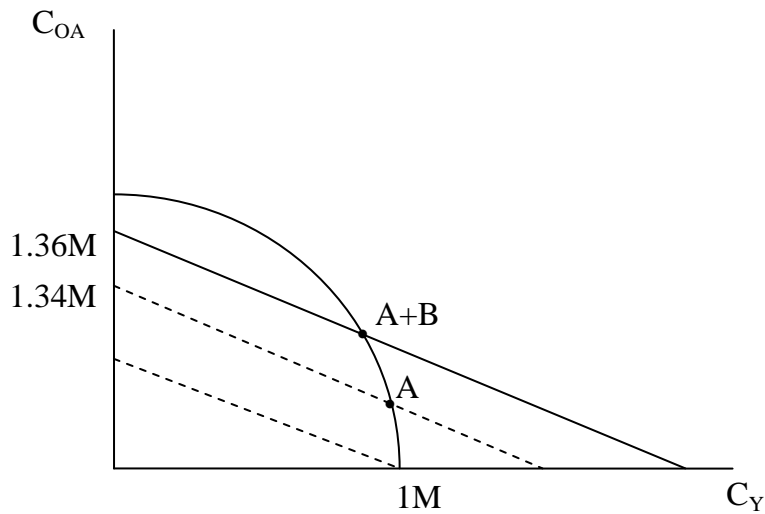
In this case,

Intercept on C_{OA} axis $= 1.2 \times 850{,}000 + 340{,}000 = 1.36$ M.

Hence, analytically the budget constraint is given by

$$C_{OA} = 1.36 \text{ M} - 1.2\, C_{Y}$$

We can see that by undertaking the first project he can push out the line representing his possible consumption, similarly when he undertakes the second project, and so on.

If he prefers more money to less, he is better off if his budget constraint is pushed out further since this allows him to consume more in both periods. Hence no matter what his preferences are, he is better off with a budget constraint that is farther out.

To see this we can represent preferences in this diagram by an indifference curve. This is the locus of combinations of C_{OA} and C_{Y} such that he is indifferent.

C_{OA}

utility increasing

C_Y
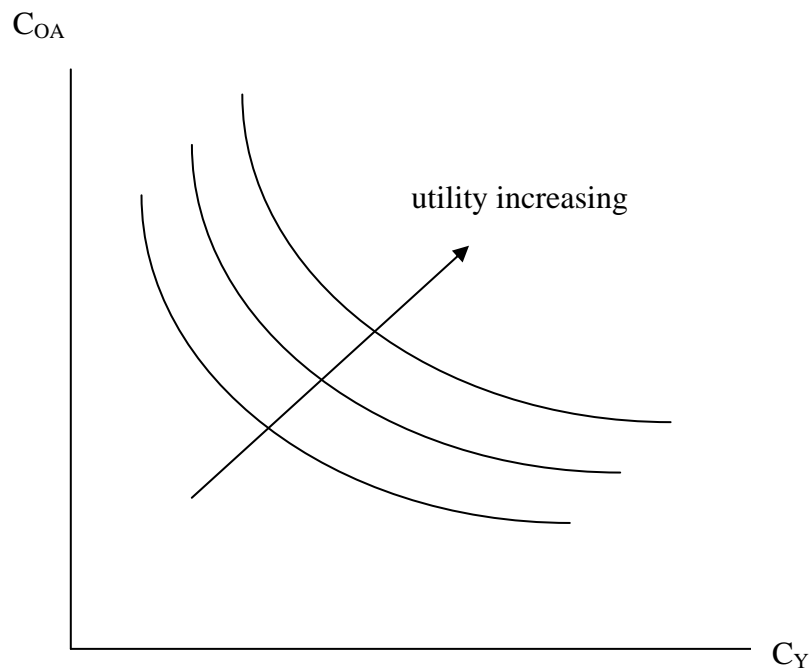
We can represent differences in preferences for consumption in old age and youth by differences in the shape of the indifference curves. Suppose, for example, somebody has a strong preference for consumption in old age. In other words, the person is a miser. Then her indifference curves will look something like this:

C_{OA}

Miser

C_Y
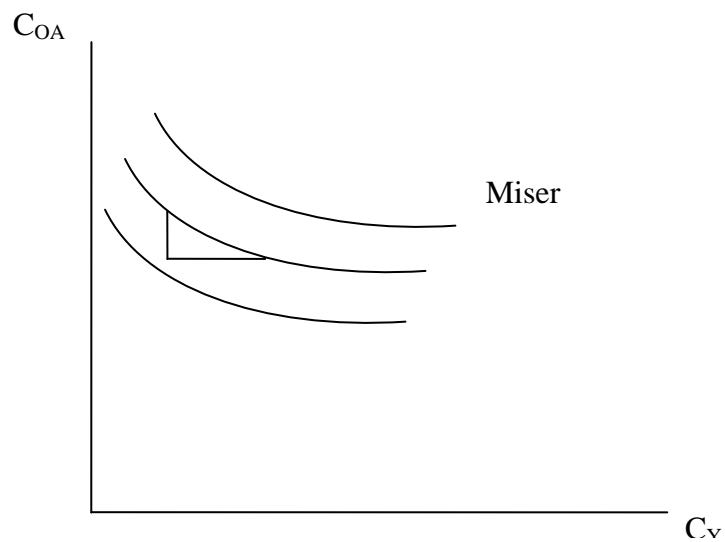
Her curves are flat for the following reason. Since she is a miser a small reduction in consumption in old age must be compensated for by a large increase in youth to make her indifferent.

Somebody who prefers consumption in his youth, who we shall call a spender, will have the type of indifference curves below. They are steep because a large reduction in consumption in old age can be compensated for by a small increase in youth to make him indifferent.

C_{OA}

Spender

C_V
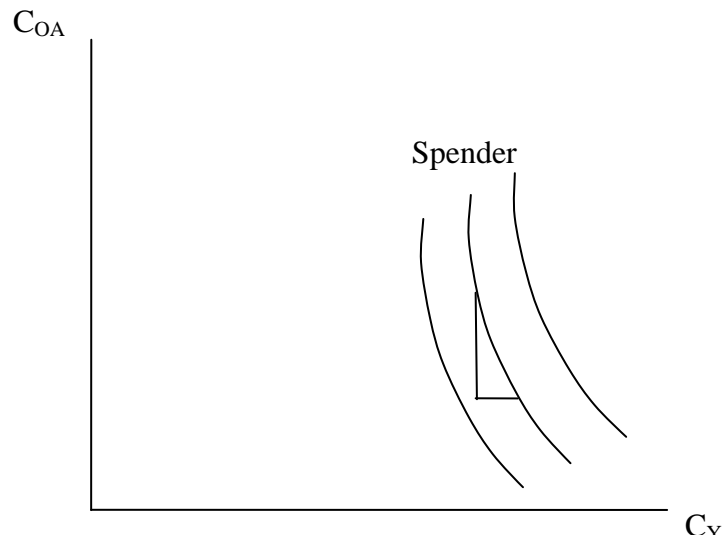
Now if we put budget constraints on these diagrams we can see that pushing a budget constraint out makes both better off.

C_{OA}

C_V
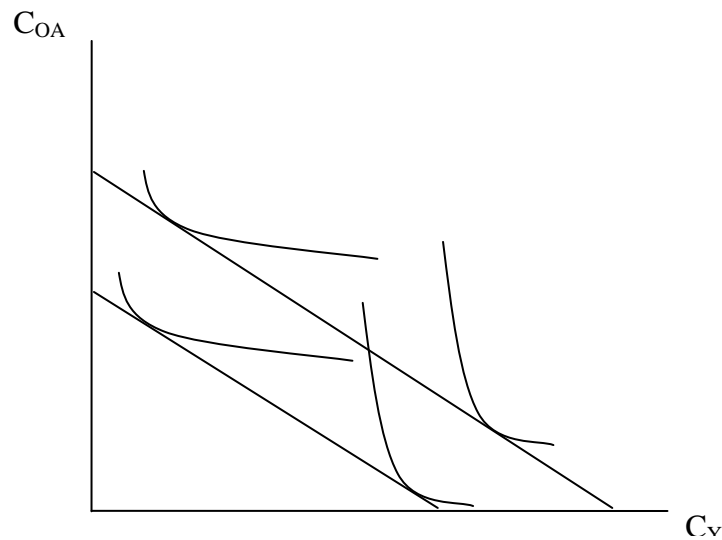
Hence no matter what Bill Ross' preferences are, he must be made better off if the budget

constraint is pushed out. Clearly the best that he can do is therefore to go on doing projects until

he can push out his budget constraint no farther. This is true no matter what his preferences are.

C_{OA}

C_V

Level of investment
he should undertake
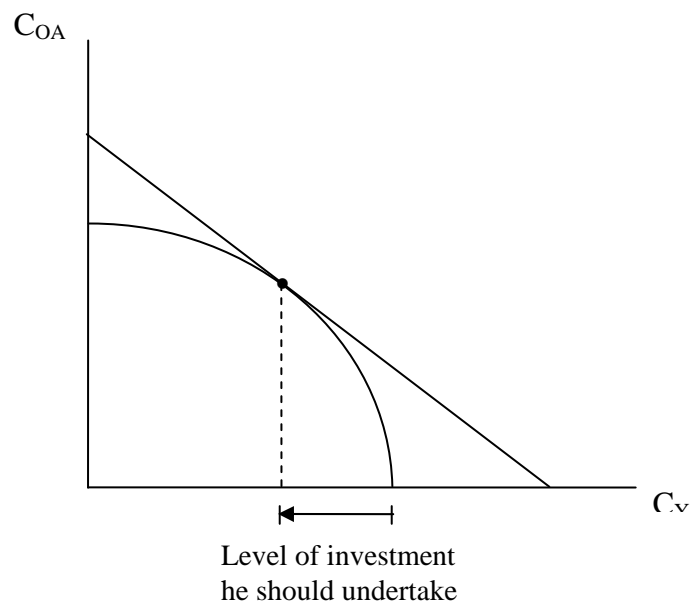
The crucial point here is that the level of investment he should undertake does not depend on his

preferences. This implies that his consumption and production decisions are separate. Let's look

at a couple of examples to illustrate how somebody actually maximizes his utility given this

separation.

In order to understand these examples, it is helpful just to review the mechanics of these diagrams. In the previous section we talked about the concept of present value and future value. If you have $100 now and the interest rate is 10% per year, then you can invest the $100 and get $110 in a year's time. The future value of $100 in 1 year's time at 10% is $110. In general

$$C_1 = (1 + r)PV \qquad \text{FV formula}$$

Another way of saying the same thing is that the present value of $110 in 1 year's time if the interest rate is 10% is $100. In general

$$PV = \frac{C_1}{1 + r} \qquad \text{PV formula}$$

OPP

ADJ
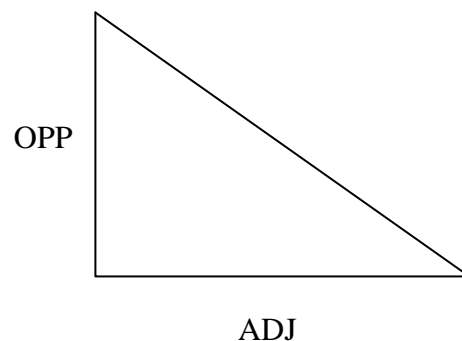
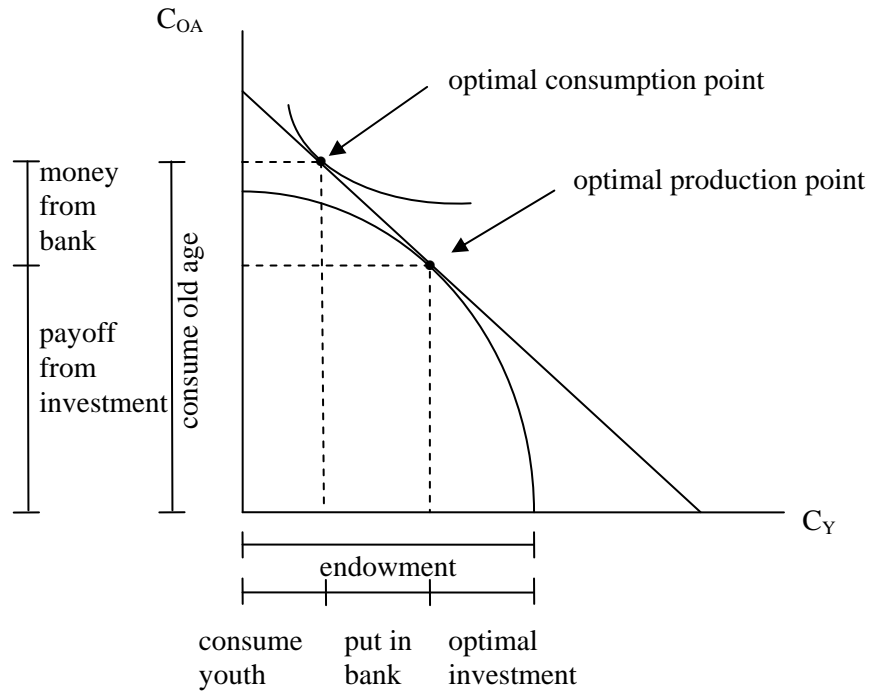$$\frac{OPP}{ADJ} = - \text{Slope} = 1 + r$$

$$OPP = (1 + r)ADJ \qquad \text{FV FORMULA}$$
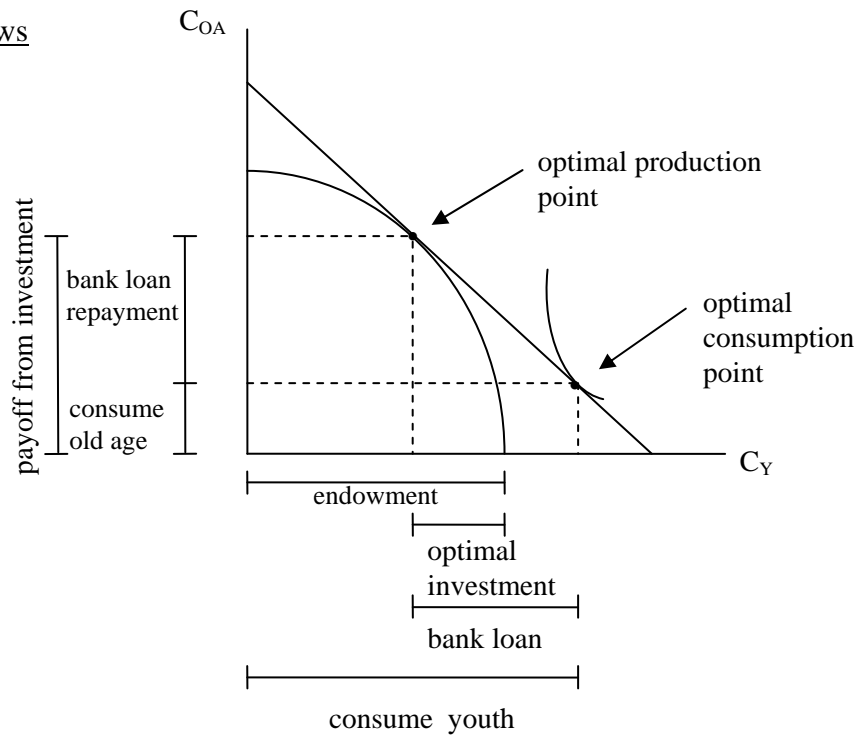
$$ADJ = \frac{OPP}{1 + r} \qquad \text{PV FORMULA}$$

Hence if we have a triangle on the budget constraint in the diagram, the vertical is a future value and the horizontal is the present value. Now let's look at the examples.
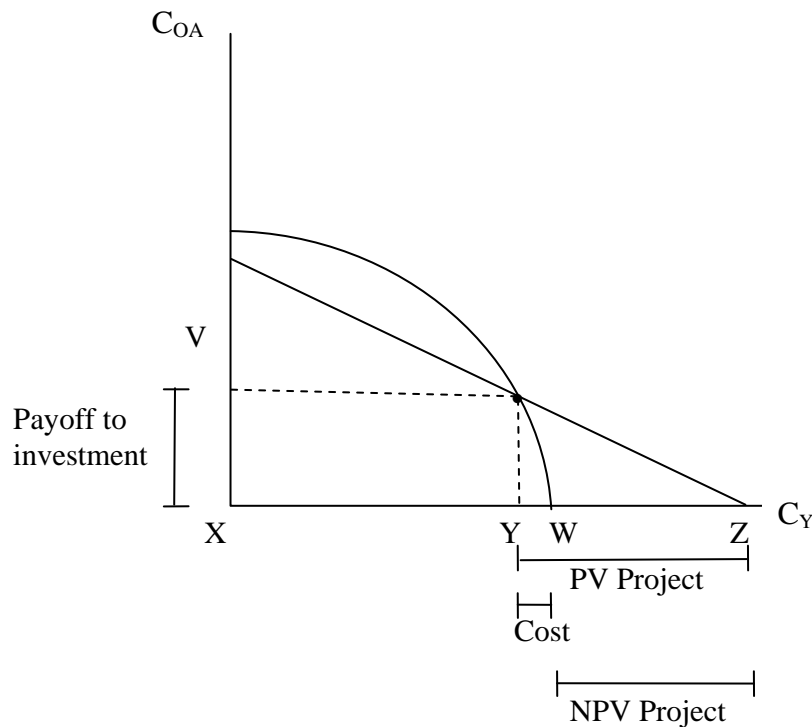
Miser - Lends

Spender – Borrows

Equivalence of Pushing Out Line to Maximization of NPV

What we're going to show next is that pushing out the budget line is equivalent to the maximization of NPV.

The thing to remember is that vertical distances are future values; horizontal distances are PV's. What is XV? It is the payoff to the investment. What therefore is YZ? It's the PV of the project. What's YW? It's the cost. Thus, what is WZ? It's the NPV of the project since NPV = PV - Cost. Hence the distance between the intercept for the production possibilities curve and the intercept on the budget constraint is NPV. Making this as large as possible allows a person to consume more in both periods, and she should do it.

Pushing the line out as far as possible is thus equivalent to making the distance WZ as large as possible and hence is equivalent to maximizing NPV.

One important thing to notice from the diagram is that NPV is a monetary measure of how much better off Bill Ross is from undertaking the investment. It measures the increase in

his wealth from the project. NPV is a measure of wealth creation.

Another important thing here is that the discount rate used to find the NPV is the

opportunity cost of capital. If Bill Ross doesn't do the project the best alternative use of his

money is to put it in the bank. We use this best alternative rate to discount so we can find out

which is the best thing to do.

The bank (or more generally the financial markets) enables people to choose their desired

consumption. The firm's role is to push out the line as far as possible.
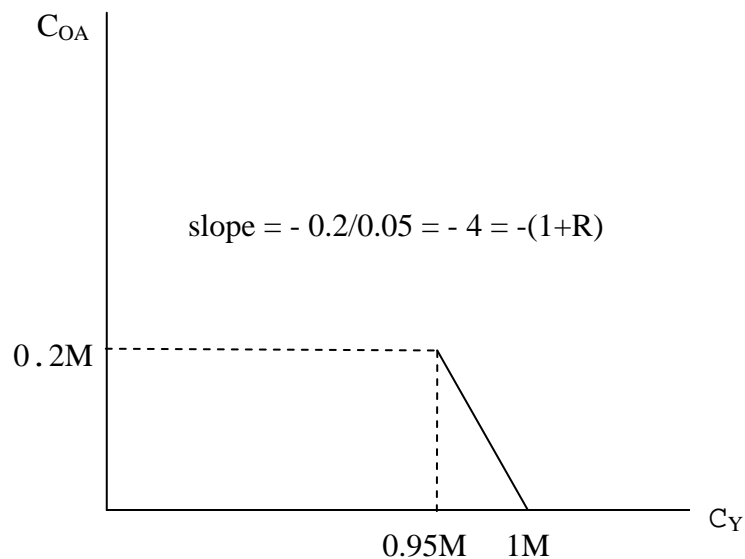
Rate of Return

Another way of looking at the problem of what the person should do is in terms of the

rate of return.

Consider Project A. The rate of return, R, this project earns is given by

$$1 + R = \frac{200,000}{50,000} = 4$$

Hence, $R = 3$ or 300%.

We can represent $1 + R$ by the slope of the line between A and 1M on the C_Y axis:
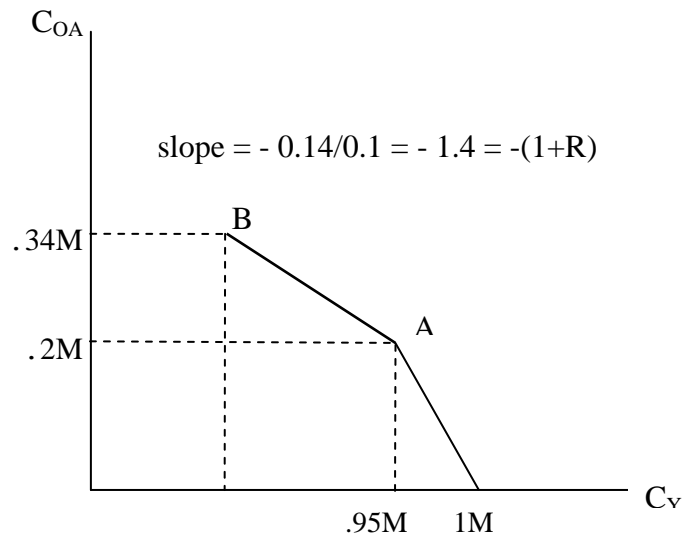
slope = - 0.2/0.05 = - 4 = -(1+R)
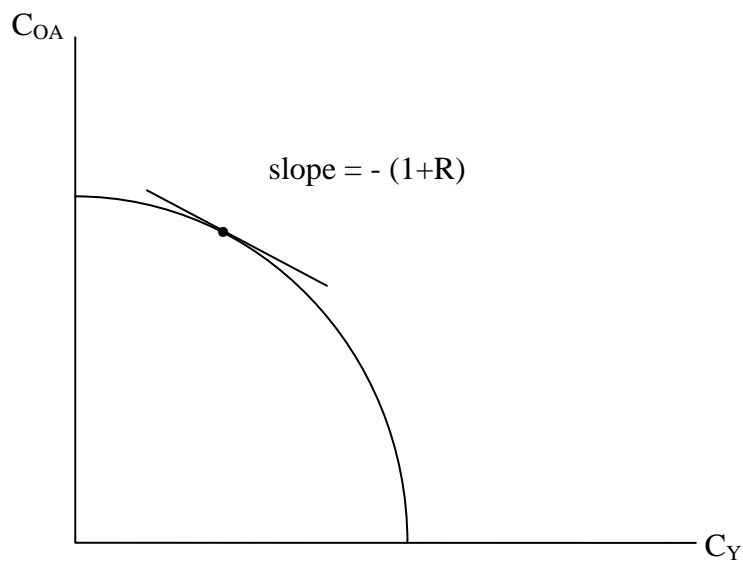
With project B the rate of return is given by

$$1 + R = \frac{140,000}{100,000} = 1.4$$

$$R = 0.4 \text{ or } 40\%.$$

Graphically, 1+R is again the slope of the line between B and A.


In the more general case where we have the curve representing the production

possibilities, the rate of return is represented by the slope of the curve.

Intuitively, it is worthwhile undertaking a project if the rate of return on it is above the rate of interest at which you can borrow or lend at the bank.

We can then see that this is equivalent to the rule we have just derived.
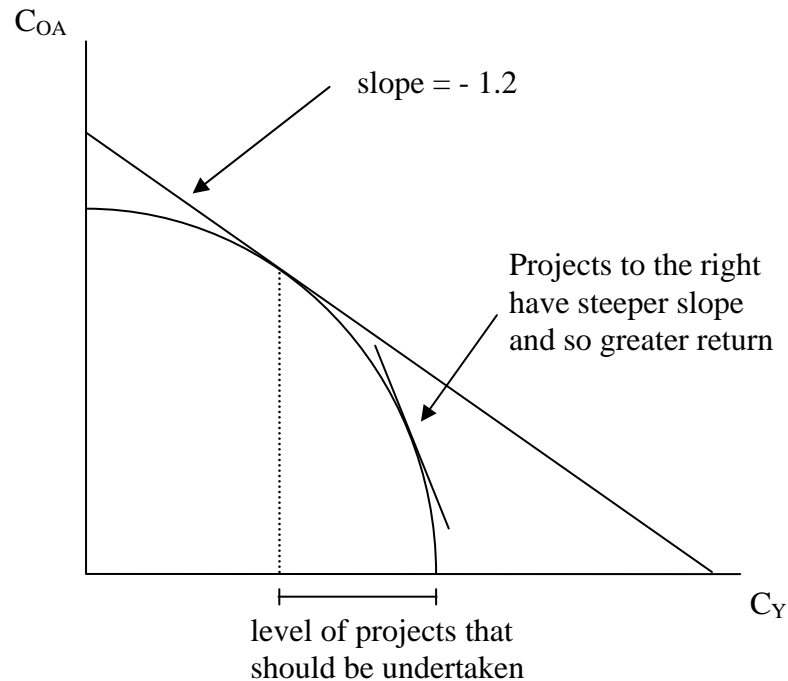
We therefore have two ways of choosing the optimal amount of investment and we have seen these are equivalent.

1. Net Present Value

    Invest so as to maximize the NPV of the investment. This is the difference between the discounted present value of the future income and the amount of the initial investment.

2. Rate of Return Rule

    Invest up to the point at which the rate of return on the investment is equal to the rate of return on alternative investments (in our case this was the bank).

Rules for Managers of Corporations

We have derived these as the rules for an individual to follow when choosing his

investments in real assets. But are they relevant for the type of firm we are interested in where
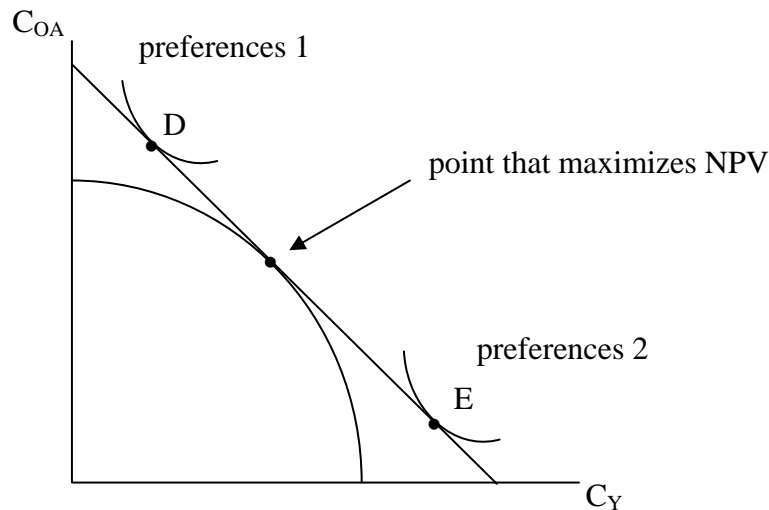
(i)   ownership and control are separated;

(ii)   there are many shareholders who may be very different?

To answer this question, we go back to our example. Here and in fact throughout Part II

we will be assuming that firms are all equity financed. Thus when we talk about borrowing and

lending this refers to shareholders borrowing and lending rather than corporations borrowing and

lending. We consider firms' borrowing decisions later in the course.

Bill Ross has been thinking a bit about the implementation of these projects and decides

they involve more work than he first thought. He therefore decides to hire managers to

implement the investment in real assets. In order to ensure that they pursue his interests all he

has to do is to tell them to *maximize NPV*, and this will ensure that he can be as well off as

possible.

The important point to note here is:

The decision on how to allocate his consumption between youth and old age is independent of

the need to maximize the NPV of the firm.

No matter whether he has preferences 1 or 2, he will produce at the point that maximizes

NPV. With preferences 1 he consumes at D. With preferences 2 he consumes at E. Thus

ownership and control can be separated. It is only necessary for the owner to tell the manager to

maximize NPV; the owner doesn't need to give a detailed description of how to run the firm.

The fact that it doesn't matter what Bill Ross' preferences are as far as the production

decision is concerned has very important implications if there is more than one shareholder.

Suppose, that in his father's will, the terms under which Bill Ross inherits, actually say that the

inheritance "is to be divided equally among his children." The $1M figure was based on the

assumption that Bill Ross was the only child. In actual fact it turns out that the father had

another brother, Rex, by a former liaison. His inheritance therefore drops to 0.5M.

Should they change the operation of the firm that Bill Ross has set up? No. Since the

NPV rule we derived was independent of the owner's preferences, it follows that both Bill and

Rex are content to leave the firm the way it is.

Why is this? What is the situation they face? The production possibilities curve is now

half what it was before.

½ production possibilities curve in the previous case
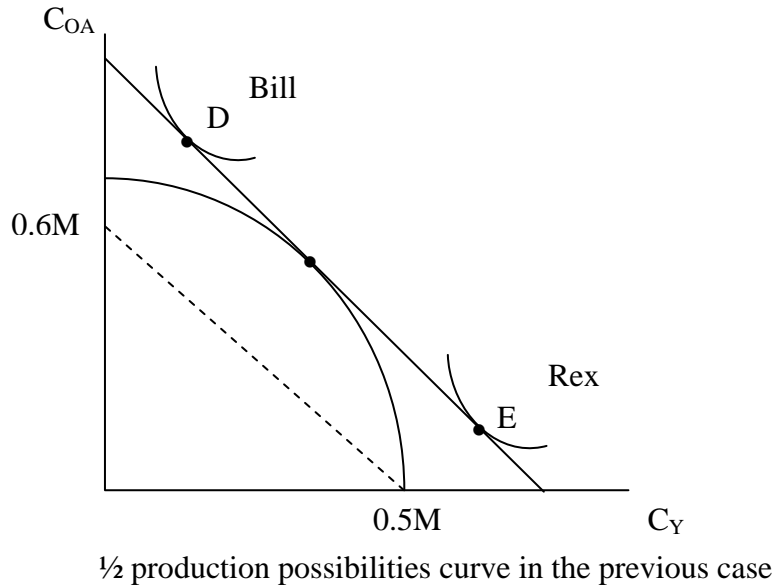
Thus even with separate ownership and control and diverse ownership it is the case that it is in the interest of the owners to maximize NPV (or equivalently to use the rate of return rule). The argument is similar for more than two shareholders.

Solution to Original Puzzle

What the firm should do is maximize NPV. It should accept any of the projects that has a positive NPV.

The old lady only cares about consumption now. What this means is that she can borrow and then use shares to repay the loan, or equivalently she can sell them. She is the ultimate spender. Similarly the little boy can deposit profits in the bank. He is the ultimate miser.

C_{OA} Old Lady's Indifference Curve   Little Boy's Indifference Curve
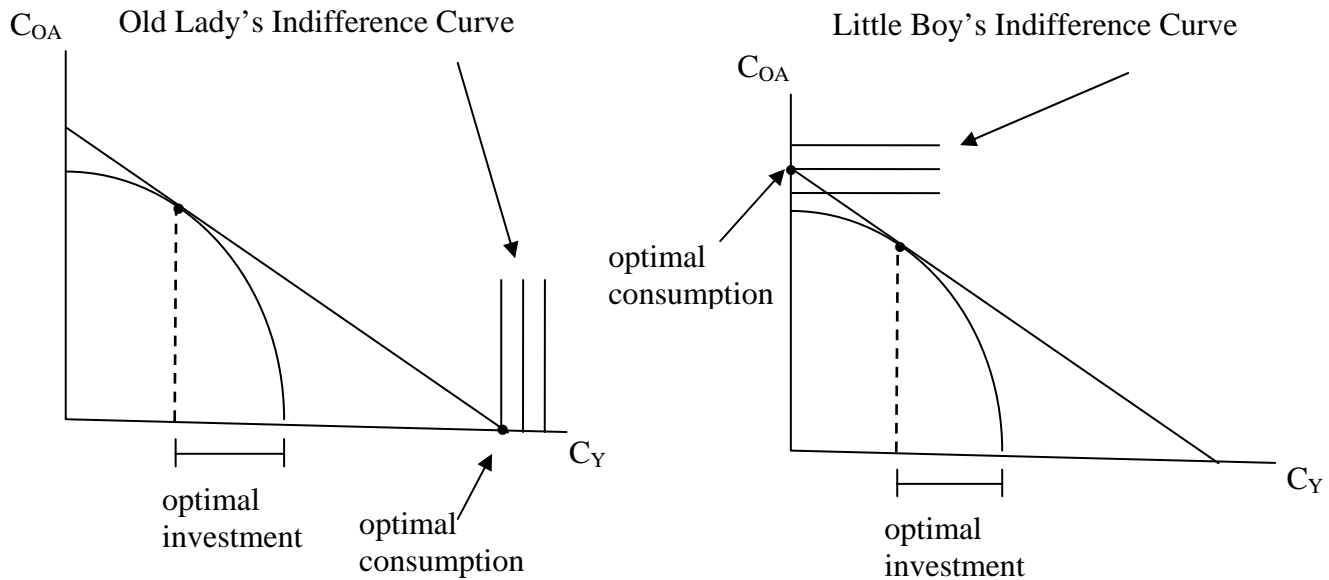
They still agree on the optimal investment policy for the firm. What Ford should do is to

maximize NPV.   The same would be true no matter how many shareholders we considered.

Significance of Separation of Ownership and Control

The fact that it is possible for many very diverse shareholders to simply tell managers to

maximize NPV is crucial for the type of capitalist economy in the U.S. It means that

shareholders don't have to be involved in the day to day affairs of the companies they own.

What they need to know is that managers have incentives to maximize NPV. As long as this is

the case they can own many different firms and can diversify their wealth among hundreds or

even thousands of companies. If shareholders had to follow the day to day affairs of all the

companies they owned shares in they would only be able to own shares in a few firms.

Assumptions Used in the Derivation of NPV Rule

Is it the case that the NPV rule is always a good rule for corporations to use? The answer is no. In our theory we have made a number of assumptions. If these assumptions do not hold then the result may not hold. It is important to know when it holds and when it doesn't hold.

There were a number of simplifying assumptions. The bank we talked about represents the whole range of financial possibilities in capital markets. We have also assumed certainty. The rate of interest at which you discount is essentially the opportunity cost of capital--this is the rate of return you can get on a similar asset. We also had only two periods, but we can clearly extend to the case where there are more than two but we would have to use more sophisticated mathematical techniques.

The crucial assumption in deriving the NPV rule is that "capital markets" are "perfect".

Perfect Capital Markets

What we mean by this is the following:

1.   There are no frictions. Examples of frictions are brokerage costs, fixed fees, and any
        type of transaction cost such as when lending and borrowing rates differ. To see what
        happens in this latter case lets start by considering the case where the two are equal (i.e.
        from the point of view of consumers)

                        lending rate   =   borrowing rate

                (i.e., put money in bank)   (i.e., loan from bank).

        First of all consider what happens at an arbitrary point on the investment curve:
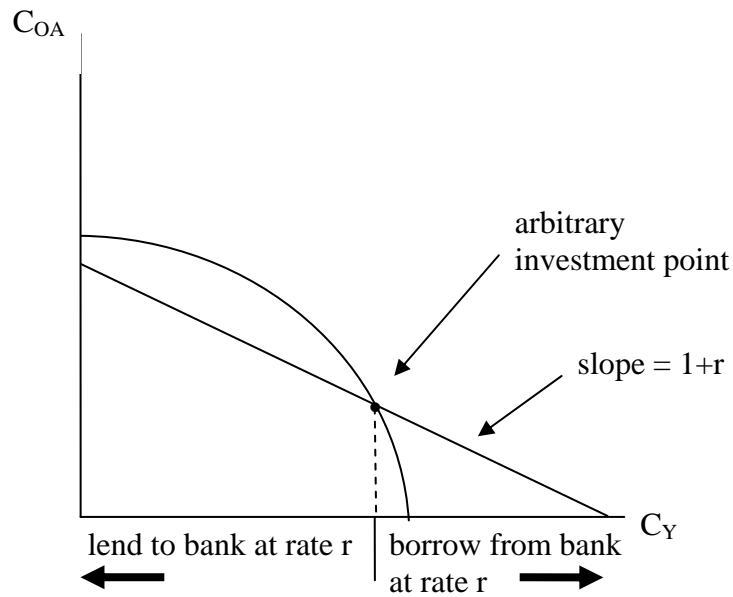
To the left of the point the person is lending to the bank; to the right the person is

borrowing from the bank.

Next consider what happens if the lending rate is less than the borrowing rate so that the

bank costs are covered.
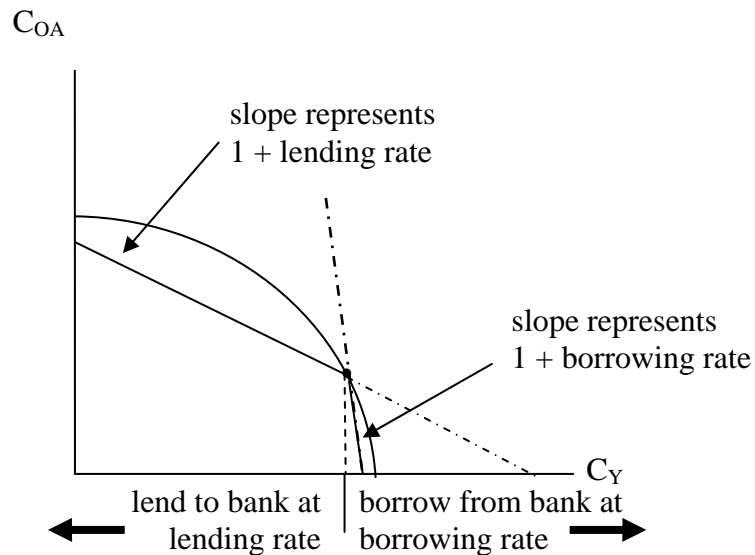
The above represents an arbitrary investment point. What would be the optimal points

for a miser and a spender? Consider first the miser. The miser wants to maximize NPV at the

lending rate.
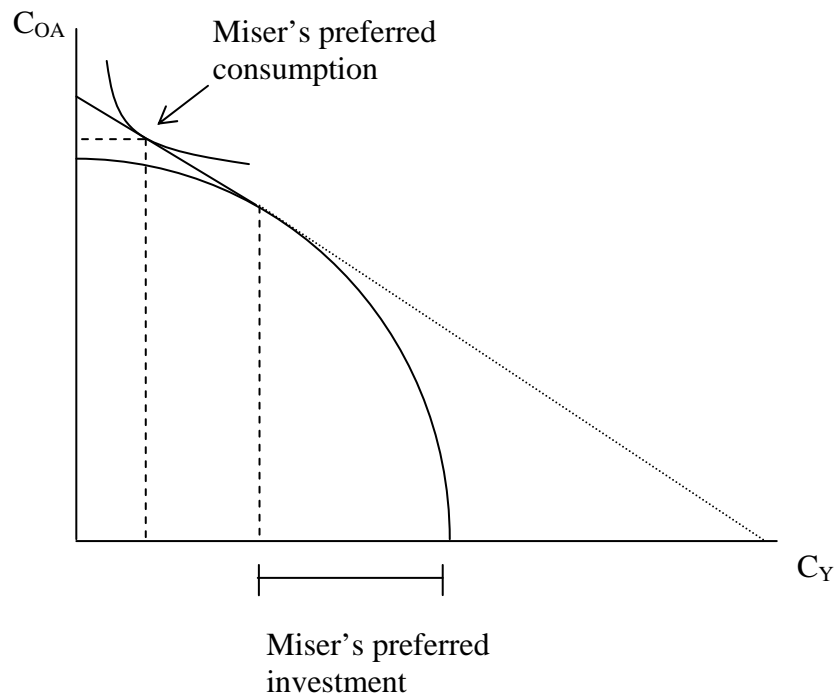
Next consider the spender. The spender wants to maximize NPV at the borrowing rate.
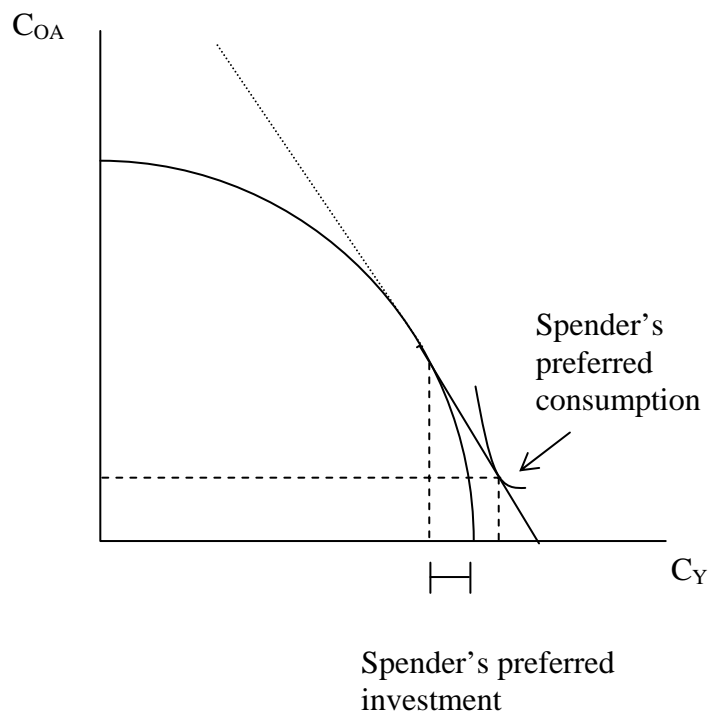
The miser and the spender clearly disagree about the level of investment the firm should

undertake. What would somebody with intermediate tastes want?
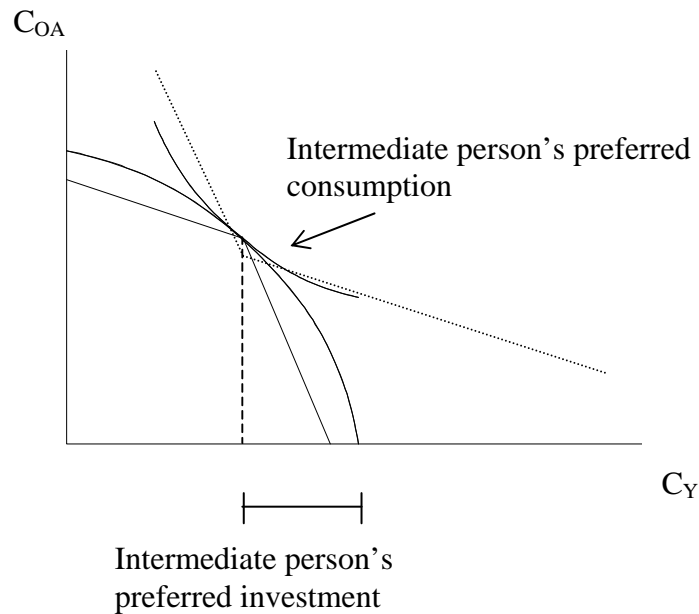
Again, they no longer agree. Shareholders must vote to determine what the firm should do.

The difference in lending and borrowing rates is just one example of a friction.

Whenever there are transaction costs there will be a friction.

2.   Information freely available. If this isn't satisfied, we could, for example, have a case

        where one shareholder didn't know about the existence of the bank, and so we again have

        problems.

C_{OA}

SH2 ignorant of bank

SH2 knows about bank

C_Y
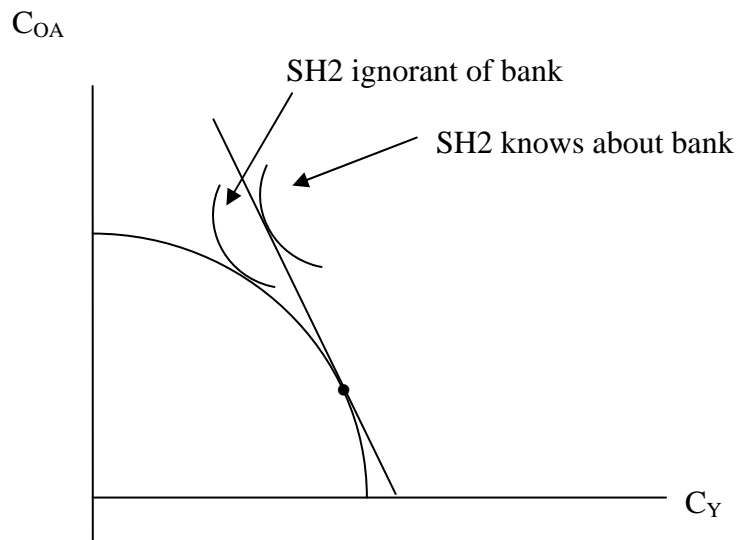
Again the shareholders must vote.

3.   <u>No firm affects interest rates.</u> If a firm (or any person) has a significant effect on interest

rates, then stockholders would like the firm to take this into account when making

investment decisions.

      For example, if the firm invests more, so that the demand for funds rises, then the

rate of interest, r, may rise from r_1 to r_2, and this may make some shareholders better off and

some worse off.
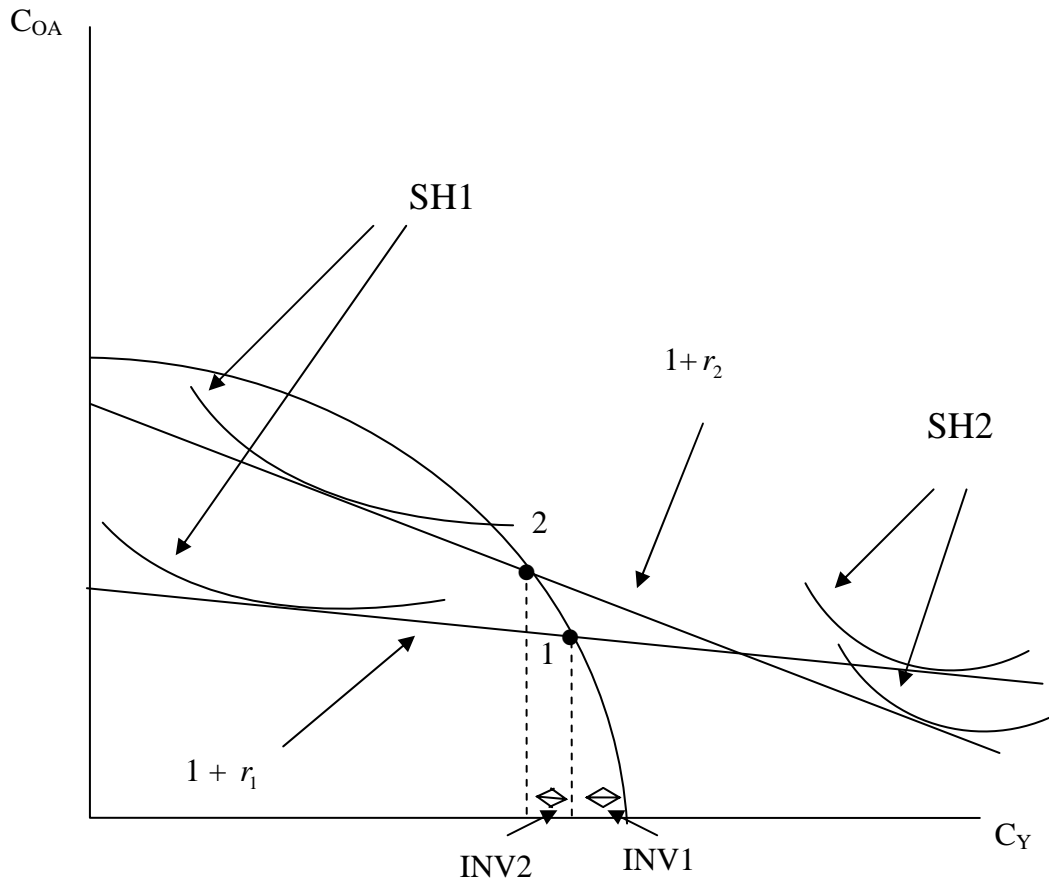
Situation 1 (r_1):   One possible level of investment and corresponding interest rate.

Situation 2 (r_2):   Higher level of investment and higher corresponding interest rate.

Once again they no longer agree. Shareholder 1 prefers Situation 2 while Shareholder 2 prefers Situation 1. Once again shareholders must vote on what the firm should do.

Assessment of Assumptions

Clearly assumptions 1 to 3 aren't entirely satisfied. However, it is the case that in many economies, particularly in the U.S., they come pretty close to being satisfied. The reason that assumptions 1 and 2 come close to being satisfied is that rich people and institutional holders

own most of the stocks in the U.S. For these investors the assumptions are reasonable. As far as the third assumption is concerned no corporations are large enough to affect interest rates so this assumption is satisfied. It is for these reasons that we can take the NPV rule as a reasonable guide for managers to follow at least in the U.S. In a number of other countries, particularly emerging economies such as Brazil, India and Korea and even a number of Western European countries such as Germany capital markets are not as perfect. In these cases it is not clear that maximization of NPV is a sensible goal for corporations with many diverse owners. In such cases shareholders must get involved in the running of the corporation and vote on what it should do. This is why family firms, which avoid these issues, are so much more prevalent in such countries.

Other Aspects of Maximizing NPV

Corporate Governance: Do Managers Pursue Shareholders Interests?

The analysis above has assumed that managers act in the interests of shareholders. One of the big debates in recent years has been concerned with whether this is in fact the case. This is the issue of *corporate governance*. It has been widely pointed out that in large corporations there is a tendency for managers to pursue their own interests. This is what is known as an *agency problem*. In order to minimize this agency problem managers' interests must be aligned with those of shareholders.

One way to do this is to ensure managers have a stake in the firm in terms of direct ownership of shares and/or stock options. If managers have a substantial portion of their wealth in the shares of the company they will clearly have an incentive to create wealth for shareholders by accepting positive NPV projects. Stock options have a similar effect. These allow managers

to buy shares at a fixed price called the exercise price. If the stock price rises above the exercise price the options clearly become valuable. The managers can buy the shares at the exercise price and sell them at the higher market price.

Bonuses are another way that managers can be rewarded for pursuing shareholders' interests. These are usually based on accounting measures of performance. Since these are not as good a measure of shareholder wealth as the stock price itself they are often not as effective as direct ownership of shares or stock options.

Another way of ensuring managers pursue shareholders' interests is to have an active *market for corporate control*. If managers pursue their own interests there is an incentive for an outside raider to mount a hostile takeover bid and buy the firm. The raider can then fire the managers and increase the value of the firm.

Should Firms be for Shareholders or All Stakeholders or Who?

Even if the assumptions of the model are satisfied and managers pursue shareholders interests it is not clear that maximizing NPV is the sole criterion that firms should pursue. A number of authors have argued that firms should take into account other "stakeholders" in the firm such as workers, residents around the firms' plants and so on. These are issues on which there is no wide agreement. You will be discussing this issue at much greater length in the legal studies and public policy and management core course LGST/PPMT 621.

Different countries have very different views on what the corporation's aim should be. The German system of codetermination involves workers in large companies having the right to elect up to half the directors to the supervisory board. In Japan managers have traditionally viewed the interests of shareholders to be secondary to those of employees and customers. The

appendix (at the end of the section) outlines the corporate philosophy of Asahi Breweries, which is one of the major Japanese brewing companies. You can see that shareholders do not get mentioned until Section 6 under Social Responsibilities.

Other evidence for the differing views of the role of the firm in different countries are given in the attached figures. These are based on interviews with executives from large companies in the five countries. In the first, where the issue is whose company it is, it can be seen that Japanese, German and French executives countries view their firms as being for all stakeholders while the U.S. and U.K. executives view their firms as being for shareholders. Similarly, when the issue is job security or dividends almost all Japanese executives and the majority of German and French managers believe job security should come first. Almost all U.S. and U.K. executives believe dividends should come first.

These interviews are from a few years ago. How much have things changed recently? In Germany, there was some discussion of these issues in recent years. Franz Müntefering, who was at the time Chairman of Germany's Social Democratic Party, famously commented "Financial investors remain anonymous. They have no face, they descend upon companies like locusts, destroy everything and move on." These comments led to a heated political discussion on the role of financial investors. Josef Ackermann, Chief Executive of Deutsche Bank, was severely criticized when he announced at the same time that Deutsche's profits were up and the number of employees had been cut.

In Japan, there have been several attempts at hostile takeovers in recent years. Examples are Oji Paper's attempted acquisition of Hokuetsu Paper, and Steel Partners attempt to acquire Bull-Dog Sauce. However, none has really succeeded. It will be interesting to see how things develop in Japan going forward.

In France, during the last Presidential campaign the right-wing candidate Nicolas Sarkozy criticized DaimlerChrysler, for insisting that EADS, the parent company of Airbus pay dividend even though it was restructuring. He said "When I hear that a shareholder, Daimler, is asking for dividends to be paid on its capital in a company that is laying off 10,000 people, I cannot accept it" (Financial Times, March 9, 2007, p.17).

During the crisis the experience of countries with different corporate governance with regard to unemployment has been very different. In Germany, which suffered a large drop in GDP, unemployment is lower in the fall of 2011 than at the beginning of the crisis. The U.S. had a much smaller drop in GDP but unemployment is still almost twice as high as at the start of the crisis. Arguably part of this difference is due to a different focus of the firms.

The issue of objective function of corporations is thus very much in the news internationally. The globalization of capital markets provides strong incentives for firms to pursue the interests of shareholders. On the other hand this has clashed with traditional views about the role of companies in many countries.

For the remainder of the course, we shall be taking the objective of the corporation to be the maximization of NPV or in other words the creation of shareholder wealth. The main point of this section has been to show the justification of this from the firm perspective.

Appendix for Section 2

ASAHI BREWERIES, LTD.

Corporate Philosophy of Asahi Breweries, Ltd.

We at Asahi Breweries, Ltd., through our business activities including alcoholic and nonalcoholic beverages, food and pharmaceuticals, wish to contribute to the health and well-being of people the world over. By thus contributing to society as a whole, the company seeks to attain the trust and confidence of the consumer and develop still further.

1. Consumer Orientation
Identifying the best interests of consumers, we endeavor to meet their demands by creating products suited for contemporary tastes and lifestyles.

2. Quality First
Open to consumer opinion of our products, we consistently enhance quality level and extend technological capabilities in order to market the finest products in the industry.

3. Respect for Human Values
Our Company firmly believes that human beings are the core of the business, and follows the principle of human values through developing human resources and implementing fair personnel management. Each employee is encouraged to fully utilize his or her own potential, and work to realize an open, positive thinking corporate culture.

4. True Partnership Between Labor and Management
Our Company aims to strengthen harmonious relations between labor and management based on mutual understanding and trust. Both parties work hand in hand for corporate development as well as the welfare of all employees.

5. Cooperation with Business Associates
We seek to build strong relations with all our business associates and affiliates in a spirit of co-existence and co-prosperity based in mutual trust. At the same time, we are determined to accept and fulfil our responsibilities as the core of the Asahi group of companies.

6. Social Responsibilities
We at Asahi, through securing and expanding the base of our operations, desire to fulfill our responsibilities to stockholders and the local communities in which we operate. Also in carrying out business activities, we sincerely observe the moral principles of management based on social standards.

Figure 2

Whose Company Is It?

%
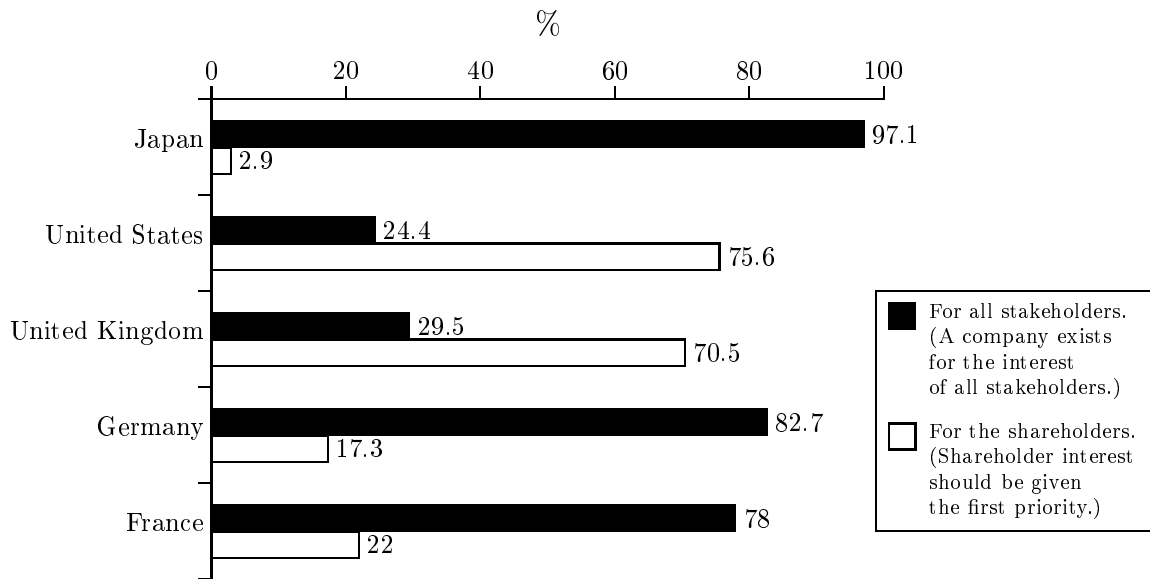
Number of firms surveyed: Japan, 68; United States, 82; United Kingdom, 78;
Germany, 110; France, 50.

Figure 3

Job Security or Dividends

%

| | 0 | 20 | 40 | 60 | 80 | 100 |

Japan: 97.1 / 2.9

United States: 10.8 / 89.2

United Kingdom: 10.7 / 89.3

Germany: 59.1 / 40.9

France: 60.4 / 39.6

■ Job security more important.
□ Dividends more important.

Number of firms surveyed: Japan, 68; United States, 83; United Kingdom, 75;
Germany, 105; France, 68.
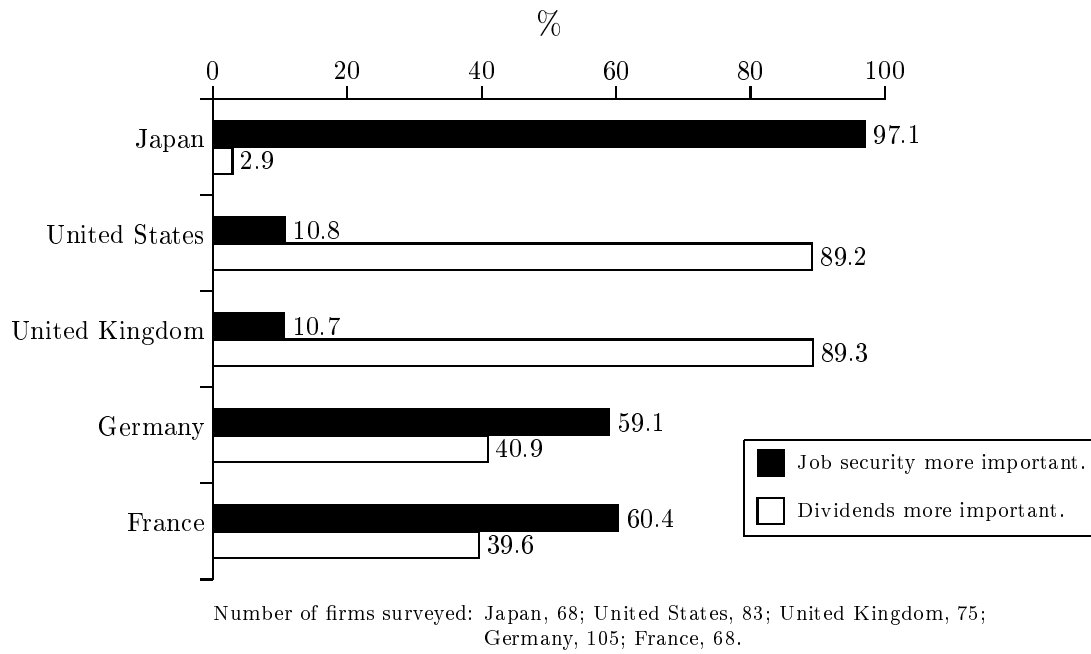
Source: Masaru Yoshimori, "Whose Company Is It? The Concept of the Corporation in Japan and the West."
Long Range Planning, Vol. 28, No. 4, pp. 33–44, 1995.

From: Institute of Fiscal and Monetary Policy (1996), Chart III-4-6, p. 84.

UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

LECTURE NOTES

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

QUARTER 1 - WEEK 2 (part 2)

Th: 9/15/11

Section 3: Calculating Present Values

Read Chapter 2 BMA

Motivation

Calculation of PV

      The first question that is usually asked is whether it is necessary to understand where the formulas that you use to calculate PV come from. The answer is that you don't need to but it's helpful if you do. Essentially a large part of what we'll be doing in the coming weeks is discounting streams of cash flows. You can always use a basic formula but this can be tedious and having short-cut formulas can save time and effort. Often it's quicker to derive a formula than to solve a problem using the basic formula - you'll see an example of this in the homework problems. Also knowing where the formulas come from help you understand the standard assumptions about the timing of cash flows.

      In the last section we used the formula

$$PV = \frac{C_1}{1+r}$$

In this section we're first going to look at how to calculate more complicated PV's when there are many periods and hence many cash flows to consider.

    Similarly to the one-period case, if we have a cash flow C_2 two periods from now then its present value is given by

$$PV = \frac{C_2}{(1+r)^2}$$

We can go on doing this for three or more years. It follows that for t years

$$PV = \frac{C_t}{(1+r)^t} = C_t \times DF(t \text{ years}, r\%)$$

where the *discount factor* DF is given by

$$DF(t \text{ years}, r\%) = \frac{1}{(1+r)^t}$$

Now, present values are all measured in dollars today. What do we know about $100 today plus $60 today? It is equal to $160. In other words, we can sum present values.

$$PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \ldots + \frac{C_T}{(1+r)^T}$$
$$= \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

Time-varying discount rates

In the expression above we are implicitly assuming that the interest rate and hence the discount rate is the same in every period. However, there is no reason why this should be so. The discount rate can vary over time. The graph of interest rates for different periods against the periods is known as the *term structure*. It is usually upward sloping as shown.
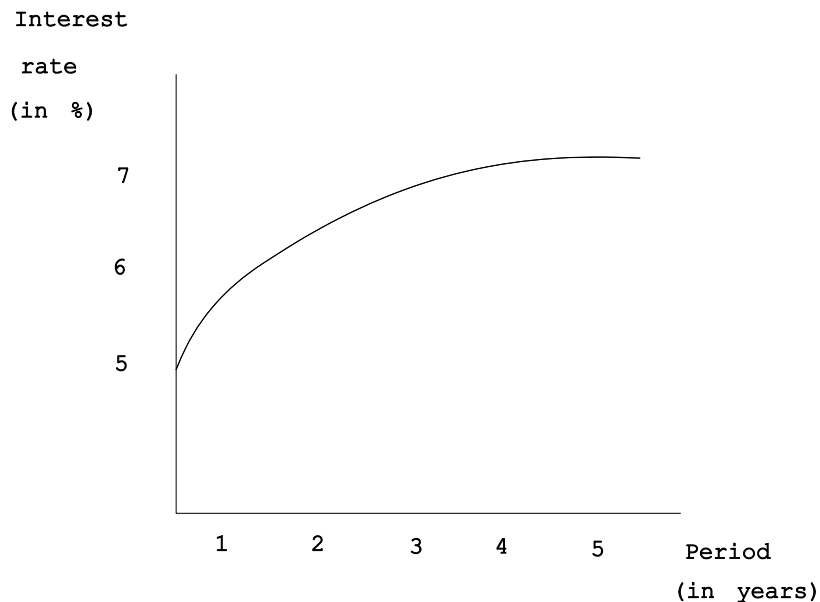
The determinants of the shape of the term structure such as inflation expectations are considered in macroeconomics courses. For the moment what we are interested in is how the term structure affects our calculation of PV. There are two different but equivalent ways of thinking about what happens when interest rates are different for different periods of time.

Method 1: One possibility is to think in terms of the *reinvestment rate*. Suppose you are investing for two periods, for the first one you invest at r_1 and for the second one you invest at $r_{1,2}$.

$$t = \quad 0 \qquad\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad 2$$

Rate: -------- r_1 for period 1 -------> -------- $r_{1,2}$ period 2 --------->

Investment of   1   becomes $1 + r_1$ becomes $(1 + r_1)(1 + r_{1,2})$
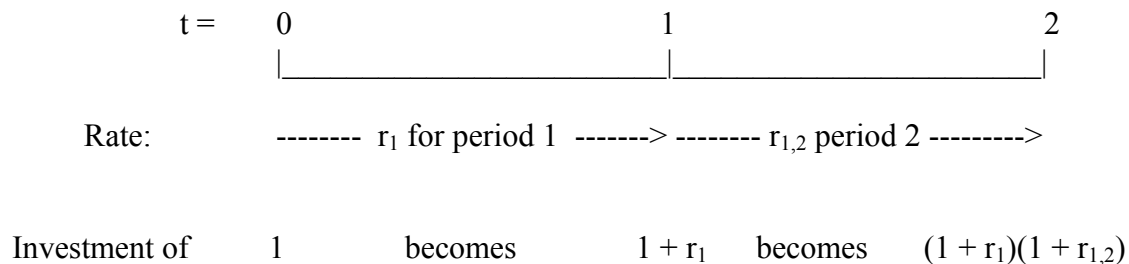
Here the present value of a cash flow C_2 two periods from now is

$$PV = \frac{C_2}{(1+r_1)(1+r_{1,2})} \tag{1}$$

Method 2: The second possibility is to use the two-period interest rate or discount rate r_2.

$$t = \quad 0 \qquad\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad 2$$

Rate: ------------- r_2 per period for both periods -------------->

Investment of   1   becomes $(1 + r_2)^2$
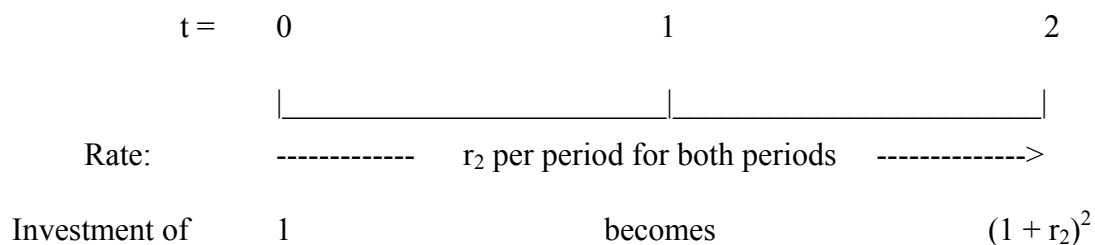
Here the present value of a cash flow C_2 two periods from now is

$$PV = \frac{C_2}{(1 + r_2)^2} \qquad (2)$$

These are just two equivalent ways of thinking about the same thing. The reason they are equivalent is that it turns out it must be the case that

$$(1 + r_1)(1 + r_{1,2}) = (1 + r_2)^2$$

so the reinvestment rate is given by

$$(1 + r_{1,2}) = \frac{(1 + r_2)^2}{(1 + r_1)}$$

To see this it is important to realize that the rates we are dealing with are rates on <u>zero-coupon bonds</u> (z.c.b.). In other words they are bonds which only have a payment at the final date, they don't have any payments at the intermediate dates.

The payments from <u>buying</u> zero coupon bonds are

Zero Coupon Bonds	t=0	1	2	3
1 year	-1	$+(1 + r_1)$		
2 year	-1		$+(1 + r_2)^2$	
3 year	-1			$+(1 + r_3)^3$

The payments from <u>selling</u> zero coupon bonds are just the negative of these (since selling is the opposite of buying):

Zero Coupon Bonds	t=0	1	2	3
1 year	+1	$-(1 + r_1)$		
2 year	+1		$-(1 + r_2)^2$	
3 year	+1			$-(1 + r_3)^3$

Suppose a firm <u>sells a 1 year</u> and <u>buys a 2 year</u> what are the net cash flows?

	t=0	1	2
Sell 1 year z.c.b.	+1	$-(1 + r_1)$	
Buy 2 year z.c.b.	-1		$+(1 + r_2)^2$
Net cash flows	0	$-(1 + r_1)$	$+(1 + r_2)^2$

The firm is effectively reinvesting $(1 + r_1)$ at date 1 and receiving $(1 + r_2)^2$ at date 2. The reinvestment rate is

$$(1 + r_{1,2}) = \frac{(1 + r_2)^2}{(1 + r_1)} \qquad (3)$$

as we required. Hence by selling a 1 year zero coupon bond and buying a 2 year zero coupon bond you can effectively lock in a reinvestment rate of $r_{1,2}$.

Given (3) is equivalent to $(1 + r_1)(1 + r_{1,2}) = (1 + r_2)^2$ it doesn't matter whether we use Method 1 or Method 2, both are equivalent. Notationally, Method 2 is easier and so this is the one that is usually used.

<u>Example</u>

Suppose the 1 year rate $r_1 = 5\%$ and the 2 year rate $r_2 = 6\%$.

(i) What is the reinvestment rate from date 1 to date 2, i.e. $r_{1,2}$?

(ii) What is the PV of 100 at date 2 using Method 1?

(iii) What is the PV of 100 at date 2 using Method 2?

<u>Solution</u>

(i) The reinvestment rate is found from (3):

$$(1 + r_{1,2}) = \frac{(1 + r_2)^2}{(1 + r_1)} = \frac{1.06^2}{1.05} = 1.0701$$

In other words, $r_{1,2} = 7.01\%$.

(ii)   Method 1:

$$PV \text{ of } 100 = \frac{100}{(1.05)(1.0701)}$$

$$= \ 89.00$$

(iii)   Method 2:

$$PV \text{ of } 100 = \frac{100}{(1.06)^2}$$

$$= \ 89.00$$

The two methods are equivalent as they should be.

It is important to understand how the different rates should be interpreted. In the context of the example $r_1 = 5\%$ is the 1 year rate that can be obtained now at date 0 and r_2 is the 2 year rate that can be obtained now. Similarly, $r_{1,2}$ is the rate between date 1 and 2 that can be guaranteed now. It may or may not be equivalent to the 1 year rate that is available after a year has passed.

It follows from what we have done that the term structure summarizes the whole interest rate structure as it stands today. Strictly speaking the term structure involves rates on zero coupon bonds. How can we find these in practice? They can be found from Treasury strips since these are just zero coupon bonds. Their yields are quoted in the Wall Street Journal, for example. Note these are not the same as yields on ordinary Treasuries since these have coupon payments during each year rather than being zero coupon. These coupon bond yields are usually fairly close and so are sometimes used as an approximation.

We have considered what happens with cash flows two periods away but you can see this can easily be extended to cash flows for t periods away, we simply discount C_t at the t year rate r_t

where r_t is the rate on a t year zero coupon bond so

$$PV = \frac{C_t}{(1 + r_t)^t}$$

In the general case where we have a stream of cash flows $C_1, C_2, \ldots, C_T$:

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \frac{C_3}{(1 + r_3)^3} + \ldots + \frac{C_T}{(1 + r_T)^T}$$

$$= \sum_{t=1}^{T} \frac{C_t}{(1 + r_t)^t}$$

where r_t is the t-period interest rate. If we want to calculate NPV, we just take into account the

initial investment, which is usually negative.

$$NPV = C_0 + \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \ldots + \frac{C_T}{(1 + r_T)^T}$$

$$= \sum_{t=0}^{T} \frac{C_t}{(1 + r_t)^t}$$

The discount factors, $1/(1 + r_1)$, $1/(1 + r_2)^2$, etc., can easily be found with calculators.

Example

What is the NPV of the following cash flows: -63,234 at t = 0, -21,230 at t = 1, and

+134,632 at t = 2 if the 1 and 2 year rates of interest are r_1 = 7% and r_2 = 8%, respectively?

(Note that there is no reason in general why we shouldn't have negative cash flows after the first

period--investment may be spread out and take some time, for example.)

Solution

In present value and net present value problems it is helpful to draw a time line; this is especially true for complex problems. To illustrate in the context of this simple example:

t =   0   1   2

                        |_____|_____|

                   -63,234   -21,230   134,632

Discounting at the given discount rates,

$$\text{NPV} = -63{,}234 + \frac{-21{,}230}{1.07} + \frac{134{,}632}{(1.08)^2}$$

$$= 32{,}350$$

If this was a project, should we accept it? Yes.

In practice the term structure is often flat enough that for many purposes, particularly in corporate finance, it is a reasonable enough approximation to use a constant discount rate. This is what is usually done.

Perpetuities

During the Napoleonic wars, the British government amassed an enormous debt in terms of the values of the day, which consisted of a number of different issues of securities. They decided to consolidate this debt by making a single issue of perpetuities, i.e., bonds with no termination date, which just promised to pay a fixed sum every year forever. Since their purpose

was to consolidate existing debt they were called "consols". Over 190 years later these are still going strong--you can still own consols.

How can we value perpetuities? If we assume a constant rate of interest, we can derive a simple formula to value them since the stream of payments is a perpetuity. The standard assumptions are that the current date is $t = 0$ and the first payment of C is paid 1 period from now at $t = 1$; the second payment of C is paid 2 periods from now at $t = 2$ and so on forever:

t =	0	1	2	3		∞

|_____|_____|_____|_____| . . . |____|

C   C   C   C
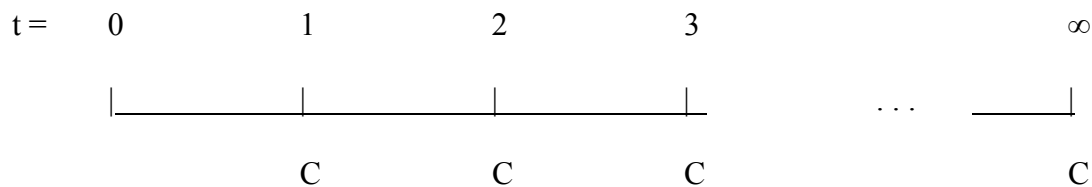
$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} \cdots$$

As we saw in the first section and is shown in Appendix at the end of the section as Result 1 this simplifies to the following:

$$PV\ Perpetuity = \frac{C}{r}$$

How can we use this formula to value consols?

Example

Suppose there is a consol paying £5 per annum and the interest rate is 10%. What's the value of the consol?

$$\text{Value} = 5/0.10 = £50$$

Growing perpetuities

We can use the same trick to value growing perpetuities that we used to value perpetuities. Suppose we expect the payments to grow at a rate of g forever and the nominal rate of interest is r. The standard assumptions are that the current date is $t = 0$ and the first payment of C is paid 1 period from now at $t = 1$; the second payment of $C(1 + g)$ is paid 2 periods from now at $t = 2$; the third payment of $C(1 + g)^2$ is paid 3 periods from now at $t = 3$ and so on forever:

t =	0	1	2	3	
		C	C(1+g)	$C(1+g)^2$	. . .
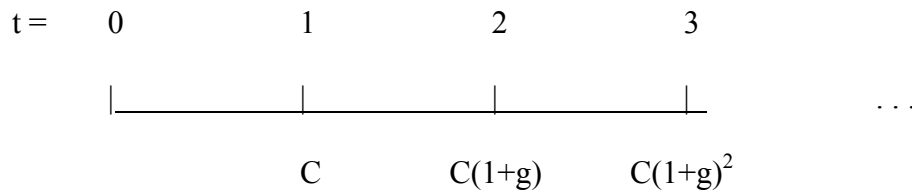
$$PV = \frac{C}{1 + r} + \frac{C(1 + g)}{(1 + r)^2} + \frac{C(1 + g)^2}{(1 + r)^3} + \dots$$

It can be seen from Appendix 1, Result 2 that this simplifies to the following:

$$\text{PV Growing Perpetuity} = \frac{C}{r - g}$$

Example

Suppose we have a growing perpetuity initially paying £5 with a growth rate of $g = 5\%$ and interest rate $r = 10\%$. Then

$$PV = \frac{5}{0.10 - 0.05} = £100$$

Annuities

There are a number of assets, such as mortgages, annuities and so on, which involve a fixed payment for a specified number of years. An annuity is like a perpetuity except that it does not go on forever; it consists of a payment C for a fixed number of years T. The standard assumptions are that the current date is t = 0 with the first payment of C being paid 1 period from now at t = 1, the second payment of C is paid 2 periods from now at t = 2 and so on with the last payment being at period t = T:

```
t =    0              1              2              3                          T
       |_____|_____|_____|          . . .      _____|
                       C              C              C                               C
```
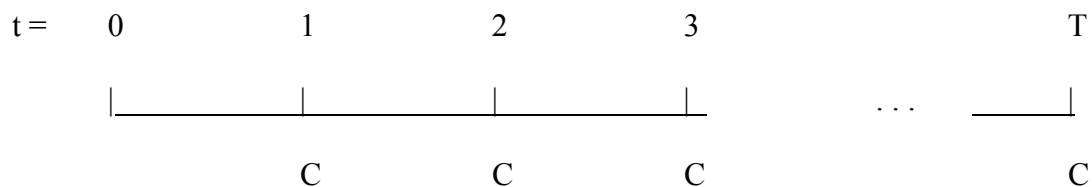
$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + ... + \frac{C}{(1+r)^T}$$

It can be seen from Appendix 1, Result 3 that this simplifies to the following:

$$PV = C \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^T}\right) = C \times AF(T \text{ years}, r\%)$$

where AF(T years, r%) is the *annuity factor* for T years at r%.

Example

Somebody tells you she has a 10%, 30 year mortgage and her annual payment is $6,000 per annum. What is the value of her mortgage?

Solution

$$PV = 6,000 \times \frac{1}{0.1}\left[1 - \frac{1}{(1+0.1)^{30}}\right]$$

Now
$$AF(30\text{yrs},10\%) = \frac{1}{0.1}\left(1 - \frac{1}{(1+0.1)^{30}}\right) = 9.427$$

$$PV = 6,000 \times 9.427$$

$$= \$56,562$$

One important thing to notice which is applicable to all the formulas we will be learning is that you can switch them around.

Example

Suppose you want to find the annual payment on a 30 year, 10% mortgage of $100,000.
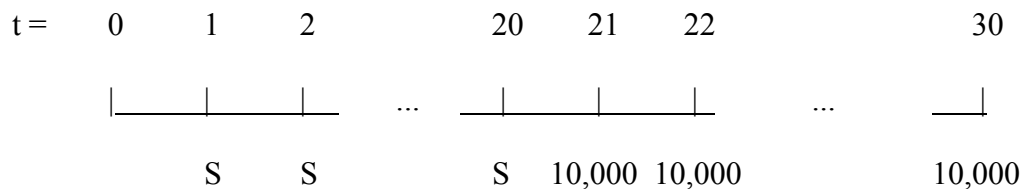
Solution

$$C = \frac{PV}{\frac{1}{r}\left(1 - \frac{1}{(1+r)^{T}}\right)}$$

$$= \frac{100,000}{9.427} = \$10,608 \text{ per annum}$$

The examples that we have been doing up to now have been very simple. You will get a lot of practice in the problem sets and exams of more difficult questions but to prepare you for those let's first do a more complex example.

<u>Example</u>

Jacqui wants to retire twenty years from now. She wants to save enough so that she can have a pension of $10,000 a year for ten years. How much must she save each year, denoted S, during the first twenty years to achieve her goal if the interest rate is 5%? Assume all cash flows occur at the end of the year so that it is possible to use the standard formulas. The first date at which S is saved is date 1 and the first date at which the $10,000 pension is received is date 21.

<u>Solution</u>

```
t =    0     1     2              20    21    22                    30

   |_____|_____|_     ...   __|_____|_____|_     ...     __|

         S     S              S   10,000 10,000          10,000
```

The first thing to do in these problems is to draw a time line so that you can conceptualize the stream of cash flows. Discounting simply moves money backwards along the time line. Taking future values moves it forward. Both of these account for interest that would be earned.

We need to choose a reference date that we are going to compare cash flows at. Often we use date 0. We could choose any date but date 20 is an obvious one in this problem.

PV of pension at date 20 = 10,000 × AF(10 yrs, 5%)

$$= 10,000 \times 7.722$$

$$= \$77,220$$

Jacqui needs to ensure that she has saved enough during the first twenty years to have this amount at date 20.

PV of savings of S for 20 years at date $0 = S \times AF(20 \text{ yrs}, 5\%)$

$$= S \times 12.462$$

This is in date 0 dollars. In order to be able to compare this with the 77,220 at date 20 we need to change it from date 0 dollars into date 20 dollars by taking a future value.

FV of savings of S for 20 years at date $20 = S \times 12.462 \times (1.05)^{20}$

$$= S \times 33.066$$

Hence to have enough savings to provide the pension Jacqui needs

$$S \times 33.066 = 77,220$$

Solving gives,

$$S = 77,220/33.066$$

$$= \$2,335$$

Jacqui needs to save \$2,335 each year for twenty years to give her a pension of \$10,000 a year for ten years.

Compound Interest

So far we have been assuming interest is paid once a year or, more generally, once a period. Often interest is paid more frequently than this. What happens if interest is paid more often, once every six months, say, in which case it is said to be compounded twice?

For example, suppose the interest rate is 10% a year compounded twice. This means 5% interest is paid every six months. After one year \$1 becomes

$$1 \times (1 + 0.05)^2 = 1.1025$$

That is, we have earned interest on the interest.

If we have 10% compounded four times, we pay 2.5% every quarter. After a year $1 becomes

$$1 \times (1 + 0.025)^4 = 1.1038$$

In general if compounding takes place q times a year and r is the interest rate/year,

$$C_1 = PV\left(1 + \frac{r}{q}\right)^q$$

After t periods,

$$C_t = PV\left(1 + \frac{r}{q}\right)^{qt}$$

If compounding takes place continuously so that interest is paid on interest all the time, so to speak, we get a particularly simple expression. In order to derive this it is necessary to recall that

$$e = \lim_{m \to \infty}\left(1 + \frac{1}{m}\right)^m$$

Taking our formula for discrete compounding and rearranging it we get

$$C_t = PV\left[\left(1 + \frac{1}{q/r}\right)^{\frac{q}{r}}\right]^{rt}$$

Letting m = q/r, we can see that as q approaches infinity and we compound continually

$$\lim_{m \to \infty} C_t = PV \left[\lim_{m \to \infty} \left(1 + \frac{1}{m} \right)^m \right]^{rt}$$

$$= PVe^{rt}$$

Alternatively this can be written

$$PV = \frac{C_t}{e^{rt}} = C_t e^{-rt}$$

In many circumstances it is much easier to calculate and manipulate these expressions than their discrete time counterparts. One example is the pricing of options using the Black and Scholes model that we will be considering later on.

Example 1

Suppose you invest $200 at 12% continuously compounded for two years. What do you receive at the end of the two years?

Solution

$$rt = 0.12 \times 2 = 0.24$$

$$e^{0.24} = 1.271$$

$$C_2 = 200 \times 1.271 = \$254.2$$

Example 2

What is the PV of $300 in 1 year's time if the interest rate is 5% continuously compounded?

Solution

$$rt = 0.05$$

$$e^{0.05} = 1.051$$

$$PV = 300/(1.051) = \$285.44$$

Relationship between continuously compounded and annual rates

If you think back to our original example of investing $1 at 10% compounded twice then in that case you ended up with $1.1025. The effective annual rate is 10.25%. In other words 10% compounded twice is equivalent to an annual rate of 10.25%. Similarly for any compound rate, including continuously compounded rates, there is an equivalent annual rate. How do we go about switching from one to the other?

To distinguish between annual rates and continuous rates, I'm going to use the following notation from now on:

r - annual rate

r_c - continuously compounded rate

Suppose that you invest $1 at an annual rate of r then at the end of the year you have

$$C_1 = 1 + r$$

If you invest $1 at a continuously compounded rate of r_c then at the end of the year you have

$$C_1 = e^{r_c}$$

We want to find the value of r such that it is equivalent to r_c; in other words, we wish to find the value of r and r_c such that the C_1 you end up with is the same as when you invest at r. Therefore equating the C_1 you get

$$e^{r_c} = 1 + r$$

$$r = e^{r_c} - 1$$

or, taking natural logs of the previous equation,

$$r_c = \log_e(1 + r)$$

Example 3

(i)   If $r_c = 0.10$, what is r?

$$r = e^{0.10} - 1 = 1.105 - 1 = 0.105 \text{ or } 10.5\%$$

(ii) If $r = 0.10$, what is r_c?

$$r_c = \log_e 1.1 = 0.0953 \text{ or } 9.53\%$$

This brings us to the end of basic results concerning discounting. Next we turn to using these ideas to value bonds and stocks.

Section 3

APPENDIX ON PRESENT VALUE FORMULAS

Result 1: Perpetuities

A perpetuity is a constant payment C every period forever. The standard assumptions are that the current date is t = 0 and the first payment of C is paid 1 period from now at t = 1; the second payment of C is paid 2 periods from now at t = 2 and so on, forever. The PV of the perpetuity is then

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ ... \tag{1}$$

Multiplying through by 1 + r

$$(1+r)PV = C + \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ ... \tag{2}$$

Subtracting (1) from (2)

$$rPV = C$$

$$PV = \frac{C}{r}$$

Result 2: Growing Perpetuities

A growing perpetuity is like a perpetuity except the payment grows at a rate of g per period instead of being constant through time. The standard assumptions are that the current date is t = 0 and the first payment of C is paid 1 period from now at t = 1; the second payment of C(1 + g) is paid 2 periods from now at t = 2; the third payment of $C(1 + g)^2$ is paid 3 periods from now at t = 3 and so on forever. The PV of a growing perpetuity is then

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots \tag{3}$$

Multiplying through by $\dfrac{1+r}{1+g}$

$$(1+r)PV\frac{1}{1+g} = \frac{C}{1+g}\frac{1}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \ldots \tag{4}$$

Subtracting (3) from (4)

$$\left(\frac{1+r}{1+g} - 1\right)PV = \frac{C}{1+g}$$

Multiplying through by 1 + g

$$(r - g)PV = C$$

$$PV = \frac{C}{r-g}$$

Result 3: Annuities

An annuity is like a perpetuity except that it does not go on forever; it consists of a payment C for a fixed number of years T. The standard assumptions are that the current date is t = 0 and the first payment of C is paid 1 period from now at t = 1; the second payment of C is paid 2 periods from now at t = 2 and so on with the last payment being at period t = T. The PV of an annuity is then

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + ... + \frac{C}{(1+r)^T}$$

The PV of an annuity is found by splitting it into the difference between two perpetuities:

$$PV = \left(\frac{C}{1+r} + \frac{C}{(1+r)^2} + ... \right) - \left(\frac{C}{(1+r)^{T+1}} + \frac{C}{(1+r)^{T+2}} + ... \right)$$

$$= \frac{C}{r} - \frac{C}{(1+r)^T} \left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + ... \right)$$

$$= \frac{C}{r} \left(1 - \frac{1}{(1+r)^T} \right)$$

UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

LECTURE NOTES

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

QUARTER 1 - WEEK 3 (part 1)

Tu: 9/20/11

Section 4: The Valuation of Bonds and Stocks

Read Chapters 3 and 4 BMA

Motivation Example

Firms A and B have the following stock prices, earnings per share and price/earnings

ratios.

	Price	EPS	Price/EPS
Firm A	$100	$10	10
Firm B	$100	$5	20

There is no uncertainty associated with either stock.

Which stock would you buy?

It seems as though Firm A is the better buy since you get $10 earnings for every $100

invested whereas with Firm B you get only $5 for every $100 invested. Is this argument correct?

(i) Present Values and Market Values

So far we have been considering present values. Why are we interested in present

values? As we'll see next, present values must be equal to market values in a competitive

market. We shall consider the relationship between market values and present values using the

following example, which is also a useful exercise in calculating PV.

Example 1

A company called the Wensum Bridge Company is to be formed at the end of 20X0 for

the sole purpose of building a bridge across the river Wensum. The contract for the construction

of the bridge involves the Wensum Bridge Company receiving payments from the local township

of $1,089,000 at the end of 20X1 and $1,320,000 at the end of 20X2, at which time the bridge

will be finished and the company liquidated. The firm must pay out a total of $630,000 at the

end of 20X1 for materials and to its employees for building the bridge. Similarly, at the end of

20X2 it must pay out $780,000. After the 20X1 and 20X2 payments for materials and to

employees the remaining cash flow will be paid out in dividends.

All cash flows occur with certainty. Investors opportunity cost of capital is 10%. If there

are 10,000 shares, what is the price of each share at the end of 20X0?

Year	20X0	20X1	20X2
		_____	_____	
(a) Township Payments	$0	$1,089,000	$1,320,000
(b) Construction Costs	$0	$630,000	$780,000
(c) Dividends = (a)-(b)	$0	$459,000	$540,000
(d) PV Divs = (c)/1.1^t	$0	$417,273	$446,281

Solution

$$\text{Total PV of firm} = 417,273 + 446,281$$

$$= \$863,554$$

$$\text{PV each share} = 863,554/10,000$$

$$= \$86.36$$

Suppose these shares were being sold in the market at a price of $80. What would you

do? You should borrow money at 10% and buy as many as possible. You can use the dividend

payments to pay back the $80 you borrow for each share. Since you receive the equivalent of $86.36 in today's money, you make $6.36 on each share in terms of today's money.

Let's see exactly how this would work. The stock will pay dividends per share of 459,000/10,000 = 45.9 in 20X1 and 540,000/10,000 = 54 in 20X2.

	20X0	20X1	20X2
Amount borrowed	80		
Amount owed at year's end with interest		$80 \times 1.1 = 88$	
Use 20X1 dividends per share to reduce debt		$88 - 45.9 = 42.1$	
Amount owed at year's end with interest			$42.1 \times 1.1 = 46.31$
Use 20X2 dividends to pay off debt so profit is			$54 - 46.31 = 7.69$
PV of 7.69 in 20X2	$\dfrac{7.69}{1.1^2} = 6.36$		

The key point is that when you discount the dividends you are effectively taking account of the interest on the loan. You don't need to go through all of these calculations. You can simply compare the discounted present value of the dividends with the market price and know if you can make money by buying it. Whenever price is less than PV, you can make money in this way. This is a "money machine". But everybody will do this, and so the price will be driven up until it is equal to PV. Hence price cannot be below PV.

If the price of the stock is $90 and you own some, what should you do? You should sell them because in this way you get $90 whereas if you held them for their dividends, you would only get $86.36. Everybody who owns the stock will do this provided price is greater than PV, and hence prices cannot be above PV.

If you don't own the stock how could you make money? You could *short sell* it. This

means you borrow the share and sell it. When the price goes down you buy it back at the lower price and give it back to the person you borrowed it from.

In equilibrium it must therefore be the case that

$$\text{Price} = \text{PV}.$$

This is an example of an *arbitrage* argument. If $P \neq PV$, then people have incentives to buy or sell the stock until $P = PV$. The idea of arbitrage is a crucial one in finance and we will be using it a great deal in the coming weeks.

(ii) Valuation of Bonds and Stocks

A large amount of the activity of brokers and analysts and so on is essentially calculating PV's to see whether bonds and stocks are over- or under-valued. We're next going to spend some time devising formulae and looking at the way the PV's of bonds and stocks are calculated.

Bonds

Calculating the PV of bonds is straightforward. You simply discount the interest payments and terminal repayment of the debt at the opportunity cost. This is given by the yield on a similar bond. Note that the yield on a bond is the rate of return that would be earned on it if it was held to maturity, i.e. it is the discount rate implicit in the price of the bond. For example, for a two-year bond with annual coupon payments C (the first a year away), it is the value of y such that

$$P_{\text{Bond}} = \frac{C}{1+y} + \frac{C}{(1+y)^2} + \frac{\text{Principal}}{(1+y)^2}$$

Example 2

The Buggy Company has just issued a seven-year $1,000 bond. Interest payments of $80 are made at the end of each year, and the $1,000 principal must be paid at the end of the seventh year. If similar bonds are now yielding 6%, what price will the bond sell for?

Solution

t =	0	1	2		6	7
				...		
Interest		80	80		80	80
Principal						1,000

Cash flows

End of year	Amount	Discount/Annuity Factors	PV
1-7	$80	AF(7 yrs, 6%) = 5.582	446.56
7	$1,000	DF(7 yrs, 6%) = 0.665	665.00
			$1,111.56

Hence, the price of the bond = $1,111.56.

What do you think there would be if the bond sold at a price different from $1,111.56? As in Example 1 there would be arbitrage.

Stock Price and Dividends

The main difference in calculating the PV's of stocks rather than bonds is the difference in the profiles of earnings.

The cash payoff to stocks comes in two forms:

(1) Cash dividends

(2) Capital gains or losses

The price now is given by the present value of the dividend plus the present value of the price expected to obtain in 1 year.

Notation:

P_0 - current price of share

P_1 - price of share in 1 year's time

DIV_1 - Expected Dividend per share in 1 year's time

r - is the opportunity cost of the expected return on similar securities

Hence,

$$P_0 = \frac{DIV_1 + P_1}{1 + r}$$

But what determines next year's price, P_1? The same formula.

$$P_1 = \frac{DIV_2 + P_2}{1 + r}$$

Substituting for P_1 in the previous equation,

$$P_0 = \frac{DIV_1}{1 + r} + \frac{DIV_2}{(1+r)^2} + \frac{P_2}{(1+r)^2}$$

We can continue similarly. For H periods

$$P_0 = \sum_{t=1}^{H} \frac{DIV_t}{(1+r)^t} + \frac{P_H}{(1+r)^H}$$

Usually common stocks have no date at which they run out--they're unending.

$$P_0 = \sum_{t=1}^{\infty} \frac{DIV_t}{(1+r)^t}$$

Hence the price of a share is equal to the PV of future dividends.

Special Cases

   This basic formula can be difficult to use because of the problem of estimating DIV_t for each period. One assumption that is often made to allow stock prices to be calculated is that dividends will grow at a constant rate. In this case,

$$DIV_t = DIV_{t-1}(1 + g)$$

so that the dividend at $t = 1$ is DIV_1, the dividend at $t = 2$ is $DIV_1(1 + g)$, the dividend at $t = 3$ is $DIV_1(1 + g)^2$ and so on.

t =   0   1   2   3

   |_____|_____|_____|____ ...

              DIV_1 $DIV_1(1 + g)$ $DIV_1(1 + g)^2$

   Then

$$P_0 = \frac{DIV_1}{1+r} + \frac{DIV_1(1+g)}{(1+r)^2} + \frac{DIV_1(1+g)^2}{(1+r)^3} + ...$$

This is a growing perpetuity so we can use our formula for that:
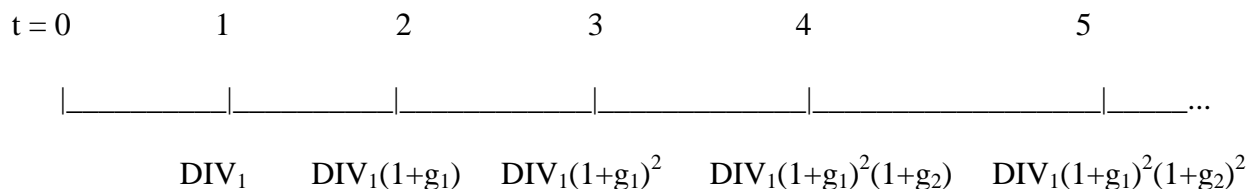
$$P_0 = \frac{DIV_1}{r - g}$$

You have to be very careful using this formula which assumes constant growth because firms have a life cycle. When they are young, they grow very quickly. Then they slow down and grow more moderately. Then they cease growth. Finally, they may shrink or go bankrupt. You have to exercise judgment as to whether present growth rates are likely to continue, and if they are not, you must determine what they will fall to and adjust the formula appropriately. Otherwise you may end up with some ridiculous answers. For example, very new firms sometimes grow at 50% per year. Clearly this can't continue for long. Good valuation requires great skill in choosing the right figures--it is not simply a matter of plugging in values. An example where a firm goes through two stages of growth is given next.

Example 3

Suppose stock A has $DIV_1 = \$10$ at date 1. This grows at $g_1 = 20\%$ for two years and $g_2 = 5\%$ forever thereafter. What is the current price at date 0 if the discount rate, r, is 0.10?

Solution

The first step is to develop an appropriate formula. The time line for the problem is as follows.

t = 0	1	2	3	4	5
	DIV_1	$DIV_1(1+g_1)$	$DIV_1(1+g_1)^2$	$DIV_1(1+g_1)^2(1+g_2)$	$DIV_1(1+g_1)^2(1+g_2)^2$

Let $DIV_3 = DIV_1(1 + g_1)^2$:

t = 0   1   2   3   4   5

|_____|_____|_____|_____|_____|_____|_____...

DIV_1 $DIV_1(1+g_1)$ DIV_3 $DIV_3(1+g_2)$ $DIV_3(1+g_2)^2$
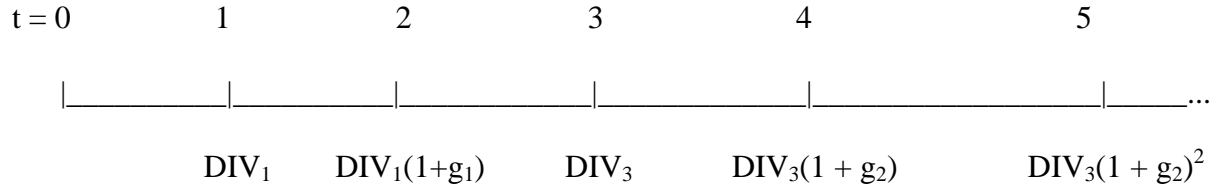
$$P_0 = \frac{DIV_1}{1+r} + \frac{DIV_1(1+g_1)}{(1+r)^2} + \frac{DIV_3}{(1+r)^3} + \frac{DIV_3(1+g_2)}{(1+r)^4} + \frac{DIV_3(1+g_2)^2}{(1+r)^5} + ..$$

$$P_0 = \frac{DIV_1}{1+r} + \frac{DIV_1(1+g_1)}{(1+r)^2} + \frac{1}{(1+r)^2}\left[\frac{DIV_3}{(1+r)} + \frac{DIV_3(1+g_2)}{(1+r)^2} + ...\right]$$

Since the last term in brackets is simply a growing perpetuity we can use the usual formula to

simplify

$$P_0 = \frac{DIV_1}{1+r} + \frac{DIV_1(1+g_1)}{(1+r)^2} + \frac{1}{(1+r)^2}\left[\frac{DIV_3}{r-g_2}\right]$$

We can then solve the original problem by substituting the numbers,

$$DIV_3 = 10 \times (1.2)^2 = 14.4$$

$$P_0 = \frac{10}{1.1} + \frac{12}{1.1^2} + \frac{1}{1.1^2}\frac{(14.4)}{0.05} = \$257$$

The basic methodology for valuing stocks is always the same as in this example. First

identify the stages of growth. The final stage is where the growth has settled down and you can

apply a growing perpetuity. The value of the stock when this final stage of growth starts is

usually called the "terminal value" or "horizon value." You find the value of the stock today by

evaluating the terminal value, discounting it back to the present and adding it to the discounted

stream of dividends occurring in the previous stages. If you have stages of growth with a constant growth rate then you can use a growing annuity to evaluate these stages.

The other thing to worry about when stocks are valued is when the first dividend is paid. The standard formulas assume the first dividend is paid at date 1. One way of phrasing this is to say a dividend "has just been paid" because that means DIV_0 has already been paid and the next one will be DIV_1. It will be paid one period from now at date 1.

Just been paid

```
0   Now                                              1
```

DIV_1
paid

DIV_0
paid

$$P_0 = \frac{DIV_1}{1+r} + \frac{DIV_2}{(1+r)^2} + ...$$

If the first dividend is to be paid at date 0 then this can be phrased as a dividend "is just about to be paid." In this case DIV_0 will be paid immediately and needs to be included in the valuation of the stock. Since it is paid immediately it does not need to be discounted back.

Just about to be paid

Now   0

                             1

DIV$_0$
paid

DIV$_1$
paid

$$P_0^* = DIV_0 + \frac{DIV_1}{1+r} + \frac{DIV_2}{(1+r)^2} + ...$$

Note that before the dividend is paid the price P_0^* is higher than the price P_0 just after it is paid. When the stock goes ex dividend so that the owner no longer has the right to receive the dividend the price drops by $P_0^* - P_0 = DIV_0$.

Suppose a dividend will be paid a while from now then stock price moves as follows.

Stock
price

DIVt
paid

DIV$_{t+1}$
paid

DIV$_{t+2}$
paid

DIV$_{t+3}$
paid

DIV$_{t+4}$
paid
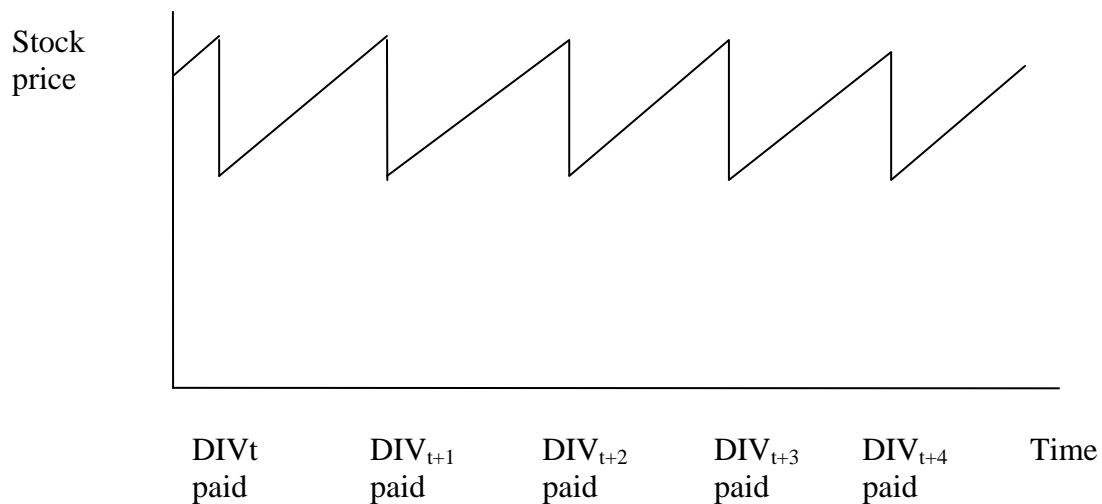
Time

Market Capitalization Rates

   A concept that is sometimes useful is the notion of <u>market capitalization rates</u>. The market

capitalization rate of a stock is the discount rate such that if the stock's dividends were

discounted at that rate you would obtain the stock's current market price. In other words, the

market capitalization rate is the discount rate implicit in the stock's current market price. It is

like the yield on a bond. For example, for a stock with dividends DIV_1, DIV_2, and so on, it is the

value of r_m such that

$$P_{Stock} = \frac{DIV_1}{1 + r_m} + \frac{DIV_2}{(1 + r_m)^2} + ...$$

where P_{Stock} is the current market price of the stock.

        For example, consider a very simple example of a stock which has $P_{Stock} = \$35$ and is

expected to have dividends next year of $1.75 per share and these are expected to grow

subsequently at a constant rate of 4% per year for ever. What is the market capitalization rate?

        In the case of a stock with a constant rate of growth of dividends

$$P_{Stock} = \frac{DIV_1}{r_m - g}$$

Rearranging,

$$r_m = \frac{DIV_1}{P_{Stock}} + g$$

$$= \frac{1.75}{35} + 0.04 = 0.09 \text{ or } 9\%$$

        In general, we would expect the market capitalization rate to be equal to investors'

opportunity cost of capital. Why is this? - Arbitrage. In Example 1 we argued that if the price of

the stock was below its present value (evaluated at investors' opportunity cost of capital) it would

be bid up to its present value. Similarly, here we can argue that if a stock's market capitalization rate is above investors' opportunity cost of capital, arbitrage will ensure that this situation won't persist for long. It will be driven to equality with investors' opportunity cost of capital. Similarly, if the market capitalization rate is below investors' opportunity cost of capital this won't persist for long because of arbitrage.   Thus market capitalization rates allow us to view arbitrage from a slightly different angle.

Market capitalization rates can sometimes also be useful in helping you to find out what a discount rate for a company is. Provided you have estimates of the growth of dividends then you can always back out the market capitalization rate. There are various ways to find the dividend growth rate. You can use direct estimates or alternatively you can try to estimate it from how much the profitability of the investments retained earnings are ploughed back into will allow dividends to grow. However, we will not go into this in detail since we will argue in Sections 7 and 8 on risk that a better way to get discount rates is to use an asset pricing model.

Stock Price and Earnings Per Share

So far we've looked at stock price and dividends. One of the statistics that is often used in the financial press and conversation is the ratio of the stock price to earnings per share (P/E ratio). The motivation example at the beginning of the section is an illustration of how it's used: investors are advised to buy stocks with low P/E ratios. This is often referred to as "value investing". In the early 1990's there were a number of studies showing that in the long run investing in low P/E ratios or "value stocks" had given high returns. In the tech boom value stocks did terribly. Since then the relative performance has been mixed. In the last part of this

section we shall develop the relationship between price and EPS and reconsider our motivation

example in this light. We will try and provide some perspective on how P/E ratios should be

interpreted.

In the previous sections we looked at price and dividends. The difference between

dividends and EPS is that the former is what's actually paid to stockholders, whereas the latter

includes retained earnings.

We start by considering a firm that does not retain any earnings at all. We can think of it

consisting of a single plant. It does not plow back any earnings and simply produces a constant

stream of earnings, which are all paid out in dividends.

t =	0	1	2	3	. . .
		_____	_____	_____		. . .
PLANT EARNINGS		EPS	EPS	EPS	
= DIVIDENDS		EPS	EPS	EPS	

Hence, since $DIV_t = EPS$, and our previous theory is applicable, so

$$P_0 = EPS \sum_{t=1}^{\infty} \frac{1}{(1+r)^t}$$

$$= \frac{EPS}{r}$$

Rearranging,

$$\frac{P_0}{EPS} = \frac{1}{r}$$

Hence, the price/earnings ratio is equal to one over the implicit discount rate in the special case of a static firm. If you know the P/E for this type of firm, you can work out the market capitalization rate (the rate you would have to discount the cash flows at to obtain the market price). Thus a price/earnings ratio of 20 is the same as a 5% market capitalization rate.

Now suppose our static company uses its period 1 earnings of EPS to invest in a new plant in period 1 instead of paying a dividend. Thus $DIV_1 = 0$. This new plant earns EPS_{NEW} every period from period 2 onwards. The firm still has the original plant that is still earning EPS in every period. From the second period onwards the firm pays out all its earnings as dividends so that from period 2 onwards $DIV_t = EPS + EPS_{NEW}$.

t =	0	1	2	3	. . .
		_____|_____|_____|	. . .
OLD PLANT		EPS	EPS	EPS	
NEW PLANT		-EPS	EPS_{NEW}	EPS_{NEW}	
DIVIDENDS		0	$EPS+EPS_{NEW}$	$EPS+EPS_{NEW}$	

In this case,

$$P_0 = \sum_{t=2}^{\infty} \frac{EPS + EPS_{NEW}}{(1+r)^t}$$

We can rewrite this in the following way separating it into two terms one representing the cash flows from the old plant and the other the cash flows from the new plant

$$P_0 = \sum_{t=2}^{\infty} \frac{EPS}{(1+r)^t} + \sum_{t=2}^{\infty} \frac{EPS_{NEW}}{(1+r)^t}$$

OLD PLANT   NEW PLANT

We are going to write this expression in a slightly different form so we can interpret what is going on. Adding and then subtracting EPS/(1 + r) we get

$$P_0 = \sum_{t=2}^{\infty} \frac{EPS}{(1+r)^t} + \frac{EPS}{1+r} - \frac{EPS}{1+r} + \sum_{t=2}^{\infty} \frac{EPS_{NEW}}{(1+r)^t}$$

Combining the first two terms,

$$P_0 = \sum_{t=1}^{\infty} \frac{EPS}{(1+r)^t} - \frac{EPS}{1+r} + \sum_{t=2}^{\infty} \frac{EPS_{NEW}}{(1+r)^t}$$

From our first case with no investment we know the first term is just EPS/r. The second and third terms taken together are just the NPV of the investment in the new plant in terms of time 0 dollars. Hence we can rewrite the expression as

$$P_0 = \frac{EPS}{r} + NPV_1$$

where NPV_1 is the NPV of the new plant which is built in the first period. NPV_1 is the NPV of the *growth opportunity* represented by the new plant.

We have considered a very simple case where we build a plant at date 1 using all the earnings from the original plant. Now if you go through the same exercise you can see that a similar conclusion would hold if we used only a proportion of the old plant's earnings to build the new plant. For example, if you used half the earnings at date 1 to build the new plant then everything would be similar except instead of adding and subtracting EPS/(1 + r) so that we could get our formula we would add and subtract 0.5[EPS/(1 + r)]. We would still get NPV_1 as the final term but now it would be the NPV of the smaller plant. Similarly, if we did the same

exercise using earnings at date 2 to build a plant we would get the same but with NPV_2 which

would be the NPV of the date 2 investment. In other words, we can similarly add other possible

projects and go through the same process. We will end up with the formula

$$P_0 = \frac{EPS}{r} + NPVGO$$

where NPVGO is the net present value of all the growth opportunities of the firm. (Note that

BMA uses a slightly different terminology and calls this the present value of growth

opportunities.)

Hence, dividing through by EPS,

$$\frac{P_0}{EPS} = \frac{1}{r} + \frac{NPVGO}{EPS}$$

Usually firms only invest in projects with positive NPVGO since otherwise they could make a

better profit from investing in the best alternative opportunity. Hence, usually the P/E ratio is an

overestimate of 1/r (or equivalently the inverse of the P/E ratio is an underestimate of r).

You often hear P/E ratios used as in the motivation example: stock B is selling at a

higher P/E ratio than stock A, so it's a worse buy than stock A. This is not a good argument.

You can see this in the context of the motivation example. Since both stocks have no uncertainty

associated with them they will have the same market capitalization rate r. If they did not what

would there be? – Arbitrage.

If there is no uncertainty so we know all the cash flows accruing to a stock how do we

value it? We discount them back at the opportunity cost. If the price is too low what do people

do - they buy and this drives the price up just as in Example 1 with the Wensum Bridge

Company. As we saw above, another way of saying the price is too low is that the market

capitalization rate is too high. In this case arbitrage drives the market capitalization rate to equal

the discount rate. When there is no uncertainty we would expect arbitrage to ensure that the

market capitalization rate on stocks is the same. Given the stocks have the same market

capitalization rate, the reason for the difference in P/E ratios must be that firm B has more

growth opportunities.

What happens if there is risk? In this case, as we will soon see, we can use a similar

argument but now the stocks must be in the same risk class if they are to have market

capitalization rates equated. We'll be returning to this soon.

When stock prices accurately reflect the discounted cash flows then we have *efficient*

markets. This is one of the central ideas in finance and we will consider it in depth in the second

part of the course.

The important point here is that a low P/E ratio probably does not mean a stock is a good

buy, it means the firm has few growth opportunities. It's what is termed an "income stock." If a

firm has a high P/E ratio it probably doesn't mean it is a bad buy it means it has a lot of growth

opportunities - it's what is termed a "growth stock". If these differences occurred because a

stock was over- or under-priced what would happen? There would be arbitrage.

To take an extreme example of a high P/E ratio, what are the earnings of biotechnology

firms early in their history when they are just engaged in research? They are zero but the stock

price is positive so the P/E ratio is infinite. This just reflects the fact that biotech firms have

many growth opportunities. It does not mean they are a bad buy.

Going back to our motivation example the reason the P/E ratios differ is probably

because of a difference in growth opportunities. The market capitalization rates are probably the

same (assuming the stocks have the same risk) so the stocks are equally good buys.

One final point to make here is that we are thinking of EPS as an accurate measure of cash flow. In fact EPS is an accounting figure, and this may mean many different things in different companies depending on the conventions and rules the companies adopt. In general you have to be very careful of accounting figures since often they don't correspond to the notions that financial economists use. If different firms use different accounting conventions then clearly their P/E ratios will be different. You must be careful to be using earnings figures calculated on a similar basis.

UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

LECTURE NOTES

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

WEEK 3 (part 2) and WEEK 4 (part 1)

Th: 9/22/11 and Tu: 9/27/11

Section 5: A Comparison of Investment Criteria

Read Chapter 5 BMA

Motivation

Consider a project with the following cash flows:

C_0	C_1	C_2	C_3
-925	+1000	+1400	-1500
Initial Investment			Clean-up Costs

Internal Rate of Return = 4.6%

Opportunity Cost of Capital = 10%

Should we accept this project?

Introduction

In Section 2 we looked at the NPV rule and showed that it was a good rule for the managers of corporations to follow. NPV is a measure of wealth created for shareholders so maximizing NPV is like maximizing the wealth of shareholders. We had a simple model, but you can show that in much more complex environments the same rule still holds, i.e., positive NPV corresponds to pushing out the budget constraint, and makes shareholders as well off as possible. The NPV rule always works. We can always go through the equivalent exercise of seeing how much better off we would be compared to putting the money in the bank.

In addition to the NPV rule, we also had the rate of return rule. In that case they were equivalent, but in general which should we use? Why go to the bother of calculating NPV when

we could find out whether to invest or not by comparing rate of return with opportunity cost? As

the motivation example illustrates it's not always easy to get the correct answer with the rate of

return rule.

In this section we are going to argue for the superiority of the NPV rule over the rate of

return rule or, in its more general form, the internal rate of return (IRR) rule and also over

various other traditional methods of project appraisal.

We will start with why it is better than the IRR. We shall argue that they are equivalent

provided the IRR rule is properly applied, but it is more difficult to apply the IRR rule, and

hence we should use the NPV rule.

In the one period case we considered before, the one period rate of return R was given by

$$1+R = \frac{\text{Payoff}}{\text{Investment}} \Rightarrow 1+R = \frac{C_1}{-C_0} \Rightarrow C_0 + \frac{C_1}{1+R} = 0$$

In the more general case with more than one period it is not entirely clear what is meant by the

rate of return, so instead we use the more general concept of IRR.

What is the IRR? The IRR extends the last form given above for the rate of return. It is

the rate you discount at such that the discounted cash flow (DCF) is zero. The DCF is a function

of the rate, i, and the cash flows:

$$DCF(i) = C_0 + \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + ... + \frac{C_T}{(1+i)^T}$$

The IRR is the value of i such that DCF = 0, i.e.:

$$DCF(IRR) = C_0 + \frac{C_1}{1+IRR} + \frac{C_2}{(1+IRR)^2} + ... + \frac{C_T}{(1+IRR)^T} = 0$$
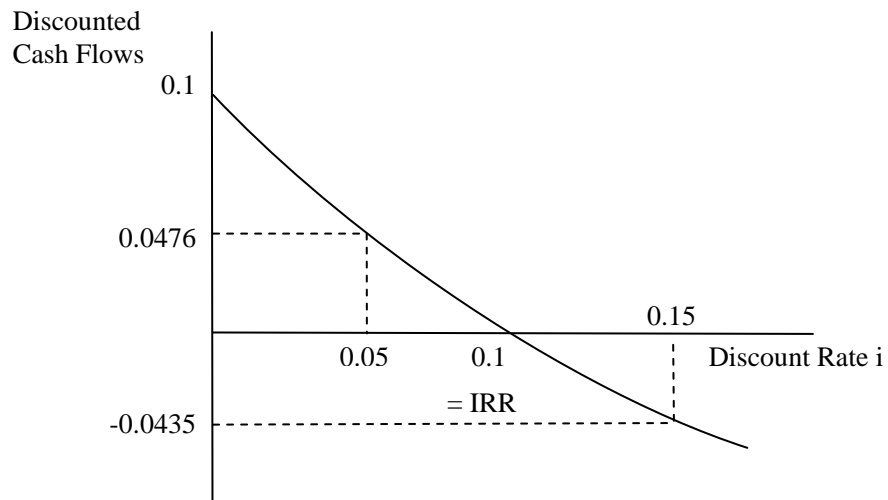
Finding the IRR

The easiest way to find IRR is a calculator. One question that may occur to you is how is the calculator finding it? To see this, it is helpful to consider what is going on graphically. We can calculate the DCF at a number of values of the discount rate, join them up and then read off the value of the discount rate at which DCF = 0. This is the IRR. For example, suppose we have the following cash flows:

Example 1 C_0 C_1

                                   -1   +1.1

In this case we have the following values of DCF for various values of i:

i	DCF(i)
0	-1 + 1.1 = 0.1
0.05	-1 + (1.1/1.05) = 0.0476
0.1	-1 + (1.1/1.1) = 0
0.15	-1 + (1.1/1.15) = -0.0435

Hence the IRR in this case is 10% since this is the value of i at which DCF is 0.

When you use a calculator what it is effectively doing is the same thing. It is finding where DCF cuts the axis.

IRR Rules

Now that we know how to extend the definition of rate of return to many periods how do we extend our simple rate of return rule from Section 2? An obvious extension would be:

Internal Rate of Return Rule (1)

Accept a project provided IRR is above the opportunity cost of capital.

In some cases this rule is perfectly correct, but we have to be careful because in other cases it has to be adapted in order to lead to the same decisions as the NPV rule.

Cases where IRR Rule (1) is the same as the NPV Rule

Let's consider a case like our example where the DCF is a downward sloping function of i.

DCF

$i > IRR$
$NPV < 0$

REJECT

IRR

i

$i < IRR$
$NPV > 0$
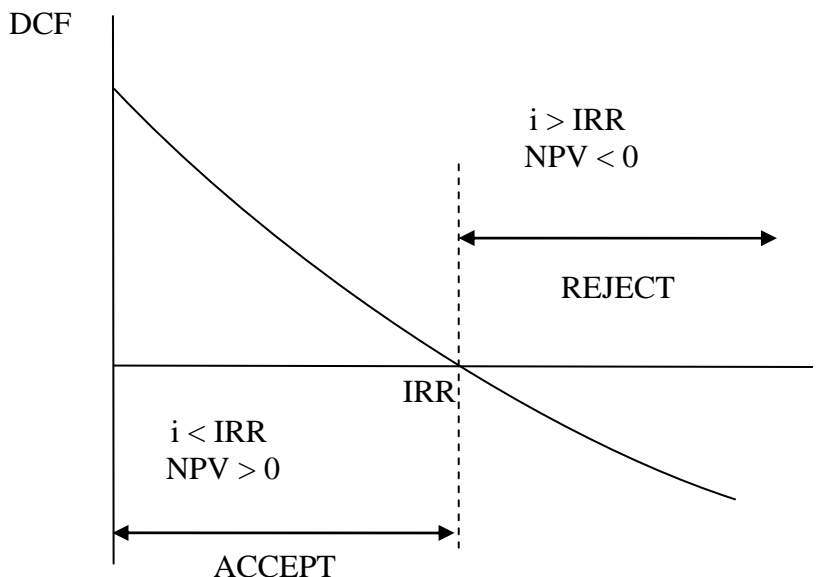
ACCEPT

When does the IRR rule (1) say that we should accept projects? It says we should accept when the IRR is above the opportunity cost of capital. For example, if the IRR is 10% and the opportunity cost of capital is 5% we should accept the project. In other words, we should accept at points to the left of the IRR. We should reject when the IRR is below the opportunity cost of capital. For example, if the IRR is 10% and the opportunity cost of capital is 15% we should reject the project. In terms of the diagram we should reject to the right of the IRR.

When does the NPV rule say that we should accept projects? The NPV of a project is just the DCF where the rate used to discount is the opportunity cost of capital. Keeping with the same example, when the opportunity cost of capital is 5% it can be seen from the diagram that the NPV is positive. Hence we should accept. Similarly, we should accept whenever the opportunity cost of capital is such that NPV is positive. When the opportunity cost of capital is 15% the NPV is negative and we should reject the project. Similarly, we should reject whenever NPV is negative.

In this case the IRR rule (1) is the same as the NPV rule. In fact it can be seen from the diagram that whenever the DCF of a project is a downward sloping function of i, the NPV and the IRR rule (1) give the same result.

When Must IRR Rule (1) Be Adapted to Correspond to the NPV Rule

Now that we have seen that the two rules are the same when DCF is a downward sloping function of i, when do you think they will differ? Suppose DCF is an upward sloping function of i:

DCF

NPV < 0 => REJECT
i < IRR => ACCEPT ?

IRR

i

NPV > 0 => ACCEPT
i > IRR => REJECT ?
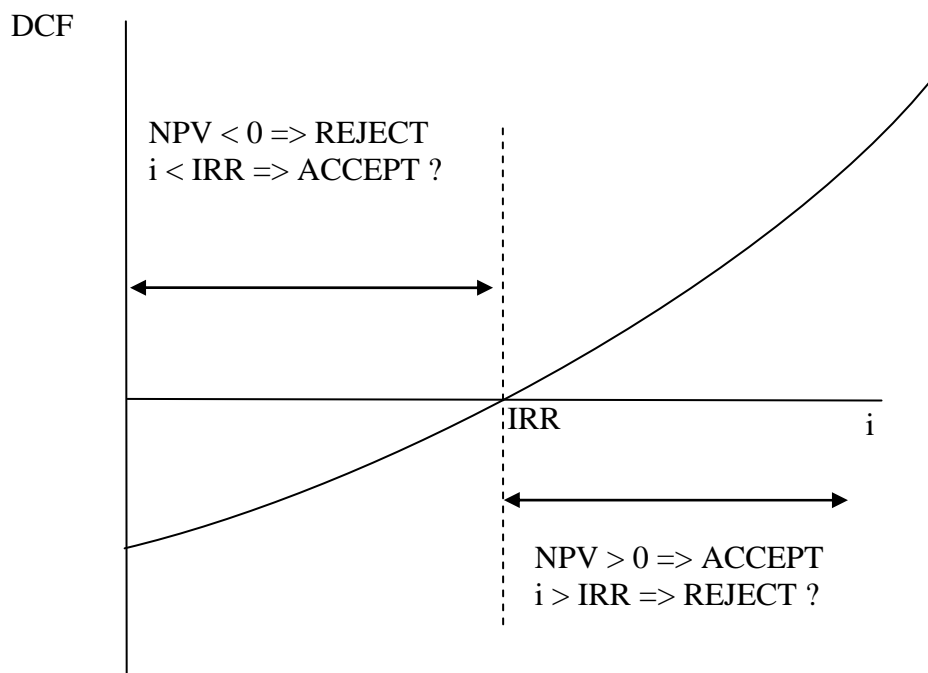
The IRR rule (1) again says we should accept the project to the left of the IRR. However,

you can see that in this case the NPV is negative so that the NPV rule would say that we should

reject the project. To the right of IRR the IRR rule (1) would say that we should reject but NPV

is positive so that the NPV rule would say we should accept. We know from Section 2 that the

NPV wealth created so in this case IRR rule (1) must be wrong. What is happening intuitively

here? To see this, let's look at a very simple example where DCF is upward sloping.

Example 2	C_0	C_1
	+1	-1.1

Notice that this is the same as our original example except that the signs are reversed. Hence the

numerical values of DCF at various values of i will be the same but the signs will be reversed.

i	DCF
0	$+1 - 1.10 = -0.1$
0.05	$+1 - (1.1/1.05) = -0.0476$
0.10	$+1 - (1.1/1.1) = 0$
0.15	$+1 - (1.1/1.15) = 0.0435$

In this case we have the type of relationship depicted above. What is happening? You receive 1 now and pay back 1.1 next period; in other words, you are "borrowing" and the IRR is like a borrowing rate. Should you try to borrow at a high rate or at a low rate? At a low rate. If you can borrow at less than the opportunity cost of capital then clearly you can make a profit (i.e. $NPV > 0$) and this is what you should do--but this is what the NPV rule is saying.

Consider our original example where DCF is downward sloping:

C_0	C_1
-1	+1.1

In this case you are laying out $1 now in return for a payment of $1.1 next period. It is like "lending" and the IRR is like the lending rate. You want to lend at as high a rate of interest as possible. If the rate you can lend at is greater than the opportunity cost of capital, you can make a profit so that $NPV > 0$. Again you can see that the NPV rule is intuitively right.

IRR rule (1) needs to be adapted to take account of whether you are "lending" or "borrowing". This gives IRR rule (2).

IRR Rule (2)

(i) If DCF is a downward sloping function of i, accept the project provided IRR is above the opportunity cost of capital.

(ii) If DCF is an upward sloping function of i, the project should be accepted if the IRR is less than the opportunity cost of capital.

The IRR rule (2) will then be equivalent to the NPV rule.

In order to use IRR rule (2) we have to know whether a project is a "lending" or "borrowing" project. In what types of situation is it likely to be the case that NPV is a declining function of the discount rate i?

$$DCF = C_0 + \frac{C_1}{1+i} + \frac{C_2}{(1+i)^2} + ... + \frac{C_T}{(1+i)^T}$$

where T is the terminal date of the project.

There is one obvious case where this is downward sloping (which is what we need for the two rules to be equivalent). This is where the first cash flow is an outlay on the investment project so $C_0 < 0$ and the cash flows in all the subsequent periods are positive so $C_1, C_2, ... C_T > 0$. As i goes up, the positive terms in the DCF expression get smaller since they are divided by $(1+i)^t$ term whereas the negative C_0 term stays the same. Hence DCF gets smaller as i gets bigger and the expression is downward sloping. In this case the project is guaranteed to be a lending project.

However, if C_1 or C_2 or any C_t up to $C_T < 0$, then the possibility arises that DCF is

upward sloping and the project is a borrowing one. The reason is that there are now negative

terms that become smaller in magnitude as i rises so DCF may go up. Thus it is not always that

easy to discover which case you are dealing with. For example, consider the following situation

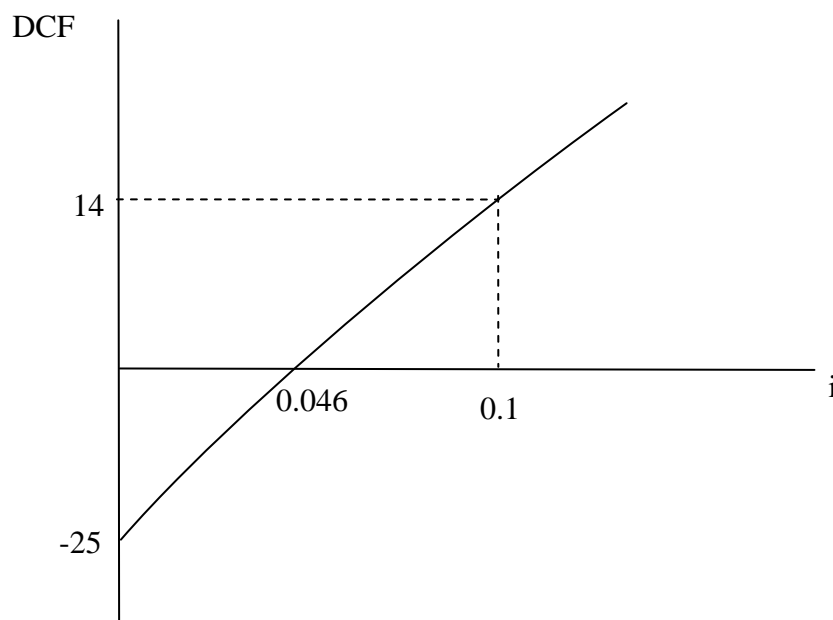that we looked at in the motivation example.

C_0	C_1	C_2	C_3
-925	+1000	+1400	-1500
Initial Investment			Clean-up Costs

Such a profile of cash flows may arise, for example, with many kinds of environmental

remediation where you have leave the site as you find it. What is the relationship between DCF

and i in this case?

$i = 0\%:$ $DCF = -925 + 1000 + 1400 - 1500 = -25 < 0$

$i = 10\%:$ $DCF = -925 + \dfrac{1000}{1.1} + \dfrac{1400}{1.1^2} - \dfrac{1500}{1.1^3} = 14 > 0$

Hence we have something like this:

We then know it's a borrowing situation, and we should apply part (ii) of rule (2).

To summarize, if the first cash flow is negative and all the subsequent ones are positive then you can use IRR rule (1). However, if there are any negative cash flows after time 0 you have to use rule (2) and check whether it's a lending or borrowing project. This takes a fairly long time to do.

With the NPV rule we don't have to determine the situation we are in--we just calculate NPV. What NPV is measuring in terms of money today is the amount you are better off if you use the 925 to do the project compared to just putting the 925 in the bank. Consider this way of thinking for our motivation example. Let's start by considering what you end up with at date 3 first for just putting the money in the bank and second putting it in the project and taking the cash flows from that and putting them in the bank.

t =	0	1	2	3
(i) Put the 925 in the bank				$925 \times 1.1^3 = 1231$

(ii) Put the 925 in project

Project cash flow at date 1	1000			$1000 \times 1.1^2 = 1210$
Project cash flow at date 2		1400		$1400 \times 1.1 = 1540$
Project cash flow at date 3				-1500
Total at date 3 from project				1250

Wealth created by project relative to bank in date 3 money $= 1250 - 1231 = 19$

"   "   "   "   "   "   date 0   " $= 19/1.1^3 = 14 = $ NPV

The great advantage of NPV is it does this calculation automatically for you. It tells you immediately how much wealth is created for shareholders by a project.
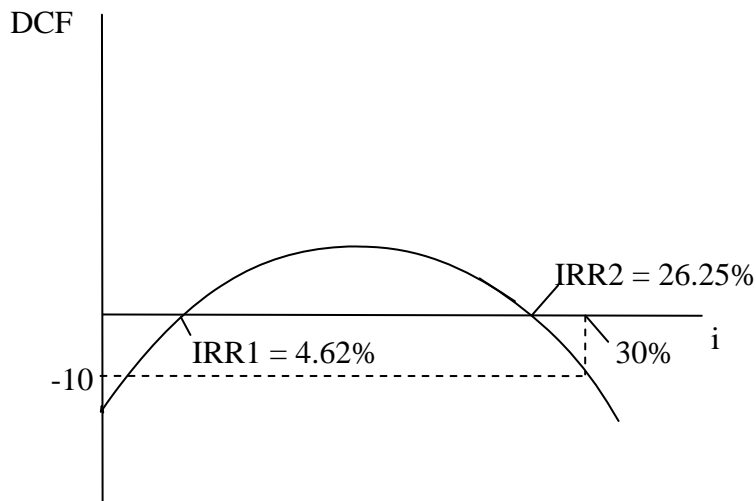
Multiple roots

The "lending"-"borrowing" problem is a disadvantage for the IRR rule. Moreover, it is not the only one. There is also the problem of multiple roots that we come to next.

Let us continue with our motivation example

$$i = 30\% \qquad DCF = -925 + \frac{1000}{1.3} + \frac{1400}{1.3^2} - \frac{1500}{1.3^3} = -10 < 0$$

We thus have the following situation

In this case there are two values of i which are such that DCF = 0. Which is the internal rate of return? We can devise a rule to deal with this type of case, but it is complicated.

IRR Rule (3)

(a) If there is just one IRR use IRR rule (2).

(b) Given two IRR: IRR_1, and IRR_2, there are two possible situations.

(i)   If at IRR_1, DCF is upward sloping and at IRR_2 DCF is downward sloping (i.e. the curve

        is n-shaped), then we should accept the project if IRR_1 < Opportunity cost of capital <

        IRR_2. Otherwise, we should reject it.
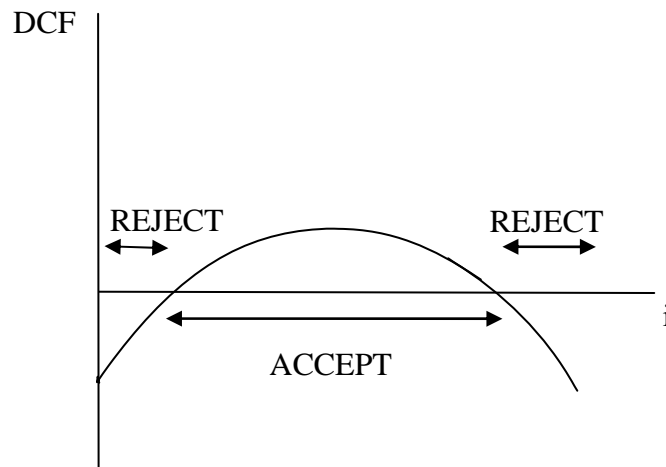
(ii)   If at IRR_1, DCF is downward sloping and at IRR_2, DCF is upward sloping (i.e. the curve

        is u-shaped), we should accept if the opportunity cost of capital < IRR_1 or > IRR_2.

        Otherwise we should reject it.
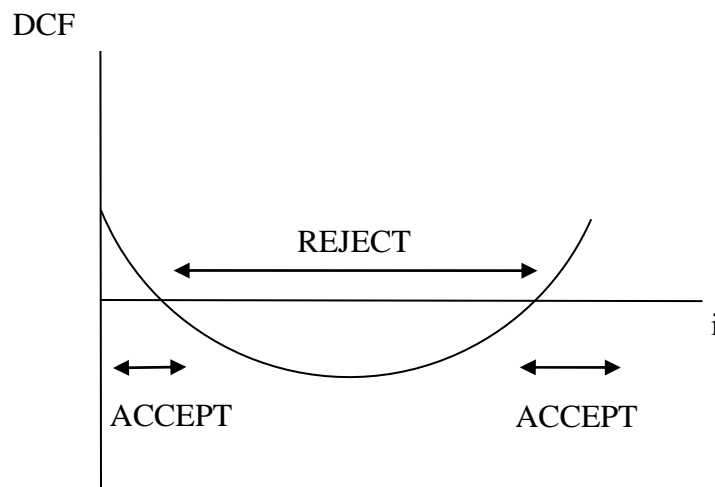
This is obviously an immensely complicated rule to apply. In fact, the problem is even more serious than this example indicates since in general we may have many more than two values at which DCF = 0, especially if the project is long-lived and there are many alterations between +ve and -ve cash flows. For each of these possibilities we will have to develop a version of our IRR rule.

In situations where the first cash flow is negative and subsequent cash flows are positive it is fine to use IRR otherwise it is simpler just to calculate NPV.

Mutually Exclusive Projects

Our problems with IRR are not at an end yet. Investment decisions are often concerned with mutually exclusive projects, that is, ones in which you are choosing among different ways of doing the same thing. In other words, in some circumstances you can only do one project, you can't do all of them. An example would be replacing a machine on a production line; there may be several alternative machines you could potentially use but you can only use one of them to replace the old one.

It would seem that the appropriate rule in this type of situation is:

"Naive" IRR Rule:

Choose the mutually exclusive project with the highest IRR.

Consider projects A and B depicted below, which are mutually exclusive.

DCF

IRR A   IRR B   i

A   B
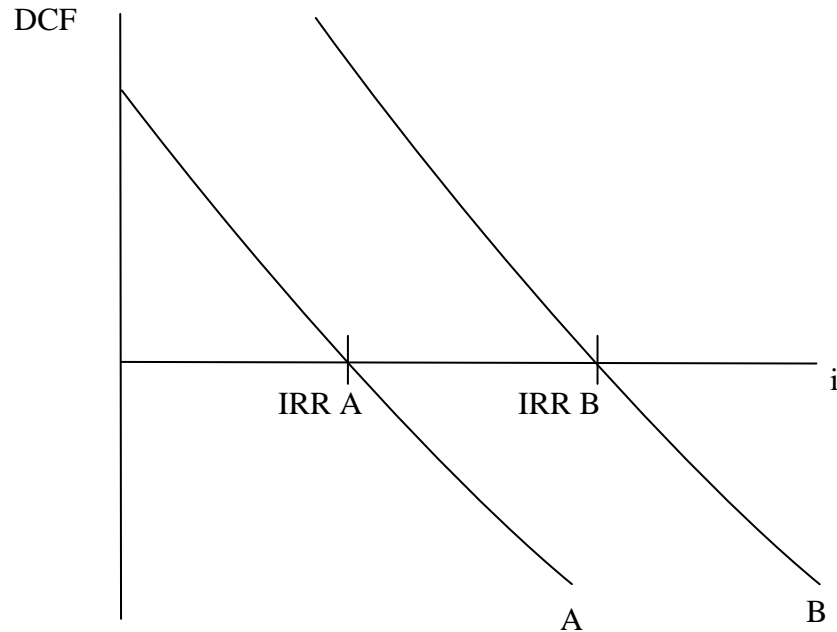
In this case there is no problem--B is clearly superior to A. It always has a higher NPV and its

IRR is higher so that the naive rule works. When will the naive rule fail to work? Consider the

following situation:

DCF

IRR B

i^* i

IRR A

B

A
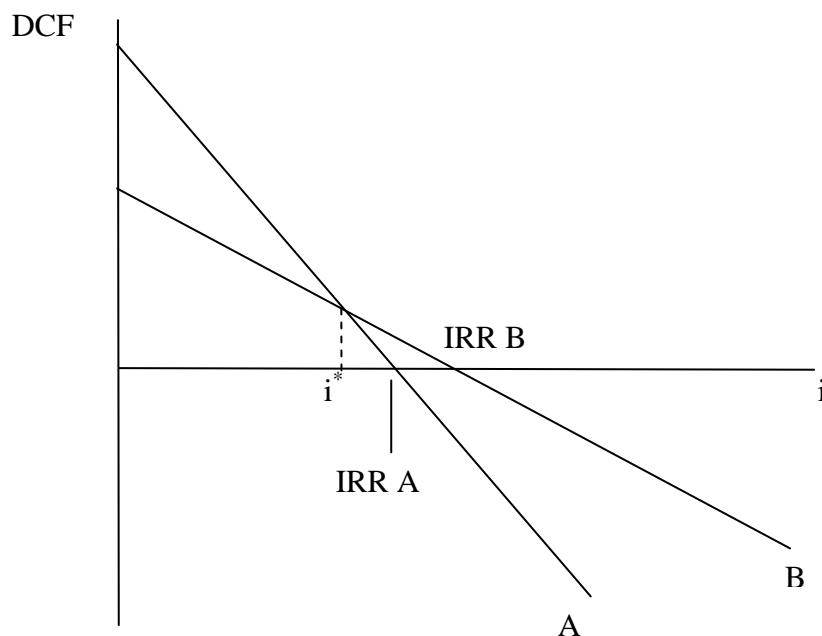
Which is superior, A or B? In terms of NPV, B is superior when the opportunity cost of capital is above the crossing point i* and inferior when it is below i*.

In terms of the naive IRR rule, however, B would appear to be superior regardless of i. What is going on here intuitively? To see this let's consider an example.

Example

Project	C_0	C_1	IRR	NPV at 10%	NPV at 20%
A	-1,000	+1,200	20	91	0
B	-10	+30	200	17	15

It can be shown that in this example i* = 18.18.

If you graph A and B for NPV against i for this example you get the type of situation shown above with 10% < i* = 18.18%.

What is happening is a question of <u>relative</u> versus <u>absolute</u> returns. A has a low relative return but a high absolute return; B has a high relative return but a low absolute return. Given they're mutually exclusive you can do one A or one B, you can't do both or multiple B. What's important is which gives you the highest absolute return since that's the amount that your budget constraint is pushed out or in other words the money that actually goes into your pocket. You may be better off investing a large amount at a low return than a small amount at a high return.

It is possible to fix up the IRR rule by looking at the IRR on incremental cash flows. This is rather painful and so we will not go through it here. I will leave you to read the details in the book. It is much simpler just to use NPV.

One important point to note is that the analysis above assumes perfect capital markets. This implies that it is possible for the firm to borrow as much as it would like to. If capital markets are imperfect and the firm is limited in the amount it can borrow there is said to be "capital rationing." The way that capital budgeting decisions should be made in this case is considered further below.

IRR and the Term Structure of Interest Rates

So far we have been assuming that there is just one opportunity cost of capital, which is the same for every period. However, as we discussed in Section 3 this may not be the case. The term structure may not be flat.
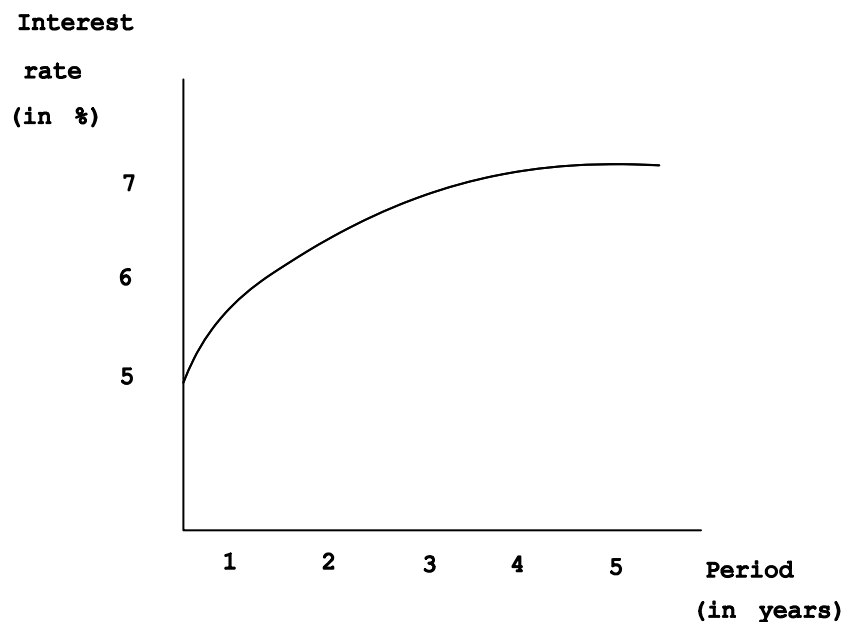
In this case the opportunity cost of capital for different periods differs, and we must calculate NPV using the formula

$$NPV = C_0 + \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + ...$$

We discount C_1 at the opportunity cost of capital for 1 year, C_2 at the opportunity cost of capital

for 2 years, and so on.

Now if we use the IRR method, which opportunity cost of capital do we compare it to?

In a situation where the yield curve is not approximately flat we have to compare the project IRR

with the expected yield to maturity offered by a traded security that

   (1) offers the same time pattern of cash flows;

   (2) is equivalent in risk to the project.

This is obviously a difficult task. Again the NPV rule is much simpler.

Next we consider the advantages and disadvantages of a number of other investment

criterion that are used by firms.

Other Investment Criteria: Profitability Index and Payback

Profitability Index

Profitability index (PI) = NPV of project
                            Initial investment

Naive Criterion: Accept project if PI > 0.

It can be seen that only projects with a positive NPV are accepted, so it is again like the

NPV rule.   However, as with the IRR, the profitability index ranks mutually exclusive projects

in a different way from NPV. The problem, as with IRR, is that they are both <u>relative</u> measures

of profitability, whereas NPV is an <u>absolute</u> measure, and it is the latter which is relevant here

since this determines the amount the budget constraint of shareholders can be shifted outward.

Consider the example we used with IRR and mutually exclusive projects again.

Project	C_0	C_1	NPV at 10%	PI
A	-1,000	+1,200	91	0.091
B	-10	+30	17	1.7

Here we should choose A since this pushes out the budget constraints of shareholders the farthest. A naive profitability rule which chose the project with the highest PI would give an incorrect answer; it would say choose B. However, we can again adapt the PI method to deal with this by looking at the PI of the incremental investment similarly to IRR. Again, appropriately adapted, PI, like IRR, is not incorrect but NPV is easier to use.

Payback

The payback rule requires that the initial outlay on any project should be recoverable within some specified cutoff period. The payback period is calculated by counting the number of years it takes before forecasted cash flows equal the initial investment.

Example

Project	C_0	C_1	C_2	C_3	Payback	NPV @ 10%
A	-300	+300	0	0	1 year	-27
B	-300	+100	+100	+500	3 years	252

With the payback method we will accept A if the cutoff period is 1 year or over and B if it is 3 years or over. With NPV, however, we would reject A and accept B. Thus payback can give incorrect answers to the capital budgeting problem, unlike IRR and PI which were just more difficult to use.

Why doesn't payback give the same results as NPV? There are three main problems.

(i) The first difference is clearly that with payback no account is taken of the difference in the value of $'s at different dates. To try and deal with this we could use the discounted payback method, which takes into account the time value of money, but this still suffers from the other two defects.

(ii) The second defect is that the cutoff date is fairly arbitrary. If you use the same cutoff, regardless of the project, then you reject too many long lived projects and accept too many short lived ones.

There are certain special cases where we can calculate a formula for the optimal cut off period that corresponds to maximizing NPV, e.g., if flows are spread evenly across the life of the project, but in general this is not possible.

(iii) The third defect is that it ignores all flows after the cutoff date so there is a bias against long lived projects.

Does payback have any advantages? One advantage of payback is its ease of calculation compared to NPV. With short time horizons where things are constantly being replaced and the projects are small (e.g., office supplies and equipment) it may be worthwhile to use the payback method because of its ease of computation. In the majority of cases we shouldn't use it though since, unlike IRR and PI, it gives incorrect answers.

There are various other criteria that are used for investment decisions. Many are accounting based such as the book rate of return. These other methods are either difficult to use or do not correspond to creating wealth for shareholders. I won't go through the deficiencies here but you can read about them. NPV is better because it is relatively easy to use and directly measures what we are interested in namely how much wealth is created for shareholders.

Capital Rationing

We have argued above that one of the problems with IRR and PI is that they are relative measures rather than absolute measures. We suggested it is the absolute amount of money received that shareholders are interested in so it can be better to take a large investment with a low rate of return than a small investment with a high rate of return. Implicit in this argument is the assumption that capital markets are perfect and the firm can invest as much as it wants to. Capital rationing occurs when the firm has limitations on the capital it can invest that prevent it from undertaking all investment opportunities with positive net present value that are available. We next consider how investment decisions should be made in this case.

There are a number of reasons why firms may operate with capital rationing. Management may believe a rapid expansion will overtax the organization and use a capital constraint as a proxy in restricting this growth. This is an example of soft rationing, where the constraint is internally applied.

Because of problems of fraud, people may be reluctant to give any particular firm too much money. This is an example of hard rationing, which is externally imposed by the market.

In these types of situations, how should the firm make its investment decisions? It is first

necessary to check that the assumption of perfect capital markets for shareholders is not violated. Unless this is true, maximizing NPV will not remain a valid objective for the firm. Assuming there are perfect capital markets for shareholders, the firm should choose its investment, I, to maximize its NPV subject to the constraint that I is not more than I^* where I^* is the maximum amount of capital that can be used.

We want to maximize the NPV of the I^* we are allowed to invest and get the "biggest bang for our buck". We need to select the projects that will maximize NPV per dollar. One natural intuition is that we should rank by some relative measure such as PI or IRR and accept projects until we have exhausted the capital constraint. As we shall see, this method sometimes works and sometimes does not.

Consider what happens if you have two projects, A and B, which require equal investments I_A and I_B, respectively, and which have 1+PI's as follows. Plot 1+Profitability Index (1+PI) against Investment (I).
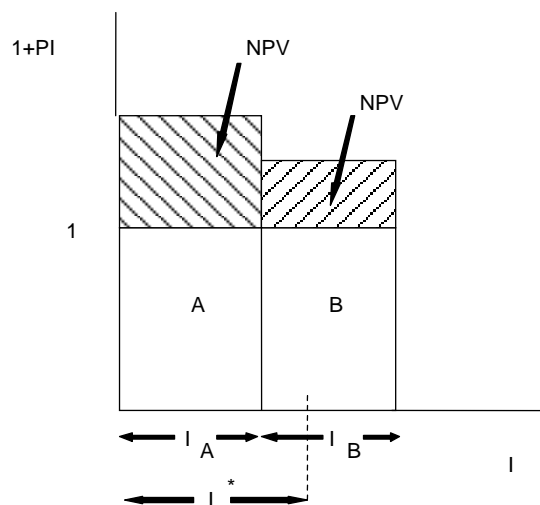
In interpreting this diagram it is helpful to remember that each bar represents a project:

Total Area in each bar = (1+PI) x I

$$= \left(1+\frac{NPV}{I}\right)I = I + NPV = I + PV - I$$

$$= PV$$

Also   Unshaded Area = I x 1

$$= I$$

Therefore   Shaded Area = PV - I

$$= NPV$$

Hence the objective of firms is to choose the projects that maximize the total shaded area. In the diagram above, if the firm is constrained to I*, so that it can do A or B but not both, it should clearly do A. Ranking by PI and accepting until the constraint is satisfied works in this case.

If there are three projects A, B and C as below which all require the same investment what should the firm do?

1+PI

1

A   B   C
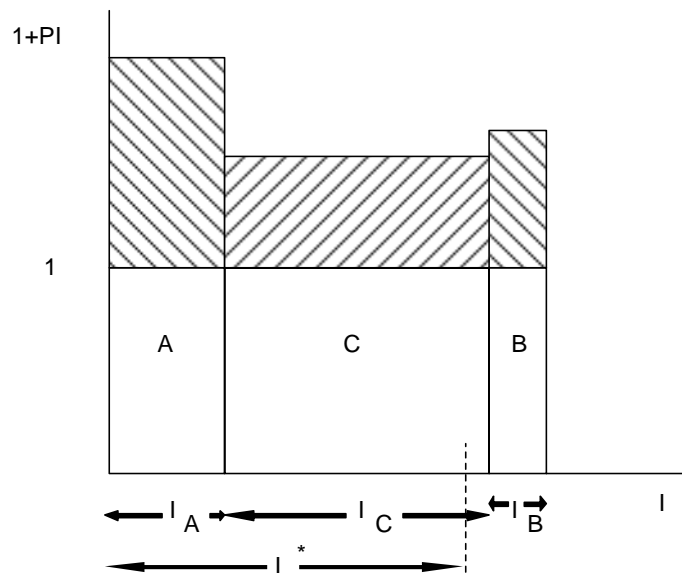
I$_A$ I$_B$ I$_C$ I

I*
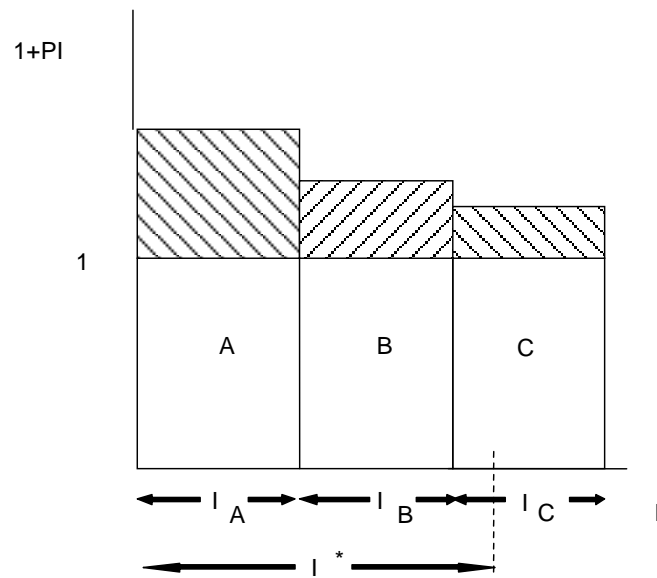
Clearly it should do A and B. Again the rule of ranking by PI and accepting until the

constraint is exhausted works. Is it always the case that the rule works? No! If the projects

require different investments the rule will not work.

1+PI

1

A   B   C

I$_A$ I$_B$ I$_C$ I

I*
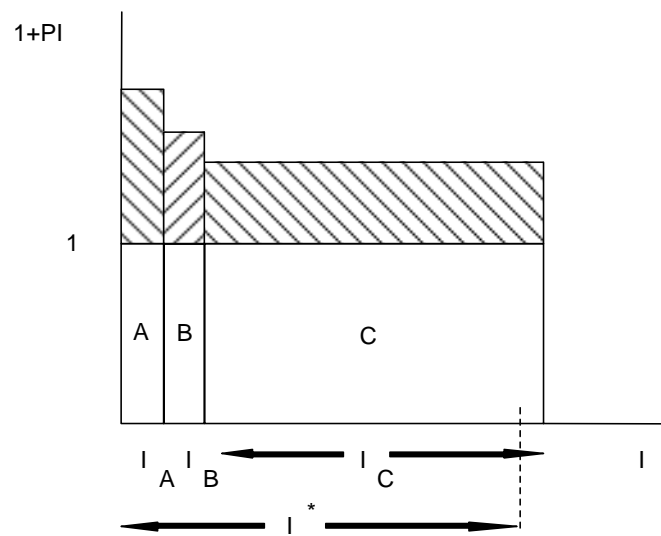
In this case it should do A and C.

It's not even clear that it will always be best to do the most profitable project.

C produces more NPV than A. You can't do both A and C, so you do C. You can still do B and add more NPV.

Thus ranking by PI and choosing the projects with the highest return until you exhaust the capital constraint can often give incorrect solutions. It is only if all investments are the same that it definitely works. Essentially what we should do if investments are unequal is try all possible combinations of projects that satisfy the capital constraint and choose the combination with the highest NPV.

With 3 projects this is not too difficult, but as the number increases, the number of possible combinations becomes large, and it may involve a substantial amount of computation. A technique that does this is integer programming. This is more sophisticated than simple trial and error: it chooses its combinations in a systematic way to increase NPV on each new combination. When it gets to a point where no more combinations increase NPV, it stops. You

will be considering integer programming and other optimization methods more in OPIM 621.

UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

LECTURE NOTES

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

QUARTER 1 - WEEK 4

Tu: 9/27/11 and Th: 9/29/11

Section 6: Practical Aspects of the NPV Rule

Read Chapter 6 BMA

Motivation

We know how to calculate present values given cash flows but how do you actually evaluate

projects? How do you use accounting data when applying the NPV rule?   For example, how do

you do problems like the following example?

Example 1

The Pierpont Company is thinking of building a plant to make trumpets. The plant and

equipment will cost $1 million. It will last for five years and will have no salvage value at the

end of that time. The costs of running the plant are expected to be $100,000 per year. The

revenues from selling the trumpets are expected to be $375,000 per year. All cash flows occur at

the end of the year. The firm uses straight line depreciation. Its corporate tax rate is 35 percent

and the opportunity cost of capital for this project is 10 percent.   The projected income

statement for the project is as follows.

Revenues	$375,000
Operating Expenses	-$100,000
Net Operating Income	$275,000
Depreciation	-$200,000
Taxable Income	$75,000
Taxes	-$26,250
Net Income	$48,750

Should the firm build the plant?

<u>Introduction</u>

How do we apply the NPV rule? Discount cash flows. This is all right as far as it goes, but so far we have not said exactly what we mean by cash flows--what should we include and what should we not include from accounting data? This is what we're going to consider next. There are four general rules.

Rule 1: Work with cash flow after taxes, not net income. This is the proper basis for capital budgeting analysis.

The points to remember are:

(i)   Always estimate cash flows on an after-corporate tax basis, since this is what shareholders are interested in. Hence if the corporate tax rate is denoted t_c the effect of additional gross revenues R is $(1-t_c)$R after tax and the effect of additional gross costs C is $(1-t_c)$C after tax.

(ii)   You have to transform accounting data into cash flows. This is not always easy. For example, such things as depreciation are not cash flows although the tax savings due to the tax deductibility of depreciation are. Suppose the corporate tax rate is $t_c = 35\%$. If you have a $100 depreciation deduction it reduces taxable income by $100 and hence taxes by $35. The value of a tax shield is t_c x Depreciation. Note that it is equivalent to subtract out depreciation from Net Operating Income (NOI) then subtract taxes and finally add back depreciation. This can be seen as follows.

$$(1-t_c)(\text{NOI - Depreciation}) + \text{Depreciation}$$

$$= (1-t_c)\text{NOI} - (1-t_c)\text{Depreciation} + \text{Depreciation}$$

$$= (1-t_c)\text{NOI} + t_c \text{ x Depreciation}$$

(iii)   Interest and dividends are not cash flows - they are taken account of by

discounting. If they were subtracted out again there would be double counting.

Rule 2: Ensure cash flows are recorded at the right time. Revenues and costs include cash you have not received or paid out. You need to adjust for this.

It is important to record cash flows at the time the cash is received or paid out. When you receive cash, for example, you can earn interest on the money.

Include working capital needs. Accounting revenues and costs include items where you have not received or paid out the cash. In order to find cash flow you need to adjust for these items. This is what working capital does. When a new project is undertaken it may take a while for customers to pay, i.e. the accounts receivable will increase. On the other hand, the payments associated with the cost of the project may be delayed, i.e. the accounts payable increases. This offsets the accounts receivable. The difference between them is a working capital need and this must be financed. Total working capital taking into account all these kinds of factor is short term assets minus liabilities.

Rule 3: Only marginal cash flows are analyzed (those which occur on the margin because you invest in the project).

Estimate all the marginal charges that arise if the project is undertaken compared to what happens if it is not undertaken. In particular

(i)   Include all incidental effects. Include all externalities on other parts of the

business. If revenues or costs in any part of the business are affected then care

should be taken to include these incidental marginal effects.

(ii)   Forget sunk costs. Sunk costs are expenditures that have already been made.

These are irreversible and should not be included in the cash flows. There's

nothing you can do about them--bygones are bygones.

(iii)   Include opportunity costs. If a firm already owns land that it is considering using

in a project, it should take into account the fact that it could sell the land, raise cash

and invest the money if it does not go ahead with the project. Thus there is an

opportunity cost to the land. The proper comparison is with or without, not before

or after.

(iv)   Be careful in the allocation of overhead Fixed costs are allocated by accountants but

they may not correspond to marginal costs borne by the firm. Be careful to adjust

for this.

Rule 4: Be consistent in the treatment of inflation

It is important first to clarify the distinction between nominal and real.

Nominal: The actual number of dollars and the actual interest rates.

Real: Taking into account inflation so that the figures are, in some sense, what would

have happened in the absence of inflation.

The important thing with nominal and real is that, in the context of capital budgeting,

there are two ways of thinking about the same thing. But you must be consistent and stick with

one or the other. Either discount nominal cash flows at a nominal discount rate, or discount real

cash flows at a real discount rate. Provided you do this, it doesn't matter whether you work in

real or nominal terms. Usually in the U.S., Europe and Japan people deal in nominal terms since

this is the form in which the data is presented. However, in high inflation countries such as

Eastern Europe and most Latin American countries until recently it's often easier to work in real

terms since you usually have a much better feel for the numbers and pick up errors more quickly.

 For example, if inflation was 20% per year and an entry level car now costs $10,000, how much

would it cost in five years time? Would it be $20,000 or $25,000. Is either of these figures

unreasonable? It's difficult to tell. In fact it would be around $25,000. What about in ten years

time? Would it be $50,000 or $60,000? In fact it would be about $62,000. If you work in real

terms you might pick up more easily what is reasonable. It becomes difficult to pick out

mistakes if the magnitudes are unfamiliar.

To see how to switch from one method to the other, consider the following example:

Example   Nominal Cash Flows

$$C_0 \qquad C_1 \qquad C_2$$

$$-1000 \qquad 800 \qquad 1700$$

Projected rate of inflation = 40% p.a.

Nominal interest rate = 50% p.a.

Nominal Calculations

We can just discount the nominal cash flows at the nominal discount rate.

$$NPV = -1000 + \frac{800}{1.5} + \frac{1700}{1.5^2}$$

$$= 289$$

Real Calculations

Alternatively, we could work in real terms, which is what would have happened in the

absence of inflation. Then we must convert both the cash flows and the discount rate to real

terms by removing the effect of inflation. We do this by deflating or in other words dividing by

(1 + inflation rate) for each period there has been inflation.

Real cash flow at date 1 = Nominal cash flow at date 1
(1 + inflation rate)

Similarly for t periods,

Real cash flow at date t = Nominal cash flow at date t
(1 + inflation rate)^t

Using this in our example:

Cash Flows	C_0	C_1	C_2
Nominal	-1000	800	1700
/(1 + inflation rate)^t	1	1.4	$(1.4)^2$
= Real	-1000	571.4	867.3

To find the real discount rate we must also take out the effects of inflation by dividing by

(1+inflation rate):

$$1 + \text{real discount rate} = \frac{1 + \text{nominal discount rate}}{1 + \text{inflation rate}}$$

Note that we do this rather than just subtract the inflation rate from the nominal discount rate

because we must take account of inflation on the principal and interest so

$$(1+\text{real discount rate})(1+\text{inflation rate}) = (1 + \text{nominal discount rate})$$

This is like interest on the interest when compounding.

For our example, we have

$$1 + r_{real} = \frac{1.5}{1.4} = 1.0714$$

so
$$r_{real} = 0.0714 \text{ or } 7.14\%$$

Our NPV calculation in real terms is then

$$NPV = -1000 + \frac{571.4}{1.0714} + \frac{867.3}{1.0714^2} = 289$$

To summarize, whether we work in real or nominal doesn't matter. However, we must

not mix them. That's the main thing you have to worry about as far as inflation is concerned. In

practice it's slightly more difficult to implement than this. In particular, you must remember that

the tax code is written in nominal terms in the U.S. and most other countries, so beware if you're

given data (e.g., examination questions) in real terms. In particular be careful to deal with

depreciation properly: work out depreciation in nominal terms since that is how it's specified in

the tax code and then convert to real terms.

Examples

As these general guidelines indicate, capital budgeting is a complicated business. It

requires careful, diligent work and a thorough understanding of the tax code in the country you

are working in. To give you some idea of how to do it, we're going to consider some examples.

Example 1

The Pierpont Company is thinking of building a plant to make trumpets. The plant and equipment will cost $1 million. It will last for five years and will have no salvage value at the end of that time. The costs of running the plant are expected to be $100,000 per year. The revenues from selling the trumpets are expected to be $375,000 per year. All cash flows occur at the end of the year. The firm uses straight line depreciation. Its corporate tax rate is 35 percent and the opportunity cost of capital for this project is 10 percent. The projected income statement for the project is as follows.

Revenues	$375,000
Operating Expenses	-$100,000
Net Operating Income	$275,000
Depreciation	-$200,000
Taxable Income	$75,000
Taxes	-$26,250
Net Income	$48,750

Should the firm build the plant?

Solution

The first thing to do is to create a cash flow table

Date:	0	1	2	3	4	5

Plant
Cost   -1M

After-tax $0.65 \times (375{,}000 - 100{,}000)$
Operating   =
Income   178,750 ⟶

Depreciation $0.35 \times 200{,}000$
Tax Shield   =70,000 ⟶

$$NPV = -1M + (178{,}750 + 70{,}000) \times AF(5yrs, 10\%)$$

$$= -57{,}042$$

Hence taking into account the opportunity cost of capital gives a negative NPV so the firm should not undertake the project.

An alternative and in this case perhaps a simpler way to do the calculation in this example is to simply add back the depreciation.

	Revenues	$375,000
	Operating Expenses	-$100,000
	Net Operating Income	$275,000
	Depreciation	-$200,000
	Taxable Income	$75,000
	Taxes	-$26,250
	Net Income	$48,750
Add back	Depreciation	+200,000
	Cash flow years 1-5	248,750

Calculating NPV,

$$NPV = -1M + 248{,}750 \times AF(5yrs, 10\%) = -57{,}042$$

Example 2

The following information is from the projected income statement for the Madison

Company for the next five years.

Revenues	100,000/year
Operating Expenses	-50,000/year
Net Operating Income	50,000/year
Depreciation	-30,000/year
Taxable	20,000/year
Taxes	-8,000/year
Net Income	12,000/year

In an attempt to improve this projected performance, the firm is considering replacing one

of their assets. The replacement is not expected to affect revenues, but the company's operating

expenses would be reduced by 10%. The old machine was purchased 3 years ago for $42,000.

At that time it was estimated to have an eight year economic life. It could now be sold for

$25,000. Annual straight-line depreciation on this asset is 1/6 of the firm's total annual

depreciation. The new machine costs $30,000, has a five year economic life, and has no

expected salvage value. If the firm's required rate of return is 16% should it purchase the new

machine?

Solution

The question that we investigate here is whether the NPV of selling the old machine and

replacing it with the new machine is positive. When answering these questions it's helpful to

make out a chart of the cash flows to give you an idea what's going on. (See subsequent pages

for explanations of cash flows.)

Cash flow at end of year:	0	1	2	3	4	5

Cash Flows Associated
  with Old Machine:

(i)   Sale of old machine   25,000

(ii) Loss of tax shield   -0.4 x 5000
      from depreciation of   = -2,000 $\longrightarrow$
      old machine

(iii) Loss of expected salvage value   -2,000

(iv) Tax benefit from
      book loss when
      machine is sold   +800

Cash Flows Associated
  with New Machine

(v)   Cost   -30,000

(vi) Tax benefit from
      the tax shield $\dfrac{30{,}000 \times 0.4}{5}$
      of depreciation   = 2,400 $\longrightarrow$

(vii) Net savings resulting
      from reduction in
      operating expenses   3,000 $\longrightarrow$

Calculations

Old Machine : Purchased 3 years ago for $42,000

(i)   Sale of old machine

        From question: $25,000

(ii)   Loss of tax shield from depreciation of old machine

        Annual depreciation on machine =

     (1/6) x firm's total depreciation = 30,000/6 = 5,000

Tax rate, t = 8,000/20,000 = Taxes/Taxable Income = 0.4

Annual tax shield from depreciation = 0.4 x 5,000 = 2,000

Note: To see why we multiply depreciation by the tax rate to get the tax shield, suppose before-tax revenue is $10,000. If there is no depreciation deduction then the taxes paid are $4,000 so the after-tax cash flow is 10,000 - 4,000 = $6,000. Suppose now that the depreciation deduction is $1000. Before-tax revenue is still $10,000 but now only $9,000 of it is taxable. Hence the taxes are 0.4 x $9,000 = $3,600 so the after-tax cash flow is 10,000 - 3,600 = $6,400. Hence the overall effect on after-tax cash flow of the $1,000 depreciation deduction is the reduction in taxes of 0.4 x $1000 = $400.

(iii) Loss of expected salvage value

8 year life and original purchase price of $42,000 =>

Total accumulated depreciation = 8 x 5,000 = 40,000

Expected salvage value = 42,000 (cash) - 40,000 = 2,000

Note: You are selling the asset at book value so there is no loss or gain and therefore no tax.

(iv)   Tax benefit from book loss (or gain) when machine is sold

Accumulated depreciation = 3 years x 5,000 = 15,000

Book value = 42,000 (cost) - 15,000 = 27,000

Sale value = 25,000

Book loss   = 27,000 - 25,000 = 2,000

Tax benefit = 0.4 x 2000 = 800

Note: The reason we multiply by the tax rate here is similar to that with the tax shield from depreciation.

New Machine: 5 year life; no salvage

(v)   Cost

From question: $30,000

(vi)   Tax benefit from depreciation shield

Annual depreciation = 30,000/5 = 6,000

Tax shield = 0.4 x 6,000 = 2,400

(vii) Net Savings resulting from reduction in operating expenses

Reduction in operating expenses if new machine is purchased

= 0.10 x 50,000 = 5,000

Net reduction in operating expenses after tax

= 0.6 x 5,000 = 3,000

Note: Here we multiply by one minus the tax rate since the before-tax cash flow is reduced by taxes at rate t. This contrasts with depreciation where only taxes are reduced and the before-tax cash flow remains the same.

Calculation of NPV from Cash Flows

$$NPV = 25,000 + 800 - 30,000 - 2,000 \ DF(5 \ yrs, 16\%)$$

$$+ \ (-2,000 + 2,400 + 3,000) \ AF(5 \ yrs, 16\%)$$

$$= 5,980 > 0$$

Therefore, the firm should sell the old machine and purchase the new machine.

What was nice about the example above? The remaining life of the old machine was the same as the life of the new machine. We look next at how to solve problems where this is not the case.

Comparing Mutually Exclusive Projects with Different Lives

With most projects there are many ways of doing things, a variety of different machines often do the same thing, and so on. For example, in deciding on the production of its new cars GM has a number of different welding robots it can choose among. One particular problem that arises in this type of case is that machines performing the same task may have different lives. How should we compare the machines in such cases?

Example 3

Machines A and B have identical capacity. They do the same job but have different lives: A lasts 2 years, while B lasts 1 year. The benefits of the machines are the same.

The cost of these machines at dates 0, 1 and 2 in real terms are

Machine	t=0	1	2	PV at 10%
A	20	6	4	28.76
B	10	5		14.55

B has the lower PV of cost--should we take it then? Not necessarily because we have to replace it a year earlier, and our calculations haven't taken the cost in the extra year into account. Notice that this problem wouldn't have occurred if they had the same lives.

One suggestion in dealing with this is to assume that the machines are replaced by identical ones until we reach a point where the machines wear out together.

Machine	t=0	1	2	PV at 10%
A	20	6	4	28.76
B	10	5+10=15	5	27.76

In this case investing in type B machines can be seen to be best.   We can always use this method, but for more complicated lives there may be a lot of tedious calculation. For example, with lives of 8 and 9 years we would need to calculate $8 \times 9 = 72$ periods ahead.

There is a simpler way, however. We can calculate an annuity equivalent. An annuity equivalent is like an average, but it takes into account the time value of money.

For A:

Machine	t=0	1	2	PV at 10%
A	20	6	4	28.76

Now what annual payment over the two years would be equivalent to 28.76?

$$\text{PV annuity} = \text{Annuity Payment} \times \text{AF(2 yrs 10\%)}$$

Hence rearranging and substituting

$$\text{Annuity Payment} = \frac{\text{PV Annuity}}{\text{AF(2 yrs, 10\%)}} = \frac{28.76}{1.736} = 16.57$$

The constant annual cost or annuity equivalent of the machines is 16.57.

	t=0	1	2	PV at 10%
A	20	6	4	28.76
Annuity		16.57	16.57	28.76

Hence 16.57 is the annuity equivalent of the cost stream of A.

For B,

$$\text{Annuity Payment} = 14.55/0.909 = 16.005$$

Hence the annuity equivalent, or the equivalent annual cost, of B is 16.005.

	t=0	1	PV at 10%
B	10	5	14.55
Annuity		16.005	14.55

In this case $16.005 < 16.57$, so again B is preferable.

The annuity equivalent allows us to calculate the constant annual cost of the machines, and we can compare them on this basis.

Do you think this would be a good methodology for personal computers? Probably not. Why? What's the implicit assumption here about technological change? We're implicitly assuming there's no technological change. If there were rapid technological change you'd have to take that into account by going back to our original overlapping method.

It is important to work in real terms except when inflation is negligible otherwise you will not be comparing like with like.

Working Capital

As discussed above, accounting revenues and costs do not reflect the amount of cash received or paid out. When a new project is undertaken it may take a while for customers to pay on new sales, i.e. the accounts receivable will increase. The amount of cash that the firm will need will be increased. This may be offset to some extent by an increase in accounts payable. The key point here is that revenues and costs don't correspond to cash received. Working capital is the adjustment for this.

The amount of inventories will also need to be increased. In other words, short term assets will need to be increased. The money for this must come from somewhere. Again working capital is the adjustment for this.

By definition,

Working capital = short term assets - short term liabilities

The important point to remember in doing problems is that the cash flows associated with working capital are the changes in working capital. When working capital requirements go up there is a negative cash flow, i.e. you need to provide funds; when working capital requirements go down there is a positive cash flow, i.e. funds are freed up. For example, consider a project which lasts for five years. Suppose the total working capital requirements for it at dates 0-5 are as follows.

t=	0	1	2	3	4	5
	1M	1.2M	1.2M	1.3M	0.8M	0M

The cash flows associated with these requirements are as follows.

t=	0	1	2	3	4	5
	-1M	-0.2M	0M	-0.1M	+0.5M	+0.8M

Notice that at the last date when the business is wound up the inventories and other short term assets are liquidated so the working capital is recovered.

In order to illustrate how this works in the context of a decision consider the following example.

Example 4

The Wharton Olympics Corporation is considering whether to build a new bakery to make cherry pies. The current date is 12/31/X5. The new bakery will be built over two years and will be ready to start production on 1/1/X8 and will cease production on 12/31/X9. The investment for the bakery requires an outlay of $2.5M on 12/31/X5. This expenditure can be depreciated using straight line depreciation over the two years the bakery is producing. The total salvage value of all the plant and equipment on 12/31/X9 is expected to be $800,000. The land the bakery will be built on could be rented out for $200,000 per year before taxes for the four years from 12/31/X5-12/31/X9 while the bakery is being built and is in production. The rental for each year is paid at the end of the year so for year 1 it is paid at date 1 (not date 0) and so on. The bakery will produce 1.2 million cherry pies per year. These can be sold at $7 per pie. Raw material costs are $1.20 per pie and the total labor costs are $420,000 per year. These revenues and costs are expected to be the same for the two years the bakery is in production. The total working capital required 12/31/X7 to allow inventories to be financed during the first year of production is $300,000. For the second year the total working capital needs will be 10% higher. When the plant ceases production all the working capital can be recovered. The firm has a corporate tax rate of 35 percent. The opportunity cost of capital for the project if it is all equity financed is 12 percent. Assume all cash flows occur at year's end and that the firm has other profitable ongoing operations.

Solution

The table of cash flows for the problem is as follows.

Item	12/31/X5	12/31/X6	12/31/X7	12/31/X8	12/31/X9
1. Investment	-2.5M				
2. Salvage Value					0.8M
3. Depr. tax shield [i.e. 0.35(2.5M-0.8M)/2]				297,500	297,500
4. After tax land rental [i.e. -0.65(200,000)]		-130,000	-130,000	-130,000	-130,000
5. Revenues [i.e. 7(1.2M)]				8.4M	8.4M
6. Raw materials [i.e. 1.2(1.2M)]				1.44M	1.44M
7. Labor costs				0.42M	0.42M
8. Before-tax operating profit [i.e. line 5 - line 6 - line 7]				6.54M	6.54M
9. After-tax operating profit [i.e. 0.65 x line 8]				4.251M	4.251M
10. Total working capital needs			300,000	330,000	0
11. Working capital cash flow			-300,000	-30,000	+330,000
12. After tax cash flows [i.e. sum of lines 1-4, 9 and 11]	-2.5M	-130,000	-430,000	4.3885M	5.5485M

$$\text{NPV} = -2.5\text{M} - \frac{0.13\text{M}}{1.12} - \frac{0.43\text{M}}{1.12^2} + \frac{4.3885\text{M}}{1.12^3} + \frac{5.5485}{1.12^4} = \$3.69\text{M}$$

Finally, consider two more examples to give you some more experience.

Example 5

Nippon Auto intends to replace one of its plants in the US. It has two mutually exclusive options. The first, which is codenamed Plan A, is to build a plant on its existing site in Indiana. The plant will be built on land the company already owns. It is estimated that the land could be currently sold for an after tax amount of $10 million. Under Plan A the cost of the new plant would be $100 million to be paid now. It is expected to have a life of 15 years and a salvage value of $25 million. The land can be sold after the fifteen years for an after tax amount of $12 million. Revenues and costs at the end of the first year are expected to be $30 million and $6 million, respectively. Both are expected to stay constant for the life of the plant. The second option, which is codenamed Plan B, is to build a new plant in Alabama. If they choose to do this they will have to buy the land for an after tax amount of $10 million. The plant will cost $70 million and last for 10 years. At the end of that time it will have a salvage value of zero and the land can be sold for an after tax amount of $10 million. Revenues and costs at the end of the first year are expected to be $27 million and $6 million, respectively. Both are expected to stay constant for the life of the plant. All figures are in nominal terms and are stated in before tax terms unless otherwise indicated. The firm uses straight line depreciation and has a tax rate of 35 percent. It has profitable ongoing operations and an opportunity cost of capital of 12 percent. Which plant should be chosen if it is anticipated that it will be replaced by a plant with identical cash flows and this will be repeated for the foreseeable future?

Solution

Since the plants have different lives and are expected to be replaced with identical plants this problem needs to be approached on an equivalent annual basis.

Plan A

Period	0	1	...	15
Land	-10			12
Plant	-100			
Salvage				25
Depreciation	0.35(100-25)/15			
tax shield	= 1.75	---------------------------->		
After tax	0.65(30-6)			
operating income	= 15.6	--------------------------->		

$NPV_A = -10 - 100 + (12+25)DF(15 \text{ years}, 12\%) + (1.75+15.6)AF(15 \text{ years}, 12\%)$

    = \$14.93 \text{ million}

Equivalent annual NPV_A = 14.93 million/AF(15 years, 12%) = \$2.19 million

Plan B

Period	0	1	...	10
Land	-10			10
Plant	-70			
Salvage				0
Depreciation	0.35(70)/10			
tax shield	= 2.45	------------------>		
After tax	0.65(27-6)			
operating income	= 13.65	------------------>		

$NPV_B = -10 - 70 + (10)DF(10 \text{ years}, 12\%) + (2.45+13.65)AF(10 \text{ years}, 12\%)$

    = \$14.19 \text{ million}

Equivalent annual NPV_B = 14.19 million/AF(10 years, 12%) = \$2.51 million

B should be chosen since it has a higher equivalent annual NPV.

Example 6

Kannon Textile is thinking of building a new plant, with a capacity of 5,000 garments per year. The site of the proposed plant is land which it owns, which is currently unused.

It could do this project immediately, in which case the plant and equipment will cost $150,000 (real) on 12/31/X5 and the plant will be ready to start production on 1/1/X6.

Another alternative is to wait a year, in which case a new type of equipment will be on the market which will only cost $100,000 (real) on 12/31/X6 and will be ready to start production on 1/1/X7. Both plants would be expected to be in production for 5 years and have a salvage value of $50,000 (real). The price of garments is currently $40 (real), and the cost of labor and materials in both types of plant is $12 (real) per garment. The prices and costs are expected to remain unchanged in real terms for as long as either plant would be in operation.

A neighboring firm has offered to rent the land the plant would be built on at $120,000 (real) per year for any year the company is prepared to let it. The firm has a tax rate of 40% and uses straight-line depreciation. Revenue is received and costs are paid at year's end. The expected rate of inflation is 6%. What should the company do if its real discount rate for this type of project is 8%?

Solution

Work in real terms because much of the data is given in real terms.

Look first at the alternative of building plant now:

	0	1	2	3	4	5
Plant	-150,000					
Salvage value						50,000
Units		5,000	-----	-----	-----	----->
Unit price		40	-----	-----	-----	----->
Unit cost		12	-----	-----	-----	----->

After-tax op. inc.   (1-0.4)(40-12)5,000
                                = 84,000 --->

After-tax opportunity   -(1-0.4)120,000
cost of land   = -72,000 --->

| Depr. tax shield* | | 6,270 | 5,917 | 5,582 | 5,266 | 4,968 |

* The tax code is written in nominal terms so we have to work out nominal depreciation and then translate to real as follows.

Nominal salvage value $= 50{,}000 \times (1.06)^5 = 66{,}900$

Nominal depreciation $= \dfrac{150{,}000 - 66{,}900}{5} = 16{,}620$

Nominal tax shield $= 0.4 \times 16{,}620 = 6{,}648$

Real tax shield $= \dfrac{6{,}648}{(1.06)^t}$

NPV = -150,000 + (84,000 - 72,000) × AF(5 years, 8%) + 50,000 × DF(5 years, 8%)

  + 6,270 × DF(1 year, 8%) + 5,917 × DF(2 years, 8%) + 5,582 × DF(3 years, 8%)

  + 5,266 × DF(4 years, 8%) + 4,968 × DF(5 years,8%)

  = -45,472.

Now look at the option of building a plant a year from now.

The only differences are the purchase price and the depreciation tax shields.

Nominal purchase price $= 100,000(1.06) = 106,000$

Nominal salvage value $= 50,000(1.06)^6 = 70,926$

Nominal depreciation $= \dfrac{106,000 - 70,926}{5} = 7,015$

Nominal tax shield $= 7,015(0.4) = 2,806$

Real tax shield $= \dfrac{2,806}{(1.06)^t}$

	1	2	3	4	5	6
Depr. tax shield		2,497	2,356	2,223	2,097	1,978

Real PV (at t = 0) of depr. tax shields $= 2,497 \times DF(2 \text{ years, } 8\%) + 2,356 \times DF(3 \text{ years, } 8\%)$

$\qquad + 2,223 \times DF(4 \text{ years, } 8\%) + 2,097 \times DF(5 \text{ years, } 8\%) + 1,978 \times DF(6 \text{ years, } 8\%)$

$\qquad = \$8,319.$

Therefore,

NPV (at t = 0)

$\quad = -100,000 \times DF(1 \text{ year}, 8\%) + (84,000 - 72,000) \times AF(5 \text{ years, } 8\%) \times DF(1 \text{ year}, 8\%)$

$\quad + 50,000 \times DF(6 \text{ years}, 8\%) + \$8,319$

$\quad = -\$8,402$

The company should not do either project. It should rent the land.

Note that sometimes a quicker way to do the calculations for the tax shield is to work in nominal terms and use the nominal annuity factor.

Nominal discount rate $= 1.08 \times 1.06 - 1 = 0.1446$ or 14.46%

In the case where the plant is built now

PV of tax shield at date $0 = 6,648 \times$ AF(5 years, 14.46%)

NPV = -150,000 + (84,000 - 72,000) $\times$ AF(5 years, 8%) + 50,000 $\times$ DF(5 years, 8%)

    + 6,648 $\times$ AF(5 years, 14.46%)

   = -45,472.

For the alternative where the plant is built 1 year from now

PV of tax shield = 2,806 $\times$ AF(5 years, 14.46%) $\times$ DF(1 year, 14.46%)

           = 8,319

This may be a slightly quicker method in some circumstances.

Conclusion

      This brings us to the end of Section 6 of the course. By this time I hope you have a good understanding of how to do NPV calculations. If you don't understand what's going on, now is the time to find out. If you have any problems, come and see me.

      So far we have sidestepped the question of uncertainty and this is what we come to next.

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UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

FNCE 601

CORPORATE FINANCE

Franklin Allen

Fall 2011

QUARTER 1 - WEEK 5

Tu: 10/4/11 and Th: 10/6/11

Section 7: Measuring Risk

Read Chapter 7 in BMA and Reading for Section 7 on Webcafe

Motivation

In this section and the next we're going to talk about risk. We've got a lot of work before we get results, so I'll give two motivation examples to start with.

Motivation Example 1

Consider the following two investments.

(i) A lone prospector intends to search for gold in the Rockies and is issuing 100 shares to finance his expenses. The evidence suggests that if he strikes gold he will hit it big and the payoff on the project will be $100. However, the probability of this is only 10 percent. There is a 90 percent chance he will find no gold in which case the payoff on the project will be $0. Hence the expected payoff is $10.

(ii) Another investment is 100 shares of an electric utility. If the economy does well a lot of electricity will be used, the utility's profits will be high and its stock will yield a gross payoff of $15. The probability of this happening is 50 percent. The other possibility is that the economy does badly and not much electricity is used. In this case the payoff on the firm's stock is $5. The probability of this happening is 50 percent so the expected payoff is again $10.

Both of these investments have the same expected return. Which of them is more risky in a financial sense?

Motivation Example 2

Suppose T-bills are yielding 4%. Is it ever worthwhile investing in a risky stock yielding a total return (i.e. including dividends and capital gains) of 2%?

What we're going to do over the next two weeks or so is derive a formula for risk adjusted discount rates of the form

$$r = r_F + \beta(r_M - r_F)$$

where

r_F - risk free rate (e.g., T-bill rate)

r_M - return on market portfolio (e.g., a value-weighted portfolio of all stocks on the NYSE)

β = Measure of risk = $\dfrac{\text{Cov (Stock, Market portfolio)}}{\text{Var (Market portfolio)}}$

Notice that what's important in the measure of risk is covariance with the market portfolio. You can see from this why we get the results that we do in the motivation examples. Understanding why covariances matter is the core concept in understanding risk. It is not a simple idea and we have a lot of ground to cover before you have a full intuitive understanding of the CAPM and these examples.

Section 7(i): Historical and Statistical Review

Before we go on to deal with risk and asset pricing risk we should start with a brief historical and statistical review.

The appendix at the end of the section starts with historical data for the US. The first figure (from the *Stocks, Bonds, Bills and Inflation Yearbook for 2010* published by Ibbotson

Associates) shows that there is a vast difference between the returns on different investments. The second thing to notice is the amount you would have had at the end of 2009 if you had invested $1 at the beginning of 1926 in various financial instruments.   Small stocks in particular have done very well. Common stocks have also done well. Both have done vastly better than bonds and bills in the long run. The table on the next page shows this data in a tabular rather than diagrammatic form.

Jeremy Siegel in his book *Stocks for the Long Run* has argued that in the short run stocks and small stocks are much more risky but in the long run (meaning 20 years or more) they are not so risky and almost always do better than bonds in this period. This view has been criticized by many because it is based on long run data from one country, the US. In a recent study Dimson, Marsh and Staunton (2006) have shown that there is a large difference in equity returns across countries in the long run. For the period 1900-2005 the annualized real return varies from 2.40% for Belgium (the worst) to 7.80% for Sweden (the best). The US at 6.52% seems to have done significantly better than the rest of the world at 5.23%. The Reading for Section 7 on Webcafe contains more information on this topic.

How can these figures be understood? For example, why do different asset classes have different risks and returns? In order to measure the relationship between risk and return and develop models which can help us understand what is going on we need some formal statistical measures.

Statistics Review

Random Variable

Our starting point is the notion of a random variable. As an example, we can think of the return on a stock. If there is uncertainty, for example, because of various possible levels of GDP, which determine whether the firm's products are profitable or not, then the firm may have a number of different returns on its stock depending on what actual incomes and so on are.

The possible outcomes are described by a random variable, which we can call X. For the moment we shall look at discrete random variables for the purposes of illustration. Discrete random variables are those in which the number of possibilities is finite rather than infinite. We denote a specific realization of the random variable X by x_i.

State	Realization
State 1	x_1 - High return 3%
State 2	x_2 - Low return 1%

Probability

There are a number of notions as to what probability represents. Perhaps the easiest one to understand is that of relative frequency. In the long run, it measures the proportion of times we get high returns, and the proportion we get low returns.

State	p_i
State 1	0.50
State 2	0.50

Note probabilities have the following properties in general

$$p_i \geq 0 \quad \text{for all } p_i$$

$$\sum_{i=1}^{n} p_i = 1$$

where there are n possible events.

Expectation

      In order to be able to consider the notion of means, variances and so on it is necessary to define the expectation of a random variable.

$$EX = \sum_{i=1}^{n} p_i x_i$$

$$EX^2 = \sum_{i=1}^{n} p_i x_i^2$$

In general, we have

$$E[f(X)] = \sum_{i=1}^{n} p_i f(x_i)$$

Two useful formulas are the following:

If a is a constant

$$E(aX) = \sum_{i=1}^{n} a p_i x_i = a \sum_{i=1}^{n} p_i x_i = aEX$$

$$E(X + Y) = \sum_{i=1}^{n} (p_i x_i + p_i y_i)$$

$$= \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i y_i = EX + EY$$

The Arithmetic Mean or Average

The first notion we are interested in is the arithmetic mean or average. This is given by

$$\text{Mean} = EX = \sum p_i x_i = \overline{X}$$

e.g.,

$$EX = 0.5(3) + 0.5(1) = 2\%$$

If you look at the second table on the handout you can see that it gives estimates for the arithmetic mean of the returns on various financial instruments. Why are these arithmetic means of interest? They give the best estimate of the mean return if we had invested for a year during that period.

Variance

We are not only interested in the average value, we are also interested in how much returns vary. For example, if you invested the money for your tuition this semester last year you presumably would have liked to know how variable the returns would likely have been. If you had invested in short term government instruments you could have been fairly sure of meeting your tuition payment. If you had put it in stocks you might have made a lot of money or you could have lost a lot of money. To measure variability we use variance

$$\text{Variance} = \sigma_X^2 = E(X - EX)^2 = \sum_{i=1}^{n} p_i (x_i - EX)^2$$

e.g.,

$$\text{Var } X = 0.5(3 - 2)^2 + 0.5(1 - 2)^2 = 1$$

Suppose instead of x_1 and x_2 being 3 and 1 we have

$$x_1{}^* = 4; \, x_2{}^* = 0$$

so that

$$\text{Var } X^* = 0.5(4 - 2)^2 + 0.5(0 - 2)^2 = 4$$

We have spread the distribution out or made it more variable and so have a higher variance. The diagram below illustrates this.
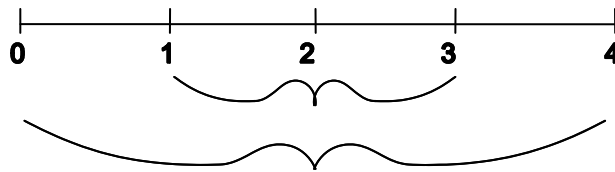
Standard deviation

The problem with variance is that it isn't in the same units as the mean--it's in $\%^2$. In the above example with outcomes of 4 and 0 this means that although the average absolute deviation from the mean is 2 the variance is 4. It is often useful to work with standard deviation that is just the square root of variance. This means that its units are the same as those of the mean.

$$\text{Standard Deviation} = \sqrt{\text{Variance}} = \text{SD } X = \sigma_X$$

In our example with $x_1{}^* = 4$ and $x_2{}^* = 0$

$$\text{SD } X^* = \sqrt{4} = 2$$

Since standard deviation has the advantage that the units are the same as the units of mean, we will mostly be using standard deviation as a measure of risk.

Covariance

When we construct portfolios of stocks one of the important things of interest is how the stocks move together. For example, suppose two stocks move in opposite directions so that when one has high returns the other has low returns and vice-versa. Then a portfolio of the two stocks would tend to have a low variance since the movements in returns would be offsetting. If on the other hand the two stocks' returns move in the same direction the variance of the portfolio will be high since there will be no offsetting movement.

A measure of how random variables move together is covariance. If we have two random variables, X and Y say, their covariance is defined as follows.

$$Cov(X, Y) = \sigma_{XY} = E(X - EX)(Y - EY)$$

$$= \sum_{i=1}^{n} p_i (x_i - EX)(y_i - EY)$$

State	x_i	y_i	p_i
State 1	3	6	0.5
State 2	1	2	0.5

$$EX = 2 \qquad EY = 4$$

$$Cov(X,Y) = \sigma_{XY} = 0.5(3 - 2)(6 - 4) + 0.5(1 - 2)(2 - 4) = 2$$

Note that in this case the variables move together, so that X is above its mean when Y is above its mean. This gives a positive term in the calculation of covariance. When X is below its mean Y is also below its mean so we have a negative times a negative which gives a positive. Hence covariance is positive. This happens whenever two variables move together. What happens if the two variables move in opposite directions?

Example 2

State	x_i	y_i	p_i
State 1	3	2	0.5
State 2	1	6	0.5

$$EX = 2 \qquad EY = 4$$

$$COV(X,Y) = \sigma_{XY} = 0.5(3 - 2)(2 - 4) + 0.5(1 - 2)(6 - 4) = -2$$

In this case we have a negative covariance. This is because when X is above its mean Y is below its mean and vice-versa so in each term we have a positive times a negative which gives a negative. Thus when variables move in opposite directions they have a negative covariance.

If they are independent and don't tend to move around together or in opposite directions, then we get some positive terms and some negative terms. When there is no systematic relationship at all these positive and negative terms tend to cancel out and we get a zero covariance. When two random variables are independent they have zero covariance.

Correlation

One of the problems with covariance is that its magnitude depends on the units of measurement. It is not possible to simply deduce from the size of the number how much they move together. A concept that allows us to do this is the correlation coefficient.

$$\text{Correlation Coefficient} = \text{Corr}(X,Y) = \rho_{XY} = \frac{\text{Cov}(X,Y)}{\text{SD X} \times \text{SD Y}}$$

Now it can be shown

$$-1 \le \rho_{XY} \le +1$$

If $\rho_{XY} = +1$, they are perfectly correlated and move in the same direction in proportion.

If $\rho_{XY} = -1$, they are perfectly negatively correlated, they move in opposite directions and in proportion.

If $\rho_{XY} = 0$, then on average they don't move together or in opposite directions.

Example 1

$$\text{Var } Y = 0.5(2)^2 + 0.5(-2)^2 = 4; \text{ SD } Y = 2$$

$$\rho_{XY} = \frac{2}{1 \times 2} = +1$$

In this case, they're perfectly correlated: they move in the same direction and in proportion. When X is 1 above its mean, Y is 2 above its mean; when X is 1 below its mean, Y is 2 below its mean.

Example 2

$$\rho_{XY} = \frac{-2}{1 \times 2} = -1$$

In this example they're perfectly negatively correlated. When X is 1 above its mean, Y is 2 below its mean; when X is 1 below its mean, Y is 2 above its mean.

Example 3

State	x_i	y_i	p_i
State 1	1	9	0.5
State 2	2	11	0.3
State 3	3	25	0.2

$$EX = 1.7 \qquad EY = 12.8$$

$$SD\ X = 0.781 \qquad SD\ Y = 6.161$$

$$Cov(X,Y) = 0.5(1 - 1.7)(9 - 12.8) + 0.3(2 - 1.7)(11 - 12.8)$$

$$+ 0.2(3 - 1.7)(25 - 12.8) = 4.34$$

$$\rho_{XY} = \frac{4.34}{0.781 \times 6.161} = 0.902$$

In this case, in state 1 when X is $1 - 1.7 = -0.7$ from the average, Y is $9 - 12.8 = -3.8$ from the average; in state 2 when X is $2 - 1.7 = +0.3$ from the average, Y is $11 - 12.8 = -1.8$ from the average; in state 3 when X is $3 - 1.7 = +1.3$ from the average, Y is $25 - 12.8 = +12.2$ from the average. Thus when one goes up, the other tends to go up, but doesn't always do so. Hence ρ_{XY} is positive but less than one.

Evidence on Covariance and Correlation

As far as the empirical evidence is concerned there are two important conclusions that can be drawn from the data:

(i) Stock returns are serially uncorrelated or in other words if stock returns are high one year then you can't use this information to predict whether returns in the subsequent year will be high or low. This evidence will be important when we talk about market efficiency later in the course. However, for the moment the one conclusion that is relevant as far as models of asset pricing is concerned is:

(ii) Within a period most stocks are positively correlated to the market portfolio (e.g. a value-weighted portfolio of all the stocks on the NYSE).

Regression

Suppose we have two random variables and we plot them on a diagram. Regression theory allows us to fit a line y = a + bx representing how the two variables are related.
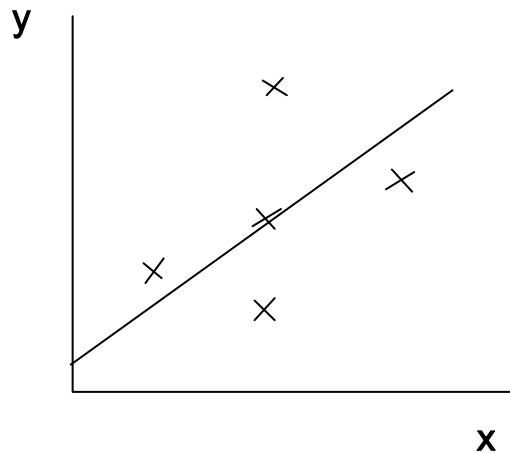
The estimate of the slope of the line, b*, is given by

$$b^* = \frac{Cov(y, x)}{Var\ x}$$

What does this remind you of? If you remember initially when we talked about β, we said it was Cov(Stock, Market) /Variance(Market). As we will be seeing one interpretation of β is that it is the slope of the regression when you plot the returns on a stock against returns on the market.

Results Needed for Theories of Asset Pricing

Diversification

Suppose you have a stock X_1 with payoff distribution:

Payoff	Prob.
0	0.5
2	0.5

so   Mean = 1; Variance = 1; S.D. = 1

The stock can be thought of as like a coin toss. With tails (T) the payoff is 0 and with Heads (H) the payoff is 2.

Next suppose you have another stock just like X_1 which we will call X_2. It has the same payoff distribution and the same price. It's payoffs are independent of X_1's. Instead of using your money to buy 1 share of X_1, suppose instead you buy 1/2 a share of X_1 and 1/2 a share of X_2. In other words, you buy a portfolio of stocks. What is the mean and standard deviation of the portfolio?

State	Payoff	Prob.
1	2T: 0	0.25
2	TH: $(1/2) \times 0 + (1/2) \times 2 = 1$	0.25
3	HT: $(1/2) \times 2 + (1/2) \times 0 = 1$	0.25
4	2H: $(1/2) \times 2 + (1/2) \times 2 = 2$	0.25

$$\text{Mean} = 1$$

$$\text{Variance} = 0.25(0\text{-}1)^2 + 0.5(1\text{-}1)^2 + 0.25(2\text{-}1)^2 = 0.5$$

$$\text{S.D.} = 0.707$$

The mean stays the same but because you can spread your investment across the two stocks the variance and S.D. go down. This is because the heads and tails cancel out in states 2 and 3. It is this basic idea that underlies most analysis of risk in finance. If you add more stocks and split your investment between them you keep on getting a reduction in variance and standard deviation. Eventually, it follows from what is known as the *law of large numbers* that all the risks go as long as the stocks are independent and so have zero covariance and correlation.

Nonzero covariance and correlation

Next consider what happens if the two stocks X_1 and X_2 are perfectly negatively correlated i.e. $\rho = -1$. In that case what happens to the portfolio mean, variance and standard deviation?

State	Payoff	Prob.
1	2T: 0	0
2	TH: $(1/2) \times 0+(1/2) \times 2 = 1$	0.5
3	HT: $(1/2) \times 2+(1/2) \times 0 = 1$	0.5
4	2H: $(1/2) \times 2+(1/2) \times 2 = 2$	0

$$\text{Mean} = 1$$

$$\text{Variance} = 0(0\text{-}1)^2 + 1(1\text{-}1)^2 + 0(2\text{-}1)^2 = 0$$

$$\text{S.D.} = 0$$

Here you can never get two tails or two heads you always get one head and one tail. Risk is completely eliminated. In terms of the probability distribution you are putting all the weight on the mean at 1.

What would happen if you had a negative correlation with $-1 < \rho < 0$? This case lies in between the two cases above. For example, if having obtained 1 T the probability of a second T is 0.25 and similarly having obtained 1 H the probability of a second is also 0.25 it can be shown the distribution of payoffs is as below and it can be straightforwardly checked the correlation is -0.5.

State	Payoff	Prob.
1	2T: 0	0.125
2	TH: $(1/2) \times 0 + (1/2) \times 2 = 1$	0.375
3	HT: $(1/2) \times 2 + (1/2) \times 0 = 1$	0.375
4	2H: $(1/2) \times 2 + (1/2) \times 2 = 2$	0.125

$$\text{Mean} = 1$$

$$\text{Variance} = 0.125(0\text{-}1)^2 + 0.75(1\text{-}1)^2 + 0.125(2\text{-}1)^2 = 0.25$$

$$\text{S.D.} = 0.5$$

Here with $\rho = -0.5$ the weight put on the extremes of 2T and 2H is half way between the cases where $\rho = -1$ and $\rho = 0$. As a result the degree of risk in terms of the standard deviation of the portfolio is also between the two cases above.

With positive correlation, the analysis is similar. With $\rho = +1$ all the weight is at the extremes so the probability of 2T and 2H is 0.5 and there is no benefit from diversification. With $\rho = +0.5$, the probability weight on 2T and 2H is 0.375 and the weight on HT and TH is 0.125.

In conclusion, except in the case where $\rho = +1$ there is always some benefit to spreading your investment among different stocks. As correlation goes down this benefit increases. As we will see this is the basic idea behind diversification. We shall see below we can extend it to a wide range of situations with many stocks and differing probability distributions of returns.

Means and Variances of Portfolios

        In the case above it was quite a simple task to take the individual stock returns combine them with the (1/2) and (1/2) weights on the two stocks and work out the payoffs of the portfolio in each state. From these portfolio payoffs we calculated the mean, variance and standard deviation of the portfolio. As the number of states becomes larger this procedure of calculating portfolio payoffs state by state and then calculating the statistics for the portfolio payoff distribution becomes more and more cumbersome.

        An alternative that is considerably easier is just to use the means and standard deviations of the individual stocks and their covariances or correlations and use them to calculate the mean and standard deviation of the portfolios. There are some useful formulas which allow us to do this. We will be using them a lot in the coming weeks. They are derived in the appendix at the end of the section. These formulas look fearsome but there are only two basic things to remember.

1. MEANS ADD.

        In other words if you invest half your money in stock X_1 and the other half in stock X_2, the mean return of the portfolio is simply half the mean of stock X_1 and half the mean of stock X_2.

2. VARIANCES DON'T ADD - COVARIANCES MATTER.

In other words if we take the same example of half our money in stock X_1 and half in stock X_2, the variance of the portfolio is <u>not</u> just half the variance of stock X_1 and half the variance of stock X_2. The reason is that correlations play an important role too as the simple examples above demonstrated. This is why there is a correlation or covariance term when calculating the variance of a portfolio.

For the moment the thing to remember is means add but variances and also standard deviations don't. We will come back to this later on and consider examples to illustrate it.

Statistical Review: Conclusion

This is all the statistics you'll basically need to know. The main thing is to have an intuitive understanding of what they mean. Arithmetic mean is a measure of how much you can expect to receive if you hold a stock for a year. The variance and standard deviation are measures of how variable the returns are likely to be. The higher the variance or standard deviation the greater the variation. Covariance and correlation are measures of whether two variables move together or in opposite directions. If they move together covariance and correlation are positive; if they move in opposite directions covariance and correlation are negative and if they are independent covariance and correlation are zero. The lower the correlation the greater the reduction in risk from putting stocks together in a portfolio.

Appendix: Section 7(i)

Results Needed for Theories of Asset Pricing

Note: To derive the formulas we need the following.

If π is a constant

$$E(\pi X) = \sum_{i=1}^{n} \pi p_i x_i = \pi \sum_{i=1}^{n} p_i x_i = \pi EX$$

Also

$$E(X + Y) = \sum_{i=1}^{n} (p_i x_i + p_i y_i)$$

$$= \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i y_i = EX + EY$$

Mean and Variance of a Portfolio of 2 assets

Suppose that you put π_1 in stock X_1 and π_2 in stock X_2.

$$\text{Return on the portfolio} = X = \pi_1 X_1 + \pi_2 X_2$$

This implies

$$\text{Mean of portfolio} = EX$$

$$= \pi_1 EX_1 + \pi_2 EX_2$$

i.e. **MEANS ADD**

Using the definition of variance $E(X - EX)^2$ gives

$$\text{Variance of Portfolio} = E[\pi_1 X_1 + \pi_2 X_2 - (\pi_1 EX_1 + \pi_2 EX_2)]^2$$

$$= E[\pi_1(X_1 - EX_1) + \pi_2(X_2 - EX_2)]^2$$

$$= E[\pi_1^2(X_1 - EX_1)^2 + \pi_2^2(X_2 - EX_2)^2 + 2\pi_1\pi_2(X_1 - EX_1)(X_2 - EX_2)]$$

This simplifies to

Variance of Portfolio

$$= \pi_1^2 E(X_1 - EX_1)^2 + \pi_2^2 E(X_2 - EX_2)^2 + 2\pi_1\pi_2 E(X_1 - EX_1)(X_2 - EX_2)$$

$$= \pi_1^2 \text{ Var } X_1 + \pi_2^2 \text{ Var } X_2 + 2\pi_1\pi_2 \text{ Cov}(X_1, X_2)$$

$$= \sum_{i=1}^{2} \pi_i^2 \text{Var } X_i + \sum_{i=1}^{2} \sum_{i \neq j, j=1}^{2} \pi_i \pi_j \text{Cov}(X_i, X_j)$$

or writing it in full

$$= \begin{array}{ll} \pi_1^2 \text{ Var } X_1 & + \pi_1\pi_2 \text{ Cov}(X_1, X_2) \\ + \pi_1\pi_2 \text{ Cov}(X_1, X_2) & + \pi_2^2 \text{ Var } X_2 \end{array}$$

Notice that in this format which will be useful later on the diagonal terms are variances and the off-diagonal terms are covariances.

Note also

VARIANCES DON'T ADD – COVARIANCES MATTER

<u>Mean and Variance of a portfolio of N assets</u>

Suppose that you put π_i in stock X_i for $i = 1,...,N$. The formulas for mean and variance are just extensions of the case with 2 assets.

$$\text{Mean of portfolio} = \pi_1 EX_1 + \pi_2 EX_2 + \ldots + \pi_N EX_N$$

MEANS ADD

$$\text{Variance of portfolio} = \sum_{i=1}^{N} \pi_i^2 \text{ Var } X_i + \sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \pi_i \pi_j \text{Cov}(X_i, X_j)$$

$$= \begin{array}{llll}
\pi_1^2 \text{ Var } X_1 & + \pi_1 \pi_2 \text{Cov}(X_1, X_2) & \ldots & + \pi_1 \pi_N \text{Cov}(X_1, X_N) \\
+ \pi_2 \pi_1 \text{Cov}(X_1, X_2) & + \pi_2^2 \text{Var } X_2 & \ldots & + \pi_2 \pi_N \text{Cov}(X_2, X_N) \\
\ldots & \ldots & \ldots & \ldots \\
+ \pi_N \pi_1 \text{Cov}(X_1, X_N) & + \pi_N \pi_2 \text{Cov}(X_2, X_N) & \ldots & + \pi_N^2 \text{ Var } X_N
\end{array}$$

Again the diagonal terms are variances and the off-diagonal terms are covariances.

VARIANCES DON'T ADD

Graph 2-1: Wealth Indices of Investments in the U.S. Capital Markets

Index (Year-End 1925 = $1.00)

—	Small Company Stocks ($16,054.70 YE10)
—	Large Company Stocks ($2,982.24 YE10)
—	LT Govt Bonds ($92.94 YE10)
—	T-Bills ($20.55 YE10)
----	Inflation ($12.23 YE10)

$100,000.0

$10,000.0

$1,000.0

$100.0

$10.0

$1.0

1925   1935   1945   1955   1965   1975   1985   1995   2005   2010

Year-end
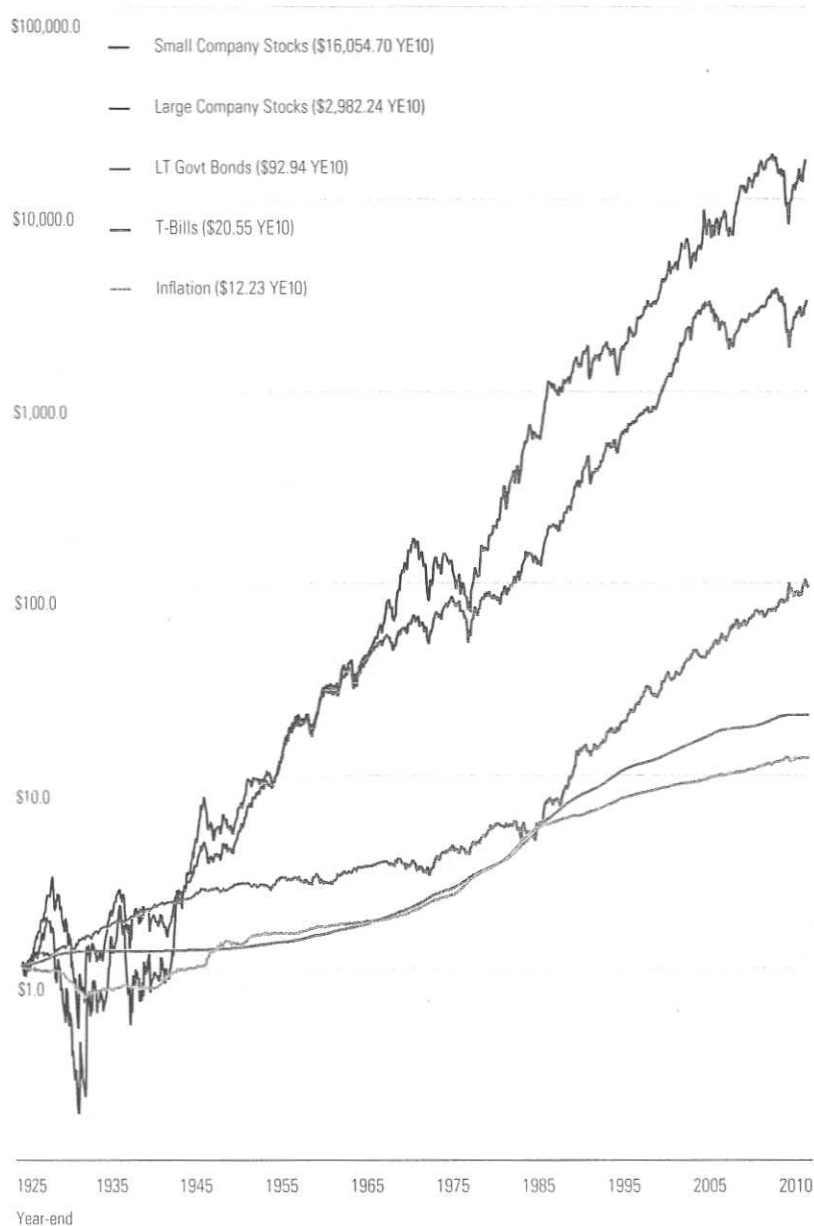
Data from 1925–2010.

Table 2-1: Basic Series: Summary Statistics of Annual Total Returns

Series	Geometric Mean (%)	Arithmetic Mean (%)	Standard Deviation (%)	Distribution (%)
Large Company Stocks	9.9	11.9	20.4	
Small Company Stocks*	12.1	16.7	32.6	
Long-Term Corporate Bonds	5.9	6.2	8.3	
Long-Term Government Bonds	5.5	5.9	9.5	
Intermediate-Term Government Bonds	5.4	5.5	5.7	
U.S. Treasury Bills	3.6	3.7	3.1	
Inflation	3.0	3.1	4.2	

-90   0   90
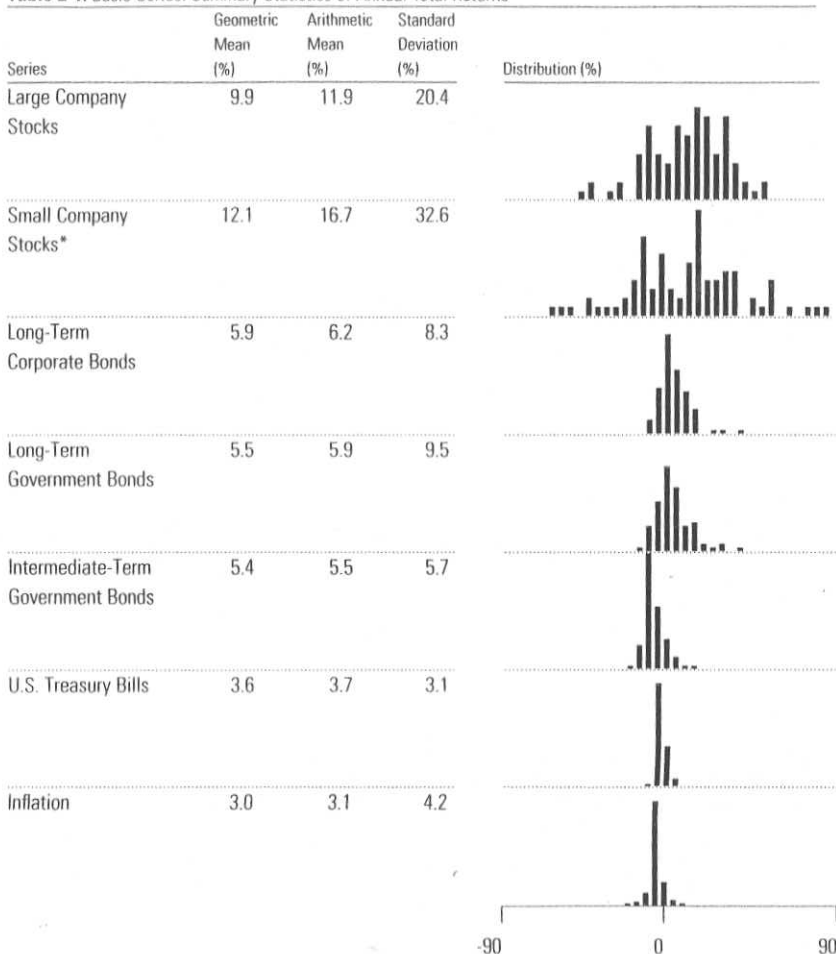
Data from 1926–2010. * The 1933 Small Company Stocks Total Return was 142.9 percent.

Real Equity Returns in 17 Countries from 1900-2005

Country	Annualized Returns (% p. a.)	Amount in 2005 from investment of 1 in 1900
Belgium	2.40	12
Italy	2.46	13
Germany	3.09	25
France	3.60	42
Spain	3.74	49
Norway	4.28	85
Switzerland	4.48	104
Japan	4.51	107
Ireland	4.79	143
World ex-U.S. (USD)	5.23	222
Denmark	5.25	227
The Netherlands	5.26	229
UK	5.50	292
World (USD)	5.75	375
Canada	6.24	612
US	6.52	809
South Africa	7.25	1,668
Australia	7.70	2,599
Sweden	7.80	2,868

Source: Adapted from Table 1 in E. Dimson, P. Marsh and M. Staunton (2006): "The Worldwide Equity Premium: A Smaller Puzzle" (Reading for Section 7).

Section 7(ii) A Theory of Asset Pricing: Introduction

What we will do is develop a theory that allows us to price stocks. The approach we take

follows a similar path to the standard theory of price determination for consumer goods in

microeconomics.

The starting point for the theory of price determination for consumer goods is a theory of

consumer choice. This is developed using utility theory. Consumers have indifference curves

over the available goods.
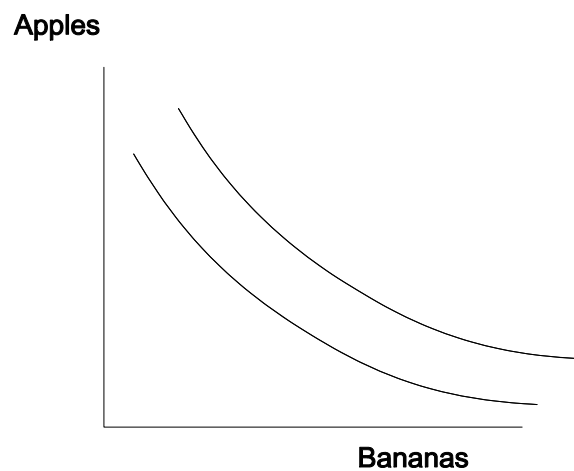
Apples

Bananas

They are constrained by their budget constraint, which gives the locus of the points that are

feasible for them. They choose the combination of apples and bananas that maximizes their

utility--this is given by the highest utility curve that touches the budget constraint.

Apples

Optimal Point

Demand
for Apples

Budget
Constraint

Demand
for Bananas

Bananas
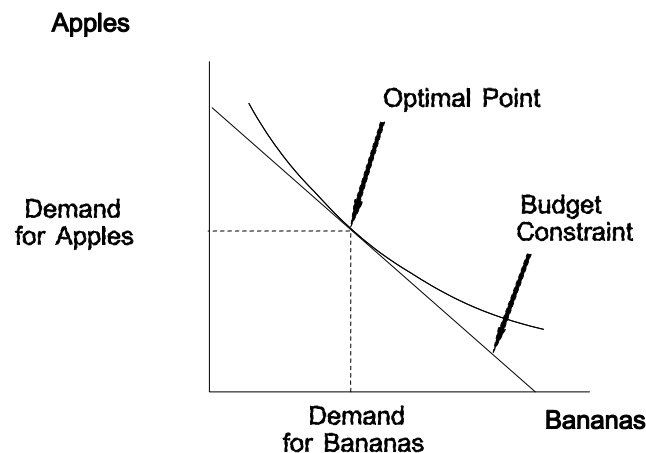
Then we go on to find aggregate demand. Given, in addition, a theory of supply, we get a theory

of price.

P

D   S

P_e

S   D

Q_e Q
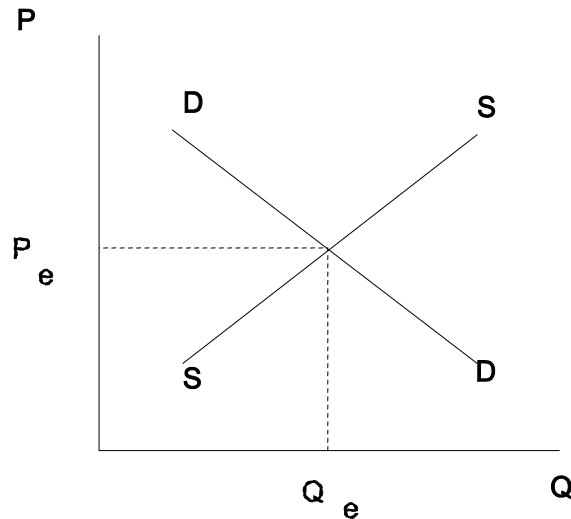
What we want to do in this lecture is to develop a theory for pricing stocks whose yields are

uncertain. In order to do this we have to develop a **theory of investor behavior** where the things

individuals are choosing between are risky stocks or portfolios. We will show that we get

indifference curves over mean and standard deviation that are analogous to the indifference

curves we get in ordinary consumer theory. We also need to develop the equivalent of a budget

constraint that will be called the **portfolio locus**.

We develop the theory of investor behavior in Section 7(iii). The analysis of portfolio

loci and the theory of asset demand that results are considered in Section 7(iv). In Section 8 we

combine this theory with the assumption that there is a fixed supply of stocks to develop a theory

of pricing risky assets, namely the CAPM.

Section 7(iii): Mean-Standard Deviation Analysis

Mean-Standard Deviation Analysis

It is helpful to start with some assumptions and terminology. We will be thinking of somebody who receives their income from stocks and bonds.

Assumption 1: Utility depends on how many dollars of income you receive.

Assumption 2: Every individual is made better off if expected income is increased.

Risk averse: Somebody is risk averse if an increase in the standard deviation of their income (holding expected income the same) makes them worse off.

Risk neutral: Somebody is risk neutral if an increase in the standard deviation of their income (holding expected income the same) does not make them worse off or better off.

Risk loving: Somebody is risk loving if an increase in the standard deviation of their income (holding expected income the same) makes them better off.

What are the implications of these assumptions for the shape of indifference curves of a risk averse person if we plot mean income against the standard deviation of income?

MEAN
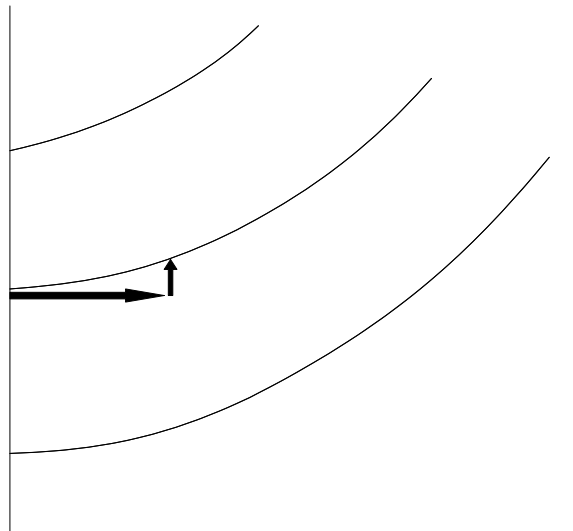
STANDARD
DEVIATION

It follows that if somebody is risk averse then for that person to be kept indifferent (i.e for utility

to remain the same), an <u>increase</u> in the standard deviation of income must be accompanied by an

<u>increase</u> in <u>mean</u> income. Hence we get the type of indifference curves shown above.

   The more risk averse a person is the greater the slope of the indifference curves. This is

because a given increase in standard deviation requires a larger increase in mean income the

more risk averse the person is. Similarly the flatter the indifference curves the less risk averse

the person is because a given rise in standard deviation requires a smaller increase in expected

income to make the person indifferent again. These cases are shown below.

When the person is very risk averse:

MEAN
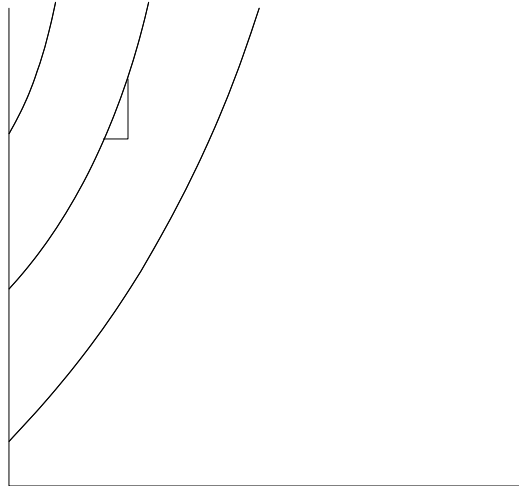
**STANDARD
DEVIATION**

When the person is not very risk averse

MEAN
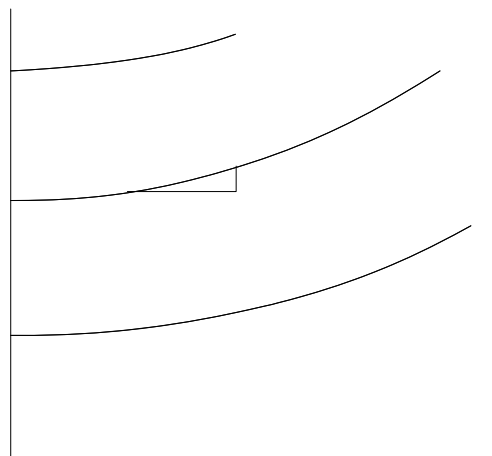
**STANDARD
DEVIATION**

What do you think happens if a person is risk neutral? In this case the indifference

curves are flat since the person's expected utility is unaffected by a change in standard deviation;

it only depends on the level of expected income.

MEAN
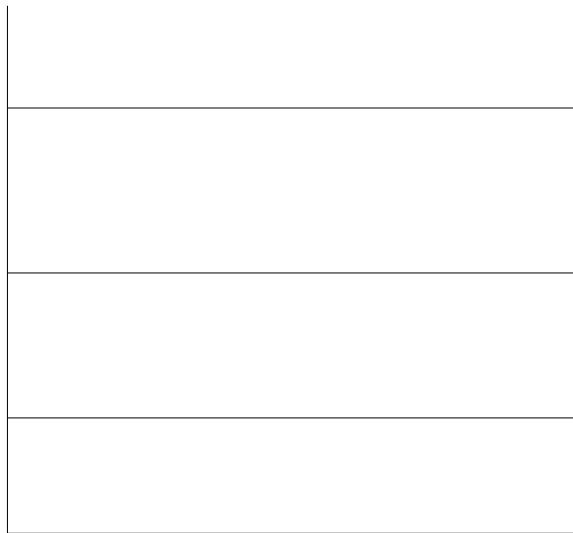
STANDARD
DEVIATION

What happens if a person is risk-loving? In this case the curves are downward sloping.

Investors like risk so an increase in standard deviation requires a fall in expected income to

restore the investor to their original level of expected utility as shown below.

MEAN

STANDARD
DEVIATION

In what follows we shall be assuming risk aversion so that indifference curves are upward sloping. The justification for this is that most people appear to be willing to pay to avoid risks by buying insurance. There are problems with the assumption of risk aversion though. For example, why do people gamble? In this case they are actually paying to take on a risk which suggests they are risk loving. The argument that is usually made to explain this is that it's the utility of going to the casino and actually seeing the screens change and the dice roll and so on that makes gambling attractive rather than the shape of people's utility functions. These factors are not that important in the context of investing in the stock market. Risk free assets usually have a lower return than risky assets. This is consistent with people being risk averse.

Section 7(iv): Diversification and the Measurement of Risk

Read Chapters 7 and 8 BMA

Introduction

        Suppose we have two stocks, A and B, and preferences as below. Which should be

chosen? Clearly we should choose B.
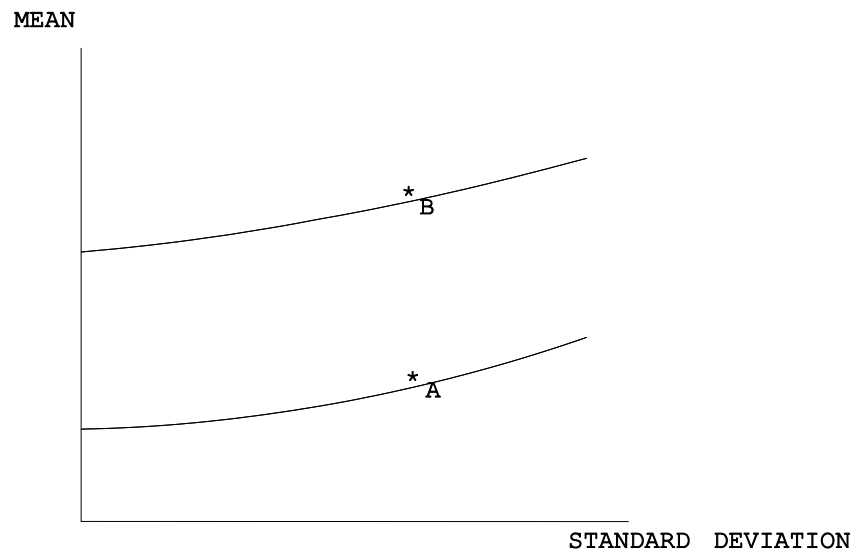
However, is this the best we can do? No, we may do better by combining the two assets in a

portfolio.

A Two Stock Portfolio

Consider the following two risky stocks:

	Prob	A	B	0.5(A + B)=D
State 1	0.25	1	5	3
" 2	0.25	2	4	3
" 3	0.25	3	3	3
" 4	0.25	4	2	3

$$EA = 2.5 \quad EB = 3.5$$

$$Var\ A = 1.25 \quad Var\ B = 1.25$$

$$SD\ A = 1.12 \quad SD\ B = 1.12$$

$$Cov\ (A,B) = -1.25$$

```
MEAN

 3.5                              *. B

 3.0  *
      D

 2.5                              *  A

                             1.12   STANDARD  DEVIATION
```
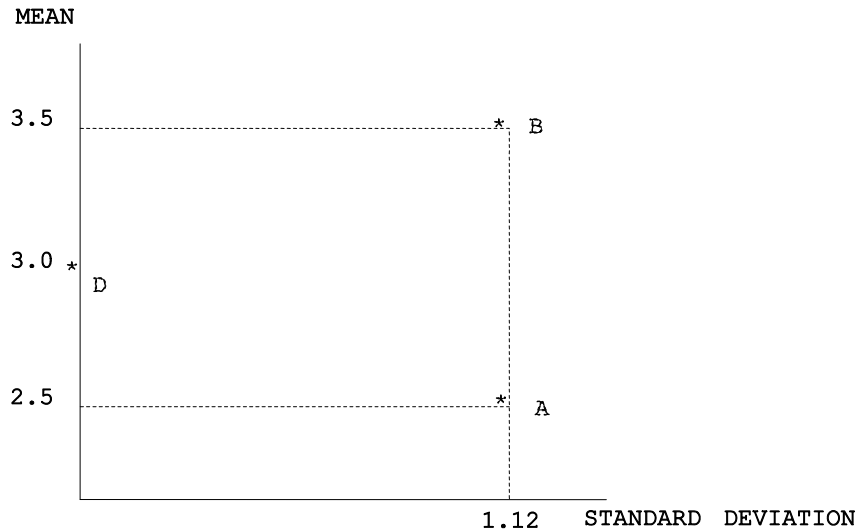
Suppose we have $1 to invest. We could invest it all in A or all in B, or we could invest 0.5 in A

and 0.5 in B and get 3 in every state. In other words, by taking a combination of A and B we

have produced a third possibility, D.

In these cases it may well be that by taking a combination of A and B, such as D, we can be better off than by taking A or B alone because the movements in the stocks tend to offset each other.
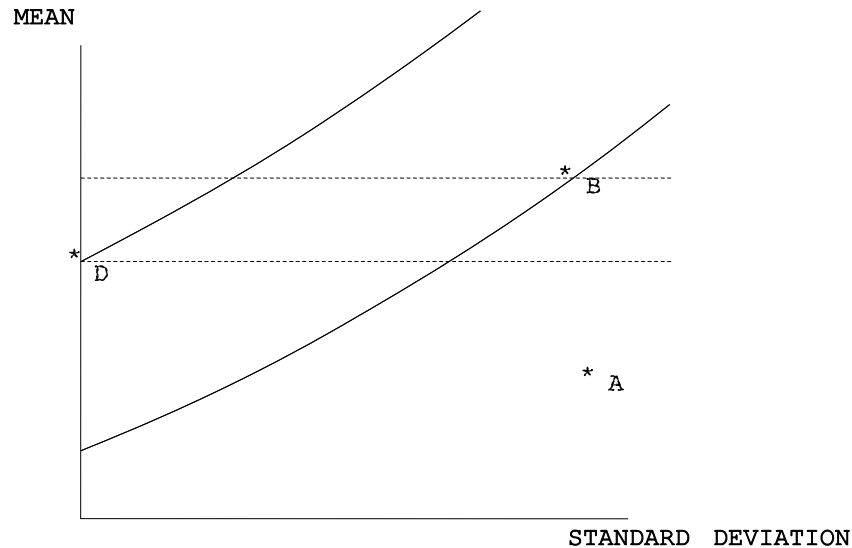
This principle is known as <u>diversification</u>. The example we are considering is rather special because it allows all risk to be eliminated--the variables always move in exactly opposite directions: they are perfectly negatively correlated, i.e.,

$$\rho_{AB} = \frac{\text{Cov}(A,B)}{SD_A\ SD_B} = \frac{-1.25}{(1.12)(1.12)} = -1$$

Of course 0.5 and 0.5 is not the only combination of A and B we could utilize. For example, we could have 0.25 in A and 0.75 in B. Where would this lie on our diagram? In the 0.5 in A and 0.5 in B case it was fairly obvious the portfolio had mean of 3 and SD of 0. In the 0.25 in A and 0.75 in B case, we could calculate the portfolio payoff in each state and find the

mean and standard deviation of these payoffs. A quicker method is to use our formulae for the mean and standard deviation of a portfolio with two stocks which we discussed in Section 7(i) to find the mean and standard deviation of this portfolio.

Remember that means add so that:

Mean of portfolio(0.25 in A, 0.75 in B) $= 0.25 \times EA + 0.75 \times EB$

$$= 0.25 \times 2.5 + 0.75 \times 3.5$$

$$= 3.25$$

Variances do not add they depend on covariance:

Var of portfolio(0.25 in A, 0.75 in B)

$$= 0.25^2 \text{Var A} + 0.75^2 \text{Var B} + 2 \times 0.25 \times 0.75 \times \text{Cov(A,B)}$$

$$= 0.25^2 \times 1.25 + 0.75^2 \times 1.25 + 2 \times 0.25 \times 0.75 \times (-1.25)$$

$$= 0.3125$$

SD of portfolio(0.25 in A, 0.75 in B) $= (0.3125)^{0.5} = 0.56$

The point with 0.25 in A and 0.75 in B has mean 3.25 and SD 0.56. It is labeled $\pi = 0.25$ in the diagram. Similarly, we can calculate for the portfolio with 0.75 in A and 0.25 in B that the mean is 2.75 and the standard deviation is 0.56 and it is shown by the point labeled $\pi = 0.75$. In general, we can use our formulae to find the mean and SD for all the other possible values combinations of A and B. We can then use these numbers to plot the portfolio locus between A and B on our mean-standard deviation diagram.
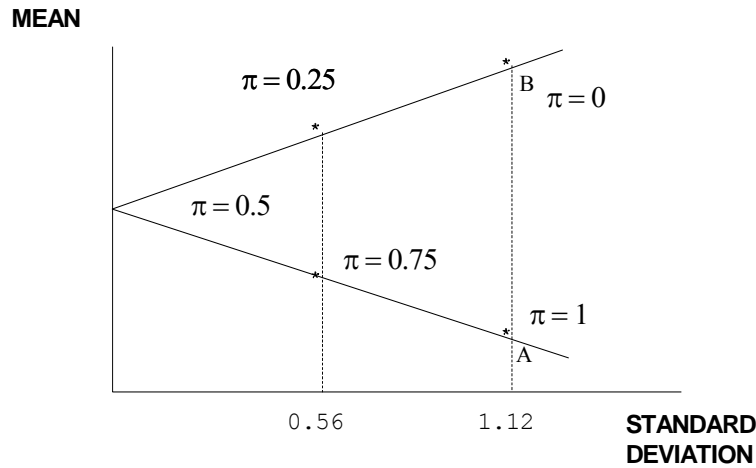
MEAN

We can go on to do this in other cases to see that we get a similar type of result--namely, diversification "bows out" the possible means and standard deviations we can obtain between A and B even if correlation is not -1.

Diversification when correlation is not -1

Next consider the following two risky stocks:

Prob	A	B
0.25	1	3
0.25	2	5
0.25	3	2
0.25	4	4

$$EA = 2.5 \qquad EB = 3.5$$

$$\text{Var A} = 1.25 \quad \text{Var B} = 1.25$$

$$\text{SD A} = 1.12 \qquad \text{SD B} = 1.12$$

$$\text{Cov (A,B)} = 0.25(1\text{-}2.5)(3\text{-}3.5) + 0.25(2\text{-}2.5)(5\text{-}3.5) + 0.25(3\text{-}2.5)(2\text{-}3.5) + 0.25(4\text{-}2.5)(4\text{-}3.5)$$

$$= 0$$

In this case it can be seen that the means and standard deviations of A and B are the

same as before. Now, however, the correlation is

$$\rho_{AB} = \frac{Cov(A,B)}{SD_A SD_B} = 0$$

Whereas with a correlation of -1, the movements in A and B always cancelled out, now it can be

seen that A and B cancel out some of the time but some of the time they do not. Overall, there is

still a benefit from diversification.

Mean of $0.5(A + B) = 3$

Variance of $0.5(A + B) = 0.625$

SD of $0.5(A + B) = (0.625)^{0.5} = 0.791$

We can see this for mixtures of A and B other than 0.5 and 0.5 using our formulas from the

Appendix to Section 7(i) and plotting the portfolio locus.
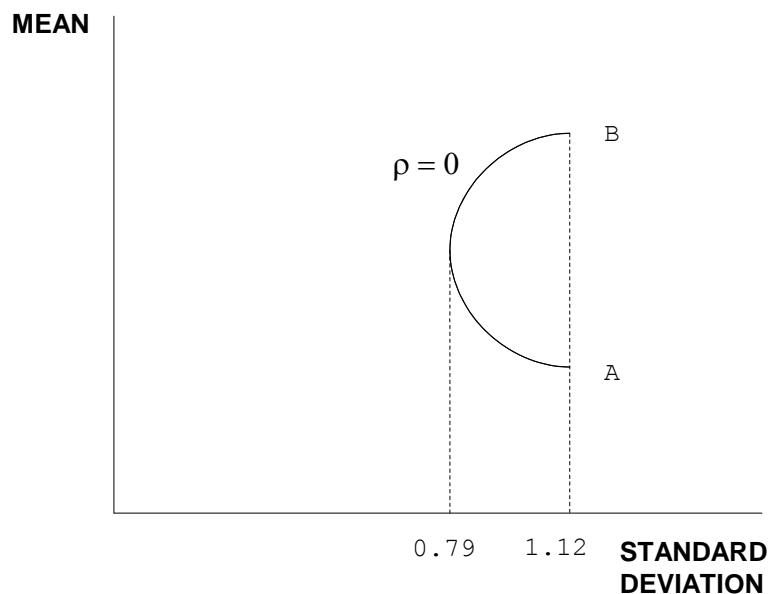
 The portfolio locus is still bowed out but not by as much as before. As we increase correlation

it becomes less bowed out. With positive correlation there is still some canceling out and a benefit to diversification. For example with $\rho_{AB} = 0.6$ we would have the situation below.
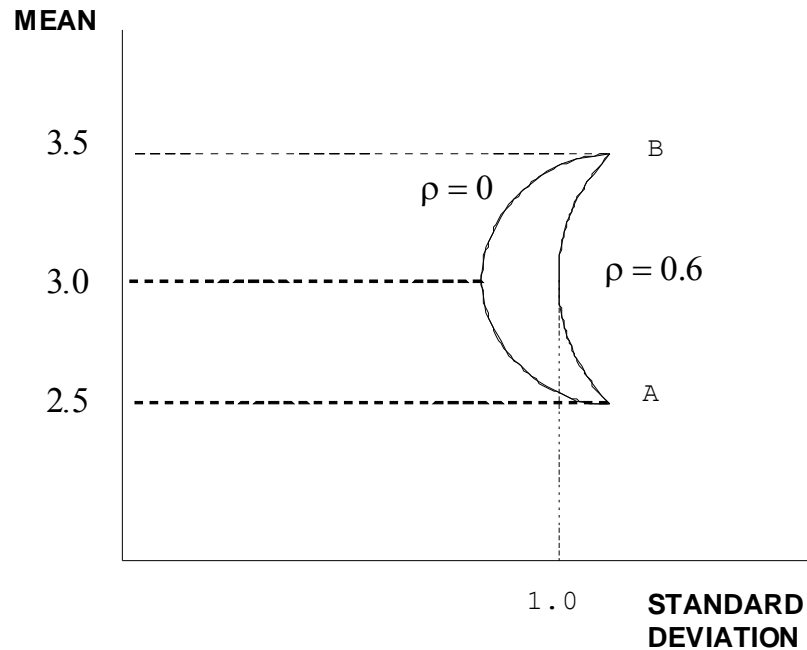
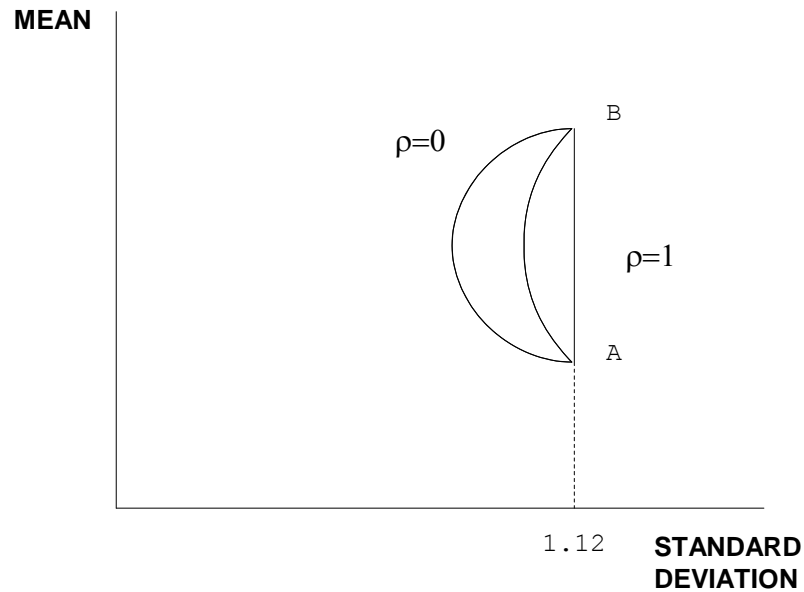Finally, consider the following case.

Prob	A	B
0.25	1	2
0.25	2	3
0.25	3	4
0.25	4	5

$$EA = 2.5 \quad EB = 3.5$$

$$\text{Var A} = 1.25 \quad \text{Var B} = 1.25$$

$$\text{SD A} = 1.12 \quad \text{SD B} = 1.12$$

$$\text{Cov (A,B)} = 1.25$$

Since A and B move proportionately it follows that

$$\rho_{AB} = 1.0$$

Here there is no benefit from diversification and we simply get a straight line between the two

points.

Relationship between ρ and diversification locus

It follows from this analysis that in general the portfolio locus is bowed out and the degree to which this happens depends on the correlation ρ between the two stocks. Thus we have:
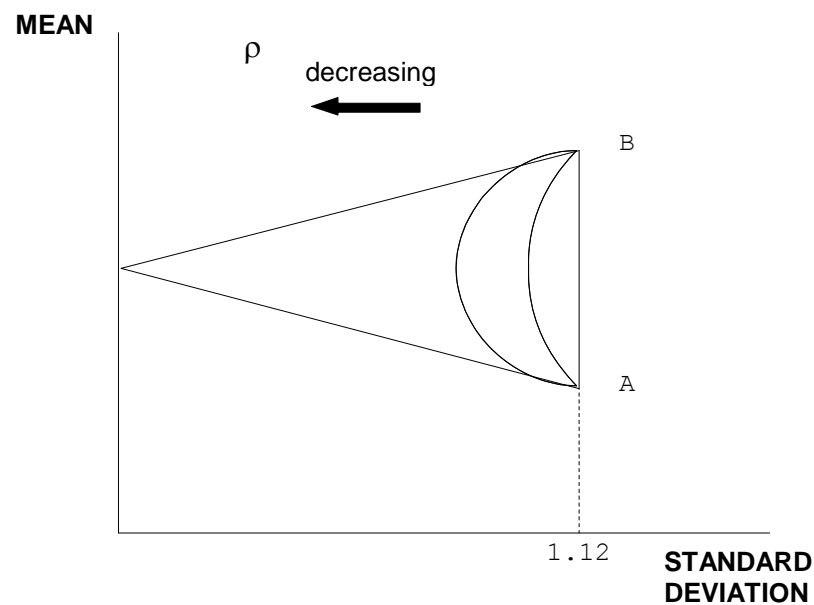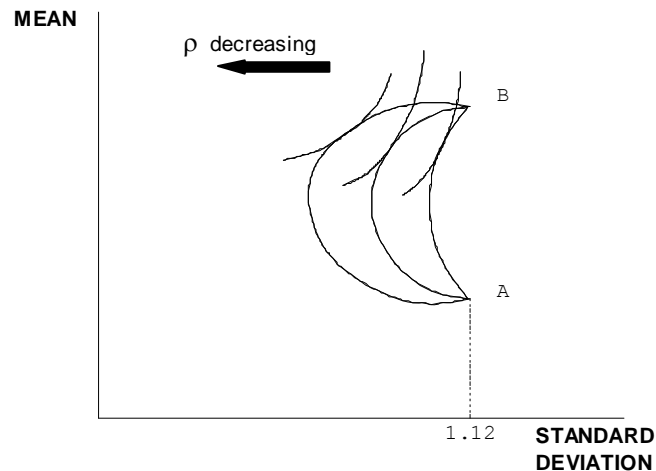
Choosing the Optimal Portfolio

Given a portfolio locus, an investor will choose the portfolio that allows her to get on the

highest indifference curve. This will correspond to the tangency point as shown below.
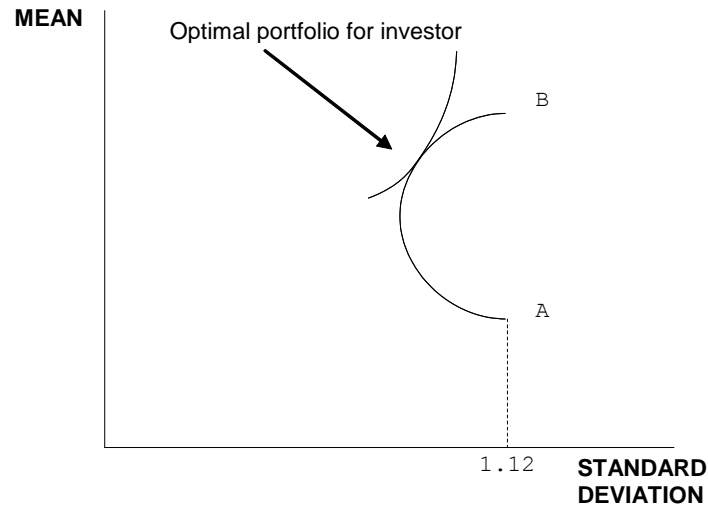

As ρ falls, the portfolio locus is farther to the left and investors are able to attain higher

indifference curves and are better off.

General Case Where There are N Stocks: $X_1, \ldots, X_N$

So far we have just been dealing with 2 stocks. Suppose there are N stocks and we can invest π_1 to π_N in these respectively. Then we can diversify among all these stocks. We can draw in all the possible means and standard deviations obtainable from varying the π_i. We are interested in minimizing standard deviation for a given expected return because this allows us to get on the highest possible indifference curve. If we look at the northwest frontier, we get the set of portfolios that have this property. This is called the portfolio efficiency locus or efficiency locus for short.

Efficiency Locus
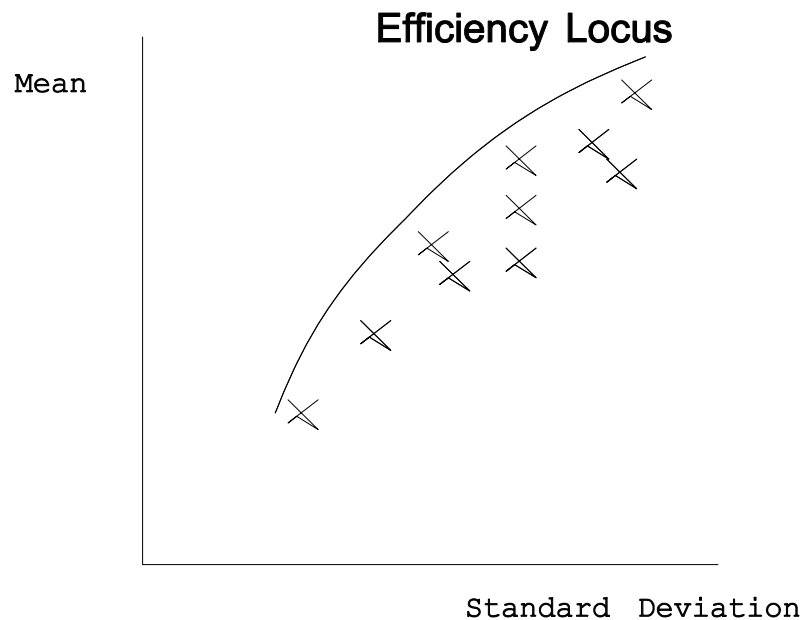
How much would we normally expect to be able to achieve by diversification? To see this we need to consider the mean and variance of a portfolio with N stocks.

We know from Section 7(i) that for a portfolio with π_i in X_i for $i = 1,\ldots,N$ the mean and the variance are given by

Mean of portfolio $= \pi_1 EX_1 + \pi_2 EX_2 + \ldots + \pi_N EX_N$

$$\text{Variance of portfolio} = \sum_{i=1}^{N} \pi_i^2 \, \text{Var } X_i \; + \; \sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \pi_i \pi_j \, \text{Cov}(X_i, X_j)$$

In order to get some idea of the orders of magnitude of these terms, consider the case where $\pi_i = 1/N$ so that you are putting the same proportion in each stock. Then

$$\text{Variance of portfolio} = \frac{1}{N^2} \sum_{i=1}^{N} \text{Var } X_i \; + \; \frac{1}{N^2} \sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \text{Cov}(X_i, X_j)$$

Now since there are N variances

$$\text{average variance of stocks} = \frac{\sum_{i=1}^{N} \text{Var } X_i}{N}$$

and since there are N^2 - N covariances

$$\text{average covariance of stocks} = \frac{\sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \text{Cov}(X_i, X_j)}{N^2 - N}$$

It follows that

$$\text{Variance of portfolio} = \frac{N}{N^2} \times \text{average var iance} + \frac{N^2 - N}{N^2} \times \text{average cov ariance}$$

$$= \frac{1}{N} \times \text{average var iance} + \left(1 - \frac{1}{N}\right) \times \text{average cov ariance}$$

As N gets large, $1/N \to 0$ so the first term goes to zero, and the second term approaches the average covariance. Hence the risk of a well-diversified portfolio depends on the average covariance of the stocks included.

Variance of portfolio = average covariance of stocks

It's thus the covariances which are the important determinants of the riskiness of a portfolio rather than the variances.

On the whole would we expect stocks to have a positive or negative covariance? Because of the business cycle, we would expect firms to have a <u>positive</u> covariance. For example, when there is a depression, both the auto and the steel industry have low returns and their stocks therefore have a positive covariance. In fact we know from what we did in the historical review that they do.

The riskiness of stocks due to the business cycle is known as <u>market risk</u> or <u>systematic risk</u>, and it can't be diversified away.

Market risk is not the only type of risk associated with stocks considered individually. There may be changes in returns due to factors only affecting the industry, such as increased foreign competition, a change in tastes and so on. Or there may be changes due to factors only affecting a single firm, such as a serious breakdown in a plant or a good or bad decision by a firm's management. This type of risk is called <u>unique risk</u> or <u>unsystematic risk</u>.

The riskiness of a stock thus has two components: market and unique risk. It is unique risk that can be diversified away. N used not to have to be very large before unique risk

becomes a very small contributor to total risk (around 20 if diversified across industries). In

recent years, however, volatility of returns have changed in such a way that around 50 stocks are

needed.   If we graph portfolio standard deviation against the number of securities, we usually

get a relationship of the following type:
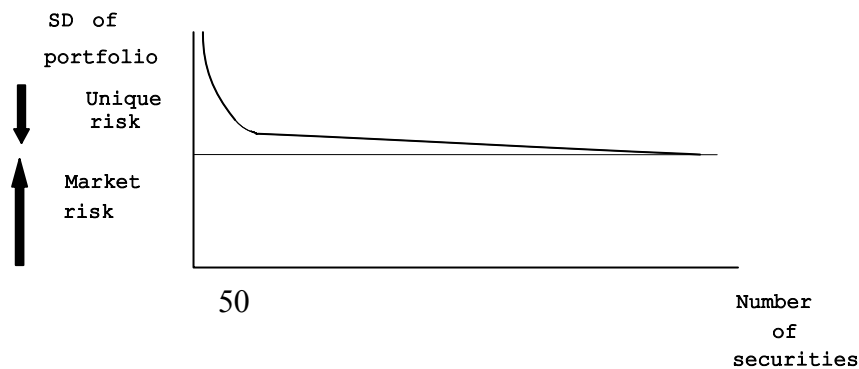
Hence it is <u>market or systematic risk</u> which is important for portfolio risk.

Even though a stock may have a high variance, it may be much more desirable than a

security with a lower variance and the same expected return because it has a lower <u>market risk</u>.


Measurement of the Market Risk of a Security: Beta (β)

In order to get rid of unique risk we need to hold stocks in portfolios. If we are holding

stocks in portfolios we know from above that the standard deviation of a stock is no longer the

appropriate measure of risk. What we need is a measure of risk that is appropriate in a portfolio

context. As we will see, it turns out that a very useful measure of risk is β where this is the

covariance of the stock with the market divided by the variance of the market:

$$\beta_X = \frac{\text{Cov}(X, \text{Market})}{\text{Var } M} = \frac{\text{Cov}(X, \text{Market})}{SD_M^{\,2}}$$

Note that X can be a stock or a portfolio.

Why is beta defined relative to the "market portfolio" which is all the stocks in the market? The reason this is a useful benchmark is because it is the portfolio where the amount of unique risk that is diversified away is maximized. On our diagram of unique and market risk it is the farthest point out on the right hand side.
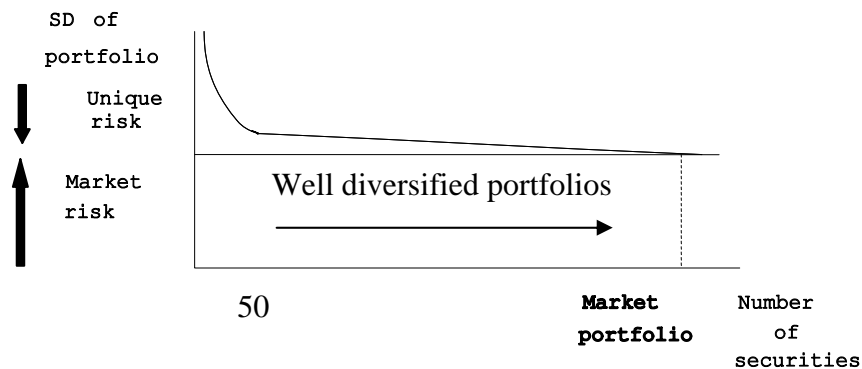
We denote the market portfolio by M. In this case π_1 is the value of the total amount of stock 1 relative to the total market value. For example, if the total value of stock 1 is 50 billion and the total value of the market portfolio is 10 trillion, then π_1 = 50 billion/10 trillion.

As a practical matter, when we have gone beyond 50 stocks the added diversification achieved by going to the market does not reduce variance by a great deal. Portfolios that have a very small amount of unique risk are called "well-diversified portfolios". These are all the portfolios out to the right on our diagram.

Well-diversified portfolios

Formally we define a well-diversified portfolio to be a portfolio with almost no unique risk; in technical terms it is a portfolio which has a correlation of approximately 1 with the

market portfolio. Using the definition of correlation and taking the approximation to be exact

$$\mathrm{Corr(port,M)} = \frac{\mathrm{Cov(port,M)}}{\mathrm{SD_{port}SD_M}} = 1$$

Hence for a well-diversified portfolio

$$\mathrm{SD_{port}} = \frac{\mathrm{Cov(port,M)}}{\mathrm{SD_M}}$$

or multiplying top and bottom by $\mathrm{SD_M}$ and using the definition of $\beta = \mathrm{Cov(port,M)}/(\mathrm{SD_M})^2$

$$\mathrm{SD_{port}} = \frac{\mathrm{Cov(port,M)}}{\mathrm{SD_M^2}}\mathrm{SD_M} = \beta_{port}\mathrm{SD_M}$$

In words, the standard deviation of a well-diversified portfolio is the beta of that portfolio times the standard deviation of the market portfolio. Hence, for well-diversified portfolios the beta is a measure of the market risk of that portfolio. The standard deviation of the market, $\mathrm{SD_M}$, is essentially a scaling factor.

It follows from the appendix that the beta of a portfolio is just the weighted sum of the betas of the stocks making up that portfolio, i.e. BETAS ADD:

$$\beta_{port} = \sum_{i=1}^{N} \pi_i \beta_i$$

Substituting this in the previous expression gives

$$\mathrm{SD_{port}} = \sum_{i=1}^{N} \pi_i \beta_i \mathrm{SD_M}$$

Thus β_i measures the contribution of a stock to the standard deviation of a well-

diversified portfolio. One of the points that we have stressed so far is that whereas means of individual stocks add to give the mean of a portfolio this is not the case for standard deviations and variances. We can see from the formula above that:

FOR WELL-DIVERSIFIED PORTFOLIOS:

BETAS ADD TO GIVE THE SD

Since it is always good to diversify away risk and hold well-diversified portfolios this means beta is a very useful measure of risk. This can be seen easily in our diagram.
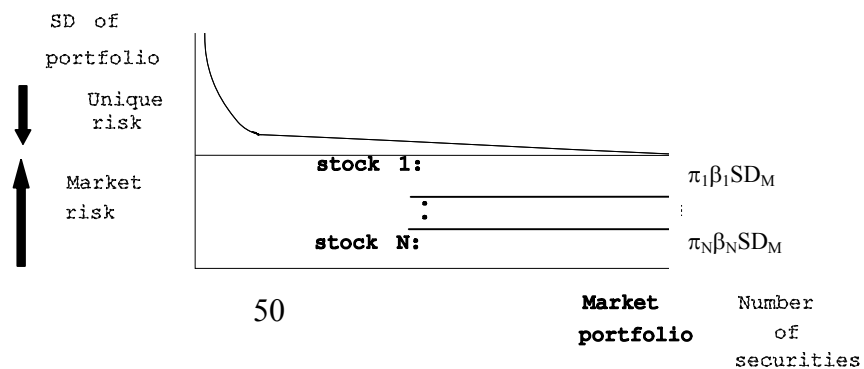

For any well-diversified portfolio β is therefore a measure of the market risk for a stock and indicates how much standard deviation it contributes to a well-diversified portfolio.

Since beta is an absolute measure its magnitude has a meaning. What is the beta of the market portfolio? Setting port = M in the formula we can find this:

$$\beta_M = \frac{Cov(M,M)}{Var\ M} = \frac{Var\ M}{Var\ M} = 1$$

Thus stocks with a $\beta > 1$ tend to be sensitive to movements in the market--they magnify these movements. Stocks with a $\beta < 1$ are relatively insensitive to movements in the market.

To see this in graphical terms we can plot excess returns (i.e. returns in excess of the risk free rate) on stock i against excess returns on the market.

Excess
return on
stock i

High beta stock

Excess return on
market portfolio
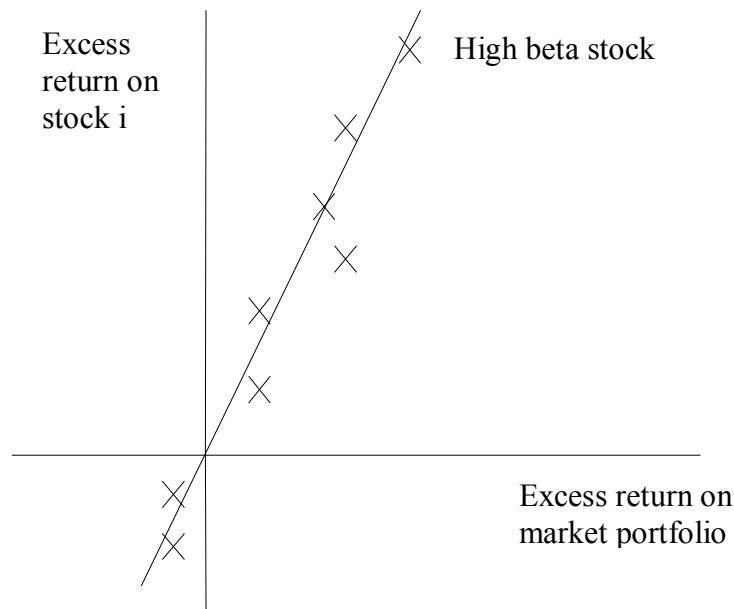
It follows from regression theory that if we fit a straight line to the data then the slope is given by

$$Slope = \frac{Cov(X_i, M)}{Var\ M} = \beta_i$$

In other words, if the stock has a high beta as shown in the diagram above, it means that it is sensitive to movements in the market and magnifies them. A small positive return on the market corresponds to a large positive return on the stock; a small negative return on the market

corresponds to a large negative return on the stock. The diagram below shows the relationship between the stock's return and the market return for a stock with low beta. Here the stock dampens movements in the market. A large return in the market only corresponds to a small return on the stock.

Excess
return on
stock i

Low beta stock

Excess return on
market protfolio
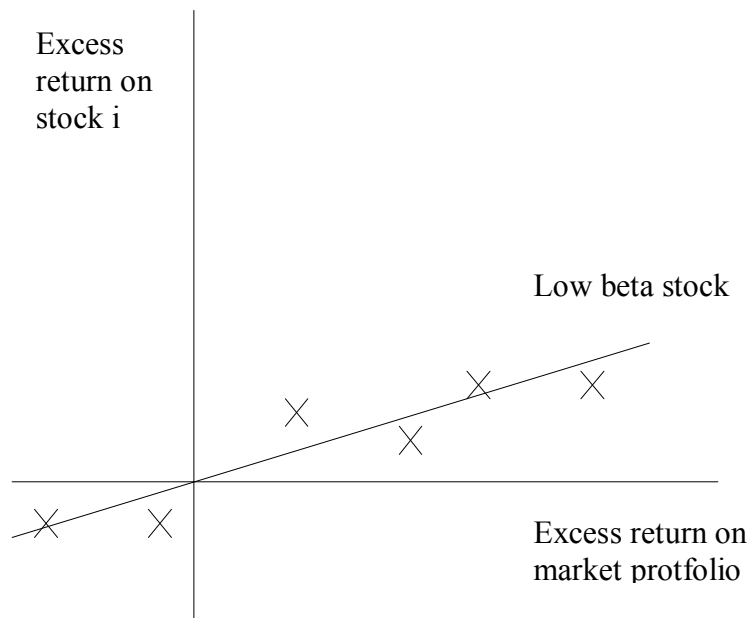
β is thus a measure of a stock's market risk, which is the risk that matters to investors since the unique risk can be diversified away. Thus what we mean by the riskiness of an asset is its β. By assets in the same risk class we mean assets with the same beta.

Now we have found an appropriate measure of the risk of stocks and portfolios we can derive a relationship between risk and return. This is the Capital Asset Pricing Model, which we turn to next.

The beta of a portfolio

Suppose you have a portfolio of i = 1, ... , N stocks with π_i in stock i with return X_i.

What is the beta of the portfolio given that the beta of stock i is β_i?

The return on the portfolio is $\sum \pi_i X_i$.. Hence from the definition of beta

$$\beta_{port} = \frac{Cov(\sum \pi_i X_i, M)}{Var\ M}$$

Using the definition of covariance

$$\beta_{port} = \frac{E\left[\sum \pi_i (X_i - EX_i)(M - EM)\right]}{Var\ M}$$

$$= \frac{\sum \pi_i E(X_i - EX_i)(M - EM)}{Var\ M}$$

Using the definition of beta

$$\beta_{port} = \sum \pi_i \beta_i$$

BETAS ADD

UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

FNCE 601

CORPORATE FINANCE

LECTURE NOTES

Franklin Allen


Fall 2011

QUARTER 1 - WEEK 6

Tu: 10/11/11 and Th: 10/13/11

Section 8: The Capital Asset Pricing Model

Read Chapters 7 and 8 BMA

The Capital Asset Pricing Model (CAPM)

We next turn to the derivation of the CAPM. We are going to assume that capital

markets are well functioning and that everybody has the same beliefs about the means and

standard deviations of all stocks. Everybody's perception of the efficiency locus of all the stocks

in the market is therefore the same. In other words, every investor's perception of the north-west

boundary of the attainable portfolios is the same. It is helpful to think of each portfolio on this

boundary as corresponding to a mutual fund.
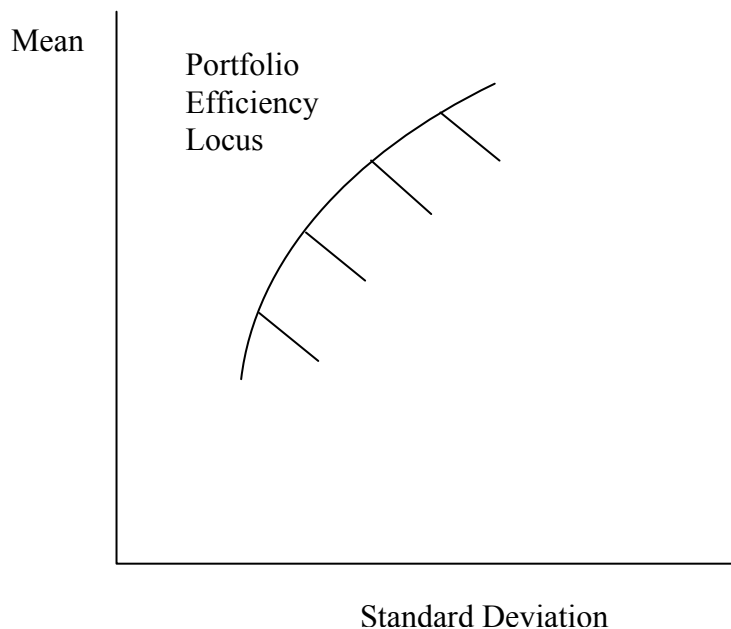
If this is the situation investors face, then the portfolio or mutual fund each chooses will

depend on his or her preferences. People such as Person 1 who are fairly risk averse will choose

mutual funds such as X with low return and low risk; people such as Person 2 who are not very

risk averse will choose mutual funds such as Y with high return and high risk.
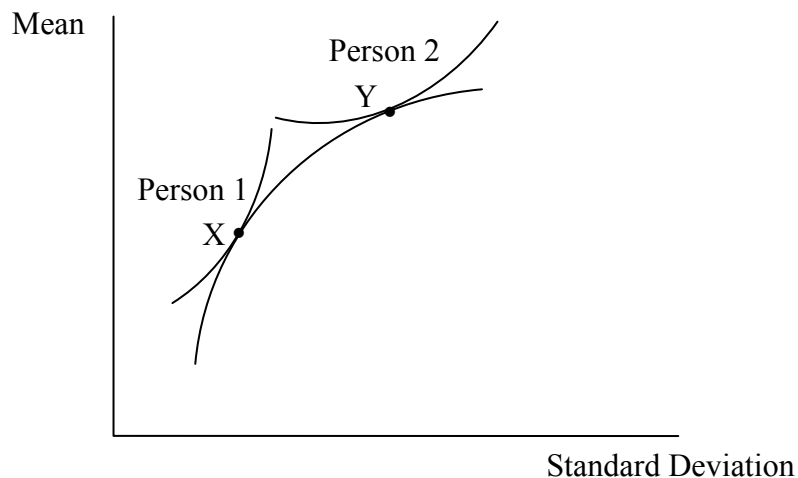
In this case we cannot say very much about the prices of stocks. In order to do so we would need to know everybody's preferences. This would be very difficult.

Now suppose that there is a risk-free asset, something like a T-bill, and this yields r_f with certainty. Can Person 1, for example, do better than X now that the risk-free asset is there?
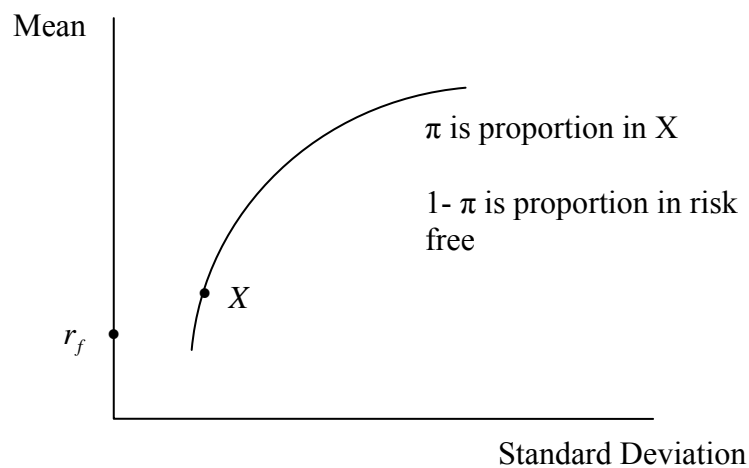
Suppose we hold a portfolio with a proportion π in the mutual fund X which has a mean

return of r_X and variance Var X and $1 - \pi$ invested in the risk-free asset which has a mean return

of r_f and a variance of 0. The covariance of the risk-free asset and X is also 0 since $r_f - Er_f = 0$.

What are the mean and standard deviation of this portfolio? Using the formulas for two-security

portfolios from Section 7(iv) with $\pi_1 = \pi$ and $\pi_2 = 1 - \pi$:

$$Er_{port} = \pi r_X + (1 - \pi)r_f$$

$$Var_{port} = \pi^2 Var\ X + (1 - \pi)^2 \times 0 + 2\pi(1 - \pi) \times 0 = \pi^2 Var\ X$$

$$SD_{port} = \pi SD_X$$

The expected return is just a weighted average of r_f and r_X as you'd expect. The formula

$SD = \pi SD_X$ simply says that if only one of the stocks in a portfolio varies the standard deviation

of the portfolio is the proportion of that stock in the portfolio times the stock's standard

deviation.

As we vary π what do we trace out? To see this suppose $r_f = 0.05$, $r_X = 0.15$, and $SD_X = $

2. Now consider the following.

$\pi = 0.25$ $\quad r_{port} = 0.075$ $\quad SD_{port} = 0.5$

$\pi = 0.50$ $\quad r_{port} = 0.100$ $\quad SD_{port} = 1.0$

$\pi = 0.75$ $\quad r_{port} = 0.125$ $\quad SD_{port} = 1.5$

Hence we trace out a straight line.

Mean

0.15 ------------------- X

0.125 -----------

0.1 --------

0.075 ----

0.05

0.5   1   1.5   2   Standard Deviation
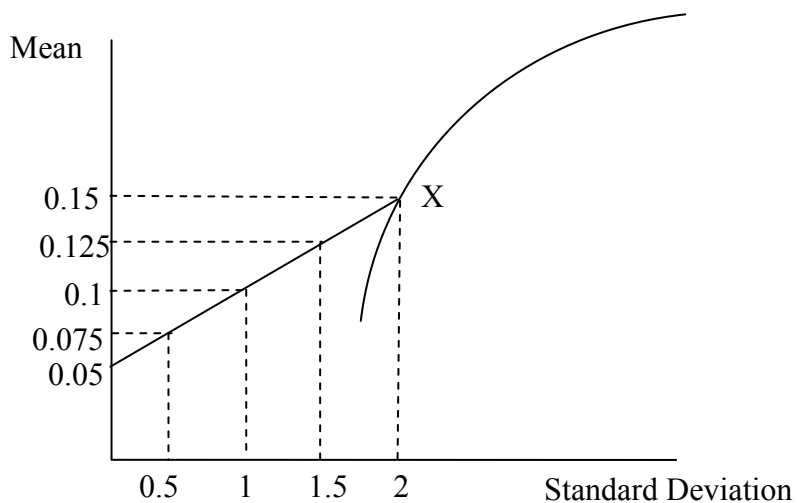
Is Person 1 better off investing in a combination of the risk-free asset and X? Yes, he can

reach a higher indifference curve if he buys a combination of the risk-free asset and X than if he

just invests in X.

Mean

X

Standard Deviation
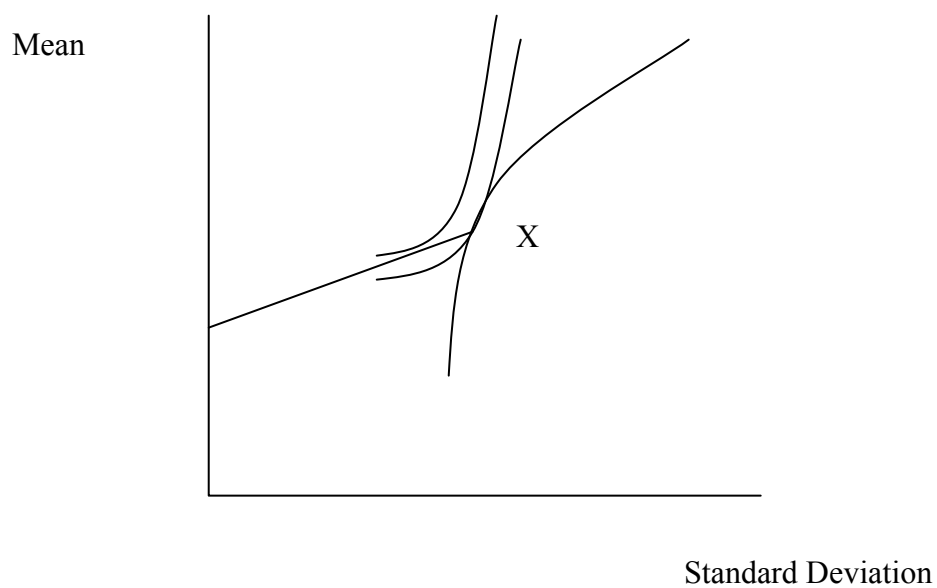
Similarly, if Person 1 invests in a combination of the risk-free asset and mutual fund Y he can

achieve the points on the straight line between r_f and Y. This allows him to do better than

holding a combination of X and the risk-free asset.

Mean

Y

X

Standard deviation
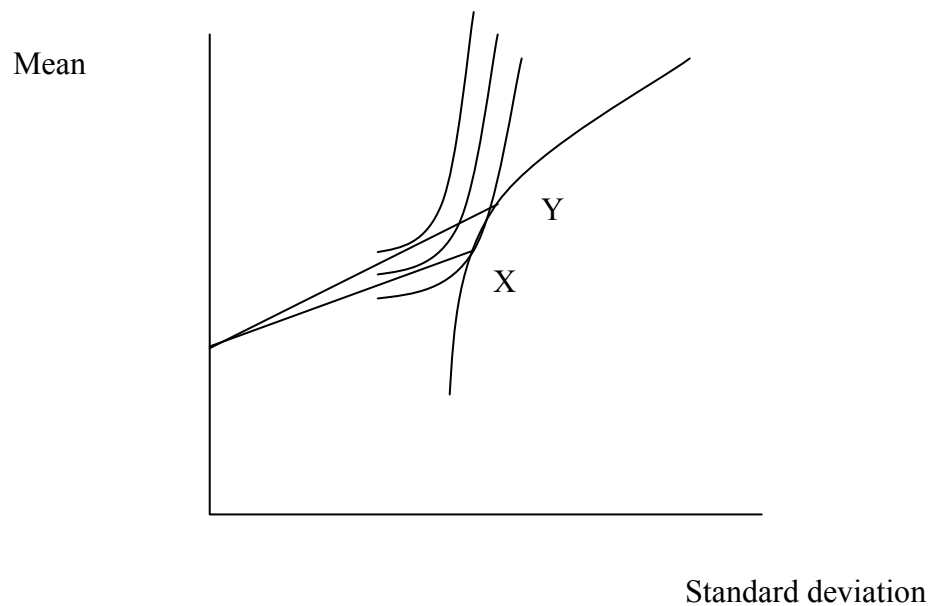
Can he do any better than this? Yes, he can go on pushing out the portfolio line until it is

tangent to the efficiency locus. Let us call this tangency point M. He will then choose a

combination of the risk-free asset and mutual fund M corresponding to the tangency of his

indifference curves with this line. This is the best Person 1 can do.
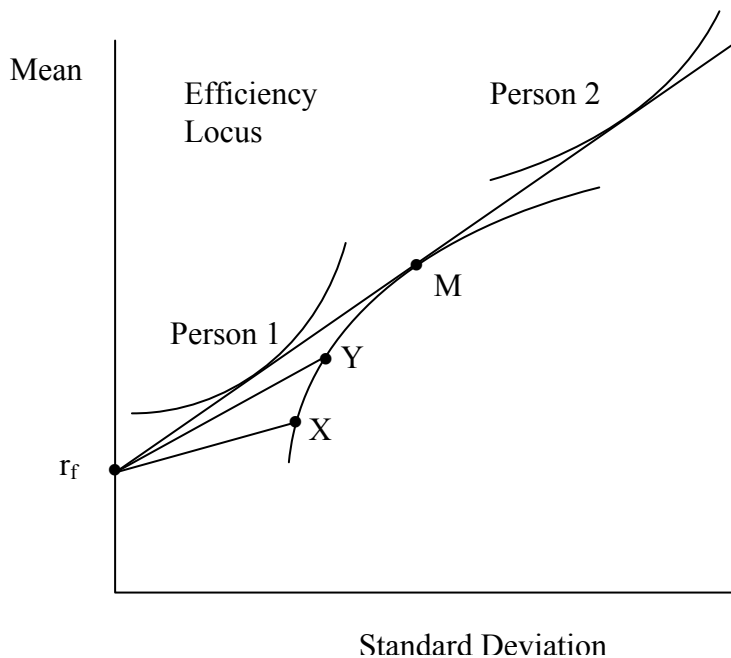
What about Person 2, what is the best she can do? Again she will want to push the line out as far as possible by choosing a combination of the risk-free asset and mutual fund M. Since the tangency of her indifference curves with the line is at a different point than person 1's, she will have a different proportion in the risk-free asset and mutual fund M than Person 1, but she will nevertheless be holding the same two assets.

Notice that people can hold portfolios on the line to the right of M. How? At M they invest all they have in the portfolio M, i.e., $\pi = 1$. To the left of M they put some in the risk-free asset and $\pi < 1$, i.e., they lend. To the right what do you think they do? They borrow to invest in portfolio M; i.e., they lever themselves and get a higher expected return in exchange for more risk, i.e. $\pi > 1$.

Mean

Borrowing

Lending

$\pi > 1$

$\pi < 1$

$\pi = 1$

M

Standard Deviation
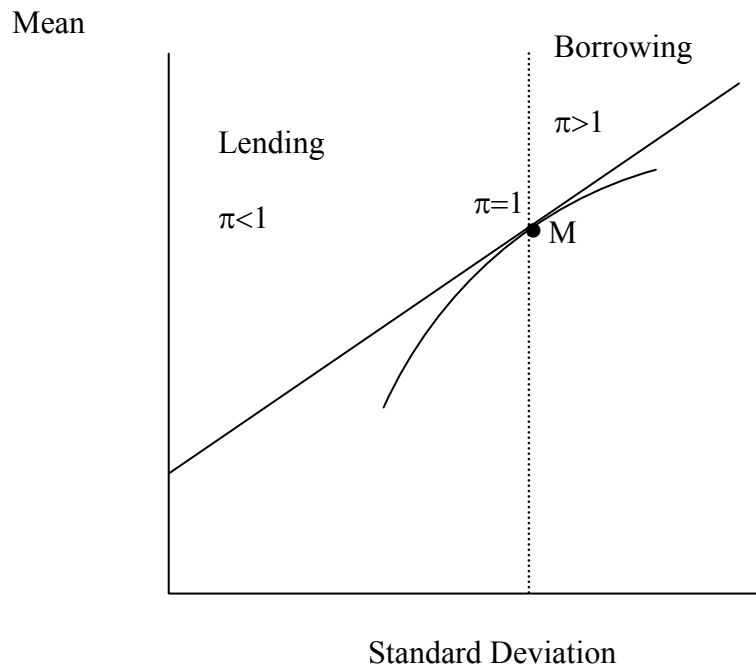
Everybody will therefore hold a portfolio consisting of some combination of the risk-free asset and the mutual fund M with the proportions depending on their preferences. Fairly risk averse people will put most of their money in the risk-free asset and only a small proportion in the mutual fund M. Less risk averse people will put more in the mutual fund M and less in the risk-free asset. People who are fairly risk neutral will borrow and put this money together with their own funds in M.

Why is risk so high when you borrow? Consider somebody who has $100 wealth. The person's portfolio for various values of π will be as follows.

π	Risk-free asset ($)	M ($)
0	100	0
0.5	50	50
1.0	0	100
1.5	-50	150
2.0	-100	200

As mentioned, with $\pi > 1$ the portfolio involves borrowing and investing more than your wealth in M. Let's compare $\pi = 1$ with $\pi = 2$ for the cases where M halves and when it doubles.

<u>Portfolio value when</u>

	<u>M doubles</u>	<u>M halves</u>
$\pi = 1$	$200	$50
$\pi = 2$	$400 - 100 = \$300$	$100 - 100 = \$0$

It can be seen that risk increases significantly when you borrow. With $\pi = 1$ there's only $200 - 50 = 150$ in variation. However, with $\pi = 2$ there's $300 - 0 = 300$ in variation. This is why risk goes up the more you borrow.

Given that everybody owns the same risky mutual fund M, containing all the stocks in the market in the same ratios, what do you know about M? In equilibrium, we can show M must be a replica of the total market; it must have the same proportions in each stock as the total market. When we introduced β, we talked about riskiness in relation to the market portfolio. It is the same market portfolio here.

Why must the mutual fund that everybody holds have the same proportions as the market portfolio?

To see this let M denote the portfolio that everybody holds and suppose that there are n units in total demanded by people. Then if markets are to clear demand = supply requires:

$$nM = \text{Market portfolio}$$

or
$$M = \frac{1}{n}\text{Market Portfolio}$$

In other words everybody holds a fraction of the market portfolio.

To summarize, all investors will hold some combination of the risk-free asset and the market portfolio M. Their preferences will determine the mix of these two assets in their portfolios. The more risk averse they are the more of the risk-free asset they will hold in their portfolios. The key point here is that everybody faces the same risk-return trade-off. Another important thing to notice is that we know something about the market portfolio; we can observe it so we know r_M and SD_M.

The line through r_f which is tangent to the efficiency locus is called the Capital Market Line (CML). Consider an arbitrary point (SD_p, r_p) on the capital market line that has a proportion π_p in M and $1 - \pi_p$ in the risk-free asset as shown below.
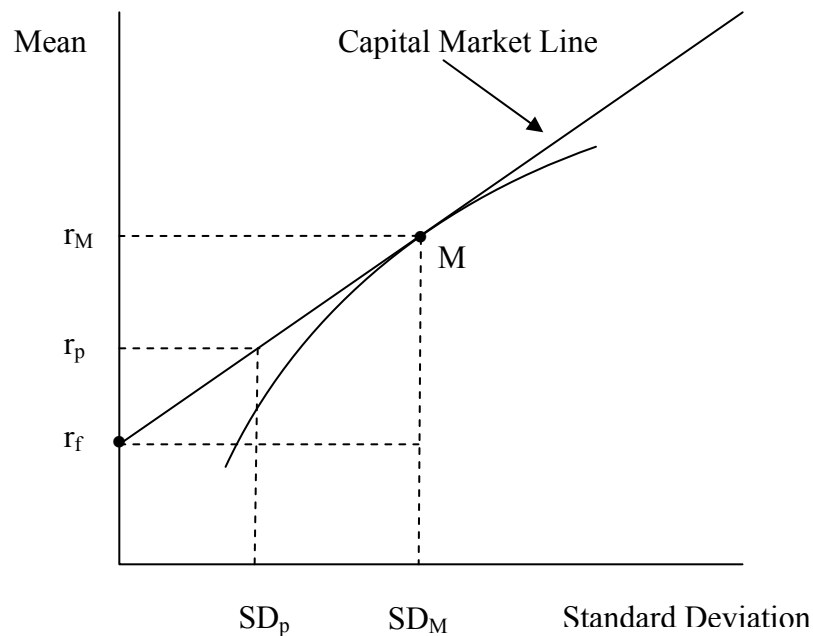
It can be seen using the standard formula that the expected return on this portfolio is

$$r_p = \pi_p r_M + (1-\pi_p)r_f$$

$$= r_f + \pi_p(r_M - r_f) \tag{1}$$

This can be changed to a more useful form by deriving the relationship between π_p and β_p. It was shown in the Appendix for Section 7(iv) last time that $\beta_{port} = \sum \pi_i \beta_i$. In this case the portfolio of interest consists of π_p in M and $1 - \pi_p$ in the risk-free asset. Now it follows from the definition of beta that $\beta_M = 1$. Also since $r_f - r_f = 0$ it follows that the covariance of the risk-free asset with the market is zero and hence $\beta_{risk\text{-}free} = 0$. Thus

$$\beta_p = \pi_p \times 1 + (1 - \pi_p) \times 0 = \pi_p$$

$$\tag{2}$$

Intuitively, the beta of a portfolio consisting of M that has a beta of 1 and the risk-free asset that has a beta of 0 is just the proportion invested in M. The more you put in the market the higher the beta.

Putting (1) and (2) together gives

$$r_p = r_f + \beta_p(r_M - r_f) \tag{3}$$

What this says is that the expected return on a portfolio along the capital market line with risk β_p is the risk free rate plus β_p times the risk premium on the market portfolio $(r_M - r_f)$. For example, if somebody puts half their money in the market portfolio, she has half of the risk of the market and half of the risk premium. We've scaled everything relative to the market so it all works out neatly.

One more step remains before we have the capital asset pricing model. We are going to argue that the risk premium on any asset or security with riskiness β will be given by $\beta (r_M - r_f)$.

The argument is essentially an <u>arbitrage</u> one. You would never hold an asset with riskiness β which had a risk premium lower than a portfolio on the capital market line with the same risk β. Similarly, you would always buy an asset with an expected return higher than that on a portfolio on the capital market line with the same β.

In equilibrium, we know that everybody must be holding a portfolio consisting of the risk free asset and M. This portfolio is well diversified (i.e. its correlation with the market is 1) so its standard deviation $SD_{port} = \beta_{port}SD_M$. Hence if we hold the beta of a portfolio constant its standard deviation stays the same.

For a situation to be an equilibrium, it must be the case that nobody can make themselves better off. Suppose we have a stock, IBM say, such that its return is r_{IBM} and its risk is β_{IBM}. Now suppose we choose a portfolio with $\beta_p^* = \beta_{IBM}$ on the capital market line (i.e. this portfolio has $\pi_p^* = \beta_{IBM}$) and the expected return on this portfolio is above the expected return on IBM.

(i) r_{IBM} β_{IBM}

                              $\wedge$ $\|$

                         r_p^* β_p^*

How could we improve our position given this situation? Clearly we would sell IBM and hold the portfolio instead. This would give us a portfolio with the same risk as before but a higher return. Everybody will want to sell IBM but what happens if everybody sells IBM? Its price falls and as a result its expected return rises. At what point will there cease to be an arbitrage opportunity? When $r_{IBM} = r_p^*$.

(ii) Similarly if we have the following situation

$$r_{IBM} \qquad \beta_{IBM}$$

$$\vee \qquad \qquad \|$$

$$r_p^* \qquad \beta_p^*$$

we would buy IBM and sell the portfolio corresponding to the point (r_p^*, β_p^*). But if everybody

does this the price of IBM rises and r_{IBM} falls until $r_{IBM} = r_p$. Hence in equilibrium

$$r_{IBM} = r_p^* \quad \beta_{IBM} = \beta_p^*$$

Given (3) is satisfied for r_p^* and β_p^*, it follows that it must be satisfied for r_{IBM} and β_{IBM}.

 Similarly, it must be satisfied for any other asset or portfolio. Hence we have:

$$r = r_f + \beta (r_M - r_f)$$

This relationship between risk and return is the <u>Capital Asset Pricing Model</u> and you

should memorize it now. It is extremely important that you know it. It provides us with a way

of pricing risky securities. It also provides us with a way of arriving at an opportunity cost of

capital for an asset which we can use to calculate NPV. If we know r_f and r_M and can calculate

the β for an asset, we can find the appropriate rate of return at which to discount using the

CAPM model.

To see intuitively what is going on here it is helpful to write the CAPM in the form:

$$r - r_f = \beta (r_M - r_f)$$

This says that the risk premium (i.e. the amount over and above the risk free rate is) is $\beta \times$ the

risk premium on the market. In other words, the expected premium on a stock depends on its

contribution to the risk of a portfolio which is its β. In fact it is just $\beta \times$ the risk premium on the

market.

How Good a Model is the CAPM?

It is important to stress that the CAPM is a theory about the real world; it is not necessarily a description of the real world. In order to evaluate the usefulness of the CAPM we must thus try and determine how much or how little the theory corresponds to the real world.

There are two ways in which we can go about this; the first is to consider the plausibility of the assumptions the model is based on and the second is to see if its predictions about the real world hold true.

We start by reviewing the explicit and implicit assumptions we have made when deriving the CAPM.

Assumptions of the CAPM

1. Mean-SD Analysis: We started out by assuming that individuals preferences can be represented in terms of the mean and standard deviation of returns.

2. Identical Beliefs: All investors have identical subjective estimates of the means, the variances and covariances of returns of all assets. We needed this to ensure that their view of the efficiency locus is the same. Suppose their views differed; then they would not all hold the same portfolio of risky assets, and so they would not hold the market portfolio.
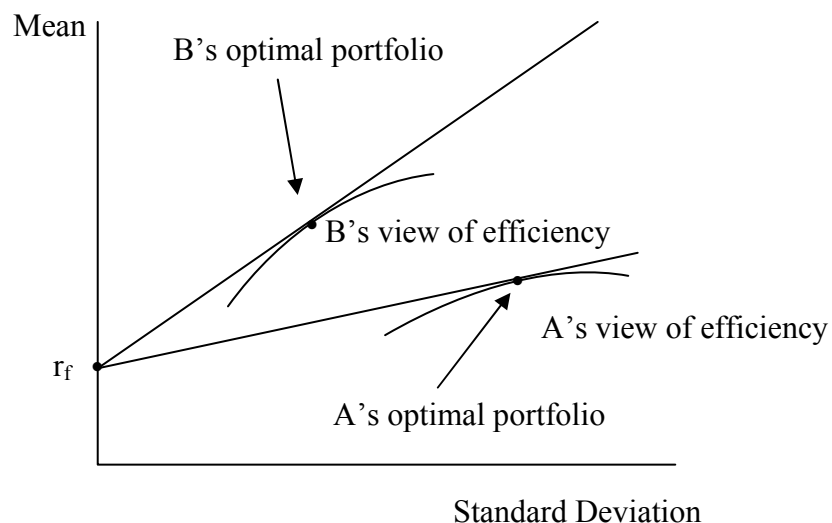
3. <u>Unlimited borrowing and lending:</u> All investors can borrow or lend an unlimited amount at an exogenously given risk-free rate of interest. This was also required to ensure that everybody held the same portfolio of risky assets so that we know that it is M which is of interest. Suppose, for example, some people can't borrow (i.e., in terms of the diagram below they can't be on the dotted line). Then we may have the following situation.
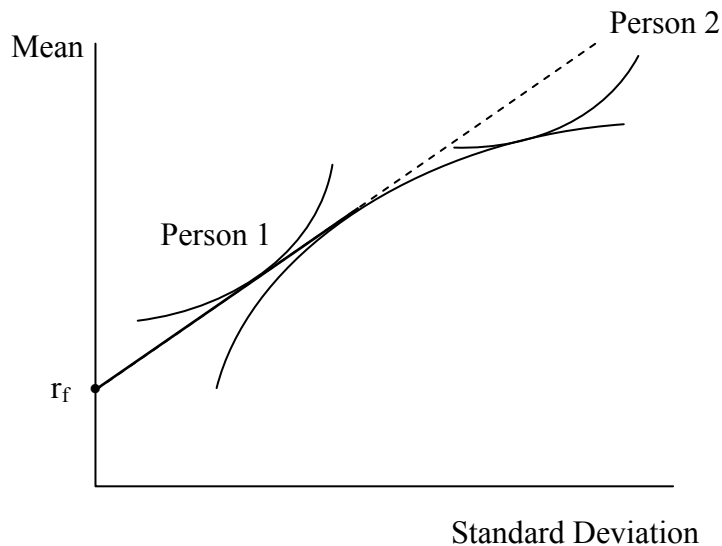
Depending on their preferences, people may hold different mutual funds.

4. <u>Perfect markets:</u> All assets are perfectly divisible, perfectly liquid so that they can be sold immediately at the market price, markets are perfectly competitive and there are no transactions costs. If these conditions didn't hold, then the opportunities faced by small and large investors would be different and we, again, wouldn't have everybody holding the same mutual fund M.

5. Same after-tax returns: Taxes must be such that after-tax returns ensure that everybody

    faces the same efficiency locus. If this isn't the case then we run into the same type of

    problems that we had in 2 when people had different subjective estimates of the means and

    variances.

    These assumptions are rather strong, which indicates that the model is not likely to be a

perfect description of the real world. But are the divergences from these assumptions such that

they make the model totally useless? To get a better feel for this we turn next to empirical

results to see how serious the strength of the assumptions is.

Empirical Tests of the CAPM

    When deriving the CAPM we argued that it must be the case that the expected return on

any asset, security or portfolio must be such that

$$r = r_f + \beta(r_M - r_f)$$

otherwise people wouldn't hold it or would hold more of it until its price changed so that this

condition was satisfied.

    If you plot r against β, this is called the Security Market Line (SML). What are the

intercept and slope? The intercept is r_f and the slope is $(r_M - r_f)$.
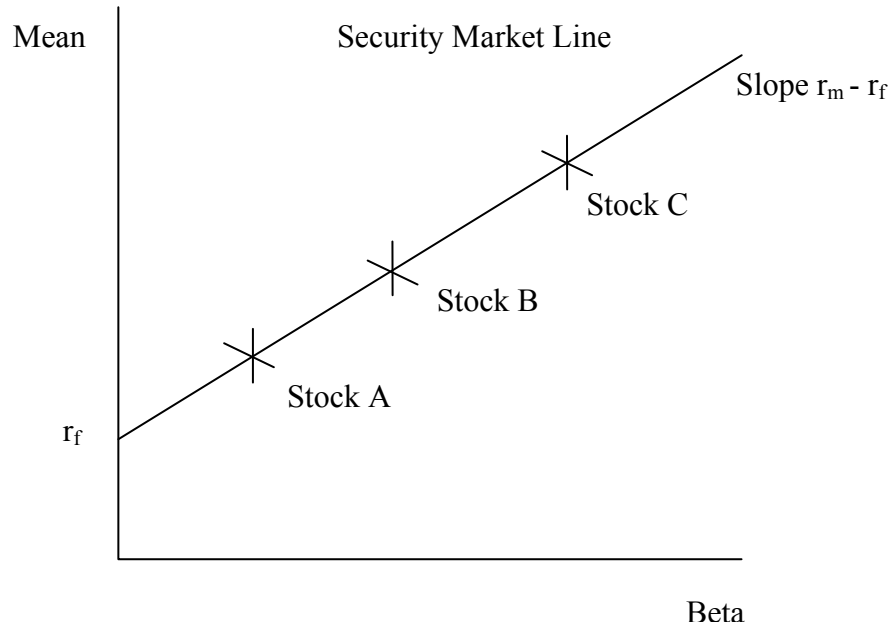
It is important not to confuse the capital market line and the security market line. The capital market line is the line between r_f and M and the axes are mean and standard deviation. It represents portfolios consisting of the risk-free asset and the market mutual fund M. In contrast, the security market line is the relationship between a stock's expected return and its beta. The axes are mean and beta.

When the CAPM holds, all stocks lie on the security market line as shown. In addition, all portfolios and other assets will lie on the security market line.

If the CAPM was a good representation of the real world, we would expect that if we measured the expected return of various assets and plotted them against their β's, we would get the SML.

$$r = r_f + \beta(r_M - r_f)$$

i.e., we would get a straight line with intercept r_f and slope $(r_M - r_f)$.

In a classic study, Fama and MacBeth did this for a number of large portfolios. They used large portfolios to try to minimize the idiosyncratic risk associated with individual securities. They did this for a number of periods and got the results shown on the diagram at the end of the section. Obviously the points do not fit perfectly, but on the other hand, they are not completely different from a straight line.

We won't go any further here. If you go on and do more finance, then you'll go into the CAPM and its validity in much more detail. For the moment a reasonable conclusion to draw from what we've done is that the CAPM is not ideal but is simple to use and better than nothing. There have been a number of other asset pricing theories such as Ross' Arbitrage Pricing Theory (APT) and the Fama-French three factor model. None of these have been sufficiently successful to supplant the CAPM.

Explaining the Motivation Examples

We initially started off with the puzzles:

1. Which is more risky, investing in gold prospecting or electric utilities? Answer: Electric utilities.

2. Would you ever invest in a security whose average return was less than that on T-bills? Answer: Yes.

The key to understanding these answers is the difference between unique risk and market risk. Unique risk can be <u>diversified</u> away by holding stocks in a portfolio but market risk cannot. The gold mining example has a lot of unique risk but no market risk. Since the unique risk can be diversified away it is not very risky. The electric utilities have market risk. This cannot be

diversified away so they are risky in a financial sense.

In the second example a stock could yield less than T-bills if it had <u>negative</u> market risk. If it moves in the opposite direction to other stocks then it can reduce the risk of a portfolio. This means it is very valuable and has a high price or equivalently a low return.

We showed the formal model that captures the notion of diversification and the difference between unique risk and market risk is the CAPM. For every asset

$$r = r_f + \beta(r_M - r_f)$$

What's important in determining a stock's return is its β because this is what contributes to the variance of the portfolio. Something with negative covariance with the market is valuable--it cuts down risk: when everything else is down, it's up and vice versa. This is good, so it's valuable, has a high price and therefore a low return.

<u>Understanding the historical data</u>

The CAPM also gives us some insight into the historical data that we looked at initially. If you think back to the chart on what your wealth would have been if you'd invested $1 in various instruments you will remember that small stocks had the highest returns, followed by large stocks, long-term government bonds and T-bills. As you might expect this follows the ordering of the stocks betas. Small firm stocks have a larger beta than those of large firms which in turn have a larger beta than long term government bonds and the zero beta of T-bills. Now although the ordering is correct the magnitudes of betas are such that they cannot fully explain the differences in returns. Small stocks have in fact earned even higher returns than their betas would suggest. Once again the empirical evidence is only partially consistent with the CAPM.

Constructing arbitrage portfolios

An important step in our derivation of the CAPM was the arbitrage argument. It is helpful in understanding portfolio theory to consider this notion further. Let us start by reviewing the simple case of a riskless arbitrage where all securities are risk free.

Example 1: Suppose that the borrowing and lending rate is 7%. You discover a risk free one-year bond which costs $1,000. At the end of the year it will pay a coupon of $80 and principal of $1,000. It therefore yields 8%. How can you make money? Is the price likely to stay at $1,000?

Clearly, an arbitrage opportunity exists here. The thing to do to make money is to borrow $1,000 at 7% and invest it in the bond that yields 8%. At the end of the year you will receive $1,080 and owe the bank $1,070 so you will make a profit of $10. Everybody will try to do this so the price will go up until the yield on the bond falls to 7% and there are no more money making opportunities.

$$P = \frac{1,080}{1.07} = \$1,009.35$$

Arbitrage opportunities such as this will be short-lived. It will not be necessary to wait the full year. Once the price rises to $1,009.35 (which will be almost immediately) you can repay your $1,000 loan and keep the $9.35 profit. Notice that this is the same as the $10 you would have received at the end of the year if you had held the position for the whole year since 10/1.07 = $9.35.

Example 2: Suppose there are two risk-free one-year bonds A and B that you discover. Bond A has a current price of $1,000. At the end of the year it will pay a coupon of $50 and principal of $1,000. It therefore yields 5%. Bond B also has a current price of $1,000. At the end of the year it will pay a coupon of $60 and principal of $1,000. It therefore yields 6%. How can you make money? Will this opportunity persist?

Similarly to Example 1, what you would like to do here is borrow at 5% and invest at 6%. Investing at 6% is no problem. All you have to do is buy bond B. How can you borrow at 5%? You can do something called go short in bond A. What this means is that instead of borrowing money you borrow bond A - you have a negative amount of it. At the end of the time you have agreed to borrow it for you have to give it back. If any interest becomes due on the bond during this time you must pay that interest to the person that lent it to you. In this example you go short in bond A. After you have borrowed it you can sell it at $1,000, take the money and invest in bond B. Investing in bond B is called going long in bond B.

At the end of the year you receive $60 interest from bond B. You must pay $50 of this to the person you borrowed bond A from. You can use the $1,000 principal repayment on bond B to pay back to buy back bond A for $1,000 and return it to the person you borrowed it from. You make a risk free profit of $60-$50 = $10. Clearly, the original prices will not persist.

Let's suppose that all the similar bonds are yielding 6% so it is A that is mispriced. The price of A will quickly adjust to

$$P_A = \frac{1,050}{1.06} = \$990.57$$

You can sell B for its $1,000 price, use $990.57 of it to buy back A and close out your short position (i.e. return A to the person you borrowed it from). You are left with a profit of 1,000-990.57 = $9.43. Again this is equivalent to the $10 profit you would have made if you had held the position for a year since 10/1.06 = $9.43.

One important thing to notice is that these arguments abstract from practical aspects of short selling such as transaction costs. These costs are explained in the Appendix. These costs pose limits to arbitrage. For large players at least these costs are usually argued to be small.

Example 3

Suppose bonds A and B are one-year bonds. Bond A yields 8.4% and bond B yields 7.1%. How can you make money?

Here you should go short in B and use the proceeds to buy bond A.

All the arbitrages so far have been in terms of riskless bonds. What happens if we are dealing with stocks that are risky and are in a CAPM world, for example? Suppose we are told that stock A has an expected return of 14% and another stock B has an expected return of 20%. Is there a risk free arbitrage opportunity here? The answer is that we can't tell without knowing the riskiness of the stocks or in other words their beta.

Example 4

	Expected return	Beta	Unique risk
Stock A	14%	0.8	None
Stock B	20%	2.0	None

$r_F = 10\%$ and $r_M = 15\%$.

Does an arbitrage opportunity exist?

To see whether an arbitrage opportunity exists here we need to look at whether the different betas account for the different returns. One way to do this is to look and see whether the returns satisfy the CAPM. According to the CAPM it should be the case that

$r_A = 0.10 + 0.8(0.15 - 0.10) = 0.14$

$r_B = 0.10 + 2.0(0.15 - 0.10) = 0.20$

These correspond to the actual expected returns and so both points lie on the security market line. The difference in expected returns is just due to the difference in risk.
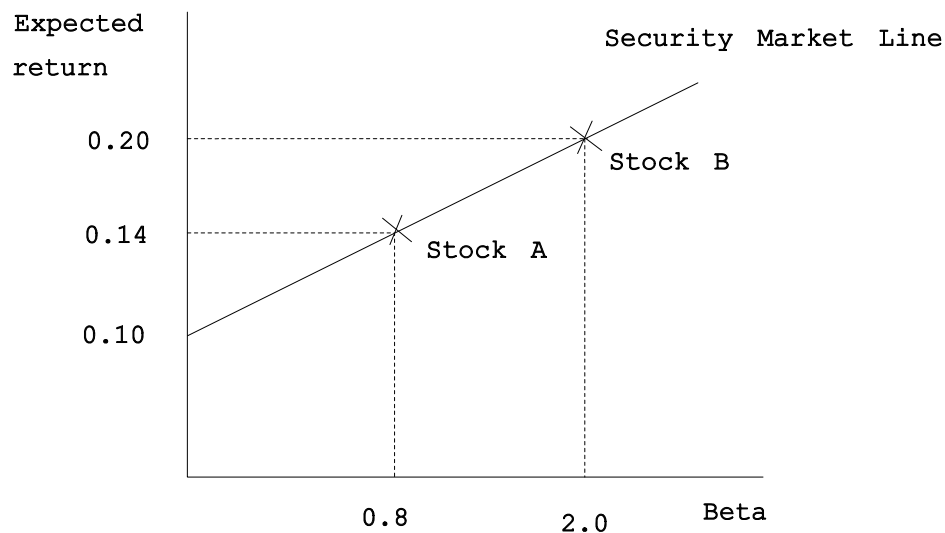
Suppose we changed Example 4 so that the stocks were no longer on the security market line.

Example 5

	Expected return	Beta	Unique risk
Stock A	15%	0.8	None
Stock B	18%	2.0	None

$r_F = 10\%$ and $r_M = 15\%$.

Does an arbitrage opportunity exist?

If we plot stocks A and B we can see that they no longer lie on the security market line.
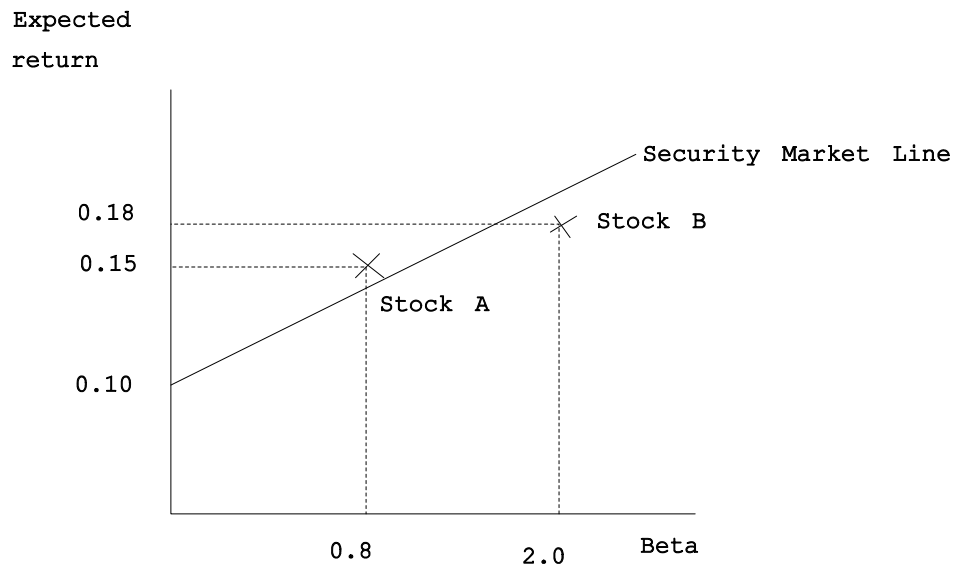
Stock A lies above the line and so has a relatively high yield for its risk. Stock B lies below the line and has a relatively low yield for its risk. We can therefore potentially construct a

portfolio that makes money without risk. Note that since the unique risk of both stocks is zero

we only have to worry about the market risk or beta. To have zero market risk we need to

construct a portfolio with zero beta. We know that

$$\beta_{portfolio} = \pi_A\beta_A + \pi_B\beta_B.$$

We need to short B because it has a relatively low yield. Suppose that we set $\pi_B = -\$1$. We are

shorting $1 worth of stock B - in other words we are borrowing it and must pay dividends to the

owner while we hold it and eventually return the share to him. Given $\pi_B = -\$1$, and the values

of π_A and π_B in the example, it follows that to have a zero risk portfolio

$$\beta_{portfolio} = \pi_A \times 0.8 + -1 \times 2.0 = 0$$

or $\qquad \pi_A = 2.0/0.8 = \$2.5$

By investing $2.5 in A and -$1 in B we have ensured it is a zero risk arbitrage. The net

investment is

$$\pi_A + \pi_B = 2.5 - 1 = \$1.5.$$

So we are effectively investing 1.5 in a zero beta, i.e. risk free, portfolio. If we were to invest in

the risk free asset we would earn a return of 0.1 x 1.5 = $0.15. In fact our portfolio earns a return

of

$$Er_{portfolio} = \pi_Ar_A + \pi_Br_B = 2.5 \times 0.15 - 1 \times 0.18 = \$0.195.$$

We have made an excess return of 0.195 - 0.15 = $0.045.

In Example 5 we have to invest something to make an above risk free return. In contrast

in Examples 1-3 we did not need to invest any of our own money - the amount we borrowed was

equal to the amount we invested. This is called a zero-investment portfolio. In order to have a

zero-risk, zero-investment portfolio in our current type of situation we need at least 3 securities.

Suppose we approach Example 5 from a slightly different direction.

Example 5'

	Expected return	Beta	Unique risk
Stock A	15%	0.8	None
Stock B	18%	2.0	None
Risk Free	10%	0.0	None

$r_M = 15\%$.

Does an arbitrage opportunity exist? Construct a zero-risk, zero-investment portfolio with a positive expected return.

We know that an arbitrage opportunity exists. To ensure our portfolio is zero-risk,

$$\beta_{portfolio} = \pi_A\beta_A + \pi_B\beta_B + \pi_{RF}\beta_{RF} = 0.$$

Since $\beta_{RF} = 0$, we know that if $\pi_B = -1$ then substituting in gives $\pi_A = 2.5$ as before. To ensure zero investment we need

$$\pi_A + \pi_B + \pi_{RF} = 0.$$

Substituting in

$$2.5 - 1 + \pi_{RF} = 0$$

or equivalently

$$\pi_{RF} = -1.5.$$

We know that the expected return is

$$Er_{portfolio} = \pi_A r_A + \pi_B r_B + \pi_{RF} r_F = 2.5 \times 0.15 - 1 \times 0.18 - 1.5 \times 0.1 = \$0.045.$$

Hence

$$\pi_A = \$2.5; \; \pi_B = -\$1 \text{ and } \pi_{RF} = -\$1.5$$

is a zero-risk, zero-investment portfolio with a positive expected return. Clearly we would not expect such an opportunity to persist for very long.

The examples above show how arbitrage portfolios can be constructed in simple situations. In fact whenever 3 securities don't lie on the security market line an arbitrage opportunity exists; it is not necessary for one of them to be the risk free asset. The examples above have the special feature that unique risk is zero. When this is not true it is necessary to work in terms of portfolios where unique risk is diversified away. The stocks that lie off the security market line must be embedded in diversified portfolios.

Understanding arbitrage is important because much of what investment banks do is to construct such trading strategies. It is this activity that ensures prices are all the time being driven into line with present values. The examples above are very simple. When we come to the section on options you will see other more complicated arbitrage strategies. The whole valuation of derivative securities of which options are one example is based on arbitrage ideas.

One thing to note is that we have ignored many of the institutional details of short selling such as collateral. For these see the Appendix.

Concluding remarks

The important lesson of this section is the notion of diversification. You should spread your investments over a number of alternatives. The CAPM is a formalization of this idea. It doesn't work that well in terms of providing exact numbers but is a useful starting point particularly for exercises such as capital budgeting where a rough and ready number is needed.

Appendix

Fama and Macbeth (1973)

Empirical Test of the CAPM
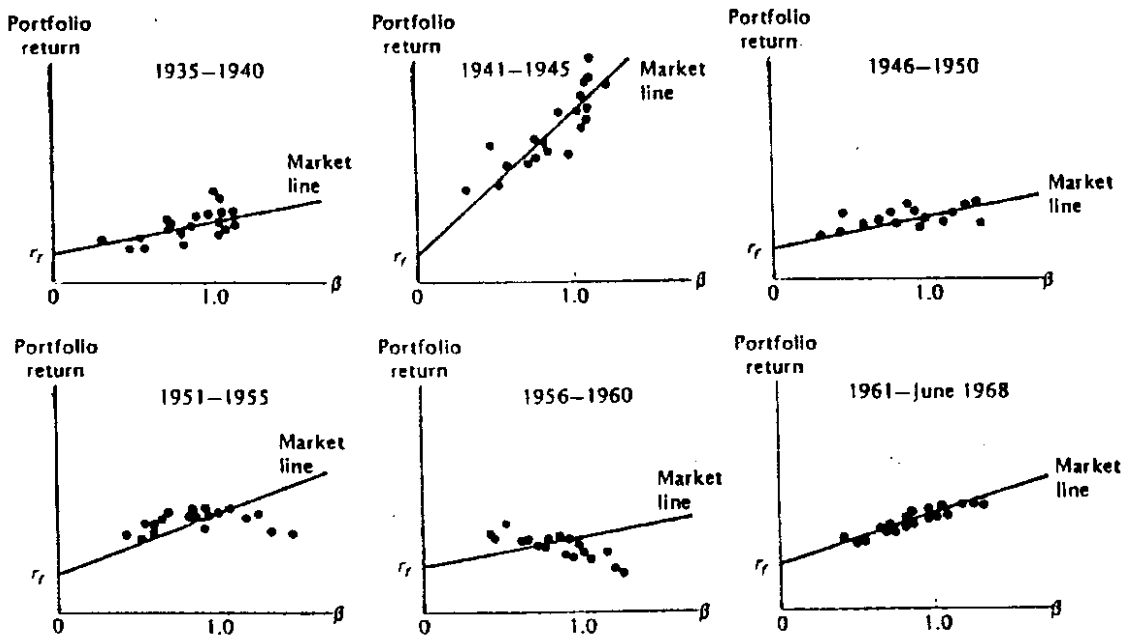

FIGURE 8-13
The capital asset pricing model states that the *expected* return from any investment should lie on the market line. The dots show the *actual* returns from portfolios with different betas. [*Source:* Results supplied by E. F. Fama and J. D. MacBeth. See their paper. "Risk. Return and Equilibrium: Empirical Tests." *Journal of Political Economy.* 81:607–636 (May 1973}.]

Institutional Details of Short Selling

(From Bodie, Kane and Marcus: *Investments*)

3.7 SHORT SALES

Normally, an investor would first buy a stock and later sell it. With a short sale, the order is reversed. First, you sell and then you buy the shares. In both cases, you begin and end with no shares.

A **short sale** allows investors to profit from a decline in a security's price. Instead of buying, an investor borrows a share of stock from a broker and sells it. Later, the short seller must purchase a share of the same stock in the market in order to replace the share that was borrowed. This is called *covering the short position*. Table 3.8 compares stock purchases to short sales.

The short seller anticipates the stock price will fall, and the share can be purchased at a lower price than it initially sold for; therefore, the short seller reaps a profit. Short sellers must not only replace the shares but also pay the lender of the security any dividends paid during the short sale.

In practice, the shares loaned out for a short sale are typically provided by the short seller's brokerage firm, which holds a wide variety of securities of its other investors in street name. The owner of the shares will not even know that the shares have been lent to the short seller. If the owner wishes to sell the shares, the brokerage firm will simply borrow shares from another investor. Therefore, the short sale may have an indefinite term. However, if the brokerage firm cannot locate new shares to replace the ones sold, the short seller will need to repay the loan immediately by purchasing shares in the market and turning them over to the brokerage house to close out the loan.

Finally, exchange rules require that proceeds from a short sale must be kept on account with the broker. The short seller, therefore, cannot invest these funds to generate income. In addition, short sellers are required to post margin (which is essentially

Table 3.8 Cash Flows from Purchasing versus Short Selling Shares of Stock

Purchase of Stock		
Time	Action	Cash Flow
0	Buy share	− Initial price
1	Receive dividend, sell share	Ending price + dividend

Profit = (Ending price + dividend) − Initial price

Note: A negative cash flow implies a cash outflow.

Short Sale of Stock		
Time	Action	Cash Flow
0	Borrow share; sell it.	+ Initial price
1	Repay dividend and buy share to replace the share originally borrowed	− (Ending price − Dividend)

Profit = Initial price − (Ending price − dividend)

collateral) with the broker to ensure that the trader can cover any losses sustained should the stock price rise during the period of the short sale.[8]

To illustrate the actual mechanics of short selling, suppose that you are bearish (pessimistic) on IBM stock, and that its current market price is $100 per share. You tell your broker to sell short 1,000 shares. The broker borrows 1,000 shares either from another customer's account or from another broker.

The $100,000 cash proceeds from the short sale are credited to your account. Suppose the broker has a 50% margin requirement on short sales. This means that you must have other cash or securities in your account worth at least $50,000 that can serve as margin (that is, collateral) on the short sale. Let us suppose that you have $50,000 in Treasury bills. Your account with the broker after the short sale will then be:

Assets		Liabilities and Owner's Equity	
Cash	$100,000	Short position in IBM stock	$100,000
T-bills	$ 50,000	(1,000 shares owed)	
		Equity	$ 50,000

Your initial percentage margin is the ratio of the equity in the account, $50,000, to the current value of the shares you have borrowed and eventually must return, $100,000:

$$\text{Percentage margin} = \frac{\text{Equity}}{\text{Value of stock owed}} = \frac{\$50,000}{\$100,000} = .50$$

---

[8] We should note that although we have been describing a short sale of a stock, bonds also may be sold short.

Suppose you are right, and IBM stock falls to $70 per share. You can now close out your position at a profit. To cover the short sale, you buy 1,000 shares to replace the ones you borrowed. Because the shares now sell for $70, the purchase costs only $70,000. Because your account was credited for $100,000 when the shares were borrowed and sold, your profit is $30,000: The profit equals the decline in the share price times the number of shares sold short. On the other hand, if the price of IBM stock goes up while you are short, you may get a margin call from your broker.

Let us suppose that the broker has a maintenance margin of 30% on short sales. This means that the equity in your account must be at least 30% of the value of your short position at all times. How far can the price of IBM stock go up before you get a margin call?

Let P be the price of IBM stock. Then the value of the shares you must return is $1,000P$, and the equity in your account is $\$150,000 - 1,000P$. Your short position margin ratio is therefore $(\$150,000 - 1,000P)/1,000P$. The critical value of P is thus

$$\frac{\text{Equity}}{\text{Value of shares owed}} = \frac{\$150,000 - 1,000P}{1,000P} = .3$$

which implies that $P = \$115.38$ per share. If IBM stock should rise above $115.38 per share, you will get a margin call, and you will either have to put up additional cash or cover your short position.

UNIVERSITY OF PENNSYLVANIA

THE WHARTON SCHOOL

FNCE 601

CORPORATE FINANCE

LECTURE NOTES

Franklin Allen

Fall 2011

QUARTER 1 - WEEK 7

Tu: 10/18/11

Section 9: Capital Budgeting and Risk

Read Chapter 9 BMA

Motivation Example 1

Macrosoft is a computer software company. It has no debt. The beta of its equity, which is equal to the beta of its assets since it has no debt, is 1.25. The risk free rate is $r_F = 5\%$ and the market risk premium is $r_M - r_F = 8\%$. Using the CAPM, the opportunity cost of capital for the firm is $5\% + 1.25 \times 8\% = 15\%$ and this is the discount rate they use for evaluating computer software projects. Macrosoft is considering a hardware project. It is considering building PC's to compete with Dell, Gateway and other PC manufacturers. Suppose the beta of the assets of these companies that engage in activities similar to this hardware project is 1.83. This corresponds to an opportunity cost of capital of $5\% + 1.83 \times 8\% = 20\%$.

Maria Fuentes is the CFO of Macrosoft. She believes that Macrosoft should use 15% as the discount rate. Her argument is that Macrosoft can raise funds from their shareholders at 15% and so this is the relevant rate. Ari Silverberg is her deputy. He argues that Macrosoft should use the rate of 20% since this compensates for the risk that Macrosoft is taking by investing in the hardware industry.

Who is correct? In evaluating the hardware project should Macrosoft use a discount rate of 15% or 20%?

Finding the discount rate for a project like the firm

Suppose that we are considering a "pure play" firm like Dell that only invests in one type of project that is the same as its existing assets. All Dell does is produce and distribute PC's. The discount rate we should use for a project is the opportunity cost of capital (or as it is often called the cost of capital). In other words, if the firm didn't do the project the shareholders could

invest in the best alternative available with the same risk and we should use the return on this as the discount rate. In Section 2 there was certainty and we simply took the best available alternative. In our initial example this was the rate at the bank.

What is the best alternative available when there is uncertainty? This depends on the risk of the project. We should choose the best alternative so that its risk is the same as that of the project. The CAPM provides a way of doing this. It provides the best available alternative in the stock market for a similar risk project. If the CAPM holds, then the opportunity cost of investing in a project with a risk of β is:

$$r = r_F + (\text{Project } \beta)(r_M - r_F)$$

This is then the discount rate for the project. If the firm doesn't do the project then it can always invest in a portfolio with the same β as the project and get a return of r.

There are a number of studies that give some idea of the market risk premium $r_M - r_F$. Ibbotson Associates publishes a book each year that gives the up-to-date figures for the US. Dimson, Marsh, and Staunton (2006) (the Reading for Section 7) also gave figures for the 17 different countries they consider based on data from 1900-2005. These are given in the table at the end of the section. The first column shows the arithmetic mean return on equities relative to bills (i.e. the short maturity risk free rate) and the second column gives it relative to bonds (i.e. the long maturity risk free rate). It can be seen that there is considerable variation in both columns. Relative to bills Italy has the highest risk premium with 10.46% while Denmark has the lowest with 4.51%. Relative to bonds Japan has the highest with 9.98% and Denmark has the lowest with 3.27%. There is considerable debate as to what figure to use, the one relative to bills or the one relative to bonds. The main thing is to be consistent. If you use the figure relative to bills, then make sure that the r_F that you use is a short-term rate (usually a T-bill rate).

If the figure you use is the one relative to bonds make sure the r_F you use is a long-term rate. People typically use the figure relative to bills and the short term rate.

For the purposes of this section we will suppose $r_M - r_F = 0.08$ and $r_F = 0.05$.

The problem is in finding the project β. For a firm with no debt this is straightforward. With an all-equity capital structure the earnings from the assets simply flow through to the equity holders. The riskiness of the assets must be equal to the riskiness of the equity. Hence for an all-equity firm we can simply get the β of the equity and this will be equal to the β of the assets or in other words the project β.

How do we measure the β of a firm's equity? We use regression techniques. We know from regression theory that if we plot the returns on a stock Y against the returns on the market and we fit a straight line by the method of least squares, its slope is given by

$$b = \frac{\text{Cov}(Y, M)}{\sigma_M^2} = \beta_{\text{Firm's Equity}}$$

Now the statistical problems in measuring betas are often significant. The easiest way to find a beta is probably to use Bloomberg. So for most companies we can find the beta of the company's stock. In the cases where we can't find a precalculated figure we must go back to the data, fit a regression line for the firm and find our β that way.

One important thing is that regressions are based on historical data. This means that the β is a measure of risk in the past. However, in analyzing a project we are looking forward so if we use the β calculated from historic data we are assuming the future will be like the past. This is a reasonable assumption in many cases but in others it is not. If you think the future riskiness of a project will be different from the past riskiness you will need to take account of this.

In our Dell example, the beta of its assets is 1.83. Hence the opportunity cost of capital for Dell is given by

$$r_{Dell} = 5\% + 1.83 \times 8\% = 20\%.$$

Dell should use a 20% when calculating whether or not a project has a positive NPV and is worth undertaking.

Example 1

Suppose Dell has a project with a cost at date 0 of $1 million. It foresees three scenarios. The first is the optimistic scenario and it occurs with probability 0.3. If the product is a big success the cash flow will be $1.6M at date 1. The second is the moderate scenario and it occurs with probability 0.5. If the project is a moderate success the cash flow will be $1.2M at date 1. The third is the pessimistic scenario and it occurs with probability 0.2. If the project is a failure the cash flow will be $0.8M at date 1.

Solution

The first step is to calculate the expected cash flow at date 1.

$$EC_1 = 0.3 \times 1.6M + 0.5 \times 1.2M + 0.2 \times 0.8M = \$1.24M$$

The next step is to calculate NPV in the usual way.

$$NPV = -1M + \frac{1.24M}{1.2} = \$0.033M$$

Suppose instead of $1.24M the project had an expected cash flow of $1.17M at date 1. Clearly it would have a negative NPV or equivalently a return of only 17%. It would not be worth undertaking because Dell would not be compensated for the project's risk. Dell (or equivalently its stockholders) would be better off to go into the stock market and buy securities with a beta of 1.83 and have an expected return of 20%.

If a firm has debt then finding the β of the assets is more complicated. When there is debt the beta of a firm's equity is different from the beta of its assets. We will consider this at length when we come to consider capital structure in Section 13 later in the course. For the moment we will focus on firms that have all equity capital structures so that the beta of the firm's equity and the beta of its assets are the same.

Finding the discount rate for a project different from the firm

We are now in a position to answer the question raised in Motivation Example 1. Suppose that a firm is investing in a project that is different from its current activities such as Macrosoft investing in hardware rather than software. What is the discount rate, or equivalently the opportunity cost of capital, for Macrosoft? The key point is that when they are investing in assets with a beta of 1.83 they must be compensated for that risk. This means that for hardware projects they should use a discount rate which is the same as Dell's, i.e. 20%. If they used the rate for software projects of 15% they might accept projects which did not compensate them for the risk they were taking. Macrosoft should use a 20% discount rate in evaluating hardware projects and a 15% discount rate in evaluating software projects.

Discount rates for Multi-division Firms

How should a multi-division firm calculate its opportunity cost of capital? The principle is again the same. The rate should reflect the risk being taken by the division. The firm should not use a rate reflecting the average for the company. Let's consider an example.

Example 2

Corporal Electric has two divisions. The first division manufactures consumer goods. The β of firms that only produce consumer goods is 1.0. The second division produces gas turbines. The β of firms that do this alone is 1.5. The two divisions each constitute 50% of the firm. The firm has no debt.

The beta of the equity is the same as the beta of the assets since there is no debt and can be found by thinking of the firm as a portfolio of two divisions.

$$\beta_E = \beta_A = 0.5 \times 1 + 0.5 \times 1.5 = 1.25$$

Assuming $r_F = 0.05$ and $r_M - r_F = 0.08$ and the CAPM holds it follows that

$$r_E = 0.05 + 1.25 \times 0.08 = 0.15.$$

What rate should the firm use in evaluating projects?

It is helpful to consider the security market line and see how projects lie relative to this. The expected return on securities with β's of 1 and 1.5 are $0.05 + 1 \times 0.08 = 0.13$ and $0.05 + 1.5 \times 0.08 = 0.17$. Suppose the firm had a project called Project A in the consumer goods division that had an IRR of 14%. (Assume there are no problems with using IRR). Should it do the project?
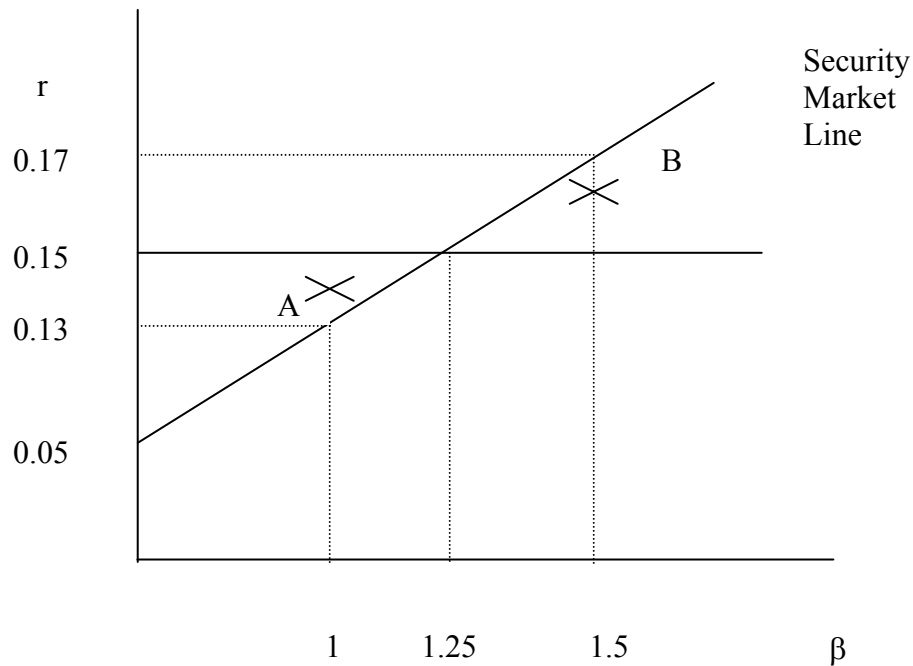
If the firm uses the company's cost of capital of 15% it will reject the project. However, if it uses the division's cost of capital it will accept the project. In fact it should use the division's cost of capital and accept the project. The capital market is only giving a return of 13% on projects with the same riskiness so doing it creates wealth for shareholders.

Suppose next that the gas turbine division has Project B with an IRR of 16%. Should the firm do it? If it used the company's cost of capital of 15% it will accept but if it uses the division's cost of capital it will reject. Again it should use the division's rate of 17% and reject the project. The shareholders would be better off using the money to invest in securities with β of 1.5 and earning 17% than investing at 16%.

If Corporal Electric used a company-wide cost of capital it would make the wrong decision. It would reject A and accept B. If divisions are significantly different they need to have different discount rates.

In multi-division firms the key thing is to always evaluate a project using the discount rate reflecting the risk of the assets that are being invested in. Finding the β for such assets involves identifying a "pure-play" firm. This is a firm that engages only in a single industry. The industry of the pure play firm and the project should obviously be the same.

Strategy and Finance

Read Chapter 11 BMA

How is it that projects like Project A can have a positive NPV and create wealth for shareholders? We argue that in capital markets all stocks must lie on the security market line. Why is it that a project like A does not have to lie on the security market line? In perfect capital markets competition among arbitrageurs ensures that all stocks lie on the security market line. If a project is to have a positive NPV it must be due to imperfections in product or input markets. When a firm has a monopoly over the product market it operates in then it will be able to have positive NPV projects. Similarly, if the firm is an oligopolist it will also have positive NPV projects available. In contrast, if a firm is in a perfectly competitive industry it will not have any positive NPV projects. Any positive NPV projects will be enthusiastically adopted by a number of firms and prices will be bid down. For example, with the internet firms in the 1990's it was quite unlikely the values in the stock market were sustainable because there were few barriers to entry so any positive NPVs would be competed away.

Intimately related to the question of identifying positive NPV projects, is the issue of strategy and finance. In order to have many profitable projects the firm must pursue an overall strategy that allows it to take advantage of barriers to entry, core competencies and so forth. Strategic management is concerned with how to find profitable areas.

Good strategy and the existence of positive NPV projects also leads to the notion of competitive response. One of the most important things in doing good project evaluation is anticipating competitors' reactions correctly. It is crucial to take account of such responses when forecasting cash flows. The existence of profitable opportunities will encourage entry and this will lead to the dissipation of profits.

Example 3

US Chemical is considering whether to build a plant to produce a specialty chemical, polyzone. It is currently in short supply in Europe and as a result its price is high. The plant would be built in the US and the product would be shipped to Europe.

US Chemical has come up with the following analysis of the project assuming the price of polyzone stays at a constant $1.20 per kilo above the cost of raw materials required to make it (amounts are in millions of $ except where otherwise stated)

Table 1

Date	0	1	2	3-10
Investment	100			
Production, millions kilos	0	0	40	80
Spread, $ per kilo	1.20	1.20	1.20	1.20
Net revenues	0	0	48	96
Production costs	0	0	30	30
Transport	0	0	4	8
Other costs	0	20	20	20
Net cash flow	-100	-20	-6	38

NPV at 8% = $63.56 million

So the plant looks like a good investment. But does it make sense to export the polyzone to Europe? Wouldn't European producers have an advantage given they don't have to transport it across the Atlantic? Couldn't they expand production and drive down prices?

Key issue: Think through why positive NPV projects exist and if these positive NPVs will be competed away.

US Chemical decides to do an analysis of the *competitive spread* (in other words, the spread at which NPV is zero for a European producer).

Table 2

Date	0	1	2	3-10
Investment	100			
Production, millions kilos	0	0	40	80
Spread, $ per kilo	0.95	0.95	0.95	0.95
Net revenues	0	0	38	76
Production costs	0	0	30	30
Other costs	0	20	20	20
Net cash flow	-100	-20	-12	26

NPV at 8% = $0

How long before European producers make the necessary expansion and drive down the spread? Suppose the management of US Chemical estimates it will be 5 years. Now we can redo the analysis for US Chemical's polyzone plant.

Table 3

Date	0	1	2	3	4	5-10
Investment	100					
Production, millions kilos	0	0	40	80	80	80
Spread, $ per kilo	1.20	1.20	1.20	1.20	1.10	0.95
Net revenues	0	0	48	96	88	76
Production costs	0	0	30	30	30	30
Transport	0	0	4	8	8	8
Other costs	0	20	20	20	20	20
Net cash flow	-100	-20	-6	38	30	18

NPV at 8% = -$9.8 million

So thinking through the effects of future competition leads to the conclusion that they shouldn't build the plant

The Effects of Changes in Technology

One of the most important factors affecting future prices is technology change. We can use similar techniques to those above to investigate the effects of the discovery of new technologies. Suppose we stay with the previous example but suppose now that the shortage of polyzone is in the US market not the European market. Thus there are no transportation costs. To keep the calculations similar let's suppose the production costs are higher.

Table 4

Date	0	1	2	3-10
Investment	100			
Production, millions kilos	0	0	40	80
Spread, $ per kilo	1.20	1.20	1.20	1.20
Net revenues	0	0	48	96
Production costs	0	0	34	38
Other costs	0	20	20	20
Net cash flow	-100	-20	-6	38

NPV at 8% = $63.56 million

This analysis assumes the technology stays the same but let's suppose that it is expected that a new technology will become available in 5 years time. This has lower costs than before of 30 and as a result spreads will be lowered to 0.95 as in Table 2. Similarly to Table 3 we can work out that now building the plant is a negative NPV investment.

This kind of analysis can lead to many interesting results. Some of the issues that can be addressed are:

- When is the best time to introduce a new technology?

- What is the effect of introducing new technologies on the value of existing plants?

- What happens if you don't introduce a new technology because of the effect on the existing value of your plants? Will one of your competitors do it instead?

The interactions between strategy and finance are key to making good investment decisions. See Chapter 12 in the book for a good analysis of these issues.

Risk Assumptions Implicit in Using CAPM to Calculate Multi-Period Discount Rates

Motivation Example 2

Ford is thinking of building a new automobile plant. The cost of this plant is $100M. The expected cash flow the firm anticipates the plant will generate in each of the three years it will be in operation is $50M. How should the firm choose its discount rate for this plant?

Time	0	1	2	3
Cash Flow	-100M	+50M	+50M	+50M

The firm is confident that economic conditions next year will not be very different from the current year so that the risk associated with the first-year estimate is fairly minimal. However, in the second year economic conditions are somewhat more uncertain so the risk associated with the second-year cash flow estimate is greater than the risk associated with the first-year estimate. Similarly, in three years time economic conditions could have changed quite significantly so that the risk associated with the cash flow estimate at this stage is even higher. The project manager argues that because the risk associated with the estimates of the project's cash flows increases through time the discount rate the firm uses should be higher the farther away the cash flow is.

Is the project manager correct?

The motivation example raises the issue of how we should deal with risk that is changing over time. One of the assumptions we made with the CAPM, was that it was based on choosing investments over one period. Many of the projects we will be interested in will obviously be for

more than one period. What are the consequences of using the CAPM to calculate discount rates for multi-period projects?

First consider the one period case:

$$PV = \frac{C_1}{1+r}$$

where
$$r = r_F + \beta(r_M - r_F)$$

- What does the r_F term adjust for? It adjusts for time.

- What does the β (r_M - r_F) term adjust for? It adjusts for risk.

Consider next the PV of a second period cash flow:

$$PV = \frac{C_2}{(1+r)^2} = \frac{1}{(1+r)} \frac{C_2}{(1+r)}$$

where r is the same as before.

- How many times are we adjusting for time? We are adjusting for time twice.

- What about risk, how many times are we adjusting for risk? We are using the risk adjustment factor once in discounting from period 2 to 1, and again in discounting from period 1 to period 0. We are adjusting for risk twice. We are therefore implicitly assuming that because the second period cash flow is two periods away, it is more risky.

In order to see this more clearly, it is helpful to develop another approach to the problem, namely, certainty equivalents.

Certainty Equivalents

When considering risky projects, we have so far been evaluating their NPV by taking the expected cash flows and by making the discount rate depend on the degree of risk. An alternative method is to adjust the cash flows to their certainty equivalents and discount at the risk free rate. This is the <u>certainty equivalent</u> approach.

The question we ask to get the certainty equivalent is "What is the smallest certain cash flow for which you would exchange the risky cash flow C_1?" This is called the certainty equivalent and is denoted CEQ_1. To get the present value you discount this at the risk free rate.

$$PV = \frac{CEQ_1}{1 + r_F}$$

Similarly for cash flows in other periods. Now we write

$$CEQ_1 = a_1\,C_1$$

where a_1 is the certainty equivalent adjustment factor.

When the CAPM holds, it must be the case that

$$\frac{CEQ_1}{1 + r_F} = \frac{C_1}{1 + r}$$

Substituting $CEQ_1 = a_1\,C_1$ gives

$$\frac{a_1 C_1}{1 + r_F} = \frac{C_1}{1 + r}$$

so
$$a_1 = \frac{1 + r_F}{1 + r}$$

Similarly, for cash flows in period t it can be shown

$$a_t = \frac{(1+r_F)^t}{(1+r)^t}$$

It is important to stress that this formula holds only when <u>the CAPM assumptions hold</u>. Since for positive β, $r_F < r$, it follows that $(1 + r_F)/(1 + r) < 1$. Hence, as t increases, a_t gets smaller. Thus in a CAPM context by assuming a constant discount rate you are implicitly assuming future cash flows are more risky. To see this in more detail consider our motivation example.

Example 4

It will be assumed that $r_F = 5\%$ and $r_M - r_F = 8\%$ and Ford is considering a project with a risk of $\beta = 1.25$ so $r = 5\% + 1.25 \times 8\% = 15\%$. The cash flows of the project are -100, +50, +50, +50.

Period	Exp CF	PV = $\frac{C_t}{(1+r)^t}$	$a_t = \frac{(1+r_F)^t}{(1+r)^t}$	CEQ_t	PV = $\frac{CEQ_t}{(1+r_F)^t}$
0	-100	-100.00	1.000	-100.00	-100.00
1	+ 50	43.48	0.913	45.65	43.48
2	+ 50	37.81	0.834	41.68	37.81
3	+ 50	<u>32.88</u>	0.761	38.06	<u>32.88</u>
		NPV = 14.16			NPV = 14.16

The third column uses the CAPM risk adjusted rate of 15% so that it adjusts for risk and time simultaneously. The fourth, fifth and sixth columns use the certainty equivalent method, which in this case since the CAPM holds gives the same answer. The fourth column gives the

certainty equivalent adjustment factors. The fifth column gives the certainty equivalents of the cash flows and the sixth discounts them at the risk free rate.

It can be seen from the fifth column that even though the expected value of each cash flow in years 1-3 is 50 the certainty equivalents are 45.65, 41.68 and 38.06 respectively. Hence by using a constant discount rate you are effectively making a much larger adjustment for risk in the later cash flows.

Explanation of Motivation Example 2

If the project were something like an automobile plant as in our motivation example, then this sort of profile would perhaps be reasonable. In 1 year's time the determinants of auto purchase, are unlikely to be very different from now. If we're in a boom now it's likely we'll be in a boom then. However, in 2 years they may be somewhat different, and in 3 years they may be very different. Cash flows 3 years away are thus riskier than in 1 year's time since it is difficult to predict the direction and magnitude of change in factors affecting demand.

By assuming a constant β you are implicitly assuming that risk grows at a constant rate: the risk borne per period will be constant, but cumulative risk will grow steadily as you look further into the future. For many applications, such as the one we discussed, this is not too bad an assumption.

Situations where constant rate CAPM assumptions are not appropriate

If risk is not increasing at a constant rate, then it is not appropriate to discount in this way. For example, risk may grow at a faster rate than is implicit in the CAPM assumptions. In some cases it may grow at a slower rate than is implicit in the CAPM assumptions. In such cases

it is necessary to adapt the certainty equivalent method to adjust the discount rates to fit the appropriate risk profile.

An example where risk grows differently than in the CAPM would be if you do research before building the plant to produce the product. At the point where the outcome of the research becomes known considerable risk will be realized. After that point risk may grow at a more reasonable rate. Another example would be when you are introducing a new product and the demand is uncertain until you have actually sold it to a large number of people. In this case risk may again be high initially but then reduce to a more moderate rate afterwards.

Suppose in Motivation Example 2 there is an issue of how strong the demand will be for the product the plant will produce. The project manager feels that instead of 0.913, 0.834 and 0.761, certainty equivalent adjustment factors of 0.700, 0.639 and 0.584 are appropriate. The first number $a_1 = 0.7$ represents the risk from the product introduction. After that the risk changes by a factor of $1.05/1.15 = 0.913$ so that $a_2 = 0.7 \times 0.913 = 0.639$ and $a_3 = 0.639 \times 0.913 = 0.584$. In other words, risk is initially increasing at a faster rate than is implicit in the CAPM assumptions. What would be the NPV in that case?

Period	Exp CF	a_t	CEQ_t	$PV = \dfrac{CEQ_t}{(1+r_F)^t}$
0	-100	1.0	-100.00	-100.00
1	+ 50	0.700	35.00	33.33
2	+ 50	0.639	31.96	28.99
3	+ 50	0.584	29.18	25.20

$$NPV = -12.48$$

In this case the project is clearly not worth doing. The increased estimates of risk have made it undesirable.

The main thing to remember here is that discounting at a constant risk adjusted rate implies that uncertainty is increasing at a steady rate, and if this is not an appropriate assumption, the result you obtain may be very misleading. In such cases you can use the certainty equivalent method to adjust appropriately.

Table of Equity Premia
Annualized Equity Premiums for 17 Countries from 1900-2005

Country	Arithmetic Mean Relative to Bills (%)	Arithmetic Mean Relative to Bonds (%)
Australia	8.49	7.81
Belgium	4.99	4.37
Canada	5.88	5.67
Denmark	4.51	3.27
France	9.27	6.03
Germany*	9.07	8.35
Ireland	5.98	5.18
Italy	10.46	7.68
Japan	9.84	9.98
The Netherlands	6.61	5.95
Norway	5.70	5.26
South Africa	8.25	7.03
Spain	5.46	4.21
Sweden	7.98	7.51
Switzerland	5.29	3.28
UK	6.14	5.29
US	7.41	6.49
Average	**7.14**	**6.08**
World ex-U.S.	**5.93**	**5.18**
World	**6.07**	**5.15**

* Germany omits 1922-23

Source: Adapted from Table 3 in E. Dimson, P. Marsh and M. Staunton (2006): "The Worldwide Equity Premium: A Smaller Puzzle" (Reading for Section 7).

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