Project 2

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Problem 1

a)

- For the UCC to be an M/M/1 queue, we need these conditions to be fulfilled:
  The UCC must have an arrival rate $\lambda > 0$ and expected treatment time $\frac{1}{\mu} > 0$. Then we will have that

  1. Interarrival times are independent and identically distributed as $\text{Exp}(\lambda)$.
  2. Treatment times are independent and identically distributed as $\text{Exp}(\mu)$.
  3. One server, and treatment times are independent of the arrival process.

  Since the arrival of patients follows a Poisson distribution with rate $\lambda > 0$ we know that the waiting time between arrivals are exponentially distributed as $\text{Exp}(\lambda)$. The task also states that we are to assume that the treatment times follow an exponential distribution with expected value $\frac{1}{\mu} > 0$ and that the treatment times are independent of the arrival times. Together this constitutes the definition of a M/M/1 queue.

- A birth and death process refers to a Markov process with a discrete state space and the state transitions can occur only between neighboring states, i.e. $i \to i+1$ or $i \to i-1$. This applies to our system because we can only have one patient arrive/leave per transition. Hence the states can be enumerated with integers and we can only move to neighboring states. Since patients arrive with rate λ, this is the birth rate. There is only one server, this means the death rate is μ.

- In the case that $\lambda < \mu$ (like in our case), we have that $L = \frac{\lambda}{\mu - \lambda}$. We use Little's law ($L = \lambda W$) to arrive at $W = \frac{1}{\mu - \lambda}$, where W is the average time a patient will spend in the UCC.

b)

- Code can be found under $1b$) in the Rmd file. The function $sim1()$ simulates 50 days.

- The estimate is computed in the function $Estimate\_W$ in the code chunk below the simulation above. Estimated average time a patient will spend in the UCC is 66 minutes.

- To compute the CI, we will use

$$\mu \in \left(\overline{X} - z\frac{\sigma}{\sqrt{n}},\ \overline{X} + z\frac{\sigma}{\sqrt{n}} \right)$$

  with the mean and standard deviation from our sample. We obtain the interval $(55.93, 61.68)$.

- The realization of $\{X(t) : t \geq 0\}$ for the time 0-12 hours is displayed below.

- To estimate the time W, we started by adding up the time spent in each state and divided it by the total time. Now we have the pi-vector. From there we used that $L = \sum_{i}^{max(state)} i\pi_i$. From here we could use Little's law to arrive at our estimate. Our computed CI is already stated. The exact value computed in 1a) is $\frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{10}-\frac{1}{12}} = 60$, which we see is inside the CI.

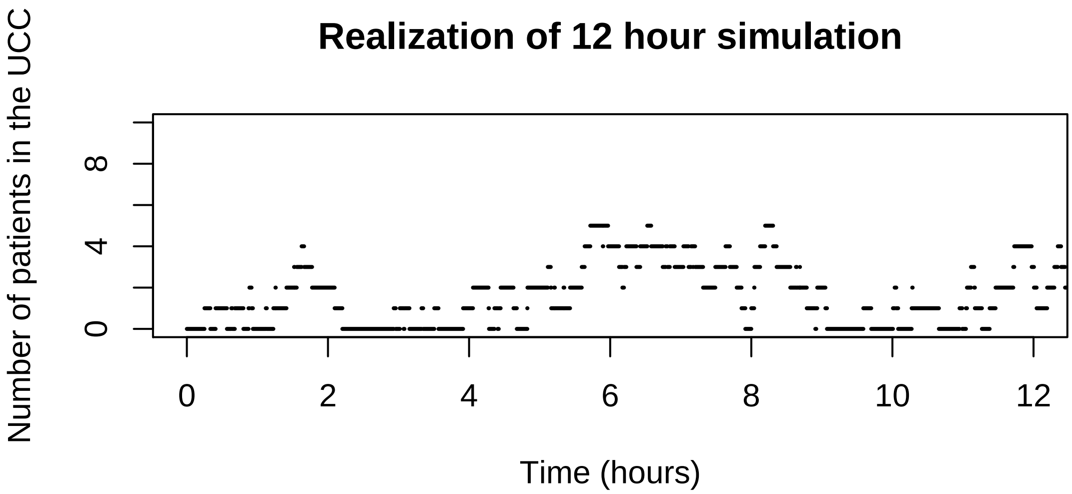

Figure 1: CAPTION

c)

- We have that $U(t) + N(t) = X(t)$. Then $\lambda_U + \lambda_N = \lambda_X$. Since urgent patients always will receive treatment before normal patients, we can consider the queue of urgent patients as independent of the normal patient. To urgent patients, the normal patients does not exist. Hence $U(t)$ is an $M/M/1$ queue for the same reasons as $X(t)$, but with another birth rate λ_U.
- The arrival rate for $U(t)$ must be the arrival rate of an arbitrary patient times the probability that the patient is urgent, i.e. $\lambda_U = p\lambda_X$.
- Since the treatment times are unchanged, we get that the long-run mean number of urgent patients in the UCC is given by $L_U = \frac{\lambda_U}{\mu-\lambda_U} = \frac{p\lambda}{\mu-p\lambda}$, where λ denotes the birth rate in $X(t)$.

d)

- In an M/M/1 queue, birth and death rates are constant functions of time. This is not the case for $N(t)$. While the birth rate is constant, the death rate is μ only when $U(t) = 0$. For $U(t) > 0$, the death rate for $N(t)$ will be zero. As a result. $N(t)$ is not an M/M/1 queue.
- We will use that $\rho_i = \frac{\lambda_i}{\mu}$. As explained in c), for the urgent patients, the normal ones does not exist, so L_U is just $\frac{\rho_U}{1-\rho_U} = \frac{p\lambda}{\mu-p\lambda}$. Even though the queue is prioritized, the total number of patients in the UCC does not change.

$$L_N + L_U = \frac{\rho_N + \rho_U}{1-\rho_N-\rho_U} \quad \Rightarrow \quad L_N = \frac{\rho_N + \rho_U}{1-\rho_N-\rho_U} - \frac{\rho_U}{1-\rho_U}$$

We insert $\frac{p\lambda}{\mu}$ for ρ_U and $\frac{(1-p)\lambda}{\mu}$ for ρ_N and obtain:

$$L_N = \frac{\frac{p\lambda+(1-p)\lambda}{\mu}}{1-\frac{p\lambda}{\mu}-\frac{(1-p)\lambda}{\mu}} - \frac{\frac{p\lambda}{\mu}}{1-\frac{p\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda} - \frac{p\lambda}{\mu-p\lambda} = \frac{\mu\lambda(1-p)}{(\mu-\lambda)(\mu-p\lambda)}$$

e)

- We have that $L_U = p\lambda W_U = \frac{p\lambda}{\mu-p\lambda} \quad \Rightarrow \quad W_U = \frac{1}{\mu-p\lambda}$.
- $L_N = (1-p)\lambda W_N = \frac{\mu\lambda(1-p)}{(\mu-\lambda)(\mu-p\lambda)} \quad \Rightarrow \quad W_N = \frac{\mu}{(\mu-\lambda)(\mu-p\lambda)}$

f)

- The plot of W_U and W_N is displayed below.
- In the cases where $p \approx 0$ or $p \approx 1$ the process will resemble $X(t)$, where all patients are equally prioritized. In the first case we will almost only have normal patients, and when a rare urgent patient arrives he/she will get treated at once, but this wont affect the queue too much since they are so rare. In the second case almost every patient is urgent. This means they can't skip to the front of the queue, they have to wait for their turn. In the rare occasion that a normal patient drops by, it is expected that he/she will have to wait six hours, because all the urgent patients are prioritized.
- We have derived the function $W_N(\lambda,\mu,p)$ earlier. By insertion we get $W_N(\lambda,\mu,p=0) = 60$ and that $W_N(\lambda,\mu,p=1) = 360$.
- We have that
$$W_N = \frac{\mu}{(\mu-\lambda)(\mu-p\lambda)} \quad \Rightarrow \quad p = \frac{\mu}{\lambda} - \frac{\mu}{\lambda(\mu-\lambda)W_N}$$
Inserting 120 minutes for W_N gives us $p = 0.6$, which we see corresponds with our plot.

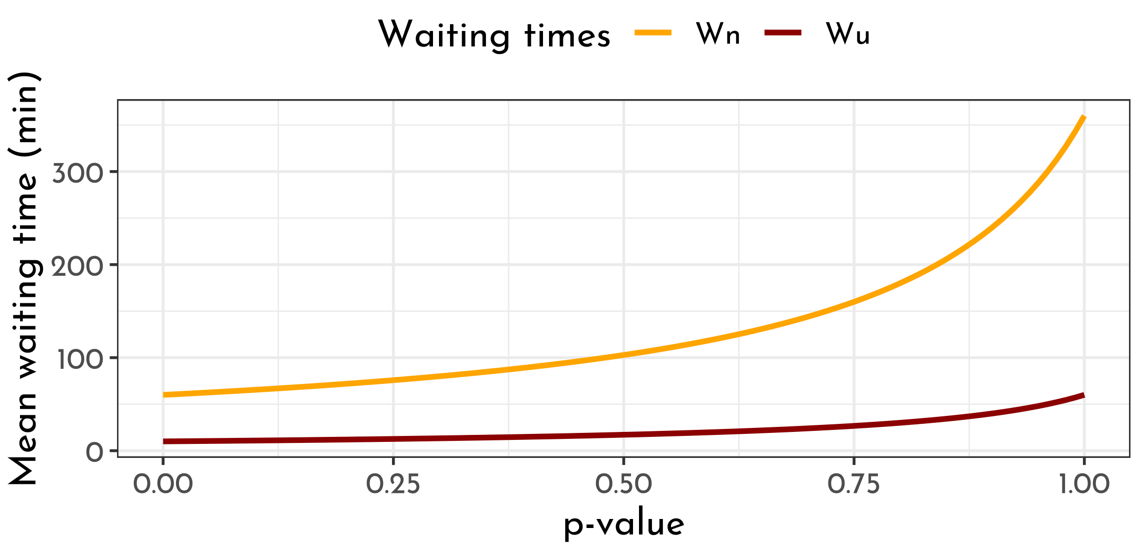

Figure 2: CAPTION

g)

- Code for simulation is in the Rmd file, the function $sim2()$.
- The estimates for W_U and W_N are 32.1381097 and 32.1381097 respectively. Code for both the estimates and the CIs are in the Rmd file below the $sim2()$ function. The computed 95
- Plot of a joint realization of $U(t)$ and $N(t)$ is displayed below.
- The estimates of W_U and W_N are computed in exactly the same way as W in task b), but we now had to look at $U(t)$ and $N(t)$ separately. The analytic solutions are $W_U(p) = 30min$ and $W_N(p) = 180min$ for $p = 0.8$. They are both within the CIs stated above.

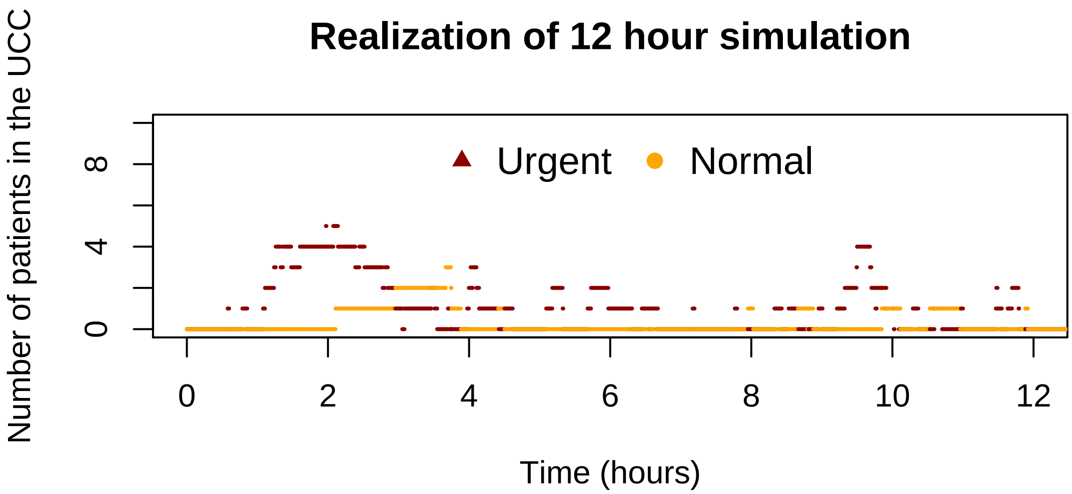

Figure 3: CAPTION

Problem 2

a)

b)

c)

Having had some formatting issues and problems with captions in figures, we decided to rather write a small paragraph here to explain what we are showing. Figure 4 and figure 5 belongs to task a). Figure 4 display the prediction of θ along with the 90% CI within the red lines. Figure 5 show the probability function that describes the probability that $Y(\theta) < 0.30$. Figure 6 and figure 7 show the same as figure 4 and 5 after adding the 6th point. By finding the $argmax$ of the probability function we know which θ-value gives the largest probability of obtaining $Y(t) < 0.30$. As a result, we would suggest for the scientists to use $\theta = 0.36$.

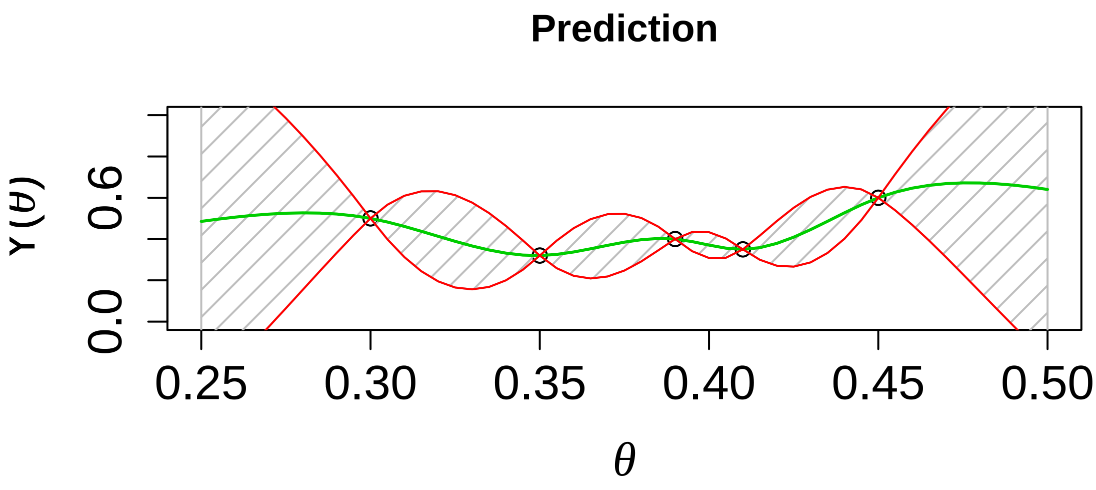

Figure 4: Probability of *heta* being less than 0.30.

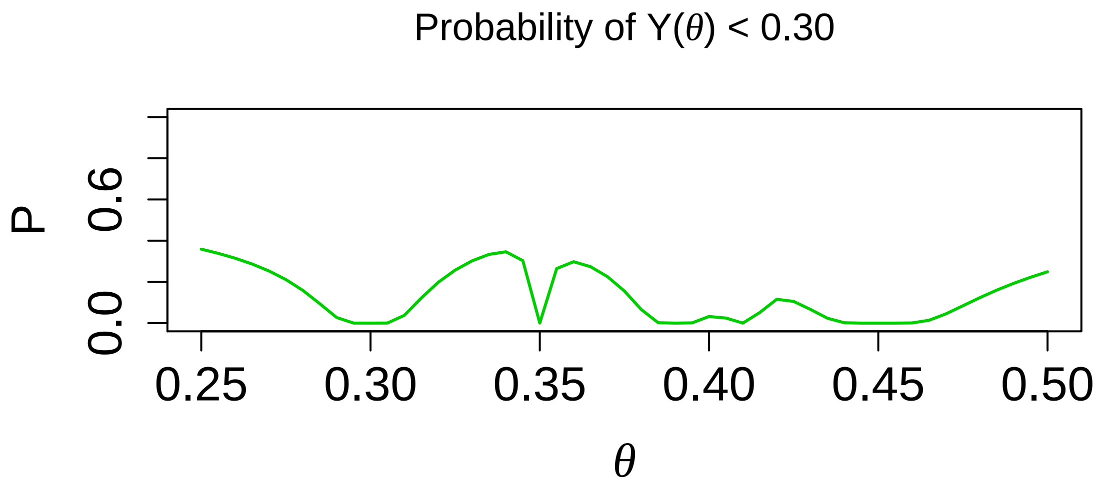

Figure 5: CAPTION

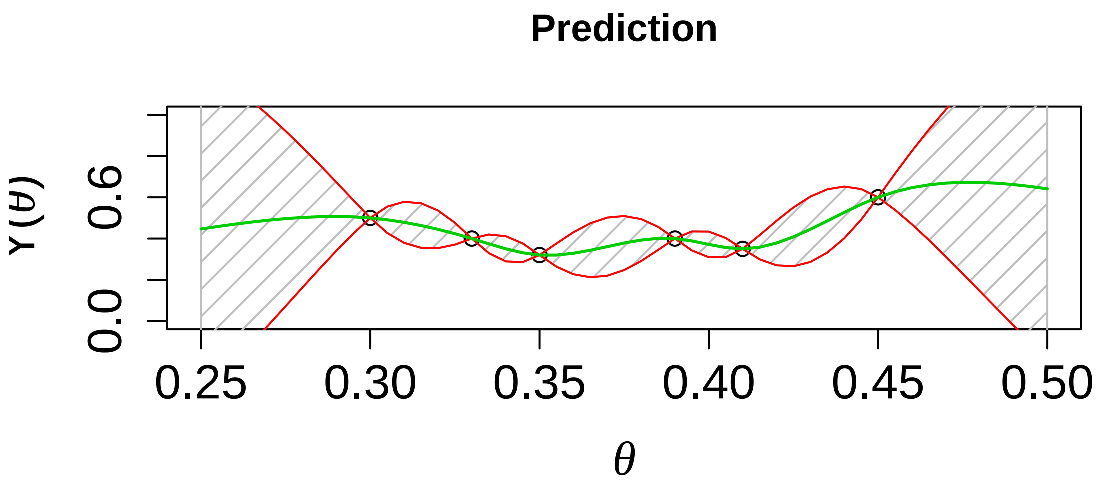

Figure 6: CAPTION

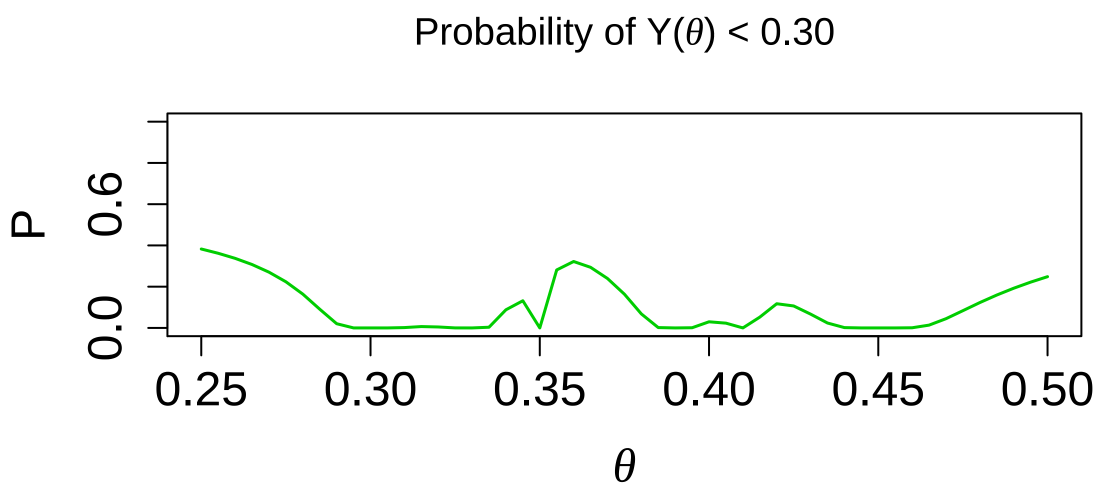

Figure 7: Probability of theta being less than 0.30. after adding additional point.